Modiwl M2 (Mecaneg 2)

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1. Light Springs and Strings.

A light spring or string is considered to have *no mass*. A spring behaves like an elastic string but it may also be *compressed*.

Let l be the **natural length** of the light spring or string, without any forces acting upon it.
Let x be the amount of the **extension**.
Let λ be the **modulus of elasticity**[1] of the light spring or string, measured in Newtons.

Hooke's law states that the tension in a light spring or string is directly proportional to the extension: $T \propto x$.
Using $\frac{\lambda}{l}$ as the constant of proportionality, we have

$$T = \frac{\lambda x}{l}.$$

The work done in extending a light spring or string from 0 to x is given by the fraction $\frac{\lambda x^2}{2l}$. As energy is the capability of doing work, we have the following formula.

Elastic Energy stored in an elastic string or spring = $\frac{\lambda x^2}{2l}$.

The work done against tension in increasing the extension from a to b is given by

$$\frac{\lambda}{2l}(b^2 - a^2).$$

Enghraifft

A light elastic spring with modulus of elasticity 160N and of natural length 0.2m hangs from a fixed point. It has a particle of mass 4kg attached to the other end. If the system is in equilibrium, calculate the extension of the spring.

[1] The modulus of elasticity measures how elastic a light spring or string is.

Ateb

Equilibrium implies that the tension in the spring is equal to the weight of the particle. Therefore

$\frac{\lambda x}{l} = 4g$

$\frac{160x}{0.2} = 4 \times 9.8$

$x = \frac{4 \times 9.8 \times 0.2}{160}$

$x = 0.049$m.

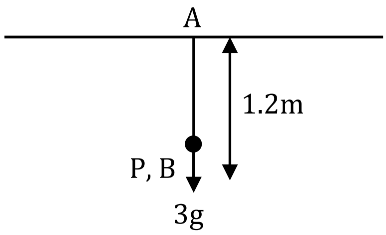
Enghraifft (Haf 2007)

The end A of a light elastic spring AB, of natural length 0.8m, is fixed. A particle P, of mass 3kg, is attached to the end B of the string. Initially, P is held at rest at the point A. It is then released and allowed to fall. The greatest extension of the string in the subsequent motion is 0.4m.

(a) Show that the modulus of elasticity of the string is 352.8N.

(b) Find the tension in the string when P is at its lowest point and deduce the magnitude of the acceleration of P in this position.

Ateb



(a) The loss in potential energy of the particle is given by the formula mgh (see Chapter 2).

$mgh = 3 \times 9.8 \times 1.2$
$= 35.28$ J.

When the string has extension 0.4m, the particle is momentarily at rest so all the lost potential energy has been converted to elastic energy in the string. Therefore

Lost potential energy = elastic energy

$35.28 = \frac{\lambda x^2}{2l}$

$35.28 = \frac{\lambda (0.4)^2}{2 \times 0.8}$

$\frac{35.28 \times 2 \times 0.8}{0.16} = \lambda$

$\lambda = 352.8$N.

(b) Using $T = \frac{\lambda x}{l}$, we have $T = \frac{352.8 \times 0.4}{0.8} = 176.4$N.

Then, using $F = ma$ to find the magnitude of the acceleration, we find that

$3g - T = 3a$

$3 \times 9.8 - 176.4 = 3a$

$a = -49$ms^{-2} (the particle is accelerating upwards).

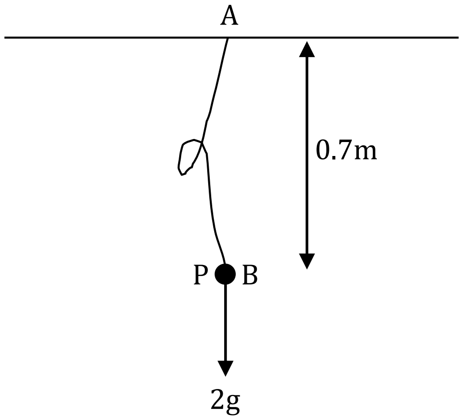
Enghraifft (Haf 2013)

The end A of a light elastic spring AB, of natural length 1.2m, and modulus of elasticity 360N, is fixed. A particle P, of mass 2kg, is attached to the end B. Initially, P is held at rest at a point which is 0.7m vertically below A. It is then released and allowed to fall.

(a) Find the greatest extension of the string in the subsequent motion. Give your answer correct to 2 decimal places.

(b) Calculate the velocity of the particle when it is 1.2m below A.

Ateb



(a) On release, the particle falls. For the first 0.5m the particle falls in freefall. During this part of the descent, Gravitational Potential Energy is converted to Kinetic Energy (See Chapter 2).

Gravitational Potential Energy Lost $= mgh$
$= 2 \times 9.8 \times 0.5$
$= 9.8$J.

This is the amount of Kinetic Energy the particle has when at a distance of 1.2m below A. For the remainder of the descent, the elastic string must be considered. At the point of greatest extension (x), all 9.8J of Kinetic Energy (KE) has been converted to Elastic Potential Energy (EPE). We also have a loss of Gravitational Potential Energy (GPE) due to the further descent – this is also coverted into EPE.

EPE at greatest extension = 9.8 + GPE lost

$\frac{\lambda x^2}{2l} = 9.8 + mgx$

$\frac{360x^2}{2\times1.2} = 9.8 + 2 \times 9.8 \times x$

$150x^2 = 9.8 + 19.6x$

$150x^2 - 19.6x - 9.8 = 0$

Using the quadratic formula, $x = \frac{19.6\pm\sqrt{(-19.6)^2-4\times150\times-9.8}}{2\times150}$.

Either x =0.329m (to 3 d.p.) or x = –0.198m (to 3 d.p.) So, to two decimal places, the greatest extension of the string in the subsequent motion is 0.33m.

(b) When the particle is 1.2m below A, we know from part (a) that it has 9.8J of Kinetic Energy.

So KE = $\frac{1}{2}mv^2$

$9.8 = \frac{1}{2} \times 2 \times v^2$

$9.8 = v^2$

$v = 3.13$ ms^{-1} to 2 d.p.

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Haf 2006 4.6ms^{-1}.

Haf 2008 (a) λ = 14.4N (b) 1.5J.

Haf 2009 (a) λ = 400N (b) v = $3\frac{1}{3}$ms^{-1}.

Haf 2010 (a) x = 0.2m (b) v = 4.2ms^{-1}.

Haf 2011 (a) T = 20N (b) v = 5.23ms^{-1} to 2 d.p.

Haf 2012 (a) x = 0.5m (b) 18.375J.

Haf 2014 (a) 15.625J (b) v = 5.25ms^{-1}.

Haf 2015 (a) x = 0.04m (b) v = 1.87ms^{-1} to 2 d.p.

Haf 2016 l = 0.8m, λ = 160N.

Haf 2018 4.02ms^{-1} to 2 d.p.

2. Work, Energy and Power.

Work

Consider a constant force applied to an object. If d is the distance travelled by the object under the action of the force, and F is the component of the force in the direction of the motion, then

$$\text{Work} = \text{Force} \times \text{Distance}, \quad \text{or} \quad \boxed{W = F \times d.}$$

Work is measured in *Joules* (J), a unit of **energy**. We say that the force "works" on the object, causing its energy to change.

Energy

Kinetic Energy is the energy objects have because they are moving. When an object is dropped it speeds up – it acquires kinetic energy. This makes us believe that the object has energy before it is dropped. This energy derives from the object's position in the earth's gravitational field. Hence this energy is known as **Gravitational Potential Energy**. When an object falls, ignoring air resistance, gravitational potential energy is converted into kinetic energy.

$$\text{Kinetic Energy} = \frac{1}{2} \times \text{Mass} \times \text{Velocity}^2, \quad \text{or} \quad \boxed{\text{KE} = \frac{1}{2}mv^2.}$$

$$\text{Gravitational Potential Energy} = \text{Mass} \times \text{Gravitational Field Strength} \times \text{Height}, \quad \text{or} \quad \boxed{\text{PE} = mgh.}$$

Recall that $g = 9.8$.

Kinetic Energy and Gravitational Potential Energy are measured in Joules (J).

Principle of Conservation of Energy

The principle of conservation of energy states that the **total energy of a closed system remains constant**. Energy may be transferred from one type to another, but the total amount of energy in the closed system must remain the same. In the module M2, types of energy include *Kinetic Energy* $\left(\frac{1}{2}mv^2\right)$, *Gravitational Potential Energy* (mgh), Elastic Potential Energy $\left(\frac{\lambda x^2}{2l}\right)$, and Work Done against Friction $(F \times d)$. Any gain in one type of energy must be counterbalanced by a loss in another type of energy.

Power

In physics *power* is the concept of the rate at which energy is converted from one form to another.

$$\text{Power} = \text{rate of conversion of energy}$$
$$\text{Power} = \frac{\text{Change in energy}}{\text{Change in time}}.$$

We can rearrange this formula to give Power × Time = Energy. But we also know that Energy, or *Work*, can be written as Energy = Force × Distance. Substituting into the first equation, we find that Power × Time = Force × Distance, or Power = Force × $\frac{\text{Distance}}{\text{Time}}$.

But as $\frac{\text{Distance}}{\text{Time}}$ = Velocity, we can deduce that $\boxed{\text{Power} = \text{Force} \times \text{Velocity.}}$

The unit of Power is *Watt* (W). One kilowatt (1kW) is equal to 1000W.

Enghraifft (Haf 2006)

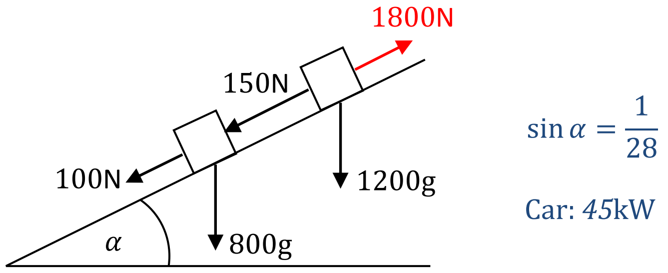
A car of mass 1200kg is towing a trailer of mass 800kg up a slope inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{28}$. The resistance to motion acting on the car is 150N and that acting on the trailer is 100N. The car's engine is working at 45kW.

