

Proof by Induction

- ① Test for a particular case (usually $n=1$)
- ② Assume the result is true for $n=k$
- ③ Show that the result is also true for $n=k+1$

Example: Show that $n^3 + 5n$ is divisible by 6 for all n .

① Case $n=1$: $n^3 + 5n = 1^3 + 5 \times 1 = 6$

Because 6 is divisible by 6 the result holds for $n=1$.

- ② Assume that the result is true for $n=k$.
Therefore 6 divides $k^3 + 5k$

③ We have to show that 6 divides $(k+1)^3 + 5(k+1)$.
But $(k+1)^3 + 5(k+1) = (k^2 + 2k + 1)(k+1) + 5k + 5$
 $= k^3 + 3k^2 + 3k + 1 + 5k + 5$
 $= k^3 + 3k^2 + 8k + 6$
 $= (k^3 + 5k) + 3k^2 + 3k + 6$
 $= (k^3 + 5k) + 6 + 3k(k+1)$

By the inductive hypothesis, $k^3 + 5k$ is divisible by 6. It follows that $(k^3 + 5k) + 6$ is also divisible by 6 as adding 6 to a number does not alter whether it is divisible by 6 or not.

Now $k(k+1)$ is always even, no matter what k is. So $k(k+1)$ is divisible by 2, and therefore $3k(k+1)$ is divisible by 6.
Conclusion: $(k^3 + 5k) + 6 + 3k(k+1)$ is divisible by 6, thus proving the hypothesis.

Past Paper Questions

HAF 2005

① Case $n=1$: $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & 2^1 - 1 \\ 0 & 2^1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \checkmark$$

② Assume that the result is true for $n=k$.

so $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}$.

③ We have to show that the result is true for $n=k+1$.

$$\begin{aligned} \text{LHS} &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^k \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^k - 1 + 2^k \\ 0 & 2(2^k) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2(2^k) - 1 \\ 0 & 2^{k+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^{k+1} - 1 \\ 0 & 2^{k+1} \end{pmatrix} \\ &= \text{RHS}. \end{aligned}$$

So the result is true for $n=k+1$. QED

Gaeaf
Haf 2006

(a) LHS = $\sum_{r=1}^{K+1} (2r+1)$

$$= \left[\sum_{r=1}^K (2r+1) \right] + (2(K+1)+1)$$
$$= (K+1)^2 + 2K+2+1$$
$$= K^2 + 2K + 1 + 2K + 3$$
$$= K^2 + 4K + 4$$
$$= (K+2)^2$$
$$= \text{RHS.}$$

So the result holds for $n=K+1$

(b) Consider the case $n=1$

$$\sum_{r=1}^n (2r+1) = (n+1)^2$$
$$(2(1)+1) = (1+1)^2$$
$$3 = 4 \quad (\text{False})$$

Because the result is false for $n=1$,
it cannot be true for all positive integers n .

Haf 2006

① Case $n=1$: $9^n - 5^n$
 $= 9 - 5$
 $= 4$ is divisible by 4, so the result is true for $n=1$.

② Assume that the result is true for $n=k$, so that $9^k - 5^k$ is divisible by 4.

③ Consider the case $n=k+1$.

Now $9^{k+1} - 5^{k+1}$
 $= 9(9^k) - 5(5^k)$
 $= 5(9^k) + 4(9^k) - 5(5^k)$
 $= 4(9^k) + 5(9^k - 5^k)$

By the inductive hypothesis, $9^k - 5^k$ is divisible by 4, and so $5(9^k - 5^k)$ is divisible by 4.

Further, $4(9^k)$ will always be divisible by 4, so $4(9^k) + 5(9^k - 5^k)$ is also divisible by 4. This proves the case $n=k+1$.

Gaeaf 2007

① Case $n=1$: $6^n + 4 = 6^1 + 4 = 10$ is divisible by 5, so the result is true for $n=1$.

② Assume that the result is true for $n=k$, so that $6^k + 4$ is divisible by 5.

