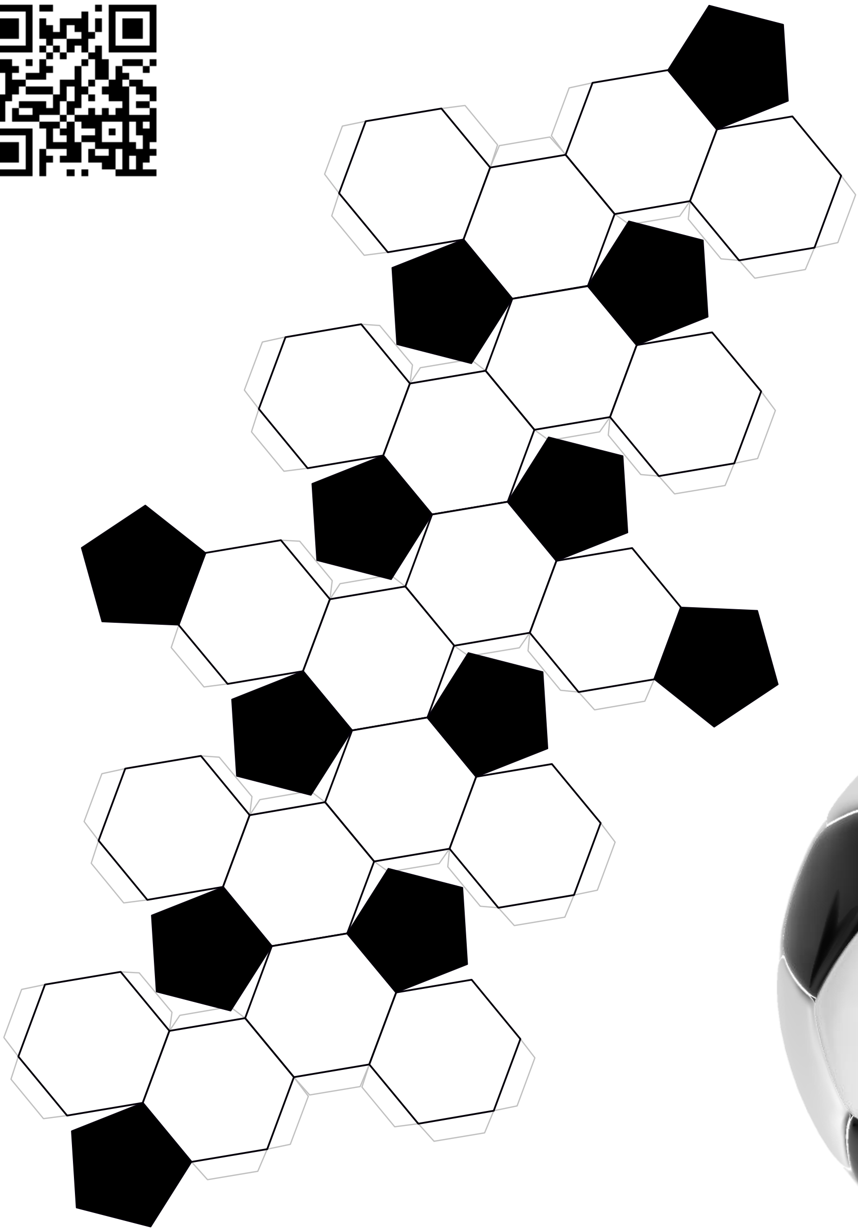
The Mathematics Department

11

Measuring Shapes 4

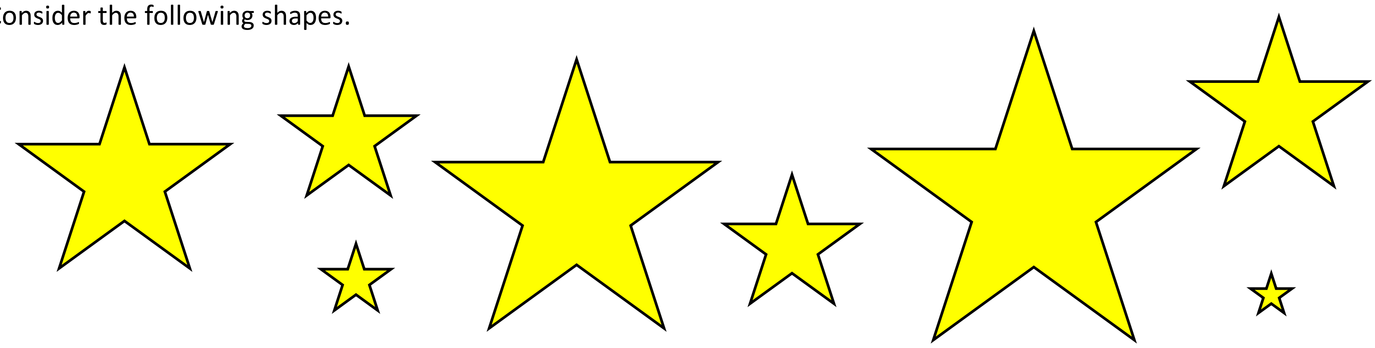
Name:

Contents



Congruent Shapes

Consider the following shapes.



The shapes are all **similar**, so that the same shape is seen each time, but only two of the shapes are **congruent**, that is to say they have the same *size*.

Congruent shapes are the same **shape**, and the same **size**.

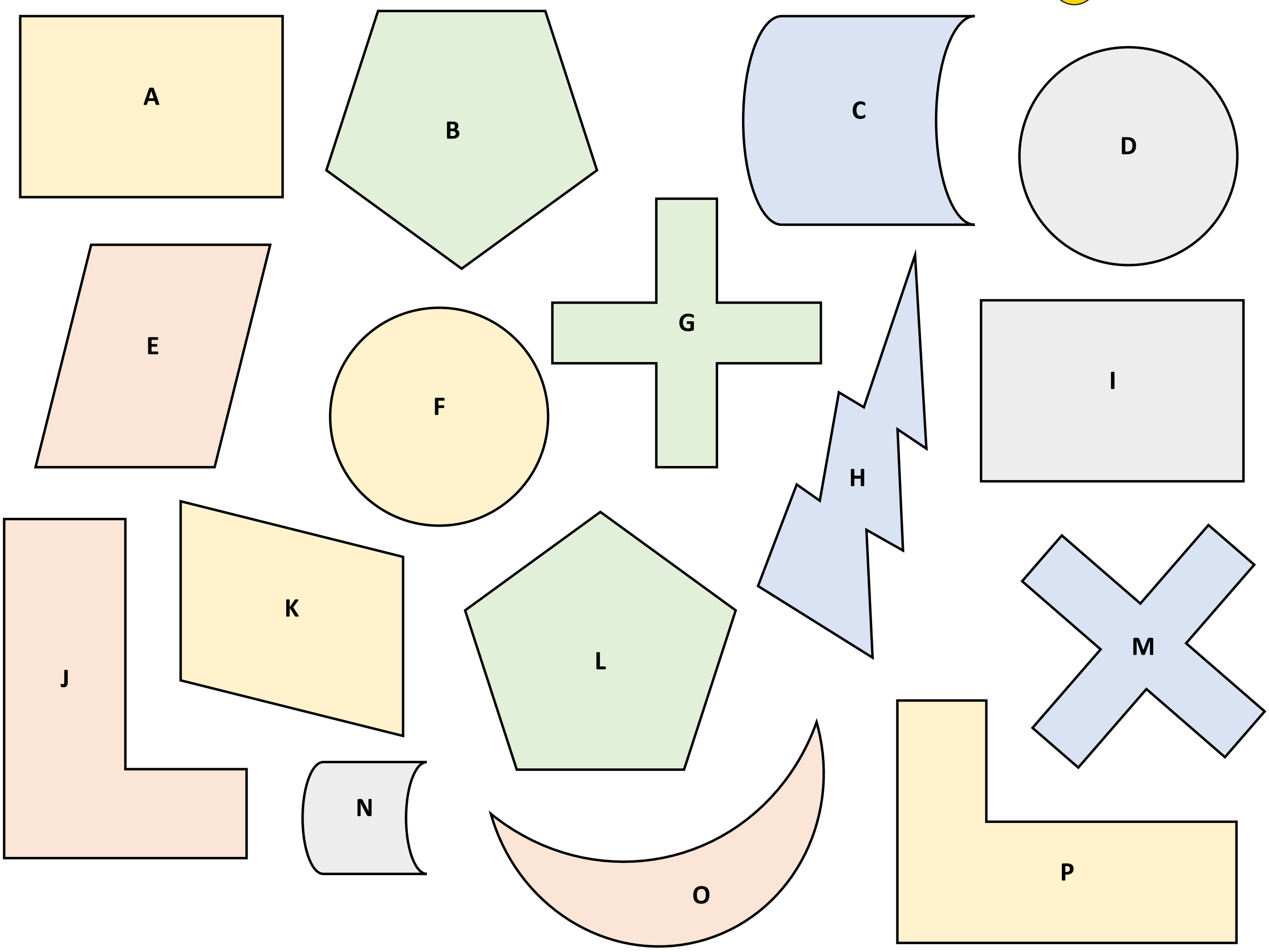
Exercise 1

Skill I

Tick the two shapes that are congruent above.

