## Arc Length of a Curve and the Curved Surface Area of a Solid of Revolution

(Haf 2006)
4. A curve has parametric equations

$$
x=\theta+\sin \theta, y=1+\cos \theta(0 \leqslant \theta \leqslant \pi) .
$$

(a) Show that

$$
\begin{equation*}
\left(\frac{\mathrm{d} x}{\mathrm{~d} \theta}\right)^{2}+\left(\frac{\mathrm{d} y}{\mathrm{~d} \theta}\right)^{2}=4 \cos ^{2}\left(\frac{\theta}{2}\right) . \tag{5}
\end{equation*}
$$

(b) Find the total length of the curve.
(c) The curve is rotated through $360^{\circ}$ about the $x$-axis. Find the curved surface area of the solid of revolution generated.
(Haf 2007)
3.


The above diagram shows the upper half of the circle with equation $x^{2}+y^{2}=a^{2}$.
(a) Show that, on this curve,

$$
\begin{equation*}
1+\left(\frac{\mathrm{d} y}{\mathrm{~d} x}\right)^{2}=\frac{a^{2}}{y^{2}} . \tag{4}
\end{equation*}
$$

(b) Hence show that the curved surface area of a sphere with radius $a$ is equal to $4 \pi a^{2}$.
6. (a) The curve $C$ has parametric equations

$$
x=\cos ^{3} \theta, y=\sin ^{3} \theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}
$$

Show that

$$
\begin{equation*}
\sqrt{\left(\frac{\mathrm{d} x}{\mathrm{~d} \theta}\right)^{2}+\left(\frac{\mathrm{d} y}{\mathrm{~d} \theta}\right)^{2}}=\frac{3}{2} \sin 2 \theta \tag{5}
\end{equation*}
$$

(b) (i) Find the arc length of $C$.
(ii) The curve $C$ is rotated through $360^{\circ}$ about the $x$-axis. Show that the curved surface area of the solid of revolution generated is given by

$$
\begin{equation*}
6 \pi \int_{0}^{\frac{\pi}{2}} \sin ^{4} \theta \cos \theta \mathrm{~d} \theta \tag{9}
\end{equation*}
$$

Hence find this curved surface area.
(Haf 2009)
4. The region $R$ is bounded by the $x$-axis, the line $x=a$ and that part of the curve $y^{2}=4 a x$ lying between the points $(0,0)$ and $(a, 2 a)$. Show that the curved surface area of the solid formed when $R$ is rotated through $360^{\circ}$ about the $x$-axis is

$$
\begin{equation*}
\frac{8(2 \sqrt{2}-1)}{3} \pi a^{2} \tag{7}
\end{equation*}
$$

(Haf 2010)
4. Find the length of the arc joining the points $(0,0)$ and $(1,1)$ on the curve having equation

$$
\begin{equation*}
y^{2}=x^{3} . \tag{7}
\end{equation*}
$$

(Haf 2011)
3. Show that the length of the arc joining the points $(2 a, 2 a)$ and $(4 a, 2 \sqrt{3} a)$ on the curve with equation $y^{2}=4 a(x-a)$ is given by the integral

$$
\int_{2 a}^{4 a} \sqrt{\frac{x}{x-a}} \mathrm{~d} x
$$

Hence evaluate this length using the substitution $x=a \cosh ^{2} u$. Give your answer in the form $k a$ where $k$ should be evaluated correct to three significant figures.
8.