③ Consider the case $n=k+1$.

$$\begin{aligned} \text{Now } 6^{k+1} + 4 &= 6(6^k) + 4 \\ &= 5(6^k) + 6^k + 4. \end{aligned}$$

By the inductive hypothesis, $6^k + 4$ is divisible by 5. Further, $5(6^k)$ will always be divisible by 5, so $5(6^k) + 6^k + 4$ is also divisible by 5.

This proves the case $n=k+1$.

Haf 2007

① Case $n=1$:

$$\sum_{r=1}^n \left[r \times \left(\frac{1}{2}\right)^r \right] = 2 - (n+2) \left(\frac{1}{2}\right)^n$$
$$1 \times \left(\frac{1}{2}\right)^1 = 2 - (3) \left(\frac{1}{2}\right)^1$$
$$\frac{1}{2} = 2 - \frac{3}{2}$$
$$\frac{1}{2} = \frac{1}{2} \quad \checkmark$$

② Assume that the result is true for $n=k$, so that

$$\sum_{r=1}^k \left[r \times \left(\frac{1}{2}\right)^r \right] = 2 - (k+2) \left(\frac{1}{2}\right)^k$$

③ Consider the case $n=k+1$.

$$\begin{aligned} \text{LHS} &= \sum_{r=1}^{k+1} \left[r \times \left(\frac{1}{2}\right)^r \right] \\ &= \sum_{r=1}^k \left[r \times \left(\frac{1}{2}\right)^r \right] + (k+1) \times \left(\frac{1}{2}\right)^{k+1} \\ &= 2 - (k+2) \left(\frac{1}{2}\right)^k + (k+1) \left(\frac{1}{2}\right)^{k+1} \\ &= 2 - k \left(\frac{1}{2}\right)^k - 2 \left(\frac{1}{2}\right)^k + k \left(\frac{1}{2}\right)^{k+1} + \left(\frac{1}{2}\right)^{k+1} \\ &= 2 - \frac{1}{2} k \left(\frac{1}{2}\right)^k - \frac{1}{2} 2 \left(\frac{1}{2}\right)^k + k \left(\frac{1}{2}\right)^{k+1} + \left(\frac{1}{2}\right)^{k+1} \\ &= 2 - 2k \left(\frac{1}{2}\right)^{k+1} - 4 \left(\frac{1}{2}\right)^{k+1} + k \left(\frac{1}{2}\right)^{k+1} + \left(\frac{1}{2}\right)^{k+1} \\ &= 2 + (-2k - 4 + k + 1) \left(\frac{1}{2}\right)^{k+1} \\ &= 2 + (-k - 3) \left(\frac{1}{2}\right)^{k+1} \\ &= 2 - (k+1+2) \left(\frac{1}{2}\right)^{k+1} \\ &= \text{RHS.} \end{aligned}$$

So the result is true for $n=k+1$. QED

Gaeaf 2008

~~Feb 2007~~

① Case $n=1$: $\sum_{r=1}^n r \times 2^r = 2^{n+1}(n-1) + 2$

$$1 \times 2^1 = 2^2(1-1) + 2$$
$$2 = 2 \quad \checkmark$$

② Assume that the result is true for $n=k$, so that

$$\sum_{r=1}^k r \times 2^r = 2^{k+1}(k-1) + 2$$

③ Consider the case $n=k+1$.

$$\begin{aligned} \text{LHS} &= \sum_{r=1}^{k+1} r \times 2^r \\ &= \left[\sum_{r=1}^k r \times 2^r \right] + (k+1) \times 2^{k+1} \\ &= 2^{k+1}(k-1) + 2 + (k+1)2^{k+1} \\ &= k2^{k+1} - 2^{k+1} + 2 + k2^{k+1} + 2^{k+1} \\ &= 2k2^{k+1} + 2 \\ &= k2^{k+2} + 2 \\ &= 2^{k+2}(k+1-1) + 2 \\ &= \text{RHS.} \end{aligned}$$

So the result is true for $n=k+1$. QED

Haf 2008

① Case $n=1$: $7^n + 5$
 $= 7 + 5$
 $= 12$ is divisible by 6, so the result
is true for $n=1$.

② Assume that the result is true for $n=k$, so that
 $7^k + 5$ is divisible by 6.

