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Pure Mathematics
Unit P3

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Pure Mathematics Unit P3

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PREFACE

This is the last of three books which cover between them most of the mathematical methods required for a modular A level course in Mathematics, whatever Examination Board the candidate is taking. Specifically, the text is based on the P3 specification of the Welsh Joint Education Committee which was introduced in 2001.

The author has many years of experience of examining and is currently the Principal Examiner for Modules P1-P3.

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Chapter 1

Binomial Expansions for Rational Indices

Introduction

In this chapter, the work given in P2 relating to binomial expansions for positive integral indices is extended to the case where the index is a rational number.

1.1 Binomial expansions for positive integral indices: another look

We saw in **P2,** that

$$(a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \cdots + \binom{n}{n}x^n \qquad (1)$$

or

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n, \qquad (2)$$

where

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}, \qquad (3)$$

or equivalently,

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1.2.3.4.\cdots(r-1)r}. \qquad (4)$$

Example 1.1

Find $\binom{7}{4}$ using the form (4) above.

$$\binom{7}{4} = \frac{7.6.5.4}{1.2.3.4} \qquad \text{(four factors on top and bottom)}$$

$$= 35.$$

Example 1.2

Expand $(a+x)^6$, expanding coefficients by means of the formulae above.

Then from (1) and (4):

$$
\begin{aligned}
(a+x)^6 &= a^6 + \frac{6}{1}a^5x + \frac{6(6-1)}{1.2}a^4x^2 + \frac{6(6-1)(6-2)a^3x^3}{1.2.3} \\
&\quad + \frac{6(6-1)(6-2)(6-3)}{1.2.3.4}a^4x^4 \\
&\quad + \frac{6(6-1)(6-2)(6-3)(6-4)}{1.2.3.4.5}ax^5 \\
&\quad + \frac{6(6-1)(6-2)(6-3)(6-4)(6-5)}{1.2.3.4.5.6}a^6 \\
&= a^6 + 6a^5x + \frac{6.5}{1.2}a^4x^2 + \frac{6.5.4}{1.2.3}a^3x^3 \\
&\quad + \frac{6.5.4.3}{1.2.33.4}a^2x^4 + \frac{6.5.4.3.2}{1.2.3.4.5}ax^5 \\
&\quad + \frac{6.5.4.3.2.1}{1.2.3.4.5.6}x^6 \\
&= a^6 + 6a^5 + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6 .
\end{aligned}
$$

We appear to have made "heavy weather" of evaluating the coefficients; after all, these could have been found by means of the ${}^nC_r$ button on most calculators. However, we believe the somewhat laboured approach will prove beneficial in the next section.

To sum up,

$$
\begin{aligned}
(1+x)^n &= 1 + nx + \frac{n(n-1)x^2}{1.2} \\
&\quad + \frac{n(n-1)(n-2)}{1.2.3}x^3 \\
&\quad + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^4 + \ldots \qquad (5)
\end{aligned}
$$

We do not consider $(a + x)^n$ at this stage.

1.2 Binomial expansions for rational indices

Let's consider (5) with the index $n = -1$.

(a) In this case the right hand side of (5) becomes

$$
\begin{aligned}
&1 + (-1)x + \frac{(-1)(-1-1)}{1.2}x^2 \\
&\quad + \frac{(-1)(-1-1)(-1-2)}{1.2.3}x^3 + \frac{(-1)(-1-1)(-1-2)(-1-3)x^4}{1.2.3.4} \\
&\quad + \cdots
\end{aligned}
$$

or, on simplifying,

$$1 - x + x^2 - x^3 + x^4 + \cdots .$$

Exercise 1.1

Find the coefficients of x^5 and x^6.

Using the formula (5) with $n = -1$, $r = 5$ and $n = -1$, $r = 6$, we find the coefficients of x^5 and x^6 are -1 and 1 respectively.

The series

$$1 - x + x^2 - x^3 + x^4 - x^5 + x^6 + \cdots$$

never terminates, coefficients being 1 (for even powers) and -1 (for odd powers).

The series is, in fact, an infinite geometric series with common ratio $r = -x$ and first term 1 whose sum is

$$\frac{1}{1-(-x)} = \frac{1}{1+x},$$

$$\frac{a}{1-r}$$

as long as $|x| < 1$.

(b) We note also that the left hand side of (5) becomes $(1+x)^{-1}$ when $n = -1$.

Combining (a) and (b), we deduce that

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 + \cdots,$$

the result deduced from (5) when $n = -1$.

It appears that the binomial expansion for $(1+x)^n$ is valid when $n = -1$, as long as $|x| < 1$. In fact, the binomial expansion for $(1+x)^n$ is valid when n is <u>any</u> rational number, whatever the value of n, as long as $|x| < 1$.

<u>Summary</u>

It should be noted that

(a) when n is a positive integer the expansion of $(1 + x)^n$ terminates (this is the case considered in **P2**);

(b) when n is a rational number (i.e. of the form $\frac{p}{q}$, where p and q are integers), but not a positive integer, the expansion of $(1 + x)^n$ has an infinite number of terms; and

(c) the expansion is valid when $|x| < 1$.

Example 1.3

Find the first four terms of the binomial expansions for

(a) $(1+x)^{\frac{1}{2}}$

(b) $(1+x)^{-\frac{3}{2}}$

(c) $(1-x)^{-2}$

(d) $(1+2x)^{-\frac{1}{2}}$

and state for which values of x the expansions are valid.

(a) For $(1+x)^{\frac{1}{2}}$, the binomial expansion is valid if $|x| < 1$.

Then setting $n = \frac{1}{2}$ in (5), we obtain

$$\begin{aligned}(1+x)^{\frac{1}{2}} &= 1 + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{1.2}x^2 \\ &\quad + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots,\end{aligned}$$

for $|x| < 1$.

(b) Here $n = -\frac{3}{2}$ and the expansion is again valid for $|x| < 1$.

Then (5) becomes

$$\begin{aligned}(1+x)^{-\frac{3}{2}} &= 1 + \left(-\frac{3}{2}\right)x + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{1.2}x^2 \\ &\quad + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)}{1.2.3}x^3 + \cdots \\ &= 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \frac{35}{16}x^3 + \cdots\end{aligned}$$

(c) Here $n = -2$ but we have $1-x$ instead of $1 + x$. We write $n = -2$ and replace x by $-x$ in (5). Before expanding we note that the expansion is valid for $|-x| < 1$, i.e.. for $|x| < 1$.

Then

$$\begin{aligned}(1-x)^{-2} &= 1 + (-2)(-x) + \frac{(-2)(-2-1)}{1.2}(-x)^2 \\ &\quad + \frac{(-2)(-2-1)(-2-2)}{1.2.3}(-x)^3 \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots\end{aligned}$$

Example 1.5

Expand $\sqrt{\dfrac{1+2x}{1-x}}$ as a series of ascending powers of x up to and including the term in x^2. State the range of validity of the expansion.

Now $\sqrt{\dfrac{1+2x}{1-x}} = (1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$.

Now $(1+2x)^{\frac{1}{2}} = 1+\left(\dfrac{1}{2}\right)(2x)+\dfrac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{1.2}(2x)^2+\cdots$

$$= 1+x-\frac{x^2}{2}+\cdots$$

and $(1-x)^{-\frac{1}{2}} = 1+\left(-\dfrac{1}{2}\right)(-x)+\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1.2}(-x)^2+\cdots$

$$=1+\frac{x}{2}+\frac{3x^2}{8}+\cdots$$

Then $\sqrt{\dfrac{1+2x}{1-x}} = (1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$

$$=\left(1+x-\frac{x^2}{2}+\cdots\right)\left(1+\frac{x}{2}+\frac{3x^2}{8}+\cdots\right)$$

$$=1+\left(\frac{x}{2}+x\right)+\left(\frac{3x^2}{8}+\frac{x^2}{2}-\frac{x^2}{2}\right)+\cdots$$

$$=1+\frac{3x}{2}+\frac{3x^2}{8}+\cdots$$

The range of validity for $(1+2x)^{\frac{1}{2}}$ is

$$|2x|<1 \text{ or } |x|<\frac{1}{2}.$$

The range of validity for $(1-x)^{-\frac{1}{2}}$ is

$$|x|<1.$$

Then the range of validity for $\sqrt{\dfrac{1+2x}{1-x}}$ must satisfy both the above requirements and is therefore $|x|<\dfrac{1}{2}$.

(d) Here $n = -\frac{1}{2}$ and we replace x by $2x$ in the expansion (5).

The expansion is valid for $|2x| < 1$, i.e. for $|x| < \frac{1}{2}$.

Then
$$(1+2x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1.2}(2x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1.2.3}(2x)^3 + \cdots$$
$$= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \cdots$$

It should be noted that (5) relates to $(1+x)^n$ and not $(a+x)^n\,(a \neq 1)$. To expand $(a+x)^n$ we first write $(a+x)^n$ as $a^n\left(1+\frac{x}{a}\right)^n$ and then we use (5) with x replaced by $\frac{x}{a}$.

Example 1.4

Find the expansion of $(2+x)^{-3}$ as far as the term in x^3. State the range of x for which the expansion is valid.

$$\text{Now } (2+x)^{-3} = 2^{-3}\left(1+\frac{x}{2}\right)^{-3}$$
$$= \frac{1}{8}\left[1 + (-3)\frac{x}{2} + \frac{(-3)(-3-1)}{1.2}\left(\frac{x}{2}\right)^2 + \frac{(-3)(-3-1)(-3-2)}{1.2.3}\left(\frac{x}{2}\right)^3 + \cdots\right]$$
$$= \frac{1}{8}\left[1 - \frac{3x}{2} + \frac{3x^2}{2} - \frac{5x^3}{4} + \cdots\right]$$
$$= \frac{1}{8} - \frac{3x}{16} + \frac{3x^2}{16} - \frac{5x^3}{32} + \cdots$$

The expansion is valid when the expansion of $\left(1+\frac{x}{2}\right)^{-3}$ is valid, i.e. when $\left|\frac{x}{2}\right| < 1$ or when $|x| < 2$.

Sometimes it is possible to express a compound function by first expressing the function in terms of more manageable functions. In such cases, care must be taken in expressing the range of validity of the expansion of the compound function.

Example 1.6

Write down the expansion of $\left(1+\frac{x}{8}\right)^{\frac{1}{3}}$ as a series of ascending powers of x up to and including the term in x^2. State the range of validity of the expansion. Use the expansion to show that

$$\sqrt[3]{7} \approx \frac{551}{228}.$$

Now

$$\left(1+\frac{x}{8}\right)^{\frac{1}{3}} = 1+\left(\frac{1}{3}\right)\left(\frac{x}{8}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{1.2}\left(\frac{x}{8}\right)^2 + \ldots$$

$$= 1+\frac{x}{24}-\frac{x^2}{576}+\cdots, \qquad (1)$$

provided $\left|\frac{x}{8}\right| < 1$, i.e. $|x| < 8$.

Now let's write $x = -1$ in (1), noting that this value of x is within the range of validity for the expansion. Then

$$\left(1-\frac{1}{8}\right)^{\frac{1}{3}} = 1-\frac{1}{24}-\frac{1}{576}+\cdots$$

$$\approx \frac{551}{576}$$

$$\therefore \left(\frac{7}{8}\right)^{\frac{1}{3}} \approx \frac{551}{576}$$

so that

$$\sqrt[3]{7} \approx 8^{\frac{1}{3}} \times \frac{551}{576} = 2\times\frac{551}{576} = \frac{551}{288}.$$

Exercises 1.2

1. Obtain the first four terms in the binomial expansion of each of the following, and state the range of values of x for which each is valid.

(a) $(1-4x)^{\frac{1}{2}}$ (b) $(1+2x)^{-2}$ (c) $\frac{1}{1-3x}$

(d) $(3+x)^{\frac{1}{2}}$ (e) $(2+x)^{-2}$ (f) $(9-4x)^{-\frac{1}{2}}$

2. Obtain the expansions up to the term in x^2 of the following, and state the range of values of x for which each is valid.

(a) $(1-x)\sqrt{1+x}$ (b) $\frac{2+x}{\sqrt{1+3x}}$ (c) $\frac{\sqrt{1+2x}}{(1-x)^4}$

(d) $\frac{1}{(1-x)(1-2x)}$

3. Find the first four terms in the expansion of $\left(x^2+\frac{1}{x^2}\right)^{\frac{1}{2}}$ in descending powers of x.

4. Expand $\sqrt{\frac{1+3x}{1-x}}$ as a series of ascending powers of x up to and including the term in x^2. By substituting $x=\frac{1}{10}$ in the expansion, find an approximation for $\sqrt{13}$, giving your answer correct to two decimal places.

5. Expand $\frac{1-2x}{\sqrt{1+2x}}$ in ascending powers of x as far as the term in x^2. The equation

$$\frac{1-2x}{\sqrt{1+2x}}=1-\frac{29x}{10}$$

has a small positive root (amongst others). Use your expansion to find an approximate value of the root.

6. When $(1+ax)^n$ is expanded in ascending powers of x the first three terms of the expansion are $1-2x+7x^2$. Find the values of a and n.

7. Find a suitable binomial expansion to find $\sqrt{0.99}$ correct to four decimal places.

8. Write down the expansion of $\left(1+\frac{x}{9}\right)^{\frac{1}{2}}$ as a series of ascending powers of x up to and including the term in x^2. State the range of values of x for which the expansion is valid.
Use your expansion to show that

$$\sqrt{8}\approx\frac{611}{216}.$$

Chapter 2

Introduction to Rational Functions, Partial Fractions

In this chapter we consider briefly functions whose denominators and numerators are both polynomials. In certain cases, we are able to express such functions in simpler form.

2.1 Rational functions

A rational function of x is a function of the form $\frac{P}{Q}$, where $P(x)$ and $Q(x)$ are polynomials in x.

Example 2.1

(a) $\frac{2x+5}{3x^2+2x+1}$ (b) $\frac{2x^3-3x^2+x+2}{x^2+2x+5}$

(c) $\frac{x^7-x^5+1}{x^7+x^6+4x+5}$ are rational functions

but

(d) $\frac{\sin x+x+3}{4x^2+2x+1}$ (e) $\frac{x^3+2x^2-x+4}{e^x+3x+2}$

are not rational functions.

The rational functions (a), (b) and (c) in Example 2.1 differ in an important respect.

In (a) the degree of the numerator (1) is <u>less</u> than the degree of the denominator (2).

In (b) the degree of the numerator (3) is <u>greater</u> than the degree of the denominator (2).

In (c) the degrees of the numerator and denominator are equal (7).

In this chapter we shall be mainly concerned with rational functions of type (a), where the degree of the numerator in less than that of the denominator. Problems involving rational functions of type (b) and (c) can always be converted into problems involving rational functions of type (a), by division.

Example 2.2

For $\dfrac{x^3+3x^2+2x-5}{x^2-4x+3}$, the degree of the numerator is greater than the degree of the denominator.

We divide out as follows.

$$\begin{array}{r l} & x+7 \\ x^2-4x+3 \big) & x^3+3x^2+2x-5 \\ & \underline{x^3-4x^2+3x} \\ & 7x^2-x-5 \\ & \underline{7x^2-28x+21} \\ & 27x-26 \end{array}$$

We terminate the division when the degree of the remainder (1) is less than the degree of the denominator (2).

Then $\dfrac{x^3+3x^2+2x-5}{x^2-4x+3}=x+7+\dfrac{27x-26}{x^2-4x+3}$.

The rational function on the right hand side is now of type (a). Rational functions of type (a) are said to be <u>proper</u> fractions. Other types of rational functions are said to be <u>improper</u> fractions.

In Example 2.2 we expressed an improper fraction as a polynomial together with a proper fraction. This process is the subject of the following exercises.

Exercises 2.1

Express, by means of division, each of the following improper fractions as a polynomial (or constant) and a proper fraction.

1. (a) $\dfrac{2x^3-3x^2+x+2}{x^2+2x+1}$ (b) $\dfrac{x^4-3x^2+2x+1}{x^3+3x^2+2}$

(c) $\dfrac{2x^3-3x^2+x+1}{x^3+x^2-x-2}$ (d) $\dfrac{12x^4-4x^3+12x^2-7}{4x^3-3x+2}$

In the next section we show that sometimes improper fractions can be expressed more simply in terms of other fractions.

2.2 Partial fractions

As mentioned earlier, we consider proper fractions. Let's start by considering how we combine fractions involving polynomial denominators.

Example 2.3

Express $\dfrac{3}{x+2} + \dfrac{4}{2x-5}$ as one fraction.

We add the fractions in the same way as we add number fractions.

$$\frac{3}{x+2} + \frac{4}{2x-5} \equiv \frac{3(2x-5)+4(x+2)}{(x+2)(2x-5)}$$
$$\equiv \frac{6x-15+4x+8}{(x+2)(2x-5)} \equiv \frac{10x-7}{(x+2)(2x-5)}.$$

> $\frac{2}{7}+\frac{5}{9} = \frac{2 \times 9 + 5 \times 7}{7 \times 9} = \frac{18+35}{63} = \frac{53}{63}$

Note in passing that there is nothing to be gained by multiplying out the bottom factors if you are not asked to do so.

It is often useful to write a complicated proper fraction in terms of simpler or so-called partial fractions. Thus in the above case, we write

$$\frac{10x-7}{(x+2)(2x-5)} \equiv \frac{3}{x+2} + \frac{4}{2x-5}.$$

The $\equiv$ sign indicates that the relationship is an identity which holds for all values of x.

Example 2.4

Write $\dfrac{2x+1}{x^2-x-6}$ in terms of partial fractions.

Note first that the denominator $x^2 - x - 6$ may be factorised as $(x+2)(x-3)$.

Let's write
$$\frac{2x+1}{x^2-x-6} \equiv \frac{A}{x+2} + \frac{B}{x-3}, \qquad (1)$$
where A and B are constants to be determined.

> **These choice of fractions will become clear.**

First, we clear the fractions by multiplying (1) throughout by $x^2 - x - 6$ or $(x+2)(x-3)$.

Then
$$2x + 1 \equiv A(x-3) + B(x+2). \qquad (2)$$

This is an identity, being true for all values of x.
Let's substitute particular values of x to find values for A and B.
Which values of x shall we use?
Let's try $x = -2$ and $x = 3$ in (2).

> **note $\frac{A}{x+2}$ becomes $\frac{A}{x+2}(x+2)(x-3)$ and similarly for $\frac{B}{x-3}$.**

$x = -2$

(2) becomes $2(-2) + 1 = A(-2-3) + B(-2+2)$,

$\therefore \quad -3 = -5A$

and $\quad A = \dfrac{3}{5}$.

> **The choice of $x = -2$ eliminates B and enables us to find the value of A.**

$x = 3$

(2) becomes $\quad 2(3) + 1 = A(3 - 3) + B(3 + 2)$

$\therefore \quad 7 = 5B$ so that $B = \frac{7}{5}$.

