Gwerthoedd Arbennig
Sin, Cos a Tan

Special Values of Sin, Cos and Tan

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Ongl $\frac{\pi}{4}$ neu 45°:

$$\sin\left(\frac{\pi}{4}\right) = \frac{\text{cyferbyn}}{\text{hypotenws}} = \frac{1}{\sqrt{2}}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\text{agos}}{\text{hypotenws}} = \frac{1}{\sqrt{2}}$$

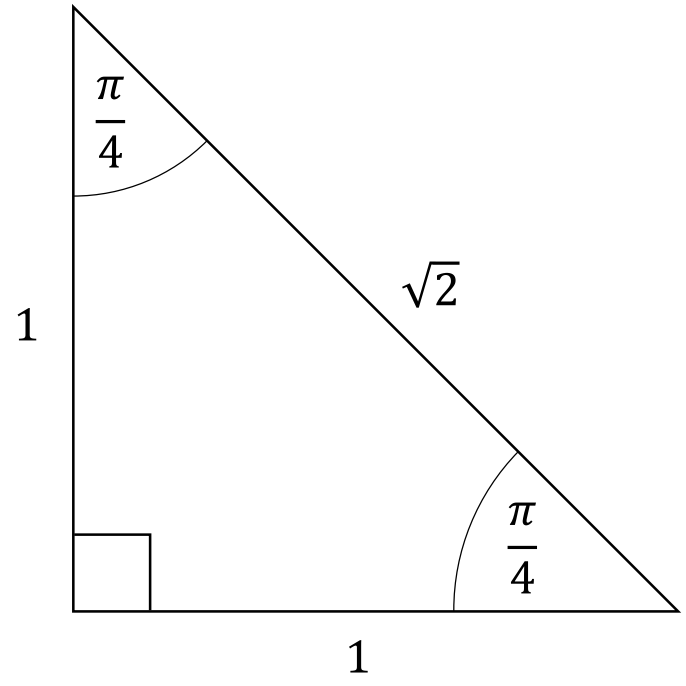
$$\tan\left(\frac{\pi}{4}\right) = \frac{\text{cyferbyn}}{\text{agos}} = \frac{1}{1} = 1$$

An angle of $\frac{\pi}{4}$ or 45°:

$$\sin\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

$$\tan\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1$$



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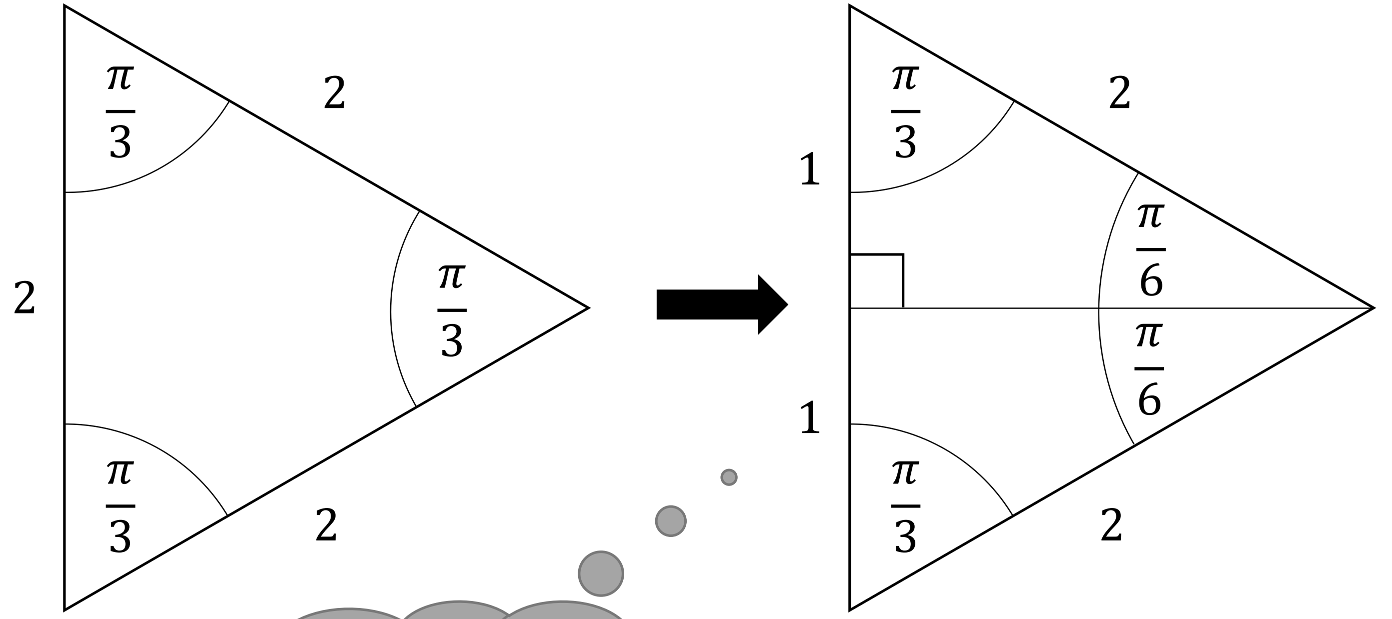
Onglau $\frac{\pi}{3}$ (60°) a $\frac{\pi}{6}$ (30°):

