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WELSH JOINT EDUCATION COMMITTEE
CYD-BWYLLGOR ADDYSG CYMRU

General Certificate of Education
Tystysgrif Addysg Gyffredinol

Summer Examination 1976
Arholiad Haf 1976

Advanced Level
Safon Uwch

(3 hours) MATHEMATICS A 1

Answer **seven** *questions only.*

1. (*a*) Solve the equations

$$xy = 4, \qquad x^2+x+y = 6. \qquad [7]$$

(*b*) Find the value of x^6+y^6 in terms of a and b where

$$x+y = a, \qquad x^2+y^2 = b^2. \qquad [4]$$

Show also that these equations give real values for x and y only if $a^2 \leqslant 2b^2$. [4]

2. (*a*) Use your tables to evaluate a^a when $a = 0{\cdot}12$. [5]

(*b*) Show by induction or otherwise that

$$5^{2n}-1$$

is divisible by 24 for all positive integers n. [5]

(*c*) In the binomial expansion of $(3+x)^n$ the coefficient of x^6 equals the coefficient of x^7. Find the value of n. [5]

3. (*a*) Prove the identities

$$\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta \equiv 4\cos\theta\cos 2\theta\cos 4\theta$$
$$\equiv \sin 8\theta/2\sin\theta. \qquad [4, 4]$$

(*b*) Solve for $0 < \theta < 360°$ the equation

$$\sec\theta - 3\tan\theta = 2. \qquad [7]$$

Turn over.

4. Matrices A, B and vectors X, K are given by

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -4 & 3 \\ 3 & -1 & 4 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 4 & 5 \\ 2 & -3 & -5 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}.$$

(i) Use a **matrix** method to solve the system of equations $\mathbf{AX} = \mathbf{K}$. [9]

(ii) Find values of the constants λ, μ such that the rows of matrix $\mathbf{B}$ are connected by the relation

$$R_3 = \lambda R_1 + \mu R_2. \qquad [3]$$

Hence explain why the system of equations $\mathbf{BX} = \mathbf{K}$ would be inconsistent (i.e. they do not possess a solution set). [3]

5. Show that the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

rotates the whole plane around the origin through an angle α anticlockwise. Show also that the transformation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

translates the whole plane from origin $(0, 0)$ to a new origin (a, b). [4, 3]

Show that the transformation

$$5x' = 3x + 4y + 5$$
$$5y' = -4x + 3y + 10$$

gives a rotation of the plane about a centre. Find the coordinates of the centre of rotation and the angle of rotation of the plane. [8]

6. (*a*) A triangle has vertices $(4, 6)$, $(-3, 5)$ and $(5, -1)$. Find the coordinates of the centre of the circumscribed circle. [8]

(*b*) Find the two possible values of the constant c if the line $2y + 3x = c$ touches the curve $2x^2 + y^2 = 2$, and give the coordinates of the respective points of contact. [7]

7. State the maximum domains of each of the functions f, g, h where

$$f(x) = \frac{2x}{x-1}, \quad g(x) = \log_e(x+1), \quad h(x) = +\sqrt{(9-x^2)}.$$

[3]

(i) Find the composite function $g \circ f$ and give its maximum domain. [3]

(ii) Find the inverse functions f^{-1} and g^{-1} and give their maximum domains. [6]

(iii) Explain why h^{-1} does not exist unless the maximum domain is restricted. State a domain of h for which h^{-1} exists. [3]

8. (*a*) Differentiate the expressions

$$(2x+3)^2e^{3x} \quad \text{and} \quad (\cos x+\sin x)/(\cos x-\sin x). \qquad [7]$$

(*b*) A variable isosceles triangle has a constant perimeter $2k$. If x denotes the length of each of the equal sides, find the area of the triangle in terms of x and k. Hence prove that the area is a maximum when the triangle is equilateral. [8]

9. (*a*) Evaluate $\displaystyle\int_1^2 \frac{dx}{4x^2-1}$. [4]