Substitution for A and B in (1) then gives

$$\frac{2x+1}{x^2 - x - 6} \equiv \frac{3}{5(x+2)} + \frac{7}{5(x-3)}.$$

In Example 2.4, the factors $x + 2$, $x - 3$ were linear polynomials, i.e. of degree one in x. The constants A and B in (1) were of degree one less than the denominator. This illustrates a general rule in partial fractions: in partial fractions the degree of the numerator (top) is one less than the degree of the bottom (denominator).

The calculation of the constants A and B in Example 2.4, where the denominator was a product of linear factors, can be streamlined. The shorter method is known as the 'cover-up rule'.

Return to Example 2.4

Given $\dfrac{2x+1}{(x+2)(x-3)} = \dfrac{A}{x+2} + \dfrac{B}{x-3}$,

we can find A and B as follows.

To find A, which relates to the $(x + 2)$ factor in the denominator:

(a) We cover up the $(x + 2)$ factor in the original fraction $\dfrac{2x+1}{(x+2)(x-3)}$,

(b) substitute the value of x given by $x + 2 = 0$, i.e. $x = -2$ into what is left uncovered, i.e. into $\dfrac{2x+1}{(x-3)}$. The result is the value of A.

Then $A = \dfrac{2(-2)+1}{-2-3} = \dfrac{-4+1}{-5} = -\dfrac{3}{5} = \dfrac{3}{5}$, as before.

Similarly, to find B we cover up $x - 3$ and substitute $x = 3$ (from $x - 3 = 0$) into what is left uncovered.

Then $B = \dfrac{2(3)+1}{(3+2)} = \dfrac{7}{5}$, as before.

Exercise 2.2

Use (a) the method of Example 2.4,

(b) the 'cover-up' rule

to find the constants A and B in the following:

$$\frac{5-8x}{(2x+1)(2x+3)} \equiv \frac{A}{2x+1} + \frac{B}{2x+3}.$$

N.B. The cover-up rule is only used when there are linear factors in the denominator, none of which are repeated.

The procedure to be adopted when a linear factor is repeated is demonstrated in the next example.

Example 2.5

Express $\dfrac{3x^2+4x+1}{(x-2)^2(x+2)}$ in terms of partial fractions.

We note that $(x-2)^2$ is a repeated linear factor and we allocate two constants to that factor as follows:

$$\frac{3x^2+4x+1}{(x-2)^2(x+2)} \equiv \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} \qquad (1)$$

≡ indicates an identity

In general, the number of constants to be determined is equal to the degree of the original numerator. Here, $(x-2)^2(x+2)$ is of degree 3 so we require 3 constants.

Clear the fractions by multiplying by $(x-2)^2(x+2)$.

$$\therefore 3x^2 + 4x + 1 = A(x-2)(x+2) + B(x+2) + C(x-2)^2. \qquad (2)$$

Do not expand the terms in brackets.

There are some obvious choices for x.

<u>$x = -2$ (eliminate A and B)</u>

$$\therefore\ 3(-2)^2 + 4(-2) + 1 = A(0) + B(0) + C(-2-2)^2$$

$$12 - 8 + 1 = C(-4)^2 = 16C$$

$$C = \frac{5}{16}.$$

<u>$x = 2$ (eliminate A and C)</u>

$$\therefore\ 3(2)^2 + 4(2) + 1 = A(0) + B(2+2) + C(0).$$

$$\therefore\ B = \frac{21}{4}.$$

No further convenient choice of numerical value of x exists. We may choose $x = 0$ or equate coefficients in the left and right hand sides.

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<u>$x = 0$ in (2)</u>

$3(0)^2 + 4(0) + 1 = A(-2)(2) + B(2) + C(-2)^2$

$\therefore \quad 1 = -4A + 2B + 4C$

so $4A = 2B + 4C - 1$

$= 2\left(\frac{21}{4}\right) + 4\left(\frac{5}{16}\right) - 1$

$= \frac{42 + 5 - 4}{4} = \frac{43}{4}.$

$\therefore \quad A = \frac{43}{16}.$

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<u>Equate the values of x^2 in (2)</u>

<u>L.H.S.</u>

$3x^2 = Ax^2 + Cx^2.$

$\therefore \quad A = 3 - C$

$= 3 - \frac{5}{16} = \frac{43}{16}.$

We prefer this method but clearly you use the method which is the more comfortable for you.

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Thus $$\frac{3x^2 + 4x + 1}{(x-2)^2(x+2)} \equiv \frac{43}{16(x-2)} + \frac{21}{4(x-2)^2} + \frac{5}{16(x+2)}.$$

For convenience the decomposition into partial fractions is summarised here.

<u>Form of partial fractions</u>

$$\frac{mx + n}{(ax+b)(cx+d)} \equiv \frac{A}{ax+b} + \frac{B}{cx+d}$$

$$\frac{kx^2 + mx + n}{(ax+b)(cx+d)^2} \equiv \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$$

Partial fractions are sometimes used in integration of rational functions, as will be shown in Chapter 8. At present, we consider the use of partial fractions in other situations.

Example 2.6

(a) Express $\frac{5x^2 + 13x + 9}{(x-1)(x+2)^2}$ in terms of partial fractions.

(b) Given that $y = \frac{5x^2 + 13x + 9}{(x-1)(x+2)^2}$, find the value of $\frac{dy}{dx}$ when $x = 2$.

(a) Let $\frac{5x^2 + 13x + 9}{(x-1)(x+2)^2} \equiv \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$

Clear the fractions by multiplying by $(x-1)(x+2)^2$.

$\therefore 5x^2 + 13x + 9 \equiv A(x+2)^2 + B(x-1)(x+2) + C(x-1).$

<u>Let $x = 1$</u> $\quad \therefore 5 + 13 + 9 = A(1+2)^2 + B(0) + C(0).$

$\therefore 27 = 9A.$

$\therefore A = 3.$

Let $x = -2$ $\quad 5(-2)^2 + 13(-2) + 9 = A(0) + B(0) + C(-2-1)$

$$\therefore 3 = -3C$$

so that $C = -1$.

Equate coefficients of x^2

$$5 = A + B,$$

since $\quad A = 3, B = 2$.

You may wish to write $x = 0$.

$$\therefore \frac{5x^2 + 13x + 9}{(x-1)(x+2)^2} \equiv \frac{3}{(x-1)} + \frac{2}{(x+2)} - \frac{1}{(x+2)^2}.$$

(b) To find $\frac{dy}{dx}$, we differentiate the partial fraction representation of y.

When $x = 2$

Then $$\frac{dy}{dx} = -\frac{3}{(x-1)^2} - \frac{2}{(x+2)^2} + \frac{2}{(x+2)^3}$$

$$= -\frac{3}{(2-1)^2} - \frac{2}{(2+2)^2} + \frac{2}{(2+2)^3}$$

$$= -3 - \frac{2}{16} + \frac{2}{64}$$

$$= -\frac{99}{32}.$$

Example 2.7

(a) Express $\frac{5+4x}{(1+2x)(1-x)}$ in terms of partial fractions.

(b) Expand $\frac{5+4x}{(1+2x)(1-x)}$ in ascending powers of x as far as the term in x^3.

(a) Let $\frac{5+4x}{(1+2x)(1-x)} \equiv \frac{A}{1+2x} + \frac{B}{1-x}$.

By the 'cover-up rule',

$1 + 2x = 0$, $x = -\frac{1}{2}$

$$A = \frac{5 + 4(-\frac{1}{2})}{1-(-\frac{1}{2})} = \frac{3}{2} = 2$$

Similarly,

$$B = \frac{5+4(1)}{1+2(1)} + \frac{9}{3} = 3.$$

$$\therefore \frac{5+4x}{(1+2x)(1-x)} = \frac{2}{1+2x} + \frac{3}{1-x}.$$

(b) Now

$$\begin{aligned}\frac{5+4x}{(1+2x)(1-x)} &= \frac{2}{1+2x} + \frac{3}{1-x} \\ &= 2(1+2x)^{-1} + 3(1-x)^{-1} \\ &= 2(1-2x+4x^2-8x^3+\cdots) \\ &\qquad +3(1+x+x^2+x^3+\cdots) \\ &= 2+3+x(-4+3) \\ &\qquad +x^2(8+3)+x^3(-16+3)+\cdots \\ &= 5-x+11x^2-13x^3+\cdots\end{aligned}$$

Exercises 2.3

1. Express the following in partial fractions:

(i) $\dfrac{2x+3}{(x+2)(x-3)}$ (ii) $\dfrac{x+7}{(x+3)(x+5)}$

(iii) $\dfrac{2x-1}{x^2-4}$ (iv) $\dfrac{3x-1}{(x+2)(2x-1)}$

(v) $\dfrac{2}{(x-1)^2(x-2)}$ (vi) $\dfrac{1}{(1-2x)(1-3x)}$

(vii) $\dfrac{5x^2+6x+7}{(x-1)(x+2)^2}$ (viii) $\dfrac{3x+1}{(x+1)^2}$

2. Find a, b, c where

$$\frac{3x^2+2x+1}{(x-1)(x+2)} = a + \frac{bx+c}{(x-1)(x+2)}$$

and express the fraction on the right hand side in terms of partial fractions.

3. Express $\dfrac{3x^2+2x+1}{x(3x-1)^2}$ in terms of partial fractions.

Hence differentiate $\dfrac{3x^2+2x+1}{x(3x-1)^2}$.

4. Write $\dfrac{4x^2-2x+2}{(x+2)(x-3)}$ as $a+\dfrac{bx+c}{(x+2)(x-3)}$, giving the values of a, b and c.

Express the last fraction in terms of partial fractions.

Hence, or otherwise, differentiate $\dfrac{4x^2 - 2x + 2}{(x+2)(x-3)}$.

5. (a) Expand $\dfrac{1}{(1+2x)(3-x)}$ in terms of partial fractions.

(b) Expand $\dfrac{1}{(1+2x)(3-x)}$ as a series of ascending powers of x as far as the term in x^2, giving the range of values of x for which the expansion is valid.

6. Express $\dfrac{2+x^2}{(2-x)^2(4+x)}$ in terms of partial fractions. Hence expand the function in a series of ascending powers of x as far as the term in x^2. For which range of values of x is the expansion valid?

Chapter 3

Trigonometry I: More Functions and Identities

The trigonometric functions sin, cos, tan and the identity $\sin^{-2}\theta + \cos^{2} \equiv 1$ were considered in **P1**. In this chapter, we consider:

(a) the reciprocal trigonometric functions cosec, sec and cot, or to give them their full names, cosecant, secant and cotangent;

(b) two additional identities;

(c) some inverse trigonometric functions.

3.1 The reciprocal trigonometric functions

Whilst most work in trigonometry is concerned with sin, cos and tan it is useful to be aware of cosec, sec and cot, and their properties.
We define the reciprocal functions by

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta},$$

$$\sec\theta = \frac{1}{\cos\theta},$$

and $$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

Note that cosec is the reciprocal of cos, not of sin.

Hence, or otherwise, differentiate $\dfrac{4x^2 - 2x + 2}{(x+2)(x-3)}$.

5. (a) Expand $\dfrac{1}{(1+2x)(3-x)}$ in terms of partial fractions.

(b) Expand $\dfrac{1}{(1+2x)(3-x)}$ as a series of ascending powers of x as far as the term in x^2, giving the range of values of x for which the expansion is valid.

6. Express $\dfrac{2+x^2}{(2-x)^2(4+x)}$ in terms of partial fractions. Hence expand the function in a series of ascending powers of x as far as the term in x^2. For which range of values of x is the expansion valid?

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3.1 The reciprocal trigonometric functions

Whilst most work in trigonometry is concerned with sin, cos and tan it is useful to be aware of cosec, sec and cot, and their properties.
We define the reciprocal functions by

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta},$$

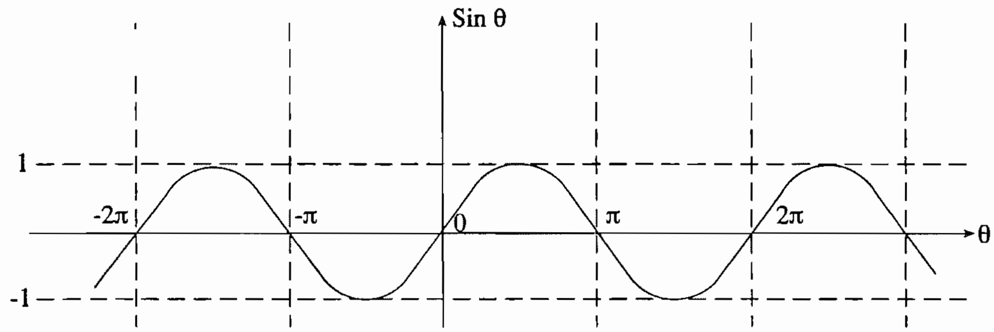
$$\sec\theta = \frac{1}{\cos\theta},$$

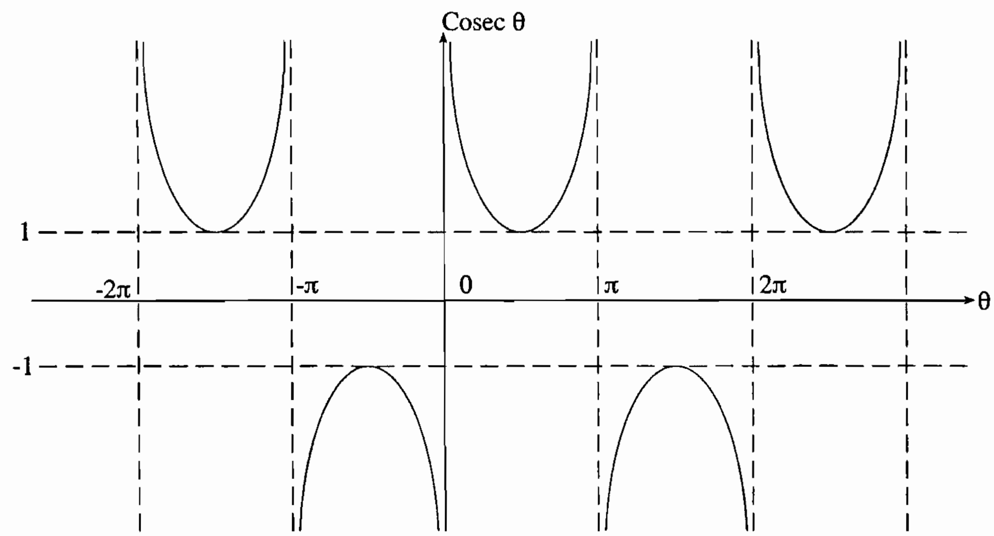
and $$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

Note that cosec is the reciprocal of cos, not of sin.

The graphs of the reciprocals are easily produced after consideration of the graphs for sin, cos and tan.

<u>$y = \sin\theta$ and $y = \text{cosec}\,\theta$</u>



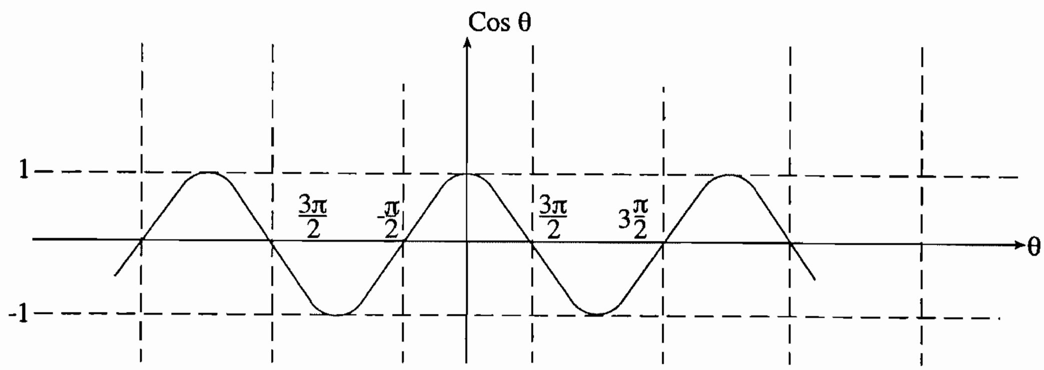


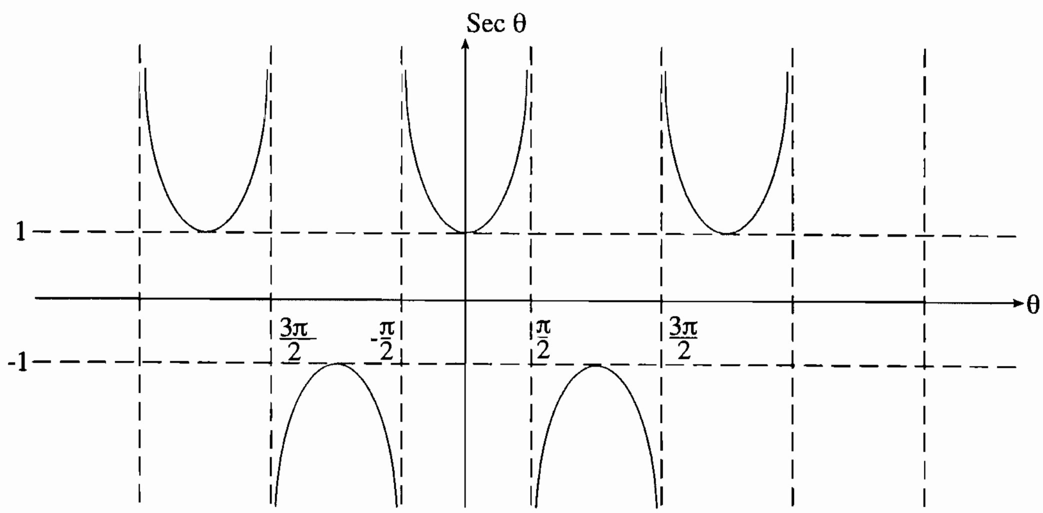
It is observed that $\sin\theta = 0$ when $\theta = k\pi$, where k is an integer. At these values of θ, cosec θ is undefined and $\text{cosec}\,\theta \to +\infty$ or $-\infty$ as $\theta \to k\pi$.

Another important difference between sin and cosec relates to their ranges: the range of sin is $[-1, 1]$; in contrast, the range of cosec is $(-\infty, -1] \cup [1, \infty)$, i.e. cosec cannot take values in the range $(-1, 1)$.

Both sin and cosec are periodic with period 2π.

$y = \cos\theta$ and $y = \sec\theta$





The graph for $y = \sec\theta$ may be obtained from the graph of $y = \operatorname{cosec}\theta$ seen previously by means of a translation $\frac{\pi}{2}$ in the negative θ direction.