Angles of $\frac{\pi}{3}$ *(60°) and* $\frac{\pi}{6}$ *(30°):*



Cychwyn efo triongl hafalochrog / *Start with an equilateral triangle*

Haneru'r triongl / *Halve the triangle*

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Ongl $\frac{\pi}{6}$ neu 30°:

$$\sin\left(\frac{\pi}{6}\right) = \frac{\text{cyferbyn}}{\text{hypotenws}} = \frac{1}{2}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\text{agos}}{\text{hypotenws}} = \frac{\sqrt{3}}{2}$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{\text{cyferbyn}}{\text{agos}} = \frac{1}{\sqrt{3}}$$

Ongl $\frac{\pi}{3}$ neu 60°:

$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{cyferbyn}}{\text{hypotenws}} = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{agos}}{\text{hypotenws}} = \frac{1}{2}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{cyferbyn}}{\text{agos}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

An angle of $\frac{\pi}{6}$ or 30°:

$$\sin\left(\frac{\pi}{6}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$$

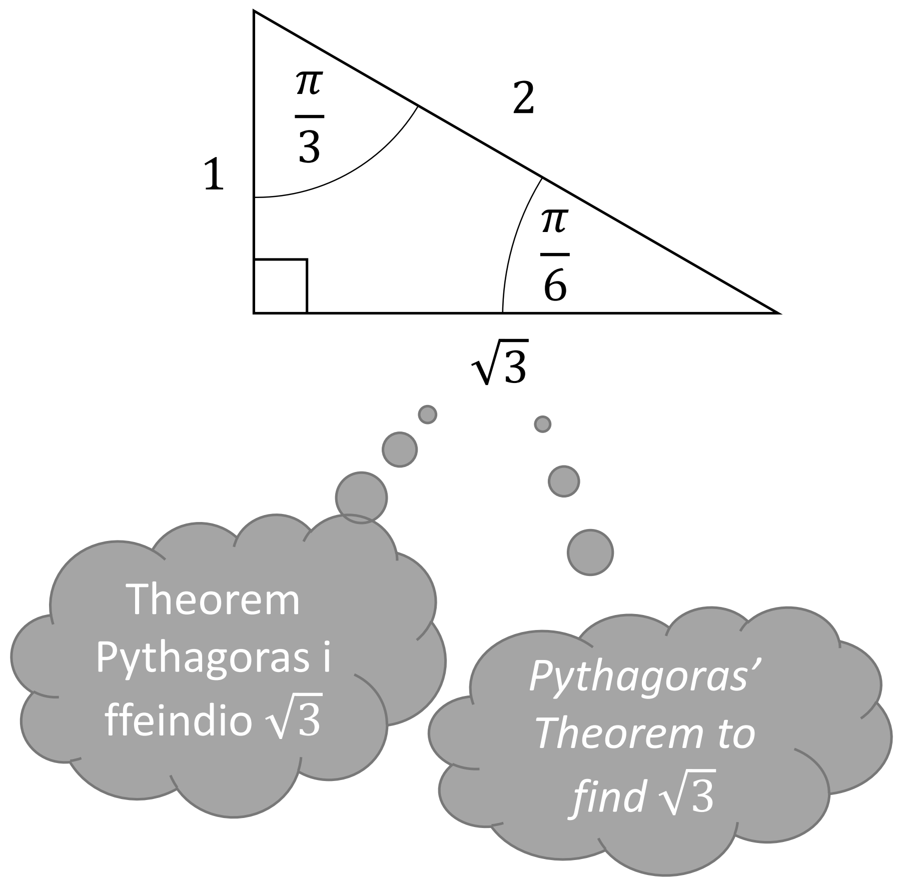
$$\tan\left(\frac{\pi}{6}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}}$$

