

Logarithmic Differentiation

Graef 2006

9) $y = x^{\frac{1}{x}}$

$$\ln(y) = \ln(x^{\frac{1}{x}})$$

$$\ln(y) = \frac{1}{x} \ln(x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(\frac{1}{x} \ln(x)\right)$$

$$\frac{dy}{dx} \frac{1}{y} = \frac{d}{dx}\left(\frac{1}{x} \ln(x)\right)$$

$$\frac{dy}{dx} \frac{1}{y} = \frac{1}{x} \left(\frac{1}{x}\right) + -x^{-2} \ln(x)$$

$$\frac{dy}{dx} \frac{1}{y} = \frac{1}{x^2} - \frac{1}{x^2} \ln(x)$$

$$\frac{dy}{dx} = y \left(\frac{1}{x^2} - \frac{1}{x^2} \ln(x)\right)$$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{x}}}{x^2} (1 - \ln(x))$$

(a) At the stationary point, $\frac{dy}{dx} = 0$,

$$\text{So } 0 = \frac{x^{\frac{1}{x}}}{x^2} (1 - \ln(x))$$

$$\text{Either } \frac{x^{\frac{1}{x}}}{x^2} = 0 \quad \text{or } 1 - \ln(x) = 0$$

$$\ln(x) = 1$$

$$\frac{1}{x} \ln(x) = \ln(1)$$

$$x = e$$

$$\text{No solution} \quad \text{So } y = e^{\frac{1}{e}}$$

The coordinates of the stationary point is $(e, e^{\frac{1}{e}})$.

(b) We will test a value either side of $e = 2.71828 \dots$ to determine the nature of the stationary point.

$$\left. \begin{array}{l} \text{If } x = 2.7, \text{ then } \frac{dy}{dx} = 0.001337 \dots \\ \text{If } x = 2.8, \text{ then } \frac{dy}{dx} = -0.005457 \dots \end{array} \right\} \text{(TABLE MODE)}$$

This shows that $(e, e^{\frac{1}{e}})$ is a maximum point.

Haf 2006

$$9) \quad y = x^{-x}$$
$$\ln(y) = \ln(x^{-x})$$

$$\ln(y) = -x \ln(x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(-x \ln(x))$$

$$\frac{dy}{dx} \frac{1}{y} = \frac{d}{dx}(-x \ln(x))$$

$$\frac{dy}{dx} \frac{1}{y} = -x \left(\frac{1}{x}\right) + \ln(x)(-1)$$

$$\frac{dy}{dx} \frac{1}{y} = -1 - \ln(x)$$

$$\frac{dy}{dx} = y(-1 - \ln(x))$$

$$\frac{dy}{dx} = -x^{-x}(1 + \ln(x))$$

(a) At the stationary point, $\frac{dy}{dx} = 0$,

$$\text{So } 0 = -x^{-x}(1 + \ln(x))$$

Either $-x^{-x} = 0$ or $1 + \ln(x) = 0$

$$x^{-x} = 0$$

$$-x \ln(x) = \ln(0)$$

$0 =$ No solution

$$\ln(x) = -1$$

$$x = e^{-1}$$

$$\text{So } y = (e^{-1})^{-e^{-1}}$$

The coordinate of the stationary point is $\left(\frac{1}{e}, \left(\frac{1}{e}\right)^{-\frac{1}{e}}\right)$

$$(b) \quad (i) \quad \frac{d^2y}{dx^2} = \frac{d}{dx}(-x^{-x}(1 + \ln(x)))$$

$$\frac{d^2y}{dx^2} = -x^{-x} \left(\frac{1}{x}\right) + (-x^{-x}(1 + \ln(x)))(1 + \ln(x))$$

~~$$\frac{1}{x^{-x}} \frac{d^2y}{dx^2} = -\frac{1}{x} + (1 + \ln(x))^2$$~~

$$x^{-x} \frac{d^2y}{dx^2} = -(x^{-x})^2 \left(\frac{1}{x}\right) + (x^{-x})^2 (1 + \ln(x))^2$$

$$y \frac{d^2y}{dx^2} = -y^2 \left(\frac{1}{x}\right) + (x^{-x}(1 + \ln(x)))^2$$

$$y \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 - \frac{y^2}{x}, \quad \text{QED}$$

$$(ii) \quad \frac{d^2y}{dx^2} = \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - \frac{y}{x}$$

At the stationary point, $\frac{dy}{dx} = 0$, so

$$\frac{d^2y}{dx^2} = -\frac{y}{x}$$

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{1}{e}\right)^{-\frac{1}{e}}}{\frac{1}{e}}$$

$$\frac{d^2y}{dx^2} = -3.927014 \dots$$

As $\frac{d^2y}{dx^2} < 0$, the point $\left(\frac{1}{e}, \left(\frac{1}{e}\right)^{-\frac{1}{e}}\right)$ must be a maximum point.

Gaeaf 2007

$$b) \quad y = x^{-\ln(x)}$$

$$\ln(y) = \ln(x^{-\ln(x)})$$

$$\ln(y) = -\ln(x) \ln(x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(-\ln(x) \ln(x))$$

$$\frac{dy}{dx} \frac{1}{y} = \frac{d}{dx}(-\ln(x) \ln(x))$$

$$\frac{dy}{dx} \frac{1}{y} = -\ln(x) \left(\frac{1}{x}\right) + \left(-\frac{1}{x}\right) \ln(x)$$

$$\frac{dy}{dx} \frac{1}{y} = -\frac{2}{x} \ln(x)$$

$$\frac{dy}{dx} = y \left(-\frac{2}{x} \ln(x)\right)$$

$$\frac{dy}{dx} = x^{-\ln(x)} \left(-\frac{2}{x}\right) \ln(x)$$

(a) At the stationary point, $\frac{dy}{dx} = 0$.

$$\text{So } 0 = x^{-\ln(x)} \left(-\frac{2}{x}\right) \ln(x)$$

Either $x^{-\ln(x)} = 0$ or $\left(-\frac{2}{x} = 0\right)$ or $\ln(x) = 0$

$$-\ln(x) \ln(x) = \ln(0)$$

$$-2 = 0$$

$$x = e^0$$

No solution

No solution

$$x = 1$$

$$\text{So } y = 1^{-\ln(1)}$$

$$y = 1$$

The coordinate of the stationary point is (1, 1).