③ Consider the case $n=k+1$.

Now $7^{k+1} + 5$
 $= 7(7^k) + 5$
 $= 6(7^k) + 7^k + 5$.

By the inductive hypothesis, $7^k + 5$ is divisible by 6. Further, $6(7^k)$ will always be divisible by 6, so $6(7^k) + 7^k + 5$ is also divisible by 6.

This proves the case $n=k+1$.

Gaeaf 2009

① Case $n=1$: LHS =

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

RHS =

$$\begin{bmatrix} 1 & 2 \times 1 & 2 \times 1^2 \\ 0 & 1 & 2 \times 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

So the result is true for $n=1$.

② Assume that the result is true for $n=k$, so that

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & 2k & 2k^2 \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{bmatrix}$$

③ Consider the case $n=k+1$.

$$\begin{aligned} \text{LHS} &= \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2k & 2k^2 \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+0+0 & 2+2k+0 & 2+4k+2k^2 \\ 0+0+0 & 0+1+0 & 0+2+2k \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2+2k & 2k^2+4k+2 \\ 0 & 1 & 2+2k \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2(k+1) & 2(k+1)^2 \\ 0 & 1 & 2(k+1) \\ 0 & 0 & 1 \end{bmatrix} \\ &= \text{RHS.} \end{aligned}$$

So the result is true for $n=k+1$. QED

Haf 2009

① Case $n=1$:

$$\sum_{r=1}^1 \frac{1}{r(r+1)} = \frac{1}{1+1}$$
$$\frac{1}{1(1+1)} = \frac{1}{2}$$
$$\frac{1}{1(2)} = \frac{1}{2}$$
$$\frac{1}{2} = \frac{1}{2} \checkmark$$

② Assume that the result is true for $n=k$, so that

$$\sum_{r=1}^k \frac{1}{r(r+1)} = \frac{k}{k+1}.$$

③ Consider the case $n=k+1$.

$$\begin{aligned} \text{LHS} &= \sum_{r=1}^{k+1} \frac{1}{r(r+1)} \\ &= \left(\sum_{r=1}^k \frac{1}{r(r+1)} \right) + \frac{1}{(k+1)(k+1+1)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2) + 1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\ &= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} \\ &= \frac{k+1}{(k+1)+1} \\ &= \text{RHS.} \end{aligned}$$

So the result is true for $n=k+1$. QED

Graaf 2010

close fact

⑥ (a) ① Case $n=1$: $1 \times 1! = (1+1)! - 1$ ②

$$\begin{aligned}1 \times 1 &= 2! - 1 \\1 &= 2 \times 1 - 1 \\1 &= 2 - 1 \\1 &= 1\end{aligned}\quad \checkmark$$

② Assume the result is true for $n=k$, so that

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! = (k+1)! - 1.$$

③ Consider the case $n=k+1$.

$$\begin{aligned}LHS &= 1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! + (k+1) \times (k+1)! \\&= [1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k!] + (k+1) \times (k+1)! \\&= (k+1)! - 1 + (k+1)(k+1)! \\&= [(k+1)!](1+(k+1)) - 1 \\&= [(k+1)!](k+2) - 1 \\&= (k+2)! - 1 \\&= RHS\end{aligned}$$

So the result is true for $n=k+1$. QED.

(b) [Topic: induction]

Haf 2010

Q10C (Ans)

⑤ Case $n=1$: $4^{2 \times 1} - 1 = 16 - 1 = 15$ is divisible by 15, so the result is true for $n=1$.

Assume that the result is true for $n=k$, so that $4^{2k} - 1$ is divisible by 15 for all the integers k .

Consider the case $n=k+1$.

Now $4^{2(k+1)} - 1 = 4^{2k+2} - 1 = 4^{2k}(4^2) - 1 = 16(4^{2k}) - 1 = 15(4^{2k}) + 4^{2k} - 1$

By the inductive hypothesis, $4^{2k} - 1$ is divisible by 15. Further, $15(4^{2k})$ will always be divisible by 15, as it is a multiple of 15. So $15(4^{2k}) + 4^{2k} - 1$ will also be divisible by 15, as it is the sum of two numbers divisible by 15. This proves the case $n=k+1$. QED.

