GCE AS/A LEVEL

WJEC GCE AS/A Level in

FURTHER MATHEMATICS

APPROVED BY QUALIFICATIONS WALES

SAMPLE ASSESSMENT MATERIALS

Teaching from 2017

This Qualifications Wales regulated qualification is not available to centres in England.

AS Further Mathematics Unit 1: Further Pure Mathematics A
General instructions for marking GCE Mathematics

1. The mark scheme should be applied precisely and no departure made from it. Marks should be awarded directly as indicated and no further subdivision made.

2. Marking Abbreviations
The following may be used in marking schemes or in the marking of scripts to indicate reasons for the marks awarded.

cao = correct answer only
MR = misread
PA = premature approximation
bod = benefit of doubt
oe = or equivalent
si = seen or implied
ISW = ignore subsequent working

F.T. = follow through ($\surd$ indicates correct working following an error and $\not\surd$ indicates a further error has been made)

Anything given in brackets in the marking scheme is expected but, not required, to gain credit.

3. Premature Approximation
A candidate who approximates prematurely and then proceeds correctly to a final answer loses 1 mark as directed by the Principal Examiner.

4. Misreads
When the data of a question is misread in such a way as not to alter the aim or difficulty of a question, follow through the working and allot marks for the candidates' answers as on the scheme using the new data.
This is only applicable if a wrong value, is used consistently throughout a solution; if the correct value appears anywhere, the solution is not classed as MR (but may, of course, still earn other marks).

5. Marking codes

- 'M' marks are awarded for any correct method applied to appropriate working, even though a numerical error may be involved. Once earned they cannot be lost.
- 'm' marks are dependant method marks. They are only given if the relevant previous 'M' mark has been earned.
- 'A' marks are given for a numerically correct stage, for a correct result or for an answer lying within a specified range. They are only given if the relevant M/m mark has been earned either explicitly or by inference from the correct answer.
- 'B' marks are independent of method and are usually awarded for an accurate result or statement.
- 'S' marks are awarded for strategy
- 'E' marks are awarded for explanation
- 'U' marks are awarded for units
- 'P' marks are awarded for plotting points
- 'C' marks are awarded for drawing curves

AS Further Mathematics Unit 1: Further Pure Mathematics A

Solutions and Mark Scheme

Qu. No.	Solution	Mark	AO	Notes
1.	When $n = 1$, $4^n + 2 = 6$ which is divisible by 6 so the proposition is true for $n = 1$.	B1	AO2	
	Assume the proposition to be true for $n = k$ so that $4^k + 2$ is divisible by 6 and equals $6N$	M1	AO2	
	Consider (for $n = k + 1$)	M1	AO2	
	$4^{k+1} + 2 = 4 \times 4^k + 2$	A1	AO2	
	$= 4 \times (6N - 2) + 2$	A1	AO2	
	$= 24N - 6$	A1	AO2	
	This is divisible by 6 so true for $k \Rightarrow$ true for $k + 1$. Since true for $n = 1$, the result is proved by induction.	A1	AO2	
		[7]		
2.(a)	$\frac{-1+7\mathrm{i}}{2+\mathrm{i}} = \frac{(-1+7\mathrm{i})(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})}$	M1	AO3	
	$= \frac{5+15\mathrm{i}}{5}$	A1	AO3	
	$= 1 + 3\mathrm{i}$	A1	AO3	
	$2(x + \mathrm{i}y) + \mathrm{i}(x - \mathrm{i}y) = 1 + 3\mathrm{i}$	M1	AO3	FT from line above
	$2x + y = 1$	A1	AO3	
	$x + 2y = 3$	A1	AO3	
	$(x = -\frac{1}{3}; y = \frac{5}{3})$			
	$z = \frac{-1+5\mathrm{i}}{3}$	A1	AO1	
(b)	$\lvert z\rvert = \frac{\sqrt{26}}{3} = 1.70 \ (1.699673171)$	B1	AO1	
	$\tan^{-1}(-5) = -1.3734...$	B1	AO1	
	$\arg(z) = -1.3734 + \pi = 1.77 \ (1.768191887)$	B1	AO1	
	$z = 1.70(\cos 1.77 + \mathrm{i}\sin 1.77)$	B1	AO1	
		[11]		

Qu. No.	Solution	Mark	AO	Notes
3.	$S_n = \sum_{r=1}^{n} r(r+1)(r+3)$	M1	AO1	
	$= \sum_{r=1}^{n} (r^3 + 4r^2 + 3r)$	A1	AO1	
	$= \frac{n^2(n+1)^2}{4} + 4\frac{n(n+1)(2n+1)}{6} + 3\frac{n(n+1)}{2}$	A1	AO1	
	$= \frac{n(n+1)}{12}\left(3n(n+1) + 8(2n+1) + 18\right)$	A1	AO1	
	$= \frac{n(n+1)}{12}\left(3n^2 + 19n + 26\right)$	A1	AO1	
	$= \frac{n(n+1)(n+2)(3n+13)}{12}$	A1 **[6]**	AO1	
4.(a)	$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\beta\gamma + \gamma\alpha + \alpha\beta)$	M1	AO2	
	$= 4^2 - 2\times 14$ $= -12$	A1	AO2	
	A cubic equation either has 3 real roots or 1 real root. Since the sum of squares is negative, the 3 roots cannot all be real so there is just 1 real root	B1	AO2	
(b)	A second root is $1 - 3i$,	B1	AO1	
	since complex roots occur in conjugate pairs.	E1	AO2	
	The third root is 2	B1	AO1	
	since the sum of the 2 complex roots is 2 and the sum of the 3 roots is 4,	E1 **[7]**	AO2	

Qu. No.	Solution	Mark	AO	Notes
5.	Putting $z = x + iy$,	M1	AO3	
	$\|x - 3 + iy\| = 2\|x + i(y + 1)\|$	A1	AO3	
	$(x-3)^2 + y^2 = 4\left(x^2 + (y+1)^2\right)$	m1	AO3	
	$x^2 - 6x + 9 + y^2 = 4x^2 + 4y^2 + 8y + 4$			
	$3x^2 + 3y^2 + 6x + 8y - 5 = 0$	A1	AO3	
	This is the equation of a circle			
	$x^2 + y^2 + 2x + \frac{8}{3}y = \frac{5}{3}$	M1	AO1	
	$(x+1)^2 + \left(y + \frac{4}{3}\right)^2 = \frac{5}{3} + 1 + \frac{16}{9}$	m1	AO1	
	Centre = $\left(-1, -\frac{4}{3}\right)$	A1	AO1	
	Radius = $\sqrt{\frac{5}{3} + 1 + \frac{16}{9}}$	M1	AO1	
	= 2.11 $\left(\frac{2\sqrt{10}}{3}\right)$	A1	AO1	
		[9]		

Qu. No.	Solution	Mark	AO	Notes
6.(a)	Reflection matrix = $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	B1	AO1	
	Translation matrix = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$	B1	AO1	
	Rotation matrix = $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	B1	AO1	
	$\mathbf{T} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	M1	AO1	
	$= \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	A1	AO1	
	$= \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	A1	AO1	
(b)	Fixed points satisfy $\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$	M1	AO1	
	$-x+2=x; y+1=y$	A1	AO1	
	The second equation is inconsistent so there are no fixed points.	A1	AO2	
		[9]		