(b) We will test a value either side of 1 to determine the nature of the stationary point.

$$\left. \begin{array}{l} \text{If } x = 0.9, \text{ then } \frac{dy}{dx} = 0.2315497631 \dots \\ \text{If } x = 1.1, \text{ then } \frac{dy}{dx} = -0.1717241815 \dots \end{array} \right\} \text{(TABLE MODE)}$$

This shows that (1, 1) is a maximum point.

Hat 2007

$$y = x^x$$

$$\ln(y) = \ln(x^x)$$

$$\ln(y) = x \ln(x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln(x))$$

$$\frac{dy}{dx} \frac{d}{dy}(\ln(y)) = \frac{d}{dx}(x \ln(x))$$

$$\frac{dy}{dx} \frac{1}{y} = (x) \left(\frac{1}{x}\right) + \ln(x)(1)$$

$$\frac{dy}{dx} \frac{1}{y} = 1 + \ln(x)$$

$$\frac{dy}{dx} = y(1 + \ln(x))$$

$$\frac{dy}{dx} = x^x(1 + \ln(x))$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^x(1 + \ln(x)))$$

$$\frac{d^2y}{dx^2} = (x^x) \left(\frac{1}{x}\right) + (1 + \ln(x)) (x^x(1 + \ln(x)))$$

$$\frac{d^2y}{dx^2} = (x^x)(x^{-1}) + x^x(1 + \ln(x))^2$$

$$\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = x^{x-1} + x^x(1 + \ln(x))^2$$

$$\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = x^x(1 + \ln(x))^2 + x^{x-1}, \quad \text{QED}$$

$$\frac{d^2y}{dx^2}$$

Gaeaf 2008

9) (a) $y = x^{-\sqrt{x}}$

$$\ln(y) = \ln(x^{-\sqrt{x}})$$

$$\ln(y) = -\sqrt{x} \ln(x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(-\sqrt{x} \ln(x))$$

$$\frac{dy}{dx} \frac{d}{dy}(\ln(y)) = \frac{d}{dx}(-\sqrt{x} \ln(x))$$

$$\frac{dy}{dx} \frac{1}{y} = (-\sqrt{x}) \frac{1}{x} + \ln(x) \left(-\frac{1}{2} x^{-\frac{1}{2}}\right)$$

$$\frac{dy}{dx} \frac{1}{y} = -x^{\frac{1}{2}} x^{-1} - \frac{\ln(x)}{2\sqrt{x}}$$

$$\frac{dy}{dx} \frac{1}{y} = -\frac{1}{\sqrt{x}} - \frac{\ln(x)}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{y}{\sqrt{x}} \left(1 + \frac{1}{2} \ln(x)\right)$$

$$\frac{dy}{dx} = -\frac{x^{-\sqrt{x}}}{\sqrt{x}} \left(1 + \frac{1}{2} \ln(x)\right)$$

(b) At the stationary point, $\frac{dy}{dx} = 0$.

$$\text{So } 0 = \frac{-x^{-\sqrt{x}}}{\sqrt{x}} \quad \text{or} \quad 0 = 1 + \frac{1}{2} \ln(x)$$

$$0 = -x^{-\sqrt{x}}$$

$$-2 = \ln(x)$$

$$\underline{\underline{x = e^{-2}}}$$

$$\ln(0) = -\sqrt{x} \ln(-x)$$

No solution

We will test a value either side of $e^{-2} = 0.13533 \dots$ to determine the nature of the stationary point.

$$\text{If } x = 0.1, \text{ then } \frac{dy}{dx} = 0.990935 \dots$$

$$\text{If } x = 0.2, \text{ then } \frac{dy}{dx} = -0.896879 \dots$$

This shows that the stationary point is a maximum point.

Haf 2008

$$8) \quad y = x^{\cos(x)}$$

$$\ln(y) = \ln(x^{\cos(x)})$$

$$\ln(y) = \cos(x) \ln(x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(\cos(x) \ln(x))$$

$$\frac{dy}{dx} \frac{d}{dy}(\ln(y)) = \frac{d}{dx}(\cos(x) \ln(x))$$

$$\frac{dy}{dx} \frac{1}{y} = \cos(x) \left(\frac{1}{x} \right) - \sin(x) \ln(x)$$

$$\frac{dy}{dx} = y \left(\frac{\cos(x)}{x} - \sin(x) \ln(x) \right)$$

$$\frac{dy}{dx} = x^{\cos(x)} \left(\frac{\cos(x)}{x} - \sin(x) \ln(x) \right)$$

(b) At the stationary point, $\frac{dy}{dx} = 0$.

$$\text{So } 0 = x^{\cos(x)} \quad \text{or} \quad 0 = \frac{\cos(x)}{x} - \sin(x) \ln(x)$$

$$\ln(x) = \frac{\cos(x)}{\sin(x) x}$$

No solution

$$\sin(x) \ln(x) = \frac{\cos(x)}{x}$$

$$\tan(x) \ln(x) = \frac{1}{x}$$

$$x \tan(x) \ln(x) = 1$$

Thus if α is the x -coordinate of the stationary point we have

$$\alpha \tan(\alpha) \ln(\alpha) = 1$$

Thus $\alpha \ln(\alpha) \tan(\alpha) = 1$. QED.

$$(ii) \quad \text{When } \alpha \approx 1.27, \quad \alpha \ln(\alpha) \tan(\alpha) = 0.978538573 \dots$$

$$\text{When } \alpha = 1.28, \quad \alpha \ln(\alpha) \tan(\alpha) = 1.055802773 \dots$$

As these values lie either side of 1,

α must lie between 1.27 and 1.28.