(a) Calculate the acceleration of the car and trailer when the speed is 25ms^{-1}.

(b) Determine the tension in the rigid tow-bar connecting the car and the trailer when the speed is 25ms^{-1}.

Ateb

We start by drawing a diagram summarising the situation.



(a) Using Power = Force × Velocity, we find that 45 × 1000 = Force × 25, so that Force = 1800N.

Using $F = ma$, applied to the whole system (positive direction = up the slope):

$1800 - 150 - 100 -$ weight component down slope $= ma$

$1550 - 2000g\sin\alpha = 2000a$

$1550 - 2000g\left(\frac{1}{28}\right) = 2000a$

$a = 0.425$ms^{-2}.

(b) Using $F = ma$, applied to the trailer (positive direction = up the slope):

Tension $-100 -$ weight component down slope $= ma$

$T - 100 - 800g\sin\alpha = 800a$

$T - 100 - 800g\left(\frac{1}{28}\right) = 800 \times 0.425$

$T = 720$N.

Enghraifft (Haf 2008)

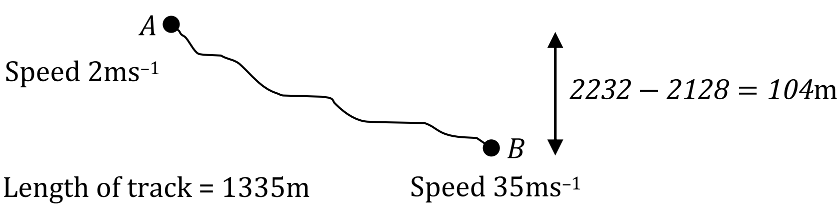
In an event in the Winter Olympic Games, a competitor pushes a sled for a short time, then jumps onto the sled at a point A when the sled has a speed of 2ms^{-1} and rides the sled downhill on a curved track. The altitude at A is 2232m, the altitude at the finish is 2128m and the length of the track from A to the finish is 1335m. The competitor has a mass of 50kg and her sled is of mass 40kg. Her speed at the finish is 35ms^{-1}.

(a) Calculate the work done against the resistance to motion from A to the finish.

(b) Assuming the resistance is constant, calculate its magnitude.

Ateb

We start by drawing a diagram summarising the situation.



(a) Loss in Potential Energy = mgh
$= (50 + 40) \times 9.8 \times 104$
$= 91728$J.
Gain in Kinetic Energy $= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$
$= \frac{1}{2} \times 90 \times 35^2 - \frac{1}{2} \times 90 \times 2^2$
$= 54945$J.
The *difference* between the gain in kinetic energy and the loss in potential energy is the work done against resistance. This is given by the sum 91728 – 53945 = 36783J.

(b) Using Work = Force × Distance, we find that 36783 = Force × 1335, so that Force = 27.5528N, to 4 d.p.

Enghraifft (Haf 2009)

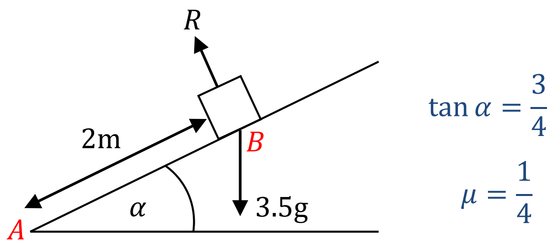
A point A is situated at the bottom of a rough plane inclined at an angle α to the horizontal where $\tan\alpha = \frac{3}{4}$. An object, of mass 3.5kg, is projected from A with speed of u ms^{-1} up the plane along a line of greatest slope of the plane. The object comes to rest at point B where AB = 2m. The coefficient of friction between the object and the plane is $\frac{1}{4}$.

(a) Calculate the work done against friction as the object travels from A to B.

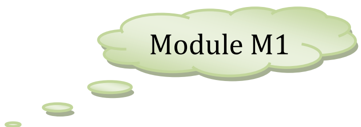
(b) By using energy considerations, find the value of u.

Ateb

We start by drawing a diagram summarising the situation.



(a) Resolving perpendicular to the plane, we see that
$R = 3.5g\cos\alpha$
$R = 3.5 \times 9.8 \times \frac{4}{5}$
$R = 27.44$N.



Using Frictional Force $= \mu R$, we see that the frictional force must be equal to $\frac{1}{4} \times 27.44 = 6.86$N.
Finally, using Work Done Against Friction = Frictional Force × Distance, we find that the required answer is obtained by calculating 6.86 × 2 = 13.72J.

(b) The Kinetic Energy at A is given by
$\frac{1}{2}mv^2 = \frac{1}{2} \times 3.5 \times u^2$
$= 1.75u^2$.
The Potential Energy at B is given by
$mgh = 3.5 \times 9.8 \times 2\sin\alpha$
$= 3.5 \times 9.8 \times 2 \times \frac{3}{5}$
$= 41.16$J.

Using the *Work–Energy Principle*, we see that KE at A = PE at B + Work Done Against Friction. Therefore
$1.75u^2 = 41.16 + 13.72$
$u^2 = 31.36$
$u = 5.6$ms^{-1}.

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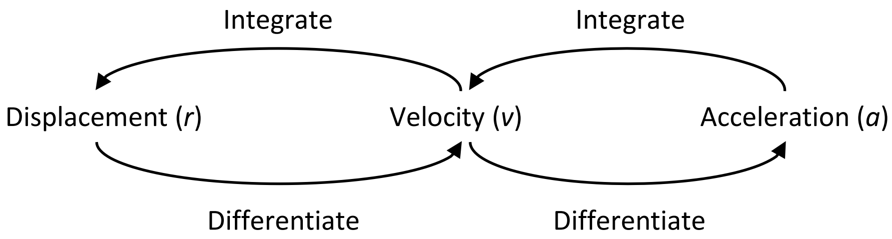
Haf 2007 (2) (a) 25ms^{-1} (b) 0.65ms^{-2} to 2 d.p. (c) 1440000J.
Haf 2008 (2) 560N.
Haf 2009 (4) (a) 30000W (b) 30ms^{-1}.
Haf 2010 (4) (a) a = 0.9ms^{-2} (b) v = 12.5ms^{-1}.
Haf 2011 (a) 32400W (b) a = 1.4125ms^{-2}.
Haf 2012 (4) (a) a = 1.02ms^{-2} (b) v = 50.7ms^{-1} to 3 s.f. (5) v = 12.7ms^{-1} to 3 s.f.
Haf 2013 (1) (a) 196J (b) $\mu = \frac{1}{6}$ (6) (a) 3000N (c) 730N.
Haf 2014 (a) 12350N (b) 6.25ms^{-1}.
Haf 2015 P = 96000W, R = 10800N.
Haf 2016 (4) (a) 4.5ms^{-1} (b) By assuming that no energy is lost to sound etc., we have 2,176,200J of energy.
(c) 13500N
(7) 7.88ms^{-1} to 2 d.p.
Haf 2017 (3) (a) a = 0.2ms^{-2} (b) v = 3.49ms^{-1} to 2 d.p. (5) 243200J.
Haf 2018 M = 600kg, T = 580N.

3. Rectilinear Motion.

In this module (M2), we can be given a particle's **displacement**, **velocity** or **acceleration** as a function of time. The following diagram summarises how we may convert between these three properties.



(Differentiation and Integration is done with respect to *time*.)

Enghraifft (Haf 2006)

A particle P, of mass 3kg, moves along the horizontal x-axis under the action of a resultant force FN. Its velocity v ms^{-1} at time t seconds is given by

$$v = 12t - 3t^2.$$

(a) Given that the particle is at the origin O when $t = 1$, find an expression for the displacement of the particle from O at time t s.

(b) Find the acceleration of the particle at time t s.

(c) Find the power of the force F when $t = 1.5$.

Ateb

(a) $r = \int 12t - 3t^2\, dt$

$r = 6t^2 - t^3 + k.$

When $t = 1$, we have $r = 0$, so that

$0 = 6(1)^2 - (1)^3 + k$

$0 = 6 - 1 + k$

$-5 = k.$

It follows that $r = 6t^2 - t^3 - 5$ is the displacement function.

(b) $a = \frac{d}{dt}(12t - 3t^2)$

$a = 12 - 6t.$

(c) When $t = 1.5$, we have $v = 12(1.5) - 3(1.5)^2 = 11.25$ms^{-1}.

Using the equation Power = Force × Velocity, with $F = ma$ to calculate the force, we have

Power = ma × Velocity

Power = 3 × (12 – 6(1.5)) × 11.25

Power = 101.25W.

Enghraifft (Haf 2008)

A particle, of mass 5kg, moves in a straight line under the action of a single force whose magnitude FN at time t s is given by

$$F = 15t^2 - 60t, \qquad t \geq 0.$$

(a) Find the acceleration of the particle when $t = 2$.

(b) The velocity of the particle at time t s is denotes by v ms^{-1}. Given that $v = 35$ when $t = 0$, find an expression for v in terms of t.

(c) Calculate the least value of the speed of the particle.

(d) Determine the distance travelled by the particle between $t = 2$ and $t = 8$.

Ateb

(a) When $t = 2$, we have $F = 15(4) - 60(2)$

$F = -60\text{N}.$

Using $F = ma$, we find that $-60 = 5a$, so that $a = -12\text{ms}^{-2}$.

(b) Rearranging $F = ma$ to obtain $a = \frac{F}{m}$, it follows that $a = \frac{15t^2 - 60t}{5}$

$a = 3t^2 - 12t.$

Integrating with respect to time in order to find the velocity, we find that $v = \int 3t^2 - 12t\, dt$

$v = t^3 - 6t^2 + k.$

When $t = 0$, we have $v = 35$, so that $35 = 0^3 - 6(0)^2 + k$

$35 = k.$

Therefore $v = t^3 - 6t^2 + 35$ is the velocity function.

(c) We need to find the minimum of the velocity function. From module C1, we remember that we need to solve the equation $\frac{dv}{dt} = 0$ in order to do this. But $\frac{dv}{dt}$ is the acceleration function, so we need to solve the equation $a = 0$, or $3t^2 - 12t = 0$. Factorising, we get $3t(t-4) = 0$, so that $t = 0$ or $t = 4$. Substituting into the second derivative $6t - 12$ to check the nature of the stationary points, we find that $(0, 35)$ is a maximum point and that $(4, 3)$ is a minimum point. Therefore the minimum speed is 3ms^{-1}.