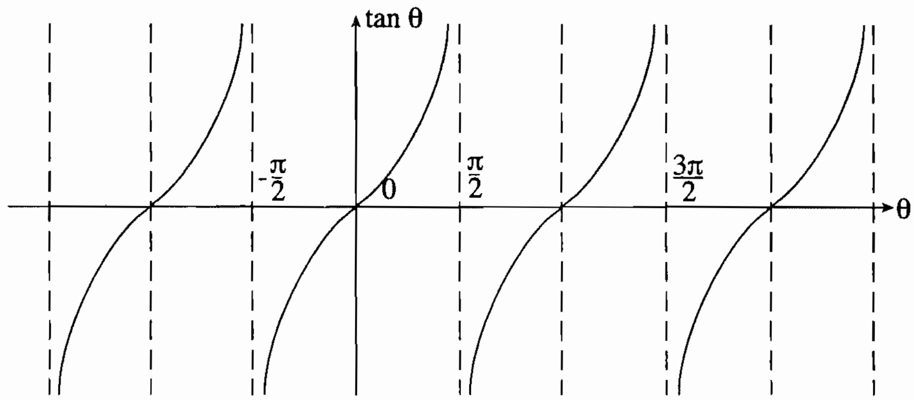
Thus the features of the $y = \operatorname{cosec}\theta$ and $y = \sec\theta$ graphs are broadly the same except for details relating to their positions on the θ axis.

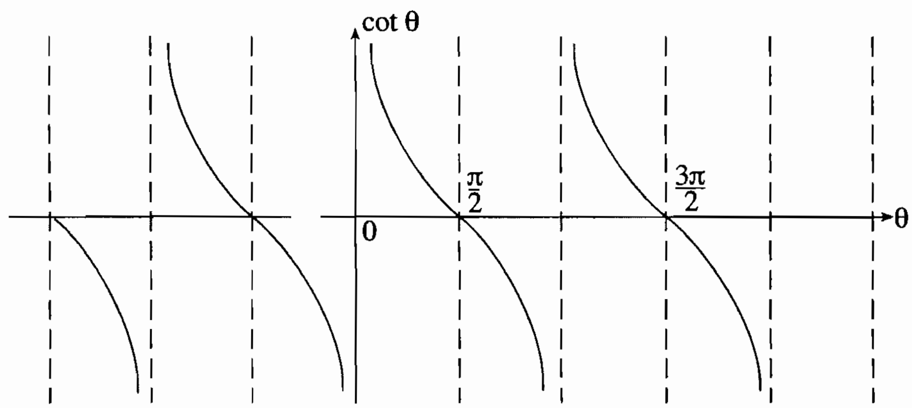
Thus, $\sec\theta$ is undefined at $\theta = (2k+1)\frac{\pi}{2}$ where k is an integer;

and $\sec\theta \to +\infty$ or $-\infty$ as $\theta \to (2k+1)\frac{\pi}{2}$.

As with cosec, the range of sec is $(-\infty, -1] \cup [1, \infty)$ and thus sec cannot take values in the range $(-1, 1)$. Also, sec is periodic with period 2π.

$y = \tan\theta$ and $y = \cot\theta$





From the graphs, it is apparent that whilst tan is discontinuous at $\theta = (2k+1)\frac{\pi}{2}$, where k is an integer, cot is discontinuous at $\theta = k\pi$. The range of cot is $(-\infty, \infty)$.

As with cosec and sec, the cot function is periodic although in contrast with those other functions the period is π.

As mentioned previously, the sin, cos, tan functions are most frequently used in problems; because, for instance, any problem involving cosec θ may be converted into a problem involving sin θ by writing cosec θ $= \dfrac{1}{\sin\theta}$.

However, the reciprocal functions are important, mainly because of their appearance in two trigonometric identities.

3.2 Two more trigonometric identities

In **P1** we showed that $\sin^2\theta + \cos^2\theta = 1$ for any angle θ. Starting with this identity, we are able to derive two further identities.

Dividing $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$, we have

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}.$$

$$\therefore \left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

or

$$\boxed{\tan^2\theta + 1 = \sec^2\theta.}$$

Similarly, dividing $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$, we obtain

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}.$$

$$\therefore 1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1}{\sin\theta}\right)^2$$

or

$$\boxed{1 + \cot^2\theta = \operatorname{cosec}^2\theta.}$$

Example 3.1

Find all the values of θ between 0° and 360° satisfying

$$\sec^2\theta = \tan\theta + 3.$$

Now $\sec^2\theta = 1 + \tan^2\theta$,

as was shown above.

Then $1 + \tan^2\theta = \tan\theta + 3$

or $\tan^2\theta - \tan\theta - 2 = 0$

$\therefore$ $(\tan\theta - 2)(\tan\theta + 1) = 0$.

$\therefore$ $\tan\theta = 2$ or -1.

Then $\theta = 63.4°, 243.4°, 135°, 315°$.

Example 3.2

Find all the values of θ between 0° and 360° satisfying

$$2\cot^2\theta - 4 = \operatorname{cosec}\theta.$$

Now $\qquad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

so that $\qquad 2(\operatorname{cosec}^2\theta - 1) - 4 = \operatorname{cosec}\theta.$

$\therefore \qquad 2\operatorname{cosec}^2\theta - \operatorname{cosec}\theta - 6 = 0.$

$\qquad (2\operatorname{cosec}\theta + 3)(\operatorname{cosec}\theta - 2) = 0$

Then $\qquad \operatorname{cosec}\theta = -\frac{3}{2}$ or 2. $\qquad$ (1)

It is easier to work in terms of sin θ.

Then since $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, we have from (1), $\sin\theta = -\frac{2}{3}$ or $\frac{1}{2}$.

$\therefore \qquad \theta = 221.8°, 318.2°, 30°, 150°.$

Example 3.3

Find $\int \tan^2 x\,dx$.

Now from **P2**, $\int \sec^2 x\,dx = \tan x + C$, as may be checked by differentiation.

Now since $1 + \tan^2 x = \sec^2 x$,

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$$

$$= \tan x - x + C.$$

> **Previously, our angle was θ, but the result holds for any angle.**

Exercises 3.1

You may need to use the factor theorem in some of the following questions.

1. Find all the values of x between 0° and 180° in the following

(a) $\sec^2 x + 1 = 3\tan x$ $\qquad$ (b) $\cot^2 x - 1 = \operatorname{cosec} x$

(c) $\tan^3 x + 4 = \sec^2 x + 3\tan x$.

2. Find all the values of θ between 0° and 360° in the following

$$4 + \tan^2\theta = \sec\theta\,(7 - \sec\theta).$$

3. Find all the values of θ between $0°$ and $360°$ satisfying

(a) $\tan\theta + 2\cot\theta = 3$

(b) $4\sin\theta + \operatorname{cosec}\theta = 4$

(c) $4\cot^2\theta + 39 = 24\operatorname{cosec}\theta$.

4. Find the values of θ between $0°$ and $180°$ satisfying

$$11 + \tan^2 2\theta = 7\sec 2\theta.$$

5. Find all the values of θ between $0°$ and $360°$ satisfying

$$2\tan^3\theta = \sec^2\theta + 13\tan\theta + 5.$$

6. Show that

$$\frac{1+\sin x}{\cos x} + \frac{\cos x}{1+\sin x} = 2\sec x.$$

Find all values of x between $0°$ and $360°$ satisfying

$$\frac{1+\sin x}{\cos x} + \frac{\cos x}{1+\sin x} = \frac{4}{\sqrt{3}}.$$

7. Integrate $\tan^2 2x$ with respect to x.

8. Given that $\cot x = \dfrac{\cos x}{\sin x}$, show that

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$$

Find (a) $\displaystyle\int \operatorname{cosec}^2 2x\,dx$

(b) $\displaystyle\int \cot^2 2x\,dx$

9. Differentiate $\operatorname{cosec} x = \dfrac{1}{\sin x}$ with respect to x, expressing your answer in terms of $\operatorname{cosec} x$ and $\cot x$.

10. (a) Differentiate $\sec x$ with respect to x giving your answer in terms of $\sec x$ and $\tan x$.

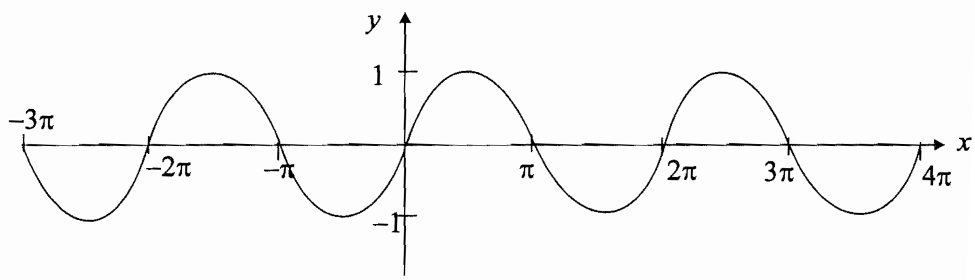
(b) Show that $\dfrac{d}{dx}[\ln(\sec x + \tan x)] = \sec x$.

3.3 Inverse trigonometric functions

Inverse functions in general were considered in **P2.** It was seen there that for a given function to possess an inverse, the function must be one-one. The requirement of one-oneness necessitates some modification of the sin, cos, tan functions when we wish to find inverse functions in those cases.

<u>$\sin^{-1}x$</u>

The graph of $y = \sin x$ for $x \in (-\infty, \infty)$ is as shown where x is measured in radians.



It is clear from the graph that the sine function $f(x) = \sin x$ is not one – one. In other words there is no unique value x corresponding to a given value of $\sin x$.

e.g. if $\sin x = \frac{1}{2}$ then $x = \frac{\pi}{6}, \frac{5\pi}{6}$ etc

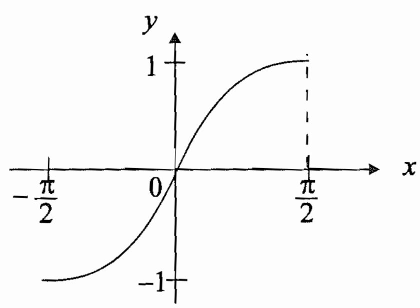
Thus $f(x) = \sin x$ does not have an inverse.

We restrict the domain to $-\frac{\pi}{2}, \frac{\pi}{2}$ and consider

$$f^*(x) = \sin x \qquad \text{for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

We obtain the following graph for $y = f^*(x)$.

Note that we use radian measure in this discussion, <u>not</u> degrees.

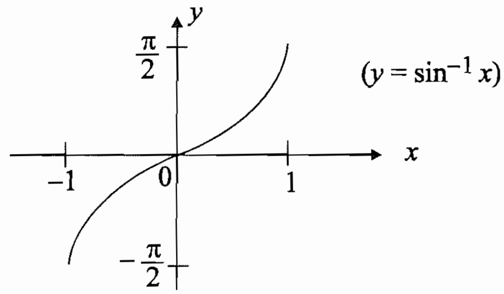


It's easy to see that f* is one–one and therefore has an inverse. We define the inverse of

$$f^*(x) = \sin x \qquad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

to be $f^{*-1}(x) = \sin^{-1} x \qquad x \in [-1, 1].$

The graph of $y = \sin^{-1} x$ is as shown.



This graph is found by reflecting $y = \sin x, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in the line $y = x$.

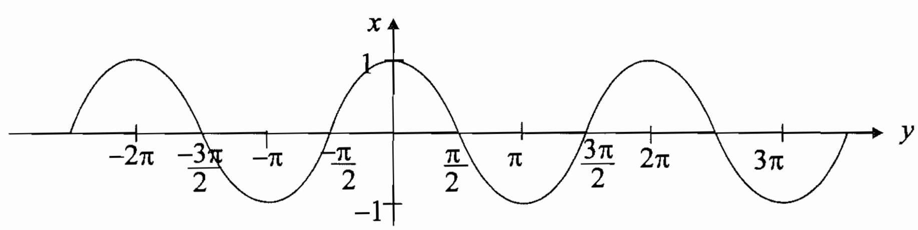
Note in passing that $\sin^{-1} x$ is an increasing function over its domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so that for

$y = \sin^{-1} x,$
$\frac{dy}{dx} > 0.$

The graph climbs to the right.

<u>$\cos^{-1} x$</u>

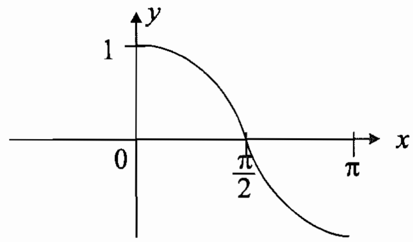
The graph of $y = \cos x$ for $x \in (-\infty, \infty)$ is as shown.



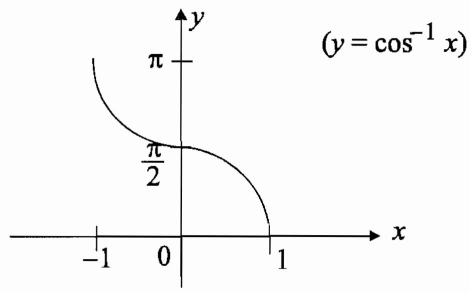
As with sin x, the function $f(x) = \cos x$ is not one-one but a one-one function may be constructed by restricting the domain. Thus we define

$$f^*(x) = \cos x \text{ for } x \in [0, \pi],$$

the graph being shown below.



The function f* has an inverse function given by $f^{*-1}(x) = \cos^{-1} x$ as shown below.



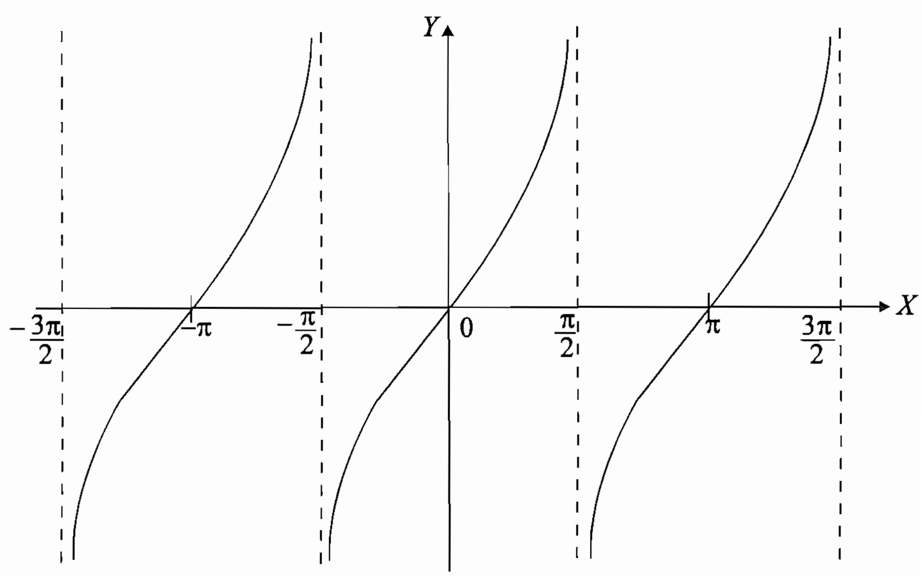
Note in passing that $\cos^{-1} x$ is a decreasing function over its domain (0, π) so that for

$$y = \cos^{-1} x, \quad \frac{dy}{dx} < 0.$$

The graph falls to the right.

<u>$\tan^{-1} x$</u>

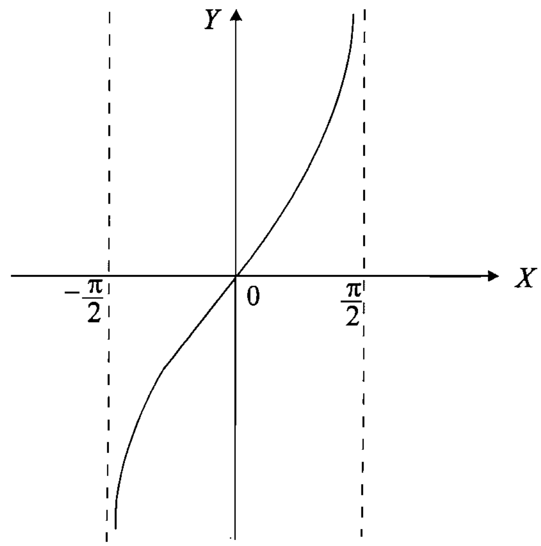
The graph of $y = \tan x$ for $x \in (-\infty, \infty)$ is shown below.



As before, the function $f(x) = \tan x$ is not one-one but a one-one function may be constructed by restricting the domain. Thus we define

$$f^*(x) \quad = \tan x \;\; \text{for} \;\; x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

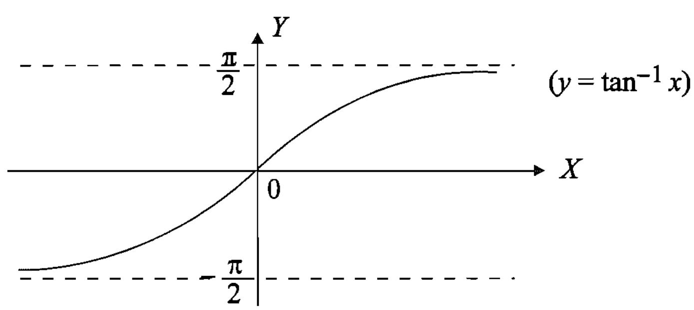
the graph being shown below.



The function f* has an inverse given by

$$f^{*-1}(x) = \tan^{-1} x \qquad x \in (-\infty, \infty).$$

The graph of $y = \tan^{-1} x$ is shown below.



Beware

Do not confuse $\sin^{-1} x$ with $\dfrac{1}{\sin x}$.

In general, distinguish between $f^{-1}(x)$ and $\dfrac{1}{f(x)}$.

Example 3.4

Find $\sin^{-1}(0.6148)$ and $\dfrac{1}{\sin(0.6148)}$.

Ensure that your calculator is in radian mode.

Then $\sin^{-1}(0.6148) = 0.6621$,

correct to four decimal places, which is equivalent to

$$\sin(0.6621) \approx 0.6148.$$

Press inverse or shift key followed by sin key.

Also $\dfrac{1}{\sin(0.6148)} = 1.7337$, correct to four decimal places.

Exercises 3.2

1. Find $\sin^{-1}\left(-\dfrac{1}{4}\right)$, $\cos^{-1}\left(-\dfrac{1}{5}\right)$, $\tan^{-1}\left(-\dfrac{1}{8}\right)$.

2. Show that $2\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$.

You don't need your calculator for Q2, 3, 4.

3. Show that $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) + \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{2}$.

4. Show that $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) + \sin^{-1}(1) = \dfrac{5\pi}{6}$.

5. Show that $2\tan^{-1}\left(\dfrac{3}{4}\right) = \tan^{-1}\left(\dfrac{24}{7}\right)$.

We return to inverse trigonometric functions in Chapter 5, specifically showing how they are differentiated.

Chapter 4

Trigonometry II: Compound Angles and Double Angles

In this chapter, we consider identities relating to $\sin(A + B)$, $\cos(A + B)$ and $\tan(A + B)$, and the particular forms of those identities when $A = B = x$, (say).

