Past Paper Questions: Vectors

H&F 2006

2) (a) $\underline{r}_A = \int \underline{v}_A \, dt$

$\underline{r}_A = \int -2\underline{i} - 2\underline{j} - 5\underline{k} \, dt$

$\underline{r}_A = -2t\underline{i} - 2t\underline{j} - 5t\underline{k} + c$

When $t = 0$ $\underline{r}_A = \underline{i} - 10\underline{k}$

so $c = \underline{i} - 10\underline{k}$

and $\underline{r}_A = \underline{i} - 10\underline{k} - t(2\underline{i} + 2\underline{j} + 5\underline{k})$

$\underline{r}_A = (1-2t)\underline{i} + (-2t)\underline{j} + (-10-5t)\underline{k}$

$\underline{r}_B = \int \underline{v}_B \, dt$

$\underline{r}_B = \int \underline{i} - 8\underline{j} - 5\underline{k} \, dt$

$\underline{r}_B = t\underline{i} - 8t\underline{j} - 5t\underline{k} + c$

When $t = 0$ $\underline{r}_B = 7\underline{i} + 9\underline{j} - 6\underline{k}$

So $c = 7\underline{i} + 9\underline{j} - 6\underline{k}$

and $\underline{r}_B = 7\underline{i} + 9\underline{j} - 6\underline{k} + t(\underline{i} - 8\underline{j} - 5\underline{k})$

$\underline{r}_B = (7+t)\underline{i} + (9-8t)\underline{j} + (-6-5t)\underline{k}$

(b) When $t = 2s$

$\underline{r}_A = -3\underline{i} - 4\underline{j} - 20\underline{k}$

$\underline{r}_B = 9\underline{i} - 7\underline{j} - 16\underline{k}$

To obtain the distance between A and B we need to find the vector taking us from A to B.

This is $-\underline{r}_A + \underline{r}_B$

$= 3\underline{i} + 4\underline{j} + 20\underline{k} + 9\underline{i} - 7\underline{j} - 16\underline{k}$

$= 12\underline{i} - 3\underline{j} + 4\underline{k}$

The magnitude of this vector is

$\sqrt{12^2 + (-3)^2 + 4^2}$

$= \sqrt{144 + 9 + 16}$

$= \sqrt{169}$

$= 13$

This is the distance between A and B.

6) (a) $\underline{r} = \cos 3t\,\underline{i} + \sin 3t\,\underline{j}$

$\underline{v} = \frac{d}{dt}(\underline{r})$

$\underline{v} = -3\sin 3t\,\underline{i} + 3\cos 3t\,\underline{j}$

(b) If the directions are perpendicular then $\underline{r} \cdot \underline{v} = 0$

Now $\underline{r} \cdot \underline{v}$

$= (\cos 3t)(-3\sin 3t) + (\sin 3t)(3\cos 3t)$

$= -3\sin 3t\cos 3t + 3\sin 3t\cos 3t$

$= 0.$

So the directions are perpendicular for all values of t.

(c) Speed of $P = |\underline{v}|$

$= \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2}$

$= \sqrt{9\sin^2 3t + 9\cos^2 3t}$

$= \sqrt{9(\sin^2 3t + \cos^2 3t)}$

$= \sqrt{9(1)}$

$= 3 \text{ ms}^{-1}$

Haf 2007

5) a) If $\underline{a}$ and $\underline{b}$ are perpendicular then $\underline{a} \cdot \underline{b} = 0$

$(2)(-1) + 13(y) + (-10)(5) = 0$

$-2 + 13y - 50 = 0$

$13y = 52$

$\underline{\underline{y = 4}}$

b) If $\underline{a}$ and $\underline{b}$ are parallel then $\underline{a} = \alpha \underline{b}$ for some α

By inspection we must have $\alpha = -2$ for the $\underline{i}$ and $\underline{k}$ components.

So $\underline{\underline{y = -6.5}}$

8) (a) $\underline{r}_A = \int \underline{v}_A \, dt$

$\underline{r}_A = \int 3\underline{i} - 2\underline{j} + 5\underline{k} \, dt$

$\underline{r}_A = 3t\underline{i} - 2t\underline{j} + 5t\underline{k} + c$

When $t=0$ $\underline{r}_A = 3\underline{j} - 140\underline{k}$

So $c = 3\underline{j} - 140\underline{k}$

and $\underline{r}_A = 3\underline{j} - 140\underline{k} + t(3\underline{i} - 2\underline{j} + 5\underline{k})$

$\underline{r}_A = (3t)\underline{i} + (3-2t)\underline{j} + (-140+5t)\underline{k}$

$\underline{r}_B = \int \underline{v}_B \, dt$

$\underline{r} = \int -2\underline{i} + 6\underline{j} + 3\underline{k} \, dt$

$\underline{r}_B = -2t\underline{i} + 6t\underline{j} + 3t\underline{k} + c.$

When $t=0$ $\underline{r}_B = -9\underline{i} - 4\underline{j} - 6\underline{k}$

So $c = -9\underline{i} - 4\underline{j} - 6\underline{k}$

and $\underline{r}_B = -9\underline{i} - 4\underline{j} - 6\underline{k} + t(-2\underline{i} + 6\underline{j} + 3\underline{k})$

$\underline{r}_B = (-9-2t)\underline{i} + (-4+6t)\underline{j} + (-6+3t)\underline{k}$

(b) To obtain the distance between A and B we need to find the vector taking us from A to B.