Qu. No.	Solution	Mark	AO	Notes
7.(a)	Putting $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, $u + \mathrm{i}v = (x + \mathrm{i}y)(x + 1 + \mathrm{i}y)$ v = imaginary part = $y(x + 1) + xy$ $= y(1 + 2x)$ u = real part = $x(x + 1) - y^2$	M1 A1 A1 A1	AO1 AO1 AO1 AO1	
(b)	$u = x(x+1) - (x+1)^2$ $= -x - 1$ $v = (x+1)(2x+1)$ $= -u(-2u - 1)$ $= 2u^2 + u$	M1 A1 M1 A1 A1 **[9]**	AO3 AO3 AO3 AO3 AO3	
8.(a)(i)	$\mathbf{AB} = (2\mathbf{i}+3\mathbf{j}+\mathbf{k}) - (\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ $= \mathbf{i}+\mathbf{j}-2\mathbf{k}$	M1 A1	AO1 AO1	
(ii)	Equation of line is $\mathrm{r} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}-2\mathbf{k})$	B1	AO1	
(b)(i)	The line cuts the plane where $1 + \lambda + 3(2 + \lambda) - 2(3 - 2\lambda) = 5$ $\lambda = \frac{1}{2}$ Point of intersection = $\left(\frac{3}{2}, \frac{5}{2}, 2\right)$	M1 A1 A1	AO1 AO1 AO1	
(ii)	Direction of normal = $(1,3,-2)$ If θ denotes the angle between L and the normal, $\cos\theta = \frac{(1,3,-2).(1,1,-2)}{\lvert(1,3,-2)\rvert\,\lvert(1,1,-2)\rvert}$ $= \frac{8}{\sqrt{14}\sqrt{6}}$ $\theta = 29.2(0593\ldots)^{\circ}$ Angle between L and plane = 60.8°. **[12]**	B1 M1 A1 A1 A1 A1	AO1 AO1 AO1 AO1 AO1 AO1	

AS Further Mathematics Unit 2: Further Statistics A

Solutions and Mark Scheme

Qu. No.	Solution	Mark	AO	Notes
1.(a)	$E(W) = E(X)E(Y) = 168$	B1	AO1	
(b)	$E(X^2) = (E(X))^2 + \text{Var}(X)$	M1	AO1	
	$= 221$	A1	AO1	
	$E(Y^2) = 153$	A1	AO1	
	$\text{Var}(W) = E(W^2) - [E(W)]^2$			
	$= E(X^2)E(Y^2) - (E(X)E(Y))^2$	M1	AO3	
	$= 221\times153 - 168^2 (= 5589)$	A1	AO1	
	SD = 74.8 (74.75961….)	A1	AO1	
		[7]		
2(a)	$E(T) = \frac{1}{2500}\int_0^{10} t^2(100 - t^2)\,\mathrm{d}t$	M1	AO3	
	$= \frac{1}{2500}\left[\frac{100t^3}{3} - \frac{t^5}{5}\right]_0^{10}$	A1	AO1	
	$= 5.33(333\ldots)$	A1	AO1	
(b)(i)	$F(t) = \frac{1}{2500}\int_0^{t} u(100 - u^2)\,\mathrm{d}u$	M1	AO3	
	$= \frac{1}{2500}\left[50u^2 - \frac{u^4}{4}\right]_0^{t}$	A1	AO1	
	$= \frac{1}{2500}\left(50t^2 - \frac{t^4}{4}\right)$ (for $0 \le t \le 10$)	A1	AO1	
	$= 1$ for $t > 10$	B1	AO1	Allow omission of $t < 0$
	($F(t) = 0$ for $t < 0$)			
(ii)	$P(T > 5) = 1 - F(5)$	M1	AO3	
	$= 0.563$ (0.5625)	A1	AO1	
(iii)	The median m satisfies $F(m) = 0.5$	M1	AO3	
	$m^4 - 200m^2 + 5000 = 0$	A1	AO3	
	$m^2 = \frac{200 \pm \sqrt{40000 - 20000}}{2}$	A1	AO1	
	(= 29.289...)			
	$m = 5.41(1961\ldots)$	A1	AO1	
		[13]		

Qu. No.	Solution	Mark	AO	Notes
3(a)	The ranks are S A B C D E F G H H 3 6 1 8 7 2 4 5 G 5 6 4 3 8 2 7 1	M1 A1 A1	AO3 AO1 AO1	Attempt to find ranks Correct values for 1st row Correct values for 2nd row
	$\sum d^2 = 64$ $r_s = 1 - \frac{6 \times 64}{8 \times 63}$ = 0.238(095238…)	B1 M1 A1	AO1 AO1 AO1	
(b)	5% 1-tail crit value = 0.6429 This suggests that there is no positive association between marks in History and marks in Geography.	B1 B1	AO1 AO3	
(c)	Because the data might not follow a bivariate normal distribution.	B1 **[9]**	AO2	
4(a)	The evidence suggests that good graduate prospects are associated with: strong research quality high entry standards.	B1 B1	AO2 AO2	Or The evidence suggests that good graduate prospects are not associated with student satisfaction
(b)	Gradient $= \frac{122.72}{1.0542}$ = 116.4(105) Intercept $= \frac{2522}{7} - 116.4105 \times \frac{22.24}{7}$ = -9.5(67) y = 116.4(105) x – 9.5(67)	M1 A1 M1 A1 B1	AO2 AO1 AO2 AO1 AO1	 Allow for using 116.4 giving -9.5(337) FT 'their' gradient and intercept
(c)	116.4 × 3 – 9.6 = 339.6	M1 A1 **[9]**	AO3 AO1	Accept 358.7(988...) if using exact values throughout

Qu. No.	Solution	Mark	AO	Notes
5(a)	H_0 : The data can be modelled by the Poisson distribution with mean 2. H_1 : The data cannot be modelled by the Poisson distribution with mean 2.	B1	AO3	
(b)	The expected frequencies are Goals scored: 0 | 1 | 2 | 3 | 4 or more Obs: 6 | 11 | 15 | 10 | 8 Exp: 6.767 | 13.534 | 13.534 | 9.022 | 7.144	B1 B1	AO3 AO3	For at least 1 correct For all correct
	Use of χ^2 stat = $\sum \frac{O^2}{E} - N$	M1	AO3	
	$= \frac{6^2}{6.767} + \frac{11^2}{13.534} + \ldots + \frac{8^2}{7.144} - 50$	A1	AO2	
	$= 0.93$	A1	AO1	
	DF = 4	B1	AO1	
	5% crit val = 9.488	B1	AO1	
	Since 0.93 < 9.488 (Accept H_0)	B1	AO2	
	We conclude that the data can be modelled by the Poisson distribution with mean 2.	B1 **[10]**	AO3	
6(a)	The number of arrivals X is Poi(7.5)	B1	AO3	
	$P(X = 5) = \frac{e^{-7.5} \times 7.5^5}{5!}$	M1	AO1	
	$= 0.109(3745\ldots)$	A1	AO1	Or straight from the calculator
(b)(i)	$P(T > t)$ = P(No customers arrive between 11am and t mins after 11am) $= e^{-0.5t}$	B1	AO2	
(ii)	The cumulative distribution function of T is $F(t) = P(T \le t)$	M1	AO3	
	$= 1 - P(T > t) = 1 - e^{-0.5t}$	A1	AO2	
	Let $f(t)$ denote the probability density function of T			
	$f(t) = F'(t)$	M1	AO2	
	$= 0.5e^{-0.5t}$	A1	AO1	
(iii)	This is the exponential distribution.	B1	AO2	
	Therefore mean = standard deviation = 1/0.5 = 2	B1 **[10]**	AO2	

<table>
<tr><th>Qu. No.</th><th>Solution</th><th>Mark</th><th>AO</th><th>Notes</th></tr>
<tr><td>7(a)(i)</td><td>H_0: There is no association between parents knowing their child's social media passwords and age of child
H_1: There is an association between parents knowing their child's social media passwords and age of child</td><td>B1</td><td>AO3</td><td>Or H_0: Parents knowing their child's social media passwords is independent of age
H_1: Parents knowing their child's social media passwords is not independent of age</td></tr>
<tr><td>(ii)</td><td>Expected values
<table><tr><th></th><th colspan="3">Age (years)</th></tr><tr><th>Parent knows password</th><th>13</th><th>14</th><th>15</th></tr><tr><td>Yes</td><td>62.79</td><td>78.71</td><td>76.50</td></tr><tr><td>No</td><td>79.21</td><td>99.29</td><td>96.50</td></tr></table>
$142\times\frac{275}{493}=79.21$ OR $275\times\frac{142}{493}=79.21$</td><td>M1
A1</td><td>AO2
AO1</td><td>Or any equivalent correct method</td></tr>
<tr><td>(iii)</td><td>Chi-squared contributions
<table><tr><th></th><th colspan="3">Age (years)</th></tr><tr><th>Parent knows password</th><th>13</th><th>14</th><th>15</th></tr><tr><td>Yes</td><td>2.779</td><td>0.175</td><td>1.180</td></tr><tr><td>No</td><td>2.203</td><td>0.139</td><td>0.935</td></tr></table></td><td>M1
A1</td><td>AO2
AO1</td><td>M1A0 for one correct χ^2 contribution
FT 'their observed values'</td></tr>
<tr><td>(iv)</td><td>2 degrees of freedom from (3 – 1) × (2 – 1)</td><td>B1</td><td>AO1</td><td></td></tr>
<tr><td>(v)</td><td>Since p-value < 0.05, Reject H_0