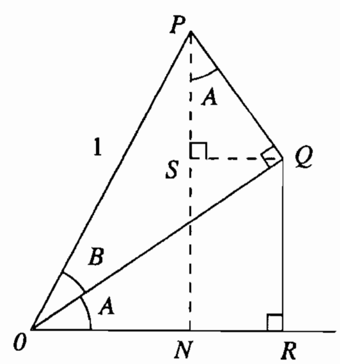
4.1 Compound angles

Given the values of A and B the values of $\sin A$, $\cos A$, $\sin B$, $\cos B$ and the other trigonometric functions may be found by means of an appropriate calculator. How does the value of $\sin(A + B)$, say, depend upon $\sin A$, $\cos A$, $\sin B$, $\cos B$? This section is concerned with exploring this dependence.

The addition formula

We derive formulae for $\cos(A + B)$ and $\sin(A + B)$, where both the angles A and B are acute.

Do not assume that $\sin(A+B) = \sin A + \sin B$. ✕



You will not be asked to produce this in an examination.

Let's consider the diagram shown where the triangles POQ, QOR are right-angled. The dotted lines are construction lines and $S\hat{P}Q = A$.

Then $$\sin(A+B) = \frac{PN}{OP} = \frac{NS + SP}{OP}$$
$$= \frac{RQ + SP}{OP}$$

$$= \frac{RQ}{OQ} \times \frac{OQ}{OP} + \frac{SP}{PQ} \times \frac{PQ}{OP}$$

$$= \sin A \cos B \; + \; \cos A \sin B.$$

Exercise 4.1

By starting with

$$\cos(A+B) = \frac{ON}{OP}$$

and writing $ON = OR - NR$, show that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

The results derived above are important and are therefore highlighted here.

$\sin(A+B) = \sin A \cos B + \cos A \sin B,$	(a)	Rules I
$\cos(A+B) = \cos A \cos B - \sin A \sin B.$	(b)	

Note in passing the occurrence of the negative sign in Rule I(b): it is tempting to assume that

$$\cos(A+B) = \cos A \cos B + \sin A \sin B \qquad \times$$

wrong

Similar results apply for sin(*A*–*B*) and cos(*A*–*B*). They may be derived from those given in Rules I by replacing B by -B and noting that

$$\sin(-B) = -\sin B,$$

$$\cos(-B) = \cos B.$$

We obtain immediately from Rules I:

$\sin(A-B) = \sin A \cos B - \cos A \sin B,$	(a)	Rules II
$\cos(A-B) = \cos A \cos B + \sin A \sin B.$	(b)	

Two further identities may be deduced from Rules I.

Now $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$ — **definition of tan**

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$ — **Rules I**

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

division of numerator and denominator by cos *A* cos *B*

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A}\frac{\sin B}{\cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

$\therefore$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \text{(a)}$$

and

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \quad \text{(b)}$$

Rules III

Replace B by $-B$ and note that $\tan(-B) = \tan B$.

Rules I – III were deduced for the case of A and B being acute. You are asked to assume that they hold for all values of A and B.

Example 4.1

Find, without using a calculator, the values of sin 75°, cos 165° and tan 15°, leaving your answers in surd form.

Now $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, $\tan 30° = \frac{1}{\sqrt{3}}$,

$\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$, $\tan 60° = \sqrt{3}$,

$\sin 120° = \frac{\sqrt{3}}{2}$, $\cos 120° = -\frac{1}{2}$,

and $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$, $\tan 45° = 1$.

Then

$$\begin{aligned}\sin 75° &= \sin(45° + 30°) \\ &= \sin 45° \cos 30° + \cos 45° \sin 30° \\ &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) \\ &= \frac{1}{2\sqrt{2}}\left(\sqrt{3}+1\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3}+1\right).\end{aligned}$$

Rules I(a)

$$\begin{aligned}\cos 165° &= \cos(120° + 45°) \\ &= \cos 120° \cos 45° - \sin 120° \sin 45° \\ &= -\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) - \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}\right) = -\frac{\sqrt{2}}{4}\left(1+\sqrt{3}\right).\end{aligned}$$

Rules I(b)

$$\begin{aligned}\tan 15° &= \tan(45° - 30°) \\ &= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} \\ &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}\end{aligned}$$

Rules III(b)

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

multiplication by $\frac{\sqrt{3}}{\sqrt{3}} = 1$

Example 4.2

If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$ where $90° < A < 180°$, and $0° < B < 90°$, find, without using the sin, cos buttons on your calculator, the value of $\sin(A + B)$.

Now $\quad \sin(A+B) = \sin A \cos B + \cos A \sin B.$

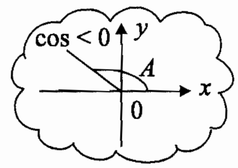
Rules I(a)

We require the values of $\cos A$ and $\sin B$.

Now $\quad \sin A = \frac{4}{5}$ and $\sin^2 A + \cos^2 A = 1.$

Then
$$\cos^2 A = 1 - \sin^2 A$$
$$= 1 - \left(\frac{4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25}.$$

$\therefore$
$$\cos A = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}.$$

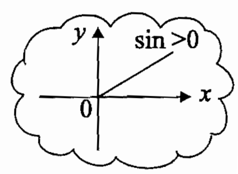


Now $\quad 90° < A < 180°$ so $\cos A < 0$.

$\therefore$
$$\cos A = -\frac{3}{5}.$$

Similarly,
$$\sin^2 B = 1 - \cos^2 B$$
$$= 1 - \left(\frac{12}{13}\right)^2 = 1 - \frac{144}{169}$$
$$= \frac{25}{169}.$$

$\therefore$
$$\sin B = \pm\sqrt{\frac{25}{169}} = \pm\frac{5}{13}.$$



Since $0° < B < 90°$, $\sin B > 0$

and $\therefore$
$$\sin B = \frac{5}{13}.$$

Thus
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$= \frac{4}{5} \times \frac{12}{13} + \left(-\frac{3}{5}\right) \times \frac{5}{13}$$
$$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}.$$

Example 4.3

Find the value of $\tan x$ if

$$2\sin(x - 45°) = \cos(x + 45°).$$

We expand the left and right hand sides using Rules II(a) and I(b).

$\therefore \quad 2(\sin x \cos 45° - \cos x \sin 45°)$

$\quad = \cos x \cos 45° - \sin x \sin 45°.$

Now $\sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}$ and therefore $\dfrac{1}{\sqrt{2}}$ may be cancelled throughout.

by multiplication by $\sqrt{2}$ throughout.

$$\therefore \quad 2\sin x - 2\cos x = \cos x - \sin x$$

$$\therefore \quad 3\sin x = 3\cos x$$

so that $\quad \dfrac{\sin x}{\cos x} = \tan x = 1.$

Example 4.4

Find, without using your calculator, the value of $\dfrac{\tan 70° - \tan 10°}{1 + \tan 70° \tan 10°}$.

From Rules III(b),

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

If $A = 70°$, $B = 10°$ we have

$$\frac{\tan 70° - \tan 10°}{1 + \tan 70° \tan 10°} = \tan(70° - 10°)$$

$$= \tan 60°$$

$$= \sqrt{3}.$$

Exercises 4.2

1. Simplify the following without the use of a calculator.

(i) $\cos 10° \cos 20° - \sin 20° \sin 10°$

(ii) $\sin 40° \cos 20° + \cos 40° \sin 20°$

(iii) $\cos\theta \cos 3\phi - \sin\theta \sin 3\phi$

(iv) $\sin 2A \cos A - \cos 2A \sin A$

(v) $\cos(A + B)\cos A + \sin(A + B)\sin A$

(vi) $\sin(A + 2B)\cos 2B - \cos(A + 2B)\sin 2B$

(vii) $\sin 60° \cos 30° + \cos 60° \sin 30°$

2. Do not use the trigonometric or inverse trigonometric function buttons on your calculator to evaluate the following.

(i) If $\sin A = \frac{3}{5}$ and A is acute, use Rules I(a) to find

$$\sin(A + A) = \sin 2A \text{ and } \cos(A + A) = \cos 2A.$$

(ii) If $\sin A = \frac{4}{5}$ and $\sin B = \frac{24}{25}$ and A and B are acute,

find the value of $\tan(A + B)$.

(iii) If $\tan(x + 45°) \tan x = 3$ find the possible values of $\tan x$.

(iv) Simplify and hence evaluate $\frac{\sqrt{3}}{2}\cos 75° + \frac{1}{2}\sin 75°$

(v) If $\tan x = \frac{\sqrt{3}+1}{1-\sqrt{3}}$ find the value of x given that $90° < x < 180°$.

3. Simplify $\sin(A + B) + \sin(A - B)$. Hence find all values of x between 0° and 360° satisfying $\sin(x + 60°) + \sin(x - 60°) = \frac{1}{\sqrt{2}}$.

4. Write down the expansion for $\tan(\theta + \theta)$. Hence find an expression for $\tan 2\theta$ in terms of $\tan \theta$.

5. Solve the equations for values of x between 0° and 360°, using your calculator whenever necessary.

(i) $2 \cos x = \sin(x + 60°)$

(ii) $\sin (x + 45°) = \sin x$

(iii) $\sin (x + 30°) = \frac{1}{2}\cos x$

(iv) $4 \cos(x + 10°) = 3 \sin (x - 10°)$.

4.2 Double angle formulae

For convenience we recap Rules I – III given in section 4.1.

$$\left.\begin{aligned} \sin (A + B) &= \sin A \cos B + \cos A \sin B, \quad (a) \\ \cos (A + B) &= \cos A \cos B - \sin A \sin B, \quad (b) \end{aligned}\right\} \text{Rules I}$$

$$\left.\begin{aligned} \sin (A - B) &= \sin A \cos B - \cos A \sin B, \quad (c) \\ \cos (A - B) &= \cos A \cos B + \sin A \sin B, \quad (d) \end{aligned}\right\} \text{Rules II}$$

$$\left.\begin{aligned} \tan (A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B}, \quad (e) \\ \tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B}. \quad (f) \end{aligned}\right\} \text{Rules III}$$

Let's write $A = B = x$ in (a), (b), (e).

Then (a) becomes

$$\sin 2x = (\sin x)(\cos x) + (\cos x)(\sin x) = 2 \sin x \cos x.$$

Similarly, (b) and (e) become

$$\cos 2x = (\cos x)(\cos x) - (\sin x)(\sin x) = \cos^2 x - \sin^2 x,$$

$$\tan 2x = \frac{\tan x + \tan x}{1 - (\tan x)(\tan x)} = \frac{2 \tan x}{1 - \tan^2 x}.$$

Alternative expressions for $\cos 2x$ are possible since

$$\cos^2 x + \sin^2 x = 1.$$

Then
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1.$$

The so-called double angle formulae are important and are summarised below.

$$\sin 2x = 2 \sin x \cos x, \quad \text{(a)}$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1, \quad \text{(b)}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \quad \text{(c)}$$

Rules IV

Exercises 4.3

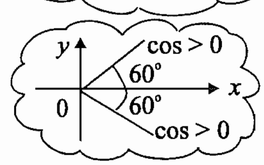
Write $A = B = x$ in Rules I - III to verify the known values of $\sin 0°$, $\cos 0°$ and $\tan 0°$.

Example 4.5

Find all the values of x between 0° and 360° satisfying

$$\sin 2x = \sin x.$$

Do not cancel sin x until you've considered the possibility that $\sin x = 0$.



Let's substitute for $\sin 2x$ from Rules IV, (a).

$\therefore \quad 2 \sin x \cos x = \sin x.$

$\therefore \quad \sin x(2 \cos x - 1) = 0.$

Then $\quad \sin x = 0$ or $2 \cos x - 1 = 0$

so $\quad \sin x = 0$ or $\cos x = \frac{1}{2}$.

Thus $x = 0°, 180°, 360°, 60°, 300°$.

Example 4.6

Find all the values of x between 0° and 360° satisfying

$3 \cos 2x - \cos x = 2.$

The presence of $\cos x$ suggests that $\cos 2x$ should be written in terms of $\cos x$.

> **Rules IV(b)**

Then $3(2 \cos^2 x - 1) - \cos x = 2.$

$\therefore \quad 6 \cos^2 x - \cos x - 5 = 0.$

$\therefore \quad (6 \cos x + 5)(\cos x - 1) = 0.$

$\therefore \quad \cos x = -\frac{5}{6}$ or 1.

> **or use the quadratic formula.**

$\cos x = -\frac{5}{6}$ gives $x = 146.44°$ or $213.56°$

> **cos < 0**
> **cos < 0**

and $\cos x = 1$ gives $x = 0°$ or $360°$.

$\therefore$ Solutions are 0°, 146.44°, 213.56°, 360°.

Example 4.7

Use the formula for $\cos (A + B)$ and the double angle formulae to show that

$$\cos 3A = 4 \cos^3 A - 3 \cos A.$$

Find all the values of x between 0° and 360° satisfying

$$\cos 3x = 12 \cos^2 x - 14 \cos x + 3.$$

Now $\cos (A + B) = \cos A \cos B - \sin A \sin B.$

> **Rules I(b)**

Let $B = 2A$.

$$
\begin{aligned}
\therefore \cos (A + 2A) &= \cos A \cos 2A - \sin A \sin 2A \\
&= \cos A(2 \cos^2 A - 1) - \sin A(2 \sin A \cos A) \\
&= 2 \cos^3 A - \cos A - 2 \sin^2 A \cos A \\
&= 2 \cos^3 A - \cos A - 2 \cos A(1 - \cos^2 A) \\
\text{giving } \cos 3A &= 4 \cos^3 A - 3 \cos A.
\end{aligned}
$$

> **Rules IV(a)(b)**

> **When you see $\sin^2$ think of $\cos^2$!**

Now let's substitute for $\cos 3x$ in the equation

$$\cos 3x = 12 \cos^2 x - 14 \cos x + 3.$$

Then $4 \cos^3 x - 3 \cos x = 12 \cos^2 x - 14 \cos x + 3.$

$\therefore \quad 4 \cos^3 x - 12 \cos^2 x + 11 \cos x - 3 = 0.$

This is a cubic equation in $\cos x$ for which we use the factor theorem.

> **Don't attempt to use the quadratic formula with a cubic equation.**

In fact, $\cos x = 1$ is a root since

$$4(1)^3 - 12(1)^2 + 11(1) - 3 = 0.$$

The left hand side of the equation may be factorised since we know that $\cos x - 1$ is a factor.

Then $(\cos x - 1)(4\cos^2 x - 8\cos x + 3) = 0$.

The left hand side factorises further to give

$$(\cos x - 1)(2\cos x - 3)(2\cos x - 1) = 0.$$

or use the fomula to solve $4\cos^2 x - 8\cos x + 3 = 0$

$\therefore \quad \cos x = 1$ or $\frac{3}{2}$ or $\frac{1}{2}$.

Now $\cos x = \frac{3}{2}$ is impossible so that

$\cos x = 1$ or $\frac{1}{2}$.

Then $x = 0°, 360°, 60°, 300°$.

Exercises 4.4

1. Write the following as simple trigonometric ratios but do not evaluate them.

(i) $2\sin 12° \cos 12°$ (ii) $\dfrac{2\tan 15°}{1-\tan^2 15°}$

(iii) $2\cos^2 24° - 1$ (iv) $2\sin \frac{1}{2}x \cos \frac{1}{2}x$

(v) $1 - 2\sin^2 20°$ (vi) $\dfrac{2\tan\frac{1}{2}x}{1-\tan^2\frac{1}{2}x}$

(vii) $\cos^2 16 - \sin^2 16$ (viii) $2\cos^2\frac{1}{2}\theta - 1$

(ix) $1 - 2\sin^2 3\theta$ (x) $\dfrac{2\tan 4x}{1-\tan^2 4x}$

2. Find, without using a calculator, the value of $\tan 2\theta$ given that $\tan\theta = -\frac{1}{2}$.

3. Find, without using the trigonometric functions on a calculator, the values of $\sin 2\theta$ and $\cos 2\theta$ in the following (given that in all cases $90° < \theta < 180°$).

(i) $\cos\theta = -\frac{3}{5}$ (ii) $\sin\theta = \frac{12}{13}$ (iii) $\sin\theta = \frac{\sqrt{3}}{2}$.

4. Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.

Hence find all values of x between 0° and 360° satisfying the following equations

(i) $\sin 3x = 2\sin x$

(ii) $\sin 3x = 1 - \sin^2 x - \sin x$.

5. Find all values of x between 0° and 360° satisfying

$$2\cos 2x + \sin x + 1 = 0.$$

6. Find all values of x between 0° and 360° satisfying

(i) $3\tan x = \tan 2x$

(ii) $4\tan x \tan 2x = 1$.

Another application of the compound angle formulae given in **4.1** relates to the harmonic form $a\cos\theta + b\sin\theta$, where a and b are contents.

4.3 The harmonic form $a \cos\theta + b \sin\theta$

The harmonic form $a \cos\theta + b \sin\theta$, where a and b are constants, can be expressed in the form $R \sin(\theta \pm \alpha)$ or $R \cos(\theta \pm \alpha)$, where $R > 0$ and α is an acute angle.

Let's assume $a > 0$, $b > 0$.

Then $R \cos(\theta - \alpha) \equiv a \cos\theta + b \sin\theta$

gives $R[\cos\theta \cos\alpha + \sin\theta \sin\alpha] \equiv a \cos\theta + b \sin\theta$.

Comparing the coefficients of $\cos\theta$ and $\sin\theta$, we have

$$R \cos\alpha = a, \qquad (1)$$

$$R \sin\alpha = b. \qquad (2)$$

Division of (2) by (1) gives $\tan\alpha = \dfrac{b}{a}$.

Squaring (1) and (2), and adding, we have

$$R^2 (\cos^2\alpha + \sin^2\alpha) = a^2 + b^2.$$

$$R^2 = a^2 + b^2$$

and $$R = \sqrt{a^2 + b^2}.$$

<u>Summary</u>

$$R \cos(\theta - \alpha) \equiv a \cos\theta + b \sin\theta,$$

where $\tan\alpha = \dfrac{b}{a}$,

$$R = \sqrt{a^2 + b^2}.$$

Since $a > 0$, $b > 0$, α is acute.

Exercise 4.5

Consider $a \cos\theta - b \sin\theta$, where $a > 0$, $b > 0$.

Show that $a \cos\theta - b \sin\theta \equiv R \cos(\theta + \alpha)$,

where $\tan\alpha = \dfrac{b}{a}$,

$$R = \sqrt{a^2 + b^2}.$$

Example 4.8

Express $\cos\theta + 2 \sin\theta$ in the form $R \cos(\theta - \alpha)$.

Then $R[\cos\theta \cos\alpha + \sin\theta \sin\alpha] \equiv \cos\theta + 2 \sin\theta$

so that $R \cos\alpha = 1$,

$R \sin\alpha = 2$.