(d) Recall from module M1 that the distance travelled is the area under the velocity-time graph. So we need to find $\int_2^8 t^3 - 6t^2 + 35\, dt$

$= \left[\frac{t^4}{4} - 2t^3 + 35t\right]_2^8$

$= \left[\frac{8^4}{4} - 2(8)^3 + 35(8)\right] - \left[\frac{2^4}{4} - 2(2)^3 + 35(2)\right]$

$= 222\text{m}.$

Atebion yr Hen Gwestiynau Arholiad

Haf 2007 (b) $a = 6t - 24$ (c) $r = t^3 - 12t^2 + 45t$ (d) 54m (e) 56m.

Haf 2009 (a) $a = 3\text{ms}^{-2}$ (b) $r = \frac{3+3\sqrt{2}}{2} \approx 3.62$m to 2 d.p.

Haf 2010 (a) $v = 3t - 2t^2 - 1$ (b) $t = ½$s or $t = 1$s (c) $^1/_{24}$m.

Haf 2011 (a) $a = 36\cos(3t) + 16\sin(2t)$ (b) $r = 4(-\cos(3t) - \sin(2t) + 1)$.

Haf 2012 $\sqrt{3}$ m

Haf 2013 (a) Between $t = 0$s and $t = 6$s (b) 144m.

Haf 2014 (b) $\frac{1}{3}$s (c) $v = -6t^{-1} - 30t + 46$; $t = \frac{1}{5}$s or $t = 1$s.

Haf 2015 (b) $v = 32t - 9t^2 + 13$; either $t = \frac{5}{9}$s or $t = 3$s.

Haf 2016 (a) $k = 1.5$ (b) 438N.

Haf 2017 (a) $r = 23$m (b) $F = 81.6$N.

Haf 2018 (a) $a = 0.8 - 0.1t$, $0 \leq t \leq 20$ (b) 7.2ms^{-1} (c) $r = 106\frac{2}{3}$m.

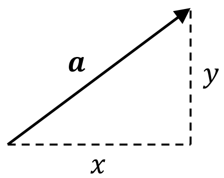
4. Vectors.

A **vector** is any object that has both *size* and *direction*. They are different from **scalar** quantities, which are specified by *size* only. A standard example of the distinction between a scalar and a vector is the distinction between *speed* and *velocity*.

A **displacement vector** is specified by its size and direction only.

A **position vector** is specified by its size and direction, and also by its point of application – that is, its starting point.

In two dimensions, a general vector $\boldsymbol{a}$ can be written as $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$, where $\boldsymbol{i}$ is a unit vector (length 1 unit) in the horizontal direction, and $\boldsymbol{j}$ is a unit vector in the vertical direction. In typed work, such as on an examination paper, vectors appear in **bold**. However, in handwriting, vectors appear underlined.



Addition and subtraction of vectors is done *component-by-component*.

Enghraifft

If $\boldsymbol{p} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{q} = -\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{r} = -3\boldsymbol{i} + \boldsymbol{j}$ are three vectors, find (a) $\boldsymbol{p} + \boldsymbol{q}$; (b) $\boldsymbol{p} - \boldsymbol{r}$; (c) $\boldsymbol{q} - \boldsymbol{r}$.

Ateb

(a) $\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{i} - \boldsymbol{j}$.
(b) $\boldsymbol{p} - \boldsymbol{r} = 5\boldsymbol{i} - 4\boldsymbol{j}$.
(c) $\boldsymbol{q} - \boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j}$.

In three dimensions, we will need a unit vector $\boldsymbol{k}$ in the third direction, so that a general vector $\boldsymbol{a}$ in three dimensions is written as $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

The **magnitude** (or size or length) of a vector $\boldsymbol{a}$ is denoted by the modulus symbol $|\boldsymbol{a}|$ and is found by use of Pythagoras' Theorem.

$$|\boldsymbol{a}| = \sqrt{x^2 + y^2 + z^2}.$$

A **unit vector** is a vector with length 1. A unit vector in the same direction as the general vector $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$ and is found by use of the equation

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}.$$

Enghraifft

Find a unit vector in the same direction as the vector $\boldsymbol{p} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$.

Ateb

$|\boldsymbol{p}| = \sqrt{2^2 + 1^2 + 2^2} = 3$.
So $\hat{\boldsymbol{p}} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|} = \frac{1}{3}(2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$
$= \frac{2}{3}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j} + \frac{2}{3}\boldsymbol{k}.$

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are *parallel* if $\boldsymbol{a} = \alpha\boldsymbol{b}$ for some number α.
Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are *perpendicular* if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, where · represents the scalar (or dot) product

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum a_i b_i.$$

Enghraifft

If $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$ and $\boldsymbol{b} = -2\boldsymbol{i} + 6\boldsymbol{j} - \boldsymbol{k}$ are two vectors, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

Ateb

$\boldsymbol{a} \cdot \boldsymbol{b} = (3 \times -2) + (4 \times 6) + (-5 \times -1) = 23.$

Integration and Differentiation of Vectors

Like addition and subtraction, integration and differentiation is done *component-by-component*.

Enghraifft

If $\boldsymbol{v} = 3t^2\boldsymbol{i} + 6t\boldsymbol{j} + 4\boldsymbol{k}$ is a velocity vector, find the acceleration and displacement vectors.

Ateb

$\boldsymbol{a} = \frac{d}{dt}(\boldsymbol{v})$

$\boldsymbol{a} = 6t\boldsymbol{i} + 6\boldsymbol{j} + 0\boldsymbol{k}.$

$\boldsymbol{r} = \int \boldsymbol{v}\, dt$

$\boldsymbol{r} = t^3\boldsymbol{i} + 3t^2\boldsymbol{j} + 4t\boldsymbol{k} + c.$

(We would need more information to find the constant of integration c.)

Enghraifft (Haf 2007)

Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given by

$\boldsymbol{a} = 2\boldsymbol{i} + 13\boldsymbol{j} - 10\boldsymbol{k},$
$\boldsymbol{b} = -\boldsymbol{i} + y\boldsymbol{j} + 5\boldsymbol{k}.$

(a) If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, find the value of y.
(b) If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, find the value of y.

Ateb

(a) If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. Therefore
$(2)(-1) + 13(y) + (-10)(5) = 0$
$-2 + 13y - 50 = 0$
$13y = 52$
$y = 4.$

(b) If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{a} = \alpha\boldsymbol{b}$ for some number α. By inspection, we must have $\alpha = -2$ for the $\boldsymbol{i}$ and $\boldsymbol{k}$ components. Therefore $y = -6.5$.

Enghraifft (Haf 2008)

A constant force $\boldsymbol{F} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ acts on a bead as it moves along a straight smooth wire from point A to point B. Point A has position vector $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and point B has position vector $3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$. Find

(a) the vector $\boldsymbol{AB}$,

(b) the work done by the force $\boldsymbol{F}$.

Ateb

(a) $\boldsymbol{AB} = -\boldsymbol{a} + \boldsymbol{b}$

$= -2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} + 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

$= \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

(b) The work done by the force $\boldsymbol{F}$ is given by calculating $\boldsymbol{F} \cdot \boldsymbol{AB}$

$= (\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}) \cdot (\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$= (1 \times 1) + (-4 \times -2) + (1 \times 1)$

$= 10\,\mathrm{J}$

Enghraifft (Haf 2008)

(a) A vehicle moves with velocity $\boldsymbol{v} = \sin(3t)\boldsymbol{i} + 2\cos(5t)\boldsymbol{j} + 3t^3\boldsymbol{k}$ at time t.
Find an expression for the acceleration of the vehicle at time t.

(b) Two vehicles A and B move on the same horizontal plane.
At time t, A is at position $(-8t-2)\boldsymbol{i} + (3t+3)\boldsymbol{j}$ and B is at position $(-16t + 11)\boldsymbol{i} + (9t - 8)\boldsymbol{j}$.
Determine the value of t when the distance between A and B is least, and calculate this distance.

Ateb

(a) $\boldsymbol{a} = \frac{d}{dt}(\boldsymbol{v})$

$\boldsymbol{a} = \frac{d}{dt}(\sin(3t)\,\boldsymbol{i} + 2\cos(5t)\,\boldsymbol{j} + 3t^3\boldsymbol{k})$

$\boldsymbol{a} = 3\cos(3t)\,\boldsymbol{i} - 10\sin(5t)\,\boldsymbol{j} + 9t^2\boldsymbol{k}$.

(b) To obtain the distance between A and B, we need to find the vector taking us from A to B.
This is the vector $-\boldsymbol{r}_A + \boldsymbol{r}_B$

$= (8t+2)\boldsymbol{i} + (-3t-3)\boldsymbol{j} + (-16t+11)\boldsymbol{i} + (9t-8)\boldsymbol{j}$

$= (-8t+13)\boldsymbol{i} + (6t-11)\boldsymbol{j}$.

The magnitude of this vector is given by $\sqrt{(-8t+13)^2 + (6t-11)^2}$

$= \sqrt{64t^2 - 208t + 169 + 36t^2 - 132t + 121}$

$= \sqrt{100t^2 - 340t + 290}$.

The distance between A and B is least when $\sqrt{100t^2 - 340t + 290}$ is least. This also happens when the *square* of the distance, that is $100t^2 - 340t + 290$, is at its least. Letting $y = 100t^2 - 340t + 290$, we therefore need to find the minimum points of y, by solving the equation $\frac{dy}{dt} = 0$.

$\frac{dy}{dt} = 0$

$200t - 340 = 0$

$t = 1.7$.

This is a minimum point because the second derivative $\frac{d^2y}{dt^2} = 200$ is always positive.
So the value of t when the distance between A and B is least is $t = 1.7$. The minimum distance is calculated as $\sqrt{100(1.7^2) - 340(1.7) + 290} = 1$ unit.

