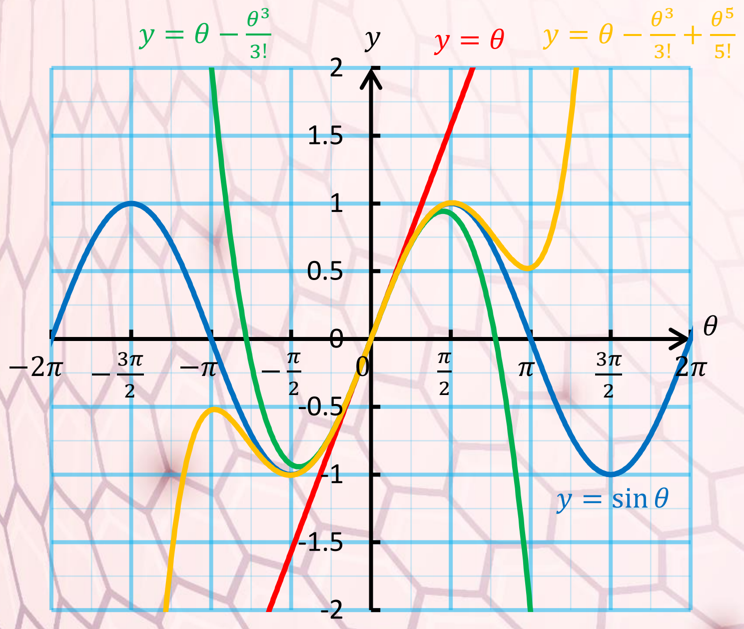
Unit 3, Workbook 9

13

Small Angle Approximations



$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \cdots$$

$$\tan(\theta) = \theta + \frac{2\theta^3}{3!} + \frac{16\theta^5}{5!} + \frac{272\theta^7}{7!} + \cdots$$

The small angle approximations use part of these **Taylor series** for sin, cos and tan.

Name:

Background

What is the work?	If you have an angle θ that is sufficiently small, then it is possible to use alternate expressions for $\sin\theta$, $\cos\theta$ and $\tan\theta$, ones that do not require a calculator in order to calculate them.
What is required before starting?	**GCSE Work:** Trigonometry, area of a sector, inequalities. **A Level Unit 3:** Measuring angles in radians, trigonometric addition formulae.
Where does this lead to?	**A Level Unit 3:** • Differentiating $\sin\theta$ and $\cos\theta$ from first principles. **Applications:** • Measuring the distance to the stars. • The "one in 60" rule when flying.

Theory

Theory

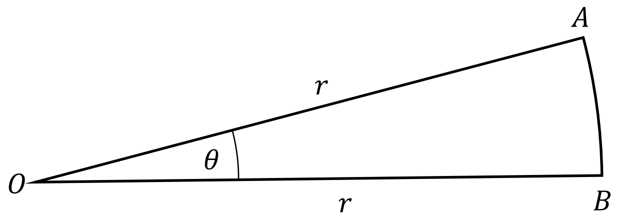
If we have a small angle measured in **radians**, then we can use the following approximations.

$$\sin\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{\theta^2}{2} \qquad \tan\theta \approx \theta$$

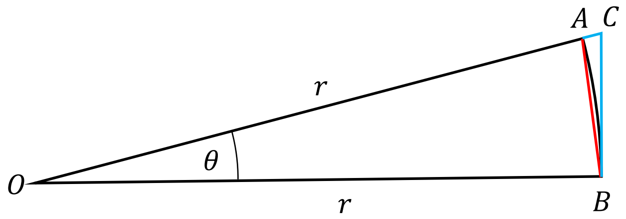
The approximations are correct to three significant figures if –0.115 < θ < 0.115 (around –6.6° < θ < 6.6°), and are correct to two significant figures if –0.241 < θ < 0.241 (around –13.8° < θ < 13.8°). The approximation for $\cos$ is the most accurate, and the approximation for $\tan$ is the least accurate.