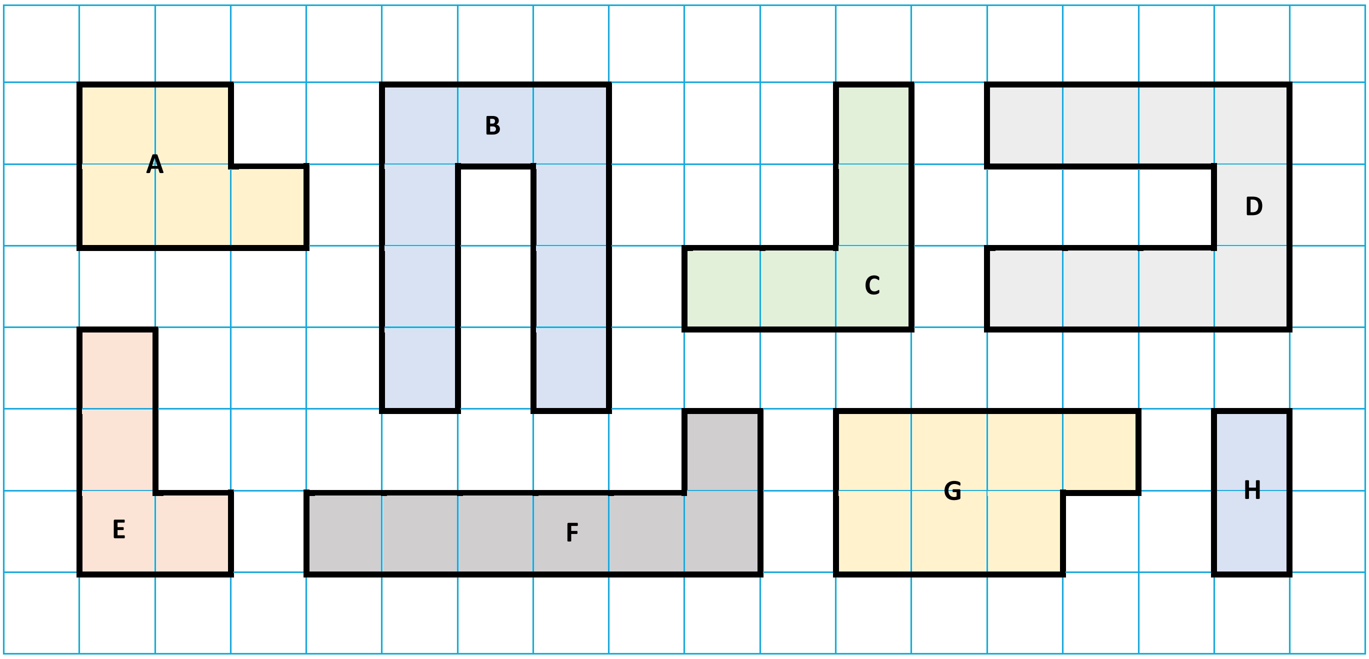
Exercise 2

Look at the following shapes. Which pairs of shapes are congruent?



Exercise 3

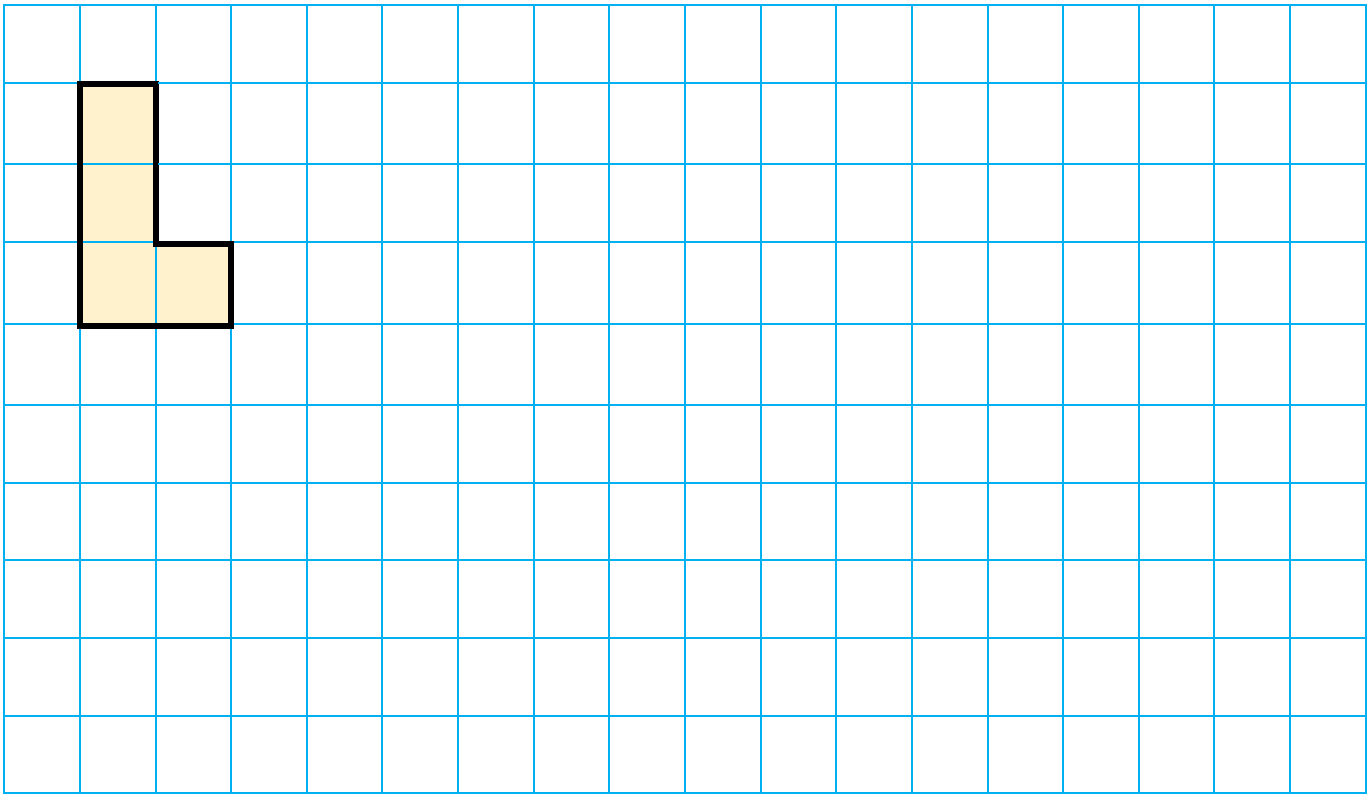
Below there is a collection of shapes drawn on a squared centimetre grid.



(a) Which two shapes are congruent?
(b) Which two shapes have an area of 5 cm²?
(c) Which two shapes have a perimeter of 12 cm?
(d) Which two shapes have an area of 7 cm²?
(e) Which two shapes have a perimeter of 10 cm?

Exercise 4

On the following grid, draw shapes that are congruent to the shown shape, but have different orientations. How many different orientations can be drawn?

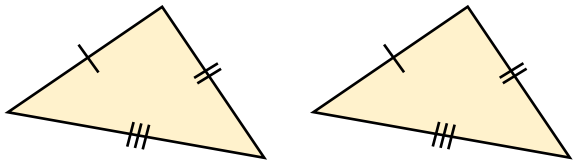


Higher Tier

Congruent Triangles Proofs

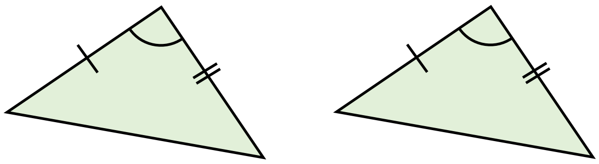
There are four ways of proving that two triangles are congruent.

(1) Side, Side, Side (SSS)



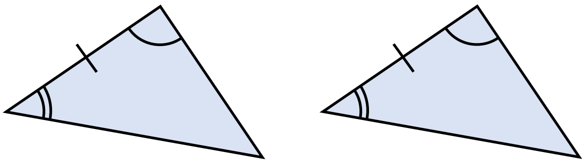
The lengths of the sides of the first triangle correspond to the lengths of the sides in the second triangle.

(2) Side, Angle, Side (SAS)



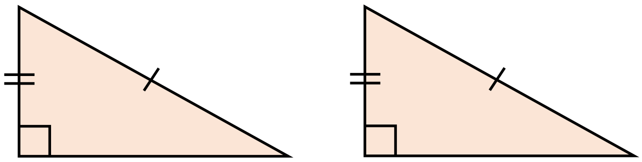
The lengths of two of the sides in the first triangle correspond to lengths of two of the sides in the second triangle, and the angles **between** the sides are equal.

(3) Angle, Side, Angle (ASA)



The size of two of the angles in the first triangle correspond to the size of two of the angles in the second triangle, and the length of the sides **between** the angles are equal.

(4) Right Angle, Hypotenuse, Side (RHS)

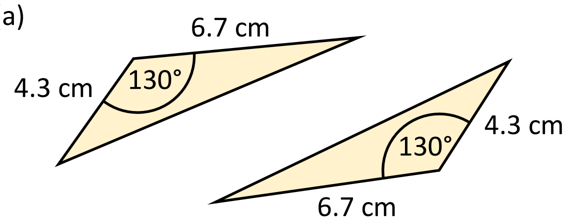


The two triangles are right-angled triangles; the lengths of the hypotenuse are equal; and the lengths of another side are equal.

Example

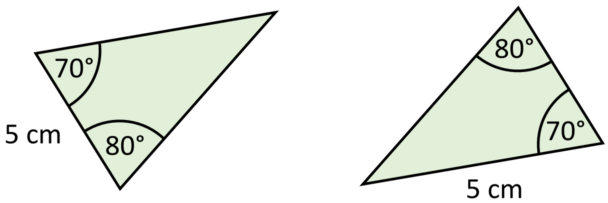
Explain, **noting your reasons**, if the following pairs of triangles are congruent or not.

(a)



Answer: The lengths of two of the sides in the first triangle are equal to the lengths of two of the sides in the second triangle (4.3 cm, 6.7 cm). The angle between the sides (130°) is also equal, so the triangles are congruent due to the SAS rule.

(b)

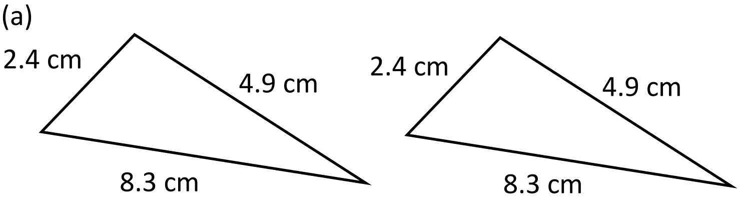


Answer: These triangles are not congruent. There is an angle of 70°, an angle of 80° and a side length of 5 cm in each triangle, but the 5 cm length is not **between** the angles in the triangle on the right. Because the longest length in a triangle is always opposite the greatest angle (80° in this case), the side between the angles must have a length less than 5 cm. So, we cannot use the ASA rule to prove that these triangles are congruent.

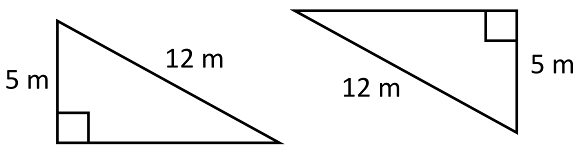
Exercise 5

Explain, **noting your reasons**, if the following pairs of triangles are congruent or not. (The diagrams are not drawn to scale.)