Atebion yr Hen Gwestiynau Arholiad

Haf 2006 (2) (a) $\boldsymbol{r}_A = (1 - 2t)\boldsymbol{i} + (-2t)\boldsymbol{j} + (-10 - 5t)\boldsymbol{k}$, $\boldsymbol{r}_B = (7 + t)\boldsymbol{i} + (9 - 8t)\boldsymbol{j} + (-6 - 5t)\boldsymbol{k}$
(b) 13 units.
(6) (a) $\boldsymbol{v} = -3\sin(3t)\boldsymbol{i} + 3\cos(3t)\boldsymbol{j}$ (c) 3 units.

Haf 2007 (8) (a) $\boldsymbol{r}_A = (3t)\boldsymbol{i} + (3 - 2t)\boldsymbol{j} + (-140 + 5t)\boldsymbol{k}$, $\boldsymbol{r}_B = (-9 - 2t)\boldsymbol{i} + (-4 + 6t)\boldsymbol{j} + (-6 + 3t)\boldsymbol{k}$
(b) $93t^2 - 558t + 18086$ (c) $t = 3$ units.

Haf 2009 (6) (a) $-16t\boldsymbol{i} + (12t - 10)\boldsymbol{j}$ (b) The acceleration $\boldsymbol{a} = -8\boldsymbol{i} + 6\boldsymbol{j}$ is constant because it does not depend on time; its magnitude is 10ms^{-2} (c) $t = 0.3$s.

Haf 2010 (2) (a) 13ms^{-1} (b) t = 0.8125s (c) $2\sqrt{13}$ms^{-1} (d) 56.31° to 2d.p.

Haf 2011 (3) (a) $\boldsymbol{F} = 0\boldsymbol{i} + 12\boldsymbol{j} + 24t^2\boldsymbol{k}$ (b) 168W.
(7) (a) 11ms^{-1} (b) 3s.

Haf 2012 (3) (a) t = 2.5s (b) 5ms^{-2} (8) (b) $(8 + 2t)\boldsymbol{i} + (7 - 4t)\boldsymbol{j}$ (c) x = 2.7, y = –4.825.

Haf 2013 (a) $\boldsymbol{r} = (6.5t^2 - 3t + 2)\boldsymbol{i} + (t^3 + 2t + 7)\boldsymbol{j}$ (b) $\boldsymbol{a} = 13\boldsymbol{i} + 6t\boldsymbol{j}$ (c) Either t = 1s or t = $\frac{7}{6}$s.

Haf 2014 (4) (a) $\boldsymbol{r}_A = (-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})t + 3\boldsymbol{i} + 5\boldsymbol{j} + 20\boldsymbol{k}$, $\boldsymbol{r}_B = (3\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k})t - 2\boldsymbol{i} + x\boldsymbol{j} + 15\boldsymbol{k}$
(b) $AB^2 = 53t^2 + (10 - 2x)t + x^2 - 10x + 75$ (c) $x = 45$.
(6) (a) 226.28N to 2 d.p. (b) $\boldsymbol{r} = -2\cos 2t\,\boldsymbol{i} + 3(\sin 5t + 1)\boldsymbol{j}$ (c) $t = \frac{\pi}{4}$, distance = 0.8787m to 4 d.p.

Haf 2015 (1) $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.
(4) (a) $\boldsymbol{v} = (4t^2 - 6t + 8)\boldsymbol{i} + (2t^3 - 5t^2 - 7)\boldsymbol{j}$ (b) 34ms^{-1}.

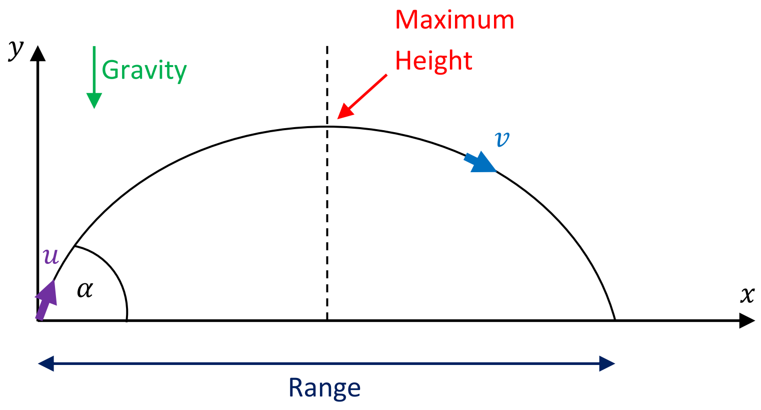
Haf 2016 (3) t = $\frac{1}{3}$s; least distance = 1.83m to 2.d.p.
(6) (a) $\boldsymbol{a} = 14\cos 2t\,\boldsymbol{i} - 18\sin 3t\,\boldsymbol{j}$ (b) $\boldsymbol{r} = 7.5\boldsymbol{i} + \boldsymbol{j}$.

Haf 2017 (a) (i) $\boldsymbol{v} = (t\cos t + \sin t)\boldsymbol{i} + (\cos t - t\sin t)\boldsymbol{j}$; $|\boldsymbol{v}| = \sqrt{t^2 + 1}$.
(ii) $3(t\cos t + \sin t)\boldsymbol{i} + 3(\cos t - t\sin t)\boldsymbol{j}$ (b) b = –3.

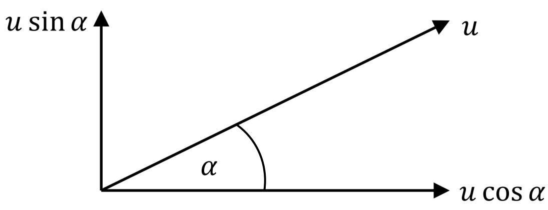
Haf 2018 (a) $8\boldsymbol{j}$ (b) $144\pi^2 + 4$J (1425.22J to 2 d.p.) (c) Force = $16(3\boldsymbol{i} - 8\pi\boldsymbol{j})$;
Perpendicular vector = $8\pi\boldsymbol{i} + 3\boldsymbol{j}$ (d) 160π W (502.65W to 2 d.p.).

5. Motion under Gravity in Two Dimensions.

A projectile is a particle travelling through space that has been launched in some way. We will consider its motion in two dimensions – *horizontally* and *vertically*. Air resistance will be **ignored**. Once launched, there is only one force acting on the particle, namely gravity.



If the initial velocity is u, projected at an angle α to the horizontal, then the initial horizontal velocity is $u_x = u\cos\alpha$ and the initial vertical velocity is $u_y = u\sin\alpha$.



At time t, the horizontal velocity will still be $u\cos\alpha$ because we assume there is no air resistance. Vertically, we must consider the effect of gravity, so that the vertical velocity at time t is given by the equation of motion

$$v = u + at$$
$$v_y = u\sin\alpha - gt$$

As there is no acceleration horizontally, the horizontal distance from the origin at time t is given by

Distance = Velocity × Time

$$S_x = (u\cos\alpha)t$$

The vertical distance from the origin at time t is given by the equation of motion

$$S = ut + \tfrac{1}{2}at^2$$
$$S_y = (u\sin\alpha)t - \tfrac{1}{2}gt^2.$$

We can summarise the above results in the following table.

Symbol	*Horizontal (x)*	*Vertical (y)*
u	$u\cos\alpha$	$u\sin\alpha$
a	0	$-g$
v	$u\cos\alpha$	$u\sin\alpha - gt$
S	$(u\cos\alpha)t$	$(u\sin\alpha)t - \frac{1}{2}gt^2$

Enghraifft (Haf 2006)

A stone is projected in a direction which makes an angle of 45° above the horizontal. It strikes a small target whose horizontal and vertical distances from the point of projection are 120m and 41.6m respectively. The target is above the level of the point of projection.
(a) Find the speed of projection and show that the time taken for the stone to reach the target is 4s.
(b) Determine, correct to two decimal places, the speed and direction of motion of the stone as it hits the target.

Ateb

(a) We are given $S_x = 120$m, $S_y = 41.6$m.

So $(u\cos\alpha)t = 120$ and $(u\sin\alpha)t - \frac{1}{2}gt^2 = 41.6$

$(u\cos 45°)t = 120$ $\qquad$ $(u\sin 45°)t - \frac{1}{2}gt^2 = 41.6$

$\frac{ut}{\sqrt{2}} = 120$ $\qquad$ $\frac{ut}{\sqrt{2}} - \frac{1}{2}gt^2 = 41.6$

Substituting from the first equation into the second equation, we obtain

$120 - \frac{1}{2}gt^2 = 41.6$

$120 - 41.6 = \frac{1}{2}gt^2$

$16 = t^2$

$t = 4$s.

Substituting back into the first equation,

$\frac{4u}{\sqrt{2}} = 120$

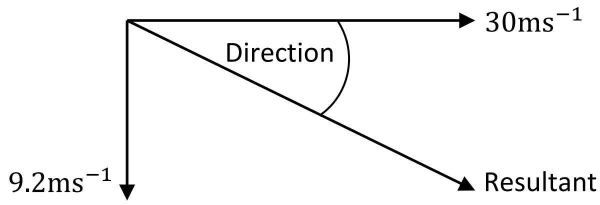
$u = 30\sqrt{2}\text{ms}^{-1}$.

$u = 42.43\text{ms}^{-1}$, correct to 2 d.p.

(b) When $t = 4s$, we have $v_x = u\cos\alpha$ and $v_y = u\sin\alpha - gt$

$v_x = 30\sqrt{2}\cos 45°$ $\qquad$ $v_y = 30\sqrt{2}\sin 45° - 9.8 \times 4$

$v_x = 30\text{ms}^{-1}$ $\qquad$ $v_y = -9.2\text{ms}^{-1}$



The resultant of these horizontal and vertical velocities is given by $\sqrt{30^2 + 9.2^2} = 31.38\text{ms}^{-1}$, to 2 d.p.
The direction of the resultant velocity is given by $\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{9.2}{30}\right) = 17.05°$, to 2 d.p.
It follows that, as the stone hits its target, it is moving with a velocity of 31.38ms^{-1} at an angle of $17.05°$ below the horizontal.

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Haf 2007 (a) 5.3ms^{-1} (b) t = 6s.

Haf 2008 (a) S_x = 24m. (b) Speed = 13.24ms^{-1}, to 2 d.p.; Direction = 25.02° below the horizontal, to 2 d.p.

Haf 2009 (a) t = 2.4s. (b) S_y = 5.376m. (c) t = $\frac{10}{7}$s.

