Unit 3 Workbook 8

13

Double Angle Formulae

$\sin(2A) = 2\sin A\cos A$

$\cos(2A) = \cos^2(A) - \sin^2(A)$
$\cos(2A) = 1 - 2\sin^2(A)$
$\cos(2A) = 2\cos^2(A) - 1$

$\tan(2A) = \dfrac{2\tan A}{1 - \tan^2 A}$

Name:

Background

What is the work?	This workbook considers the formulae for $\sin(2A)$, $\cos(2A)$ and $\tan(2A)$.
What is required before starting?	**A Level Unit 1:** Trigonometry. **A Level Unit 3:** Trigonometric Addition Formulae.
Where does this lead to?	**A Level Unit 3:** • Integration of functions that contain (e.g.) $\sin^2(x)$. **Applications:** • Solving trigonometric equations by reducing the power of the equation.

Theory

In the previous workbook, the following formulae were introducted for trigonometric addition:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Theory

We will now consider what happens above when the angles A and B are equal.

$$\sin(A+A) = \sin A \cos A + \cos A \sin A$$
$$\sin(2A) = 2\sin A \cos A$$

$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$
$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

The above formulae are recognised as the formulae for **double angles**. In the case of the formula for $\cos(2A)$, it turns out that using the identity $\sin^2 A + \cos^2 A = 1$ from the Unit 1 course makes the formula even more useful.

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\cos(2A) = (1 - \sin^2(A)) - \sin^2(A)$$
$$\cos(2A) = 1 - 2\sin^2(A)$$

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\cos(2A) = \cos^2(A) - (1 - \cos^2(A))$$
$$\cos(2A) = \cos^2(A) - 1 + \cos^2(A)$$
$$\cos(2A) = 2\cos^2(A) - 1$$

Here is a summary of the formulae for double angles. These need to be memorised.

$$\sin(2A) = 2\sin A\cos A$$

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\cos(2A) = 1 - 2\sin^2(A)$$
$$\cos(2A) = 2\cos^2(A) - 1$$

$$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

Challenge!

Show that it is possible to write (a) $\sin(2A) = \frac{2\tan A}{1+\tan^2(A)}$, (b) $\cos(2A) = \frac{1-\tan^2(A)}{1+\tan^2(A)}$.

Exercises

(C4 Summer 2010)

3. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$2\cos 2\theta = 9\cos\theta + 7.$$ [5]

(C4 Summer 2011)

3. *(a)* Find all values of x in the range $0° \leqslant x \leqslant 180°$ satisfying

$$\tan 2x = 4\tan x.$$ [5]

Top 10

(C4 Summer 2005)

4. *(a)* Find all values of θ in the range $0 \leqslant \theta \leqslant 360°$ satisfying

$$\sin 2\theta = \cos\theta.$$ [4]

(C4 Summer 2009)

2. Find all the values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying $3\sin 2\theta = 2\sin\theta$. [5]

(C4 Summer 2006)

3. Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$2 + 3\cos 2\theta = \cos\theta.$$ [6]

(C4 Summer 2012)

3. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$4\cos 2\theta = 1 - 2\sin\theta.$$ [6]

(C4 Summer 2013)

3. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$8\cos 2\theta + 6 = \cos^2\theta + \cos\theta.$$ [6]

(C4 Summer 2014)

3. *(a)* Find all values of x in the range $0° \leqslant x \leqslant 180°$ satisfying

$$\tan 2x = 3\cot x.$$ [4]

(C4 Summer 2019)

3. *(a)* Given that $\theta \neq 90°$, find all values of θ in the range $0° \leqslant \theta \leqslant 180°$ satisfying

$$5\tan 2\theta = 8\cot\theta.$$

Give your answers in degrees, correct to two decimal places. [4]

(C4 Summer 2018)

3. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$2\cos 2\theta = 3\sin^2\theta - 5\cos^2\theta + \cos\theta + 1.$$ [6]

(C4 Summer 2017)

3. *(a)* Show that the equation

$$5\cos^2\theta + 7\sin 2\theta = 3\sin^2\theta$$

may be rewritten in the form

$$a\tan^2\theta + b\tan\theta + c = 0,$$

where a, b, c are non-zero constants whose values are to be found.
Hence, find all values of θ in the range $0° \leqslant \theta \leqslant 180°$ satisfying the equation

$$5\cos^2\theta + 7\sin 2\theta = 3\sin^2\theta.$$ [6]

(C4 Summer 2016)

4. *(a)* The angle x is such that $0° \leqslant x \leqslant 180°$, $x \neq 90°$.

Given that x satisfies the equation $3\tan 2x + 16\cot^2 x = 0$,

(i) show that $3\tan^3 x - 8\tan^2 x + 8 = 0$,

(ii) find all possible values of x, giving each answer in degrees, correct to one decimal place. [8]

Revision Question

(Sample Assessment Materials, Question 8)

(b) Use an appropriate substitution to show that

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{\pi}{12} - \frac{\sqrt{3}}{8}.$$

[8]