Strong evidence to suggest there is an association between parents knowing their child's social media passwords and age</td><td>B1
B1

B1
B1</td><td>AO1
AO2

AO2
AO3</td><td>B1 for < 0.05
B1 for Reject H_0

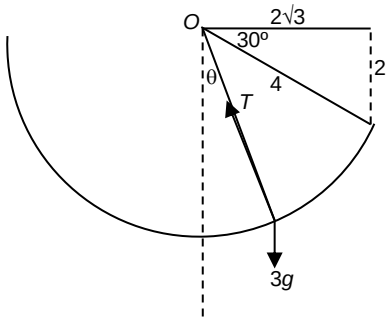
B1 for strong evidence
B1 for relating back to hypothesis</td></tr>
<tr><td>(b)</td><td>Largest contribution for 13-year-olds especially for yes
It seems more parents than expected know passwords for their 13-year-old children.</td><td>E1
E1

[12]</td><td>AO2
AO2</td><td></td></tr>
</table>

AS Further Mathematics Unit 3: Further Mechanics A

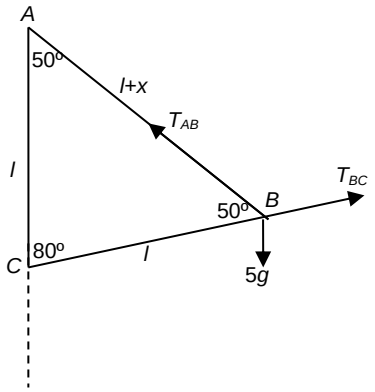
Solutions and Mark Scheme

Question Number	Solution	Mark	AO	Notes
1. (a)	Conservation of momentum	M1	AO3	Dimensionally correct
	$12 \times 600 = 1600 \times v$	A1	AO2	
	$v = \frac{9}{2}$ (ms^{-1})	A1	AO1	allow -ve
(b)	Energy considerations	M1	AO3	
	$E = 0.5 \times 12 \times 600^2 + 0.5 \times 1600 \times 4.5^2$	A1	AO2	both expressions correct, Ft v in (a)
	$E = 2160000 + 16200$			
	$E = \underline{2176200 \text{ (J)}}$	A1	AO1	cao
	Energy dissipated by eg sound of cannon firing ignored.	B1	AO3	oe
	In actual fact, quite a lot of energy would be dissipated as sound or heat or in overcoming the friction in the barrel of the cannon.	E1	AO3	
(c)	Work-energy principle $F \times d = E$	M1	AO3	Used
	$F \times 1.2 = 16200$			
	$F = \underline{13500 \text{ (N)}}$	A1	AO2	cao
	We have not taken into account friction or other resistance to motion which would stop the recoiling part anyway even if no external force is applied.	E1	AO3	
	So the calculated force is an over estimate of the force required.	B1	AO3	
		[12]		

Question Number	Solution	Mark	AO	Notes
2.				
(a)	For *P* falling freely $v^2 = u^2 + 2as$, $u = 0$, $s = 2$, $a = g$ $v^2 = 2g \times 2$ $v = 2\sqrt{g}$	M1 A1	AO3 AO1	
	When string tightens, *P* has vertical speed $2\sqrt{g}$. The component of this along the string is destroyed and *P* begins to move in a vertical circle with initial speed $2\sqrt{g}\cos 30^\circ$ $= 2\sqrt{g} \times \frac{\sqrt{3}}{2} = \sqrt{3g}$	m1 A1	AO3 AO1	
(b) (i)	Conservation of energy $\frac{1}{2}mv^2 = \frac{1}{2}m \times 3g + mg \times 4(\cos 45^\circ - \sin 30^\circ)$ $v^2 = 3g - 4g + 8g\cos 45^\circ$ $v^2 = 45.63717(165)$ $v = 6.76 \text{ ms}^{-1}$ $(6.7555289\ldots)$	M1 A1 A1 A1	AO3 AO1 AO1 AO1	dim correct equation KE PE
(ii)	N2L towards centre $T - mg\cos 45^\circ = \frac{mv^2}{4}$ $T = 3g\left(\frac{1}{\sqrt{2}}\right) + \frac{3}{4}(45.63717)$ $T = 55.02$ $(55.0168\ldots)$	M1 A1 m1 A1 **[12]**	AO3 AO2 AO1 AO1	dim correct equation substitute for v^2

Question Number	Solution	Mark	AO	Notes
3.	$\mathbf{r} = \mathbf{p} + t\mathbf{v}$	M1	AO3	Used
	$\mathbf{r}_A = (1 + 2t)\mathbf{i} + 5t\mathbf{j} - 4t\mathbf{k}$ $\mathbf{r}_B = (3 + t)\mathbf{i} + 3t\mathbf{j} - 5t\mathbf{k}$	A1	AO2	either correct, any form
	$\mathbf{r}_B - \mathbf{r}_A = (2 - t)\mathbf{i} - 2t\mathbf{j} - t\mathbf{k}$	M1	AO3	
	$AB^2 = x^2 + y^2 + z^2$	M1	AO1	
	$AB^2 = (2 - t)^2 + 4t^2 + t^2$	A1	AO1	
	$(AB^2 = 6t^2 - 4t + 4)$			cao
	Differentiate	M1	AO2	
	$\frac{dAB^2}{dt} = 2(2 - t)(-1) + 10t \ (= 12t - 4)$			at least 1 power reduced
	$-4 + 2t + 10t = 0$	m1	AO2	
	$t = \frac{1}{3}$	A1	AO1	equating to 0
	(least distance)^2 = $(2 - \frac{1}{3})^2 + 5(\frac{1}{3})^2$			cao
	least distance = $\sqrt{\frac{10}{3}}$ = <u>1.83 (m)</u>	A1 **[9]**	AO1	cao

Question Number	Solution	Mark	AO	Notes
4. (a)	$\mathbf{v} = \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$	M1	AO2	
	$\mathbf{v} = 2\mathrm{e}^{2t}\,\mathbf{i} + 2\cos(2t)\,\mathbf{j} - 2\sin(2t)\,\mathbf{k}$	A1	AO1	correct differentiation of any one term
		A1	AO1	all correct
	$\mathbf{v}.\mathbf{r} = 2\mathrm{e}^{4t} + 2\cos(2t)\sin(2t) - 2\sin(2t)\cos(2t)$	M1	AO2	
		A1	AO1	correct dot product
	$\mathbf{v}.\mathbf{r} = 2\mathrm{e}^{4t}$ which is never 0 Hence $\mathbf{v}$ and $\mathbf{r}$ are never perpendicular to each other	E1	AO2	
(b)	$v^2 = (2\mathrm{e}^{2t})^2 + (2\cos(2t))^2 + (-2\sin(2t))^2$	M1	AO2	
	$v^2 = 4\mathrm{e}^{4t} + 4\cos^2(2t) + 4\sin^2(2t)$ $v^2 = 4\mathrm{e}^{4t} + 4$	A1	AO1	
(c)	KE $= 0.5 \times 0.4 \times (4\mathrm{e}^{4t} + 4)$ KE $= 0.8(\mathrm{e}^{4t} + 1)$	B1	AO1	
(d)	WD = change in KE	M1	AO1	
	WD $= 0.8(\mathrm{e}^4 + 1) - 0.8(1 + 1)$ WD $= 0.8(\mathrm{e}^4 - 1) = 42.9$ (J)	A1	AO1	
(e)	Rate of work $= \dfrac{\mathrm{d}}{\mathrm{d}t}$ (KE)	M1	AO2	
	Rate of work $= \dfrac{\mathrm{d}}{\mathrm{d}t}(0.8(\mathrm{e}^{4t} + 1))$			
	Rate of work $= 3.2\,\mathrm{e}^{4t}$ (W)	A1	AO1	
		[13]		