Geaf 2009

1) (a) $y = 2^x$

$$\ln(y) = \ln(2^x)$$

$$\ln(y) = x \ln(2)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln(2))$$

$$\frac{dy}{dx} \frac{d}{dy}(\ln(y)) = \frac{d}{dx}(x \ln(2))$$

$$\frac{dy}{dx} \frac{1}{y} = \ln(2)$$

$$\frac{dy}{dx} = y \ln(2)$$

$$\frac{dy}{dx} = 2^x \ln(2)$$

(b) (Differentiation from first principles - different topic)

Haf 2009

9) $y = x^x e^{-2x}$

(a) (i) $\ln(y) = \ln(x^x e^{-2x})$

$$\ln(y) = \ln(x^x) + \ln(e^{-2x})$$

$$\ln(y) = x \ln(x) + (-2x)$$

$$\ln(y) = x \ln(x) - 2x$$

(ii) $\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln(x) - 2x)$

$$\frac{dy}{dx} \frac{d}{dy}(\ln(y)) = \frac{d}{dx}(x \ln(x) - 2x)$$

$$\frac{dy}{dx} \frac{1}{y} = x \left(\frac{1}{x}\right) + \ln(x)(1) - 2$$

$$\frac{dy}{dx} \frac{1}{y} = 1 + \ln(x) - 2$$

$$\frac{dy}{dx} = y (\ln(x) - 1)$$

$$\frac{dy}{dx} = x^x e^{-2x} (\ln(x) - 1)$$

Thus $f'(x) = f(x) (\ln(x) - 1)$

($a=1$, $b=-1$)

$$(b) f'(x) = f(x)(\ln(x) - 1)$$

$$f''(x) = \frac{d}{dx} (f(x)(\ln(x) - 1))$$

$$f''(x) = f'(x)(\ln(x) - 1) + f(x)\left(\frac{1}{x}\right)$$

(c) At the stationary point, $\frac{dy}{dx} = 0$.

$$\text{So } 0 = x^x e^{-2x} (\ln(x) - 1)$$

$$\text{Either } x^x e^{-2x} = 0 \quad \text{or } \ln(x) - 1 = 0$$

$$\ln(x^x e^{-2x}) = \ln(0)$$

No solution

$$\ln(x) = 1$$

$$x = e^1$$

$$x = e$$

$$\text{So } y = e^e e^{-2e}$$

$$y = e^{-e}$$

The coordinate of the stationary point is (e, e^{-e})

We will test a value either side of $e = 2.71828 \dots$ to determine the nature of the stationary point.

$$\text{If } x = 2.7, \text{ then } \frac{dy}{dx} = -0.00044532963$$

$$\text{If } x = 2.8, \text{ then } \frac{dy}{dx} = 0.00195690569$$

(TABLE MODE)

This shows that (e, e^{-e}) is a minimum point.

(Alternate answer: substitute into $f''(x)$ and discover a positive value.)

Graef 2010

$$7) y = (\operatorname{cosec} x)^x$$

$$(a) \ln(y) = \ln(\operatorname{cosec}(x))^x$$

$$\ln(y) = x \ln(\operatorname{cosec}(x))$$

$$\frac{d}{dx} (\ln(y)) = \frac{d}{dx} (x \ln(\operatorname{cosec}(x)))$$

$$\frac{dy}{dx} \frac{d}{dy} (\ln(y)) = \frac{d}{dx} (x \ln(\operatorname{cosec}(x)))$$

$$\frac{dy}{dx} \frac{1}{y} = x \left(\frac{-\operatorname{cosec}(x) \cot(x)}{\operatorname{cosec}(x)} \right) + \ln(\operatorname{cosec}(x))$$

$$\frac{dy}{dx} \frac{1}{y} = -x \cot(x) + \ln(\operatorname{cosec}(x))$$

$$\frac{dy}{dx} = y(\ln(\operatorname{cosec}(x)) - x \cot(x))$$

$$\frac{dy}{dx} = (\operatorname{cosec}(x))^x (\ln(\operatorname{cosec}(x)) - x \cot(x))$$

(b) At the stationary point, $\frac{dy}{dx} = 0$.

$$\text{So } 0 = (\operatorname{cosec}(x))^x (\ln(\operatorname{cosec}(x)) - x \cot(x))$$

$$\text{Either } (\operatorname{cosec}(x))^x = 0 \quad \text{or} \quad \ln(\operatorname{cosec}(x)) - x \cot(x) = 0$$

$$x \ln(\operatorname{cosec}(x)) = \ln(0)$$

No solution

$$\ln(\operatorname{cosec}(x)) = x \cot(x)$$

$$x = \frac{\ln(\operatorname{cosec}(x))}{\cot(x)}$$

$$x = \ln(\operatorname{cosec}(x)) \tan(x)$$

Thus if α is the x -coordinate of the stationary point we have

$$\alpha = \ln(\operatorname{cosec}(\alpha)) \tan(\alpha)$$

$$\alpha = \tan(\alpha) \ln(\operatorname{cosec}(\alpha))$$

Q.E.D.

$$(ii) \alpha_{n+1} = \tan(\alpha_n) \ln(\operatorname{cosec}(\alpha_n))$$

$$\alpha_0 = 0.5$$

$$\alpha_1 = 0.4016233912$$

$$\alpha_2 = 0.3989156193$$

$$\alpha_3 = 0.3986145464$$

$$\alpha_4 = 0.3985802327$$

$$\alpha_5 = 0.3985763112$$

$$\alpha_6 = 0.3985758629$$

$$\alpha_7 = 0.3985758117$$

$$\alpha_8 = 0.3985758058$$

Thus, to 4 d.p., the value of α is 0.3986

Half 2010

7) $y = x^{-2x}$

$\ln(y) = \ln(x^{-2x})$

$\ln(y) = -2x \ln(x)$

$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(-2x \ln(x))$

$\frac{dy}{dx} \frac{d}{dy}(\ln(y)) = \frac{d}{dx}(-2x \ln(x))$

$\frac{dy}{dx} \frac{1}{y} = (-2x) \left(\frac{1}{x}\right) - 2 \ln(x)$

$\frac{dy}{dx} \frac{1}{y} = -2 - 2 \ln(x)$

$\frac{dy}{dx} = -2y(1 + \ln(x))$

$\frac{dy}{dx} = -2x^{-2x}(1 + \ln(x))$

(b) At the stationary point, $\frac{dy}{dx} = 0$

so $0 = -2x^{-2x}(1 + \ln(x))$

Either $-2x^{-2x} = 0$ or $1 + \ln(x) = 0$

$-2x \ln(-2x) = \ln(0)$

$\ln(x) = -1$

$x = e^{-1}$

No solution

so $y = (e^{-1})^{-2e^{-1}}$

$y = e^{2e^{-1}}$

$y \approx 2.087065229 \dots$

The stationary value of $f(x)$ is $e^{2e^{-1}} \approx 2.087065229 \dots$

We will test a value either side of e^{-1} to determine the nature of the stationary point.