(*b*) Integrate $x \cos x$ and hence evaluate

$$\int_0^{\frac{1}{2}\pi} (x+\cos x)^2 \, dx. \qquad [5]$$

(*c*) A sphere of radius a is cut into unequal parts by a plane distant d from the centre. Prove that the volume of the smaller part is $\frac{1}{3}\pi(2a+d)(a-d)^2$. [6]

10. (*a*) *Sketch* the graphs of

$$y = x^2(x^2-1) \quad \text{and} \quad y = x^2/(x^2-1),$$

indicating any maxima or minima, and any horizontal or vertical asymptotes. [9]

(*b*) Find the equation of the tangent drawn from the origin to the curve $y = \log_e x$. [6]

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(3 hours) PURE MATHEMATICS **A 1**

Answer **seven** *questions.*

1. (i) Answer the following in any convenient order:

(*a*) show that the roots of the equation $x^4+4=0$ are $\pm(1\pm i)$;

(*b*) express x^4+4 as a product of linear factors with complex number coefficients;

(*c*) express x^4+4 as a product of quadratic factors with real number coefficients. [7]

(ii) Prove that if the polynomial equation $f(x)=0$ has a repeated root $x=\alpha$ then α is also a root of the equation $f'(x)=0$. [3]

Solve the equation

$$18x^3+15x^2-4x-4=0$$

given that it has a repeated root. [5]

Turn over.

2. (i) By considering $(1-2x+x^2)S_n$, or otherwise, find

$$S_n=\sum_{r=1}^{n} rx^r.$$ [5]

Show that when $x=\frac{1}{2}$, $S_n=2-(n+2)2^{-n}$, and find the least integer n such that $1{\cdot}99<S_n<2$. [4]

(ii) By using partial fractions find $\sum_{r=1}^{n} u_r$ where

$$u_r=\frac{r^2+r+1}{r^2+r}.$$ [6]

3. Reduce to echelon form, and hence find (where possible) the most general form of the solution of the system of equations

$$x+2y+kz=a$$
$$3x-y-z=b$$
$$-2x+\tfrac{1}{2}y+z=c$$

in the three separate cases

(i) $k=2$; $a=4\frac{1}{2}$, $b=10$, $c=-6$,

(ii) $k=-5$; $a=4$, $b=-2$, $c=1$,

(iii) $k=-5$; $a=4$, $b=-2$, $c=1\frac{1}{2}$. [11]

Interpret each case geometrically as a system of planes. [4]

4. Given the matrix

$$A=\begin{pmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -1 & 3 & -2 \end{pmatrix}$$

calculate A^2 and A^3 and show that

$$A^3-7A+6I=O$$

where I is the unit 3×3 matrix and O is the zero 3×3 matrix. [6]

By writing this equation in the form $AB=I$, with a suitable matrix B, or otherwise, evaluate A^{-1}. [5]

If $M=A-2I$, show that M is a singular matrix, and find a non-zero column vector X such that $MX=0$. [4]

5. A transformation T of the plane is given by the matrix equation

$$X' = AX+H$$

where

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad X' = \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad H = \begin{pmatrix} -\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}.$$

T may be described as a rotation about the origin O followed by a translation. Find the angle of the rotation. [2]

Find $K = \begin{pmatrix} a \\ b \end{pmatrix}$ so that the same transformation T is represented as a translation followed by a rotation about O in the form

$$X' = A(X+K).$$ [5]

Express the inverse transformation T^{-1} in the form

$$X = A_1X'+H_1.$$ [4]

Find the equation of the transform by T of the line $2x+y-3 = 0$. [4]

6. Two circles with centres A and B intersect at points P and Q, such that $AB^2 = AP^2+PB^2$. Show that the circles cut orthogonally (i.e. at right angles). [2]

The equations of two circles C_1 and C_2 are respectively

$$x^2+y^2+2\lambda x+3 = 0$$

$$x^2+y^2+2\mu y-3 = 0,$$

where λ and μ are non-zero constants. Show that C_1 and C_2 intersect orthogonally. [4]

Find the equation of the circle through the origin which is orthogonal to both C_1 and C_2. [5]