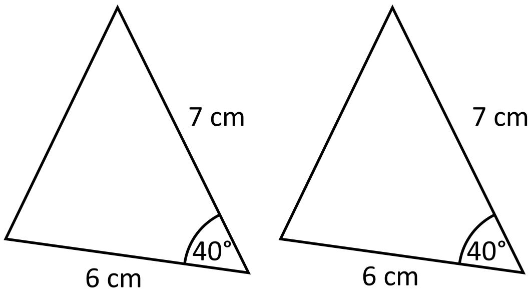
(a)



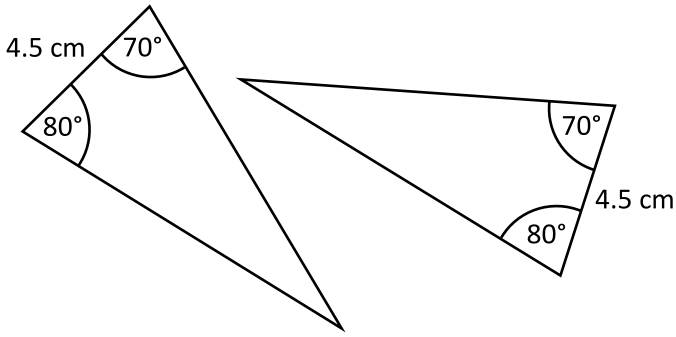
(b)



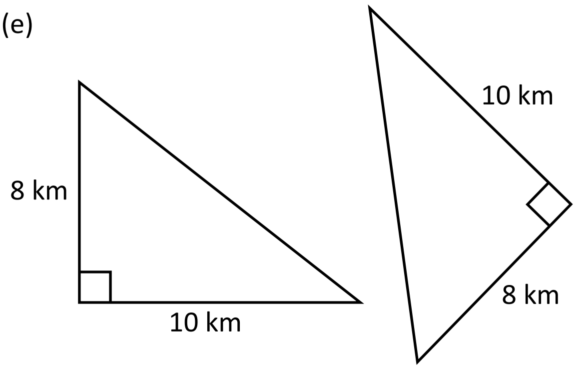
(c)


7 cm
40°
6 cm
7 cm
40°
6 cm

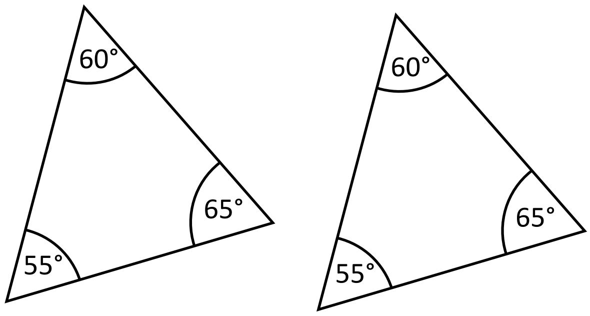
(d)


4.5 cm
70°
80°
70°
4.5 cm
80°

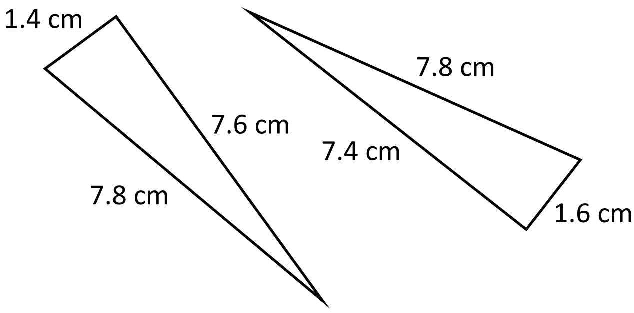
(e)


8 km
10 km
10 km
8 km

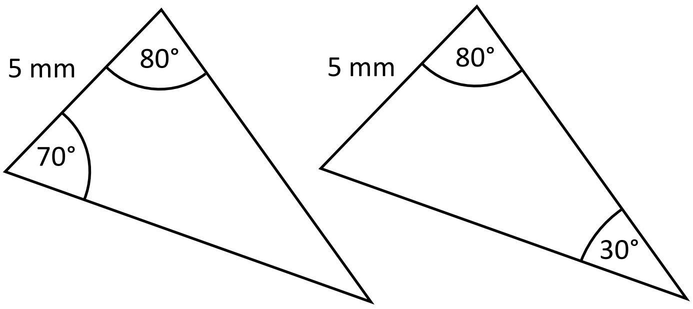
(f)


60°
65°
55°
60°
65°
55°

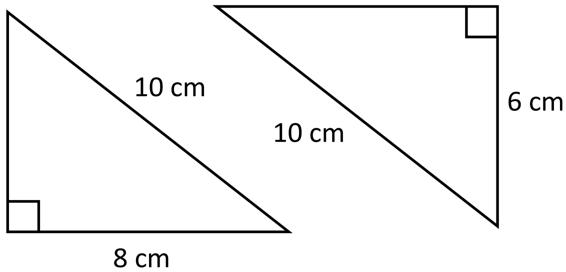
(g)


1.4 cm
7.6 cm
7.8 cm
7.8 cm
7.4 cm
1.6 cm

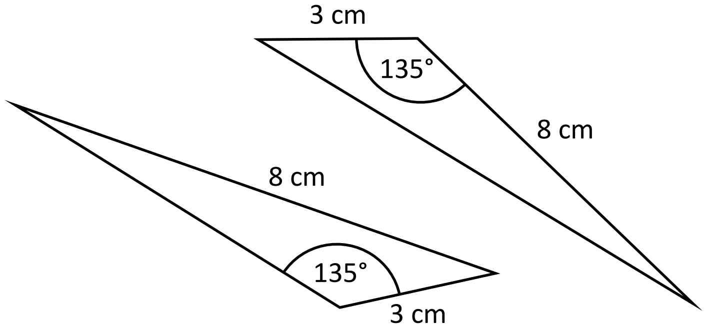
(h)


5 mm
80°
70°
5 mm
80°
30°

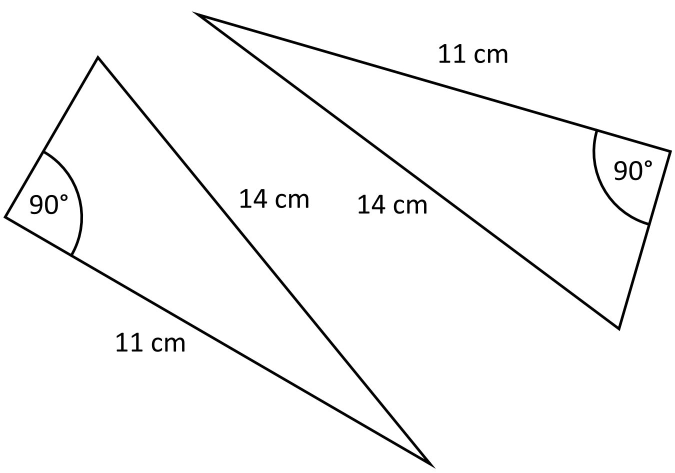
(i)


10 cm
8 cm
10 cm
6 cm

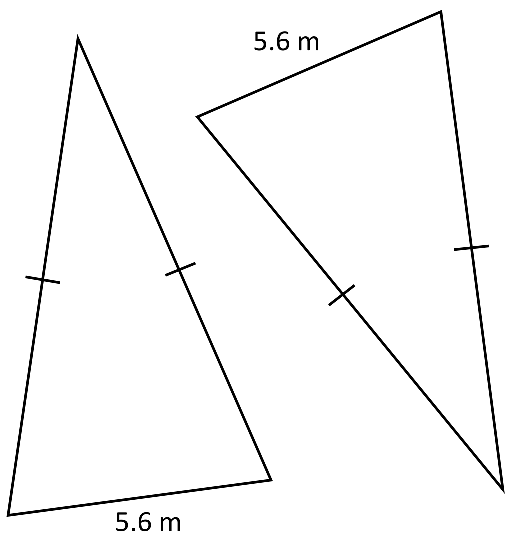
(j)


3 cm
135°
8 cm
8 cm
135°
3 cm

(k)


90°
14 cm
11 cm
11 cm
14 cm
90°

(l)


5.6 m
5.6 m

Exercise 6

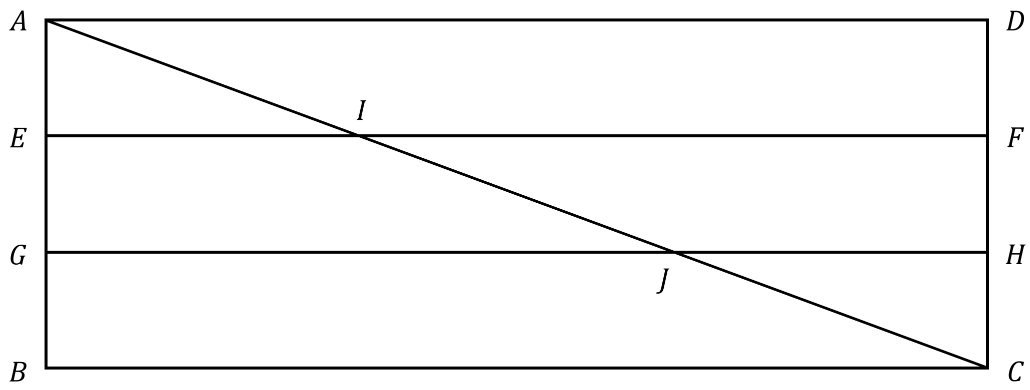
Applying

H

The following diagram shows a wooden fence, $ABCD$. The framework is strengthened by the addition of three wooden bars, EF, GH and AC.

The beams AD, EF, GH and BC are parallel to each other, with equal spaces between them.

The bar AC meets EF and GH at I and J, respectively.



(a) Name a triangle that is congruent to the triangle AGJ.
(b) Explain clearly why these triangles are congruent.

Exercise 7

I

Circle either TRUE or FALSE for the following statements.