If $x = 0.3$, then $\frac{dy}{dx} = 0.8400971467 \dots$

If $x = 0.4$, then $\frac{dy}{dx} = -0.3484620983 \dots$

This shows that $(e^{-1}, e^{2e^{-1}})$ is a maximum point.

Graef 2011

$$(7) \quad y = 2^x \times 3^{\frac{1}{x}}$$

$$\ln(y) = \ln(2^x \times 3^{\frac{1}{x}})$$

$$\ln(y) = \ln(2^x) + \ln(3^{\frac{1}{x}})$$

$$\ln(y) = x \ln(2) + \left(\frac{1}{x}\right) \ln(3)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(x \ln(2) + \left(\frac{1}{x}\right) \ln(3)\right)$$

$$\frac{dy}{dx} \frac{1}{y} = \frac{d}{dx}\left(x \ln(2) + \left(\frac{1}{x}\right) \ln(3)\right)$$

$$\frac{dy}{dx} \frac{1}{y} = \ln(2) + \frac{d}{dx}\left(x^{-1} \ln(3)\right)$$

$$\frac{dy}{dx} \frac{1}{y} = \ln(2) - x^{-2}(\ln(3))$$

$$\frac{dy}{dx} = y \left(\ln(2) - \frac{\ln(3)}{x^2} \right)$$

$$\frac{dy}{dx} = \left(2^x \times 3^{\frac{1}{x}}\right) \left(\ln(2) - \frac{\ln(3)}{x^2} \right)$$

$$\text{Felly } f'(x) = \left(2^x \times 3^{\frac{1}{x}}\right) \left(\ln(2) - \frac{\ln(3)}{x^2} \right)$$

(b) At the stationary point, $\frac{dy}{dx} = 0$.

Either $2^x \times 3^{\frac{1}{x}} = 0$ (or) $\ln(2) - \frac{\ln(3)}{x^2} = 0$

$$\ln(2^x \times 3^{\frac{1}{x}}) = \ln(0)$$

No solution

$$\ln(2) = \frac{\ln(3)}{x^2}$$

$$x^2 \ln(2) = \ln(3)$$

$$x^2 = \frac{\ln(3)}{\ln(2)}$$

$$x = \pm \sqrt{\frac{\ln(3)}{\ln(2)}}$$

As $f(x)$ is only defined

for $x > 0$ we take

$$x = \sqrt{\frac{\ln(3)}{\ln(2)}}$$

We will test a value either side of $\sqrt{\frac{\ln(3)}{\ln(2)}}$ to determine

the nature of the stationary point.

$$\sqrt{\frac{\ln(3)}{\ln(2)}} = 1.259 \text{ to 3 d.p.}$$

$$\left. \begin{array}{l} \text{If } x = 1.2, \text{ then } \frac{dy}{dx} = -0.400456 \dots \\ \text{If } x = 1.3, \text{ then } \frac{dy}{dx} = 0.24696 \dots \end{array} \right\} \text{(TABLE MODE)}$$

This shows that there is a minimum point

$$\text{when } x = \sqrt{\frac{\ln(3)}{\ln(2)}}$$

Græaf 2012

$$f(x) = (\sin x)^x$$

$$y = (\sin x)^x$$

$$\ln(y) = \ln((\sin x)^x)$$

$$\ln(y) = x \ln(\sin x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln(\sin x))$$

$$\frac{dy}{dx} \frac{d}{dy}(\ln(y)) = x \frac{d}{dx}(\ln(\sin x)) + \ln(\sin x) \frac{d}{dx}(x)$$

$$\frac{dy}{dx} \frac{1}{y} = \frac{x \cos(x)}{\sin(x)} + \ln(\sin x)$$

$$\frac{dy}{dx} = y \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right)$$

$$\frac{dy}{dx} = (\sin x)^x \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right)$$

$$\text{So } f'(x) = (\sin x)^x \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right)$$

$$\text{(We have } g(x) = \frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \text{)}$$

$$\left. \begin{array}{l} \text{(b) If } x=0.39, \text{ then } \frac{dy}{dx} = -0.01255 \dots \\ \text{If } x=0.40, \text{ then } \frac{dy}{dx} = 0.0020489 \dots \end{array} \right\} \text{(TABLE MODE)}$$

A change of sign in $\frac{dy}{dx}$ between 0.39 and 0.40 indicates that there is a stationary point between $x=0.39$ and $x=0.40$.

$$\text{(c) } f'(x) = (\sin x)^x \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right)$$

$$f''(x) = \frac{d}{dx} \left((\sin x)^x \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right) \right)$$

$$f''(x) = \frac{d}{dx} \left((\sin x)^x \right) \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right) + (\sin x)^x \frac{d}{dx} \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right)$$

$$f''(x) = (\sin x)^x \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right)^2 + (\sin x)^x \frac{d}{dx} \left(x \cot(x) + \ln(\sin x) \right)$$

$$f''(x) = (\sin x)^x \left(x \cot(x) + \ln(\sin x) \right)^2 + (\sin x)^x \left(\cot(x) + x(-\operatorname{cosec}^2(x)) + \frac{\cos(x)}{\sin(x)} \right)$$

$$\cancel{f''(x) = f'(x) + f(x)g'(x)}$$

$$f''(x) = f'(x) (x \cot(x) + \ln(\sin x)) + f(x)g'(x)$$

Therefore

$$f''(\alpha) = f'(\alpha) (x \cot(x) + \ln(\sin \alpha)) + f(\alpha)g'(\alpha)$$

But α is a stationary point, so $f'(\alpha) = 0$.