Question Number	Solution	Mark	AO	Notes
5.	Resolve vertically $T\cos\theta = mg$ N2L towards centre $T\sin\theta = \frac{mv^2}{r}$ $T\sin\theta = \frac{m\times 4.8^2}{2}$ $\tan\theta = \frac{4.8^2}{2\times 9.8}$ $\theta = 49.61(2371\ldots)^{\circ}$	M1 A1 M1 A1 m1 A1 **[6]**	AO3 AO2 AO3 AO2 AO1 AO1	
6.				
(a)	$AB = 2\times 2\cos 50^{\circ}$ Hooke's Law $T_{AB} = \frac{\lambda}{2}(4\cos 50^{\circ} - 2) = \lambda(2\cos 50^{\circ} - 1)$ $T_{AB} = 0.286\lambda$ (N)	B1 M1 A1	AO3 AO2 AO1	
(b)	$EE = \frac{1}{2}\frac{\lambda\left(4\cos 50^{\circ} - 2\right)^2}{2}$ $EE = 0.0816\lambda$ (J)	M1 A1	AO1 AO1	
(c)	For vertical equilibrium $T_{AB}\cos 50^{\circ} + T_{BC}\cos 80^{\circ} = mg$ $T_{BC}\cos 80^{\circ} = 49 - \lambda 0.286\cos 50^{\circ}$ $T_{BC} = 282.180 - 1.057\lambda$ (N) OR For horizontal equilibrium $T_{AB}\sin 50^{\circ} = T_{BC}\sin 80^{\circ}$ $T_{BC} = \lambda(2\cos 50^{\circ} - 1)\times\frac{\sin 50^{\circ}}{\sin 80^{\circ}}$ $T_{BC} = 0.222\lambda$ (N)	M1 A1 m1 A1 (M1) (A1) (m1) (A1) **[9]**	AO3 AO2 AO1 AO1 (AO3) (AO2) (AO1) (AO1)	resolve vertically Resolve horizontally

Question Number	Solution	Mark	AO	Notes
7.	N2L	M1	AO3	dim correct, all forces
	$T - mg\sin\alpha - R = ma$	A1	AO2	correct equation
	$T = \frac{P}{v}$	B1	AO3	used si
	$\frac{5P}{16} - 6000 \times 9.8 \times \frac{6}{49} - R = 6000 \times 2$	A1	AO1	correct equation in P & R
	$\frac{5P}{16} - R = 19200$			
	N2L with $a = 0$	M1	AO3	dim correct, all forces
	$T - mg\sin\alpha - R = 0$	A1	AO2	correct equation
	$\frac{3P}{16} - 6000 \times 9.8 \times \frac{6}{49} - R = 0$	A1	AO1	correct equation in P & R
	$\frac{3P}{16} - R = 7200$			
	Solving simultaneously	m1	AO1	eliminating one variable, Dep. on both M's
	$\frac{2P}{16} = 12000$			
	$P = 96000; R = 10800$	A1 **[9]**	AO1	both answers cao

A2 Further Mathematics Unit 4: Further Pure Mathematics B

Solutions and Mark Scheme

Qu. No.	Solution	Mark	AO	Notes
1.(a)	$\int_0^{\infty} \frac{\mathrm{d}x}{(1+x)^5} = -\frac{1}{4}\left[\frac{1}{(1+x)^4}\right]_0^{\infty}$	M1	AO1	
	$= -\frac{1}{4}(0-1)$	A1	AO1	
	$= \frac{1}{4}$	A1	AO1	
(b)	$\mathrm{d}u = \frac{\mathrm{d}x}{x}; [2,\infty) \to [\ln 2, \infty)$	B1	AO1	
	Integral $= \int_{\ln 2}^{\infty} \frac{\mathrm{d}u}{u}$	M1	AO1	
	$= [\ln u]_{\ln 2}^{\infty}$	A1	AO1	
	$\to \infty$ because $\ln u \to \infty$	A1	AO1	
		[7]		
2.	Attempting to complete the square	M1	AO3	Award M0 for unsupported working
	Integral $= \int_0^1 \frac{\mathrm{d}x}{\sqrt{2(x+1)^2+4}}$	A1	AO3	
	$= \frac{1}{\sqrt{2}} \int_0^1 \frac{\mathrm{d}x}{\sqrt{(x+1)^2+2}}$	A1	AO3	
	$= \frac{1}{\sqrt{2}}\left[\sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right]_0^1$	A1	AO3	
	$= \frac{1}{\sqrt{2}}\left(\sinh^{-1}\left(\frac{2}{\sqrt{2}}\right) - \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$	A1	AO3	
	$= 0.345 \ \ (0.344882\ldots)$	A1	AO3	
		[6]		

Qu. No.	Solution	Mark	AO	Notes
3.	Area $= \frac{1}{2}\int r^2 \mathrm{d}\theta$	M1	AO1	
	$= \frac{9}{2}\int_0^{\pi}(4 + 4\cos\theta + \cos^2\theta)\mathrm{d}\theta$	A1	AO1	
	$= \frac{9}{2}\int_0^{\pi}\left(\frac{9}{2} + 4\cos\theta + \frac{\cos 2\theta}{2}\right)$	A1	AO1	
	$= \frac{9}{2}\left[\frac{9}{2}\theta + 4\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi}$	A1	AO1	
	$= \frac{81\pi}{4}$	A1 **[5]**	AO1	
4.	$\lvert z\rvert = \sqrt{13}$	B1	AO3	
	$\arg(z) = \tan^{-1}1.5 = 0.98279...$	B1	AO3	
	$z = \sqrt{13}(\cos 0.98279.. + \mathrm{i}\sin 0.98279..)$	M1	AO3	
	First cube root $= 13^{1/6}(\cos 0.32759... + \mathrm{i}\sin 0.32759...)$	m1	AO3	
	$= 1.45 + 0.493\mathrm{i}$	A1	AO3	
	Second cube root $= 13^{1/6}(\cos(0.32759... + 2\pi/3) + \mathrm{i}\sin(0.32759... + 2\pi/3))$	M1	AO3	
	$= -1.15 + 1.01\mathrm{i}$	A1	AO3	
	Third cube root $= 13^{1/6}(\cos(0.32759... + 4\pi/3) + \mathrm{i}\sin(0.32759... + 4\pi/3))$	M1	AO3	
	$= -2.99 - 1.50\mathrm{i}$	A1 **[9]**	AO3	

Qu. No.	Solution	Mark	AO	Notes
5.	Rewrite the equation in the form $\cos 3\theta + 2\cos 2\theta \cos 3\theta = 0$ $\cos 3\theta(1 + 2\cos 2\theta) = 0$ Either $\cos 3\theta = 0$ $3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ Or $\cos 2\theta = -\frac{1}{2}$ $2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$	M1 A1 M1 A1 A1 M1 A1 A1 **[8]**	AO1 AO1 AO1 AO1 AO1 AO1 AO1 AO1	
6.(a)(i)	$\text{adj}(\mathbf{M}) = \begin{bmatrix} 11 & 1 & -7 \\ 1 & 1 & -1 \\ -7 & -1 & 5 \end{bmatrix}$	M1 A1	AO1 AO1	Award M1 if at least 5 correct
(ii)	$\det(\mathbf{M}) = 2\times(15–4) + 1\times(6–5) + 3\times(2–9)$ $= 2$ $\mathbf{M}^{-1} = \frac{1}{2}\begin{bmatrix} 11 & 1 & -7 \\ 1 & 1 & -1 \\ -7 & -1 & 5 \end{bmatrix}$	M1 A1 B1	AO1 AO1 AO1	
(b)	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 11 & 1 & -7 \\ 1 & 1 & -1 \\ -7 & -1 & 5 \end{bmatrix}\begin{bmatrix} 13 \\ 13 \\ 22 \end{bmatrix}$ $= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$	M1 A1 **[7]**	AO1 AO1	

