# FP3: Integru Pellach

## Haf 2006

(2) $$\int_0^{\frac{\pi}{2}} \frac{dx}{(1+3\cos x)}$$

$t = \tan\left(\frac{x}{2}\right)$

$\frac{dt}{dx} = \left(\frac{1}{2}\right)\sec^2\left(\frac{x}{2}\right)$

$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$

$\frac{2\,dt}{\sec^2\left(\frac{x}{2}\right)} = dx$

$2\cos^2\left(\frac{x}{2}\right)dt = dx$

Nawr $\cos 2x = 2\cos^2 x - 1$

Felly $\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$

Felly $(\cos x + 1)dt = dx$

Terfannau:

$[0]$ $t = \tan\left(\frac{0}{2}\right)$

$t = 0$

$\left[\frac{\pi}{2}\right]$ $t = \tan\left(\frac{\frac{\pi}{2}}{2}\right)$

$t = 1$

$\cos t = \frac{1-t^2}{1+t^2}$

$$\int_0^1 \frac{(\cos x + 1)dt}{(1+3\cos x)}$$

$$= \int_0^1 \frac{\frac{1-t^2}{1+t^2} + 1}{1 + 3\left(\frac{1-t^2}{1+t^2}\right)}\,dt$$

$$= \int_0^1 \frac{1-t^2+(1+t^2)}{(1+t^2)+3(1-t^2)}\,dt$$

$$= \int_0^1 \frac{2}{1+t^2+3-3t^2}\,dt$$

$$= \int_0^1 \frac{2}{4-2t^2}\,dt$$

$$= \int_0^1 \frac{1}{2-t^2}\,dt$$

$$= \int_0^1 \frac{1}{\sqrt{2}^2 - t^2}\,dt$$

$$= \left[\frac{1}{\sqrt{2}}\tanh^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1 + K$$

$$= \frac{1}{\sqrt{2}}\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\tanh^{-1}\left(\frac{0}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$\doteq 0.6232252401$

$\doteq 0.623$ i 3 lle degol.

## Haf 2007

(1) a) $\int_0^1 \frac{dx}{\sqrt{x^2+2x+5}}$

$x = 2\sinh\theta - 1$

$\frac{dx}{d\theta} = 2\cosh\theta$

$dx = d\theta(2\cosh\theta)$

Terfannau:

[0] $0 = 2\sinh\theta - 1$

$1 = 2\sinh\theta$

$\sinh\theta = \frac{1}{2}$

$\theta = \sinh^{-1}(\frac{1}{2})$

[1] $1 = 2\sinh\theta - 1$

$2 = 2\sinh\theta$

$\sinh\theta = 1$

$\theta = \sinh^{-1}(1)$

$$\begin{aligned} x^2+2x+5 &= (2\sinh\theta - 1)^2 + 2(2\sinh\theta - 1) + 5 \\ &= 4\sinh^2\theta - 4\sinh\theta + 1 + 4\sinh\theta - 2 + 5 \\ &= 4\sinh^2\theta + 4 \\ &= 4(\sinh^2\theta + 1) \\ &= 4\cosh^2\theta \end{aligned}$$

$$\int_{\sinh^{-1}(\frac{1}{2})}^{\sinh^{-1}(1)} \frac{(2\cosh\theta)d\theta}{\sqrt{4\cosh^2\theta}}$$

$$= \int_{\sinh^{-1}(\frac{1}{2})}^{\sinh^{-1}(1)} \frac{2\cosh\theta}{2\cosh\theta}\, d\theta$$

$$= \int_{\sinh^{-1}(\frac{1}{2})}^{\sinh^{-1}(1)} 1\, d\theta$$

$$= \Big[\theta\Big]_{\sinh^{-1}(\frac{1}{2})}^{\sinh^{-1}(1)}$$

$= \sinh^{-1}(1) - \sinh^{-1}(\frac{1}{2})$

$= 0.400161762$

$= 0.400$ ; 3 lle degol.