STATEMENT		
Every rectangle is congruent.	TRUE	FALSE
Circles with equal area are congruent.	TRUE	FALSE
Every regular pentagon is congruent.	TRUE	FALSE
Using a 100% setting, a photocopier produces congruent shapes.	TRUE	FALSE
Each triangle with a base of 5 cm and a height of 4 cm is congruent.	TRUE	FALSE
Every semicircle with a diameter of 6 cm is congruent.	TRUE	FALSE

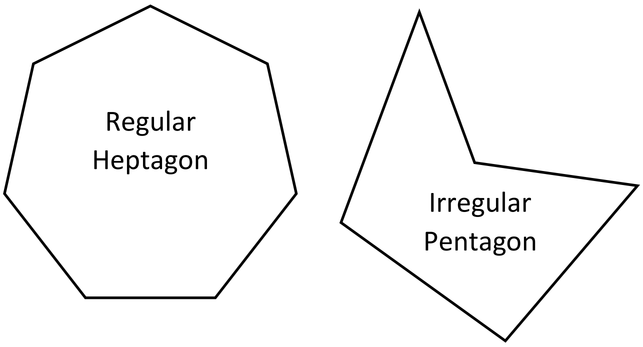
Evaluation

Key words	Corrections	I am happy with...	I need to revise...

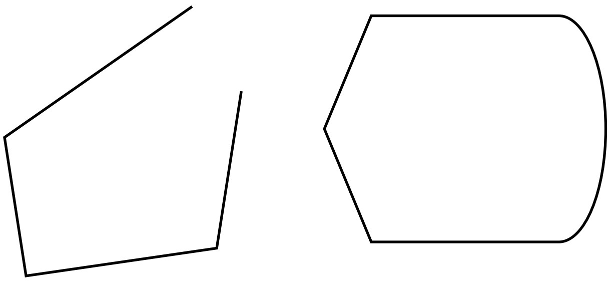
Angles in Polygons

A shape which uses straight lines only is called a **polygon**. The polygon is **regular** if its sides all have the same length and its angles are also all equal. If the polygon is not regular, then it is an **irregular** polygon.

Examples of polygons



Non-examples of polygons



Skill F

Exercise 8

Complete the following table.

Number of edges	Polygon name	Total interior angles	Interior angle for a regular polygon
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
n	Polygon with n edges		

Exercise 9 F

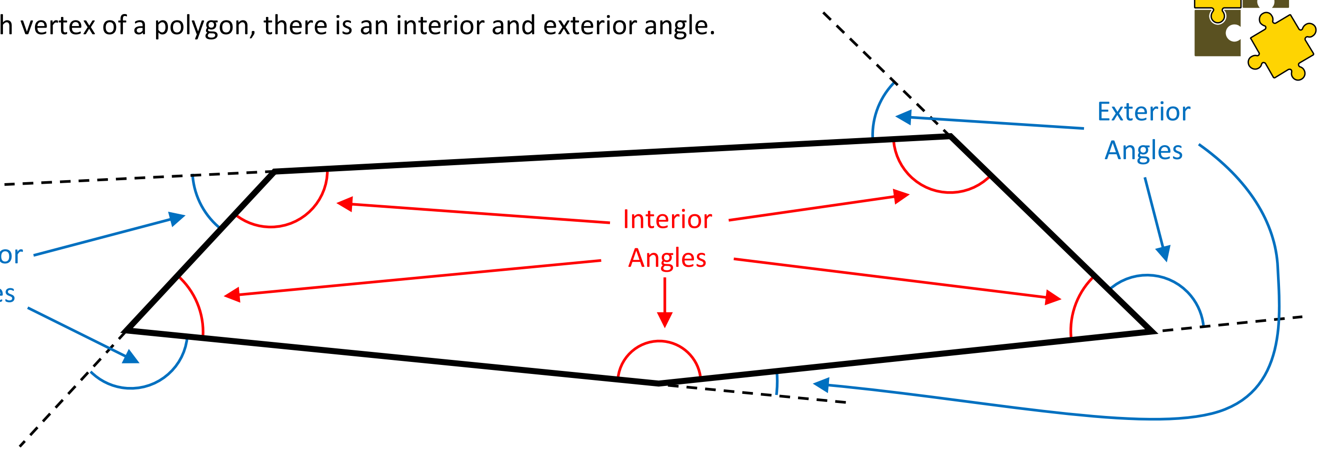
Use a ruler and a protractor to draw (a) a regular pentagon; (b) a regular hexagon; (c) a regular decagon.

Challenge!

Use an Excel spreadsheet to investigate the size of the interior angles of different regular polygons. As the number of edges increases, what happens to the interior angle? Does this pattern continue forever? What type of shape is a polygon with an ∞ number of edges?

The Exterior Angle of a Polygon

At each vertex of a polygon, there is an interior and exterior angle.



When walking along the exterior perimeter of a polygon, the exterior angle is the angle that you must turn through to continue travelling along the perimeter. For example, imagine walking around the exterior perimeter of the *Pentagon*, the headquarters of the United States Department of Defense.

The total of the interior angles is dependent upon the type of polygon. Above, the polygon is a pentagon, so the total interior angles is 540°. What is the total of the exterior angles? Again, imagine walking along the exterior perimeter. On returning to your original position, you will have rotated around a full turn, or 360°. The type of polygon is not important here, so the total exterior angles of any polygon is 360°.

Summary

For a polygon with n edges,

$$\text{Total of the exterior angles} = 360°$$
$$\text{Total of the interior angles} = 180°(n-2)$$

For any vertex in a polygon,

$$\text{Interior angle} + \text{exterior angle} = 180°$$

If the polygon is a regular polygon,

$$\text{One exterior angle} = \frac{360°}{n}$$
$$\text{One interior angle} = \frac{180°(n-2)}{n} \text{ or } 180° - \frac{360°}{n}$$

Challenge!

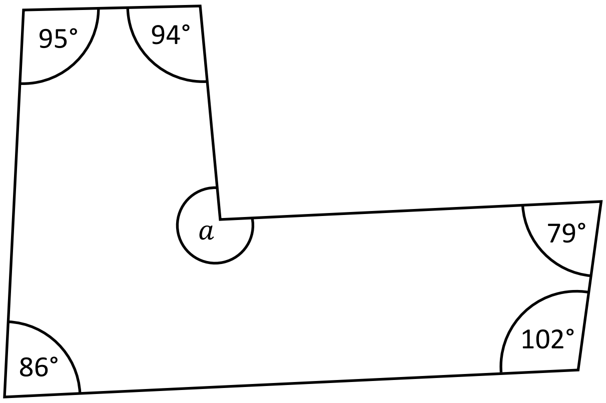
Prove that $\frac{180°(n-2)}{n} \equiv 180° - \frac{360°}{n}$.

Exercise 10

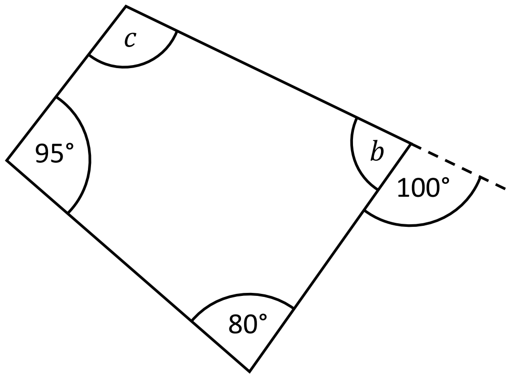
F

Calculate the size of the missing angles. (The diagrams are not drawn to scale.)

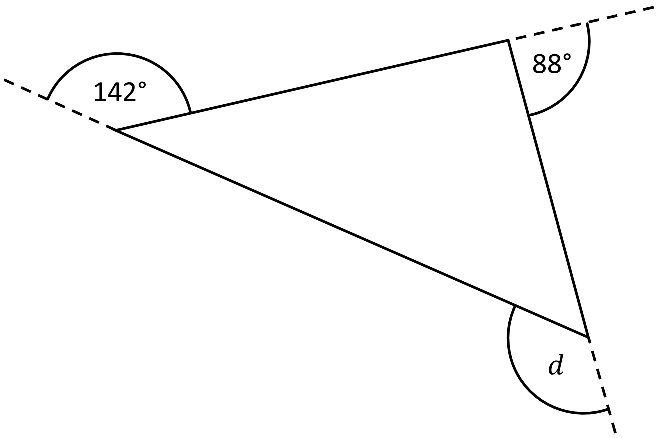
(a)



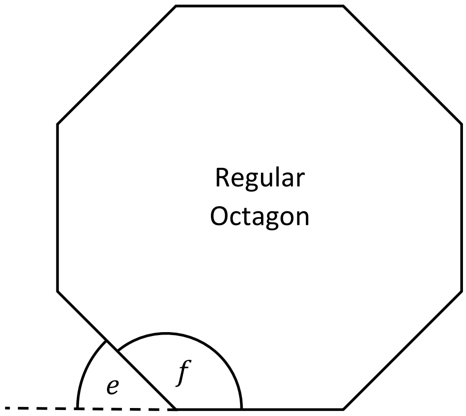
(b)



(c)



(d)



Exercise 11

F

(a) What is the total interior angles of any heptagon?

(b) What is the exterior angle of any equilateral triangle?

(c) What is the interior angle of any regular nonagon?

(d) Four of the exterior angles of a pentagon are 110°, 90°, 70°, 50°. What is the size of the fifth exterior angle?

(e) Four of the interior angles of a pentagon are 150°, 130°, 110°, 90°. What is the size of the fifth interior angle?

Exercise 12

I

(a) The size of the exterior angles of a regular polygon are 18°? How many edges does the regular polygon have?

(b) The size of the interior angles of a regular polygon are 156°. How many edges does the regular polygon have?

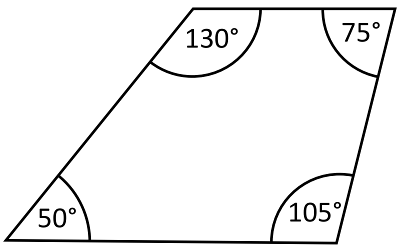
(c) Three of the exterior angles of a hexagon are 100°. The other three exterior angles are equal. Calculate the size of each of these exterior angles.

(d) Why is it not possible to draw a triangle with exterior angles 170°, 160°, 150°?

(e) Four of the six interior angles of a hexagon are 130°, 140°, 150°, 160°. The other two interior angles are equal. Calculate the size of the largest **exterior** angle of the hexagon.

Exercise 13

Draw polygons in the spaces below, showing clearly the size of each angle.

		Size of the least interior angle		
		Less	Equal	More
Total of the interior angles	More			
	Equal			
	Less			

Exercise 14

Squares (or regular quadrilaterals) tessellate, as can be seen in the picture on the right of tiles placed on a floor.

Use the ATM mats to find which two other regular polygons tessellate.

Prove that only these three regular polygons tessellate. (Hint: use the factors of 360 and the list of interior angles of regular polygons.)

Exercise 15

F

A **semi-regular tessellation** uses two or more regular polygons to fill the plane.