Qu. No.	Solution	Mark	AO	Notes
7.(a)	Let $\dfrac{8x^2+x+5}{(2x+1)(x^2+3)} = \dfrac{A}{2x+1} + \dfrac{Bx+C}{x^2+3}$	M1	AO1	
	$= \dfrac{A(x^2+3)+(Bx+C)(2x+1)}{(2x+1)(x^2+3)}$	A1	AO1	A1 each constant
	$A = 2$, $B = 3$, $C = -1$	A1 A1	AO1 AO1	
(b)	Integral $= \left(\int_2^3 \dfrac{2}{2x+1} + \dfrac{3x}{x^2+3} - \dfrac{1}{x^2+3}\right)\mathrm{d}x$	M1	AO1	Award M0 for work unsupported
	$= \left[\ln(2x+1) + \dfrac{3}{2}\ln(x^2+3) - \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right)\right]_2^3$	A1 A1 A1	AO1 AO1 AO1	A1 each integral
	$= \ln 7 + \dfrac{3}{2}\ln 12 - \dfrac{1}{\sqrt{3}}\tan^{-1}\sqrt{3} - \ln 5 - \dfrac{3}{2}\ln 7 + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$	A1	AO1	
	$= 1.035$	A1	AO1	
		[10]		
8.(a)	Capacity $= \pi\int_1^9 x^2\mathrm{d}y$	M1	AO3	
	$= \pi\int_1^9 (y-1)^{2/3}\mathrm{d}y$	A1	AO3	
	$= \pi\left[\dfrac{3}{5}(y-1)^{5/3}\right]_1^9$	A1	AO3	
	$= \dfrac{3\pi}{5}(32-0)$	A1	AO3	
	$= 60.3(1857\ldots.)$	A1	AO3	
(b)	Capacity $= \pi\int_1^a (y-1)^{2/3}\mathrm{d}y$	M1	AO3	
	$= \pi\left[\dfrac{3}{5}(y-1)^{5/3}\right]_1^a$	A1	AO3	
	$= \dfrac{3\pi}{5}(a-1)^{5/3}$	A1	AO3	
	Attempting to solve $\dfrac{3\pi}{5}(a-1)^{5/3} = 25$	M1	AO3	
	$a = 5.72$ $(5.71610...)$	A1	AO3	
		[10]		

Qu. No.	Solution	Mark	AO	Notes
9.(a)	Putting $n = 1$, the proposition gives $\cos\theta + \mathrm{i}\sin\theta = \cos\theta + \mathrm{i}\sin\theta$ which is true	B1	AO2	
	Let the proposition be true for $n = k$, ie $[\cos\theta + \mathrm{i}\sin\theta]^k = \cos k\theta + \mathrm{i}\sin k\theta$	M1	AO2	
	Consider (for $n = k + 1$) $(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = (\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + i\sin\theta)$	M1	AO2	
	$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$	A1	AO2	
	$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta)$	A1	AO2	
	$= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta$	A1	AO2	
	Therefore true for $n = k \Rightarrow$ true for $n = k + 1$ and since true for n =1 the proposition is proved by induction.	A1	AO2	
(b)(i)	Consider $\cos 5\theta + \mathrm{i}\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5$	M1	AO2	
	$= \mathrm{i}(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$ + real terms It follows equating imaginary terms that	A1	AO2	
	$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$	A1	AO2	
	$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$	A1	AO2	
	$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$	A1	AO2	
(ii)	$\dfrac{\sin 5\theta}{\sin\theta} = 16\sin^4\theta - 20\sin^2\theta + 5$	M1	AO1	
	$\to 5$ as $\theta \to 0$	A1	AO1	
		[14]		

Qu. No.	Solution	Mark	AO	Notes
10.(a)	Integrating factor $= e^{\int 2\tan x dx}$	M1	AO1	
	$= e^{2\ln\sec x}$	A1	AO1	
	$= e^{\ln\sec^2 x}$	A1	AO1	
	$= \sec^2 x$	A1	AO1	
(b)	Applying the integrating factor,			
	$\sec^2 x\frac{dy}{dx} + 2y\tan x\sec^2 x = \sin x\sec^2 x$	M1	AO1	
	$= \frac{\sin x}{\cos^2 x}$ (or $\sec x\tan x$)	A1	AO1	
	Integrating,			
	$y\sec^2 x = \sec x + C$	A1 A1	AO1 AO1	A1 each side
	$0 = \sqrt{2} + C$	M1	AO1	
	$C = -\sqrt{2}$	A1	AO1	
	The solution is $y = \cos x - \sqrt{2}\cos^2 x$	A1 **[11]**	AO1	

Qu. No.	Solution	Mark	AO	Notes
11.(a)	Let $y = \tanh^{-1} x$ so $x = \tanh y$			
	$= \dfrac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$	M1	AO2	
	$xe^{y} + xe^{-y} = e^{y} - e^{-y}$	A1	AO2	
	$e^{2y} = \dfrac{1+x}{1-x}$	A1	AO2	
	$y = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	A1	AO2	
(b)	$a\cosh x + b\sinh x \equiv r\cosh(x + \alpha)$	M1	AO2	
	$= r\cosh x\cosh\alpha + r\sinh x\sinh\alpha$	A1	AO2	
	Equating like terms,			
	$r\cosh\alpha = a$			
	$r\sinh\alpha = b$	A1	AO2	
	Dividing,			
	$\tanh\alpha = \dfrac{b}{a}$	M1	AO2	
	$\alpha = \tanh^{-1}\left(\dfrac{b}{a}\right)$	A1	AO2	
	$= \dfrac{1}{2}\ln\left(\dfrac{1+b/a}{1-b/a}\right) = \dfrac{1}{2}\ln\left(\dfrac{a+b}{a-b}\right)$			
	Squaring and subtracting the above equations,			
	$r^{2}(\cosh^{2}\alpha - \sinh^{2}\alpha) = a^{2} - b^{2}$	M1	AO1	
	$r = \sqrt{a^{2} - b^{2}}$	A1	AO1	
(c)	Here $r = 3$	B1	AO1	
	$\alpha = \dfrac{1}{2}\ln 9 = \ln 3$	B1	AO1	
	The equation simplifies to $3\cosh(x + \ln 3) = 10$	B1	AO1	
	$x + \ln 3 = (\pm)\cosh^{-1}\left(\dfrac{10}{3}\right)$	M1	AO1	
	$x = 0.775$	A1	AO1	
	or $x = -2.97$	A1	AO1	
		[17]		

Qu. No.	Solution	Mark	AO	Notes
12.(a)	$f'(x) = e^x \cos x - e^x \sin x$	B1	AO2	
	$f''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x$	B1	AO2	
	$= -2e^x \sin x$			
(b)	$f'''(x) = -2e^x \sin x - 2e^x \cos x$	B1	AO1	
	$f^{(4)}(x) = -2e^x \sin x - 2e^x \cos x - 2e^x \cos x + 2e^x \sin x$	B1	AO1	
	$(= -4e^x \cos x)$			
	$f(0) = 1,\ f'(0) = 1,\ f''(0) = 0$	B1	AO1	
	$f'''(0) = -2,\ f^{(4)}(0) = -4$	B1	AO1	
	The Maclaurin series is			
	$e^x \cos x = 1 + x - \frac{2x^3}{6} - \frac{4x^4}{24} + \ldots$	M1	AO1	
	$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \ldots$	A1	AO1	
(c)	Valid attempt at differentiating both sides,	M1	AO1	
	$e^x \cos x - e^x \sin x = 1 - x^2 - \frac{2x^3}{3} + \ldots$	A1	AO1	
	$e^x \sin x = 1 + x - \frac{x^3}{3} - 1 + x^2 + \frac{2x^3}{3} + \ldots$	A1	AO1	
	$= x + x^2 + \frac{x^3}{3} + \ldots$	A1	AO1	
(d)	Replacing $e^x \sin x$ by its series,			
	$10\left(x + x^2 + \frac{x^3}{3}\right) - 11x = 0$	M1	AO3	
	$10x^3 + 30x^2 - 3x = 0$	A1	AO3	
	$x = \frac{-30 + \sqrt{900 + 120}}{20}$	m1	AO3	
	$= 0.097$	A1	AO3	
		[16]		

