Unit 3 Workbook 7

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Trigonometric Addition Formulae

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Bhaskara was an eminent Indian mathematician in the 12th century. He published a number of books, including *Siddhantasiromani,* a book on astronomy that contained the two formulae shown on the right.

Name:

Background

What is the work?	This workbook considers the formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$.
What is required before starting?	**GCSE Work:** Angles, trigonometry, similar shapes. **A Level Unit 1:** Trigonometry.
Where does this lead to?	**A Level Unit 3:** • Double angle formulae. • Writing values for (e.g.) $\sin 15°$ without using a calculator. **Applications:** • Vibration theory. • Electrical circuit theory.

Theory

Theory

Example

The trigonometric functions are not distributive. This means that (for example) $\sin(A+B) \neq \sin A + \sin B$. We can prove this by considering $A = B = 45°$:

Left hand side $= \sin(45° + 45°)$
$= \sin(90°)$
$= 1$

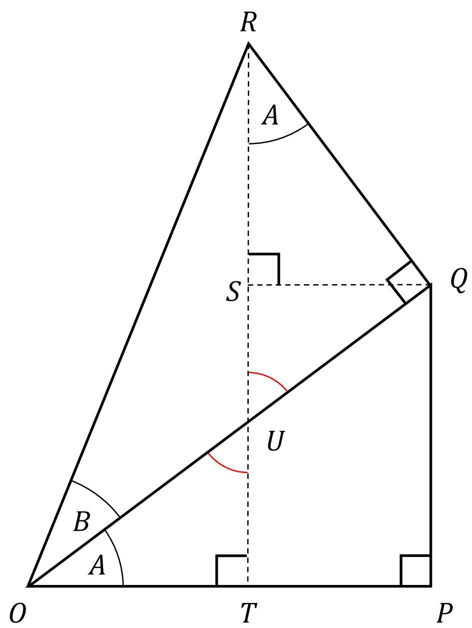
Right hand side $= \sin 45° + \sin 45°$
$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$
$= \frac{2}{\sqrt{2}}$

$1 \neq \frac{2}{\sqrt{2}}$ so it is not true that $\sin(A+B) = \sin A + \sin B$. The correct formula for $\sin(A+B)$ is

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

If A and B are acute angles, then we can prove this formula by considering the triangles on the right.

OPQ and OQR are right-angled triangles containing the angles A and B.
RT and SQ are lines forming the right angles $O\hat{T}R$ and $R\hat{S}Q$.
$O\hat{U}T = R\hat{U}Q$ therefore OTU and URQ are similar triangles.
Therefore, $U\hat{R}Q = A$.



$$\sin(A+B) = \frac{TR}{OR}$$
$$= \frac{TS+SR}{OR}$$
$$= \frac{PQ+SR}{OR}$$
$$= \frac{PQ}{OR} + \frac{SR}{OR}$$
$$= \frac{PQ}{OQ} \times \frac{OQ}{OR} + \frac{SR}{QR} \times \frac{QR}{OR}$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