This is $-\underline{r}_A + \underline{r}_B$

$= (-3t)\underline{i} + (-3+2t)\underline{j} + (140-5t)\underline{k} + (-9-2t)\underline{i} + (-4+6t)\underline{j} + (-6+3t)\underline{k}$

$= (-9-5t)\underline{i} + (-7+8t)\underline{j} + (134-2t)\underline{k}$

The magnitude of this vector is

$\sqrt{(-9-5t)^2 + (-7+8t)^2 + (134-2t)^2}$

Thus the square of the distance between A and B at time t is

$(-9-5t)^2 + (-7+8t)^2 + (134-2t)^2$

$= 81 + 90t + 25t^2$
$+ 49 - 112t + 64t^2$
$+ 17956 - 536t + 4t^2$

$= 93t^2 - 558t + 18086$

c) A and B are closest together when $93t^2 - 558t + 18086$ is at its minimum.

Let $y = 93t^2 - 558t + 18086$

$\frac{dy}{dt} = 186t - 558$

$\frac{d^2y}{dt^2} = 186$

When $\frac{dy}{dt} = 0$

we have $186t - 558 = 0$

$\underline{\underline{t = 3s}}$

This is a minimum point because $\frac{d^2y}{dt^2}$ is positive.

Haf 2008

6) a) $\overrightarrow{AB} = -\underline{a} + \underline{b}$

$= -2\underline{i} - \underline{j} - \underline{k} + 3\underline{i} - \underline{j} + 2\underline{k}$

$= \underline{i} - 2\underline{j} + \underline{k}$

b) Work done by the force $\underline{F}$

$= \underline{F} \cdot \overrightarrow{AB}$

$= (\underline{i} - 4\underline{j} + \underline{k}) \cdot (\underline{i} - 2\underline{j} + \underline{k})$

$= (1\times1) + (-4\times-2) + (1\times1)$

$= 1 + 8 + 1$

$= 10\,J$

7) (a) $\underline{a} = \frac{d}{dt}(\underline{v})$

$\underline{a} = \frac{d}{dt}\left(\sin 3t\,\underline{i} + 2\cos 5t\,\underline{j} + 3t^3\,\underline{k}\right)$

$\underline{a} = 3\cos 3t\,\underline{i} - 10\sin 5t\,\underline{j} + 9t^2\,\underline{k}$

(b) To obtain the distance between A and B we need to find the vector taking us from A to B. This is $-\underline{r}_A + \underline{r}_B$

$= (8t+2)\underline{i} + (-3t-3)\underline{j} + (-16t+11)\underline{i} + (9t-8)\underline{j}$

$= (-8t+13)\underline{i} + (6t-11)\underline{j}$.

The magnitude of this vector is

$\sqrt{(-8t+13)^2 + (6t-11)^2}$.

$= \sqrt{64t^2 - 208t + 169 + 36t^2 - 132t + 121}$

$= \sqrt{100t^2 - 340t + 290}$

The distance between A and B is least when $\sqrt{100t^2 - 340t + 290}$ is least.

This also happens when the square of the distance, that is

$100t^2 - 340t + 290$,

is at its least.

Let $y = 100t^2 - 340t + 290$

$\frac{dy}{dt} = 200t - 340$

$\frac{d^2y}{dt^2} = 200$

When $\frac{dy}{dt} = 0$

$200t - 340 = 0$

$t = 1.7s$

This is a minimum point because $\frac{d^2y}{dt^2}$ is always positive.

So the value of t when the distance between A and B is least is $t = 1.7s$.

This minimum distance is

$\sqrt{100(1.7^2) - 340(1.7) + 290}$

$= \sqrt{1}$

$= 1$ unit

Haf 2009

6) a) Momentum = mass × velocity

$= 2 \times \frac{d}{dt}\left((1-4t^2)\underline{i} + (3t^2-5t)\underline{j}\right)$

$= 2(-8t\,\underline{i} + (6t-5)\underline{j})$

$= -16t\,\underline{i} + (12t-10)\underline{j}$

b) Acceleration $= \frac{d}{dt}(-8t\,\underline{i} + (6t-5)\underline{j})$

$= -8\underline{i} + 6\underline{j}$

This is constant as it does not depend on time.

Its magnitude is given by $\sqrt{(-8)^2 + 6^2}$

$= \sqrt{64 + 36}$

$= 10 \quad ms^{-2}$

c) velocity ⊥ Acceleration ⇒

$\underline{v} \cdot \underline{a} = 0$

$(-8t\,\underline{i} + (6t-5)\underline{j}) \cdot (-8\underline{i} + 6\underline{j}) = 0$

$(-8t \times (-8)) + ((6t-5)\times 6) = 0$

$64t + 36t - 30 = 0$

$100t = 30$

$t = \frac{30}{100}$

$t = 0.3s$

Haf 2010

2 $\underline{r} = (3t^2+1)\underline{i} + (13t - 2t^2)\underline{j}$

a) $\underline{v} = 6t\underline{i} + (13-4t)\underline{j}$

Os yw $t=2$ mae $\underline{v} = 12\underline{i} + (13-8)\underline{j}$

$\underline{v} = 12\underline{i} + 5\underline{j}$

Felly buanedd P os yw $t=2$ yw $\sqrt{12^2+5^2} = \sqrt{169}$

$= 13\ ms^{-1}$

b) Rydym angen $\underline{v}\cdot(2\underline{i} - \underline{j}) = 0$

$$(6t\underline{i} + (13-4t)\underline{j})\cdot(2\underline{i} - \underline{j}) = 0$$
$$12t + (13-4t)(-1) = 0$$
$$12t - 13 + 4t = 0$$
$$16t = 13$$
$$t = \frac{13}{16}$$
$$t = 0.8125\ s$$

c) $\underline{a} = 6\underline{i} - 4\underline{j}$

Maint y cyflymiad yw $\sqrt{36+(-4)^2} = \sqrt{52}$

$= 2\sqrt{13}\ ms^{-2}$

ch) Yn defnyddio $\underline{a}\cdot\underline{b} = |\underline{a}|\,|\underline{b}|\cos\theta$ efo $\underline{a}$ = cyflymiad

