

C4: Tangiad, Normal

Haf 2005

(3)

$$4x^2 + 3xy - y^2 = 21$$

$$8x + 3x\left(\frac{dy}{dx}\right) + (3)y - 2y\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}(3x - 2y) + 8x + 3y = 0$$

$$\frac{dy}{dx}(3x - 2y) = -8x - 3y$$

$$\frac{dy}{dx} = \frac{-8x - 3y}{3x - 2y}$$

$$\frac{dy}{dx} = \frac{8x + 3y}{2y - 3x}$$

$$\text{os yw } x=2, y=1 \text{ yna } \frac{dy}{dx} = \frac{8 \times 2 + 3 \times 1}{2 \times 1 - 3 \times 2}$$

$$= \frac{19}{-4}$$

$$= -4.75$$

Hafaliad y Langiad yn (2, 1):

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -4.75(x - 2)$$

$$y - 1 = -4.75x + 9.5$$

$$y = -4.75x + 10.5$$

Haf 2006

②

$$2x^3 + 6xy^2 - y^4 = 27$$

$$6x^2 + 6x(2y \frac{dy}{dx}) + 6(y^2) - 4y^3(\frac{dy}{dx}) = 0$$

$$\frac{dy}{dx}(12xy - 4y^3) + 6x^2 + 6y^2 = 0$$

$$\frac{dy}{dx}(12xy - 4y^3) = -6x^2 - 6y^2$$

$$\frac{dy}{dx} = \frac{-6x^2 - 6y^2}{12xy - 4y^3}$$

$$\frac{dy}{dx} = \frac{3x^2 + 3y^2}{2y^3 - 6xy}$$

$$\text{os yw } x=2, y=1 \text{ yna } \frac{dy}{dx} = \frac{3(2)^2 + 3(1)^2}{2(1)^3 - 6 \times 2 \times 1} \\ = \frac{15}{-10} \\ = -1.5$$

Hafaliad y normal yn (2, 1):

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 1 = -\frac{1}{-1.5}(x - 2)$$

$$y - 1 = \frac{2}{3}(x - 2)$$

$$y - 1 = \frac{2}{3}x - \frac{4}{3}$$

$$y = \frac{2}{3}x - \frac{1}{3}$$

Haf 2007

② $x^5 + xy^2 + y^3 = 17$

$$5x^4 + x(2y \frac{dy}{dx}) + 1(y^2) + 3y^2(\frac{dy}{dx}) = 0$$

$$\frac{dy}{dx}(2xy + 3y^2) + 5x^4 + y^2 = 0$$

$$\frac{dy}{dx}(2xy + 3y^2) = -5x^4 - y^2$$

$$\frac{dy}{dx} = \frac{-5x^4 - y^2}{2xy + 3y^2}$$

os yw $x = -1, y = 3$ yna $\frac{dy}{dx} = \frac{-5(-1)^4 - 3^2}{2(-1) \times 3 + 3(3^2)}$

$$= \frac{-14}{21}$$
$$= -\frac{2}{3}$$

Itafaliad y tangiad yn $(-1, 3)$:

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -\frac{2}{3}(x - -1)$$

$$y - 3 = -\frac{2}{3}(x + 1)$$

$$y - 3 = -\frac{2}{3}x - \frac{2}{3}$$

$$y = -\frac{2}{3}x + \frac{7}{3}$$

Haf 2008

$$(2) \quad x^2 + xy + 2y^2 = 8$$

$$2x + x\left(\frac{dy}{dx}\right) + 1(y) + 4y\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}(x+4y) + 2x + y = 0$$

$$\frac{dy}{dx}(x+4y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x + 4y}$$

os yw $x = -3, y = 1$ gna $\frac{dy}{dx} = \frac{-2x - 3 - 1}{-3 + 4 \cdot 1}$

$$= \frac{5}{1}$$
$$= 5$$

Hafaliod y normal yn $(-3, 1)$:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 1 = -\frac{1}{5}(x - -3)$$