Haf 2008

(2) $\int_1^2 \sqrt{x^2-2x+2}\, dx.$

$x = 1 + \sinh\theta$

$\frac{dx}{d\theta} = \cosh\theta$

$dx = \cosh\theta\, d\theta$

Terfannau:

[1] $1 = 1 + \sinh\theta$

$\sinh\theta = 0$

$\theta = \sinh^{-1}(0)$

$\theta = 0$

[2] $2 = 1 + \sinh\theta$

$\sinh\theta = 1$

$\theta = \sinh^{-1}(1)$

$$\begin{aligned} x^2-2x+2 &= (1+\sinh\theta)^2 - 2(1+\sinh\theta) + 2 \\ &= 1 + \cancel{2\sinh\theta} + \sinh^2\theta \cancel{-2} \cancel{-2\sinh\theta} \cancel{+2} \\ &= \sinh^2\theta + 1 \\ &= \cosh^2\theta. \end{aligned}$$

$$\int_0^{\sinh^{-1}(1)} \sqrt{\cosh^2\theta}\ \cosh\theta\, d\theta$$

$$= \int_0^{\sinh^{-1}(1)} \cosh\theta(\cosh\theta)\, d\theta$$

$$= \int_0^{\sinh^{-1}(1)} \cosh^2\theta\, d\theta$$

$\cosh 2\theta = 2\cosh^2\theta - 1$

$\frac{\cosh 2\theta + 1}{2} = \cosh^2\theta$

$$= \int_0^{\sinh^{-1}(1)} \frac{\cosh 2\theta + 1}{2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\sinh^{-1}(1)} \cosh 2\theta + 1\, d\theta$$

$$= \frac{1}{2}\left[\frac{\sinh 2\theta}{2} + \theta\right]_0^{\sinh^{-1}(1)}$$

$$= \frac{1}{2}\left[\left(\frac{\sinh(2\times\sinh^{-1}(1))}{2} + \sinh^{-1}(1)\right) - \left(\frac{\sinh(0)}{2} + 0\right)\right]$$

$$= \tfrac{1}{2}[\,2.295587149 - 0\,]$$

$$= 1.147793575$$

$= 1.15$ i 2 le degol.

Haf 2009

(3) $$\int_0^2 \frac{dx}{(x^2+4)^{\frac{3}{2}}}$$

$x = 2\sinh u$

$\frac{dx}{du} = 2\cosh u$

$dx = 2\cosh u\, du$

Terfannau:

[0] $0 = 2\sinh u$

$\sinh u = 0$

$u = \sinh^{-1}(0)$

$u = 0$

[2] $2 = 2\sinh u$

$\sinh u = 1$

$u = \sinh^{-1}(1)$

$$\begin{aligned} x^2+4 &= (2\sinh u)^2 + 4 \\ &= 4\sinh^2 u + 4 \\ &= 4(\sinh^2 u + 1) \\ &= 4\cosh^2 u. \end{aligned}$$

$$\int_0^{\sinh^{-1}(1)} \frac{2\cosh u\, du}{(4\cosh^2 u)^{\frac{3}{2}}}$$

$$= \int_0^{\sinh^{-1}(1)} \frac{2\cosh u}{8\cosh^3 u} \cdot du$$

$$= \frac{1}{4}\int_0^{\sinh^{-1}(1)} \frac{1}{\cosh^2 u}\, du$$

$$= \frac{1}{4}\int_0^{\sinh^{-1}(1)} \operatorname{sech}^2 u\, du$$

$$= \frac{1}{4}\Big[\tanh u\Big]_0^{\sinh^{-1}(1)}$$

$$= \frac{1}{4}\Big[\tanh(\sinh^{-1}(1)) - \tanh(0)\Big]$$

$$= \tfrac{1}{4} \times \tanh(\sinh^{-1}(1))$$

$$= \tfrac{1}{4} \times 0.7071067812$$

$$= 0.1767766953$$

$= 0.18$ i 2 le degol

## Haf 2010

(2) $\int_0^3 \frac{x^2}{\sqrt{x^2+1}}\,dx$

$x = \sinh u$

$\frac{dx}{du} = \cosh u$

$dx = \cosh u\, du$

Terfannau:

$[0]$ $0 = \sinh u$, $u = \sinh^{-1}(0)$, $u = 0$

$[3]$ $3 = \sinh u$, $u = \sinh^{-1}(3)$

$$x^2+1 = \sinh^2 u + 1 = \cosh^2 u.$$

$$\int_0^{\sinh^{-1}(3)} \frac{\sinh^2 u}{\sqrt{\cosh^2 u}} \times \cosh u\, du$$

$$= \int_0^{\sinh^{-1}(3)} \frac{\sinh^2 u}{\cancel{\cosh u}} \times \cancel{\cosh u}\, du$$

$$= \int_0^{\sinh^{-1}(3)} \sinh^2 u\, du$$

$$\cosh 2\theta = 2\sin^2\theta + 1$$
$$\frac{\cosh 2\theta - 1}{2} = \sin^2\theta$$