Sin Proof

Let a small angle θ form the sector OAB of a circle. The area of the sector is $\frac{1}{2}r^2\theta$.



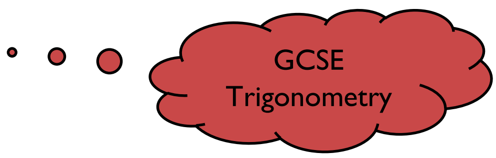
Let us add the chord AB and extend the radius OA to reach the point C so that OB and BC are perpendicular.



OCB is a right-angled triangle with base r and height $r\tan\theta$.

The area of the triangle OCB is $\frac{1}{2}r^2\tan\theta$.

The area of the isosceles triangle OAB is $\frac{1}{2}r^2\sin\theta$.



Area of triangle OAB < area of sector OAB < area of triangle OCB

$$\frac{1}{2}r^2 \sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2 \tan\theta$$

We can divide by $\frac{1}{2}r^2$ as it is always positive.

$$\sin\theta < \theta < \tan\theta$$

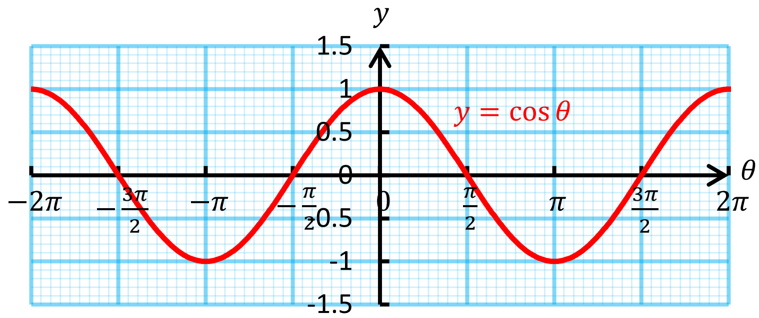
Because θ is a small positive angle, $\sin\theta$ is positive. We can therefore divide the inequality by $\sin\theta$.

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{\sin\theta}{\cos\theta} \times \frac{1}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \sec\theta$$

As θ approaches 0, $\sec\theta$ approaches 1 (consider the graphs on the right).



Therefore, as θ approaches 0, $\frac{\theta}{\sin\theta}$ lies between 1 and a number approaching 1.

Therefore, as θ approaches 0, $\frac{\theta}{\sin\theta}$ approaches 1.

This means that $\sin\theta \approx \theta$ for small values of θ.

Tan Proof

The proof for tan is very similar to the proof for sin. We use exactly the same steps until we reach the inequality

$$\sin\theta < \theta < \tan\theta$$

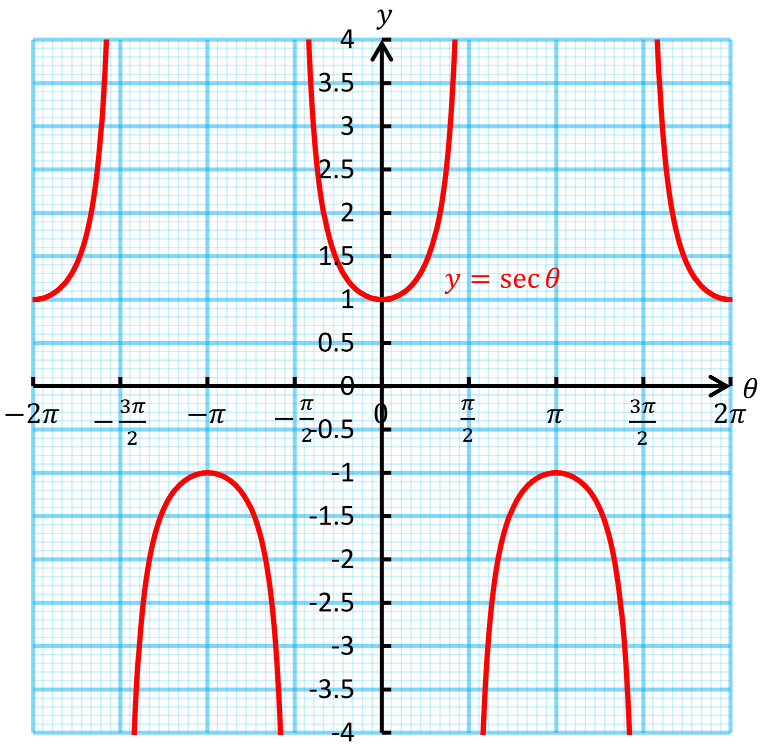
Here, we divide by $\tan\theta$ (instead of by $\sin\theta$):