The diagram shows a sketch of the part of the curve $y=2-\cosh x$ which lies above the $x$-axis.
(a) Find the total length of the curve shown.
(b) The region enclosed between the curve and the $x$-axis is rotated through $2 \pi$ radians about the $x$-axis. Find the curved surface area of the solid generated, giving your answer correct to three significant figures.
(Haf 2013)
7. (a) (i) Assuming the derivatives of $\cosh x$ and $\sinh x$, show that the derivatives of $\operatorname{cosech} x$ and coth $x$ are respectively $-\operatorname{cosech} x \operatorname{coth} x$ and $-\operatorname{cosech}^{2} x$.
(ii) Hence show that

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} x}[\ln (\operatorname{cosech} x+\operatorname{coth} x)]=-\operatorname{cosech} x \tag{6}
\end{equation*}
$$

(b) (i) Show that the length $L$ of the arc joining the points $(1,0)$ and $(e, 1)$ on the graph of $y=\ln x$ is given by

$$
\int_{1}^{\mathrm{e}} \frac{\sqrt{1+x^{2}}}{x} \mathrm{~d} x
$$

(ii) Use the substitution $x=\sinh u$ to show that

$$
L=\int_{\sinh ^{-1} 1}^{\sinh ^{-1} \mathrm{e}}(\operatorname{cosech} u+\sinh u) \mathrm{d} u
$$

(iii) Use the result in (a)(ii) to determine the value of $L$ correct to three significant figures.
(Haf 2014)
7. (a) Using the substitution $x=a \sinh \theta$, show that

$$
\begin{equation*}
\int \sqrt{x^{2}+a^{2}} \mathrm{~d} x=\frac{a^{2}}{2}\left(\sinh ^{-1}\left(\frac{x}{a}\right)+\frac{x \sqrt{x^{2}+a^{2}}}{a^{2}}\right)+\text { constant } \tag{5}
\end{equation*}
$$

(b) The equation of the curve $C$ is

$$
y=x^{2}, \quad 0 \leqslant x \leqslant 1
$$

Find the arc length of $C$.
5. The curve $C$ has parametric equations

$$
x=t+\sin t, \quad y=1-\cos t \quad(0 \leqslant t \leqslant \pi)
$$

(a) Show that

$$
\begin{equation*}
\left(\frac{\mathrm{d} x}{\mathrm{~d} t}\right)^{2}+\left(\frac{\mathrm{d} y}{\mathrm{~d} t}\right)^{2}=4 \cos ^{2} \frac{1}{2} t \tag{3}
\end{equation*}
$$

(b) (i) Find the arc length of $C$.
(ii) Find the curved surface area of the solid generated when $C$ is rotated through an angle $2 \pi$ about the $x$-axis.
(Haf 2016)
5. The curve $C$ has equation $y=\ln (1+\cos x)$.
(a) Show that

$$
\begin{equation*}
1+\left(\frac{\mathrm{d} y}{\mathrm{~d} x}\right)^{2}=\frac{2}{1+\cos x} \tag{4}
\end{equation*}
$$

(b) Find the length of the arc joining the points $(0, \ln 2)$ and $\left(\frac{\pi}{2}, 0\right)$ on $C$.

Give your answer in the form $\ln (a+b \sqrt{2})$, where $a, b$ are positive integers.
3. The curve $C$ has equation $y=x^{3}$. The arc joining the points $(0,0)$ and $(1,1)$ on $C$ is rotated through an angle $2 \pi$ about the $x$-axis. Calculate the curved surface area of the solid generated, giving your answer correct to three significant figures.
(Haf 2018)
5. (a) Show that the length $L$ of the arc joining the points $(1,2)$ and $(4,4)$ on the curve with equation $y^{2}=4 x$ is given by

$$
\begin{equation*}
\int_{1}^{4} \sqrt{\left(1+\frac{1}{x}\right)} d x \tag{4}
\end{equation*}
$$

(b) Use the substitution $x=\sinh ^{2} u$ to determine the value of $L$ correct to three significant figures.
(Haf 2019)
2. The curve $C$ has equation $y=\ln (\sec x)$. Show that the length of the arc joining the points $(0,0)$ and $\left(\frac{\pi}{3}, \ln 2\right)$ on $C$ is equal to $\ln (a+\sqrt{b})$, where $a, b$ are positive integers to be determined. [5]