$$= \int_0^{\sinh^{-1}(3)} \frac{\cosh 2u - 1}{2}\, du$$

$$= \frac{1}{2}\int_0^{\sinh^{-1}(3)} \cosh 2u - 1$$

$$= \frac{1}{2}\left[\frac{\sinh 2u}{2} - u\right]_0^{\sinh^{-1}(3)}$$

$$= \frac{1}{2}\left[\left(\frac{\sinh(2\times\sinh^{-1}(3))}{2} - \sinh^{-1}(3)\right) - \left(\frac{\sinh(0)}{2} - 0\right)\right]$$

$$= \frac{1}{2}\left(\frac{\sinh(2\times\sinh^{-1}(3))}{2} - \sinh^{-1}(3)\right)$$

$$= \tfrac{1}{2} \times 7.668386521$$

$$= 3.834193261$$

$= 3.83$ i 2 le degol

# Haf 2011

② $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}\,dx$

$t = \tan\left(\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

$2\,dt = \sec^2\left(\frac{x}{2}\right)dx$

$\frac{2\,dt}{\sec^2\left(\frac{x}{2}\right)} = dx$

$2\cos^2\left(\frac{x}{2}\right)dt = dx$

Terfannau:

$[0]$ $t = \tan\left(\frac{0}{2}\right)$

$t = 0$

$\left[\frac{\pi}{2}\right]$ $t = \tan\left(\frac{\pi}{2}\over 2\right)$

$t = 1$

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

Nawr $\cos 2\theta = 2\cos^2\theta - 1$

$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

Felly $(\cos\theta + 1)\,dt = dx$

$\left(\frac{1-t^2}{1+t^2}+1\right)dt = dx$

$\left(\frac{1-t^2}{1+t^2}+\frac{1+t^2}{1+t^2}\right)dt = dx$

$\left(\frac{1-t^2+1+t^2}{1+t^2}\right)dt = dx$

$\frac{2\,dt}{1+t^2} = dx$

$$\int_0^1 \frac{1}{2+\frac{2t}{1+t^2}}\,\frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{\frac{2}{1+t^2}}{2\left(\frac{1+t^2}{1+t^2}\right)+\frac{2t}{1+t^2}}\,dt$$

$$= \int_0^1 \frac{\frac{2}{1+t^2}}{\frac{2(1+t^2)+2t}{1+t^2}}\,dt$$

$$= \int_0^1 \frac{2}{\cancel{1+t^2}} \times \frac{\cancel{(1+t^2)}}{2(1+t^2)+2t}\,dt$$

$$= \int_0^1 \frac{\cancel{2}}{\cancel{2}(1+t^2+t)}\,dt$$

$$= \int_0^1 \frac{1}{t^2+t+1}\,dt$$

Cwblhau'r sgwâr:

$$= \int_0^1 \frac{1}{(t+\frac{1}{2})^2-\frac{1}{4}+1}\,dt$$

$$= \int_0^1 \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}\,dt$$

$$= \left[\frac{1}{\sqrt{\frac{3}{4}}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)\right]_0^1$$

$$= \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\frac{3}{2}}{\sqrt{\frac{3}{4}}}\right) - \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)$$

$$= \frac{2\sqrt{3}}{3}\left(\frac{\pi}{3}\right) - \frac{2\sqrt{3}}{3}\left(\frac{\pi}{6}\right)$$

$$= \frac{2\sqrt{3}}{3}\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi\sqrt{3}}{9}$$

$$= \frac{\pi\sqrt{3}\times\sqrt{3}}{9\sqrt{3}}$$

$$= \frac{3\pi}{9\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}} \checkmark$$

Haf 2012

(5) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{4\cos x+3}\,dx$

$t = \tan\left(\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{\sec^2\left(\frac{x}{2}\right)}{2}$

$2dt \quad \sec^2\left(\frac{x}{2}\right)dx$

$\frac{2\,dt}{\sec^2\left(\frac{x}{2}\right)} = dx$

$2\cos^2\left(\frac{x}{2}\right)dt = dx$

Terfannau:

$[0]$ $t = \tan\left(\frac{0}{2}\right)$

$t = 0$

$[\frac{\pi}{2}]$ $t = \tan\left(\frac{\frac{\pi}{2}}{2}\right)$

$t = 1$

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

Nawr $\cos 2\theta = 2\cos^2\theta - 1$

$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

Felly $(\cos\theta + 1)\,dt = dx$

$\left(\frac{1-t^2}{1+t^2} + 1\right)dt = dx$

$\left(\frac{1-t^2}{1+t^2} + \frac{1+t^2}{1+t^2}\right)dt = dx$

$\left(\frac{1-t^2+1+t^2}{1+t^2}\right)dt = dx$

$\frac{2\,dt}{1+t^2} = dx.$

$$\int_0^1 \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right)+3}\;\frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{\frac{2}{1+t^2}}{4\left(\frac{1-t^2}{1+t^2}\right)+3\left(\frac{1+t^2}{1+t^2}\right)}\,dt$$

$$= \int_0^1 \frac{\frac{2}{1+t^2}}{\frac{4(1-t^2)+3(1+t^2)}{1+t^2}}\,dt$$

$$= \int_0^1 \frac{2}{\cancel{(1+t^2)}} \times \frac{\cancel{(1+t^2)}}{4(1-t^2)+3(1+t^2)}\,dt$$

$$= \int_0^1 \frac{2}{4-4t^2+3+3t^2}\,dt$$

$$= \int_0^1 \frac{2}{7-t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{\sqrt{7}^2 - t^2}\,dt$$

$$= 2\left[\frac{1}{\sqrt{7}}\tanh^{-1}\left(\frac{t}{\sqrt{7}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{7}}\left[\tanh^{-1}\left(\frac{1}{\sqrt{7}}\right) - \tanh^{-1}\left(\frac{0}{\sqrt{7}}\right)\right]$$

$$= \frac{2}{\sqrt{7}}\tanh^{-1}\left(\frac{1}{\sqrt{7}}\right)$$

$= 0.3006198874$

$= 0.301$ i 3 ffigur ystyrlon

## Haf 2013

④ $\int_1^2 \sqrt{3+2x-x^2}\,dx$

Cwblhau'r sgwâr: $3+2x-x^2$

$= 3-(x^2-2x)$

$= 3-((x-1)^2-1)$

$= 3-(x-1)^2+1$

$= 4-(x-1)^2$

$\int_1^2 \sqrt{4-(x-1)^2}\,dx$

Dewis $x-1 = 2\sin u$

$x = 1+2\sin u$

$\frac{dx}{du} = 2\cos u$

$dx = 2\cos u\,du$

Terfannau:

[1] $1-1 = 2\sin u$

$0 = 2\sin u$

$u = \sin^{-1}(0)$

$u = 0$

[2] $2-1 = 2\sin u$

$\frac{1}{2} = \sin u$

$u = \sin^{-1}(\frac{1}{2})$

$u = \frac{\pi}{6}$

$\int_0^{\frac{\pi}{6}} \sqrt{4-(2\sin u)^2}\, 2\cos u\,du$

$= \int_0^{\frac{\pi}{6}} \sqrt{4-4\sin^2 u}\; 2\cos u\,du$

$= \int_0^{\frac{\pi}{6}} \sqrt{4(1-\sin^2 u)}\; 2\cos u\,du$

$= \int_0^{\frac{\pi}{6}} \sqrt{4\cos^2 u}\; 2\cos u\,du$

$= \int_0^{\frac{\pi}{6}} 2\cos u(2\cos u)\,du$

$= 4\int_0^{\frac{\pi}{6}} \cos^2 u\,du$

$\cos 2x = 2\cos^2 x - 1$

$\frac{\cos 2x + 1}{2} = \cos^2 x$

$= 4\int_0^{\frac{\pi}{6}} \frac{\cos 2u+1}{2}\,du$

$= 2\int_0^{\frac{\pi}{6}} \cos 2u + 1\,du$

$= 2\left[\frac{1}{2}\sin 2u + u\right]_0^{\frac{\pi}{6}}$

$= 2\left[\left(\frac{1}{2}\sin\left(\frac{\pi}{3}\right)+\frac{\pi}{6}\right) - \left(\frac{1}{2}\sin(0)+0\right)\right]$

$= 2\left[\frac{1}{2}\times\frac{\sqrt{3}}{2} + \frac{\pi}{6} - 0\right]$

$= \frac{\sqrt{3}}{2} + \frac{\pi}{3}$

$= 1.91$ i 3 ffigur ystyrlon.