If we substitute $-B$ instead of B in the formula

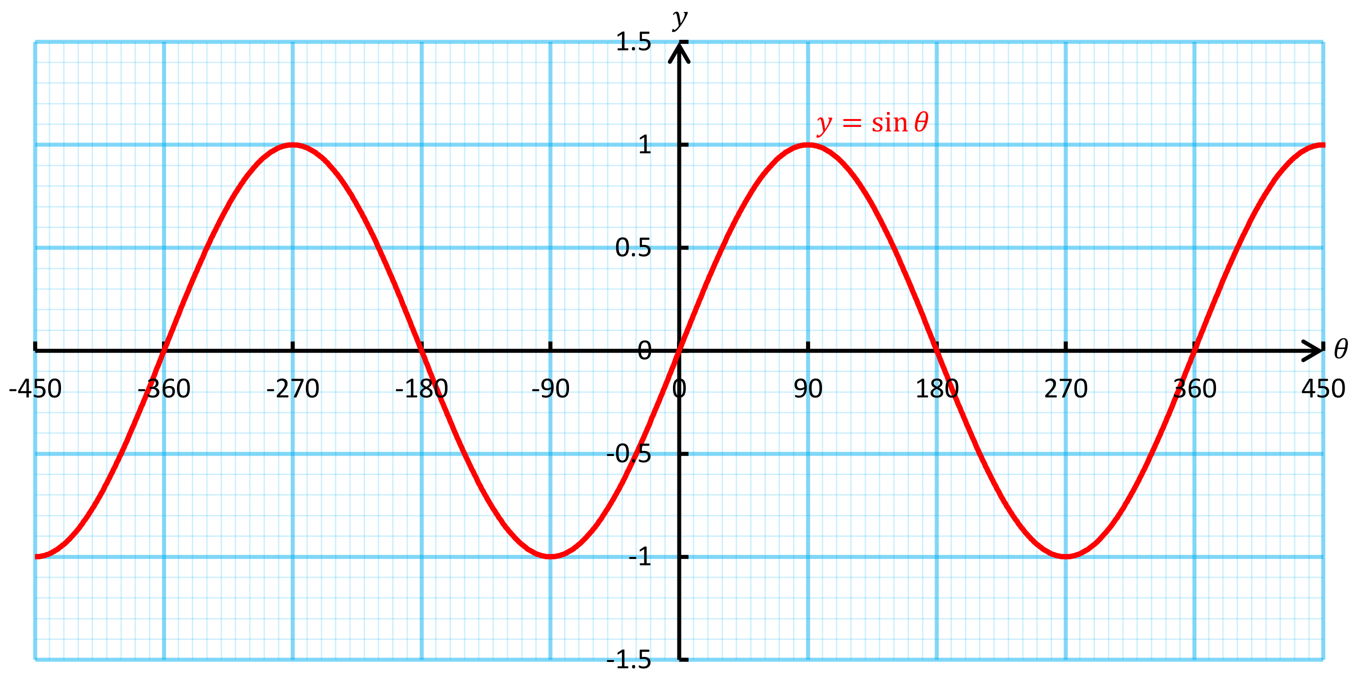
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