$\underline{b}$ = cyflymder

$$(6\underline{i} - 4\underline{j})\cdot(12\underline{i} + 5\underline{j}) = \sqrt{6^2+(-4)^2}\sqrt{12^2+5^2}\cos\theta$$
$$72 + -20 = \sqrt{52}\times\sqrt{169}\cos\theta$$
$$52 = 26\sqrt{13}\cos\theta$$
$$\theta = \cos^{-1}\left(\frac{52}{26\sqrt{13}}\right)$$

$\theta = 56.31^\circ$ yn gywir i ddau le degol.

May '11

3) $\underline{v} = 2\underline{i} + 6t\,\underline{j} + 4t^3\,\underline{k}$

$\underline{a} = \frac{d}{dt}(\underline{v})$

$\underline{a} = \frac{d}{dt}(2\underline{i} + 6t\,\underline{j} + 4t^3\,\underline{k})$

$\underline{a} = 0\underline{i} + 6\underline{j} + 12t^2\,\underline{k}$

a) Using $\underline{F} = m\underline{a}$

$\underline{F} = 2(0\underline{i} + 6\underline{j} + 12t^2\,\underline{k})$

$\underline{F} = 0\underline{i} + 12\underline{j} + 24t^2\,\underline{k}$

b) $\underline{F} \cdot \underline{v} = (0\underline{i} + 12\underline{j} + 24t^2\underline{k}) \cdot (2\underline{i} + 6t\,\underline{j} + 4t^3\,\underline{k})$

$= (0 \times 2) + (12 \times 6t) + (24t^2 \times 4t^3)$

$= 0 + 72t + 96t^5$

$= 72t + 96t^5$

When $t = 1$, $\underline{F} \cdot \underline{v} = 72(1) + 96(1^5)$

$\underline{F} \cdot \underline{v} = 168$

As Power = Force x Velocity, then the unit of $\underline{F} \cdot \underline{v}$ is Watt.

$\underline{F} \cdot \underline{v} = 168W$

May '11

7) $\underline{OA} = 2\underline{i} + 3\underline{j} + \underline{k} + t(2\underline{i} - 6\underline{j} + 9\underline{k})$

$\underline{OB} = 5\underline{i} - 8\underline{j} + 10\underline{k} + t(3\underline{i} - 6\underline{j} + 7\underline{k})$

(a) position vector of A = $\underline{OA}$

velocity vector of A = $\frac{d}{dt}(\underline{OA})$

$= \frac{d}{dt}(2\underline{i} + 3\underline{j} + \underline{k} + t(2\underline{i} - 6\underline{j} + 9\underline{k}))$

$= 2\underline{i} - 6\underline{j} + 9\underline{k}$.

The speed of particle A is given by $\sqrt{2^2 + (-6)^2 + 9^2}$

$= \sqrt{121}$

$= 11\,ms^{-1}$

(b) $\underline{AB} = \underline{AO} + \underline{OB}$

$= -2\underline{i} - 3\underline{j} - \underline{k} - t(2\underline{i} - 6\underline{j} + 9\underline{k}) + 5\underline{i} - 8\underline{j} + 10\underline{k} + t(3\underline{i} - 6\underline{j} + 7\underline{k})$

$= 3\underline{i} - 11\underline{j} + 9\underline{k} + t(\underline{i} - 2\underline{k})$

$= (3+t)\underline{i} - 11\underline{j} + (9-2t)\underline{k}$

Distance AB at time $t = \sqrt{(3+t)^2 + (-11)^2 + (9-2t)^2}$

$= \sqrt{9 + 6t + t^2 + 121 + 81 - 36t + 4t^2}$

$= \sqrt{5t^2 - 30t + 211}$

Therefore this distance squared is given by $5t^2 - 30t + 211$.

When are A and B closest together?

Need to find the minimum of the function $5t^2 - 30t + 211$.

Let $y = 5t^2 - 30t + 211$

$\frac{dy}{dt} = 10t - 30$

Minimum point when $\frac{dy}{dt} = 0$

$10t - 30 = 0$

$10t = 30$

$\underline{\underline{t = 3s}}$

The time when A and B are closest together is 3s.

(This is a minimum point because $\frac{d^2y}{dt^2} = 10$ is positive)

M2 Haf 2012

③ $\underline{r} = (t + 2t^2)\underline{i} + (1.5t^2 - 2t)\underline{j}$

(a) $\underline{v} = \frac{d}{dt}(\underline{r})$

$$\underline{v} = \frac{d}{dt}\left((t+2t^2)\underline{i} + (1.5t^2 - 2t)\underline{j} \right)$$

$$\underline{v} = (1 + 4t)\underline{i} + (3t - 2)\underline{j}$$

We require $\underline{v} \cdot (-\underline{i} + 2\underline{j}) = 0$

$$[(1+4t)\underline{i} + (3t-2)\underline{j}] \cdot [-\underline{i} + 2\underline{j}] = 0$$

$$(1+4t)(-1) + (3t-2)(2) = 0$$

$$-1 - 4t + 6t - 4 = 0$$

$$2t - 5 = 0$$

$$2t = 5$$

$$\underline{\underline{t = 2.5 \text{ s}}}$$

(b) $\underline{a} = \frac{d}{dt}(\underline{v})$

$$\underline{a} = \frac{d}{dt}\left((1+4t)\underline{i} + (3t-2)\underline{j} \right)$$

$$\underline{a} = 4\underline{i} + 3\underline{j}$$

This vector is constant (it does not depend on t).