FP3 Haf 2014

(4) $\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x}\,dx$

$t = \tan\left(\frac{x}{2}\right)$

$\frac{dt}{dx} = \left(\frac{1}{2}\right)\sec^2\left(\frac{x}{2}\right)$

$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$

$\frac{2\,dt}{\sec^2\left(\frac{x}{2}\right)} = dx$

$2\cos^2\left(\frac{x}{2}\right)dt = dx$

Nawr $\cos 2x = 2\cos^2 x - 1$

Felly $\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$

Felly $(\cos x + 1) = 2\cos^2\left(\frac{x}{2}\right)$

Felly $(\cos x + 1)dt = dx$

Terfannau:

$[0]$ $t = \tan\left(\frac{0}{2}\right)$

$t = 0$

$\left[\frac{\pi}{2}\right]$ $t = \tan\left(\frac{\frac{\pi}{2}}{2}\right)$

$t = 1$

$\cos t = \frac{1-t^2}{1+t^2}$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x}\,dx$$

$$= \int_0^1 \frac{(\cos x + 1)\,dt}{2 - \cos x}$$

$$= \int_0^1 \frac{\frac{1-t^2}{1+t^2} + 1}{2 - \frac{1-t^2}{1+t^2}}\,dt$$

$$= \int_0^1 \frac{1-t^2 + 1(1+t^2)}{2(1+t^2) - (1-t^2)}\,dt$$

$$= \int_0^1 \frac{1 - t^2 + 1 + t^2}{2 + 2t^2 - 1 + t^2}\,dt$$

$$= \int_0^1 \frac{2}{1 + 3t^2}\,dt$$

$$= \int_0^1 \frac{\frac{2}{3}}{\frac{1}{3} + t^2}\,dt$$

$$= \frac{2}{3}\int_0^1 \frac{1}{\frac{1}{3} + t^2}\,dt$$

$$= \frac{2}{3}\int_0^1 \frac{1}{\left(\sqrt{\frac{1}{3}}\right)^2 + t^2}\,dt$$

$$= \frac{2}{3}\left[\frac{1}{\sqrt{\frac{1}{3}}}\tan^{-1}\left(\frac{t}{\sqrt{\frac{1}{3}}}\right)\right]_0^1$$

$$= \frac{2}{3\sqrt{\frac{1}{3}}}\left[\tan^{-1}\left(\frac{1}{\sqrt{\frac{1}{3}}}\right) - \tan^{-1}\left(\frac{0}{\sqrt{\frac{1}{3}}}\right)\right]$$

$$= \frac{2}{3\sqrt{\frac{1}{3}}}\left[\frac{\pi}{3} - 0\right]$$

$$= \frac{2\sqrt{3}}{9}\pi$$

$(= 1.209199576)$

FP3 Haf 2015

(2) $\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx$ $\qquad u = e^{2x}$ $\quad \frac{dv}{dx} = \cos x$

$\qquad \frac{du}{dx} = 2e^{2x}$ $\quad v = \sin x$

$$\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx = \left[e^{2x}\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2e^{2x}\sin x\,dx$$

$$= \left[e^{2(\frac{\pi}{2})}\sin(\tfrac{\pi}{2}) - e^0\sin(0)\right] - \int_0^{\frac{\pi}{2}} 2e^{2x}\sin x\,dx$$

$$= e^{\pi} - 2\boxed{\int_0^{\frac{\pi}{2}} e^{2x}\sin x\,dx} \rightarrow \quad u = e^{2x} \quad \frac{dv}{dx} = \sin x \quad \frac{du}{dx} = 2e^{2x} \quad v = -\cos x$$

$$= e^{\pi} - 2\left\{\left[e^{2x}(-\cos x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2e^{2x}(-\cos x)\,dx\right\}$$

$$= e^{\pi} - 2\left\{\left[e^{2(\frac{\pi}{2})}(-\cos(\tfrac{\pi}{2})) - e^0(-\cos 0)\right] + \int_0^{\frac{\pi}{2}} 2e^{2x}\cos x\,dx\right\}$$

$$= e^{\pi} - 2\left\{\left[0 + 1\right] + 2\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx\right\}$$

$$= e^{\pi} - 2 - 4\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx.$$

Felly $\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx = e^{\pi} - 2 - 4\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx$

$$5\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx = e^{\pi} - 2$$

$$\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx = \frac{e^{\pi} - 2}{5}$$

(Felly $\underline{a = 1}$, $\underline{b = -2}$)

FP3 Haf 2016

(3) $$\int_0^{\frac{\pi}{2}} \frac{dx}{3+5\cos x}$$

$$t = \tan\left(\frac{x}{2}\right)$$

$$\frac{dt}{dx} = \frac{\sec^2\left(\frac{x}{2}\right)}{2}$$

$$2dt = \sec^2\left(\frac{x}{2}\right)dx$$

$$\frac{2dt}{\sec^2\left(\frac{x}{2}\right)} = dx$$

$$2\cos^2\left(\frac{x}{2}\right)dt = dx$$

Terfannau:

$[0]$ $t = \tan\left(\frac{0}{2}\right)$

$t = 0$

$\left[\frac{\pi}{2}\right]$ $t = \tan\left(\frac{\frac{\pi}{2}}{2}\right)$

$t = 1$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Nawr $\cos 2\theta = 2\cos^2\theta - 1$

$$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

Felly $(\cos\theta + 1)dt = dx$

$$\left(\frac{1-t^2}{1+t^2} + 1\right)dt = dx$$

$$\left(\frac{1-t^2}{1+t^2} + \frac{1+t^2}{1+t^2}\right)dt = dx$$

$$\left(\frac{1-t^2+1+t^2}{1+t^2}\right)dt = dx$$

$$\frac{2\,dt}{1+t^2} = dx$$

$$\int_0^1 \frac{1}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2\,dt}{1+t^2}$$

$$= \int_0^1 \frac{\frac{2}{1+t^2}}{3\left(\frac{1+t^2}{1+t^2}\right)+5\left(\frac{1-t^2}{1+t^2}\right)}\,dt$$

$$= \int_0^1 \frac{\frac{2}{1+t^2}}{\frac{3(1+t^2)+5(1-t^2)}{(1+t^2)}}\,dt$$

$$= \int_0^1 \frac{2}{\cancel{1+t^2}} \times \frac{\cancel{1+t^2}}{3(1+t^2)+5(1-t^2)}\,dt$$

$$= \int_0^1 \frac{2}{3+3t^2+5-5t^2}\,dt$$

$$= \int_0^1 \frac{2}{8-2t^2}\,dt$$

$$= \int_0^1 \frac{1}{4-t^2}\,dt$$

$$= \left[\frac{1}{2\times2}\ln\left|\frac{2+t}{2-t}\right|\right]_0^1$$

$$= \frac{1}{4}\ln\left|\frac{2+1}{2-1}\right| - \frac{1}{4}\ln\left|\frac{2+0}{2-0}\right|$$

$$= \frac{1}{4}\ln(3) - \frac{1}{4}\ln(1)$$

$$= \frac{1}{4}\ln(3) - 0$$

$$= \frac{1}{4}\ln(3)$$

$$= \ln\left(3^{\frac{1}{4}}\right)$$

(Felly $a = \frac{1}{4}$.)

FP3 Haf 2017

2) $\int_0^{\frac{\pi}{2}} \frac{2}{1+\sin x + 2\cos x}\, dx$

$t = \tan\left(\frac{x}{2}\right)$

$\frac{dt}{dx} = \left(\frac{1}{2}\right)\sec^2\left(\frac{x}{2}\right)$

$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$

$2dt = \sec^2\left(\frac{x}{2}\right)dx$

$\frac{2\,dt}{\sec^2\left(\frac{x}{2}\right)} = dx$

$2\cos^2\left(\frac{x}{2}\right)dt = dx$

Terfanau:

$[0]$ $t = \tan\left(\frac{0}{2}\right)$

$t = 0$

$\left[\frac{\pi}{2}\right]$ $t = \tan\left(\frac{\frac{\pi}{2}}{2}\right)$

$t = 1$

$\cos x = \frac{1-t^2}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

Nawr $\cos 2\theta = 2\cos^2\theta - 1$

$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

Felly $(\cos\theta + 1) = 2\cos^2\left(\frac{\theta}{2}\right)$

Felly $(\cos x + 1)\,dt = dx$

$\left(\frac{1-t^2}{1+t^2} + 1\right)dt = dx$

$\left(\frac{1-t^2}{1+t^2} + \frac{1+t^2}{1+t^2}\right)dt = dx$

$\left(\frac{1-t^2+1+t^2}{1+t^2}\right)dt = dx$

$\frac{2dt}{1+t^2} = dx$

Felly'r integryn yw

$$\int_0^1 \frac{2}{1+\frac{2t}{1+t^2}+2\left(\frac{1-t^2}{1+t^2}\right)}\left(\frac{2}{1+t^2}\right)dt$$

$$= \int_0^1 \frac{2}{\frac{1+t^2}{1+t^2}+\frac{2t}{1+t^2}+\frac{2(1-t^2)}{1+t^2}}\left(\frac{2}{1+t^2}\right)dt$$

$$= \int_0^1 \frac{2}{\frac{1+t^2+2t+2-2t^2}{1+t^2}}\left(\frac{2}{1+t^2}\right)dt$$