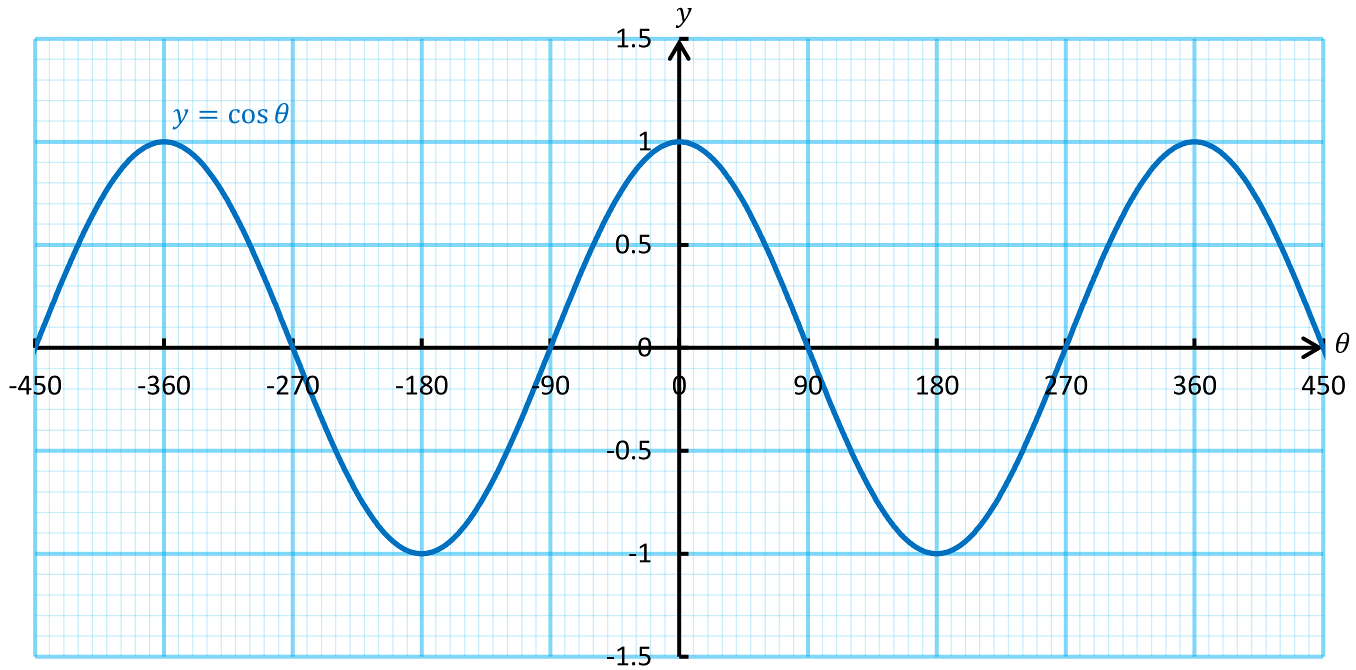
we obtain

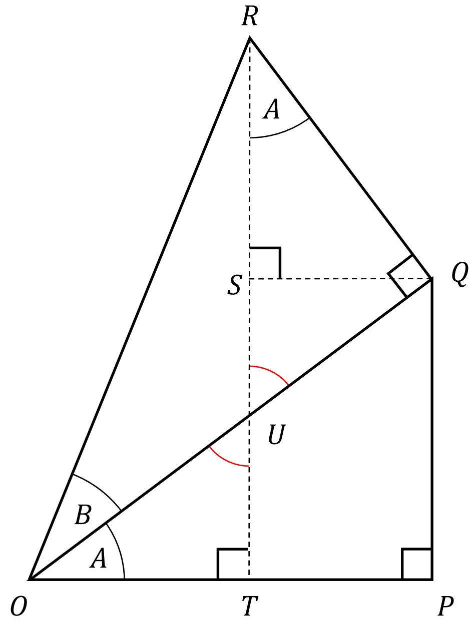
$$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$$

Considering the symmetry of the sin and cos graphs shown below, we see that $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$. Therefore,

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$







If we substitute $\left(\frac{\pi}{2} - A\right)$ instead of A in the formula

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

we obtain

$$\sin\left(\frac{\pi}{2} - A - B\right) = \sin\left(\frac{\pi}{2} - A\right)\cos B - \cos\left(\frac{\pi}{2} - A\right)\sin B$$

$$\sin\left(\frac{\pi}{2} - A - B\right) = \cos A \cos B - \sin A \sin B$$

(through the connection between the graphs of sin and cos)

$$\sin\left(O\hat{R}T\right) = \cos A \cos B - \sin A \sin B$$

(considering the triangle ORT)

$$\cos\left(\frac{\pi}{2} - O\hat{R}T\right) = \cos A \cos B - \sin A \sin B$$

(through the connection between the graphs of sin and cos)

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

(considering the triangle ORT)

Then, if we substitute $-B$ instead of B in the formula

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

we obtain

$$\cos(A - B) = \cos A \cos(-B) - \sin A \sin(-B)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

(through the symmetry of the graphs of sin and cos)

Next, remembering that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, it follows that

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$\tan(A + B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

(using the previous formulae for $\sin(A + B)$ and $\cos(A + B)$)

$$\tan(A + B) = \frac{\frac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} + \frac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B}}{\frac{\cancel{\cos A}\,\cancel{\cos B}}{\cancel{\cos A}\,\cancel{\cos B}} - \frac{\sin A \sin B}{\cos A \cos B}}$$