Therefore

$$f''(\alpha) = f(\alpha)g'(\alpha) \text{ as required.}$$

If $\alpha = 0.399$

Then $f''(\alpha) = f(\alpha)g'(\alpha)$

$$= (\sin \alpha)^\alpha (\cot \alpha + \alpha(-\operatorname{cosec}^2 \alpha)) + \cot \alpha$$

$$= (\sin 0.399)^{0.399} (\cot 0.399 + 0.399(-\operatorname{cosec}^2(0.399)) + \cot(0.399))$$

~~1.475657581~~

1.440112262

This is a positive value so $f(\alpha)$ is a minimum point.

Haf 2012

(8) (a) $F(x) = x^x$
 $y = x^x$

$\ln(y) = \ln(x^x)$

$\ln(y) = x \ln(x)$

$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln(x))$

$\frac{dy}{dx} \frac{1}{y} (\ln(y)) = x \left(\frac{1}{x} \right) + (1) \ln(x)$

$\frac{dy}{dx} \left(\frac{1}{y} \right) = 1 + \ln(x)$

$\frac{dy}{dx} = y(1 + \ln(x))$

$\frac{dy}{dx} = x^x(1 + \ln(x))$

(b) At the stationary point, $\frac{dy}{dx} = 0$

so $0 = x^x(1 + \ln(x))$

Either $x^x = 0$ or $1 + \ln(x) = 0$

$x \ln(x) = \ln(0)$

$\ln(x) = -1$

No solution $x = e^{-1} \approx 0.3679$ to 4 d.p.

The stationary value of $F(x)$ is $(e^{-1})^{e^{-1}} \approx 0.6922$ to 4 d.p.

\therefore The coordinates of the stationary points are $(0.3679, 0.6922)$ or $(e^{-1}, (e^{-1})^{e^{-1}})$.

$$c) f'(x) = x^x(1 + \ln(x))$$

$$f''(x) = \frac{d}{dx}(x^x(1 + \ln(x)))$$

$$f''(x) = \frac{d}{dx}(x^x)(1 + \ln(x)) + x^x \frac{d}{dx}(1 + \ln(x))$$

$$f''(x) = f'(x)(1 + \ln(x)) + x^x \left(-\frac{1}{x}\right)$$

$$f''(x) = x^x(1 + \ln(x))(1 + \ln(x)) + x^x \left(-\frac{1}{x}\right)$$

$$f''(x) = x^x(1 + \ln(x))^2 + x^x(x^{-1})$$

$$f''(x) = x^x(1 + \ln(x))^2 + x^{x-1}$$

$$f''(x) = x^{x-1} + x^x(1 + \ln(x))^2$$

QED.

$$\text{Now } f''(e^{-1}) = (e^{-1})^{e^{-1}-1} + (e^{-1})^{e^{-1}}(1 + \ln(e^{-1}))^2 \\ = 1.881596388 \dots$$

Therefore the stationary point $(e^{-1}, (e^{-1})^{e^{-1}})$
is a minimum point. $\left(\frac{d^2y}{dx^2} > 0\right)$

FPI lonawr 2013

(a) ⑦ $y = x^{\ln(x)}$
 $\ln(y) = \ln(x^{\ln(x)})$
 $\ln(y) = \ln(x) \ln(x)$
 $\ln(y) = (\ln(x))^2$
 $\frac{d}{dx}(\ln(y)) = \frac{d}{dx}((\ln(x))^2)$
 $\frac{dy}{dx} \frac{d}{dy}(\ln(y)) = \frac{d}{dx}((\ln(x))^2)$
 $\frac{dy}{dx} \frac{1}{y} = 2(\ln(x)) \frac{1}{x}$
 $\frac{dy}{dx} = \frac{2y \ln(x)}{x}$
 $\frac{dy}{dx} = \frac{2x^{\ln(x)} \ln(x)}{x}$
So $f'(x) = \frac{2x^{\ln(x)} \ln(x)}{x}$

(b) At the stationary point, $\frac{dy}{dx} = 0$
So $0 = \frac{2x^{\ln(x)} \ln(x)}{x}$
Either $\frac{2x^{\ln(x)}}{x} = 0$ or $\ln(x) = 0$
 $x = e^0$
 $x = 1$
No solution - need to use $\ln(0)$

Now $y = (1)^{\ln(1)}$
 $y = 1^0$
 $y = 1$

The coordinate of the stationary point is (1, 1).

→ Nature:
 $f'(0.99) = \frac{2(0.99)^{\ln(0.99)} \ln(0.99)}{0.99}$
 $= -0.02 \dots$
 $f'(1.01) = \frac{2(1.01)^{\ln(1.01)} \ln(1.01)}{1.01}$
 $= 0.0197 \dots$

So the stationary point is a minimum point.

FPI Haf 2013

$$⑦ \quad f(x) = \frac{\sqrt{1 + \sin x}}{(1 + \tan x)^2}$$

$$y = \frac{\sqrt{1 + \sin x}}{(1 + \tan x)^2}$$

$$\ln(y) = \ln\left(\frac{\sqrt{1 + \sin x}}{(1 + \tan x)^2}\right)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(\ln\left(\frac{(1 + \sin x)^{\frac{1}{2}}}{(1 + \tan x)^2}\right)\right)$$

$$\frac{dy}{dx} \cdot \frac{d}{dy}(\ln(y)) = \frac{d}{dx}\left(\ln\left(\frac{(1 + \sin x)^{\frac{1}{2}}}{(1 + \tan x)^2}\right)\right)$$

$$\frac{dy}{dx} \left(\frac{1}{y}\right) = \frac{d}{dx}\left(\ln((1 + \sin x)^{\frac{1}{2}}) - \ln((1 + \tan x)^2)\right)$$

$$\frac{dy}{dx} \left(\frac{1}{y}\right) = \frac{\frac{1}{2}(1 + \sin x)^{-\frac{1}{2}}(\cos x)}{(1 + \sin x)^{\frac{1}{2}}} - \frac{2(1 + \tan x) \sec^2 x}{(1 + \tan x)^2}$$

$$\frac{dy}{dx} \left(\frac{1}{y}\right) = \frac{\cos x}{2(1 + \sin x)^{\frac{1}{2}}(1 + \sin x)^{\frac{1}{2}}} - \frac{2 \sec^2 x}{1 + \tan x}$$

$$\frac{dy}{dx} \left(\frac{1}{y}\right) = \frac{\cos x}{2(1 + \sin x)} - \frac{2 \sec^2 x}{1 + \tan x}$$

$$\frac{dy}{dx} = y \left(\frac{\cos x}{2(1 + \sin x)} - \frac{2 \sec^2 x}{1 + \tan x} \right)$$

$$\frac{dy}{dx} = \frac{\sqrt{1 + \sin x}}{(1 + \tan x)^2} \left(\frac{\cos x}{2(1 + \sin x)} - \frac{2 \sec^2 x}{1 + \tan x} \right)$$