① Case n=1:

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^1 = \begin{bmatrix} 1 & 2^1 - 1 \\ 0 & 2^1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 - 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \checkmark$$

② Assume that the result is true for n=k, so that

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^k = \begin{bmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{bmatrix}.$$

③ Consider the case n=k+1.

$$\text{LHS} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{k+1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 1+2(2^k-1) \\ 0+0 & 0+2(2^k) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+2^{k+1}-2 \\ 0 & 2^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2^{k+1}-1 \\ 0 & 2^{k+1} \end{bmatrix}$$

$$= \text{RHS.}$$

So the result is true for n=k+1.

QED

Itaf 2011

11CS 7002

Case $n=1$: $6^1 + 4 = 10$ is divisible by 10, so the result is true for $n=1$.

Assume that the result is true for $n=k$, so that $6^k + 4$ is divisible by 10 for all positive integers k .

Consider the case $n=k+1$

Now $6^{k+1} + 4$
= $6^k(6) + 4$
= $10(b^k) - 4(b^k) + 4$
= $10(b^k) - 4(b^k - 1)$
= $10(b^k) - 4(6^k + 4 - 5)$
= $10(b^k) - 4(b^k + 4) + 20$

By the inductive hypothesis, $b^k + 4$ is divisible by 10, so that $-4(b^k + 4)$ will also be divisible by 10.

Further, $10(b^k)$ will be divisible by 10 as it is a multiple of 10; and 20 is divisible by 10 ($20 \div 10 = 2$). It follows that $10(b^k) - 4(b^k + 4) + 20$ will also be divisible by 10 as it is a sum of three numbers divisible by 10. This proves the case $n=k+1$. QED

Graef 2012

Stora fett

① Case $n=1$: $\sum_{r=1}^1 r(r+1) = \frac{1(1+1)(1+2)}{3} =$

~~Since $1(1+1) = \frac{1(2)(3)}{3}$~~

~~$1(2) = \frac{6}{3}$~~

~~$2 = 2$ ✓~~

② Assume that the result is true for $n=k$, so that

$$\sum_{r=1}^k r(r+1) = \frac{k(k+1)(k+2)}{3} \text{ for all positive integers } k.$$

③ Consider the case $n=k+1$.

$$\text{LHS} = \sum_{r=1}^{k+1} r(r+1)$$

$$= \left[\sum_{r=1}^k r(r+1) \right] + (k+1)(k+1+1)$$

$$= \left[\sum_{r=1}^k r(r+1) \right] + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \quad \text{by the inductive hypothesis}$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$$= \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$$

= RHS.

So the result is true for $n=k+1$. QED

Haf 2012

Case $n=1$: $1^3 + 2(1) = 1+2 = 3$ is divisible by 3, so the result is true for $n=1$.

Assume that the result is true for $n=k$, so that $k^3 + 2k$ is divisible by 3 for all positive integers k .

Consider the case $n=k+1$.

Now $(k+1)^3 + 2(k+1)$
= $(k+1)^2(k+1) + 2(k+1)$
= $(k^2+2k+1)(k+1) + 2(k+1)$
= $k^3+2k^2+k+k^2+2k+1 + 2(k+1)$
= $(k^3+2k) + 3k^2+k+1+2(k+1)$
= $(k^3+2k) + 3k^2+k+2k+2$
= $(k^3+2k) + 3k^2+3k+3$
= $(k^3+2k) + 3(k^2+k+1)$

By the inductive hypothesis, $k^3 + 2k$ is divisible by 3.

Further, $3(k^2+k+1)$ will be divisible by 3

as it is a multiple of 3.

So $(k^3+2k) + 3(k^2+k+1)$ will also be divisible by 3 as it is a sum of two numbers divisible by 3.

This proves the case $n=k+1$. QED.