Then $\tan\alpha = 2$

and $R = \sqrt{1^2 + 2^2} = \sqrt{5}.$

Then $\alpha = 63.4°$ and thus

$$\cos\theta + \sin\theta = \sqrt{5}\cos(\theta - 63.4°).$$

Example 4.9

Express $3\cos\theta - 4\sin\theta$ in the form $R\cos(\theta + \alpha)$.

Then $R[\cos\theta\ \cos\alpha - \sin\theta\ \sin\alpha] \equiv 3\cos\theta - 4\sin\theta$

so that $R\cos\theta = 3,$

$R\sin\alpha = 4.$

Then $\tan\alpha = \frac{4}{3}$

and $R = \sqrt{3^2 + 4^2} = \sqrt{25} = 5\,.$

Then $\alpha = 53.3°$ and thus

$$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1°).$$

Example 4.10

Solve the equation

$$5\cos\theta + 3\sin\theta = 2$$

for $0° \le \theta \le 360°$.

Now $5\cos\theta + 3\sin\theta = R\cos(\theta - \alpha),$

where $R = \sqrt{5^2 + 3^2} = \sqrt{34},$

$\tan\alpha = \frac{3}{5}.$

$\therefore$ $\alpha = 31.0°$

Thus $5\cos\theta + 3\sin\theta = 2$

becomes $\sqrt{34}\cos(\theta - 31°) = 2$

$\therefore$ $\cos(\theta - 31°) = \frac{2}{\sqrt{34}}$

$\therefore$ $\theta - 31° = 69.9, 290.1$

$\therefore$ $\theta = 100.9°, 321.1°.$

Example 4.11

Find the maximum and minimum values of

(a) $2\cos\theta + \sin\theta$.

(b) $\frac{1}{2\cos\theta + \sin\theta + 3}$.

(a) Now $2\cos\theta + \sin\theta = \sqrt{5}\cos(\theta - 26.6°)$. **check**

Now $-1 \le \cos(\theta - 26.6°) \le 1$ **true for all cosines**

so that the maximum value of $2\cos\theta + \sin\theta$ is $\sqrt{5}$ and the minimum value is $-\sqrt{5}$.

(b) $$\frac{1}{2\cos\theta + \sin\theta + 3} = \frac{1}{\sqrt{5}\cos(\theta - 26.6°) + 3}$$

The maximum and minimum values arise when the denominator is minimum and maximum respectively.

$\therefore$ The maximum value is $\frac{1}{3-\sqrt{5}}$ and the minimum value is $\frac{1}{3+\sqrt{5}}$.

Exercises 4.6

1. Find the value of R and acute α in each of the following identities.

(a) $5\cos\theta + 12\sin\theta \equiv R\cos(\theta - \alpha)$

(b) $\cos\theta - 3\sin\theta = R\cos(\theta + \alpha)$

(c) $\cos\theta + \sin\theta = R\sin(\theta + \alpha)$

(d) $3\sin\theta - 4\cos\theta = \mathrm{R}\sin(\theta - \alpha)$

2. Find the greatest and least values of the following.

(a) $\sqrt{3}\cos\theta + \sin\theta$ (b) $4\cos\theta - 3\sin\theta$

(c) $\cos\theta + 3\sin\theta$ (d) $\frac{1}{3 + \cos\theta + \sin\theta}$

(e) $\frac{1}{7 + 2\cos\theta - \sqrt{5}\sin\theta}$ (f) $(4\cos\theta + 3\sin\theta)^2$

3. Solve each of the following equations for $0° \le \theta \le 360$, giving your answers correct to one decimal place.

(a) $\cos\theta + \sqrt{3}\sin\theta = 1$ (b) $4\cos\theta - 3\sin\theta = 2$

(c) $5\cos\theta + 12\sin\theta = 7$ (d) $4\cos\theta - 7\sin\theta = 3$

(e) $2\cos\theta + 5\sin\theta = 4$ (f) $4\cos 2\theta - 9\sin 2\theta = 6$

4. Express $2\cos^2\theta + 6\sin\theta\cos\theta$ in the form $A + B\cos 2\theta + C\sin 2\theta$. Hence find all the values of θ between 0° between 0° and 180° satisfying

$$2\cos^2\theta + 6\sin\theta\cos\theta = 2.$$

5. Show that

$$-2 \le 6\cos^2\theta - 8\sin\theta\cos\theta \le 8.$$

Chapter 5

More Differentiation

Differentiation has already been considered in **P1** and **P2**. In this chapter we consider three further aspects of differentiation: differentiation of inverse functions and of functions defined implicitly or parametrically.

5.1 Differentiation of inverse functions

Let's recall how differentiation is expressed in the delta notation (**P1**).

When $y = \mathrm{f}(x)$,

let δx, δy be corresponding small increments in x and y, respectively.

We say that $$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

or equivalently, $$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0}\left(\frac{1}{\delta x/\delta y}\right).$$

Now $\delta y \to 0$ as $\delta x \to 0$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta y \to 0}\left(\frac{1}{\delta x/\delta y}\right) = \frac{1}{\lim\limits_{\delta y \to 0}\left(\dfrac{\delta x}{\delta y}\right)} = \frac{1}{{}^{\mathrm{d}x}\!/\!{}_{\mathrm{d}y}}.$$

$\therefore$ $$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{{}^{\mathrm{d}x}\!/\!{}_{\mathrm{d}y}}}$$

or $$\boxed{\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{{}^{\mathrm{d}y}\!/\!{}_{\mathrm{d}x}}.}$$ Rule I

Rule I assists us in the differentiation of inverse functions.

Example 5.1

Now if $\quad y = e^x$

then $\quad x = \ln y.$ (by definition)

Now $\quad \dfrac{dy}{dx} = e^x$

so that $\quad \dfrac{dx}{dy} = \dfrac{1}{dy/dx} \quad$ (Rule I)

$$= \frac{1}{e^x} = \frac{1}{y}.$$

Thus $\quad \dfrac{d}{dy}(\ln y) = \dfrac{1}{y}$

or $\quad \dfrac{d}{dx}(\ln x) = \dfrac{1}{x}.$

(It is convenient to express the diffⁿ rule in terms of x.)

Example 5.2

Using $\dfrac{d}{dx}(x^3) = 3x^2$ and Rule I,

show that $\dfrac{d}{dx}\left(x^{\frac{1}{3}}\right) = \dfrac{1}{3x^{\frac{2}{3}}}$.

Let $y = x^3$ so that $x = y^{\frac{1}{3}}$.

Now $\quad \dfrac{dy}{dx} = 3x^2$

and $\quad \dfrac{dx}{dy} = \dfrac{1}{dy/dx} = \dfrac{1}{3x^2} = \dfrac{1}{3y^{\frac{2}{3}}}.$ ($x = y^{\frac{1}{3}}$)

Thus $\quad \dfrac{d}{dy}\left(y^{\frac{1}{3}}\right) = \dfrac{1}{3y^{\frac{2}{3}}}$

or with a change of letter,

$$\frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3x^{\frac{2}{3}}}.$$

Exercises 5.1

1. Using Rule I and the derivative of x^5, deduce the derivative of $x^{\frac{1}{5}}$.

2. Using Rule I and the derivative of $x^2 + 1$ $(x > 0)$, deduce the derivative of $(x-1)^{\frac{1}{2}}$ $\quad (x > 1)$.

3. Using Rule I and the derivative of $(x^2 + 1)^2$ $(x > 0)$, deduce the derivative of $(x^{\frac{1}{2}} - 1)^{\frac{1}{2}}$

The inverse trigonometric functions may be differentiated by means of Rule I.

Example 5.3

Differentiation of $\sin^{-1} x$

Now if $y = \sin^{-1} x,$

$\sin y = x.$

Then $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \cos y$

so that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}y/\mathrm{d}x}$$
$$= \frac{1}{\cos y}$$
$$= \frac{1}{\pm\sqrt{1-\sin^2 y}}$$
$$= \frac{1}{\pm\sqrt{1-x^2}},$$

$\cos^2 y + \sin^2 y = 1$

since $\sin y = x.$

How do we decide on the choice of sign?

In the discussion of $\sin^{-1}x$, it was pointed out in **Chapter 3** that $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ in the domain $[-1, 1]$, so we choose the + sign.

Thus
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$
Rule II

Example 5.4

Differentiation of $\cos^{-1} x$

As before, if $y = \cos^{-1} x,$

$\cos y = x.$

Then $\dfrac{\mathrm{d}x}{\mathrm{d}y} = -\sin y$

so that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$$
$$= -\frac{1}{\sin y}$$
$$= -\frac{1}{\pm\sqrt{1-\cos^2 y}}$$

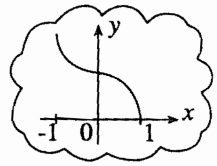
$\sin^2 y + \cos^2 y = 1$

$$= \mp \frac{1}{\sqrt{1-x^2}},$$

since $\cos y = x.$

As pointed out earlier,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x) < 0 \text{ over the domain } [-1, 1]$$



so that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}}$$

Rule III

You are asked to differentiate $\tan^{-1} x$ in the next exercise.

Exercise 5.2

Complete the boxes in the following.

If $y = \tan^{-1} x,$

$$\square = \square \, .$$

Then $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \square$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$$

$$= \frac{1}{\square}.$$

sec² y = 1 + tan²y
for any angle y
section 3.2

$$= \frac{1}{\square}.$$

$$= \frac{1}{1+x^2},$$

since $\tan y = x.$

In contrast to the discussions for $\sin^{-1} x$ and $\cos^{-1} x$, there is no ambiguity in relation to sign.

Thus

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x) = \frac{1}{1+x^2}.}$$

Rule IV

The rules II, III, IV may be used with the other rules of differentiation to differentiate more complicated functions.

Example 5.5

Differentiate the following with respect to x.

(i) $\cos^{-1}(4x)$ (ii) $\sin^{-1}\left(\frac{1}{x}\right)$

(iii) $\tan^{-1}\left(\sqrt{x-1}\right)$ (iv) $(1+x^2)\tan^{-1} x$

(i) Let $y = \cos^{-1}(4x) = \cos^{-1} u$,

where $u = 4x$.

function of a function rule

$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$,

$\frac{d}{dx}(\cos^{-1}u) = -\frac{1}{\sqrt{1-u^2}}$,

$u = 4x$, $\frac{du}{dx} = 4$.

Now
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -\frac{1}{\sqrt{1-u^2}} \times 4$$
$$= -\frac{4}{\sqrt{1-16x^2}}.$$

(ii) Let $y = \sin^{-1}\left(\frac{1}{x}\right) = \sin^{-1}(u)$,

where $u = \frac{1}{x}$.

Now
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{\sqrt{1-u^2}} \times \frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \times \left(-\frac{1}{x^2}\right)$$
$$= \frac{1}{\sqrt{\frac{x^2-1}{x^2}}} \times \left(-\frac{1}{x^2}\right)$$
$$= -\frac{1}{x\sqrt{x^2-1}}.$$

(iii) Let $y = \tan^{-1}(u)$,

where $u = \sqrt{x-1}$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{1+u^2} \times \frac{d}{dx}\left(\sqrt{x-1}\right)$$

$$= \frac{1}{1+\left(\sqrt{x-1}\right)^2} \times \frac{1}{2}(x-1)^{-\frac{1}{2}}(1)$$

$$= \frac{1}{1+x-1} \times \frac{1}{2\sqrt{x-1}}$$

$$= \frac{1}{2x\sqrt{x-1}}.$$

> To differentiate $\sqrt{x-1}$ we could write $y = u^{\frac{1}{2}}$ where $u = x - 1$.

(iv) Let $y = (x^2 + 1)\tan^{-1} x$.

By the product rule with $u = x^2 + 1$, $v = \tan^{-1} x$,

> $v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = (\tan^{-1} x)(2x) + (x^2+1)\left(\frac{1}{x^2+1}\right)$$

$$= 2x\tan^{-1} x + 1.$$

The differentiation of functions such as (i), (ii), (iii) in Example 5.5 may be streamlined if we modify Rules II, III, IV.

For instance, let's differentiate $\sin^{-1}(g(x))$, where g is some function.

Thus, if $y = \sin^{-1} u$, where $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \times g'(x)$$

so

$$\frac{d}{dx}\left(\sin^{-1}(g(x))\right) = \frac{1}{\sqrt{1-(g(x))^2}} \times g'(x). \qquad \text{Rule V}$$

Similarly,

$$\frac{d}{dx}\left(\cos^{-1}(g(x))\right) = -\frac{1}{\sqrt{1-(g(x))^2}} \times g'(x). \qquad \text{Rule VI}$$

$$\frac{d}{dx}\left(\tan^{-1}(g(x))\right) = \frac{1}{1+(g(x))^2} \times g'(x). \qquad \text{Rule VII}$$

Example 5.6

Use Rules V, VI, VII to differentiate the following with respect to x.

(i) $\sin^{-1}(1-x)$ (ii) $\cos^{-1}\left(\sqrt{x-1}\right)$ (iii) $\tan^{-1}(\cot x)$

(i) $$\frac{d}{dx}\left(\sin^{-1}(1-x)\right) = \frac{1}{\sqrt{1-(1-x)^2}}\times\frac{d}{dx}(1-x)$$

Rule V
g(x) = 1 – x

$$= \frac{1}{\sqrt{2x-x^2}}\times(-1)$$
$$= -\frac{1}{\sqrt{2x-x^2}}.$$

(ii) $$\frac{d}{dx}\left(\cos^{-1}(\sqrt{1-x})\right) = -\frac{1}{\sqrt{1-(\sqrt{1-x})^2}}\times\frac{d}{dx}\left(\sqrt{1-x}\right)$$

Rule VI
g(x) = $\sqrt{1-x}$

$$= -\frac{1}{\sqrt{1-(1-x)}}\times\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$$
$$= \frac{1}{2\sqrt{x(1-x)}}.$$

(iii) $$\frac{d}{dx}\left(\tan^{-1}(\cot x)\right) = \left(\frac{1}{1+\cot^2 x}\right)\times\frac{d}{dx}(\cot x)$$

Rule VII,
g(x) = cot x

$$= \frac{1}{\operatorname{cosec}^2 x}\times -\operatorname{cosec}^2 x$$
$$= -1,$$

(which suggests incidentally that $\tan^{-1}(\cot x) = -x + \text{constant}$).

In the following exercises you may use substitutions or equivalently Rules V, VI and VIII.

Exercises 5.3

1. Differentiate the following with respect to x, given that a is constant.

(i) $\sin^{-1}\left(\frac{x}{a}\right)$ (ii) $\cos^{-1}\left(\frac{x}{a}\right)$ (iii) $\tan^{-1}\left(\frac{x}{a}\right)$ (iv) $\sin^{-1}(x^2)$

(v) $\tan^{-1}\left(\frac{1}{x^2}\right)$ (vi) $x^2\sin^{-1}(1-x)$ (vii) $\sin^{-1}\left(\sqrt{x}\right)$ (viii) $\sin^{-1}(\cos x)$

(ix) $\sqrt{1-x^2}\cos^{-1}x$ (x) $\sqrt{\tan^{-1}x}$ (xi) $\sqrt{1-x^2}-\sin^{-1}x$

(xii) $\tan^{-1}\left(\frac{1-x}{1+x}\right)$ (xiii) $\frac{\tan^{-1}x}{1+x^2}$ (xiv) $(\sin^{-1}x)^2$ (xv) $\frac{1}{\tan^{-1}x}$

5.2 Differentiation of implicit functions

Up until now the variable y has been written in terms of x. We say in such cases that y is an explicit function of x.

for example, $y = \sin^{-1} x$, $y = x^2 + 2x + 3$.

In contrast, y and x may be related by an equation such that it is not possible or easy to write y in terms of x. Then y is said to be an **implicit** function of x. For example,

$$x^2 + y^2 + 6x + 4y - 14 = 0$$

and

$$y^3 + xy + x^2 \sin y = 1$$

both express y implicitly in terms of x.

To make progress in differentiating implicit functions we must differentiate terms such as y^2, $x\, y^3$, $x^2 \sin y$ with respect to x. We use the function of a function rule in combination with other rules where necessary.

Thus,

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}.$$

function of a function rule

Again,

$$\frac{d}{dx}(\cos y) = \frac{d}{dy}(\cos y)\frac{dy}{dx} = -\sin y\frac{dy}{dx}.$$

Also,

$$\frac{d}{dx}(xy^3) = y^3\frac{d}{dx}(x) + x\frac{d}{dx}(y^3) = y^3 + x\frac{d}{dy}(y^3)\frac{dy}{dx} = y^3 + 3xy^2\frac{dy}{dx}.$$

Product rule with $u = x$, $v = y^3$

Finally,

$$\frac{d}{dx}(x^2 \sin y) = \sin y\frac{d}{dx}(x^2) + x^2\frac{d}{dy}(\sin y)\frac{dy}{dx} = 2x \sin y + x^2 \cos y\frac{dy}{dx}.$$

The above examples may be encapsulated in the following rules.

$$\frac{d}{dx}[g(y)] = g'(y)\frac{dy}{dx}, \qquad \text{Rule VIII}$$

$$\frac{d}{dx}[f(x)g(y)] = g(y)f'(x) + f(x)g'(y)\frac{dy}{dx}. \qquad \text{Rule IX}$$

Example 5.7

Find $\dfrac{dy}{dx}$ given that

$$x^2 + 3x^2y^3 - 4y^2 - x = 0.$$

Differentiate with respect to x.

$$2x + \frac{d}{dx}(3x^2y^3) - \frac{d}{dx}(4y^2) - 1 = 0.$$

$$\therefore \quad 2x + 6xy^3 + 3x^2\frac{d}{dx}(y^3) - 8y\frac{dy}{dx} - 1 = 0$$

so $$2x + 6xy^3 + 9x^2y^2\frac{dy}{dx} - 8y\frac{dy}{dx} - 1 = 0.$$

Collecting terms in $\dfrac{dy}{dx}$, we obtain

$$(9x^2y^2 - 8y)\frac{dy}{dx} = 1 - 2x - 6xy^3$$

so $$\frac{dy}{dx} = \frac{1-2x-6xy^3}{9x^2y^2-8y}.$$

Equations of tangents and normals to curves are easily found when the curves are given by equations in which y depends implicitly upon x.

Example 5.8

Find the slopes of the tangent and normal at the point (2, 3) of the curve

$$xy^3 + y^2 - xy + x = 59.$$

The slope of the tangent at any point is given by $\dfrac{dy}{dx}$.