Show that at any point (x, y) on C_1 the slope dy/dx satisfies the differential equation

$$x^2-y^2-3+2xy\frac{dy}{dx} = 0.$$ [4]

7. The curve C, whose parametric equations are

$$x = 3+\cos 2\theta, \quad y = 1+2\sin\theta,$$

is *part* of a parabola K. Find the cartesian equation of K. Sketch K and C on one diagram. [7]

Find the equations of the tangents to K at the end points of C, and show that one of these tangents intersects the x-axis at the point $(8, 0)$. Draw both tangents on your sketch and find a set of points on the x-axis from which just one tangent to C can be drawn. [8]

8. A curve is given by the parametric equations

$$x = 3t^2, \quad y = 3t^3-t^2,$$

where $t \in (-\infty, \infty)$. Find the cubic equation in t satisfied by the points of intersection of the curve and a straight line $ax+by+c = 0$ not through the origin. Deduce that if t_1, t_2, t_3 are the parameters of these points of intersection then

$$\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3} = 0.$$ [5]

Deduce, or prove otherwise, that the tangent at the point of parameter t meets the curve again at the point of parameter u, where $u = -\frac{1}{2}t$. [2]

Show that there are two lines each of which is tangent to the curve at one point and normal to the curve at another point, and that the angle between these lines is $\tan^{-1} 3$. [8]

9. (i) Solve the equation

$$\sin 2\theta - \cos 2\theta = \sin\theta - \cos\theta,$$

giving all solutions $\theta \in [0, 2\pi)$. [6]

(ii) $ABCD$ is one face of a cube and AA', BB', CC', DD' are edges, each of length $2a$. If M is the mid-point of $C'D'$ and N is the mid-point of BC find the angle MAN (to the nearest degree) and show that the area of the triangle MAN is $\frac{1}{2}a^2\sqrt{29}$. [9]

10. (i) Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

where n is a positive integer. Find the roots of the equation

$$z^5 = 4-4i$$

in the form $r(\cos\theta + i\sin\theta)$ and plot their positions in the Argand diagram. [3, 5]

(ii) Points P, Q in the Argand diagram represent the complex numbers z, $2z^2+4$ respectively. Find the locus of Q as P describes the circle C given by $|z| = 1$. Describe carefully the motion of Q as P makes one anti-clockwise circuit of C starting from the point $z = 1$. [4, 3]

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(3 hours) PURE MATHEMATICS A2

Answer **seven** *questions.*

1. Find the stationary points of the function f given by

$$f(x) = \frac{x^2-3x+4}{x^2-1}. \qquad [4]$$

Sketch the graph of the function and from consideration of your graph, or otherwise, answer the following: [4]

(i) State the domain and range of f;

(ii) Find the solution-set of the inequality $f(x) < 1$;

(iii) State any one interval of values of x on which $f(x)$ is strictly decreasing;

(iv) State in what part of the graph there is a point of inflexion. [7]

2. A function f is defined on the interval $[0, 2]$ as follows:

$$\begin{aligned} f(x) &= 2 && \text{if } x = 0 \\ &= x && \text{if } x \in (0, 1) \\ &= 0 && \text{if } x = 1 \\ &= 3-x && \text{if } x \in (1, 2]. \end{aligned}$$

Draw the graph of f and state its range. Indicate clearly any isolated points of the graph, and show where the graph has gaps. [4]

Explain why the given function f has an inverse f^{-1}, and draw the graphs of the functions

$$f^{-1}, \quad f \circ f, \quad f \circ f^{-1}, \quad f^{-1} \circ f^{-1}. \qquad [2, 9]$$

Turn over.