An angle of $\frac{\pi}{3}$ or 60°:

$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$



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Crynodeb / *Summary:*

Ongl / *Angle*	**Sin**	**Cos**	**Tan**
0	0	1	0
$\frac{\pi}{6}$ neu / *or* 30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ neu / *or* 45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ neu / *or* 60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ neu / *or* 90°	1	0	Heb ei ddiffinio / *Not defined*

Gellir ffeindio lluosrifau gwahanol o 30° trwy ddefnyddio cymesuredd graffiau sin, cos a tan.
Other multiples of 30° can be found by using the symmetries of the graphs of sin, cos and tan.

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Brascamcanion onglau bach / *Small angle approximations*

Os oes gennym ongl fach, ac os yw'r ongl yn cael ei fesur mewn **radianau**, yna gellir defnyddio'r brasamcanion canlynol.

*If we have a small angle, and if the angle is measured in **radians**, then we can use the following approximations.*

$$\sin\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{\theta^2}{2} \qquad \tan\theta \approx \theta$$

Mae'r brasamcanion yn gywir i dri ffigur ystyrlon os yw –0.105 < θ < 0.105 (neu –6° < θ < 6°).
The approximations are correct to three significant figures if –0.105 < θ < 0.105 (or –6° < θ < 6°).

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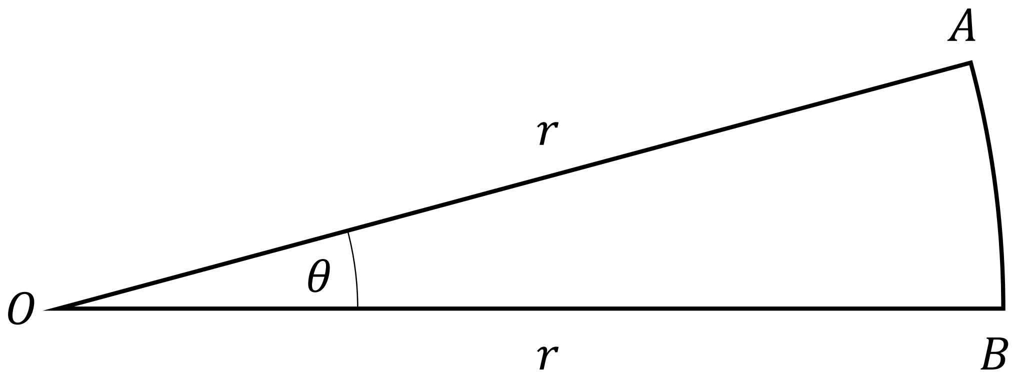
Brascamcanion onglau bach / *Small angle approximations*

Gadewch i'r ongl fach θ ffurfio sector o gylch OAB.
Arwynebedd y sector yw $\frac{1}{2}r^2\theta$.



Let the small angle θ form the sector OAB of a circle.
The area of the sector is $\frac{1}{2}r^2\theta$.