Haf 2010 (b) t = 4s (c) 25.9ms^{-1}.

Haf 2011 (a) 4.78s, to 2 d.p. (b) 28.68m, to 2 d.p. (c) Magnitude = 44.75ms^{-1}, to 2 d.p.;
Direction = 82.29° below the horizontal, to 2 d.p.

Haf 2012 (a) Horizontal $\frac{4}{5}V$; Vertical $\frac{3}{5}V$ (c) $T = \frac{12}{7}$s; $V = 8.75\text{ms}^{-1}$ (d) 13.5ms^{-1} to 1 d.p.

Haf 2013 (a) (i) t = 0.75s (ii) 4.99375m (b) 15.64ms^{-1}, to 2 d.p.

Haf 2014 (a) Horizontal 16.7ms^{-1}; Vertical 13.45ms^{-1} (b) 20.11ms^{-1}, to 2 d.p.; 33.33° to 2 d.p.
(c) 4.11m to 2 d.p.

Haf 2015 (a) The ball does **not** fall into the lake as the range (120m) is greater than 117.5m.
(b) Magnitude = 28.89ms^{-1}, to 2 d.p.; Direction = 43.36° above the horizontal, to 2 d.p.

Haf 2016 (a) 53.04m to 2 d.p. (b) 7.66m to 2 d.p.
(c) Magnitude = 24.5ms^{-1}; Direction = 30° below the horizontal.

Haf 2017 (a) v = $15\sqrt{3}\text{ms}^{-1}$ (b) t = 0.6s ✓ (c) 10.33ms^{-1} to 2 d.p.

Haf 2018 (b) θ = 27.82° to 2 d.p.; v = 36.79ms^{-1} to 2 d.p.

Derivations of Formulae

You may be expected to derive the following equations in an examination. In questions where derivation of formulae has not been requested, the quoting of these formulae will **not** gain full credit.

(a) *Greatest Height.* (This is the maximum height of the particle during its flight.)

At the greatest height of the particle, the vertical velocity, v_y, is zero.

(i) Greatest Height.

Using $v^2 = u^2 + 2aS$

$$v_y^2 = u_y^2 - 2gS_y$$
$$0 = (u\sin\alpha)^2 - 2gS_y$$
$$S_y = \frac{(u\sin\alpha)^2}{2g}.$$

(ii) Time at Greatest Height.

Using $v = u + at$

$$v_y = u_y - gt$$
$$0 = u\sin\alpha - gt$$
$$t = \frac{u\sin\alpha}{g}.$$

(b) *Time of Flight.* (This is the time taken to return to the ground.)

Method (i): Double the time at the greatest height (above) to get $\frac{2u\sin\alpha}{g}$.

Method (ii): Use the equation of motion $S = ut + \frac{1}{2}at^2$

$$S_y = u_y t - \frac{1}{2}gt^2$$
$$0 = (u\sin\alpha)t - \frac{1}{2}gt^2$$
$$0 = t(u\sin\alpha - \frac{1}{2}gt)$$

Either $t = 0$ or $t = \frac{2u\sin\alpha}{g}$.

(c) *Range*. (This is the horizontal distance travelled.)

Horizontally, we have constant velocity, so

Distance = Velocity × Time

Range = Initial Horizontal Velocity × Time of Flight

Range = $u\cos\alpha \times \left(\frac{2u\sin\alpha}{g}\right)$

Range = $\frac{u^2 \times 2\cos\alpha\sin\alpha}{g}$

Range = $\frac{u^2\sin 2\alpha}{g}$.

Double angle formulae

(d) *Equation of path.* (This is the equation of the path the particle takes during its flight.)

Consider the motion of the particle from the origin O to a general point (x, y). Vertically, using the equation of motion $S = ut + \frac{1}{2}at^2$, we find that $S_y = u_y t - \frac{1}{2}gt^2$

$$S_y = (u\sin\alpha)t - \frac{1}{2}gt^2.$$

Therefore, for the general point (x, y), we have $y = (u\sin\alpha)t - \frac{1}{2}gt^2$.

Horizontally, we have $S_x = u_x t$

$$S_x = (u\cos\alpha)t.$$

Therefore, for the general point (x, y), we have $x = (u\cos\alpha)t$, so it follows that $t = \frac{x}{u\cos\alpha}$.

Substituting for t into our equation for y, we find that

$$y = (u\sin\alpha)\left(\frac{x}{u\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{1}{2}g\left(\frac{x^2}{u^2\cos^2\alpha}\right)$$

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}.$$

Recall that $\sec\alpha = \frac{1}{\cos\alpha}$

Using $\sec^2\alpha = 1 + \tan^2\alpha$,

$$y = x\tan\alpha - \frac{gx^2(1+\tan^2\alpha)}{2u^2}.$$

This is the **equation of path**. Given a general point (x, y) on the path and the initial velocity u, this can be used to find the angle of projection α.

Enghraifft

A particle is projected at 25ms^{-1} at an angle α to the horizontal, where $\tan\alpha = \frac{4}{3}$. Find the equation of path and the direction of motion at $t = 3$s.

Ateb

Vertically, we have $S_y = u_y y - \frac{1}{2}gt^2$

$$S_y = (25\sin\alpha)t - \frac{1}{2}gt^2.$$

Horizontally, we have $S_x = u_x t$

$$S_x = (25\cos\alpha)\,t$$

$$\frac{S_x}{25\cos\alpha} = t.$$

Substituting for t from the second equation into the first equation,

$$S_y = (25\sin\alpha)\left(\frac{S_x}{25\cos\alpha}\right) - \frac{1}{2}g\left(\frac{S_x}{25\cos\alpha}\right)^2$$

$$S_y = S_x\tan\alpha - \frac{gS_x^2}{2(625)\cos^2\alpha}$$

$$S_y = S_x\left(\frac{4}{3}\right) - \frac{gS_x^2\sec^2\alpha}{1250}$$

$$S_y = \frac{4}{3}S_x - \frac{gS_x^2(1+\tan^2\alpha)}{1250}$$

$$S_y = \frac{4}{3}S_x - \frac{gS_x^2(1+\frac{16}{9})}{1250}$$

$$S_y = \frac{4}{3}S_x - \frac{49}{2250}S_x^2.$$

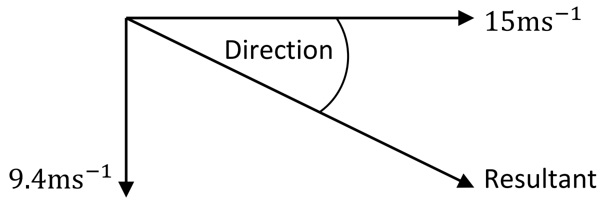
It follows that $y = \frac{4}{3}x - \frac{49}{2250}x^2$ is the equation of path.

When $t = 3$s, we have $v_x = 25\cos\alpha$ and $v_y = u_y - gt$

$v_x = 24(\frac{3}{5})$ $\qquad v_y = 25\sin\alpha - g(3)$

$v_x = 15\text{ms}^{-1}$ $\qquad v_y = 25\left(\frac{4}{5}\right) - 9.8 \times 3$

$v_y = -9.4\text{ms}^{-1}$



The direction of the resultant velocity is given by $\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{9.4}{15}\right) = 32.07°$, to 2 d.p.

It follows that, when $t = 3$s, the direction of motion is 32.07° below the horizontal.

Enghraifft

A tennis ball is projected at 40ms^{-1} at an angle 5° below the horizontal from a height of 2.4m. The ball should pass a net 12m away (the net is 0.9m high). The ball is modelled as a particle and there is negligible air resistance.

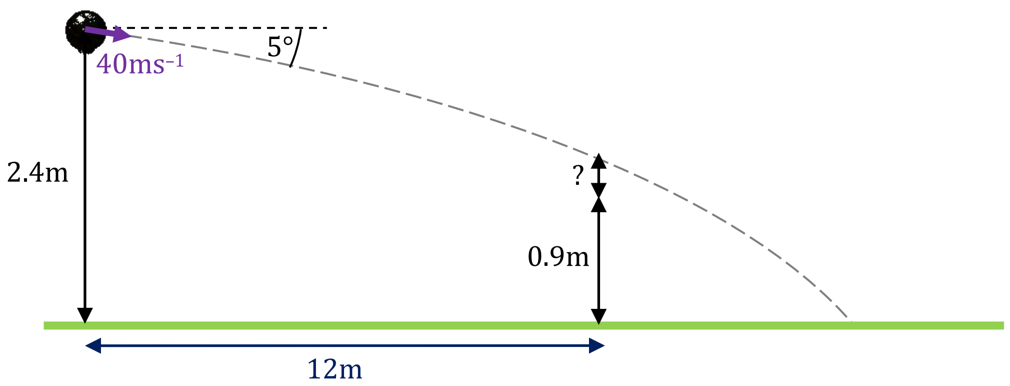
(a) Show that at $t = 0.3s$ the ball is directly above the net.

(b) Find the clearance above the net.

(c) Find the magnitude and direction of the velocity as the ball reaches the ground.

Ateb

We start by sketching a diagram of the situation as described in the question.



(a) Using $S_x = (u\cos\alpha)t$ at the net, we find that $12 = 40\cos 5° \times t$. It follows that $t = \frac{12}{40\cos 5°} = 0.3011$s, to 4 d.p. Therefore at $t = 0.3s$ the ball is more or less directly above the net.

(b) *Vertically*, we have $S_y = u_y t + \frac{1}{2}at^2$.

In projectile questions, the usual convention is to take the positive direction as upwards, so we must remember here that the vertical component of the initial velocity (u_y) will be negative.

When $t = 0.3$s, we find that $S_y = (-40\sin 5°)(0.3) - \frac{1}{2}g(0.3)^2$

$S_y = -1.4869$m, to 4 d.p.

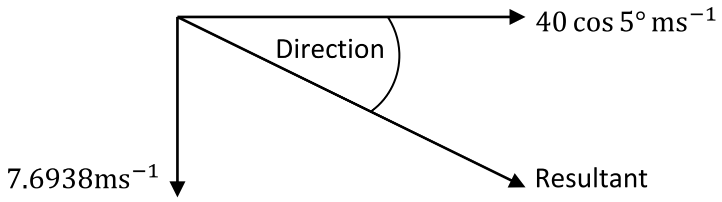
So the clearance above the net is 2.4 – 0.9 – 1.4869 = 0.0131m, to 4 d.p.