FPI Tonawur 2013

⑥ Case $n=1$: LHS = $\sum_{r=1}^1 r^3$
 LHS = 1^3
 LHS = 1.
 RHS = $\frac{1^2(1+1)^2}{4}$
 RHS = $\frac{1(2)^2}{4}$
 RHS = 1. ✓

Assume that the result is true for $n=k$, so that

$$\sum_{r=1}^k r^3 = \frac{k^2(k+1)^2}{4}.$$

Let us look at the case $n=k+1$.

$$\begin{aligned}
 \text{LHS} &= \sum_{r=1}^{k+1} r^3 \\
 &= \left(\sum_{r=1}^k r^3 \right) + (k+1)^3 \\
 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 && \text{by the inductive hypothesis} \\
 &= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
 &= \frac{(k+1)^2(k^2+4k+4)}{4} \\
 &= \frac{(k+1)^2(k^2+4k+4)}{4} \\
 &= \frac{(k+1)^2(k+2)^2}{4} \\
 &= \frac{(k+1)^2((k+1)+1)^2}{4} \\
 &= \text{RHS.}
 \end{aligned}$$

So the result, by mathematical induction, is true for $n=k+1$. QED

FPI Haf 2013

$$\textcircled{5} \quad \text{Achos } n=1: \quad \begin{aligned} \text{Mae } 7^n - 1 &= 7^1 - 1 \\ &= 7 - 1 \\ &= 6. \end{aligned}$$

Mae 6 yn rhannadwy a b felly mae'r gasodiad yn wir ar gyfer $n=1$.

Cymerwch bod y gasodiad yn wir ar gyfer $n=k$, felly mae $7^k - 1$ yn rhannadwy a b.

Edrychwn ar yr achos $n=k+1$.

$$\begin{aligned} 7^{k+1} - 1 &= 7 \times 7^k - 1 \\ &= 6 \times 7^k + 7^k - 1 \\ &= 6(7^k) + (7^k - 1) \end{aligned}$$

Yn ôl yr hypothesis anwythol mae $7^k - 1$ yn rhannadwy a b, felly $7^k - 1 = bp$ ar gyfer rhwng rif cyfan p.

$$\begin{aligned} \text{Felly } 7^{k+1} - 1 &= 6(7^k) + bp \\ 7^{k+1} - 1 &= 6(7^k + p) \end{aligned}$$

Mae $7^{k+1} - 1$ felly yn lluosrif o 6, felly'n rhannadwy a b. Mae hyn yn profi'r achos $n=k+1$, felly brwyg anwythiad mathemategol mae $7^n - 1$ yn rhannadwy a b ar gyfer pob cyfanrif positif n.

FPI Graef 2014

⑥

a) Achos $n=1$:

$$\text{Ochr chwth} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^1$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$\text{Ochr dde} = \begin{pmatrix} 1 & 3^1 - 1 \\ 0 & 3^1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 - 1 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

Mae'r ochr chwth yn hafal ir ochr dde felly mae'r hafaliad yn wir ar gyfer $n=1$

Cymenni fad yr hafaliad yn wir ar gyfer $n=k$, felly

$$\text{mae } \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{pmatrix}$$

Gradewch i ni edrych ar yr achos $n=k+1$.

$$\text{Ochr Chwth} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{k+1}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^k \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \quad \text{trwy annytiad mathemategol}$$

$$= \begin{pmatrix} 1+0 & 2+3(3^k-1) \\ 0+0 & 0+3(3^k) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2+3^{k+1}-3 \\ 0 & 3^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3^{k+1}-1 \\ 0 & 3^{k+1} \end{pmatrix}$$

= Ochr Dde ✓

Felly, erwyd anwylchiad mathemategol, mae'r hafaliad yn wir ar gyfer pob cyfrannif positif n.

b) Achos n = -1:

$$\text{Ochr Chwth} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{-1}$$

$$\text{Determinant: } \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 1 \times 3 - 0 \times 2 \\ = 3$$

$$\text{Felly } \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$\begin{aligned} \text{Ochr Dde} &= \begin{pmatrix} 1 & 3^{-1} - 1 \\ 0 & 3^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{1}{3} - 1 \\ 0 & \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \end{aligned}$$

Mae'r ochr chwth yn hafal i'r ochr dde felly mae'r hafaliad yn wir ar gyfer n = -1.