Differentiate the equation with respect to x.

$$y^3 + x\frac{d}{dx}(y^3) + \frac{d}{dx}(y^2) - y - x\frac{dy}{dx} + 1 = 0.$$

$$\therefore \quad y^3 + 3xy^2\frac{dy}{dx} + 2y\frac{dy}{dx} - y - x\frac{dy}{dx} + 1 = 0.$$

Solve for $\dfrac{dy}{dx}$.

$$\therefore \quad (3xy^2 + 2y - x)\frac{dy}{dx} = y - y^3 - 1$$

so that $$\frac{dy}{dx} = \frac{y-y^3-1}{3xy^2+2y-x}.$$

At the point (2, 3), $$\frac{dy}{dx} = \frac{3-3^3-1}{3(2)(3)^2+2(3)-2} = -\frac{25}{58}.$$

The slope of the normal is

$$-\frac{1}{\text{slope of tangent}} = -\frac{1}{-\frac{25}{58}} = \frac{58}{25}.$$

Cautionary Note

On occasions, we have observed students presenting the work incorrectly as follows.

Given $$x^2 + y^2 + 3x^3y^2 - 1 = 0,$$

$$\frac{dy}{dx} = 2x + 2y\frac{dy}{dx} + 9x^2y^2 + 6x^3y\frac{dy}{dx} = 0.$$ ✗ INCORRECT

The correct form is

$$2x + 2y\frac{dy}{dx} + 9x^2y^2 + 6x^3y\frac{dy}{dx} = 0.$$ ✓ CORRECT

Exercises 5.4

1. Find $\frac{dy}{dx}$ in the following, expressing your answer in terms of x and y.

   (i) $x^2 + y^2 = 5$ (ii) $y^4 = 8x$ (iii) $x + 2y^2 = 4$

   (iv) $\sqrt{x} + \sqrt{y} = 2$ (v) $x^2 + y^2 + 3xy = 2$ (vi) $x^3 + y^3 - 8 = 0$

   (vii) $xy + y^3 - 2 = 0$ (viii) $y \cos x + y^2 = 0$ (ix) $x^2y^3 = 8$

   (x) $xy(x + y) = 4$ (xi) $x^2 + y^2 - 2xy + 3y - 2x + 5 = 0$

2. Find the slopes of the tangent and normal to the curve

   $3y^2 + 2x^2 = 14$ at the point (1, 2).

3. Find the slope of the tangent to the curve

   $$x^3 + y^3 + 4x^2 + 3xy = 2x - 1$$

   at the point (1, − 1)

4. Find the slopes of the tangent and normal to the curve

   $$x + y + x \cos y = 2 + \frac{\pi}{2}$$

   at the point $\left(2, \frac{\pi}{2}\right)$.

5.3 Differentiation of functions defined parametrically

It is often useful to express x and y in terms of a third variable called a parameter, for example

$$x = t^3 + t, \quad y = 2t^2 + \sin t.$$

Then if values are assigned to t, corresponding values of x and y may be evaluated which can be plotted as a graph. We shall discuss briefly the graphical representation of such functions in the next chapter.

Our immediate interest is the differentiation of functions defined parametrically.

Suppose $x = f(t)$, $y = g(t)$.

Let δt, δx, δy, be corresponding small increments in t, x and y, respectively.

Then $$\frac{\delta y}{\delta x} = \left(\frac{\delta y}{\delta t}\right) \Big/ \left(\frac{\delta x}{\delta t}\right).$$

Now as $\delta t \to 0$, δx and $\delta y \to 0$.

Then assuming
$$\lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt},$$
$$\lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$
$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx},$$

and the limit of a quotient is the quotient of the limits, we obtain

We assume this non-trivial result.

$$\boxed{\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right).}$$

Thus, differentiation of y with respect to x may be achieved by first differentiation of x and y separately with respect to the parameter t.

Example 5.9

Given $\quad x = t^3 + t, \quad y = 2t^2 + \sin t,$

find $\frac{dy}{dx}$ in terms of t.

Now $$\frac{dx}{dt} = 3t^2 + 1, \quad \frac{dy}{dt} = 4t + \cos t$$

so that $$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right) = \frac{4t + \cos t}{3t^2 + 1}.$$

Slopes of tangents and normals to a curve are easily defined when the curve is represented parametrically.

Example 5.10

Given $x = at$, $y = 2at^2$, where a is a constant, find the slopes of the tangent and the normal to the curve at the point described by t.

Now $\quad x = at, y = 2at^2$

Do not attempt to eliminate t to obtain $2x^2 = ay$

so $$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right) = \frac{4at}{a} = 4t.$$

The slope of the normal is $-\frac{1}{4t}$.

Exercises 5.5

1. Find $\dfrac{dy}{dx}$ in the following cases, where a and c are constant.

(i) $x = a\cos\theta,\ y = a\sin\theta$ (ii) $x = ct,\ y = \dfrac{c}{t}$ (iii) $x = 2t^3,\ y = t^2$

(iv) $x = \sec t,\ y = \tan t$ (v) $x = at^2,\ y = 2at$ (vi) $x = 3t + t^3,\ y = 3 - t^{\frac{5}{2}}$

(vii) $x = \cos^3 t,\ y = \sin^3 t$ (viii) $x = \dfrac{t}{1-t},\ y = \dfrac{t^2}{1-t}\ (t \neq 1)$

(ix) $x = (t+1)^2,\ y = t^2 - 1$

2. For what values of t do the following curves (represented parametrically) have stationary points?

(i) $x = t^2 + 1,\ \ y = t^3 - 3t$

(ii) $x = 2t + 5\sin t,\ \ y = 2 + 5t + 5\cos t\ \ (0 \le t \le 2\pi)$

(iii) $x = t^2 + t,\ \ y = te^{-t}$.

3. Find the slopes of the tangents and normals at general points on the following curves.

(i) $x = t,\ \ y = \dfrac{1}{t+1}$ (ii) $x = t^3,\ y = t^4$

(iii) $x = \cos^2 t,\ \ y = 2 + \sin t$ (iv) $x = 2\cos t,\ \ y = 3\sin t$.

It is possible to find higher derivatives when functions are represented parametrically. We restrict our discussion to second derivatives only.

Now we know that

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right).$$

when the parameter is *t*, of course

Hopefully you'll understand the use of the boxes shortly.

Let's summarise this.

To differentiate $\boxed{y}$ with respect to x, we differentiate $\boxed{y}$ with respect to t and divide by $\dfrac{dx}{dt}$.

Now replace y in the boxes by $\dfrac{dy}{dx}$.

Then to differentiate $\boxed{\dfrac{dy}{dx}}$ with respect to x, we differentiate $\boxed{\dfrac{dy}{dx}}$ with respect to t and divide by $\dfrac{dx}{dt}$.

Thus

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\left(\frac{dx}{dt}\right).}$$

Example 5.11

Find $\dfrac{d^2y}{dx^2}$ given that $x = t^3$, $y = t^4$.

Now $\quad \dfrac{dy}{dt} = 4t^3, \quad \dfrac{dx}{dt} = 3t^2$

and
$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{4t^3}{3t^2} = \frac{4t}{3}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{4t}{3}\right)\Big/3t^2 = \frac{4}{9t^2}.$$

Example 5.12

Find and classify the stationary points on the curve represented parametrically by $x = 3t + t^3$, $y = 3t - t^3$.

We require $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Now
$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{3-3t^2}{3+3t^2} = \frac{3(1-t^2)}{3(1+t^2)} = \frac{1-t^2}{1+t^2}.$$

Also,
$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)}{3+3t^2};$$

on substituting for $\dfrac{dy}{dx}$ and $\dfrac{dx}{dt}$.

Now
$$\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right) = \frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}$$

$\dfrac{v\,du - u\,dv}{v^2}$, quotient rule

$$= \frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}$$

$$= -\frac{4t}{(1+t^2)^2}.$$

Then $$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)\bigg/\left(\frac{dx}{dt}\right)$$

$$= -\frac{4t}{(1+t^2)^2.(3+3t^2)}$$

so $$\frac{d^2y}{dx^2} = -\frac{4}{3}\frac{t}{(1+t^2)^3}.$$

For a stationary point,

$$\frac{dy}{dx} = 0$$

so that $$\frac{1-t^2}{1+t^2} = 0.$$

If $\frac{a}{b} = 0$ then $a = 0$.

$\therefore$ $$1 - t^2 = 0$$

so that $$t^2 = 1$$

and $$t = \pm 1.$$

<u>When $t = 1$</u>,

$$\frac{d^2y}{dx^2} = -\frac{4}{3}\frac{1}{(1+1^2)^3} = -\frac{4}{24} = -\frac{1}{6} < 0,$$

corresponding to a maximum point.

When $t = 1$, $x = 3(1) + (1)^3 = 4$,

$y = 3(1) - (1)^3 = 2$.

There is a maximum point at (4, 2).

<u>When $t = -1$</u>,

$$\frac{d^2y}{dx^2} = -\frac{4}{3}\frac{(-1)}{(1+(-1^2))^3} = \frac{4}{24} = \frac{1}{6} > 0,$$

corresponding to a minimum point.

When $t = -1$, $x = 3(-1) + (-1)^3 = -4$

$y = 3(-1) - (-1)^3 = -2$.

There is a minimum point at (–4, –2).

Exercises 5.6

1. A curve is given by the parametric equation $x = a\cos^3 t,\ y = a\sin^3 t$.
Show that $\dfrac{dy}{dx} = -\tan t$ and find the value of $\dfrac{d^2y}{dx^2}$ when $t = \dfrac{\pi}{6}$.

2. If $x = \dfrac{2t+1}{t}$ and $y = \dfrac{1+t}{1+2t}$ show that $\dfrac{dy}{dx} = \dfrac{t^2}{(1+2t)^2}$ and find the value of $\dfrac{d^2y}{dx^2}$ when $y = 0$.

3. Find $\dfrac{d^2y}{dx^2}$ when $x = 1 + 2\cos t,\ y = 1 + 3\sin t$.

4. If $x = \tan t,\ y = \sin t$ for $0 \le t < 2\pi$, find $\dfrac{dy}{dx}$ and show that
$$\frac{d^2y}{dx^2} = -3\cos^4 t\sin t.$$

5. Given that a curve C is represented parametrically by $x = \dfrac{t^2}{2} + 3t,\ y = t^2 - 2t$, find the value of $\dfrac{dy}{dx}$ and show that
$$\frac{d^2y}{dx^2} = \frac{8}{(t+3)^3}.$$
Show that C has only one stationary point and that this is a minimum.

6. Find and classify the stationary points on the curve represented parametrically by
$$x = t - \sin t,$$
$$y = t + 2\cos t,$$
for $0 \le t \le 2\pi$.

Chapter 6

More Coordinate Geometry

In **Chapter 5** we considered various aspects of differentiation and restated its importance in finding slopes of tangents and normals. Most of this chapter is concerned with the working of problems involving tangents and normals.

6.1 Equations of a tangent and normal to a curve

We start by considering an example.

Example 6.1

Find the equations of the tangents to the curve $y^2 = 16x$ at the points (16, 16) and (1, –4). Show that the tangents are perpendicular and find the coordinates of their point of intersection.

We find the slope of the tangent from the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$. Thus we differentiate

$$y^2 = 16x$$

with respect to x, recalling from section 5.2 that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{g}(y)) = \mathrm{g}'(y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Here $\mathrm{g}(y) = y^2$.

$$\therefore \quad 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 16$$

We don't recommend substituting $y = \pm\sqrt{16x}$.

so that $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{16}{2y} = \frac{8}{y}.$$

When $y = 16$, $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8}{16} = \frac{1}{2}.$$

The equation of the tangent at (16, 16) is then

$$y - 16 = \frac{1}{2}(x - 16)$$

so that $$2y - x - 16 = 0. \qquad (1)$$

When $y = -4$, $\quad \frac{dy}{dx} = \frac{8}{-4} = -2.$

The equation of the tangent at (1, – 4) is

$$y - (-4) = -2(x - 1)$$

giving $\quad y + 2x + 2 = 0. \qquad (2)$

The slopes of the tangents are $\frac{1}{2}$ and – 2 and since $\frac{1}{2} \times -2 = -1$ the tangents are perpendicular.

To find the point of intersection we solve equations (1) and (2).

$$2y - x - 16 = 0, \qquad (1)$$

$$y + 2x + 2 = 0. \qquad (2)$$

Then (1) × 2 + (2) gives

$$5y - 30 = 0$$

so that $\quad y = 6.$

Check in (2).
6 + 2(– 4) + 2 = 0,
as required.

Substitution for y in (1) gives

$$2 \times 6 - x - 16 = 0$$

so that $\quad x = -4.$

The point of intersection is therefore (– 4, 6).

Note that $y^2 = 16x$ was not written as $y = \pm 4\sqrt{x}$, in order to avoid the ambiguity in sign.

Example 6.2

Find the equations of the tangents to the curve

$$4x^2 + 9y^2 = 36$$

at the points $\left(\frac{12}{5}, \frac{6}{5}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$.

Find the coordinates of the point of intersection of the tangents and show that this point lies on the curve

$$4x^2 + 9y^2 = 72.$$

We find $\frac{dy}{dx}$ by differentiating

$$4x^2 + 9y^2 = 36$$

with respect to x.

$$8x + \frac{d}{dx}(9y^2) = 0.$$

Then $\quad 8x + \frac{d}{dy}(9y^2)\frac{dy}{dx} = 0.$

$\therefore \quad \frac{dy}{dx} = -\frac{8x}{18y} = -\frac{4x}{9y}.$

When $x = \dfrac{12}{5}$, $y = \dfrac{6}{5}$,

$$\frac{dy}{dx} = \frac{-4 \times \frac{12}{5}}{9 \times \frac{6}{5}} = -\frac{8}{9}.$$

The equation of the tangent is

$$y - \frac{6}{5} = -\frac{8}{9}\left(x - \frac{12}{5}\right)$$

so that $$9y - \frac{54}{5} = -8x + \frac{96}{5}.$$

$\therefore$ $$9y + 8x - \frac{150}{5} = 0$$

or $$9y + 8x - 30 = 0. \qquad (1)$$

When $x = -\dfrac{9}{5}$, $y = \dfrac{8}{5}$,

$$\frac{dy}{dx} = -\frac{4x}{9y} = -\frac{4\left(-\frac{9}{5}\right)}{9\left(\frac{8}{5}\right)} = \frac{1}{2}.$$

The equation of the tangent is

$$y - \frac{8}{5} = \frac{1}{2}\left(x + \frac{9}{5}\right).$$

so that $$2y - \frac{16}{5} = x + \frac{9}{5}.$$

$\therefore$ $$2y - x - \frac{25}{5} = 0$$

so that $$2y - x - 5 = 0. \qquad (2)$$

To find the point of intersection we solve (1) and (2),

$$9y + 8x - 30 = 0 \qquad (1)$$
$$2y - x - 5 = 0. \qquad (2)$$

(1) + 8 × (2) gives

$$9y + 16y - 30 - 40 = 0.$$

$\therefore$ $$25y = 70$$

so that $$y = \frac{14}{5}.$$

Substitution of y in (2) gives

$$\frac{28}{5} - x - 5 = 0$$

so that $$x = \frac{3}{5}.$$

The point of intersection is therefore $\left(\dfrac{3}{5}, \dfrac{14}{5}\right)$.

For this point,

$$4x^2 + 9y^2 = 4\left(\frac{3}{5}\right)^2 + 9\left(\frac{14}{5}\right)^2$$
$$= \frac{1}{25}[4 \times 9 + 9 \times 196]$$
$$= \frac{1}{25} \times 1800 = 72.$$

Whilst the Cartesian coordinate representation of curves is useful its use may often lead to difficulties when we require the equation of a tangent at a general point of a curve.

Example 6.3

Find the equation of the tangent to the curve $4x^2 + 16y^2 = 1$ at the point (x_1, y_1).

Now differentiation of

$$4x^2 + 16y^2 = 1$$

gives
$$8x + 32y\frac{dy}{dx} = 0.$$

$\therefore$
$$\frac{dy}{dx} = -\frac{8x}{32y} = -\frac{x}{4y}.$$

At (x_1, y_1),

$$\frac{dy}{dx} = -\frac{x_1}{4y_1}.$$

The equation of the tangent is

$$y - y_1 = -\frac{x_1}{4y_1}(x - x_1)$$

giving
$$4y_1y - 4y_1^2 = -x_1x + x_1^2.$$

$\therefore$
$$4y_1y + x_1x = 4y_1^2 + x_1^2. \qquad (1)$$

Equation (1) may be simplified by multiplication by 4.

(x_1, y_1) lies on the curve.

$\therefore$
$$16y_1y + 4x_1x = 16y_1^2 + 4x_1^2$$

so that the equation of the tangent is

$$16y_1y + 4x_1x = 1.$$

This form of equation is concise and therefore attractive. However, it has one disadvantage as it is written : the relationship between x_1 and y_1 is concealed. We recall that this relationship is $4x_1^2 + 16y_1^2 = 1$ and we may write

$$x_1^2 = \frac{1 - 16y_1^2}{4}$$

to obtain
$$x_1 = \frac{\pm\sqrt{1 - 16y_1^2}}{2}.$$

The equation of the tangent is therefore

$$16y_1y \pm 2\sqrt{1-16y_1^2}\,x = 1,$$

a less attractive form of the equation.

It is desirable that the relationship between the coordinates of a point lying on a curve is evident during the working of a problem.

Parametric representation is a useful aid in this respect. Let's recall that in this representation the coordinates of a point on the curve are written in terms of a parameter.

See Chapter 5

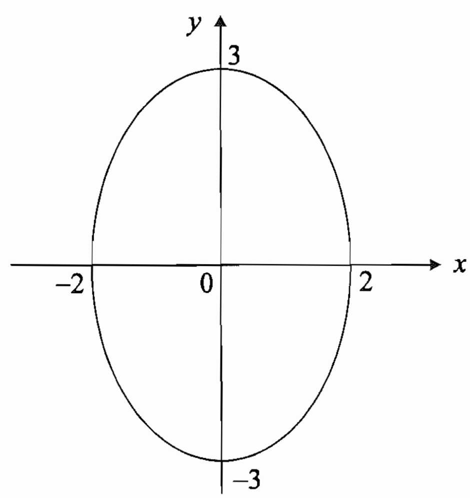
Example 6.4

Given $x = 2\cos t$, $y = 3\sin t$ we may plot the points (x, y) for various values of t. First, let's produce a table of values of x and y.

t	$x = 2\cos t$	$y = 3\sin t$
0	2	0
45°	$\sqrt{2}$	$\frac{3}{2}\sqrt{2}$
90°	0	3
135°	$-\sqrt{2}$	$\frac{3}{2}\sqrt{2}$
180°	-2	0
225°	$-\sqrt{2}$	$-\frac{3}{2}\sqrt{2}$
270°	0	-3
315°	$\sqrt{2}$	$-\frac{3}{2}\sqrt{2}$
360°	2	0

We've restricted the values of *t*, for convenience.

Values are repeated when *t* is taken to be greater than 360°.



The plot of the curve is as shown, where you are asked to assume that the complete shape is as shown.