The magnitude of the vector is $|\underline{a}| = \sqrt{4^2 + 3^2}$

$$|\underline{a}| = \sqrt{16 + 9}$$

$$|\underline{a}| = \sqrt{25}$$

$$\underline{\underline{|\underline{a}| = 5 \text{ ms}^{-2}}}$$

Haf 2012

(8) (a) $\text{Velocity} = \dfrac{\text{change in displacement}}{\text{time}}$

$$= \frac{(14-8)\underline{i} + (-5-7)\underline{j}}{3}$$

$$= \frac{6\underline{i} - 12\underline{j}}{3}$$

$$= 2\underline{i} - 4\underline{j} \quad ✓$$

(b) $(8+2t)\underline{i} + (7-4t)\underline{j}$

(c) When $t = 10s$, S is at position $(8+2\times10)\underline{i} + (7-4\times10)\underline{j}$

$= 28\underline{i} - 33\underline{j}$

and B is at position $0\underline{i} + 0\underline{j}$

The position vector of B at time t is

$x(t-10)\underline{i} + y(t-10)\underline{j}$.

When $t = 50s$, S is at position $(8+2\times50)\underline{i} + (7-4\times50)\underline{j}$

$= 108\underline{i} - 193\underline{j}$

and B is at position $x(50-10)\underline{i} + y(50-10)\underline{j}$

$= 40x\underline{i} + 40y\underline{j}$.

For intercept at $t = 50s$ we require

$40x = 108$ and $40y = -193$

$\underline{\underline{x = 2.7}}$ $\underline{\underline{y = -4.825}}$

M2 Haf 2013

2) $\underline{v} = (13t - 3)\underline{i} + (2 + 3t^2)\underline{j}$ Mass 2Kg

At time 0s, the position vector is $\underline{r} = (2\underline{i} + 7\underline{j})$ m

(a) $\underline{r} = \int \underline{v}\, dt$

$\underline{r} = \int (13t - 3)\underline{i} + (2 + 3t^2)\underline{j}\ dt$

$\underline{r} = \left(\frac{13t^2}{2} - 3t\right)\underline{i} + \left(2t + \frac{3t^3}{3}\right)\underline{j} + \underline{c}$

where $\underline{c}$ is a constant of integration.

When $t = 0$s, $\underline{r} = (2\underline{i} + 7\underline{j})$, so

$2\underline{i} + 7\underline{j} = \left(\frac{13(0)^2}{2} - 3(0)\right)\underline{i} + \left(2(0) + \frac{3(0)^3}{3}\right)\underline{j} + \underline{c}$

$2\underline{i} + 7\underline{j} = \underline{c}$

Therefore $\underline{r} = \left(\frac{13t^2}{2} - 3t\right)\underline{i} + \left(2t + \frac{3t^3}{3}\right)\underline{j} + 2\underline{i} + 7\underline{j}$

$\underline{r} = (6.5t^2 - 3t)\underline{i} + (2t + t^3)\underline{j} + 2\underline{i} + 7\underline{j}$

$\underline{r} = (6.5t^2 - 3t + 2)\underline{i} + (t^3 + 2t + 7)\underline{j}$

(b) $\underline{a} = \frac{d}{dt}(\underline{v})$

$\underline{a} = \frac{d}{dt}\left((13t - 3)\underline{i} + (2 + 3t^2)\underline{j}\right)$

$\underline{a} = 13\underline{i} + 6t\underline{j}$

(c) We require $\underline{v} \cdot (\underline{i} - 2\underline{j}) = 0$

$((13t - 3)\underline{i} + (2 + 3t^2)\underline{j}) \cdot (\underline{i} - 2\underline{j}) = 0$

$(13t - 3) \times 1 + (2 + 3t^2) \times -2 = 0$

$13t - 3 - 4 - 6t^2 = 0$

$0 = 6t^2 - 13t + 7$

$0 = (6t - 7)(t - 1)$

Either $6t - 7 = 0$ or $t - 1 = 0$

$6t = 7$ $\underline{\underline{t = 1\text{s}}}$

$\underline{\underline{t = \frac{7}{6}\text{s}}}$

M2 Haf 2014

(4) $\underline{v}_A = -\underline{i} + 2\underline{j} + \underline{k}$

$\underline{r}_A = \int \underline{v}_A \, dt$

$\underline{r}_A = \int -\underline{i} + 2\underline{j} + \underline{k} \, dt$

$\underline{r}_A = (-\underline{i} + 2\underline{j} + \underline{k})t + \underline{c}$

Now $\underline{r}_A = 3\underline{i} + 5\underline{j} + 20\underline{k}$

When $t = 0$ so

$\underline{c} = 3\underline{i} + 5\underline{j} + 20\underline{k}$

$\underline{v}_B = 3\underline{i} - 4\underline{j} + 2\underline{k}$

$\underline{r}_B = \int \underline{v}_B \, dt$

$\underline{r}_B = \int 3\underline{i} - 4\underline{j} + 2\underline{k} \, dt$

$\underline{r}_B = (3\underline{i} - 4\underline{j} + 2\underline{k})t + \underline{c}$

Now $\underline{r}_B = -2\underline{i} + x\underline{j} + 15\underline{k}$

when $t = 0$ so

$\underline{c} = -2\underline{i} + x\underline{j} + 15\underline{k}$

Therefore $\underline{r}_A = (-\underline{i} + 2\underline{j} + \underline{k})t + 3\underline{i} + 5\underline{j} + 20\underline{k}$

$\underline{r}_B = (3\underline{i} - 4\underline{j} + 2\underline{k})t - 2\underline{i} + x\underline{j} + 15\underline{k}$

b) To obtain the distance between A and B we need to find the vector taking us from A to B.