$$y - 1 = -\frac{1}{5}(x + 3)$$

$$y - 1 = -\frac{1}{5}x - \frac{3}{5}$$

$$y = -\frac{1}{5}x + \frac{2}{5}$$

Haf 2010

$$② \quad 5x^2 + 4xy - y^3 = 5$$

$$10x + 4x\left(\frac{dy}{dx}\right) + 4(y) - 3y^2\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}(4x - 3y^2) + 10x + 4y = 0$$

$$\frac{dy}{dx}(4x - 3y^2) = -10x - 4y$$

$$\frac{dy}{dx} = \frac{-10x - 4y}{4x - 3y^2}$$

$$\frac{dy}{dx} = \frac{10x + 4y}{3y^2 - 4x}$$

$$\text{OS yw } x=1, y=-2 \text{ yna } \frac{dy}{dx} = \frac{10x+4x-2}{3(-2)^2-4x}$$

$$= \frac{2}{8}$$

$$= \frac{1}{4}$$

Hafaliad y normal yn $(1, -2)$:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y + 2 = -4(x - 1)$$

$$y + 2 = -4x + 4$$

$$y = -4x + 2$$

Haf 2011

②

$$x^4 - 2x^2y + y^2 = 4$$

$$4x^3 - 2x^2 \left(\frac{dy}{dx} \right) - 4x(y) + 2y \left(\frac{dy}{dx} \right) = 0$$

$$\frac{dy}{dx} (-2x^2 + 2y) + 4x^3 - 4xy = 0$$

$$\frac{dy}{dx} (-2x^2 + 2y) = 4xy - 4x^3$$

$$\frac{dy}{dx} = \frac{4xy - 4x^3}{2y - 2x^2}$$

$$\frac{dy}{dx} = \frac{2xy - 2x^3}{y - x^2}$$

os yw $x=1, y=3$ yna $\frac{dy}{dx} = \frac{2 \times 1 \times 3 - 2 \times 1^3}{3 - 1^2}$

$$= \frac{4}{2}$$
$$= 2$$

Hafaliad y normal yn $(1, 3)$:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$y - 3 = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{7}{2}$$

Haf 2012

$$② \quad y^3 - 4x^2 - 3xy + 25 = 0$$

$$3y^2\left(\frac{dy}{dx}\right) - 8x - 3x\left(\frac{dy}{dx}\right) - 3(y) = 0$$

$$\frac{dy}{dx}(3y^2 - 3x) - 8x - 3y = 0$$

$$\frac{dy}{dx}(3y^2 - 3x) = 8x + 3y$$

$$\frac{dy}{dx} = \frac{8x + 3y}{3y^2 - 3x}$$

$$\text{OS gw } x=2, y=-3 \quad \text{yna } \frac{dy}{dx} = \frac{8x+3x-3}{3x(-3)^2-3x^2}$$
$$= \frac{7}{21}$$
$$= \frac{1}{3}$$

Hafaliad y tangiad yn (2, -3):

$$y - y_1 = m(x - x_1)$$

$$y - (-3) = \frac{1}{3}(x - 2)$$

$$y + 3 = \frac{1}{3}x - \frac{2}{3}$$

$$y = \frac{1}{3}x - \frac{11}{3}$$

Haf 2013

(2)

$$x^3 - 2xy^2 + y^3 = 5$$

$$3x^2 - 2x(2y \frac{dy}{dx}) - 2(y^2) + 3y^2(\frac{dy}{dx}) = 0$$

$$\frac{dy}{dx} (-4xy + 3y^2) + 3x^2 - 2y^2 = 0$$

$$\frac{dy}{dx} (-4xy + 3y^2) = 2y^2 - 3x^2$$

$$\frac{dy}{dx} = \frac{2y^2 - 3x^2}{3y^2 - 4xy}$$

OS yw $x=2, y=1$ yna

$$\frac{dy}{dx} = \frac{2 \times 1^2 - 3 \times 2^2}{3 \times 1^2 - 4 \times 2 \times 1}$$
$$= \frac{-10}{-5}$$
$$= 2$$