Example 6.4 showed that when given a curve in parametric representation it is possible to plot the curve by taking values of the parameter. In principle, the Cartesian equation of the curve may be derived by eliminating the parameter.

Example 6.5

Find the Cartesian equation of the curves described by the following equations.

(a) $x = 2\cos t, \quad y = 3\sin t$

(b) $x = 1 - \cos t, \quad y = t + \sin t.$

(a) $x = 2\cos t, \quad y = 3\sin t.$

The parameter may be eliminated by noting that

$$\cos^2 t + \sin^2 t = 1.$$

Substitution for $\cos t$ and $\sin t$ gives

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$$

$$\therefore \quad \frac{x^2}{4} + \frac{y^2}{9} = 1$$

multiply by 36

or $$9x^2 + 4y^2 = 36.$$

(b) $x = 1 - \cos t, \quad y = t + \sin t.$

The elimination of t is not as obvious as in (a). Use of the x equation gives

$$x = 1 - \cos t$$

so that $$\cos t = 1 - x.$$

$$\therefore \quad t = \cos^{-1}(1 - x)$$

also $$\sin t = \pm\sqrt{1 - \cos^2 t} = \pm\sqrt{1 - (1 - x)^2} = \pm\sqrt{2x - x^2}.$$

Then use of the y equation gives

$$y = \cos^{-1}(1 - x) \pm \sqrt{2x - x^2},$$

an unattractive equation.

Example 6.5 showed that derivation of the Cartesian equation from the parametric equations may or may not be straightforward and that a Cartesian equation such as that in (b) may not reveal the features of a relationship.

Bearing these points in mind, we advise that when you are given a parametric representation you should use this representation rather than seek the Cartesian equation, unless you are asked for the latter.

Before considering the next example let's recall from Chapter 5 that if $x = x(t)$, $y(t)$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\bigg/\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Example 6.6

P is the point $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$. The tangent at P meets the x-axis at A and the y-axis at B. Show that the area of triangle AOB is independent of t.

> **We repeat : stay with the parametric representation.**

Now $\quad x = ct, \quad y = \frac{c}{t}$

and $\quad \frac{\mathrm{d}x}{\mathrm{d}t} = c, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{c}{t^2}.$

$$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\bigg/\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{c}{t^2}\bigg/c = -\frac{1}{t^2}.$$

The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ is therefore

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

giving $\quad t^2y + x - 2ct = 0.$

For the point A, $y = 0$ and

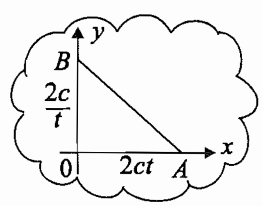
$$0 + x - 2ct = 0.$$

$$\therefore \quad x = 2ct.$$

For the point B, $x = 0$ and

$$t^2y + 0 - 2ct = 0.$$

$$\therefore \quad y = \frac{2c}{t}.$$



The area of the triangle AOB is

$$\frac{1}{2}AO \times OB = \frac{1}{2} \times 2ct \times \frac{2c}{t}$$

$$= 2c^2, \quad \text{which doesn't involve } t.$$

Example 6.7

P is the point ($2 \cos t$, $\sin t$) on the curve $x^2 + 4y^2 = 4$. The normal to the curve at P meets the x-axis at A and meets the y-axis at B.

(a) Find the coordinates of A and B, assuming that $\sin t \neq 0$, $\cos t \neq 0$.

(b) Show that C the midpoint of AB lies on the curve $16x^2 + 4y^2 = 9$.

(a) The slope of the tangent is $\dfrac{dy}{dx} = \dfrac{dy}{dt} \bigg/ \dfrac{dx}{dt}$ and the slope of the normal is

$$-\frac{1}{dy/dx} = -\frac{dx}{dt} \bigg/ \frac{dy}{dt}.$$

Slope of tangent x slope of normal = −1, the condition for perpendicular lines.

Now $x = 2 \cos t$, $y = \sin t$ so that the slope of the normal is

$$\frac{-(-2\sin t)}{\cos t} = \frac{2\sin t}{\cos t}.$$

The equation of the normal is

$$y - \sin t = \frac{2\sin t}{\cos t}(x - 2\cos t).$$

$\therefore \qquad (\cos t)y - \sin t \cos t = (2 \sin t)x - 4 \sin t \cos t$

so that $\qquad (\cos t)y - (2 \sin t)x + 3 \sin t \cos t = 0.$

<u>A</u> When $y = 0$,

$$0 - (2 \sin t)x + 3 \sin t \cos t = 0$$

so that $\qquad x = \dfrac{3}{2}\cos t,$

assuming that $\qquad \sin t \neq 0.$

<u>B</u> When $x = 0$,

$$(\cos t)y - 0 + 3 \sin t \cos t = 0$$

so that $\qquad y = -3 \sin t,$

assuming that $\qquad \cos t \neq 0.$

(b) The coordinates of the midpoint of AB, C are

$$\left(\frac{1}{2}\left(0 + \frac{3}{2}\cos t\right), \frac{1}{2}(0 - 3\sin t)\right)$$

i.e. $\qquad \left(\dfrac{3}{4}\cos t, -\dfrac{3}{2}\sin t\right).$

Thus for C, $\qquad x = \dfrac{3}{4}\cos t, \quad y = \dfrac{-3}{2}\sin t$

so that $\qquad \cos t = \dfrac{4x}{3}, \quad \sin t = -\dfrac{2y}{3}.$

Then $\cos^2 t + \sin^2 t = 1$

gives $\left(\frac{4x}{3}\right)^2 + \left(-\frac{2y}{3}\right)^2 = 1.$

$\therefore$ $\frac{16x^2}{9} + \frac{4y^2}{9} = 1$

so that $16x^2 + 4y^2 = 9.$

Example 6.8

Find the equation of the normal at the point $P(ap^2, 2ap)$ on the curve $y^2 = 4ax$. The normal at P meets the curve again at the point $Q(aq^2, 2aq)$. Show that

$$2q + pq^2 - 2p - p^3 = 0.$$

Given that $q = -3$, find the value of p.

Now $x = ap^2$, $y = 2ap$ so that the slope of the tangent is

$$\frac{dy}{dx} = \frac{dy}{dp}\bigg/\frac{dx}{dp} = \frac{2a}{2ap} = \frac{1}{p}.$$

The slope of the normal is therefore $-p$.

The equation of the normal is

$$y - 2ap = -p(x - ap^2)$$

giving $y + px - 2ap - ap^3 = 0.$ (1)

The normal cuts the curve at the point $Q(aq^2, 2aq)$.

Substitution of $x = aq^2$, $y = 2aq$ in (1) gives

$$2aq + apq^2 - 2ap - ap^3 = 0$$

which becomes $2q + pq^2 - 2p - p^3 = 0,$ (2)

on cancelling a throughout.

When $q = -3$, equation (2) becomes

$$-6 + 9p - 2p - p^3 = 0$$

or $p^3 - 7p + 6 = 0.$ (3)

By the factor theorem, $p - 1$ is a factor.

Then (3) reduces to

When $p = 1$, $p^3 - 7p + 6 = 1^3 - 7 + 6 = 0.$

$$(p - 1)(p^2 + p - 6) = 0$$

$\therefore$ $(p - 1)(p - 2)(p + 3) = 0.$

The root $p = -3$ corresponds to the point Q, so that $p = 1$ or 2.

Exercises 6.1

1. Show that the equation of the normal at the point (2, 3) on the curve $xy = 6$ is $3y - 2x - 5 = 0$. Find also the coordinates of the point at which the normal meets the curve again.

2. Show that the equation of the tangent to the curve $xy^2 = 1$ at the point $P\left(t^2, \frac{1}{t}\right)$ is $2t^3y + x - 3t^2 = 0$.

The tangent meets at the x-axis at Q. Show that the midpoint of PQ lies on the curve

$$2xy^2 = 1.$$

3. $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$, $R(ar^2, 2ar)$ are three points on the curve $y^2 = 4ax$. The tangent at P is parallel to the chord QR.

(a) Show that $q + r = 2p$.

(b) Show that the line joining P to the midpoint of QR is parallel to the x-axis.

4. Find the equation of the tangent at the point $P(5 \cos t, 3 \sin t)$ on the curve

$$9x^2 + 25y^2 = 225.$$

Show that the equation of the normal at P is

$$(3 \cos t)y - (5 \sin t)x + 16 \sin t \cos t = 0.$$

The tangent at P intersects the y-axis at R and the normal at P intersects the y-axis at S. Given that O is the origin, show that

$$OR \, . \, OS = 16.$$

5. Find the equation of the normal at the point $P(ap^2, 2ap)$ on the curve $y^2 = 4ax$. Given that this normal passes through the point $(4a, 4a)$ show that

$$p^3 - 2p - 4 = 0.$$

Show that there is only one value satisfying this equation and state this value.

6. $P(a \cos^3 t, a \sin^3 t)$ is the point on the curve C given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(a) Show that the equation of the tangent to C at P is

$$y \cos t + x \sin t = a \sin t \cos t.$$

(b) The tangent at P meets the x-axis at R and the y-axis at S. Find the maximum value of the area of triangle ORS, where O is the origin.

Chapter 7

Integration I

Integration has been considered previously in **P1** and **P2**, a number of standard forms being introduced. In this chapter, we introduce some techniques which enable us to integrate functions more complicated than the standard forms.

7.1 Standard forms

The results of the differentiation of inverse trigonometric functions carried out in **Chapter 5** enables us to extend the list of standard forms given previously in **P1** and **P2**. For convenience the standard forms are listed below. The constants of integration are omitted.

$\mathrm{g}(x)$	$\int \mathrm{g}(x)\mathrm{d}x$
x^n	$\dfrac{x^{n+1}}{n+1} (n \neq -1)$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\mathrm{cosec}^2 x$	$-\cot x$
$\mathrm{cosec}\, x \cot x$	$-\mathrm{cosec}\, x$
$\dfrac{1}{\sqrt{1-x^2}}$	$\begin{cases} \sin^{-1} x \\ -\cos^{-1} x \end{cases}$
$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
e^x	e^x
$\dfrac{1}{x}$	$\ln\lvert x\rvert$

Differentiating the right hand column gives the left hand column.

$\sin^{-1}x$ and $-\cos^{-1}x$ differ by a constant.

In **P2** we used some of the standard forms to integrate functions of the form $f(ax + b)$, where a and b are constants.

Example 7.1

In the following, the constants of integration are omitted.

(a) $$\int \cos(2x+5)\mathrm{d}x = \frac{1}{2}\sin(2x+5)$$

(b) $$\int \sin(2-4x)\mathrm{d}x = \frac{1}{4}\cos(2-4x)$$

(c) $$\int e^{5x-7}\mathrm{d}x = \frac{1}{5}e^{5x-7}$$

(d) $$\int \frac{1}{3x+2}\mathrm{d}x = \frac{1}{3}\ln|3x+2|$$

It was pointed out in **P2** that the above results only apply when the inner expression is of the form $ax + b$. Thus

$$\int \cos(3x^2+2x+7)\mathrm{d}x \neq \frac{\sin(3x^2+2x+7)}{6x+2}.$$

The type of results given in (a)-(d) above is the starting point for a further look at integration.

7.2 A first look at integration by substitution

It is clear that successful integration of functions involves the recognition of the functions as being of standard or near standard form.

Thus, to find

$$\int 7(3x-2)^4\,\mathrm{d}x$$

we recognise $(3x-2)^4$ as involving the linear expression $3x - 2$.

Then from **P2**, we may write

$$\int 7(3x-2)^4\,\mathrm{d}x = \frac{7(3x-2)^5}{5\times 3}+k = \frac{7(3x-2)^5}{15}+k.$$

$\int c(ax+b)^n\,\mathrm{d}x = \frac{c(ax+b)^{n+1}}{n+1}$, **Rule IV**

Again, to find

$$\int 2\cos(5x+7)dx$$

we recognise $\cos(5x + 7)$ as involving the linear expression $5x + 7$, and hence

$$\int 2\cos(5x+7)\mathrm{d}x = \frac{2\sin(5x+7)}{5}+k.$$

To develop further our ideas, let's reconsider the above two integrals by making use of a substitution or change of variable.

Example 7.2

Find $\int 7(3x-2)^4\,\mathrm{d}x$. (1)

Let $u = 3x - 2$. (2)

Differentiate (2) with respect to x.

$\therefore \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = 3.$

Treat $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ as a fraction

so that $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 3$

may be written $\mathrm{d}u = 3\,\mathrm{d}x$

giving $\mathrm{d}x = \dfrac{1}{3}\,\mathrm{d}u.$ (3)

Substitute for $3x - 2$ from (2) and $\mathrm{d}x$ from (3) into (1).

Then $\int 7(3x-2)^4\,\mathrm{d}x$ becomes

$$\begin{aligned}\int 7u^4\,\frac{1}{3}\,\mathrm{d}u &= \frac{7}{3}\int u^4\,\mathrm{d}u\\ &= \frac{7}{3}\int u^4\,\mathrm{d}u\\ &= \frac{7}{3}.\frac{u^5}{5} + \text{constant}\\ &= \frac{7}{15}(3x-2)^5 + \text{constant,}\end{aligned}$$

Integration is now with respect to u.

$u = 3x - 2$, from (2)

which is the answer found previously.

Example 7.3

Find $\int 2\cos(5x+7)\,\mathrm{d}x.$ (1)

Let $u = 5x + 7$ (2)

so that $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 5$

or $\mathrm{d}u = 5\,\mathrm{d}x.$

$\therefore \quad \mathrm{d}x = \dfrac{1}{5}\mathrm{d}u.$ (3)

Substitution from (2) and (3) into (1) gives

$$\begin{aligned}\int 2\cos(5x+7)\,\mathrm{d}x &= \int 2\cos u\,\frac{1}{5}\,\mathrm{d}u\\ &= \frac{2}{5}\int \cos u\,\mathrm{d}u\\ &= \frac{2}{5}\sin u + \text{constant.}\\ &= \frac{2}{5}\sin(5x+7) + \text{constant.}\end{aligned}$$

Exercises 7.1

Use a substitution of the form $u = ax + b$, where a and b are constants, to find the following integrals.

(a) $\int (2x+1)^2\,dx$ (b) $\int \frac{1}{(3x+5)^2}\,dx$ (c) $\int \sin(9x+1)\,dx$

(d) $\int \sec^2(2x+9)\,dx$ (e) $\int \operatorname{cosec}(3x+1)\cot(3x+1)\,dx$

(f) $\int \sqrt{3x-7}\,dx$ (g) $\int \frac{dx}{2x+1}$ (h) $\int \frac{1}{\sqrt{9x+2}}\,dx$

The integrations in Examples 7.2, 7.3 and Exercises 7.1 were closely related to the differentiation of a function of a function.

We recall that

$$\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$$

and both our integrals in examples 7.2, 7.3, namely

(a) $\int 7(3x-2)^4\,dx$

(b) $\int 2\cos(5x+7)\,dx$,

may be regarded as being of the form

$$\int f'(g(x))g'(x)\,dx,$$

except for the appearance of some constant factors which do not affect the general structure of the integrals.

In (a), $f'(x) = x^4$, $g(x) = 3x - 2$.
In (b), $f'(x) = \cos x$, $g(x) = 5x + 7$.

Let's therefore switch our attention to more general integrals of the form

$$\int f'(g(x))g'(x)\,dx,$$

e.g. $\int \cos(x^3 + x)(3x^2+1)dx$

where $g(x)$ is not necessarily a linear expression in x.

If we let $u = g(x)$

then $\frac{du}{dx} = g'(x)$

and $du = g'(x)\,dx$.

Then substitution for $g(x)$ and $g'(x)\,dx$ in $\int f'(g(x))g'(x)\,dx$,

gives $\int f'(u)\,du = f(u) + \text{constant}$

$= f(g(x)) + \text{constant}$.

This is an important result and is therefore displayed.

> Substitution Rule
>
> Given an integral of the form
>
> $$\int f'(g(x))g'(x)\,dx, \qquad \text{(A)}$$
>
> write $u = g(x)$
>
> $du = g'(x)\,dx$.

Before we attempt some examples, let's note the form of the integral

$$f'(g(x))g'(x)$$

in (A).

The term $g'(x)$ is obtained by differentiation of the inner part of $f'(g(x))$.

Example 7.4

Find $\int (3x^2 + 1)\cos(x^3 + x + 1)\,dx$.

i.e. $g(x) = x^3 + x + 1$

Now $\cos(x^3 + x + 1)$ is a composite function and

$\dfrac{d}{dx}(x^3 + x + 1) = 3x^2 + 1$ is also present in the integral.

Let $\quad u = (x^3 + x + 1)$

so that $\quad \dfrac{du}{dx} = 3x^2 + 1$

or $\quad du = (3x^2 + 1)\,dx.$ ← **These two give du.**

Then $\quad \int (3x^2 + 1)\cos(x^3 + x + 1)\,dx$

$$= \int \cos u\,du = \sin u + k$$

$$= \sin(x^3 + x + 1) + k.$$

Example 7.5

Use a substitution to find

$$\int 4x^3\sqrt{x^4 + 1}\,dx$$

$g(x) = x^4 + 1$

Let $\quad u = x^4 + 1$

so that $\quad du = 4x^3\,dx.$

Then the integral becomes

$$\int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du$$

$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + k$$

$$= \frac{2}{3}u^{\frac{2}{3}} + k$$

$$= \frac{2}{3}\left(x^4 + 1\right)^{\frac{3}{2}} + k.$$

Example 7.6

Use a substitution to find

$$\int (x^2 + 1) \sec^2(x^3 + 3x + 1)\, dx$$

Here we note that the inner term

$$g(x) = x^3 + 3x + 1$$

so that $\quad g'(x) = 3x^2 + 3.$

Whilst f '(x) is not present in the integral we note that $x^2 + 1 = \frac{1}{3}g'(x)$.

Let $\quad u = x^3 + 3x + 1.$

$\therefore \quad \frac{du}{dx} = 3x^2 + 3$

so that $\quad du = (3x^2 + 3)\, dx = 3(x^2 + 1)\, dx.$

Then $\quad (x^2 + 1)\, dx = \frac{1}{3}\, du.$

The integral then becomes

$$\int \sec^2 u \ \frac{1}{3}\, du = \frac{1}{3} \int \sec^2 u\, du = \frac{1}{3} \tan u + k = \frac{1}{3} \tan (x^3 + 3x + 1) + k.$$

Example 7.7

Find $\int x^4 e^{x^5+1}\, dx$.

Here we note that when we differentiate $x^5 + 1$ we almost obtain x^4, except for a missing constant.

$g(x) = x^5 + 1$

Let $\quad u = x^5 + 1$

so that $\quad \frac{du}{dx} = 5x^4$

and $\quad x^4\, dx = \frac{1}{5} du.$

The integral becomes

$$\int e^u \frac{1}{5}\, du = \frac{1}{5} \int e^u\, du = \frac{1}{5} e^u + k = \frac{1}{5} e^{x^5+1} + k.$$

Example 7.8

Find $\displaystyle\int \frac{x}{x^2+4}\,dx$.