This is $-\underline{r}_A + \underline{r}_B$

$= -(-\underline{i} + 2\underline{j} + \underline{k})t - 3\underline{i} - 5\underline{j} - 20\underline{k}$
$\quad + (3\underline{i} - 4\underline{j} + 2\underline{k})t - 2\underline{i} + x\underline{j} + 15\underline{k}$

$= (4i - 6\underline{j} + \underline{k})t - 5i + (x-5)\underline{j} - 5\underline{k}$

$= (4t-5)\underline{i} + (-6t + x - 5)\underline{j} + (t-5)\underline{k}$

The magnitude of this vector is

$\sqrt{(4t-5)^2 + (-6t+x-5)^2 + (t-5)^2}$

$= \sqrt{16t^2 - 40t + 25 + (-6t+x-5)(-6t+x-5) + t^2 - 10t + 25}$

$= \sqrt{17t^2 - 50t + 50 + 36t^2 - 6tx + 30t - 6tx + x^2 - 5x + 30t - 5x + 25}$

$= \sqrt{53t^2 + 10t + 75 - 12tx + x^2 - 10x}$

$= \sqrt{53t^2 + (10-12x)t + x^2 - 10x + 75}$

Therefore $AB^2 = 53t^2 + (10-12x)t + x^2 - 10x + 75$

(c) The distance between A and B is least when AB^2 is least.

Let $y = AB^2$

$y = 53t^2 + (10 - 12x)t + x^2 - 10x + 75$

$\frac{dy}{dt} = 106t + 10 - 12x$

We know that the shortest distance occurs at $t = 5$. Therefore when $t=5$ we must have $\frac{dy}{dt} = 0$ as this is a minimum point.

Substituting $\frac{dy}{dt} = 0$, $t=5$:

$0 = 106 \times 5 + 10 - 12x$

$0 = 530 + 10 - 12x$

$12x = 540$

$\underline{\underline{x = 45}}$

M2 Haf 2014

(6) $\underline{v} = 4\sin 2t\,\underline{i} + 15\cos 5t\,\underline{j}$

(a) $\underline{a} = \frac{d\underline{v}}{dt}$

$\underline{a} = 2(4\cos 2t)\underline{i} - 5(15\sin 5t)\underline{j}$

$\underline{a} = 8\cos 2t\,\underline{i} - 75\sin 5t\,\underline{j}$

When $t = \frac{3\pi}{2}$s, $\underline{a} = 8\cos(2\times\frac{3\pi}{2})\underline{i} - 75\sin(5\times\frac{3\pi}{2})\underline{j}$

$\underline{a} = -8\underline{i} + 75\underline{j}$

The magnitude of the acceleration is $\sqrt{(-8)^2+75^2}$

$= 75.42545989\cdots \text{ms}^{-2}$

Using $F=ma$, the magnitude of the force is

$3\times 75.42545989\cdots$

$= 226.28\text{N}$ to 2 d.p.

(b) $\underline{r} = \int \underline{v}\,dt$

$\underline{r} = \int 4\sin 2t\,\underline{i} + 15\cos 5t\,\underline{j}$

$\underline{r} = -2\cos 2t\,\underline{i} + 3\sin 5t\,\underline{j} + \underline{c}$

When $t=0$, $\underline{r} = -2\underline{i} + 3\underline{j}$, so

$-2\underline{i}+3\underline{j} = -2\cos(2\times 0)\underline{i} + 3\sin(5\times 0)\underline{j} + \underline{c}$

$\cancel{-2\underline{i}} + 3\underline{j} = \cancel{-2\underline{i}} + \underline{c}$

$\underline{c} = 3\underline{j}$

So the position vector of the particle at time ts is

$\underline{r} = -2\cos 2t\,\underline{i} + 3\sin 5t\,\underline{j} + 3\underline{j}$

$\underline{r} = -2\cos 2t\,\underline{i} + 3(\sin 5t + 1)\underline{j}$

(c) We require the $\underline{i}$ component of $\underline{r}$ to be zero.

$$-2\cos 2t = 0$$
$$\cos 2t = 0$$
$$2t = \cos^{-1}(0)$$
$$2t = \frac{\pi}{2}$$
$$\underline{\underline{t = \frac{\pi}{4}}}$$

$\cos^{-1}(0) = 90°$

When $t = \frac{\pi}{4}$, $\underline{r} = -2\cos\left(2 \times \frac{\pi}{4}\right)\underline{i} + 3\left(\sin\left(5 \times \frac{\pi}{4}\right) + 1\right)\underline{j}$

$$\underline{r} = 0\underline{i} + 3\left(-\frac{\sqrt{2}}{2} + 1\right)\underline{j}$$

$$\underline{r} = \frac{6 - 3\sqrt{2}}{2}\underline{j}$$

So the distance of the particle from the origin when it crosses the y-axis for the first time is $\frac{6-3\sqrt{2}}{2} \approx \underline{0.8787\text{ m}}$ to 4 d.p.