Os yw $x = \frac{\pi}{4}$, yna

$$\frac{dy}{dx} = \frac{\sqrt{1 + \sin\left(\frac{\pi}{4}\right)}}{(1 + \tan\left(\frac{\pi}{4}\right))^2} \left(\frac{\cos\left(\frac{\pi}{4}\right)}{2(1 + \sin\left(\frac{\pi}{4}\right))} - \frac{2}{\cos^2\left(\frac{\pi}{4}\right)(1 + \tan\left(\frac{\pi}{4}\right))} \right)$$

$$\frac{dy}{dx} = -0.5856319699$$

$$\frac{dy}{dx} = -0.586 \quad \text{i 3 ffigur ystyrbon}$$

FPI Gaeaf 2014

⑧ $f(x) = \left(\frac{1}{x}\right)^{\sqrt{x}}$ ar gyfer $x > 0$

a) $y = \left(\frac{1}{x}\right)^{\sqrt{x}}$

$$\ln(y) = \ln\left[\left(\frac{1}{x}\right)^{\sqrt{x}}\right]$$

$$\ln(y) = \sqrt{x} \ln\left(\frac{1}{x}\right)$$

$$\ln(y) = x^{\frac{1}{2}} \ln\left(\frac{1}{x}\right)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(x^{\frac{1}{2}} \ln\left(\frac{1}{x}\right)\right)$$

$$\frac{dy}{dx} \times \frac{d}{dy}(\ln(y)) = \frac{d}{dx}\left(x^{\frac{1}{2}} \ln(x^{-1})\right)$$

$$\frac{dy}{dx} \times \frac{1}{y} = x^{\frac{1}{2}} \left(\frac{-x^{-2}}{x^{-1}}\right) + \frac{1}{2} x^{-\frac{1}{2}} \ln(x^{-1})$$

$$\frac{dy}{dx} \times \frac{1}{y} = x^{\frac{1}{2}} \left(\frac{-x}{x^2}\right) + \frac{1}{2\sqrt{x}} \ln\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} \times \frac{1}{y} = x^{\frac{1}{2}} \left(\frac{-1}{x}\right) + \frac{1}{2\sqrt{x}} \ln\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} \times \frac{1}{y} = -x^{-\frac{1}{2}} + \frac{1}{2\sqrt{x}} \ln\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = y \left(-\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x}} \ln\left(\frac{1}{x}\right)\right)$$

$$\frac{dy}{dx} = y \left(\frac{\ln\left(\frac{1}{x}\right) - 2}{2\sqrt{x}}\right)$$

$$\frac{dy}{dx} = \left(\frac{1}{x}\right)^{\sqrt{x}} \left(\frac{\ln\left(\frac{1}{x}\right) - 2}{2\sqrt{x}}\right)$$

$$\frac{dy}{dx} = \left(\frac{1}{x}\right)^{\sqrt{x}} \left(\frac{\ln(1) - \ln(x) - 2}{2\sqrt{x}}\right)$$

$$\frac{dy}{dx} = \left(\frac{1}{x}\right)^{\sqrt{x}} \left(\frac{-\ln(x) - 2}{2\sqrt{x}}\right)$$

Felly $g(x) = \frac{-\ln(x) - 2}{2\sqrt{x}}$

b) Pwyntiau arhosol: $f'(x) = 0$

$$\text{Felly } \left(\frac{1}{x}\right)^{\sqrt{x}} \left(\frac{-\ln(x) - 2}{2\sqrt{x}}\right) = 0$$

$$\text{Unai } \left(\frac{1}{x}\right)^{\sqrt{x}} = 0 \quad \text{neu } \frac{-\ln(x) - 2}{2\sqrt{x}} = 0$$

$$\ln\left(\left(\frac{1}{x}\right)^{\sqrt{x}}\right) = \ln(0)$$

Dim datrysiaid

$$-\ln(x) - 2 = 0$$

$$-\ln(x) = 2$$

$$\ln(x) = -2$$

$$x = e^{-2}$$

$$\text{Nawr } y = \left(\frac{1}{e^{-2}}\right)^{\sqrt{e^{-2}}}$$

$$y = (e^2)^{\sqrt{e^{-2}}}$$

Felly cyfesurynnau'r pwynt arhosol yw $(e^{-2}, (e^2)^{\sqrt{e^{-2}}})$.
I dri ffigur gstyrlon: (0.135, 2.09)

c) Amnewid $x = 0.1$ i mewn i $f'(x)$:

$$f'(0.1) = \left(\frac{1}{0.1}\right)^{\sqrt{0.1}} \left(\frac{-\ln(0.1) - 2}{2 \times \sqrt{0.1}}\right)$$

$$= 0.9909 \dots$$

Amnewid $x = 0.2$ i mewn i $f'(x)$:

$$f'(0.2) = \left(\frac{1}{0.2}\right)^{\sqrt{0.2}} \left(\frac{-\ln(0.2) - 2}{2 \times \sqrt{0.2}}\right)$$

$$= -0.8968 \dots$$

Mae $f'(x)$ yn bositif ar gyfer $x < e^{-2}$ ac yn negatïf ar gyfer $x > e^{-2}$ felly maximum yw'r pwynt arhosol.