For example, the tessellation shown on the right uses squares (black and white) and octagons (red and white).

Use the ATM mats to find the **eight** semi-regular tessellations. Record the tessellations in the following table. (The first row has been completed for you.)

Tessellation	Polygons	Angles around any point
1	Octagon, Octagon, Square	135° + 135° + 90° = 360°
2		
3		
4		
5		
6		
7		
8		

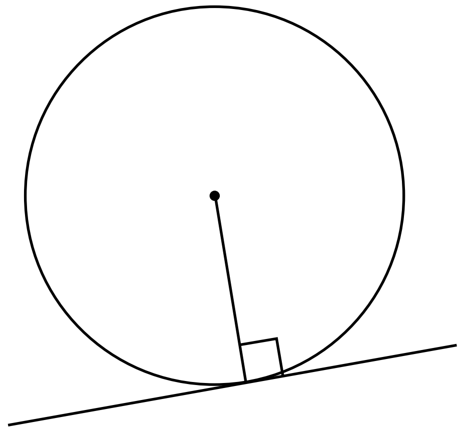
Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Circle Theorems

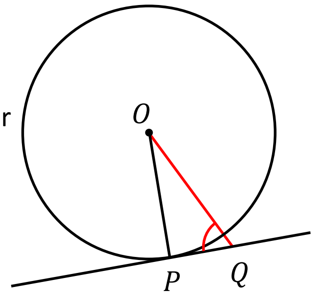
A number of facts related to angles in a circle must be learnt. (You do not have to learn the proofs.)

(1) Tangent and radius meet at a right angle.



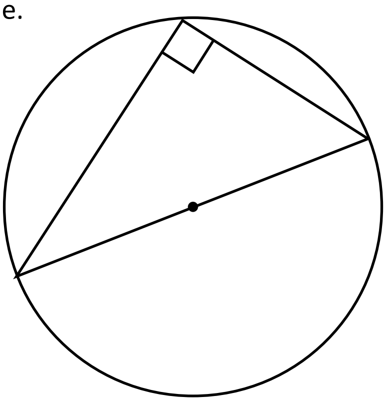
Proof

Assume that the tangent and radius do not meet at a right angle. Then we can draw the perpendicular line from the centre O to the point Q on the tangent (a point that is outside the circle), so that the angle $O\hat{Q}P = 90°$.



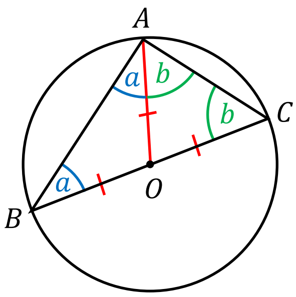
It follows that the triangle OQP is a right-angled triangle where the radius OP is the hypotenuse of the triangle. But we see that the line OQ must be longer than the line OP (as Q lies outside the circle). This goes against the mathematical fact that the hypotenuse of a right-angled triangle is the longest side, so tangent and radius must meet at a right angle.

(2) The angle in a semicircle is a right angle.



Proof

Split the triangle into two isosceles triangles, by adding a radius from the centre O to the vertex A.



The total of the angles of triangle ABC is $180°$, so

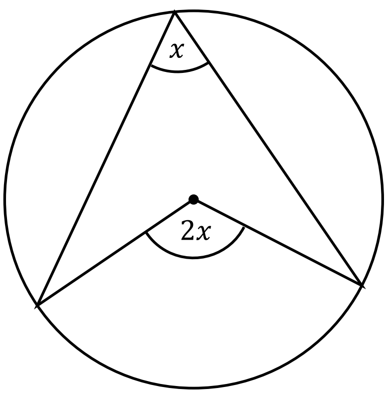
$a + a + b + b = 180°$

$2a + 2b = 180°$ [Collect like terms]

$a + b = 90°$ [Divide by 2]

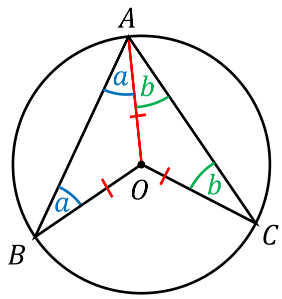
So, the angle $B\hat{A}C$ is a right angle.

(3) The angle in the centre of a circle is twice the angle on the circumference.



Proof

Split the quadrilateral into two isosceles triangles, by adding a radius from the centre O to the vertex A.



In triangle ABO, $A\hat{O}B = 180° - 2a$.
In triangle ACO, $A\hat{O}C = 180° - 2b$.

Using the angles around the centre O,

$B\hat{O}C = 360° - A\hat{O}B - A\hat{O}C$

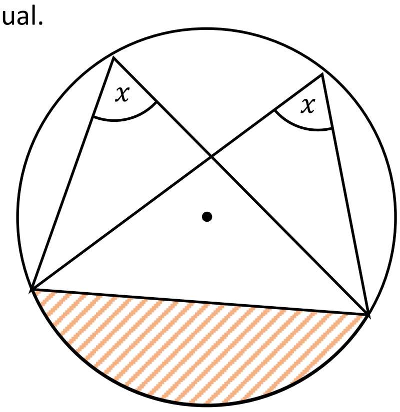
$B\hat{O}C = 360° - (180° - 2a) - (180° - 2b)$

$B\hat{O}C = 2a + 2b$

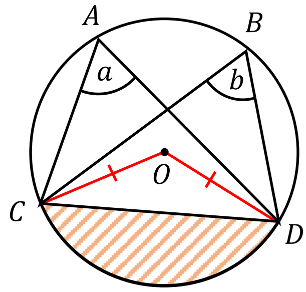
$B\hat{O}C = 2(a + b)$

So, the angle in the centre of the circle ($B\hat{O}C$) is twice the angle on the circumference ($B\hat{A}C$ or $a + b$).

(4) Angles in the same segment are equal.



Proof

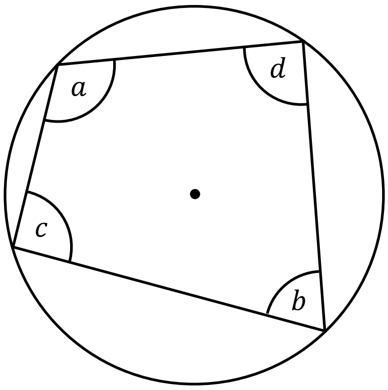


Add two radii from the centre O to the vertices C and D.

From the previous circle theorem, the size of angle $C\hat{O}D$ is twice the angle $C\hat{A}D$. But we can also state that the angle $C\hat{O}D$ is twice the angle $C\hat{B}D$.

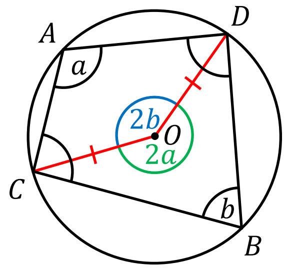
It follows that the angles $C\hat{A}D$ and $C\hat{B}D$ are equal. So, angles in the same segment are equal.

(5) Opposite angles in a cyclic quadrilateral sum to 180°.



$a + b = 180°$
$c + d = 180°$

Proof



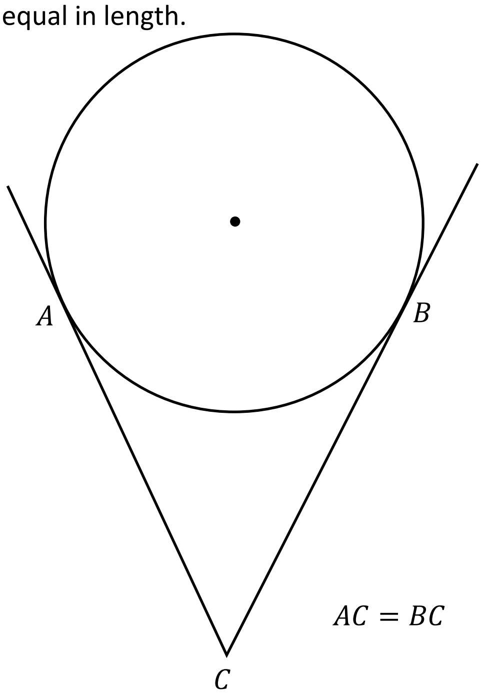
Add two radii from the centre O to the vertices C and D.

Because the angle in the centre is twice the angle on the circumference, we can state that $C\hat{O}D = 2b$, and $C\hat{O}D$ reflex $= 2a$.

Angles around any point sum to 360°, so
$2a + 2b = 360°$
$a + b = 180°$ [Divide by 2]

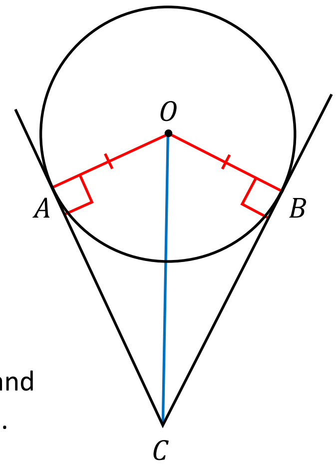
So, opposite angles in a cyclic quadrilateral sum to 180°.

(6) Tangents from an external point are equal in length.



$AC = BC$

Proof (Higher Tier)



Add two radii from the centre O to the vertices A and B.
Then, add a line from the centre O to the vertex C.

Because tangent and radius meet at a right angle, we have $O\hat{A}C = O\hat{B}C = 90°$.

The two right-angled triangles OAC and OBC share the same hypotenuse OC.

We have $OA = OB$, because they are both radii.

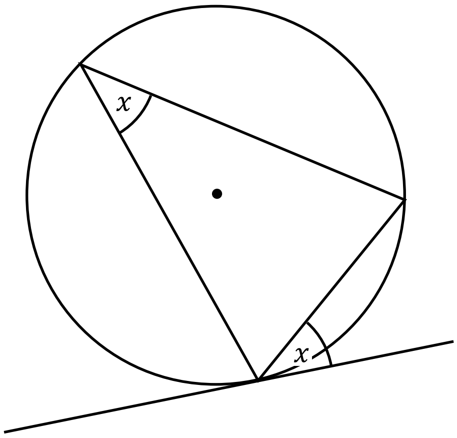
Using the RHS rule, we can state that the triangles OAC and OBC are congruent.

It follows that $AC = BC$, and so tangents from an external point are equal in length.