M2 Haf 2015

1) $\underline{x} = \sin\theta\,\underline{i} + 2\cos 2\theta\,\underline{j}$
$\underline{y} = 2\underline{i} - \underline{j}$

If the vectors are perpendicular then $\underline{x}\cdot\underline{y} = 0$

$$[\sin\theta\,\underline{i} + 2\cos 2\theta\,\underline{j}]\cdot[2\underline{i} - \underline{j}] = 0$$
$$\sin\theta \times 2 + 2\cos 2\theta \times -1 = 0$$
$$2\sin\theta - 2\cos 2\theta = 0$$

Now $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

Therefore $2\sin\theta - 2(1 - 2\sin^2\theta) = 0$
$2\sin\theta - 2 + 4\sin^2\theta = 0$
$4\sin^2\theta + 2\sin\theta - 2 = 0$
$2\sin^2\theta + \sin\theta - 1 = 0$ [Divide by 2]
$(2\sin\theta - 1)(\sin\theta + 1) = 0$

Either $2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$
$2\sin\theta = 1$ $\quad$ $\sin\theta = -1$
$\sin\theta = \frac{1}{2}$ $\quad$ $\theta = \sin^{-1}(-1)$
$\theta = \sin^{-1}(\frac{1}{2})$ $\quad$ $\theta = -90^\circ, 270^\circ$
$\theta = 30^\circ, 150^\circ$

S | A
T | C

S | A
T | C

In radians, between 0 and 2π,
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

M2 Haf 2015

4) $\underline{F} = (4t - 3)\underline{i} + (3t^2 - 5t)\underline{j}$

a) $\underline{F} = m\underline{a}$ so $\underline{a} = \dfrac{\underline{F}}{m}$

$$\underline{a} = \frac{\underline{F}}{0.5}$$

$$\underline{a} = \frac{(4t-3)\underline{i} + (3t^2-5t)\underline{j}}{0.5}$$

$$\underline{a} = (8t-6)\underline{i} + (6t^2-10t)\underline{j}$$

$$\underline{v} = \int \underline{a}\,dt$$

$$\underline{v} = \int (8t-6)\underline{i} + (6t^2-10t)\underline{j}$$

$$\underline{v} = \left(\frac{8t^2}{2} - 6t\right)\underline{i} + \left(\frac{6t^3}{3} - \frac{10t^2}{2}\right)\underline{j} + \underline{c}$$

$$\underline{v} = (4t^2 - 6t)\underline{i} + (2t^3 - 5t^2)\underline{j} + \underline{c}.$$

When $t = 0$, $\underline{v} = 8\underline{i} - 7\underline{j}$, so

$$8\underline{i} - 7\underline{j} = (4\times0^2 - 6\times0)\underline{i} + (2\times0^3 - 5\times0^2)\underline{j} + \underline{c}$$

$$8\underline{i} - 7\underline{j} = \underline{c}$$

Therefore $\underline{v} = (4t^2 - 6t)\underline{i} + (2t^3 - 5t^2)\underline{j} + 8\underline{i} - 7\underline{j}$

$$\underline{v} = (4t^2 - 6t + 8)\underline{i} + (2t^3 - 5t^2 - 7)\underline{j}$$

b) When $t = 3s$, the velocity is

$$\underline{v} = (4\times3^2 - 6\times3 + 8)\underline{i} + (2\times3^3 - 5\times3^2 - 7)\underline{j}$$

$$\underline{v} = (36 - 18 + 8)\underline{i} + (54 - 45 - 7)\underline{j}$$

$\underline{v} = 26\underline{i} + 2\underline{j}$ ← This is the velocity just before receiving the impulse

Impulse = Change in Momentum

Impulse = $m\underline{v} - m\underline{u}$

$$2\underline{i} - 9\underline{j} = 0.5\underline{v} - 0.5(26\underline{i} + 2\underline{j})$$

$$2\underline{i} - 9\underline{j} = 0.5\underline{v} - 13\underline{i} - \underline{j}$$

$$2\underline{i} + 13\underline{i} - 9\underline{j} + \underline{j} = 0.5\underline{v}$$

$$15\underline{i} - 8\underline{j} = 0.5\underline{v}$$

$$\underline{v} = \frac{15\underline{i} - 8\underline{j}}{0.5}$$

$$\underline{v} = 30\underline{i} - 16\underline{j}$$

So the speed of the particle immediately after the impulse is

$$\sqrt{30^2 + (-16)^2}$$

$$= \sqrt{1156}$$

$$= \underline{\underline{34\,ms^{-1}}}$$

M2 Haf 2016

(3) At time 0s, A is at $\underline{i}$ m and B is at $3\underline{i}$ m.
A has constant velocity $2\underline{i} + 5\underline{j} - 4\underline{k}$ ms^{-1}
B has constant velocity $\underline{i} + 3\underline{j} - 5\underline{k}$ ms^{-1}

At time t, position vector of A = $\underline{OA}$
$\underline{OA} = \underline{i} + t(2\underline{i} + 5\underline{j} - 4\underline{k})$
$= (2t+1)\underline{i} + 5t\underline{j} - 4t\underline{k}$

Position vector of B = $\underline{OB}$
$\underline{OB} = 3\underline{i} + t(\underline{i} + 3\underline{j} - 5\underline{k})$
$= (t+3)\underline{i} + 3t\underline{j} - 5t\underline{k}$

When is $\underline{AB}$ the least?
$\underline{AB} = \underline{AO} + \underline{OB}$
$= (-2t-1)\underline{i} - 5t\underline{j} + 4t\underline{k} + (t+3)\underline{i} + 3t\underline{j} - 5t\underline{k}$
$= (2-t)\underline{i} - 2t\underline{j} - t\underline{k}$

Distance AB at time $t = \sqrt{(2-t)^2 + (-2t)^2 + (-t)^2}$
$= \sqrt{4 - 4t + t^2 + 4t^2 + t^2}$
$= \sqrt{6t^2 - 4t + 4}$
This is least when the square of the distance, $6t^2 - 4t + 4$, is least.
Let $y = 6t^2 - 4t + 4$
$\frac{dy}{dt} = 12t - 4$
Minimum point when $\frac{dy}{dt} = 0$
$12t - 4 = 0$
$12t = 4$
$t = \frac{4}{12}$
$t = \frac{1}{3}$ s