(dividing each term by $\cos A \cos B$)

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(cancelling and using $\tan\theta = \frac{\sin\theta}{\cos\theta}$)

Finally, if we substitute $-B$ instead of B in the formula

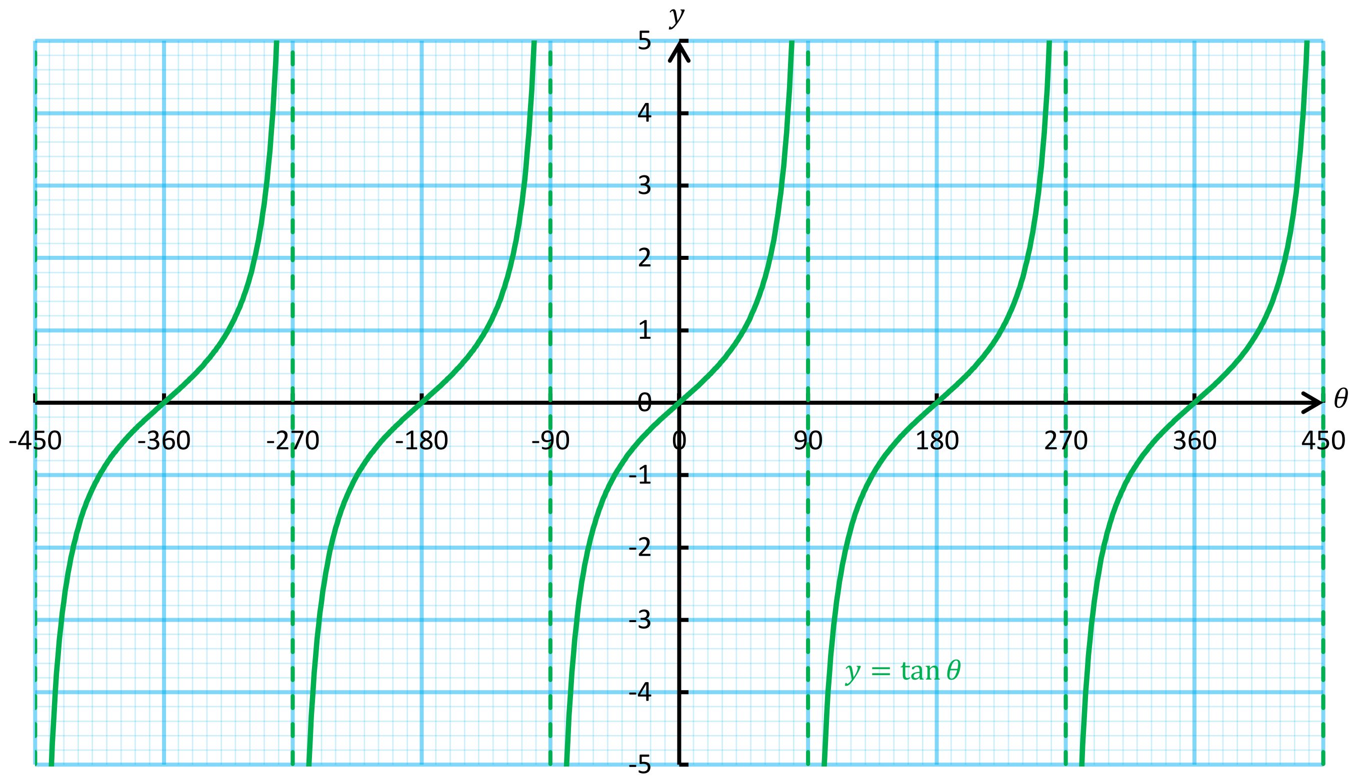
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

we obtain

$$\tan(A-B) = \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(through the symmetry of the tan graph shown below, where $\tan(-B) = -\tan B$)