$$= \int_0^1 \frac{2\cancel{(1+t^2)}}{3+2t-t^2}\left(\frac{2}{\cancel{1+t^2}}\right)dt$$

$$= \int_0^1 \frac{4}{3+2t-t^2}\,dt$$

$$= -4\int_0^1 \frac{1}{t^2-2t-3}\,dt$$

$$= -4\int_0^1 \frac{1}{(t-1)^2-1-3}\,dt$$

$$= -4\int_0^1 \frac{1}{(t-1)^2-4}\,dt$$

$$= 4\int_0^1 \frac{1}{4-(t-1)^2}\,dt \qquad \left(\text{er mwyn defnyddio } \int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)\right)$$

$$= 4\left[\frac{1}{2}\tanh^{-1}\left(\frac{t-1}{2}\right)\right]_0^1$$

$$= 2\left[\tanh^{-1}\left(\frac{t-1}{2}\right)\right]_0^1$$

$$= 2\left[\tanh^{-1}\left(\frac{1-1}{2}\right)-\tanh^{-1}\left(\frac{0-1}{2}\right)\right]$$

$$= 2\left(\tanh^{-1}(0)-\tanh^{-1}\left(-\tfrac{1}{2}\right)\right)$$

$$= 2(0--0.5493061443)$$

$$= 1.098612289$$

$$= \ln 3.$$

(Felly $N=3$.)

# FP3 Haf 2018

3) $\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx$

$u = e^{2x}$ $\quad \frac{dv}{dx} = \sin(3x)$

$\frac{du}{dx} = 2e^{2x}$ $\quad v = -\frac{1}{3}\cos(3x)$

$$\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx = \left[e^{2x}\left(-\frac{1}{3}\cos(3x)\right)\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} 2e^{2x}\left(-\frac{1}{3}\cos(3x)\right)dx$$

$$= \left[e^{\frac{2\pi}{3}}\left(-\frac{1}{3}\cos\left(\frac{3\pi}{3}\right)\right) - e^0\left(-\frac{1}{3}\cos(0)\right)\right] + \frac{2}{3}\int_0^{\frac{\pi}{3}} e^{2x}\cos(3x)\,dx$$

$$= e^{\frac{2\pi}{3}}\left(-\frac{1}{3}(-1)\right) - e^0\left(-\frac{1}{3}(1)\right) + \frac{2}{3}\int_0^{\frac{\pi}{3}} e^{2x}\cos(3x)\,dx$$

$$= \frac{1}{3}e^{\frac{2\pi}{3}} + \frac{1}{3} + \frac{2}{3}\boxed{\int_0^{\frac{\pi}{3}} e^{2x}\cos(3x)\,dx}$$

$u = e^{2x}$ $\quad \frac{dv}{dx} = \cos(3x)$

$\frac{du}{dx} = 2e^{2x}$ $\quad v = \frac{1}{3}\sin(3x)$.

Felly $\int_0^{\frac{\pi}{3}} e^{2x}\cos(3x)\,dx = \left[e^{2x}\left(\frac{1}{3}\sin(3x)\right)\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} 2e^{2x}\left(\frac{1}{3}\sin(3x)\right)dx$

$$= \left[e^{\frac{2\pi}{3}}\left(\frac{1}{3}\sin\left(\frac{3\pi}{3}\right)\right) - e^0\left(\frac{1}{3}\sin(0)\right)\right] - \frac{2}{3}\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx$$

$$= [0 - 0] - \frac{2}{3}\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx$$

$$= -\frac{2}{3}\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx.$$

Felly

$$\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx = \frac{1}{3}e^{\frac{2\pi}{3}} + \frac{1}{3} + \frac{2}{3}\left[-\frac{2}{3}\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx\right]$$

$$\left(1+\frac{4}{9}\right)\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx = \frac{1}{3}e^{\frac{2\pi}{3}} + \frac{1}{3}$$

$$\frac{13}{9}\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx = \frac{1}{3}e^{\frac{2\pi}{3}} + \frac{1}{3}$$

$$\frac{13}{3}\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx = e^{\frac{2\pi}{3}} + 1$$

$$\int_0^{\frac{\pi}{3}} e^{2x}\sin(3x)\,dx = \frac{3}{13}\left(1 + e^{\frac{2\pi}{3}}\right)$$

(Felly $a = \frac{3}{13}$, $b = \frac{2}{3}$.)