(This is a minimum point because $\frac{d^2y}{dt^2} = 12$ is positive)

The least distance is $\sqrt{6t^2 - 4t + 4}$

$$= \sqrt{6\left(\frac{1}{3}\right)^2 - 4\left(\frac{1}{3}\right) + 4}$$

$$= \sqrt{6\left(\frac{1}{9}\right) - \frac{4}{3} + 4}$$

$$= \sqrt{\frac{2}{3} - \frac{4}{3} + \frac{12}{3}}$$

$$= \sqrt{\frac{10}{3}}$$

≈ 1.83 m to 2 d.p.

M2 Haf 2016

(6) $\underline{v} = 7\sin 2t\,\underline{i} + 6\cos 3t\,\underline{j}$

a) $\underline{a} = \frac{d}{dt}(\underline{v})$

$\underline{a} = 7(2)\cos 2t\,\underline{i} + 6(-3)\sin 3t\,\underline{j}$

$\underline{a} = 14\cos 2t\,\underline{i} - 18\sin 3t\,\underline{j}$

b) $\underline{r} = \int \underline{v}\,dt$

$\underline{r} = \int 7\sin 2t\,\underline{i} + 6\cos 3t\,\underline{j}$

$\underline{r} = \frac{7\cos 2t}{-2}\underline{i} + \frac{6\sin 3t}{3}\underline{j} +$ constant vector

$\underline{r} = -3.5\cos 2t\,\underline{i} + 2\sin 3t\,\underline{j} +$ constant vector

When $t = 0$, $\underline{r} = 0.5\underline{i} + 3\underline{j}$ $\therefore$

$0.5\underline{i} + 3\underline{j} = -3.5\cos(2\times 0)\,\underline{i} + 2\sin(3\times 0)\underline{j} +$ constant vector

$0.5\underline{i} + 3\underline{j} = -3.5\underline{i} + 0\underline{j} +$ constant vector

constant vector $= 4\underline{i} + 3\underline{j}$

$\therefore \quad \underline{r} = (4 - 3.5\cos 2t)\underline{i} + (3 + 2\sin 3t)\underline{j}$

When $t = \frac{\pi}{2}$,

$\underline{r} = (4 - 3.5\cos(2\times\frac{\pi}{2}))\underline{i} + (3 + 2\sin(3\times\frac{\pi}{2}))\underline{j}$

$\underline{r} = (4 - 3.5(-1))\underline{i} + (3 + 2(-1))\underline{j}$

$\underline{r} = 7.5\underline{i} + \underline{j}$

Atebion M2 Haf 2017

1) $\underline{r} = t\sin t\,\underline{i} + t\cos t\,\underline{j}$

a) i) $\underline{v} = \frac{d\underline{r}}{dt}$

$\underline{v} = \frac{d}{dt}\left(t\sin t\,\underline{i} + t\cos t\,\underline{j}\right)$

$\underline{v} = (t\cos t + \sin t)\underline{i} + (-t\sin t + \cos t)\underline{j}$

$\underline{v} = (t\cos t + \sin t)\underline{i} + (\cos t - t\sin t)\underline{j}$

The speed of P is given by the magnitude of $\underline{v}$.

$|\underline{v}| = \sqrt{(t\cos t + \sin t)^2 + (\cos t - t\sin t)^2}$

$|\underline{v}| = \sqrt{(t^2\cos^2 t + \cancel{2t\cos t\sin t} + \sin^2 t) + (\cos^2 t - \cancel{2t\sin t\cos t} + t^2\sin^2 t}$

$|\underline{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t + \sin^2 t + \cos^2 t}$

$|\underline{v}| = \sqrt{t^2(\cos^2 t + \sin^2 t) + (\sin^2 t + \cos^2 t)}$

$|\underline{v}| = \underline{\underline{\sqrt{t^2+1}}}$

ii) Momentum = mass × velocity

$= 3\underline{v}$

$= 3(t\cos t + \sin t)\underline{i} + 3(\cos t - t\sin t)\underline{j}$

b) At time $t = \frac{\pi}{6}$, $(b\,\underline{i} + \sqrt{3}\,\underline{j})\cdot\underline{r} = 0$

$(b\underline{i} + \sqrt{3}\underline{j})\cdot(t\sin t\,\underline{i} + t\cos t\,\underline{j}) = 0$

$bt\sin t + \sqrt{3}t\cos t = 0$

$b(\frac{\pi}{6})\sin(\frac{\pi}{6}) + \sqrt{3}(\frac{\pi}{6})\cos(\frac{\pi}{6}) = 0$

$b\left(\frac{\pi}{12}\right) + \sqrt{3}\left(\frac{\pi}{6}\right)\left(\frac{\sqrt{3}}{2}\right) = 0$

$b(\frac{\pi}{12}) + 3(\frac{\pi}{12}) = 0$

$b + 3 = 0$ $[\div \frac{\pi}{12}]$

$\underline{\underline{b = -3}}$

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M2 Haf 2018

2) $\underline{r} = (3t^2+1)\underline{i} + (t\cos 4t)\underline{j}$ metrau

a) Momentwm = Mās × cyflymder

$\underline{v} = \frac{d\underline{r}}{dt}$

$\underline{v} = (6t)\underline{i} + (1\cos 4t + t(-4\sin 4t))\underline{j}$

$\underline{v} = 6t\underline{i} + (\cos 4t - 4t\sin 4t)\underline{j}$

Momentwm = $8 \times \underline{v}$

$\underline{p} = 8(6t\underline{i} + (\cos 4t - 4t\sin 4t)\underline{j})$

$\underline{p} = 48t\underline{i} + (8\cos 4t - 32t\sin 4t)\underline{j}$

Os yw $t = 0$ yna

$\underline{p} = 48(0)\underline{i} + (8\cos(4\times 0) - 32(0)\sin(4\times 0))\underline{j}$