FPI Haf 2014

⑧ Achos $n=1$: Ochr chwith = $\sum_{r=1}^1 (r \times 2^{r-1})$

$$\begin{aligned} &= 1 \times 2^{1-1} \\ &= 1 \times 2^0 \\ &= 1 \times 1 \\ &= 1. \end{aligned}$$

$$\begin{aligned} \text{Ochr dde} &= 1 + 2^1(1-1) \\ &= 1 + 2^1(0) \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

Maer ochr chwith yn hafal i'r ochr dde felly mae'r hafaliad yn wir ar gyfer $n=1$.

Cymwnn fod yr hafaliad yn wir ar gyfer $n=K$, fel bod $\sum_{r=1}^K (r \times 2^{r-1}) = 1 + 2^K(K-1)$.

Gadewch i ni edrych ar yr achos $n=K+1$.

$$\begin{aligned} \text{Ochr chwith} &= \sum_{r=1}^{K+1} (r \times 2^{r-1}) \\ &= \left[\sum_{r=1}^K (r \times 2^{r-1}) \right] + ((K+1) \times 2^{K+1-1}) \\ &= \left[\sum_{r=1}^K (r \times 2^{r-1}) \right] + (K+1) \times 2^K \\ &= 1 + 2^K(K-1) + (K+1) \times 2^K \quad \text{bu anuythiad mathemategol} \\ &= 1 + 2^K(K-1 + K+1) \\ &= 1 + 2^K(2K) \\ &= 1 + 2^{K+1}(K) \\ &= 1 + 2^{K+1}((K+1)-1) \\ &= \text{Ochr Dde } \checkmark \end{aligned}$$

Felly, trwy anuythiad mathemategol, mae'r hafaliad yn wir ar gyfer pob cyfanrif positif n .

FPI Haf 2015

8) $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$$\begin{aligned} a) A^2 &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} & 2A - I \\ &= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} & = 2 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \quad \checkmark \end{aligned}$$

$$\begin{aligned} b) \text{ Case } n=1: \quad LHS &= A^1 & RHS &= 1A - (1-1)I \\ &= A & &= A - 0I \\ & & &= A. \end{aligned}$$

The LHS is equal to the RHS so the equation is true for $n=1$.

(The case $n=2$ has been proven in part (a)).

Assume that the equation is true for $n=k$, so that
 $A^k = kA - (k-1)I$

Let us look at the case $n=k+1$.

$$\begin{aligned} LHS &= A^{k+1} \\ &= A(A^k) \\ &= A(kA - (k-1)I) \quad \text{by the inductive hypothesis} \\ &= KA^2 - (k-1)AI \\ &= KA^2 - (k-1)A \\ &= K(2A - I) - (k-1)A \quad \text{from part (a)} \\ &= 2KA - KI - KA + A \\ &= (k+1)A - KI \\ &= (k+1)A - ((k+1)-1)I \\ &= RHS. \end{aligned}$$

so, by mathematical induction, the equation is true for all positive integers n .

FPI Itaf 201b

(7) $x_{n+1} = 2x_n - n + 1, \quad x_1 = 3$

Let us prove that $x_n = 2^n + n$.

(1) Case $n=1$: $x_1 = 2^1 + 1$
 $= 2 + 1$
 $= 3 \quad \checkmark$

(2) Assume that the result is true for $n=k$,
so that $x_k = 2^k + k$.

(3) Let us look at the next step ($k+1$).

$$\begin{aligned} x_{k+1} &= 2x_k - k + 1 && \text{by definition} \\ &= 2(2^k + k) - k + 1 && \text{by the inductive hypothesis} \\ &= 2^{k+1} + 2k - k + 1 \\ &= 2^{k+1} + k + 1 \end{aligned}$$

which agrees with $x_n = 2^n + n$
for $n=k+1$.

Therefore, by mathematical induction,
 $x_n = 2^n + n$ for all positive integers n .

FPI Haf 2017

6) Achos $n=1$: Mae $q^n - 1 = q^1 - 1$
 $= q - 1$
 $= 8.$

Mae 8 yn rhannadwy ag 8 felly mae'r gosodiad yn wir ar gyfer $n=1$.