A2 Further Mathematics Unit 5: Further Statistics B

Solutions and Mark Scheme

Qu. No.	Solution	Mark	AO	Notes
1(a)(i)	Upper quartile = $\mu + 0.6745\sigma$ = 32 + 0.6745 × 4 = 34.7 This is the time that is exceeded on 25% of the days.	M1 A1 E1	AO3 AO1 AO2	
(ii)	Let $T = X_1 + X_2 + X_3 + X_4 + X_5$ Then $E(T) = 160$ $\text{Var}(T) = 5\text{Var}(X)$ $\text{Var}(T) = 80$ $P(T > 170) = 0.132$	 B1 M1 A1 B1	 AO3 AO3 AO1 AO1	
(b)	Consider $U = X - 2Y$ $E(U) = -4$ $\text{Var}(U) = \text{Var}(X) + 4\text{Var}(Y)$ = 32 We require P(U > 0) = 0.240	M1 A1 M1 A1 M1 A1 **[13]**	AO3 AO1 AO3 AO1 AO3 AO1	
2(a)	$\Sigma x = 691,\ \Sigma x^2 = 47762.32$ $\hat{\mu} = 69.1$ $s^2 = \sum \frac{x^2}{n-1} - \frac{\left(\sum x\right)^2}{n(n-1)}$ = 1.58 DF = 9 t value = 2.262 Standard error = $\frac{s}{\sqrt{n}} = \frac{\sqrt{1.58}}{\sqrt{10}}$ Confidence limits = $\bar{x} \pm t \times \frac{s}{\sqrt{n}}$ $= 69.1 \pm 2.262 \times \frac{\sqrt{1.58}}{\sqrt{10}}$ leading to [68.2,70.0]	 B1 M1 A1 B1 B1 B1 M1 A1 A1	 AO1 AO3 AO1 AO1 AO1 AO1 AO3 AO1 AO1	
(b)	The value of μ either lies in the interval or it does not, there is no question of a probability being involved. EITHER The confidence interval is an observed value of a random interval which contains μ with probability 0.95. OR If the process is carried out a large number of times, we would expect 95% of the confidence intervals obtained to contain μ. **[11]**	E1 E1	AO2 AO2	

Qu. No.	Solution	Mark	AO	Notes
3(a)	H_0 : The petrol consumptions of models A and B are the same	B1	AO3	B0 for saying that the **mean** petrol consumption is the same
	H_1 : The petrol consumptions of models A and B are not the same	B1	AO3	For correctly identifying the alternative hypothesis as two-sided
(b)	From tables upper crit value = 31 Therefore lower crit value = 36 – 31 = 5 The critical region is $(U \geq 31) \cup (U \leq 5)$	B1 B1 B1	AO1 AO2 AO2	
(c)	Use of the formula $U = \sum \sum z_{ij}$ U = 1 + 6 + 2 +6 + 6 + 3 = 24	M1 A1	AO3 AO1	
	The conclusion is that there is no difference in petrol consumption of the two models because 24 is not in the critical region.	B1 B1 **[9]**	AO3 AO2	
4(a)	$\hat{p} = \frac{1242}{1800} = 0.69$	B1	AO3	
	ESE = $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $= \sqrt{\frac{0.69 \times 0.31}{1800}}$ = 0.0109(0107…)	M1 A1	AO1 AO1	
	95% confidence limits are $\hat{p} \pm z \times \text{ESE}$ $0.69 \pm 1.96 \times 0.0109..$ giving [0.669,0.711]	M1 A1 A1	AO3 AO2 AO1	
(b)(i)	$\hat{p} = \frac{0.672 + 0.732}{2} = 0.702$ Number of people = 0.702 × 1000 = 702	B1 B1	AO3 AO1	
(ii)	$0.732 - 0.672 = 2z\sqrt{\frac{0.702 \times 0.298}{1000}}$ z = 2.07417.. Prob from tables = 0.98077 or 0.98097 from calc Confidence level = 96.2%	M1 A1 A1 A1 **[12]**	AO3 AO1 AO1 AO2	

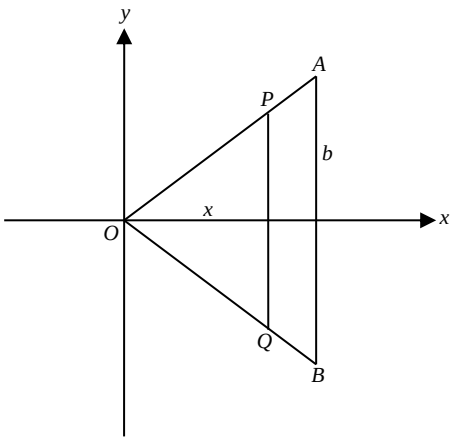
Qu. No.	Solution	Mark	AO	Notes
5(a)	$H_0 : \mu_M = \mu_F; H_1 : \mu_M \neq \mu_F$	B1	AO3	
(b)	Let X= male weight, Y=female weight			
	$(\sum x = 39.2; \sum y = 46.6)$			
	$\bar{x} = 4.9;$	B1	AO1	
	$\bar{y} = 4.66$	B1	AO1	
	SE of diff of means= $\sqrt{\frac{0.5^2}{8} + \frac{0.5^2}{10}}$	M1	AO2	
	$= 0.237\ldots$	A1	AO1	Award m0 if no working seen
	Test statistic = $\frac{4.9 - 4.66}{0.237...}$	m1	AO1	
	$= 1.01$	A1	AO1	From calculator, prob = 0.1558 FT 'their' test statistic
	Prob from tables = 0.1562	A1	AO1	From calculator, p-value = 0.3116
	p-value = 0.3124	B1	AO2	FT 'their' p-value
	Insufficient evidence to conclude that there is a difference in mean weight between males and females.	B1	AO3	
		[10]		
6(a)	The differences are 5 –2 8 10 –6 12 –4 7 9 1	B1	AO3	
	The signs may be omitted at this stage.			
	The ranks are 4 2 7 9 5 10 3 6 8 1	M1 A1	AO3 AO1	Attempting to rank absolute values All correct
	W = Sum of positive ranks	M1	AO3	
	= 4 + 7 + 9 +10 + 6 + 8 + 1 = 45	A1	AO1	
	The critical value is 44.	B1	AO1	
(b)	The conclusion at this significance level is that Method B gives on average a higher reading than Method A	B1	AO3	
	because 45 > 44	E1	AO2	
		[8]		

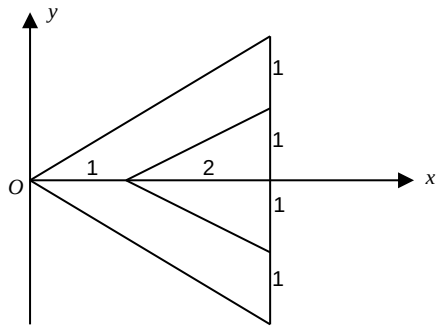
Qu. No.	Solution	Mark	AO	Notes
7(a)	$E(X) = \theta + 3(1-3\theta) + 5\times 2\theta$ $= 2\theta + 3$ Var(X) = $\theta + 9(1-3\theta) + 25\times 2\theta - (2\theta+3)^2$ $= \theta + 9 - 27\theta + 50\theta - 4\theta^2 - 12\theta - 9$ $= 4\theta(3-\theta)$	M1 A1 M1 A1	AO1 AO1 AO2 AO2	
(b)(i)	Consider $E(V) = \dfrac{E(\bar{X})-3}{2}$ $= \dfrac{2\theta+3-3}{2} = \theta$ (Therefore V is unbiased)	M1 A1	AO2 AO2	
(ii)	Var(V) = $\dfrac{\text{Var}(\bar{X})}{4}$ $= \dfrac{\theta(3-\theta)}{n}$	M1 A1	AO3 AO1	
(c)	Y is B(n,θ) So $E(Y) = n\theta$ $E(W) = E\left(\dfrac{Y}{n}\right) = \theta$ (Therefore W is unbiased) Var(W) = $\dfrac{\text{Var}(Y)}{n^2}$ $= \dfrac{\theta(1-\theta)}{n}$	M1 A1 A1 M1 A1	AO3 AO2 AO2 AO2 AO1	
(d)	$\dfrac{\text{Var}(V)}{\text{Var}(W)} = \dfrac{\theta(3-\theta)}{n} \div \dfrac{\theta(1-\theta)}{n}$ $= \dfrac{(3-\theta)}{(1-\theta)}$ It follows that W is the better estimator since it has the smaller variance **[17]**	M1 A1 B1 B1	AO3 AO1 AO2 AO2	