(c) We require $S_y = -2.4$m. Using $v_y^2 = u_y^2 + 2aS_y$

$$v_y^2 = (-40\sin 5°)^2 - 2 \times 9.8 \times -2.4$$

$$v_y^2 = 59.19$$

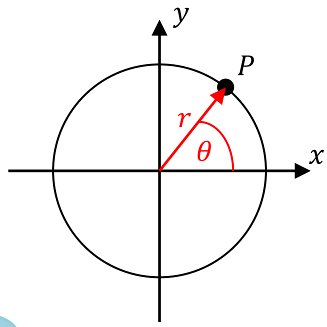
$v_y = \pm 7.6938\text{ms}^{-1}$, to 4 d.p.



The resultant of these horizontal and vertical velocities is given by $\sqrt{(40\cos 5°)^2 + 7.6938^2}$ = 40.58ms⁻¹, to 2 d.p. The direction of the resultant velocity is given by $\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{7.6938}{40\cos 5°}\right) = 10.93°$, to 2 d.p. It follows that the ball travels at a velocity of 40.58ms⁻¹ at an angle of 10.93° below the horizontal as it reaches the ground.

6. Motion in a Horizontal Circle.

Consider a particle P travelling in a horizontal circle (the diagram on the right should be seen as lying flat on a table with the viewer seeing the diagram from above). Let the radius of the circle be denoted by r and, further, let the position of P at time t be such that P makes an angle θ with the x-axis. With this information, we can construct the displacement vector $\boldsymbol{r}$ of P at time t, namely



$$\boldsymbol{r} = r\cos\theta\,\boldsymbol{i} + r\sin\theta\,\boldsymbol{j}.$$

Module M1: resolving forces

The velocity of P at time t is given by differentiation:

See Chapters 3 and 4

Module C3: chain rule

$$\boldsymbol{v} = \frac{d}{dt}(r\cos\theta\,\boldsymbol{i} + r\sin\theta\,\boldsymbol{j})$$
$$\boldsymbol{v} = r\left(\frac{d}{dt}\cos\theta\right)\boldsymbol{i} + r\left(\frac{d}{dt}\sin\theta\right)\boldsymbol{j}$$
$$\boldsymbol{v} = r\left(\frac{d\theta}{dt}\frac{d}{d\theta}\cos\theta\right)\boldsymbol{i} + r\left(\frac{d\theta}{dt}\frac{d}{d\theta}\sin\theta\right)\boldsymbol{j}$$
$$\boldsymbol{v} = r\frac{d\theta}{dt}(-\sin\theta)\boldsymbol{i} + r\frac{d\theta}{dt}\cos\theta\,\boldsymbol{j}$$
$$\boldsymbol{v} = r\frac{d\theta}{dt}(-\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}).$$

Let $\omega = \frac{d\theta}{dt}$ be defined as the **angular velocity**, that is the rate of change of the angle θ with respect to time. Thus

$$\boldsymbol{v} = r\omega(-\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}).$$

The *speed* v of P at time t is given by[2]

$$v = \sqrt{(-r\omega\sin\theta)^2 + (r\omega\cos\theta)^2}$$
$$v = \sqrt{r^2\omega^2(\sin^2\theta + \cos^2\theta)}$$
$$v = \sqrt{r^2\omega^2}$$
$$v = r|\omega|$$

where $|\omega|$ is the **angular speed**, with unit rad s^{-1}.

By convention, ω is positive for anticlockwise motion and negative for clockwise motion. Thus, **for anticlockwise motion**, we have $\omega = |\omega|$ and thus

v = radius × angular velocity

and

v = radius × angular speed.

The *acceleration* $\boldsymbol{a}$ of P at time t is given by

$$\boldsymbol{a} = \frac{d}{dt}(r\omega(-\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}))$$
$$\boldsymbol{a} = \frac{d\theta}{dt}\frac{d}{d\theta}(r\omega(-\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}))$$

[2] Note the difference between the notation for speed (v) and velocity ($\boldsymbol{v}$): here we use a **bold** letter for the velocity; when writing we would underline so that the velocity would be written as $\underline{v}$.

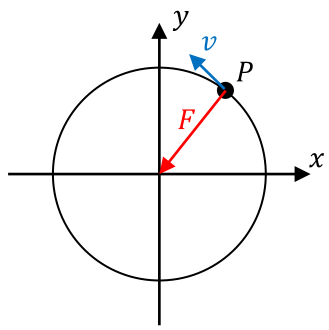
$$\boldsymbol{a} = \omega \frac{d}{d\theta}(r\omega(-\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}))$$
$$\boldsymbol{a} = \omega(-r\omega\cos\theta\,\boldsymbol{i} - r\omega\sin\theta\,\boldsymbol{j})$$
$$\boldsymbol{a} = -r\omega^2(\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}).$$

The *magnitude* a of the acceleration is given by

$$a = \sqrt{(-r\omega^2\cos\theta)^2 + (-r\omega^2\sin\theta)^2}$$
$$a = \sqrt{r^2\omega^4\cos^2\theta + r^2\omega^4\sin^2\theta}$$
$$a = \sqrt{r^2\omega^4(\cos^2\theta + \sin^2\theta)}$$
$$\boxed{a = r\omega^2.}$$

Alternatively, since $v = r\omega$ so that $\omega = \frac{v}{r}$ and therefore $\omega^2 = \frac{v^2}{r^2}$, we have $\boxed{a = \frac{v^2}{r}.}$

Motion in a horizontal circle with uniform angular speed



When an object P is moving in a circle with constant speed v, a force is required to keep it in this circle. This force is directed towards the centre of the circle and is called a **centripetal force**.

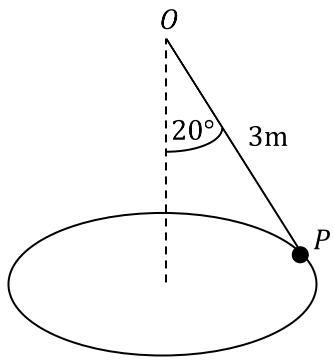
Using $F = ma$, we find that $\boxed{F = \frac{mv^2}{r}}$ or $\boxed{F = mr\omega^2.}$

It follows that $\boxed{a = \frac{v^2}{r}}$ or $\boxed{a = r\omega^2.}$

The centripetal force is often provided by a *tension* in a string or the *reaction* of a track or surface.

Enghraifft

Let the particle P shown in the diagram move in a horizontal circle with constant speed u ms^{-1}. Find u.



Ateb

Let T be the tension in the string. The vertical component of T is in equilibrium with the weight of P. It follows that $T\cos 20° = mg$. The horizontal component of T is equal to the centripetal force, so that $T\sin 20° = \frac{mv^2}{r}$. Solving these equations simultaneously, we find that

$$\frac{mg}{\cos 20°} = \frac{mv^2}{r\sin 20°}$$
$$\frac{r\sin 20° g}{\cos 20°} = v^2$$
$$r\tan 20° g = v^2.$$

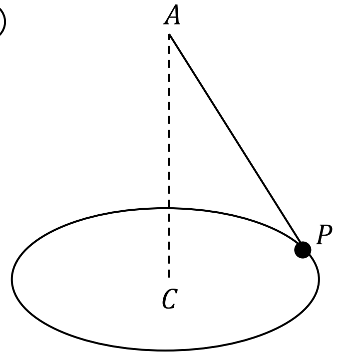
We can calculate r by using trigonometry: $\sin 20° = \frac{r}{3}$, so that $3\sin 20° = r$. It follows that $(3\sin 20°)\tan 20° g = v^2$, so we have $v = 1.91$ms^{-1} to 2 decimal places.

Enghraifft (Haf 2006)

The diagram shows a small body P, of mass 3kg, attached by means of a light inextensible string of length 1.3m, to a fixed point A. The point C is vertically below A, and P describes a horizontal circle, with centre C and radius 0.5m, with a uniform angular speed of ω radians per second about C.

(a) Find the tension in the string.

(b) Calculate, correct to two decimal places, the value of ω.



Ateb

(a) The vertical component of the tension is equal to the weight.

$T\cos\theta = mg$

$T\left(\frac{1.2}{1.3}\right) = 3g$

$T = 31.85$ N.

(b) The horizontal component of the tension is equal to the centripetal force.

$T\sin\theta = mr\omega^2$

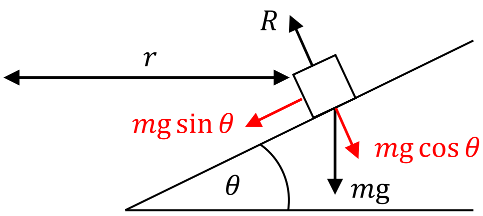
$31.85\left(\frac{0.5}{1.3}\right) = 3 \times 0.5 \times \omega^2$

$\frac{49}{6} = \omega^2$

$\omega = 2.86$ rad s^{-1} to 2 decimal places.

Banked Tracks

Consider a vehicle with mass m travelling on a banked track, which is at an angle θ to the horizontal. The motion of the vehicle is a horizontal circle with radius r.



If the vehicle travels at a constant speed v then, ignoring friction down the slope, the only forces acting are the normal reaction R and the gravitational force mg.

Horizontally, $R\sin\theta = \frac{mv^2}{r}$. *Vertically*, $R\cos\theta = mg$.

These two equations can be rearranged to give the equation $v^2 = gr\tan\theta$. This value for v is the value for no side slip (no tendency to slip sideways).

Enghraifft (Haf 2009)

A car, of mass 1000kg, is travelling in a horizontal circle of radius 250m on a track which is banked at an angle α to the horizontal. When the car is travelling at 28ms^{-1}, it has no tendency to slip sideways. Calculate the value of α.

Ateb

Method 1

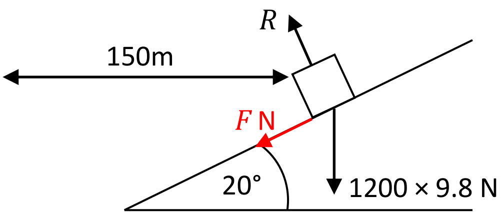
Knowing there is no tendency to slip sideways, we simply substitute into the equation $v^2 = gr\tan\theta$ to give $28^2 = 9.8 \times 250 \times \tan\alpha$, and so $\alpha = 17.74°$ to two decimal places.