Hafaliad y normal yn $(2, 1)$:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$\therefore y - 1 = -\frac{1}{2}x + 1$$

$$y = -\frac{1}{2}x + 2$$

C4 Haf 2014

①

$$3x^3 - 5xy^2 + 2y^4 = 15$$

$$9x^2 - 5x(2y \frac{dy}{dx}) - 5(y^2) + 8y^3 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (8y^3 - 10xy) = 5y^2 - 9x^2$$

$$\frac{dy}{dx} = \frac{5y^2 - 9x^2}{8y^3 - 10xy}$$

$$\text{Os yw } x=1, y=2 \text{ yna } \frac{dy}{dx} = \frac{5(2^2) - 9(1^2)}{8(2^3) - 10(1)(2)}$$

$$= \frac{20 - 9}{64 - 20}$$

$$= \frac{1}{4}$$

Felly graddiant y tangiad yn $(1, 2)$ yw $\frac{1}{4}$
 Graddiant y normal yn $(1, 2)$ yw -4
 (negatif y cilydd)

Hafaliad y normal i C yn P:

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -4(x - 1)$$

$$y - 2 = -4x + 4$$

$$\underline{y = -4x + 6}$$

C4 Haf 2015

2) C: $x^4 + 3x^2y - 2y^2 = 34$

a) Differ: :

$$4x^3 + 6xy + 3x^2\left(1 \times \frac{dy}{dx}\right) - 4y\left(\frac{dy}{dx}\right) = 0$$

$$4x^3 + 6xy + \left(\frac{dy}{dx}\right)(3x^2 - 4y) = 0$$

$$\frac{dy}{dx}(3x^2 - 4y) = -4x^3 - 6xy$$

$$\frac{dy}{dx} = \frac{-4x^3 - 6xy}{3x^2 - 4y}$$

$$\frac{dy}{dx} = \frac{4x^3 + 6xy}{4y - 3x^2} \quad \checkmark$$

(lluosi Eop a gwaelod
efo -1)

b) Mae graddiant llinell sy'n baratol i'r echelin-y yn anfeidrol (∞). Er mwyn i $\frac{dy}{dx}$ fod yn anfeidrol mae'n rhaid bod enwadur y ffracciwn yn sero. Felly rydym angen

$$4y - 3x^2 = 0$$

$$4y = 3x^2$$

$$y = \frac{3}{4}x^2$$

Yn amnewid i mewn i C:

$$x^4 + 3x^2\left(\frac{3}{4}x^2\right) - 2\left(\frac{3}{4}x^2\right)^2 = 34$$

$$x^4 + \frac{9}{4}x^4 - 2\left(\frac{9}{16}x^4\right) = 34$$

$$x^4 + \frac{9}{4}x^4 - \frac{18}{16}x^4 = 34$$

$$\frac{17}{8}x^4 = 34$$

$$x^4 = 16$$

$$x = \pm \sqrt[4]{16}$$

$$x = \pm 2$$

Felly unai $x = 2$ neu $x = -2$

Yn amnewid yn ôl i $y = \frac{3}{4}x^2$:

$$\begin{aligned} \text{Unai } y &= \frac{3}{4}(2^2) & \text{neu } y &= \frac{3}{4}(-2)^2 \\ y &= 3 & y &= 3 \end{aligned}$$

Felly mae'r tangiad i c yn barael ir echelin-y
yn y pwyntiau (2, 3) a (-2, 3)

C4 Haf 2016

(3) C: $x^4 + 2x^3y - 3y^4 = 16$

a) Differ: $4x^3 + 6x^2y + 2x^3(\frac{dy}{dx}) - 12y^3(\frac{dy}{dx}) = 0$
 $\frac{dy}{dx}(2x^3 - 12y^3) = -4x^3 - 6x^2y$

$$\frac{dy}{dx} = \frac{-4x^3 - 6x^2y}{2x^3 - 12y^3}$$

$$\frac{dy}{dx} = \frac{2x^3 + 3x^2y}{-x^3 + 6y^3}$$

$$\frac{dy}{dx} = \frac{2x^3 + 3x^2y}{6y^3 - x^3} \quad \checkmark$$

[Lloesi Eop a
gwaelod efo $-\frac{1}{2}$]

b) Rydym ailen $\frac{dy}{dx} = -2$

$$\frac{2x^3 + 3x^2y}{6y^3 - x^3} = -2$$

$$2x^3 + 3x^2y = -2(6y^3 - x^3)$$

$$2x^3 + 3x^2y = -12y^3 + 2x^3$$

$$x^2y = -4y^3$$

$$x^2y + 4y^3 = 0$$

$$y(x^2 + 4y^2) = 0$$

Naill ai $y=0$

neu $x^2 + 4y^2 = 0$



Yr unig ddatrysiaid wrth $x=0, y=0$

gan fod hun yn sum ddaus gwâr ag
yn bositif ar gyfer unrhyw wellhoeddol
arall o x ag y.