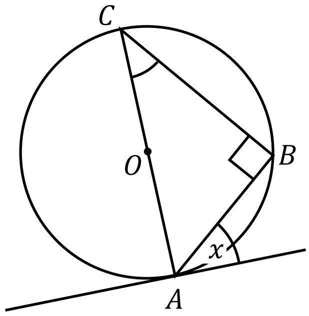
The final two theorems appear in the higher tier only.

Higher Tier

(7) The angle between chord and tangent is equal to the angle in the alternate segment.



Proof



Because angles in the same segment are equal, we can choose to prove the case where the line AC is a diameter to the circle.

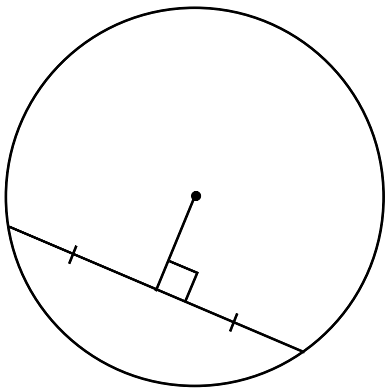
Tangent and radius meet at a right angle, so $C\hat{A}B = 90° - x$.

The angle in a semicircle is a right angle, so $A\hat{B}C = 90°$.

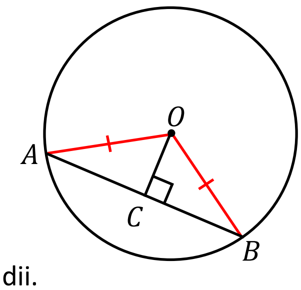
Using the triangle ABC,
$A\hat{C}B = 180° - A\hat{B}C - C\hat{A}B$
$A\hat{C}B = 180° - 90° - (90° - x)$
$A\hat{C}B = x$

So, the angle between chord and tangent is equal to the angle in the alternate segment.

(8) The perpendicular from the centre to a chord bisects the chord.



Proof



Add two radii from the centre O to the vertices A and B.

The triangles OAC and OBC are right-angled triangles.

The hypotenuse of both triangles are equal in length, since they are both radii.

The two triangles also share a side OC.

Using the RHS rule, we can state that the triangles OAC and OBC are congruent.

It follows that $AC = BC$, and so the perpendicular from the centre to a chord bisects the chord.

Exercise 16

I

Take some time to become familiar with the circle theorems. Here are some ideas:

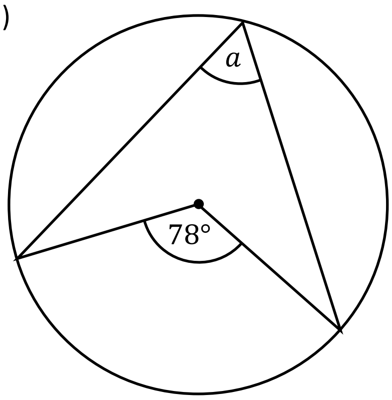
- Try to re-create the circle theorems using the GeoGebra software.
- Draw examples of the circle theorems in your revision book.
- Try to re-create the circle theorems using paper plates, string and colouring materials.
- Verify that the theorems are true by drawing examples using a compass, ruler and protractor.

Skill I

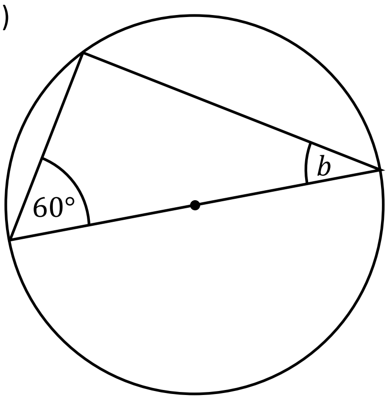
Exercise 17

Use the circle theorems to find the size of the marked angles in the following diagrams. (The diagrams are not drawn to scale.)

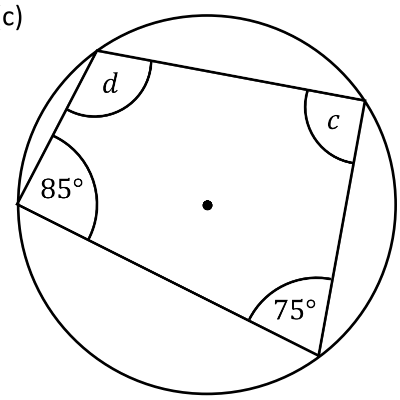
(a)



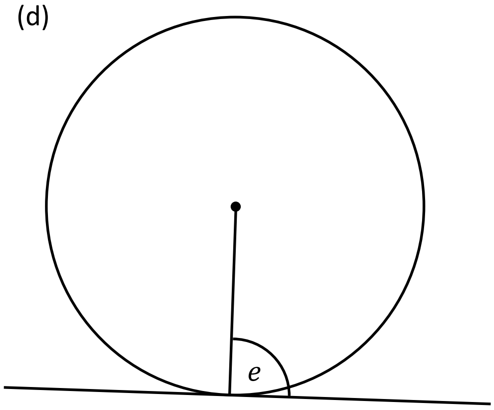
(b)



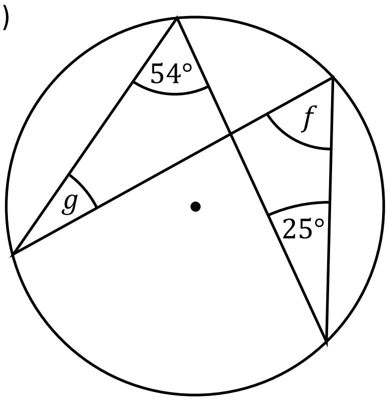
(c)



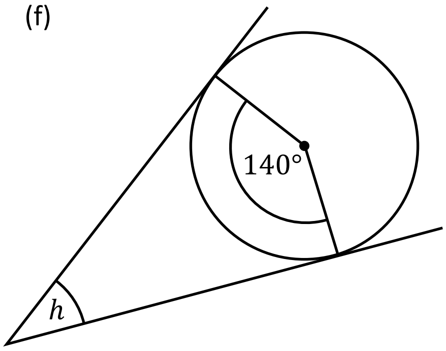
(d)



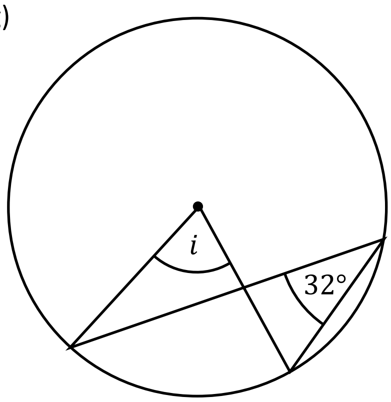
(e)



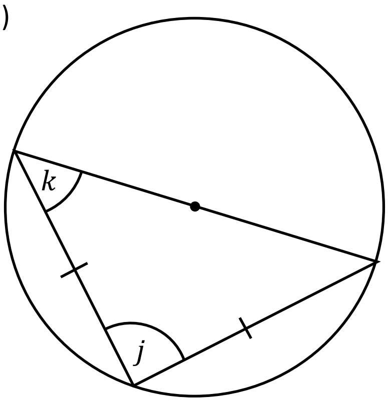
(f)



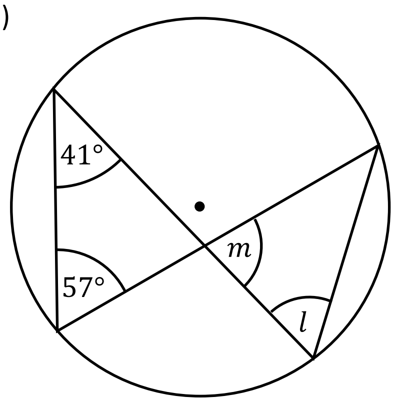
(g)



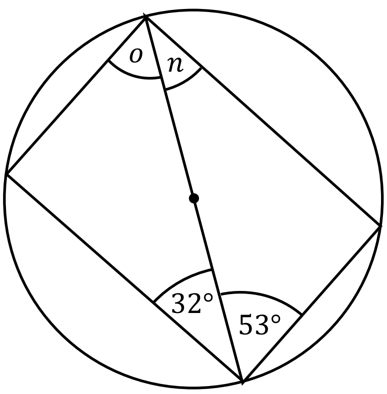
(h)



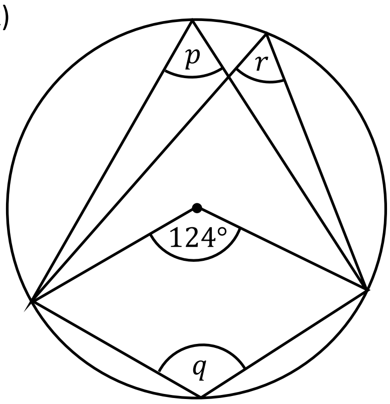
(i)



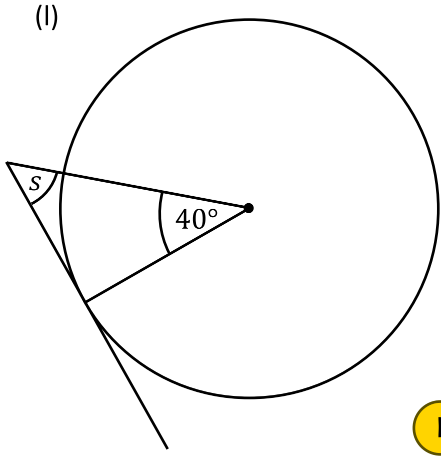
(j)



(k)



(l)



Exercise 18

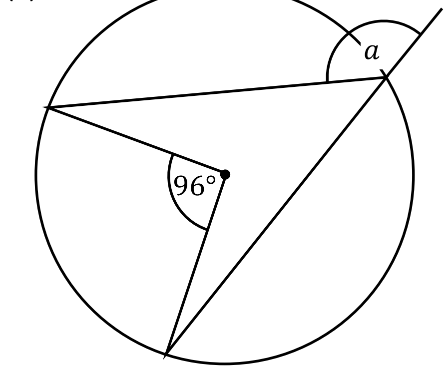
I

For each question in Exercise 17 above, note which circle theorem(s) you used in finding the missing angle(s).

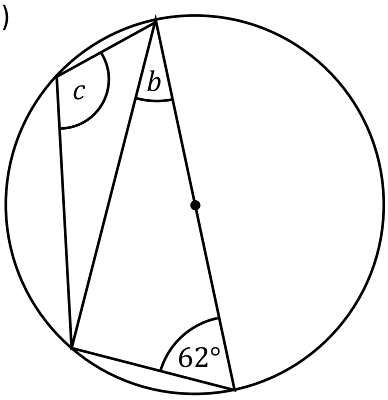
Exercise 19 I

Use the circle theorems to find the size of the marked sides or angles in the following diagrams.
(The diagrams are not drawn to scale.)