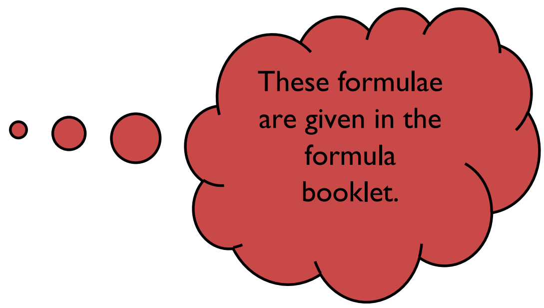


To summarise:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$



Exercises

(C4 Summer 2009)

3. *(a)* Express $\cos\theta + \sqrt{3}\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

(b) Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$\cos\theta + \sqrt{3}\sin\theta = 1.$$ [4]

(C4 Summer 2006)

4. *(a)* Express $4\sin x + 3\cos x$ in the form $R\sin(x + \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$. [3]

(b) Hence find the greatest value of

$$\frac{1}{4\sin x + 3\cos x + 7} \quad .$$ [2]

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(C4 Summer 2013, Question 3)

(b) Express $\sqrt{15}\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.
Hence find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$\sqrt{15}\cos\theta - \sin\theta = 3.$$

[6]

(C4 Summer 2015, Question 3)

(b) (i) Express $\sqrt{13}\sin\theta - 6\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(ii) Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$\sqrt{13}\sin\theta - 6\cos\theta = -4.$$

[6]

(C4 Summer 2015)

3. *(a)* Find all values of x in the range $0° \leqslant x \leqslant 180°$ satisfying

$$\tan(x + 45°) = 8\tan x.$$

[5]

(C4 Summer 2008)

3. *(a)* Express $3\cos x + 2\sin x$ in the form $R\cos(x - \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$. [3]

(b) Find all values of x between 0° and 360° satisfying

$$3\cos x + 2\sin x = 1.$$ [3]

(C4 Summer 2010, Question 3)

(b) (i) Express $5\sin x - 12\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(ii) Use your results to part (i) to find the least value of

$$\frac{1}{5\sin x - 12\cos x + 20}.$$

Write down a value for x for which this least value occurs. [6]

(C4 Summer 2014, Question 3)

(b) (i) Express $21\sin\theta - 20\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(ii) Use your results to part (i) to find the greatest value of

$$\frac{1}{21\sin\theta - 20\cos\theta + 31}.$$

Write down a value for θ for which this greatest value occurs. [6]

(C4 Summer 2012, Question 3)

(b) (i) Express $8\sin x + 15\cos x$ in the form $R\sin(x + \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(ii) Find all values of x in the range $0° \leqslant x \leqslant 360°$ satisfying

$$8\sin x + 15\cos x = 11.$$

(iii) Find the greatest possible value for k so that

$$8\sin x + 15\cos x = k$$

has solutions. Give a reason for your answer. [7]

(C4 Summer 2016, Question 4)

(b) Express $24\cos\theta - 7\sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

Hence, find the range of values of k for which the equation

$$24\cos\theta - 7\sin\theta = k$$

has no solutions. [5]

(C4 Summer 2005, Question 4)

(b) Find all values of θ in the range $0 \leqslant \theta \leqslant 360°$ satisfying

$$4\sin\theta + \cos\theta = 2,$$

giving your answers in degrees correct to one decimal place. [6]

(C4 Summer 2007)

3. Find all values of x in the range $0° \leqslant x \leqslant 360°$ satisfying the equation $4\cos x + 2\sin x = 3$. [7]

Revision Question

(C4 Summer 2017, Question 3)

(b) (i) Express $\sqrt{5}\cos\phi + \sqrt{11}\sin\phi$ in the form $R\cos(\phi - \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(ii) Use your result to part (i) to find the least value of

$$\frac{1}{\sqrt{5}\cos\phi + \sqrt{11}\sin\phi + 6}.$$

Write down a value for ϕ for which this least value occurs. [6]