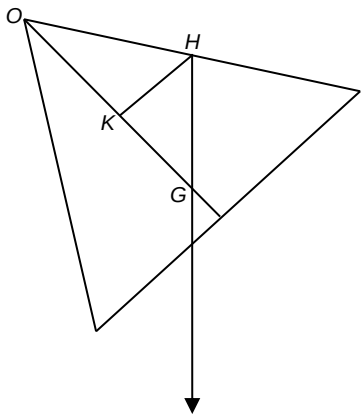
A2 Further Mathematics Unit 6: Further Mechanics B

Solutions and Mark Scheme

Question Number	Solution	Mark	AO	Notes
1. (a)	N2L on ball, upwards positive $-0.01v^2 - 0.4g = 0.4a$	M1 A1	AO3 AO2	dim correct correct equation
	$0.4v\dfrac{dv}{dx} = -3.92 - 0.01v^2$			
	$40v\dfrac{dv}{dx} = -(392 + v^2)$	A1	AO2	convincing
(b)	$40\displaystyle\int \frac{v}{392+v^2}\,dv = -\int dx$	M1	AO2	separate variables
	$20\ln(392 + v^2) = -x + C$	A1 A1	AO1 AO1	$\ln(392 + v^2)$ everything correct
	When $t = 0$, $v = 17$, $x = 0$ $20\ln(392 + 17^2) = C$ $C = 20\ln(681)$	m1 A1	AO2 AO1	use of initial conditions
	$x = 20\ln(681) - 20\ln(392 + v^2)$ $x = 20\ln\left(\dfrac{681}{392+v^2}\right)$ $\dfrac{x}{20} = \ln\left(\dfrac{681}{392+v^2}\right)$			
	$\left(\dfrac{681}{392+v^2}\right) = e^{0.05x}$ $681 = (392 + v^2)\,e^{0.05x}$	m1	AO1	
	$v^2 = 681e^{-0.05x} - 392$			
	$v = \sqrt{681e^{-0.05x} - 392}$	A1	AO1	
(c)	At greatest height $v = 0$	M1	AO2	
	$x = 20\ln\left(\dfrac{681}{392}\right) = 11.05$	A1	AO1	cao
(d)	Speed of ball when it returns to O is less than 17 ms^{-1}. This is because energy is lost in overcoming air resistance.	B1 E1	AO2 AO2	
		[14]		

Question Number	Solution	Mark	AO	Notes
2. (a)				
	Let ρ be mass per unit volume. By symmetry, c of m lies on Ox. Divide cone into slices parallel to base. Consider slice PQ, distance x from O and of thickness δx.	M1	AO2	
	By similar triangles, radius of slice is $\frac{bx}{h}$.			
	Mass of slice = $\frac{\pi b^2 x^2}{h^2}\rho\delta x$ acting x from O.			
	Mass of cone = $\frac{\pi b^2 h}{3}\rho$ acting at $\bar{x}$ from O.			
	Take moments about y axis	m1	AO2	
	$\frac{\pi b^2 h}{3}\rho\bar{x} = \int_0^h \frac{\pi b^2 x^2}{h^2}\times x\rho\,\mathrm{d}x$	A1	AO2	
	$\frac{1}{3}h\bar{x} = \frac{1}{h^2}\left[\frac{1}{4}x^4\right]_0^h$			
	$\bar{x} = \frac{3}{h^3}\frac{h^4}{4}$			
	$\bar{x} = \frac{3h}{4}$	A1	AO2	

Question Number	Solution	Mark	AO	Notes
2 (b)				
	Shape mass distance			
	C_1 $\frac{\pi}{3}(2)^2\times 3\rho$ $\frac{3}{4}\times 3$	B1	AO1	
	C_2 $\frac{\pi}{3}\times 1^2\times 2\rho$ $1+\frac{3}{4}\times 2$	B1	AO1	
	Rem. $\frac{\pi}{3}\rho(12-2)$ $\bar{h}$	B1	AO1	
	Take moments about y axis	M1	AO3	
	$\frac{\pi}{3}\rho\times 10\times\bar{h} = \frac{\pi}{3}\times 12\times\rho\times\frac{9}{4}$			
	$-\frac{\pi}{3}\times 2\rho\times\frac{5}{2}$	A1	AO1	
	$\bar{h}=\frac{11}{5}$	A1	AO1	

Question Number	Solution	Mark	AO	Notes
2. (c)	 Draw *HK* perpendicular to *OG*. *OH* = $\frac{\sqrt{13}}{3}$, *OG* = $\frac{11}{5}$ Angle *HOK* = θ, $\tan\theta = \frac{2}{3}$ $\sin\theta = \frac{2}{\sqrt{13}}$, $\cos\theta = \frac{3}{\sqrt{13}}$ $HK = OH\sin\theta = \frac{\sqrt{13}}{3} \times \frac{2}{\sqrt{13}} = \frac{2}{3}$ $KG = \frac{11}{5} - OH\cos\theta = \frac{11}{5} - \frac{\sqrt{13}}{3} \times \frac{3}{\sqrt{13}}$ $KG = \frac{6}{5}$ $\tan\alpha = \frac{2}{3} \div \frac{6}{5} = \frac{2}{3} \times \frac{5}{6} = \frac{5}{9}$	B1 B1 B1 B1 B1 **[15]**	AO3 AO3 AO3 AO3 AO3	

Question Number	Solution	Mark	AO	Notes
3. (a)	Using N2L	M1	AO3	
	$-0.2 - 0.03v = 9\dfrac{\mathrm{d}v}{\mathrm{d}t}$	A1	AO2	
	$900\dfrac{\mathrm{d}v}{\mathrm{d}t} = -(20 + 3v)$	A1	AO2	
(b)	$900\displaystyle\int \frac{\mathrm{d}v}{20+3v} = -\int \mathrm{d}t$	M1	AO2	sep. var.
	$900 \times \dfrac{1}{3}\ln(20 + 3v) = -t\ (+ C)$	A1 A1	AO1 AO1	$\ln(20 + 3v)$ all correct
	When t =0, v = 20 C = 300 ln 80	m1	AO2	used
	Therefore $t = 300\ln(80) - 300\ln(20 + 3v)$ $t = 300\ln\left(\dfrac{80}{20+3v}\right)$	A1	AO1	
(c)	When body is at rest, v=0 T = 300 ln(80) – 300 ln(20) $T = 300\ln(4)$	m1	AO2	used
	T = <u>416 s</u>	A1	AO1	cao
		[10]		

Question Number	Solution	Mark	AO	Notes
4. (a)	$\bar{x}$ = 4 (cm)	B1	AO1	
(b)	Shape mass distance			
	GPQE 64 4	B1	AO1	
	EFG 8π $8+\frac{16}{3\pi}$	B1	AO3	
	APB π $\frac{8}{3\pi}$			
	CQD π $\frac{8}{3\pi}$	B1	AO1	either *APB* or *CQD*
	ABCDEFG $64+6\pi$ $\bar{y}$	B1	AO1	areas
	Moments about *BC*	M1	AO3	
	$(64+6\pi)\bar{y} = 64\times 4 + 8\pi\times(8+\frac{16}{3\pi})$			
	$-2\pi\times\frac{8}{3\pi}$	A1	AO1	
	$\bar{y}$ = 5.967 (cm) (correct to 3 d.p.)	A1	AO1	
(c)	If hanging in equilibrium, vertical passes through centre of mass.	M1	AO3	correct triangle
	$\theta = \tan^{-1}\left(\frac{8-5\cdot 967}{4}\right)$	A1	AO1	
	θ = 26.94(1954…)°	A1	AO1	
		[11]		