$$\frac{\sin\theta}{\tan\theta} < \frac{\theta}{\tan\theta} < \frac{\tan\theta}{\tan\theta}$$

$$\frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} < \frac{\theta}{\tan\theta} < 1$$

$$\cos\theta < \frac{\theta}{\tan\theta} < 1$$

As θ approaches 0, $\cos\theta$ approaches 1 (consider the graph on the right).



Therefore, as θ approaches 0, $\frac{\theta}{\tan\theta}$ lies between a number approaching 1 and 1.

Therefore, as θ approaches 0, $\frac{\theta}{\tan\theta}$ approaches 1.

This means that $\tan\theta \approx \theta$ for small values of θ.

Cos Proof

We use the double angle identity $\cos 2\theta \equiv 1 - 2\sin^2\theta$ to find the approximation for $\cos$.

In the identity, we halve the angles, so that 2θ changes to be θ, and θ changes to be $\frac{\theta}{2}$.

$$\cos\theta \equiv 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

If $\frac{\theta}{2}$ is a small angle, then we can use the approximation $\sin\left(\frac{\theta}{2}\right) \approx \frac{\theta}{2}$, so that

$$\cos\theta \approx 1 - 2\left(\frac{\theta}{2}\right)^2$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

Summary

If θ is a small angle, then

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan\theta \approx \theta$$

Exercises

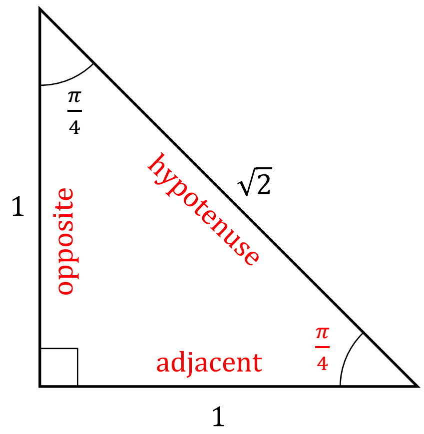
Exercise 1

If θ is a small angle, find an approximation for the expression $\frac{\sin 3\theta}{1+\cos 2\theta}$.

Exact Values

Consider a right-angled triangle where the length of the shortest sides are 1 unit.

It is possible to use this triangle to calculate the exact values of sin, cos and tan for the angle $\frac{\pi}{4}$ (or 45°).

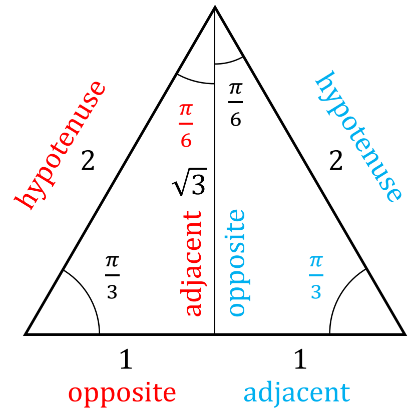


$$\sin\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos\left(\frac{\pi}{4}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \tan\left(\frac{\pi}{4}\right) = 1$$

Next, consider an equilateral triangle where the length of the sides are 2 units.