$\underline{p} = 0\underline{i} + (8-0)\underline{j}$

$\underline{p} = 8\underline{j}$

b) Egni cinetig $= \frac{1}{2}mv^2$

Os yw $t = \pi$, mae

$\underline{v} = 6\pi\underline{i} + (\cos(4\pi) - 4\pi\sin(4\pi))\underline{j}$

$\underline{v} = 6\pi\underline{i} + (1 - 4\pi(0))\underline{j}$

$\underline{v} = 6\pi\underline{i} + \underline{j}$

Maint y cyflymder yw $\sqrt{(6\pi)^2 + 1^2}$

$= 18.87606311$

Felly egni cinetig $= \frac{1}{2}mv^2$

$= 0.5 \times 8 \times (18.8760631\ldots)^2$

$= 4 \times ((6\pi)^2 + 1^2)$

$= 4 \times (36\pi^2 + 1)$

$= 144\pi^2 + 4\ J$

$= \underline{1425.22\ J}$ i 2 le degol

c) Grym = Mās × Cyflymiad

$= 8 \times \underline{a}$

$= 8 \times \frac{d}{dt}(\underline{v})$.

Nawr $\frac{d}{dt}(\underline{v}) = \frac{d}{dt}\left(6t\underline{i} + (\cos 4t - 4t\sin 4t)\underline{j}\right)$

$= 6\underline{i} + (-4\sin 4t - 4\sin 4t - 4t(4\cos 4t))\underline{j}$

$= 6\underline{i} + (-8\sin 4t - 16t\cos 4t)\underline{j}$

Felly Grym $= 8(6\underline{i} + (-8\sin 4t - 16t\cos 4t)\underline{j})$

$= 48\underline{i} + (-64\sin 4t - 128t\cos 4t)\underline{j}$

Os yw $t = \pi$ y grym yw

$48\underline{i} + (-64\sin(4\pi) - 128\pi\cos(4\pi))\underline{j}$

$= 48\underline{i} + (-64(0) - 128\pi(1))\underline{j}$

$= 48\underline{i} - 128\pi\underline{j}$

$= 16(3\underline{i} - 8\pi\underline{j})$

Fector a fyddai'n berpendicwlar i'r grym yma fyddai

$8\pi\underline{i} + 3\underline{j}$.

GWIRIO: $(16(3\underline{i} - 8\pi\underline{j})) \cdot (8\pi\underline{i} + 3\underline{j})$

$= 16 \times 3 \times 8\pi + 16 \times -8\pi \times 3$

$= 384\pi - 384\pi$

$= 0$ ✓

ch) Gwaith = grym × pellter

Cyfradd gwaith $= \dfrac{\text{grym} \times \text{pellter}}{\text{amser}}$

Cyfradd gwaith = grym × cyflymder

Meun fectorau:

Cyfradd gwaith $= \underline{\text{grym}} \cdot \underline{\text{cyflymder}}$

$= (16(3\underline{i} - 8\pi\underline{j})) \cdot (6\pi\underline{i} + \underline{j})$

$= 16 \times 3 \times 6\pi - 16 \times 8\pi \times 1$

$= 288\pi - 128\pi$

$= 160\pi$

$\approx \underline{502.65\ W}$ i 2 le degol

M2 Haf 2019

6) $\underline{OA} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + t(\mathbf{i} - 6\mathbf{j} + 5\mathbf{k})$

$\underline{OB} = 4\mathbf{i} - 8\mathbf{j} + 7\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

a) $\mathbf{v}_A = \frac{d}{dt}(\mathbf{r}_A)$

$= \frac{d}{dt}(4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + t(\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}))$

$= \mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$

Buanedd A $= \sqrt{1^2 + (-6)^2 + 5^2}$

$= \sqrt{62}\ \text{ms}^{-1}$

$(= 7.87\ \text{ms}^{-1}$; 2 le degol$)$

b) $\underline{AB} = -\underline{OA} + \underline{OB}$

$= \cancel{-4\mathbf{i}} - 3\mathbf{j} + 2\mathbf{k} - t(\mathbf{i} - 6\mathbf{j} + 5\mathbf{k})$

$\quad \cancel{+4\mathbf{i}} - 8\mathbf{j} + 7\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

$= -11\mathbf{j} + 9\mathbf{k} + t(2\mathbf{i} + 5\mathbf{j} - \mathbf{k})$

$= 2t\mathbf{i} + (5t-11)\mathbf{j} + (9-t)\mathbf{k}$

Pellter AB $= \sqrt{(2t)^2 + (5t-11)^2 + (9-t)^2}$

(Pellter AB)$^2 = (2t)^2 + (5t-11)^2 + (9-t)^2$

$AB^2 = 4t^2 + 25t^2 - 110t + 121 + 81 - 18t + t^2$

$AB^2 = 30t^2 - 128t + 202$ ✓

Pryd mae'r pellter AB ar ei leiaf?

Angen ffeindio pwynt minimum $30t^2 - 128t + 202$.

Differu: $60t - 128$

Pwynt arhosol: $60t - 128 = 0$

$60t = 128$

$t = \frac{32}{15}$ s

$(t = 2.1\dot{3}\text{ s})$