Question Number	Solution	Mark	AO	Notes
5. (a)	$\mathbf{r}_A = 11\mathbf{i} + 6\mathbf{j} + (2\mathbf{i} + 7\mathbf{j})t$ $\mathbf{r}_B = 7\mathbf{i} + 10\mathbf{j} + (5\mathbf{i} + 4\mathbf{j})t$ If particles collide, $\mathbf{r}_A = \mathbf{r}_B$ for some value of *t.* For **i** component $11 + 2t = 7 + 5t$ $t = \frac{4}{3}$ For **j** component $6 + 7t = 10 + 4t$ $t = \frac{4}{3}$ Since the value for *t* for both components are equal, the particles collide. Conservation of momentum $m(2\mathbf{i} + 7\mathbf{j}) + 2m(5\mathbf{i} + 4\mathbf{j}) = 3m(x\mathbf{i} + y\mathbf{j})$ $12\mathbf{i} + 15\mathbf{j} = 3x\mathbf{i} + 3y\mathbf{j}$ $x = 4, y = 5$ $x\mathbf{i} + y\mathbf{j} = 4\mathbf{i} + 5\mathbf{j}$ (Ns)	M1 A1 M1 A1 A1 M1 A1 m1 A1	AO3 AO1 AO2 AO2 AO2 AO3 AO2 AO2 AO1	
(b)	**I** = change in momentum $\mathbf{I} = 2m(4\mathbf{i} + 5\mathbf{j}) - 2m(5\mathbf{i} + 4\mathbf{j})$ $\mathbf{I} = m(-2\mathbf{i} + 2\mathbf{j})$ $\mathbf{I} = 2m(-\mathbf{i} + \mathbf{j})$ (Ns)	M1 A1	AO3 AO1	used
(c)	Loss in KE = $\frac{1}{2}m(4 + 49) + \frac{1}{2}2\text{m}(25 + 16)$ $- \frac{1}{2} \times 3m(16 + 25)$ Loss in KE = $6m$ (J)	 M1 A1 **[13]**	 AO3 AO1	

Question Number	Solution	Mark	AO	Notes
6. (a)	At equilibrium			
	$12g = \dfrac{\lambda \times 0\cdot 05}{0\cdot 75}$	M1	AO3	use of Hooke's Law
	λ = <u>1764 (N)</u>	A1	AO1	
(b)	Consider a displacement x from the equilibrium position. Apply N2L			
	$12g - T = 12\,\ddot{x}$	M1	AO3	
	$12g - \dfrac{\lambda(0\cdot 05 + x)}{0\cdot 75} = 12\ \ddot{x}$	A1	AO3	ft λ
	$\ddot{x} = -(14)^2 x$			
	Therefore is SHM (with $\omega = 14$).	A1	AO2	
	Amplitude = <u>0.05 (m)</u>	B1	AO1	
	Period = $\dfrac{2\pi}{\omega} = \dfrac{\pi}{7}$ s	B1	AO1	
(c)	Maximum speed = $a\omega$	M1	AO3	used
	$= 0.05 \times 14$			
	= <u>0.7 (ms^{-1})</u>	A1	AO1	ft a
(d)	Use of $v^2 = \omega^2(a^2 - x^2)$			
	with $\omega = 14$, $a = 0.05$(c), $x = 0.03$	M1	AO3	
	$v^2 = 14^2(0.05^2 - 0.03^2)$	A1	AO2	ft a
	$=14^2 \times 0.04^2$			
	v = <u>0.56 (ms^{-1})</u>	A1	AO1	cao
(e)	Displacement from Origin = x			
	$x = 0.05\cos(14t)$	M1	AO3	(Accept ±)
	When $t = 1.6$			
	$x = 0.05\cos(14 \times 1.6)$	A1	AO2	ft a (Accept ±)
	$x = (-)$<u>0.046 (m)</u>	A1	AO1	cao
(f)	The seat is modelled as a particle.	B1	AO3	
	The spring is assumed to be light.	B1	AO3	
		[17]		

APPENDIX

ASSESSMENT OBJECTIVE WEIGHTINGS

GCE FURTHER MATHEMATICS

Level	AO1	AO2	AO3	TOTAL
AS	108	49	53	210
	51%	23%	25%	
Total mark for assessment objectives must be in the range	95 - 115	42 - 63	42 - 63	
	(45% - 55%)	(20% - 30%)	(20% - 30%)	

Level	AO1	AO2	AO3	TOTAL
A2 - Unit 4 & Unit 5	103	43	54	200
	52%	22%	27%	
Total mark for assessment objectives must be in the range	90 - 110	40 - 60	40 - 60	
	(45% - 55%)	(20% - 30%)	(20% - 30%)	

Level	AO1	AO2	AO3	TOTAL
A2 - Unit 4 & Unit 6	101	47	52	200
	51%	24%	26%	
Total mark for assessment objectives must be in the range	90 - 110	40 - 60	40 - 60	
	(45% - 55%)	(20% - 30%)	(20% - 30%)	

Level	AO1	AO2	AO3	TOTAL
A LEVEL (Unit 5 option)	211	92	107	410
	51%	22%	26%	
Total mark for assessment objectives must be in the range	185 - 225	82 - 123	82 - 123	
	(45% - 55%)	(20% - 30%)	(20% - 30%)	

Level	AO1	AO2	AO3	TOTAL
A LEVEL (Unit 6 option)	209	96	105	410
	51%	23%	26%	
Total mark for assessment objectives must be in the range	185 - 225	82 - 123	82 - 123	
	(45% - 55%)	(20% - 30%)	(20% - 30%)	

AS FURTHER MATHEMATICS
Unit 1: Further Pure Mathematics A (70 marks)

Question Number	AO1	AO2	AO3	TOTAL
1	0	7	0	**7**
2	5	0	6	**11**
3	6	0	0	**6**
4	2	5	0	**7**
5	5	0	4	**9**
6	8	1	0	**9**
7	4	0	5	**9**
8	12	0	0	**12**
9				**0**
10				**0**
				0
				0
				0
				0
TOTAL	**42**	**13**	**15**	**70**
Total mark for assessment objectives must be in the range	39 - 45	11 - 17	11 - 17	

AS FURTHER MATHEMATICS
Unit 2: Further Statistics A (70 marks)

Question Number	AO1	AO2	AO3	TOTAL
1	6	0	1	**7**
2	8	0	5	**13**
3	6	1	2	**9**
4	4	4	1	**9**
5	3	2	5	**10**
6	3	5	2	**10**
7	4	6	2	**12**
8				**0**
9				**0**
10				**0**
				0
				0
				0
				0
TOTAL	**34**	**18**	**18**	**70**
Total mark for assessment objectives must be in the range	28 - 35	16 - 22	16 - 22	

AS FURTHER MATHEMATICS
Unit 3: Further Mechanics A (70 marks)

Question Number	AO1	AO2	AO3	TOTAL
1	2	3	7	**12**
2	7	1	4	**12**
3	4	3	2	**9**
4	8	5	0	**13**
5	2	2	2	**6**
6	5	2	2	**9**
7	4	2	3	**9**
8				**0**
9				**0**
10				**0**
				0
				0
				0
				0
TOTAL	**32**	**18**	**20**	**70**
Total mark for assessment objectives must be in the range	28 - 35	16 - 22	16 - 22	

A2 FURTHER MATHEMATICS
Unit 4: Further Pure Mathematics B (120 marks)

Question Number	AO1	AO2	AO3	TOTAL
1	7	0	0	**7**
2	0	0	6	**6**
3	5	0	0	**5**
4	0	0	9	**9**
5	8	0	0	**8**
6	7	0	0	**7**
7	10	0	0	**10**
8	0	0	10	**10**
9	2	12	0	**14**
10	11	0	0	**11**
11	8	9	0	**17**
12	10	2	4	**16**
				0
				0
				0
				0
				0
				0
TOTAL	**68**	**23**	**29**	**120**
Total mark for assessment objectives must be in the range	63 - 74	20 - 31	20 - 31	

A2 FURTHER MATHEMATICS
Unit 5: Further Statistics B (80 marks)

Question Number	AO1	AO2	AO3	TOTAL
1	6	1	6	**13**
2	7	2	2	**11**
3	2	3	4	**9**
4	6	2	4	**12**
5	6	2	2	**10**
6	3	1	4	**8**
7	5	9	3	**17**
8				**0**
9				**0**
10				**0**
				0
				0
				0
				0
TOTAL	**35**	**20**	**25**	**80**
Total mark for assessment objectives must be in the range	28 - 36	20 - 28	20 - 28	

A2 FURTHER MATHEMATICS
Unit 6: Further Mechanics B (80 marks)

Question Number	AO1	AO2	AO3	TOTAL
1	6	7	1	**14**
2	5	4	6	**15**
3	4	5	1	**10**
4	8	0	3	**11**
5	4	5	4	**13**
6	6	3	8	**17**
7				**0**
8				**0**
9				**0**
10				**0**
				0
				0
				0
				0
TOTAL	**33**	**24**	**23**	**80**
Total mark for assessment objectives must be in the range	28 - 36	20 - 28	20 - 28	