(b) $f(x) = (\sec x)^x$ or gylfer $(0, \frac{\pi}{2})$

$$y = (\sec x)^x$$

$$\ln(y) = \ln[(\sec x)^x]$$

$$\ln(y) = x \ln(\sec x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln(\sec x))$$

$$\frac{dy}{dx} \times \frac{1}{y} = \frac{d}{dx}(x \ln(\sec x))$$

$$\frac{dy}{dx} \times \frac{1}{y} = (x) \frac{\sec x \tan x}{\sec x} + (1) \ln(\sec x)$$

$$\frac{dy}{dx} \times \frac{1}{y} = x \tan x + \ln(\sec x)$$

$$\frac{dy}{dx} = y(x \tan x + \ln(\sec x))$$

$$\frac{dy}{dx} = (\sec x)^x (x \tan x + \ln(\sec x))$$

FPI Haf 2015

9) $f(x) = 2^x \sin x$ if $x \in (0, \pi)$

Let $y = 2^x \sin x$

$$\ln(y) = \ln(2^x \sin x)$$

$$\ln(y) = \ln(2^x) + \ln(\sin x)$$

$$\ln(y) = x \ln(2) + \ln(\sin x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln(2) + \ln(\sin x))$$

$$\frac{dy}{dx} \cdot \frac{d}{dy}(\ln(y)) = \frac{d}{dx}(x \ln(2) + \ln(\sin x))$$

$$\frac{dy}{dx} \left(\frac{1}{y} \right) = \ln(2) + \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = y (\ln(2) + \cot x)$$

$$\frac{dy}{dx} = 2^x \sin x (\ln(2) + \cot(x))$$

So $f'(x) = 2^x \sin x (\ln(2) + \cot(x))$

b) At the stationary point, $\frac{dy}{dx} = 0$.

$$\text{So } 0 = 2^x \sin x (\ln(2) + \cot(x))$$

Either $2^x \sin x = 0$

$$\ln(2^x \sin x) = \ln(0)$$

↑
x

No solution.

~~S/A~~
~~T/C~~

or $\ln(2) + \cot(x) = 0$

$$\cot(x) = -\ln(2)$$

$$\tan(x) = -\frac{1}{\ln(2)}$$

$$x = \tan^{-1} \left(\frac{-1}{\ln(2)} \right)$$

$$x = -0.96 \text{ rad}$$

or $x = 2.18 \text{ rad}$ to 2 d.p.

FPI Haf 2016

⑧ $f(x) = x^{\sin x}$ defined on the domain $(0, \frac{\pi}{2})$

a) Let $y = x^{\sin x}$

$$\ln(y) = \ln(x^{\sin x})$$

$$\ln(y) = \sin(x) \ln(x)$$

$$\frac{d}{dx} [\ln(y)] = \frac{d}{dx} [\sin(x) \ln(x)]$$

$$\frac{dy}{dx} \times \frac{d}{dy} [\ln(y)] = \frac{d}{dx} [\sin(x) \ln(x)]$$

$$\frac{dy}{dx} \frac{1}{y} = \sin(x) \left(\frac{1}{x} \right) + \cos(x) \ln(x)$$

$$\frac{dy}{dx} = y \left(\frac{\sin(x)}{x} + \cos(x) \ln(x) \right)$$

$$\frac{dy}{dx} = x^{\sin x} \left(\frac{\sin(x)}{x} + \cos(x) \ln(x) \right)$$

$$\therefore f'(x) = x^{\sin x} \left(\frac{\sin(x)}{x} + \cos(x) \ln(x) \right)$$

b) At the stationary point, $\frac{dy}{dx} = 0$

$$\text{So } 0 = x^{\sin x} \left(\frac{\sin(x)}{x} + \cos(x) \ln(x) \right)$$

Either $x^{\sin x} = 0$

$$\sin(x) \ln(x) = \ln(0)$$

No solution

or $\frac{\sin(x)}{x} + \cos(x) \ln(x) = 0$

$$\frac{\sin(x)}{x} = -\cos(x) \ln(x)$$

$$\sin(x) = -x \cos(x) \ln(x)$$

$$\sin(x) + x \cos(x) \ln(x) = 0$$

When $x = 0.35$, $\sin(x) + x \cos(x) \ln(x) = -0.00226 \dots$

When $x = 0.36$, $\sin(x) + x \cos(x) \ln(x) = 0.00805 \dots$

The change of sign indicates the x -coordinate of the stationary point lies between 0.35 and 0.36

FPI Haf 2017

7) $f(x) = (\tan x)^{\tan x}$ if $x \in (0, \frac{\pi}{2})$

a) Let $y = (\tan x)^{\tan x}$

$$\ln(y) = \ln((\tan x)^{\tan x})$$

$$\ln(y) = \tan x \ln(\tan x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(\tan x \ln(\tan x))$$

$$\frac{dy}{dx} \cdot \frac{d}{dy}(\ln(y)) = \cancel{\tan x} \left(\frac{\sec^2 x}{\cancel{\tan x}} \right) + \sec^2 x \ln(\tan x)$$

$$\frac{dy}{dx} \left(\frac{1}{y} \right) = \sec^2 x + \sec^2 x \ln(\tan x)$$

$$\frac{dy}{dx} = y \sec^2 x (1 + \ln(\tan x))$$

$$\frac{dy}{dx} = (\tan x)^{\tan x} \sec^2 x (1 + \ln(\tan x))$$

$$\therefore f'(x) = (\tan x)^{\tan x} \sec^2 x (1 + \ln(\tan x))$$

so that $g(x) = (\tan x)^{\tan x} \sec^2 x$.

b) At the stationary point, $\frac{dy}{dx} = 0$.

so $0 = (\tan x)^{\tan x} \sec^2 x (1 + \ln(\tan x))$

Either $(\tan x)^{\tan x} \sec^2 x = 0$ or $1 + \ln(\tan x) = 0$

Either $(\tan x)^{\tan x} = 0$ or $\sec^2 x = 0$

$\ln(\tan x)^{\tan x} = \ln(0)$

No solution

$\frac{1}{\cos^2 x} = 0$

$\frac{1}{0} = \cos^2 x$

No solution

$\ln(\tan x) = -1$

$\tan x = e^{-1}$ s/A

$x = \tan^{-1}\left(\frac{1}{e}\right)$ 1/c

$x = 0.35$ to 2d.p.