OND, nid yw'r pwnt $(0;0)$ ar y
gronlin C gan fod amnewid $x=0,$
 $y=0$ i meun i ochr chwith C
ddim yn rhoi llo. Felly nid yw
 $x=0, y=0$ yn ddatrysiaid!

os yw $y=0$, yna brwy amnewid i meun i C:

$$x^4 + 2x^3(0) - 3(0^4) = 16$$

$$x^4 = 16$$

$$x = \pm \sqrt[4]{16}$$

$$x = \pm 2$$

Felly'r ddau bwynt ar C ble mae graddiant y
tangriad yn -2 yw $(2,0)$ a $(-2,0)$.

C4 Haf 2017

2) $y^6 - 3x^4 - 9x^2y + 48 = 0$

a) Differi: $6y^5 \left(\frac{dy}{dx} \right) - 12x^3 - 9x^2 \left(\frac{dy}{dx} \right) - 18xy = 0$

$$\frac{dy}{dx} (6y^5 - 9x^2) = 12x^3 + 18xy$$

$$\frac{dy}{dx} = \frac{12x^3 + 18xy}{6y^5 - 9x^2}$$

$$\frac{dy}{dx} = \frac{\beta(4x^3 + 6xy)}{\beta(2y^5 - 3x^2)}$$

$$\frac{dy}{dx} = \frac{6xy + 4x^3}{2y^5 - 3x^2} \quad \checkmark$$

b) Mae $y=0$ ym mhob pwynnt ble mae C yn croesir echelin- x . Yn amnewid $y=0$ i mewn i hafaliad y gronlin:

$$0^6 - 3x^4 - 9x^2(0) + 48 = 0$$

$$-3x^4 + 48 = 0$$

$$48 = 3x^4$$

$$16 = x^4$$

$$x = \pm \sqrt[4]{16}$$

$$x = \pm 2$$

Mae C yn croesir echelin- x yn y pwynbiau $(-2, 0)$ a $(2, 0)$.

Os yw $x = -2$ mae

$$\frac{dy}{dx} = \frac{6(-2)(0) + 4(-2)^3}{2(0)^5 - 3(-2)^2}$$

$$\frac{dy}{dx} = \frac{8}{3}$$

Os yw $x = 2$ mae

$$\frac{dy}{dx} = \frac{6(2)(0) + 4(2)^3}{2(0)^5 - 3(2)^2}$$

$$\frac{dy}{dx} = -\frac{8}{3}$$

C4 Haf 2018

2) C: $x^2 - y^3 - 3xy + 1 = 0$ $P = (-2, -1)$.

a) Differ: $2x - 3y^2 \left(\frac{dy}{dx} \right) - 3x(1) \left(\frac{dy}{dx} \right) - 3y = 0$

$$2x - 3y = 3y^2 \left(\frac{dy}{dx} \right) + 3x \left(\frac{dy}{dx} \right)$$

$$2x - 3y = (3y^2 + 3x) \left(\frac{dy}{dx} \right)$$

$$\frac{2x - 3y}{3y^2 + 3x} = \frac{dy}{dx}$$

os yw $x = -2$, $y = -1$, yna $\frac{dy}{dx} = \frac{2(-2) - 3(-1)}{3(-1)^2 + 3(-2)}$

$$\frac{dy}{dx} = \frac{-1}{-3}$$

$$\frac{dy}{dx} = \frac{1}{3}$$

Hafaliad y tangiad i C yn y pwnt P:

$$y - y_1 = m(x - x_1)$$

$$y - -1 = \frac{1}{3}(x - -2)$$

$$y + 1 = \frac{1}{3}(x + 2)$$

$$3(y + 1) = x + 2$$

$$3y + 3 = x + 2$$

$$3y + 1 = x$$

$$\underline{x = 3y + 1}$$



b) Hafaliadau cydamlserol:

$$x^2 - y^3 - 3xy + 1 = 0 \quad \text{---} \quad 1$$

$$x = 3y + 1 \quad \text{---} \quad 2$$

Yn amneuriad am x o ②; ①:

$$(3y+1)^2 - y^3 - 3(3y+1)y + 1 = 0$$

$$(3y+1)(3y+1) - y^3 - 3(3y+1)y + 1 = 0$$

$$9y^2 + 3y + 3y + 1 - y^3 - (9y+3)y + 1 = 0$$

$$9y^2 + 6y + 1 - y^3 - (9y^2 + 3y) + 1 = 0$$

$$\cancel{9y^2} + 6y + 1 - y^3 - \cancel{9y^2} - 3y + 1 = 0$$

$$-y^3 + 3y + 2 = 0$$

$$y^3 - 3y - 2 = 0$$

Mae $x = -2, y = -1$ yn ddatrysiaid i'r hafaliadau cydamlserol, felly mae $(y+1)$ yn ffactor o $y^3 - 3y - 2 = 0$.

$$\begin{array}{r} & y^2 + y - 2 \\ y+1 &) y^3 & -3y - 2 \\ y^3 + y^2 & \hline -y^2 - 3y - 2 \\ -y^2 - y & \hline -2y - 2 \\ -2y - 2 & \hline \end{array}$$

$$\begin{aligned} \text{Felly } y^3 - 3y - 2 &= (y+1)(y^2 + y - 2) \\ &= (y+1)(y-2)(y+1) \end{aligned}$$

$$\text{Datrys } y^3 - 3y - 2 = 0$$

$$(y+1)(y-2)(y+1) = 0$$

$$\text{Naill ai } y+1=0 \text{ neu } y-2=0$$

$$y = -1$$

$$\underline{y = 2}$$

[P]

$$\rightarrow \text{Felly } x = 3y + 1$$

$$x = 3(2) + 1$$

$$\underline{x = 7}$$

$$\text{Felly } \underline{Q = (7, 2)}$$

14 May 2019

2) $x^6 + 6x^3y + 16y^2 = 28 \quad -\textcircled{1}$

a) $6x^5 + 6x^3 \frac{dy}{dx} + 18x^2y + 32y \frac{dy}{dx} = 0$

$$\frac{dy}{dx}(6x^3 + 32y) = -6x^5 - 18x^2y \quad (\div 2)$$

$$\frac{dy}{dx}(3x^3 + 16y) = - (3x^5 + 9x^2y)$$

$$\begin{aligned}\frac{dy}{dx} &= - \frac{(3x^5 + 9x^2y)}{3x^3 + 16y} \\ &= - \frac{(3x^5 + 9x^2y)}{16y + 3x^3}\end{aligned}$$

QED.

b) //^{er} ir edelux pan fo $\frac{dy}{dx} = 0$

Felly gan mai graddunkt y normal i C yw

$$\frac{16y + 3x^3}{3x^5 + 9x^2y}$$

$$\text{mai } 16y + 3x^3 = 0$$

$$\Rightarrow 16y = -3x^3$$

$$y = -\frac{3x^3}{16}$$

Un ammenia i $\textcircled{1}$:

$$x^6 + 6x^3 \left(-\frac{3x^3}{16} \right) + 16 \left(-\frac{3x^3}{16} \right)^2 = 28$$

$$x^6 - \frac{18x^6}{16} + \frac{x^6}{16} = 28$$

$$16x^6 - 18x^6 + 9x^6 = 28 \times 16$$

$$7x^6 = 448$$

$$x^6 = 64 \Rightarrow x = 2 \text{ neu } x = -2$$

$$x = 2 \Rightarrow y = \frac{-3(2)^3}{16} = -\frac{3}{2}$$

$$x = -2 \Rightarrow y = \frac{-3(-2)^3}{16} = \frac{3}{2}$$

Felly cyfesurynnau y pwyseddau he mae'r
normal yn hanner i'r eddylun x yw:

$$\underline{(2, -\frac{3}{2})} \text{ a } \underline{(-2, \frac{3}{2})}$$