Method 2

Horizontally, we have $R\sin\alpha = \frac{1000(28)^2}{250}$, so that $R\sin\alpha = 3136$. *Vertically*, we have $R\cos\alpha = 1000g$, so that $R\cos\alpha = 9800$. Dividing the first equation by the second equation, we find that $\tan\alpha = \frac{3136}{9800}$, so (as before) $\alpha = 17.74°$ to two decimal places.

Enghraifft

A car, of mass 1200kg, is travelling in a horizontal circle of radius 150m on a track which is banked at an angle 20° to the horizontal. If the car is travelling at a constant speed of 25ms^{-1}, find the frictional force F on the car.



Ateb

Horizontally, we have

Left forces = Centripetal force

$R\sin 20° + F\cos 20° = \frac{mv^2}{r}$

$R\sin 20° + F\cos 20° = 5000.$

$R\sin 20° = 5000 - F\cos 20°$

$R = \frac{5000}{\sin 20°} - \frac{F\cos 20°}{\sin 20°}$

Vertically, we have

Up forces = Down forces

$R\cos 20° = F\sin 20° + mg$

$R\cos 20° = F\sin 20° + 11760.$

$R = \frac{F\sin 20°}{\cos 20°} + \frac{11760}{\cos 20°}$

Equating the above two equations, we find that

$\frac{5000}{\sin 20°} - \frac{F\cos 20°}{\sin 20°} = \frac{F\sin 20°}{\cos 20°} + \frac{11760}{\cos 20°}$

$5000 - F\cos 20° = \frac{F\sin^2 20°}{\cos 20°} + \frac{11760\sin 20°}{\cos 20°}$

$5000\cos 20° - F\cos^2 20° = F\sin^2 20° + 11760\sin 20°$

$5000\cos 20° - 11760\sin 20° = F\sin^2 20° + F\cos^2 20°$

$5000\cos 20° - 11760\sin 20° = F(\sin^2 20° + \cos^2 20°)$

$5000\cos 20° - 11760\sin 20° = F$

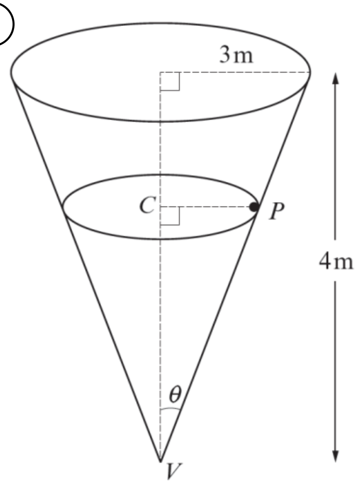
$F = 676.31$ N to two decimal places.

Enghraifft (Haf 2013)

The diagram shows a hollow cone, of base radius 3m and height 4m, which is fixed with its axis vertical and vertex V downwards. A particle P, of mass M kg, moves in the horizontal circle with centre C on the smooth inner surface of the cone with constant speed $\sqrt{\frac{8g}{3}}$ ms^{-1}, where g ms^{-2} is the acceleration due to gravity.

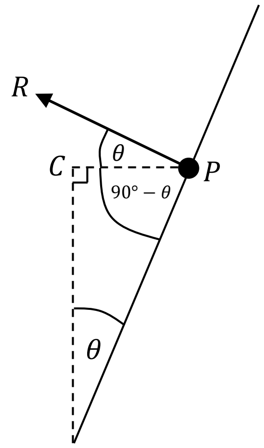
(a) Show that the normal reaction of the surface of the cone on the particle is $\frac{5Mg}{3}$ N.

(b) Calculate the length of CP and hence determine the height of C above V.



Ateb

The horizontal component of the reaction, $R\cos\theta$, is equal to the centripetal force. So $R\cos\theta = \frac{mv^2}{r}$.
The vertical component of the reaction, $R\sin\theta$, is equal to the weight of the particle. So $R\sin\theta = Mg$.



(a) Using $R\sin\theta = Mg$, we find that

$$R\left(\frac{3}{5}\right) = Mg$$
$$3R = 5Mg$$
$$R = \frac{5Mg}{3}\text{ N, as required.}$$

(b) Using $R\cos\theta = \frac{mv^2}{r}$, we find that

$$\left(\frac{5Mg}{3}\right)\times\frac{4}{5} = \frac{M\left(\sqrt{\frac{8g}{3}}\right)^2}{r}$$
$$\frac{20Mg}{15} = \frac{M\left(\frac{8g}{3}\right)}{r}$$
$$\frac{20Mg}{15} = \frac{8Mg}{3r}$$
$$\frac{20}{15} = \frac{8}{3r}$$
$$r = \frac{8\times15}{20\times3}$$
$$r = 2\text{m.}$$

So, using similar triangles, the height of C above V is $4\times\frac{2}{3} = \frac{8}{3}$m.

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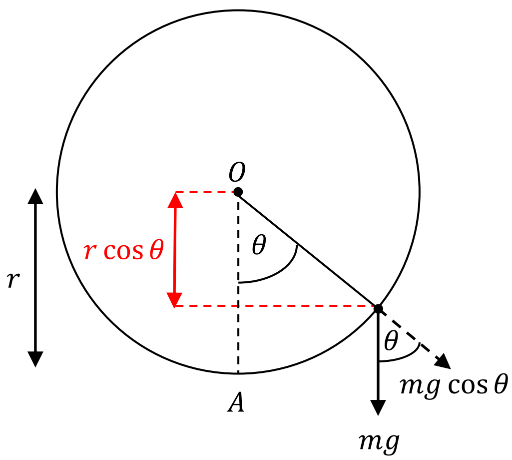
Haf 2007 (a) $\omega = 7.5$ rad s^{-1} (b) T = 18N.
Haf 2008 (a) 43.60° to 2 d.p. (b) T = 54.13N to 2 d.p. (c) 0.62m to 2 d.p.
Haf 2010 19.5ms^{-1}.
Haf 2011 (a) 3ms^{-1} (b) T = 7.5N.
Haf 2012 (a) θ = 1.23 rad to 2 d.p. (b) 3.75m.
Haf 2015 R = 12022.7N to 1 d.p., v = 12.9ms^{-1} to 1 d.p.
Haf 2016 v = 3.36ms^{-1}, ω = 2.1 rad s^{-1}.
Haf 2017 (a) $\cos\theta = \frac{g}{l\omega^2}$ ✓ (b) (i) $\cos\theta = \frac{1}{10}$ (ii) $l = \frac{10}{3}$ ✓ (iii) $\lambda = 1764$N (iv) $\frac{98}{3}$ J.
Haf 2018 (4) Least value = $\frac{7\sqrt{7}}{5}$rad s^{-1}; Greatest value = $\frac{7\sqrt{23}}{5}$ rad s^{-1}.
(7) (a) (i) θ = 26.10° to 2 d.p. (ii) T = 32.74N to 2 d.p. (b) l = 0.68m to 2.d.p.

7. Motion in a Vertical Circle.

Let us consider a particle, attached either to a string or a light rod, moving in a vertical circle. This particle will experience a *centripetal force* during its motion. This is the resultant of the tension in the string or rod and the radial component of the weight:

$$\boxed{\frac{mv^2}{r} = T - mg\cos\theta}$$

where v is the tangential speed, and θ is the angle with the line OA.



Note that, when the particle is immediately below O, then we have a special case of the formula: $\boxed{\frac{mv^2}{r} = T - mg.}$

Similarly, when the particle is immediately above O, then we have the formula $\boxed{\frac{mv^2}{r} = T + mg.}$

Energy considerations

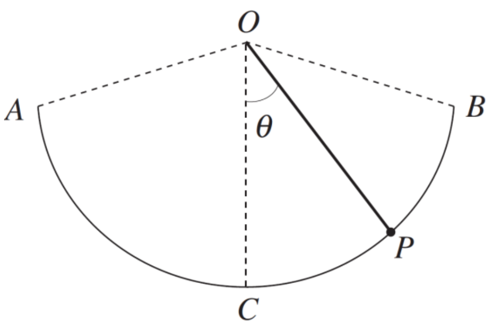
By the principle of conservation of energy, the sum of the particle's *potential energy* and *kinetic energy* is always constant. At a general point during the motion, say at an angle θ to the line OA, the energy is given by

$$\frac{1}{2}mv^2 + mgh$$
$$= \boxed{\frac{1}{2}mv^2 + mg(r - r\cos\theta)}$$

where $r - r\cos\theta$ is the height above the bottom of the circle, and v is the tangential speed.

Enghraifft (Haf 2006)

One end of a light rod of length l m is attached to a fixed point O and the other end is attached to a particle P of mass m kg. The particle P is set in motion so that it moves back and forth along the minor arc AB of a vertical circle with centre O and radius l m, as shown in the diagram.



When P is at its lowest point C, its speed is u ms^{-1} and the tension in the rod is $2mg$ N.

(a) Show that $u = \sqrt{gl}$.

(b) The speed of P when OP makes an angle θ with the vertical is denoted by v ms^{-1}.
Show that $v^2 = gl(2\cos\theta - 1)$.

(c) Find the greatest value of θ.

(d) Find the value of θ when the tension in the rod is mg N.

Ateb

(a) At the lowest point C, we know that $\frac{mv^2}{r} = T - mg$. Substituting in the values from the question, we find that $\frac{mu^2}{l} = 2mg - mg$

$\frac{mu^2}{l} = mg$

$u^2 = gl$

$u = \sqrt{gl}$.

(b) By the principle of conservation of energy, the total energy at C is equal to the total energy at P. Therefore Kinetic Energy at C + Potential Energy at C = Kinetic Energy at P + Potential Energy at P

$\frac{1}{2}mu^2 + mg(l - l\cos 0°) = \frac{1}{2}mv^2 + mg(l - l\cos\theta)$

$\frac{1}{2}mu^2 + mg(l - l) = \frac{1}{2}mv^2 + mg(l - l\cos\theta)$

$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mg(l - l\cos\theta)$

$\frac{1}{2}u^2 = \frac{1}{2}v^2 + g(l - l\cos\theta)$

$u^2 = v^2 + 2g(l - l\cos\theta)$.