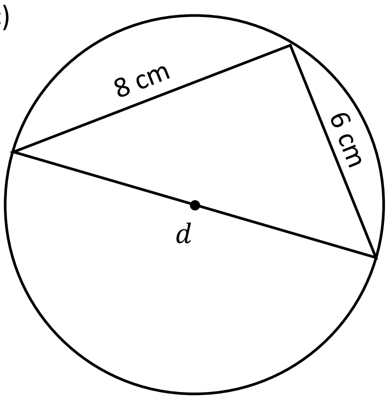
(a)



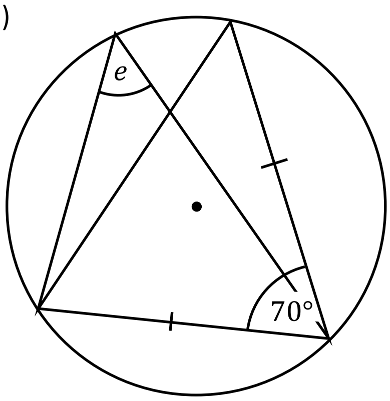
(b)



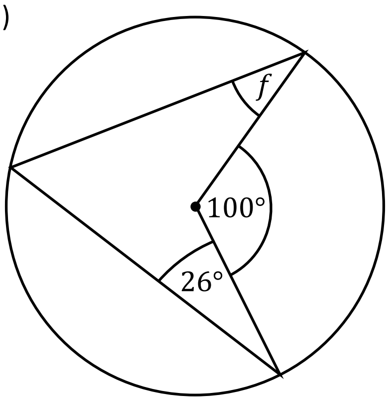
(c)



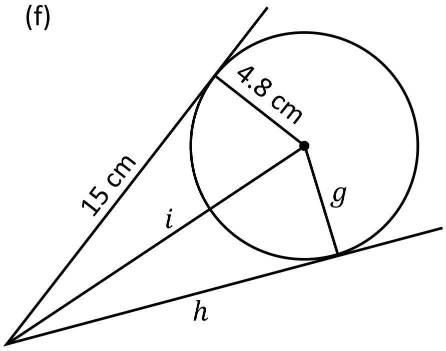
(d)



(e)



(f)



Exercise 20 I

For each question in Exercise 19 above, note which circle theorem(s) you used in finding the missing value(s).

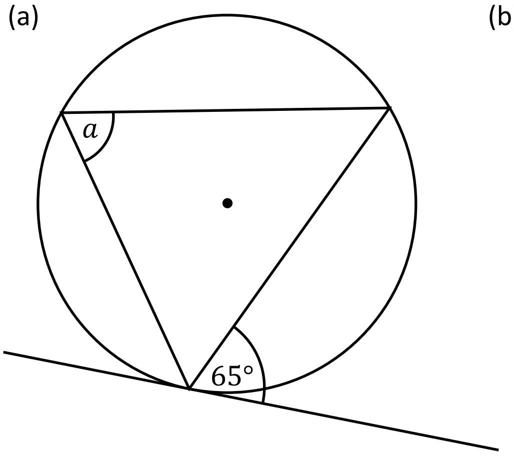
Challenge!

Imagine a circle with radius 1 m. 6 equilateral triangles are placed in the circle, with one vertex of each triangle in the centre of the circle, and the two other vertices on the circumference. The triangles do not overlap. What is the difference between the area of the circle and the area of the six equilateral triangles?

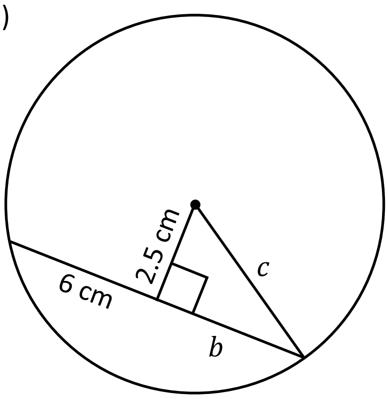
Exercise 21 H

Use the circle theorems to find the size of the marked sides or angles in the following diagrams. (The diagrams are not drawn to scale.)

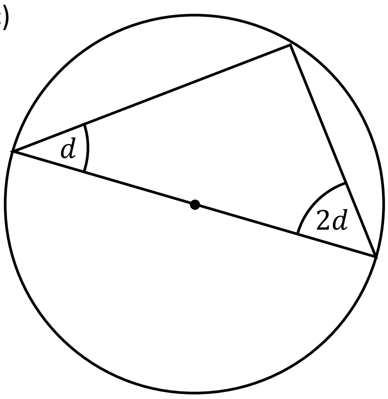
(a)



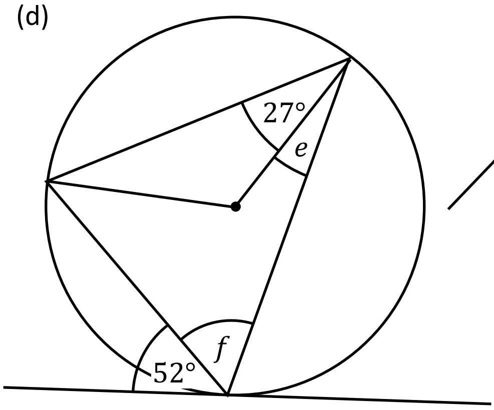
(b)



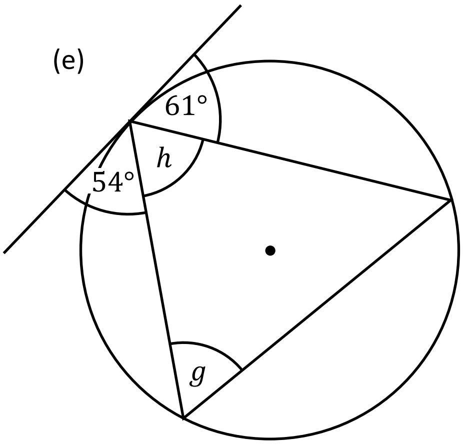
(c)



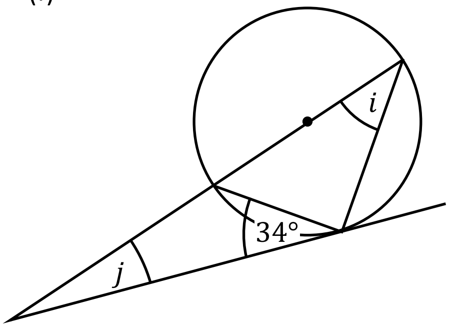
(d)



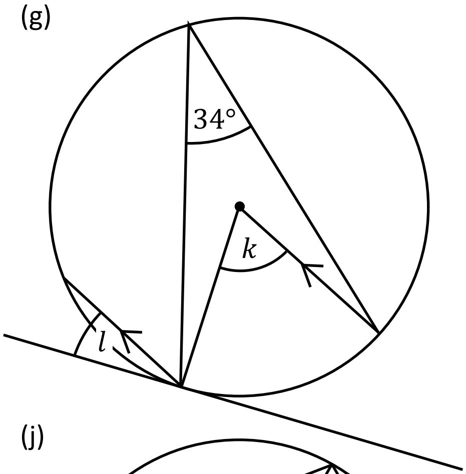
(e)



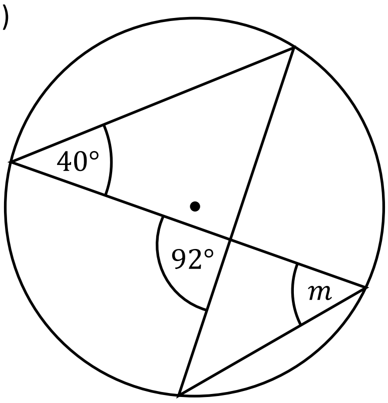
(f)



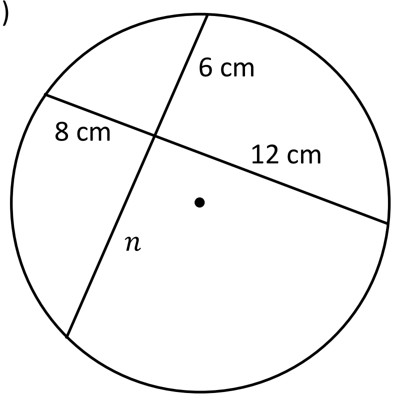
(g)



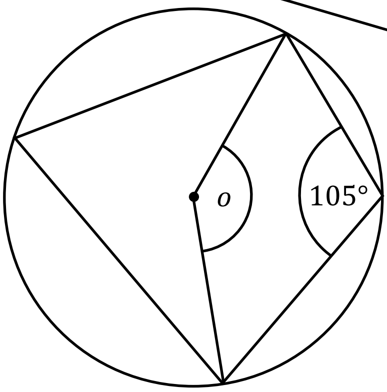
(h)



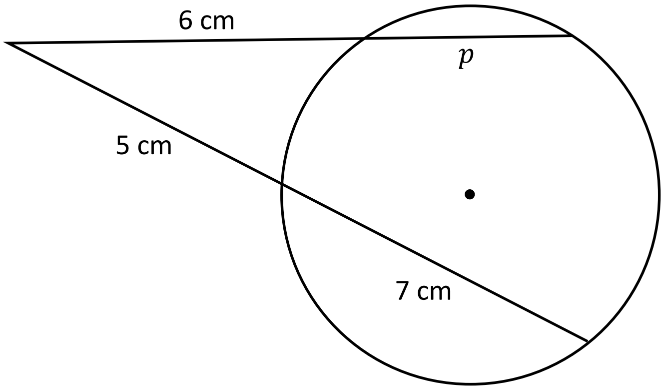
(i)



(j)



(k)



Exercise 22

H

For each question in Exercise 21 above, note which circle theorem(s) you used in finding the missing value(s).

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Transformations

Over the years, we have seen four types of transformation.