By splitting this triangle in half, it is possible to calculate the exact values of sin, cos and tan for the angles $\frac{\pi}{6}$ (or 30°) and $\frac{\pi}{3}$ (or 60°).



$$\sin\left(\frac{\pi}{6}\right) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos\left(\frac{\pi}{6}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan\left(\frac{\pi}{6}\right) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos\left(\frac{\pi}{3}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan\left(\frac{\pi}{3}\right) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Summary

	sin	cos	tan
0	0	1	0
$\frac{\pi}{6}$ (or 30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (or 45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (or 60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (or 90°)	1	0	Not defined

We can find the value of sin, cos and tan for other multiples of $\frac{\pi}{6}$ (or 30°) by using the **symmetry** in the graphs of sin, cos and tan.

Exercise 2

If θ is a small angle, show that $\tan\left(\frac{\pi}{4}+\theta\right) \approx \frac{1+\theta}{1-\theta}$.

Exercise 3

If θ is sufficiently small so that you can ignore θ^2, show that $4\sin\left(\frac{\pi}{4}-\theta\right) \approx 2\sqrt{2}(1-\theta)$.

Exericse 4

Given that $1° \approx 0.017$ radian, find values for each of the following without using the trigonometric functions on your calculator.

(a) $\tan(61°)$

(b) $\sin(29°)$

Top 5

(Sample Assessment Materials)

1. Find a small positive value of x which is an approximate solution of the equation.

$$\cos x - 4\sin x = x^2 .$$ [4]

(Unit 3 Summer 2018)

07 Use small angle approximations to find the small negative root of the equation

$$\sin x + \cos x = 0{\cdot}5.$$

[3]

(Unit 3 Summer 2024)

12. (a) Given that θ is small, show that $2\cos\theta + \sin\theta - 1 \approx 1 + \theta - \theta^2$. [2]

(Edexcel Pure Mathematics 1 [9MA0/01] Summer 2018)

1. Given that θ is small and is measured in radians, use the small angle approximations to find an approximate value of

$$\frac{1 - \cos 4\theta}{2\theta \sin 3\theta}$$

(3)

4. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation $y = f(x)$ where $x \in \mathbb{R}$

Given that

- $f'(x) = 2x + \frac{1}{2}\cos x$
- the curve has a stationary point with x coordinate α
- α is small

(a) use the small angle approximation for $\cos x$ to estimate the value of α to 3 decimal places.

(3)

Identities

(Unit 3 Summer 2019)

0 9 **a)** Given that α and β are two angles such that $\tan\alpha = 2\cot\beta$, show that

$$\tan(\alpha + \beta) = -(\tan\alpha + \tan\beta).$$ [2]

(Unit 3 Summer 2022)

1 7 a) Prove that

$$\cos(\alpha - \beta) + \sin(\alpha + \beta) \equiv (\cos\alpha + \sin\alpha)(\cos\beta + \sin\beta).$$ [2]

b) i) Hence show that $\dfrac{\cos 3\theta + \sin 5\theta}{\cos 4\theta + \sin 4\theta}$ can be expressed as $\cos\theta + \sin\theta$.

ii) Explain why $\dfrac{\cos 3\theta + \sin 5\theta}{\cos 4\theta + \sin 4\theta} \neq \cos\theta + \sin\theta$ when $\theta = \dfrac{3\pi}{16}$. [3]

(Unit 3 Summer 2023)

0 6 **a)** Using the trigonometric identity $\cos(A+B) = \cos A\cos B - \sin A\sin B$, show that the **exact** value of $\cos 75°$ is $\dfrac{\sqrt{6}-\sqrt{2}}{4}$. [3]

(Edexcel Pure Mathematics 1 [9MA0/01] Summer 2022)

14. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that

$$2\sin(x-60°)=\cos(x-30°)$$

show that

$$\tan x = 3\sqrt{3}$$

(4)

(b) Hence or otherwise solve, for $0 \leqslant \theta < 180°$

$$2\sin 2\theta = \cos(2\theta + 30°)$$

giving your answers to one decimal place.

(4)