From part (a) of the question, we know that $u = \sqrt{gl}$, so that $u^2 = gl$. Therefore

$gl = v^2 + 2g(l - l\cos\theta)$

$v^2 = gl - 2g(l - l\cos\theta)$

$v^2 = gl(1 - 2 + 2\cos\theta)$

$v^2 = gl(2\cos\theta - 1)$.

(c) The maximum value of θ will occur when $v = 0$. Therefore

$0^2 = gl(2\cos\theta - 1)$

$0 = 2gl\cos\theta - gl$

$gl = 2gl\cos\theta$

$1 = 2\cos\theta$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = 60°$.

(d) At a general point, we have the formula $\frac{mv^2}{r} = T - mg\cos\theta$. Substituting in the values from the question, we find that $\frac{mv^2}{l} = mg - mg\cos\theta$

$mv^2 = mgl - mgl\cos\theta$

$v^2 = gl(1 - \cos\theta)$.

Substituting for v^2 from part (b) of the question,

$gl(2\cos\theta - 1) = gl(1 - \cos\theta)$

$2\cos\theta - 1 = 1 - \cos\theta$

$3\cos\theta = 2$

$\theta = \cos^{-1}\left(\frac{2}{3}\right)$

$\theta = 48.19°$ to two decimal places.

Enghraifft (Haf 2007)

A particle, of mass 3kg, is attached to one end of a light rod of length 0.9m. The other end of the rod is freely pivoted at a fixed point O. The particle moves in a vertical circle with centre O, such that its speed at the lowest point of its path is three times its speed at the highest point of its path.

(a) Show that the speed of the particle at the lowest point of its path is 6.3ms^{-1}.

(b) Calculate the thrust in the rod when the particle is at the highest point of its path.

(c) If a string replaced the rod, state, with a reason, whether the particle would still move in complete circles.

Ateb

(a) Let the tangential speed at the highest point be denoted by v, so that the tangential speed at the lowest point is $3v$. By the principle of conservation of energy, the total energy at the lowest point is equal to the total energy at the highest point. Therefore

KE at lowest point + PE at lowest point = KE at highest point + PE at highest point

$\frac{1}{2}m(3v)^2 + mg(0.9 - 0.9\cos 0°) = \frac{1}{2}mv^2 + mg(0.9 - 0.9\cos 180°)$

$\frac{1}{2}(9v^2) = \frac{1}{2}v^2 + 1.8g$

$9v^2 = v^2 + 3.6g$

$8v^2 = 3.6g$

$v = \sqrt{0.45g}$

$v = 2.1$ms^{-1}.

It follows that the speed at the lowest point is given by 3 × 2.1 = 6.3ms^{-1}.

(b) At the highest point, we have a special case of the formula $\frac{mv^2}{r} = T - mg\cos\theta$, namely $\frac{mv^2}{r} = T + mg$. Substituting in the values from the question, we find that $\frac{3(2.1)^2}{0.9} = T + 3g$, so that $T = -14.7$N. (So the rod exerts a thrust of 14.7N upwards.)

(c) The object would not move in complete circles as T, at the highest point, was negative; i.e. the rod exerted a thrust which a string cannot exert.

Enghraifft

The diagram shows a 'figure-of-eight' ride at a funfair. The ride is designed so that a passenger at the highest point of the track feels a force from the seat of at least half of the person's weight.

(a) Find the minimum speed at the highest point.

(b) Find the minimum speed at the lowest point.

Ateb

(a) At the highest point, we have $\frac{mv^2}{r} = T + mg$. If the force provided by the seat must be at least half of the weight, then the limit *is* half of the weight. Therefore we are looking to solve the equation

$\frac{mv^2}{r} = \frac{1}{2}mg + mg$

$\frac{v^2}{r} = \frac{3}{2}g$

$v^2 = \frac{3gr}{2}$

$v^2 = \frac{3\times 9.8\times 12}{2}$

$v^2 = 176.4$

$v = 13.28$ms^{-1} to two decimal places.

(b) By the principle of conservation of energy, the total energy at the lowest point is equal to the total energy at the highest point. Therefore

KE at lowest point + PE at lowest point = KE at highest point + PE at highest point

$\frac{1}{2}mv^2 + mg(12 - 12\cos 0°) = \frac{1}{2}m(176.4) + mg(12 - 12\cos 180°)$

$\frac{1}{2}v^2 + g(12 - 12\cos 0°) = \frac{1}{2}(176.4) + g(12 - 12\cos 180°)$

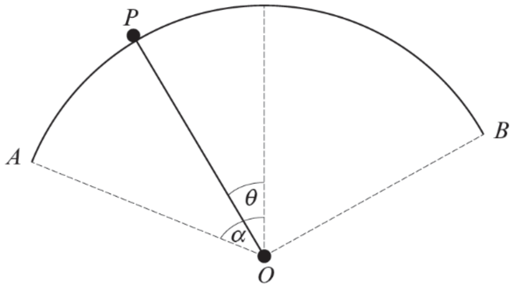
$\frac{1}{2}v^2 = 88.2 + 235.2$

$v^2 = 646.8$

$v = 25.43$ms^{-1} to two decimal places.

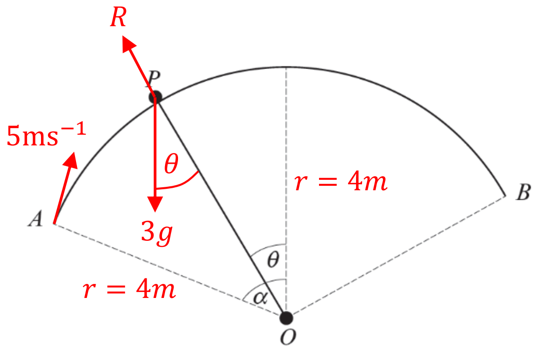
Enghraifft (Haf 2013)

The diagram shows a particle of mass 3 kg at a point P on the smooth outer surface AB of a sphere centre O and radius 4m. The points O, A, P and B are in the same vertical plane. Initially, the particle is held at rest at the point A, where OA makes an angle α with the upwards vertical and $\cos \alpha = 0.8$. The particle is then projected with velocity 5 ms^{-1} in a direction which is perpendicular to OA, so that the particle moves along the arc AB. When the particle is at P, OP makes an angle θ with the upwards vertical.



(a) Find, in terms of θ, the speed of the particle at P.
(b) Determine, in terms of θ, the reaction between the particle and the sphere at P.

Ateb



(a) By the principle of conservation of energy:
KE at A + PE at A = KE at P + PE at P
(Let us measure the height for the PE above O.)

$$\frac{1}{2}mv^2 + mgh = \frac{1}{2}mv^2 + mgh$$
$$\frac{1}{2} \times 3 \times 5^2 + 3 \times 9.8 \times r\cos\alpha = \frac{1}{2} \times 3 \times v^2 + 3 \times 9.8 \times r\cos\theta$$
$$37.5 + 29.4 \times 4 \times 0.8 = 1.5v^2 + 29.4 \times 4 \times \cos\theta$$
$$131.58 = 1.5v^2 + 117.6\cos\theta$$
$$87.72 = v^2 + 78.4\cos\theta$$
$$v^2 = 87.72 - 78.4\cos\theta$$
$$v = \sqrt{87.72 - 78.4\cos\theta}$$

(b) The centripetal force is the resultant of the radial component of the weight and the normal reaction.

$$\frac{mv^2}{r} = mg\cos\theta - R$$
$$\frac{3v^2}{4} = 3 \times 9.8 \times \cos\theta - R$$
$$\frac{3}{4}(87.72 - 78.4\cos\theta) = 29.4\cos\theta - R$$
$$R = 29.4\cos\theta - \frac{3}{4}(87.72 - 78.4\cos\theta)$$
$$R = 29.4\cos\theta - 65.79 + 58.8\cos\theta$$
$$R = 88.2\cos\theta - 65.79$$

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Haf 2008 (b) $T = 58.8\cos\theta + 44.4$.

Haf 2009 (a) $v = \sqrt{41.8 + 39.2\cos\theta}$ (b) $R = 104.5 + 147\cos\theta$ (c) At the top, when $\theta = 180°$, R is negative (which is impossible), so the particle will leave the circle before reaching the top.

Haf 2010 (a) $v = \sqrt{120 + 49\cos\theta}$ (b) $T = 144 + 88.2\cos\theta$ (c) Yes because *T* is always positive (the minimum value of *T*, which is 55.8N, occurs when $\theta = 180°$).

Haf 2011 (b) $T = 61.2 + 88.2\cos\theta$ (c) *P* does not describe a complete circle because, when $\theta = 180°$ (for example) the tension is negative. (d) A light rigid rod can exert a negative tension (thrust) so potentially *P* will move in a complete circle. As the KE at *A,* 24J, is greater than the GPE difference between the bottom and the top (23.52J), then the particle will move in complete circles when attached to a light rod.

Haf 2012 (a) u = 2.8ms^{-1}; $v^2 = -15.68 + 23.52\cos\theta$ (b) $T = 88.2\cos\theta - 39.2$ (c) T = 49N; T = 19.6N.

Haf 2014 (b) (i) $T = mg(\frac{34}{15} - 3\cos\theta)$ (ii) $\theta = 139.07°$ to 2 d.p.

Haf 2015 (a) (i) $v^2 = 9.32 + 15.68\cos\theta$ (ii) $T = 34.95 + 88.2\cos\theta$ (b) $\theta = 113.3°$, to 1 d.p. In its subsequent motion P will cease to move in a vertical circle and will behave initially as a projectile.

Haf 2016 (a) $v^2 = g(9 - 8\cos\theta)$ (b) $R = mg(3\cos\theta - 2.25)$; $\cos\theta = 0.75$, $v^2 = 29.4$.

Haf 2017 (a) $v^2 = u^2 + 15.68\cos\theta - 7.84$ (b) $T = 6.25u^2 + 147\cos\theta - 49$
(c) $u = 5.6$ ms^{-1} (ch) $u = 4.85$ms^{-1} to 2 d.p.

Dr. Gareth Evans, fersiwn 17 Awst 2018.