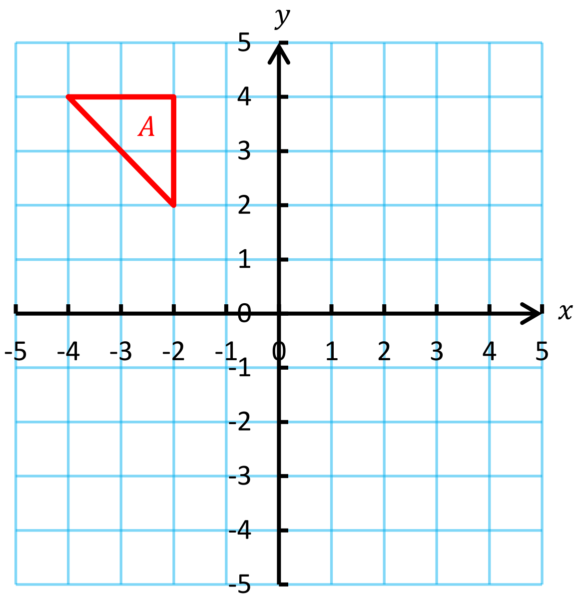
Year 7	Year 8	Year 9	Year 10
Translation (moving a shape)	Rotation (turning a shape)	Reflection (reflecting a shape)	Enlargement (changing the size of a shape)

In this chapter, we will revise these transformations, before combining them in different ways.

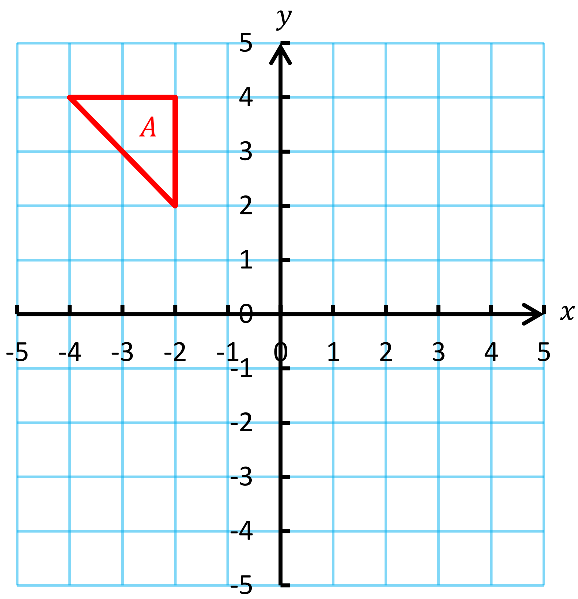
Exercise 23

Revision I

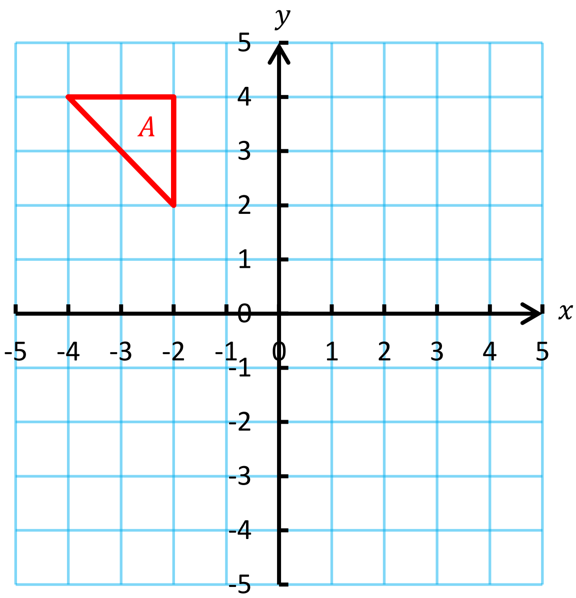
(a) Translate the triangle A using the column vector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$.



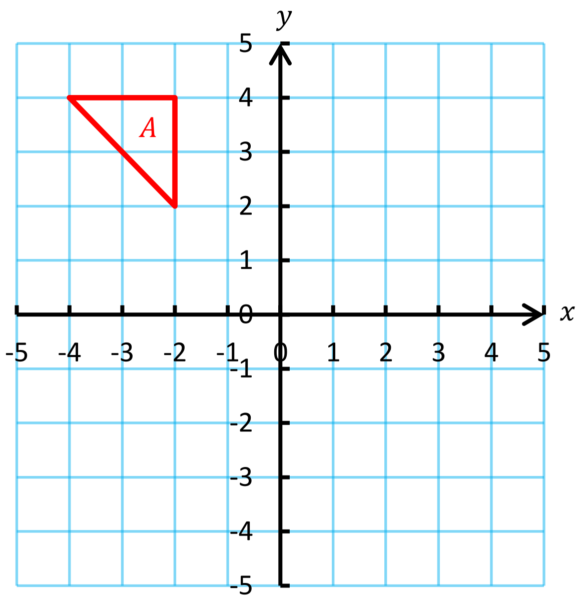
(b) Rotate the triangle A 90° clockwise around the point (–1, 1).



(c) Reflect the triangle A in the line $y = 1$.



(d) Enlarge the triangle A using scale factor 2 and centre of enlargement (–5, 3).



Skill I

Exercise 24

(a) Translate the triangle A using the column vector $\begin{pmatrix}-2\\-3\end{pmatrix}$. Label the new triangle B.

(b) Reflect the triangle B in the line $x = 4$. Label the new triangle C.

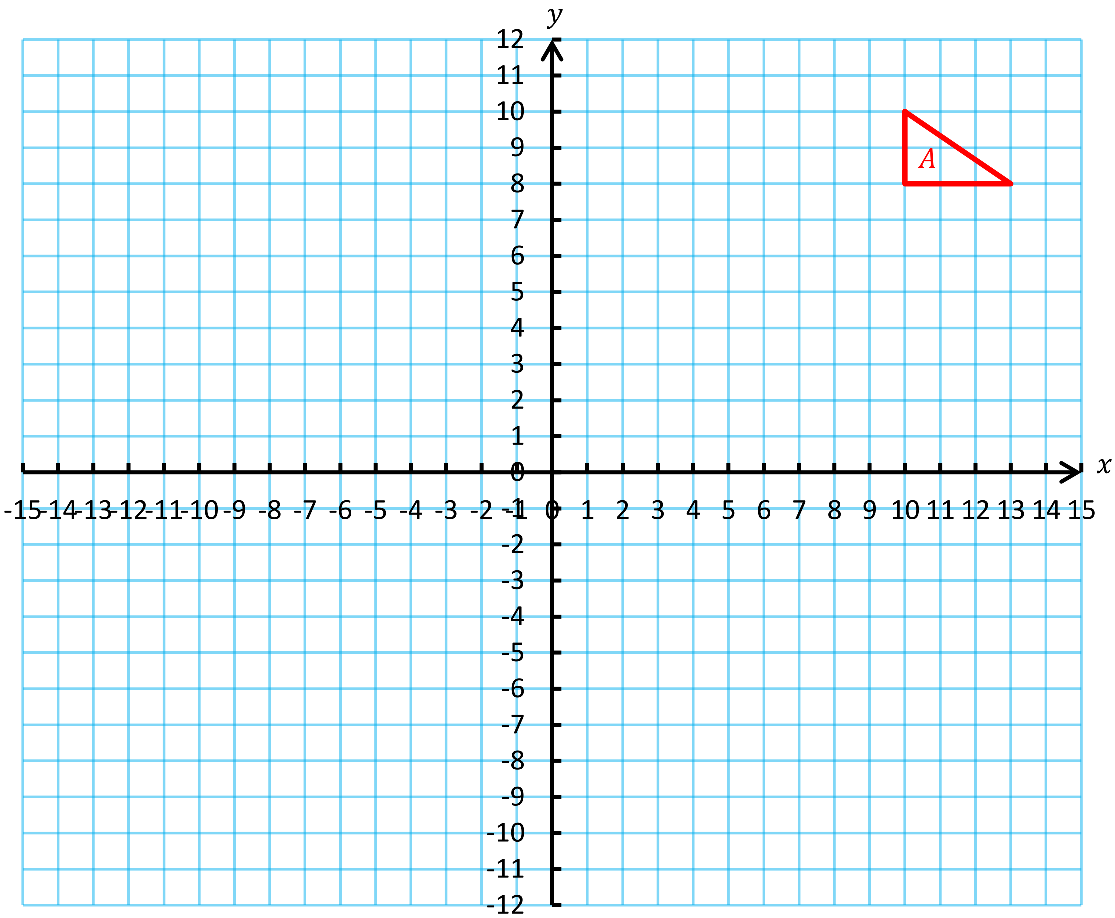
(c) Rotate the triangle C 90° anticlockwise around the point (–4, 3). Label the new triangle D.

(d) Reflect the triangle D in the line $y = 1$. Label the new triangle E.

(e) Enlarge the triangle E using scale factor 3 and centre of enlargement (–10, –3). Label the new triangle F.

(f) Reflect the triangle F in the line $x = 4$. Label the new triangle G.

(g) Translate the triangle G using the column vector $\begin{pmatrix}-20\\11\end{pmatrix}$. Label the new triangle H.



Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Mesur

Siapiau 4

Reflection

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I can **recognise congruent shapes**.			1, 2	
I can calculate the **total interior angles** of any polygon.			3, 7	
I can calculate the **interior angle of any regular polygon**.			6, 7	
I know the **total exterior angles of any polygon**.			3, 5	
I can calculate the **exterior angle of any regular polygon**.			4	
I know the **connection between the interior and exterior angles** for any vertex in a polygon.			3, 5	
I know when regular polygons **tessellate**, and when they do not tessellate.			7	
I can recite the names of the **circle theorems**.			8, 9, 10	
I can **use the circle theorems** to find missing angles and sides.			8, 9, 10	
I can **combine the four transformations** to transform different shapes.			11	

Intermediate Tier

Mesur

Siapiau 4

Reflection

Name: ....

Percentage in the test: ....

	I know this.	**I need to revise this.**	**Question in the test:**	**Correct in the test?**
I can **recognise congruent shapes**.			1	
I can use the **SSS, SAS, ASA and RHS rules to prove when two triangles are congruent.**			2, 3	
I can calculate the **total interior angles** of any polygon.			4, 6, 8	
I can calculate the **interior angle of any regular polygon**.			7, 8	
I know the **total exterior angles of any polygon**.			4	
I can calculate the **exterior angle of any regular polygon**.			5	
I know the **connection between the interior and exterior angles** for any vertex in a polygon.			4, 6	
I know when regular polygons **tessellate**, and when they do not tessellate.			8	
I can recite the names of the **circle theorems**.			9, 10, 11	
I can **use the circle theorems** to find missing angles and sides.			9, 10, 11	
I can **combine the four transformations** to transform different shapes.			12	

Higher Tier