FP3 Haf 2019

3) $$\int_0^1 \frac{x^2}{(x^2+1)^{\frac{3}{2}}}\,dx$$

$x = \sinh\theta$

$\frac{dx}{d\theta} = \cosh\theta$

$dx = \cosh\theta\, d\theta$

Terfannau:

[0] $0 = \sinh\theta$, $\theta = \sinh^{-1}(0)$, $\theta = 0$

[1] $1 = \sinh\theta$, $\theta = \sinh^{-1}(1)$

$$\begin{aligned} x^2+1 &= (\sinh\theta)^2 + 1 \\ &= \sinh^2\theta + 1 \\ &= \cosh^2\theta \end{aligned}$$

$$\int_0^{\sinh^{-1}(1)} \frac{\sinh^2\theta}{(\cosh^2\theta)^{\frac{3}{2}}} \cosh\theta\, d\theta$$

$$= \int_0^{\sinh^{-1}(1)} \frac{\sinh^2\theta}{\cosh^3\theta} \cosh\theta\, d\theta$$

$$= \int_0^{\sinh^{-1}(1)} \frac{\sinh^2\theta}{\cosh^2\theta}\, d\theta$$

$$= \int_0^{\sinh^{-1}(1)} \tanh^2\theta\, d\theta$$

$$= \int_0^{\sinh^{-1}(1)} (1-\operatorname{sech}^2\theta)\, d\theta$$

$$= \left[\theta - \tanh\theta\right]_0^{\sinh^{-1}(1)}$$

$$= (\sinh^{-1}(1) - \tanh(\sinh^{-1}(1))) - (0 - \tanh(0))$$

$$= \sinh^{-1}(1) - \tanh(\sinh^{-1}(1)) - 0$$

$$= 0.1742668058$$

$= \underline{0.174}$ i 3 ffigur ystyrlon

# M3 Haf 2019

5) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}\,dx$

$t = \tan\left(\frac{x}{2}\right)$

Terfannau:

$[0]\quad t = \tan\left(\frac{0}{2}\right)$

$t = 0$

$\left[\frac{\pi}{2}\right]\quad t = \tan\left(\frac{\frac{\pi}{2}}{2}\right)$

$t = 1$

$\sin x = \dfrac{2t}{1+t^2}$

$\cos x = \dfrac{1-t^2}{1+t^2}$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

$$2dt = \sec^2\left(\frac{x}{2}\right)dx$$

$$\frac{2\,dt}{\sec^2\left(\frac{x}{2}\right)} = dx$$

$$2\cos^2\left(\frac{x}{2}\right)dt = dx$$

Nawr $\cos 2\theta = 2\cos^2\theta - 1$

$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

Felly $(\cos\theta + 1)\,dt = dx$

$$\left(\frac{1-t^2}{1+t^2} + 1\right)dt = dx$$

$$\left(\frac{1-t^2}{1+t^2} + \frac{1+t^2}{1+t^2}\right)dt = dx$$

$$\left(\frac{1-t^2+1+t^2}{1+t^2}\right)dt = dx$$

$$\frac{2}{1+t^2}\,dt = dx$$

$$\int_0^1 \frac{1}{2+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{\frac{2}{1+t^2}}{2\left(\frac{1+t^2}{1+t^2}\right) + \frac{2t}{(1+t^2)}}\,dt$$

$$= \int_0^1 \frac{\frac{2}{1+t^2}}{\frac{2(1+t^2)+2t}{1+t^2}}\,dt$$

$$= \int_0^1 \frac{2}{\cancel{1+t^2}} \times \frac{\cancel{(1+t^2)}}{2(1+t^2)+2t}\,dt$$

$$= \int_0^1 \frac{\cancel{2}}{\cancel{2}(1+t^2+t)}\,dt$$

$$= \int_0^1 \frac{1}{t^2+t+1}\,dt$$

Cwblhau'r Sgwâr:

$$= \int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2 - \frac{1}{4} + 1}\,dt$$

$$= \int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}}\,dt$$

$$= \left[\frac{1}{\sqrt{\frac{3}{4}}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)\right]_0^1$$

$$= \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\frac{3}{2}}{\sqrt{\frac{3}{4}}}\right) - \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)$$

$$= \frac{2\sqrt{3}}{3}\left(\frac{\pi}{3}\right) - \frac{2\sqrt{3}}{3}\left(\frac{\pi}{6}\right)$$

$$= \frac{2\sqrt{3}}{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{2\sqrt{3}}{3}\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi\sqrt{3}}{9}$$

(Felly $a=3$, $b=9$)