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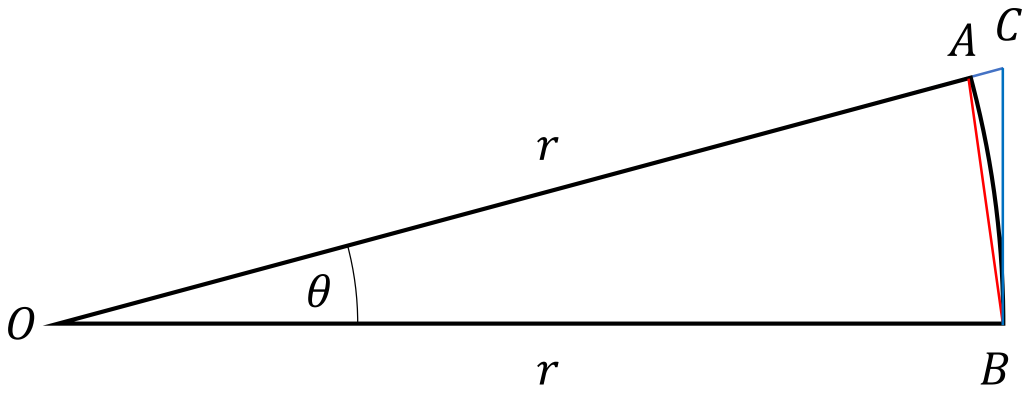
Brascamcanion onglau bach / *Small angle approximations*

Gadewch i ni ychwanegu'r cord AB ag ymestyn y radiws OA i gyrraedd y pwynt C fel bod OB a BC yn berpendicwlar.



Let us add the chord AB and extend the radius OA to reach the point C so that OB and BC are perpendicular.

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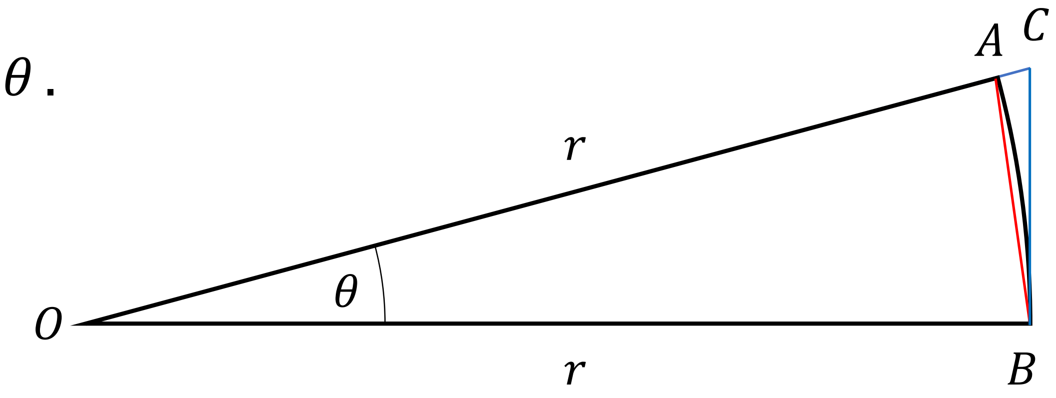
Brascamcanion onglau bach / *Small angle approximations*

Mae OCB yn driongl ongl sgwâr efo sail r ag uchder $r\tan\theta$.
Arwynebedd triongl OCB yw $\frac{1}{2}r^2\tan\theta$.
Arwynebedd y triongl isosgeles OAB yw $\frac{1}{2}r^2\sin\theta$.



OCB is a right-angled triangle with base r and height $r\tan\theta$.
The area of the triangle OCB is $\frac{1}{2}r^2\tan\theta$.
The area of the isosceles triangle OAB is $\frac{1}{2}r^2\sin\theta$.

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Mae arwynebedd triongl OAB < arwynebedd sector OAB < arwynebedd triongl OCB

$$\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$$

Gallwn rannu efo $\frac{1}{2}r^2$ gan ei fod o hyd yn bositif.

$$\sin\theta < \theta < \tan\theta$$

Gan fod θ yn ongl fach bositif, mae $\sin\theta$ yn bositif. Felly gallwn rannu'r anhafaledd efo $\sin\theta$.

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{\sin\theta}{\cos\theta} \times \frac{1}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \sec\theta$$

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Now area of triangle OAB < *area of sector* OAB < *area of triangle* OCB

$$\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$$

We can divide by $\frac{1}{2}r^2$ *as it is always positive.*

$$\sin\theta < \theta < \tan\theta$$

Because θ *is a small positive angle,* $\sin\theta$ *is positive. We can therefore divide the inequality by* $\sin\theta$.