FPI Haf 2018

9) $f(x) = (\sin x)^x$

a) $y = (\sin x)^x$

$$\ln(y) = \ln((\sin x)^x)$$

$$\ln(y) = x \ln(\sin x)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln(\sin x))$$

$$\frac{dy}{dx} \times \frac{d}{dy}(\ln(y)) = x \frac{d}{dx}(\ln(\sin x)) + \ln(\sin x) \frac{d}{dx}(x)$$

$$\frac{dy}{dx} \times \frac{1}{y} = x \frac{\cos(x)}{\sin(x)} + \ln(\sin x)$$

$$\frac{dy}{dx} = y \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right)$$

$$\frac{dy}{dx} = (\sin x)^x \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right)$$

$$\text{Felly } f'(x) = (\sin x)^x \left(\frac{x \cos(x)}{\sin(x)} + \ln(\sin x) \right)$$

$$\text{fel bod } g(x) = x \frac{\cos(x)}{\sin(x)} + \ln(\sin x).$$

b) i) $g(0.1) = 0.1 \times \frac{\cos(0.1)}{\sin(0.1)} + \ln(\sin(0.1))$

$$= -1.307587873$$

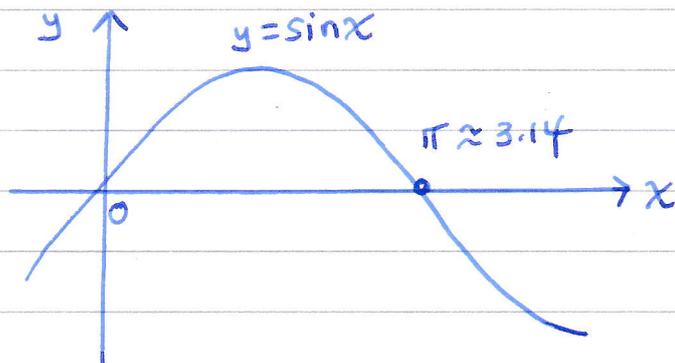
$$g(1) = 1 \times \frac{\cos(1)}{\sin(1)} + \ln(\sin(1))$$

$$= 0.4694888697$$

$$g(1.6) = 1.6 \times \frac{\cos(1.6)}{\sin(1.6)} + \ln(\sin(1.6))$$

$$= -0.04716565301$$

Beth yw $f(x)$ ar y parth $(0, 2)$?



Mae $(\sin x)$ yn bositif ar y parth $(0, 2)$.

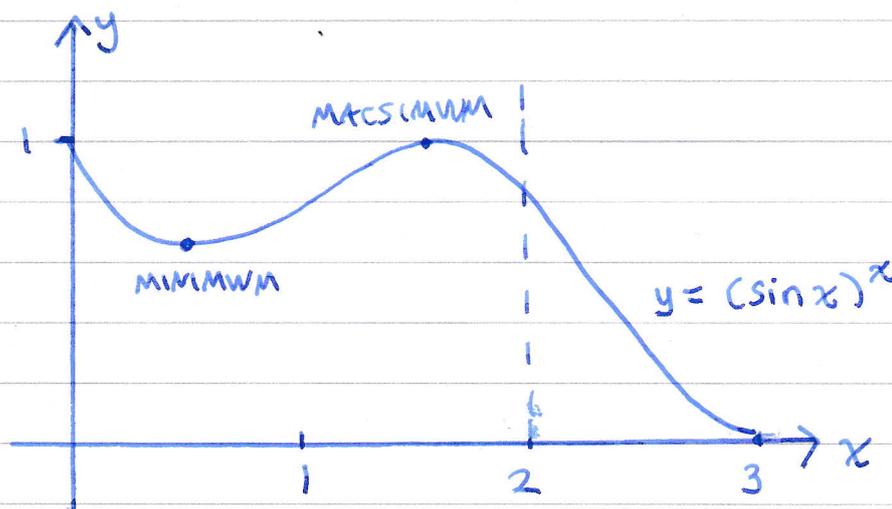
Felly mi fydd $(\sin x)^x$ yn bositif ar y parth $(0, 2)$.

Bydd arwydd $f(x)g(x)$ ar y parth $(0, 2)$ yn dibynnu ar arwydd $g(x)$. Ond mae gwerthoedd $g(0.1)$, $g(1)$ a $g(1.6)$ yn dangos bod $g(x)$ yn newid arwydd (o leiaf) dwywaith yn y parth $(0, 2)$.

Felly mi fydd (o leiaf) 2 werth ar gyfer $f'(x)$ ar y parth $(0, 2)$ ble fydd $f'(x) = 0$.

Felly mae gan $f(x)$ (o leiaf) 2 bwynt arhoso! ar y parth $(0, 2)$.

Braslun efo cyfrifiannell graffigol:



FP1 May 2019

7. a) Gadeuch is $y = 2^x$

* mae $\ln 2$ yn gysonyn

Felly $\ln y = x \ln 2$

$$\frac{1}{y} \frac{dy}{dx} = \ln 2$$

$$\frac{dy}{dx} = y \ln 2$$

$$= 2^x \ln 2$$

QED.

b) i) $f'(x) = 0$ os yw y pwnt yn bwynt arhosol.

$$f'(x) = 2^x \ln 2 - 2$$

Felly $2^x \ln 2 - 2 = 0$

$$2^x \ln 2 = 2$$

$$2^x = \frac{2}{\ln 2}$$

$$\log_2 2^x = \log_2 \left(\frac{2}{\ln 2} \right)$$

$$x = \log_2 \left(\frac{2}{\ln 2} \right)$$

$$\approx 1.53 \text{ (2 lle degol).}$$

$$\begin{aligned} \text{ii) } f''(x) &= 2^x(\ln 2) + 2^x \ln 2 (\ln 2) \\ &= 2^x (\ln 2)^2 \end{aligned}$$

> 0 felly mae'r pwnt yn MINIMUM.

$$\text{c) } \int_1^2 f(x) dx = \int_1^2 (2^x - 2x) dx$$

$$= \left[\frac{2^x}{\ln 2} - x^2 \right]_1^2$$

$$= \left(\frac{2^2}{\ln 2} - 2^2 \right) - \left(\frac{2^1}{\ln 2} - 1^2 \right)$$

$$= \frac{4}{\ln 2} - 4 - \frac{2}{\ln 2} + 1$$

$$= \frac{2}{\ln 2} - 3$$

$$\approx -0.115$$

(3 fig. gte.)