Cymerwch bod y gosodiad yn wir ar gyfer $n=k$, felly mae $q^k - 1$ yn rhannadwy ag 8.

Edrychwn ar yr achos $n=k+1$.

$$\begin{aligned} q^{k+1} - 1 &= q \times q^k - 1 \\ &= 8 \times q^k + q^k - 1 \\ &= 8(q^k) + (q^k - 1) \end{aligned}$$

Yn ôl yr hypothesis anwythol, mae $q^k - 1$ yn rhannadwy ag 8, felly $q^k - 1 = 8p$ ar gyfer rhyw rif cyfan p.

$$\begin{aligned} \text{Felly } q^{k+1} - 1 &= 8(q^k) + 8p \\ q^{k+1} - 1 &= 8(q^k + p) \end{aligned}$$

Mae $q^{k+1} - 1$ felly yn lluosrif o 8, felly'n rhannadwy ag 8. Mae hyn yn profi'r achos $n=k+1$, felly trwy anwythiad mathemategol mae $q^n - 1$ yn rhannadwy ag 8 ar gyfer pob cyfanrif positif n.

FPI Haf 2018

7) Achos $n=1$: Ochr chwith = $\sum_{r=1}^1 r^2$
 $= 1^2$
 $= 1$

$$\text{Ochr dde} = \frac{1(1+1)(2 \times 1 + 1)}{6}$$

$$= \frac{1 \times 2 \times 3}{6}$$

$$= 1$$

Mae'r ddwy ochr yn 1 fel y rydym wedi profi'r achos $n=1$.

Cymerwn bod y gosodiad yn wir ar gyfer $n=k$, fel bod

$$\sum_{r=1}^k r^2 = \frac{k(k+1)(2k+1)}{6}$$

Gadewch i ni ystyried yr achos $n=k+1$.

$$\begin{aligned} \text{Ochr chwith} &= \sum_{r=1}^{k+1} r^2 \\ &= \left(\sum_{r=1}^k r^2 \right) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{Enwyr hypothesis anuyfhol} \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} \end{aligned}$$

$$\begin{aligned}
 &= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\
 &= \frac{(k+1)(2k+3)(k+2)}{6} \\
 &= \frac{(k+1)(k+2)(2k+3)}{6} \\
 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\
 &= \text{Ochr Pde.}
 \end{aligned}$$

Trwy anwythiad mathemategol gallwn ddweud fod

$$\sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}$$

ar gyfer pob cyfanrif positif n.

FP1 Mai 2019

$$\begin{aligned}8) \text{ Achos } n=1: \text{ Mae } 5^n - (-1)^n &= 5^1 - (-1)^1 \\&= 5 + 1 \\&= 6\end{aligned}$$

Mae 6 yn rhannadwy ag 6 iello
mae'r gosodiad yn wr ar gyfer
 $n=1$.

Cyrraedd bod y gosodiad yn wr ar gyfer
 $n=K$, iello mae $5^K - (-1)^K$ yn rhannadwy
ag 6.

Edrychwr ar yr achos $n=K+1$

$$\begin{aligned}5^{K+1} - (-1)^{K+1} &= 5 \times 5^K - (-1)(-1)^K \\&= 6 \times 5^K - 5^K + (-1)^K \\&= 6 \times 5^K - (5^K - (-1)^K)\end{aligned}$$

Yn ôl yr hypothesis amwythol, mae $5^K - (-1)^K$ yn rhannadwy ag 6, iello $5^K - (-1)^K = 6p$ ar gyfer unrhyw rôl cyfan p.

$$\begin{aligned}\text{Ielly } 5^{K+1} - (-1)^{K+1} &= 6(5^K) - 6p \\&= 6(5^K - p)\end{aligned}$$

Mae $5^{K+1} - (-1)^{K+1}$ iello yn morri o 6,
iello'n rhannadwy ag 6. Mae hyn yn parhau
yr achos $n=K+1$, iello brwyg amwythiad
mathemategol mae $5^n - (-1)^n$ yn rhannadwy
ag 6 ar gyfer pob agramrif positif n.

QED.