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{\sin\theta}{\cos\theta} \times \frac{1}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \sec\theta$$

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Fel mae θ yn agosau at 0, mae $\sec\theta$ yn agosau at 1.

Felly, wrth i θ agosau at 0, mae $\frac{\theta}{\sin\theta}$ yn gorwedd rhwng 1 a rhif sy'n agosau at 1.

Felly, wrth i θ agosau at 0, mae $\frac{\theta}{\sin\theta}$ yn agosau at 1.

Mae hyn yn golygu bod $\sin\theta \approx \theta$ ar gyfer gwerthoedd bach o θ.

Mae'n bosib dangos bod $\tan\theta \approx \theta$ trwy rannu'r anhafaleddau efo $\tan\theta$ (yn lle $\sin\theta$).

Gallwn ddefnyddio'r unfathiant ongl ddwbl $\cos\theta \equiv 1 - 2\sin^2\frac{\theta}{2}$ i ddarganfod brasamcan ar gyfer $\cos\theta$. Os yw $\frac{\theta}{2}$ yn fach, mae

$$\cos\theta \approx 1 - 2\left(\frac{\theta}{2}\right)^2$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}.$$

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As θ *approaches 0,* $\sec\theta$ *approaches 1.*

Therefore, as θ *approaches 0,* $\frac{\theta}{\sin\theta}$ *lies between 1 and a number approaching 1.*

Therefore, as θ *approaches 0,* $\frac{\theta}{\sin\theta}$ *approaches 1.*

This means that $\sin\theta \approx \theta$ *for small values of* θ*.*

It is possible to show that $\tan\theta \approx \theta$ *by dividing the inequality by* $\tan\theta$ *(instead of* $\sin\theta$*).*

We can use the double angle identity $\cos\theta \equiv 1 - 2\sin^2\frac{\theta}{2}$ *to find an approximation for* $\cos\theta$*.*

If $\frac{\theta}{2}$ *is small, then*

$$\cos\theta \approx 1 - 2\left(\frac{\theta}{2}\right)^2$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}.$$

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Ymarfer 1

(a) Os yw θ yn ongl fach, darganfyddwch frasamcan ar gyfer y mynegiad $\frac{\sin 3\theta}{1+\cos 2\theta}$.

(b) Os yw θ yn ongl fach, dangoswch fod $\tan\left(\frac{\pi}{4}+\theta\right) \approx \frac{1+\theta}{1-\theta}$.

(c) Os yw θ yn ddigon bach fel y gallwch anwybyddu θ^2, dangoswch fod $4\sin\left(\frac{\pi}{4}-\theta\right) \approx 2\sqrt{2}(1-\theta)$.

(ch) O wybod bod $1° \approx 0.017$ radian, darganfyddwch werth ar gyfer $\tan(61°)$ heb ddefnyddio'r ffwythiant $\tan$ ar eich cyfrifiannell.

(d) Darganfyddwch werth bach positif o x sydd yn fras ddatrysiad i'r hafaliad $\cos x - 4\sin x = x^2$.

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Exercise 1

(a) *If* θ *is a small angle, find an approximation for the expression* $\frac{\sin 3\theta}{1+\cos 2\theta}$.

(b) *If* θ *is a small angle, show that* $\tan\left(\frac{\pi}{4}+\theta\right) \approx \frac{1+\theta}{1-\theta}$.

(c) *If* θ *is small enough so that you can ignore* θ^2, *show that* $4\sin\left(\frac{\pi}{4}-\theta\right) \approx 2\sqrt{2}(1-\theta)$.

(d) *Given that* $1° \approx 0.017$ *radian, find a value for* $\tan(61°)$ *without using the* tan *function on your calculator.*

(e) *Find a small positive value of* x *which is an approximate solution of the equation* $\cos x - 4\sin x = x^2$.

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Atebion: / *Answers:*

(a) $\frac{3\theta}{2(1-\theta^2)}$

(ch) [*or (d)*] 1.802

(d) [*or (e)*] 0.230 radian