3. Find the stationary points of the function f given by

$$f(x) = x^3-4x^2+5x+1$$

and sketch its graph. [6]

Find the image $f([-1, 1])$ of the interval $[-1, 1]$ and the inverse image $f^{-1}([1, 3])$ of the interval $[1, 3]$. [3]

State the intervals of values of x on which f is (i) strictly increasing, (ii) strictly decreasing. [2]

Using any numerical or graphical method, find the real root of $f(x) = 0$ correct to one decimal place. [4]

4. (i) Classify the functions, defined on the real numbers by the following, as *even*, *odd* or *neither* even *nor* odd:

$$e^x \sin x, \quad \frac{2x}{x^2+1}, \quad x \sin x. \qquad [3]$$

(ii) Sketch the graph of the function f defined by

$$f(x) = x|x| \quad \text{where } x \in [0, 2]. \qquad [2]$$

Sketch the graphs of, and also define by formulae, functions g and h such that

(*a*) g has domain $[-2, 2]$, g is even, and $g(x) = f(x)$ if $x \in [0, 2]$,

(*b*) h has domain $[-2, 2]$, h is odd, and $h(x) = f(x)$ if $x \in [0, 2]$. [6]

(iii) Show that the function given by

$$\phi(x) = |x+2|+|x-2|$$

is an even function, and sketch its graph. [4]

5. Prove that

$$\sinh 3\theta = 3 \sinh \theta + 4 \sinh^3 \theta. \qquad [4]$$

By considering $f'(x)$, explain briefly why the equation

$$f(x) = x^3+3x-24 = 0$$

has exactly one real root. Solve the equation by using the substitution $x = k \sinh \theta$, choosing k to give an equation in $\sinh 3\theta$. Give your answer to two significant figures. [3, 8]

6. Functions f, g, h are defined as follows:

$$f(x) = 4x \quad \text{if } 0 \leqslant x \leqslant 1,$$
$$= 4 \quad \text{if } 1 < x < 2,$$
$$= 12-4x \quad \text{if } 2 \leqslant x \leqslant 3,$$
$$g(x) = f'(x),$$
$$h(x) = \int_0^x f(t)\,dt.$$

Find formulae for $g(x)$ and $h(x)$ corresponding to those given for $f(x)$, and sketch graphs of f, g, h, one vertically below another so that values of x correspond. [8,7]

7. Express $$f(x) = \frac{4}{(1-x)^3(1+x)}$$

in partial fractions. Hence [4]

(i) evaluate $\int_2^3 f(x)\,dx$ in terms of logarithms; [4]

(ii) find $f'(x)$ and $f''(x)$ and deduce a formula for

$$f^{(n)}(x) = \frac{d^n f(x)}{dx^n}$$

where n is a positive integer. (You need not give a formal proof of your formula.) [5]

Show that

$$f^{(n)}(0) = (n+1)!(n+3) \quad \text{if } n \text{ is odd.}$$ [2]

8. (i) Prove that if $y = \sin(\lambda \sin^{-1}x)$, where $x \in (-1, 1)$ and λ is constant, then

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \lambda^2 y = 0.$$ [5]

(ii) Evaluate $$\int_0^{\frac{1}{4}\pi} \sin^3\theta \sec^2\theta\,d\theta,$$

giving your answer to two significant figures. [4]

(iii) Show that

$$\int_0^1 x^2 \tan^{-1}x\,dx = \frac{\pi}{12} - \frac{1}{6} + \frac{\log 2}{6}.$$ [6]

9. (i) Differentiate the following with respect to x, simplifying your answers as far as possible. In each case state an interval of values of x (as large as possible) for which the result holds.

(*a*) $\dfrac{xe^{-x}}{\sqrt{(1-x^2)}}$, (*b*) $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$, (*c*) $\int_0^x \sin(t^2)\,dt$.

[9]

(ii) Show that

$$\int_1^2 \sqrt{(x^2-1)}\,dx = \sqrt{3} - \tfrac{1}{2}\log(2+\sqrt{3}).$$ [6]

10. Sketch the graph of the function given by

$$y = \log_e(\tfrac{3}{4}) - \log_e(1-x^2)$$

and state its domain. [5]

Show that if s is the arc-length of the curve then

$$\frac{ds}{dx} = \frac{1+x^2}{1-x^2}$$

and that the arc-length below the x-axis is $-1+2\log 3$.

[5, 5]