Here we note that when we differentiate $x^2 + 4$ we obtain $2x$ which is $2 \times$ the top of the fraction.

$$
\begin{array}{lrcl}
\text{Let} & u & = & x^2 + 4 \\
\text{so that} & \dfrac{du}{dx} & = & 2x \\
\text{giving} & x\,dx & = & \dfrac{1}{2}\,du.
\end{array}
$$

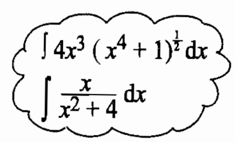
The integral becomes

$$
\begin{aligned}
\int \frac{1}{u}\frac{1}{2}\,du &= \frac{1}{2}\int \frac{1}{u}\,du \\
&= \frac{1}{2}\ln|u| + \text{constant} \\
&= \frac{1}{2}\ln|x^2+4| + \text{constant}.
\end{aligned}
$$

Examples 7.5 and 7.8 represent special situations where the integrals are of the form

constant $\times \int (g(x))^n g'(x)\,dx$,

constant $\times \displaystyle\int \frac{g'(x)}{g(x)}\,dx$.



These cases often turn up and you may find it useful to regard them as standard forms. Use of the substitution $u = g(x)$ then gives

$$
\boxed{
\begin{aligned}
\int (g(x))^n g'(x)\,dx &= \frac{(g(x))^{n+1}}{n+1} + k, \qquad &\text{(B)} \\
\int \frac{g'(x)}{g(x)}\,dx &= \ln|g(x)| + k. \qquad &\text{(C)}
\end{aligned}
}
$$

Exercises 7.2

The following integrals are of the form constant $\times \int f(g(x))\,g'(x)\,dx$. Use substitutions, or if appropriate the standard forms (B) and (C), to find the following integrals.

(a) $\int x^2(x^3+1)^4\,dx$ (b) $\displaystyle\int \frac{x^3}{x^4+2}\,dx$ (c) $\displaystyle\int \frac{x}{(x^2+3)^2}\,dx$

(d) $\int x^2 \sec^2(x^3+1)\,dx$ (e) $\displaystyle\int \frac{x}{\sqrt{1+3x^2}}\,dx$ (f) $\int (2x+1)\sin(x^2+x+5)\,dx$

(g) $\int e^x(1 + e^x)^3 \, dx$ (h) $\int\sqrt{1-\sin x}\ \cos x \, dx$ (i) $\int\sqrt{x^2+4x+1}(x+2)\, dx$

(j) $\int\frac{x^{\frac{1}{2}}}{1+x^{\frac{3}{2}}}\, dx$ (k) $\int\frac{2x+1}{x^2+x-3}\, dx$ (l) $\int (\sin(2x) + 4)^3 \cos(2x) \, dx$

(m) $\int\frac{\ln x}{x}\, dx$ (n) $\int\frac{\sin x}{1-\cos x}\, dx$ (o) $\int\frac{e^x+1}{e^x+x}\, dx$

(p) $\int \tan^2 x \sec^2 x \, dx$ (q) $\int \sin^3 x \cos x \, dx$ (r) $\int \mathrm{cosec}^2 x \cot x \, dx$

Detach one of the cosec terms.

7.3 A second look at integration by substitution

In the previous section the usefulness of the substitution $u = g(x)$ was highlighted when the integral is essentially of the form $\int f'(g(x))\, g'(x)\, dx$.

In this section examples of other situations are given where substitutions are useful.

Example 7.9

By writing $u = 5 - 3x$, integrate $x(5 - 3x)^{20}$ with respect to x.

We must find $\int x(5 - 3x)^{20} \, dx$. (1)

Let $u = 5 - 3x$. (2)

Then $\frac{du}{dx} = -3$

so that $dx = -\frac{1}{3}du$. (3)

Inspection of the integral in (1) shows that we must substitute for x, $(5 - 3x)^{20}$ and dx. The last two substitutions will use (2) and (3). For x we note that

$u = 5 - 3x$

so that $x = \frac{5-u}{3}$. (4)

Substitution from (2), (3), (4) into (1) gives

Remember the final integral must involve u only.

$$\int\left(\frac{5-u}{3}\right) u^{20}.-\frac{1}{3}\, du = -\frac{1}{9}\int(5-u)u^{20}\, du$$

$$= \frac{1}{9}\int(u^{21}-5u^{20})\, du$$

$$= \frac{u^{22}}{9\times 22}-\frac{5u^{21}}{9\times 21}+k$$

$$= \frac{(5-3x)^{22}}{198}-\frac{5(5-3x)^{21}}{189}+k.$$

Example 7.10

Find $\int \frac{2-3x}{\sqrt{1+x}}\,\mathrm{d}x$ by writing $u = \sqrt{1+x}$.

Now	$u = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$	(1)
gives	$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+x}}$.	
$\therefore$	$\mathrm{d}x = 2\sqrt{1+x}\,\mathrm{d}u$.	(2)

Now we must remove all traces of x from the integral.

What about the term $2-3x$?

Now	$u = \sqrt{1+x}$	
so that	$u^2 = 1 + x$	
and	$x = u^2 - 1$.	
The	$2 - 3x = 2 - 3(u^2 - 1)$	
so that	$2 - 3x = 5 - 3u^2$.	(3)

Substitution from (2), (3) into the integral gives

$$\int \frac{5-3u^2}{\sqrt{1+x}} 2\sqrt{1+x}\,\mathrm{d}x = \int 2(5-3u^2)\,\mathrm{d}u$$

$$= 2\int 5 - 3u^2)\,\mathrm{d}u$$

$$= 2(5u - u^3) + k$$

$$= 10u - 2u^3 + k$$

$$= 10\sqrt{1+x} - 2(1+x)^{\frac{3}{2}} + k.$$

We repeat: make sure that the final integral involves *u* terms only.

Example 7.11

Now we know that

$$\int \frac{\mathrm{d}x}{1+x^2} = \tan^{-1} x + k$$

or

$$\int \frac{\mathrm{d}u}{1+u^2} = \tan^{-1} u + k.$$

(a) By writing $x = au$, find

$$\int \frac{\mathrm{d}x}{a^2+x^2} \qquad (a \text{ is a constant}).$$

(b) By writing $u = 2x$, find

$$\int \frac{\mathrm{d}x}{1+4x^2}.$$

(a)	$x = au$	(1)
so that	$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{a}$	
or	$\mathrm{d}x = a\,\mathrm{d}u$	(2)

Substitute from (1) and (2) into $\int \frac{dx}{a^2 + x^2}$.

The integral becomes

$$\int \frac{1}{a^2 + a^2u^2} a\,du = a \int \frac{1}{a^2(1+u^2)} du$$

$$= \frac{1}{a} \int \frac{du}{1+u^2}$$

$$= \frac{1}{a} \tan^{-1} u + k$$

$$= \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + k.$$

again, only u terms are in the integral

$x = au$, $u = \frac{x}{a}$.

(b)

$$u = 2x \qquad (1)$$

so that $\frac{du}{dx} = 2$

or $dx = \frac{1}{2} du. \qquad (2)$

Substitute from (1) and (2) into $\int \frac{dx}{1+4x^2}$.

The integral becomes

$$\int \frac{1}{1+u^2} \frac{1}{2} du = \frac{1}{2} \int \frac{du}{1+u^2}$$

$$= \frac{1}{2} \tan^{-1}(u) + k$$

$$= \frac{1}{2} \tan^{-1}(2x) + k.$$

$u = 2x$

The occurrence of square roots in integrals may often be removed by trigonometric substitutions. The essence of such transformations is that

$$\sin^2 u + \cos^2 u = 1,$$

$$1 + \tan^2 u = \sec^2 u,$$

for any value of u.

Example 7.12

Use the substitution $x = 3 \sin u$ to find

$$\int \frac{dx}{\sqrt{9 - x^2}}.$$

This could also be worked by using $x = 3u$ and using $\int \frac{dx}{\sqrt{1-u^2}} = \sin^{-1} u + \text{constant}$

Now $x = 3 \sin u$

so that $\frac{dx}{du} = 3 \cos u.$

$\therefore \qquad dx = 3\cos u\, du. \qquad (1)$

x = 3 sin u

Also $\sqrt{9-x^2} = \sqrt{9-9\sin^2 u}$
$= \sqrt{9(1-\sin^2 u)}$
$= \sqrt{9\cos^2 u}$.

$\therefore \qquad \sqrt{9-x^2} = 3\cos u. \qquad (2)$

When you see $\sin^2 u$, think of $\cos^2 u$.

Substitution from (1) and (2) into the integral gives

$$\int \frac{3\cos u}{3\cos u}\, du = \int du$$
$$= \int 1\, du$$
$$= u + k$$
$$= \sin^{-1}\left(\frac{x}{3}\right) + k.$$

The integral involves *u* only.

The 1 has been inserted for the sake of greater clarity.

$x = 3\sin u$
$\therefore \sin u = \frac{x}{3}$
$\therefore u = \sin^{-1}(\frac{x}{3})$.

Notice that the trigonometric substitution enabled us to remove the square root from the integrand

Example 7.13

Use the substitution $x = 5\sin u$ to find $\int \frac{x^2}{\sqrt{25-x^2}}\, dx$.

Now $\qquad x = 5\sin u$

so that $\qquad \frac{dx}{du} = 5\cos u$

and $\qquad dx = 5\cos u\, du. \qquad (1)$

Also $\qquad \sqrt{25-x^2} = \sqrt{25-25\sin^2 u}$
$= \sqrt{25(1-\sin^2 u)}$
$= \sqrt{25\cos^2 u}$.

$\therefore \qquad \sqrt{25-x^2} = 5\cos u. \qquad (2)$

Substitution from (1) and (2) into the integral gives

$$\int \frac{(5\sin u)^2}{5\cos u} 5\cos u\, du = \int 25\sin^2 u\, du$$
$$= 25\int \sin^2 u\, du.$$

The integral $\int \sin^2 x\, dx$ requires the use of a double angle formula.

Now $\qquad \cos 2u = 1 - 2\sin^2 u$

so that $\qquad \sin^2 u = \frac{1-\cos 2u}{2}$.

See Section 4.2

$$\therefore \qquad 25\int \sin^2 u\, du = 25\int \frac{1-\cos 2u}{2}\, du$$

$$= \frac{25}{2}\int(1-\cos 2u)\,du$$

$$= \frac{25}{2}\left(\int 1\,du - \int \cos 2u\,du\right)$$

$$= \frac{25u}{2} - \frac{25}{2}.\frac{\sin 2u}{2} + k$$

$$= \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) - \frac{25}{2}.\frac{2\sin u\cos u}{2} + k$$

$$= \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) - \frac{x}{2}\sqrt{25-x^2} + k.$$

$$= \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) - \frac{25}{2}.\frac{x}{5}.\frac{\sqrt{25-x^2}}{5} + k$$

$x = 5\sin u$
$u = \sin^{-1}\left(\frac{x}{5}\right)$
$\sqrt{25-x^2} = 5\cos u$
$\therefore \cos u = \frac{\sqrt{25-x^2}}{5}.$

Exercises 7.3

Note that $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + k.$

1. Use the substitutions to find the following indefinite integrals.

(a) $\int \frac{x^2}{(x^3+1)^2}\,dx \quad (u = x^3 + 1)$

(b) $\int \frac{dx}{\sqrt{4-x^2}} \quad (x = 2u)$

(c) $\int \frac{x}{\sqrt{1-x^2}}\,dx \quad (u = 1 - x^2)$

(d) $\int \frac{x}{\sqrt{1-x^2}}\,dx \quad (x = \sin u)$

(e) $\int \frac{x}{\sqrt{2x+1}}\,dx \quad (u = 2x + 1)$

(f) $\int \frac{\cos x}{\sqrt{1+2\sin x}}\,dx \quad (u = 1 + 2\sin x)$

(g) $\int \frac{dx}{4+x^2} \quad (x = 2\tan u)$

(h) $\int \frac{x(x^2-2)}{x^2+4}\,dx \quad (u = x^2 + 4)$

(i) $\int x\sqrt{x^2+1}\,dx \quad (u = x^2 + 1)$

(j) $\int \frac{dx}{\sqrt{1-4x^2}} \quad (u = 2x)$

(k) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx \quad (u = e^x)$

(l) $\int \tan x\,dx \quad \left(u = \cos x \text{ and note that } \tan x = \frac{\sin x}{\cos x}\right)$

(m) $\int \frac{x}{\sqrt{x^2+1}}\,dx \quad (u = x^2 + 1)$

(n) $\int \frac{x}{\sqrt{x^2+1}}\,dx \quad (x = \tan u)$

(o) $\int \sin^4 x\cos x\,dx \quad (u = \sin x)$

(p) $\int \sqrt{25-x^2}\,dx \quad (x = 5\sin u)$

2. Find the following integrals by making use of suitable substitutions.

(a) $\int \frac{x}{\sqrt{4-x^2}}\, dx$ (two possible substitutions) (b) $\int \sqrt{9-x^2}\, dx$

(c) $\int \sqrt{1+\sin x}\cos x\, dx$ (d) $\int \frac{dx}{9+x^2}$

(e) $\int \frac{dx}{1+9x^2}$ (f) $\int \cos^5 x \sin x\, dx$

(g) $\int \sec^2 x \tan x\, dx$ (h) $\int \frac{x}{\sqrt{9+x^2}}\, dx$ (two possible substitutions).

3. Use the substitutions to find the following integrals.

(a) $\int \frac{5x}{\sqrt{x-2}} dx$ $(u = \sqrt{x-2})$

(b) $\int \frac{x}{\sqrt{1-x^4}} dx$ $(u = x^2)$

(c) $\int x(x-1)^6$ $(u = x-1)$

(d) $\int (x-1)(x+2)^5 dx$ $(u = x+2)$

(e) $\int \frac{x^2}{\sqrt{1-x^2}} dx$ $(x = \sin u)$

(f) $\int \frac{1}{x^2\sqrt{4-x^2}} dx$ $(x = 2\sin u)$

Chapter 8

Integration II

In this chapter we introduce some additional techniques for finding indefinite integrals and also reconsider definite integrals as introduced in **P1.**

8.1 Integration using partial fractions

Partial fractions were introduced in Section 2.2. When partial fractions are used to find integrals it turns out that two standard integrals appear frequently. For that reason we give further consideration to those types of integral.

Example 8.1

Find the following integrals.

(a) $\int \frac{4}{3+5x}\,\mathrm{d}x$ (b) $\int \frac{6}{(2x+3)^2}\,\mathrm{d}x$

It will be useful if you're able to find quickly the integrals of the above types. For that reason we summarise some general results (the constant of integration is omitted for convenience).

$$\int \frac{c\,dx}{ax+b} = \frac{c}{a}\ln|ax+b|. \qquad \text{(A)}$$

$$\int \frac{c}{(ax+b)^2}\,\mathrm{d}x = -\frac{c}{a(ax+b)}. \qquad \text{(B)}$$

substitution
u* = *ax* + *b

We recommend that you know thoroughly the above results.
Then returning to the integrals (a) – (d), we obtain by means of the formula:

(a) $$\int \frac{4}{3+5x}\,\mathrm{d}x = \frac{4}{5}\ln|3+5x|.$$

Again we omit the constant of integration

(b) $$\int \frac{6}{(2x+3)^2}\,\mathrm{d}x = \frac{-6}{2(2x+3)} = \frac{-3}{(2x+3)}.$$

Exercises 8.1

Find as quickly as possible, the following integrals.

(a) $\int \frac{1}{3-2x}\,\mathrm{d}x$ (b) $\int \frac{6}{(4x-7)^2}\,\mathrm{d}x$ (c) $\int \left[\frac{2}{3x+2}+\frac{5}{(3x+2)^2}\right]\mathrm{d}x$

We are now ready to use partial fractions to find definite integrals. You are advised to refer to Section 2.2 to remind yourself of the various forms of fraction.

Example 8.2

Use partial fractions to find the indefinite integral $\int \frac{4x-1}{(x-2)(2x+3)}\,\mathrm{d}x$.

We split the integrand into partial fractions as suggested.

Two linear factors in denominator

Let $\frac{4x-1}{(x-2)(2x+3)} = \frac{A}{x-2}+\frac{B}{2x+3}$, (1)

where A and B are constants.

Clear the fractions in (1) by multiplying throughout by $(x-2)(2x+3)$.

$\therefore \quad 4x-1 \equiv A(2x+3)+B(x-2).$ (2)

Let $x = 2$ in (2).

$\therefore \quad 4(2)-1 = A(2(2)+3)+B(0)$

so that $\quad 7 = 7A$

$\therefore \quad A = 1.$

A suitable choice of x-values eliminates one of the coefficients.

Let $x = -\frac{3}{2}$ in (2).

$\therefore \quad 4\left(-\frac{3}{2}\right)-1 = A(0)+B\left(-\frac{3}{2}-2\right)$

so that $\quad -7 = B\left(-\frac{7}{2}\right).$

$\therefore \quad B = 2.$

Using the partial fractions in the integrand, we obtain

Integrate term by term and use result (A) twice.

$$\int\left[\frac{1}{x-2}+\frac{2}{2x+3}\right]\mathrm{d}x = \ln|x-2| + 2\,.\,\frac{1}{2}\ln|2x+3| + k$$

$$= \ln|(x-2)(2x+3)| + k,$$

on combining logs.

Example 8.3

Use partial fractions to find

$$\int \frac{7x^2+2}{(x-1)(x+2)^2}\,\mathrm{d}x\,.$$

Let $$\frac{7x^2+2}{(x-1)(x+2)^2} \equiv \frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}, \quad (1)$$

One linear factor, one repeated linear factor.

where A, B and C are constants.

Clear the fractions by multiplying (1) throughout by $(x-1)(x+2)^2$.

$\therefore \quad 7x^2+2 \equiv A(x+2)^2+B(x-1)(x+2)+C(x-1). \quad (2)$

Let $x = 1$ in (2).

$\therefore \quad 7(1)^2+2 = A(1+2)^2+B(0)+C(0)$

so that $\quad 9 = 9A.$

$\therefore \quad A = 1.$

Let $x = -2$ in (2).

$\therefore \quad 7(-2)^2+2 = A(0)+B(0)+C(-2-1)$

so that $\quad 30 = -3C$

$\therefore \quad C = -10.$

There is no further obvious value of x to substitute.

Let $x = 0$ in (2).

$\therefore \quad 7(0)^2+2 = A(2)^2+B(-1)(2)+C(-1)$

so that
$$\begin{aligned} 2 &= 4A-2B-C \\ &= 4(1)-2B-(-10) \\ &= 14-2B. \end{aligned}$$

$A = 1$, $C = -10$

$\therefore \quad 2B = 14-2 = 12$

so that $\quad B = 6.$

Using the partial fractions in the integral, we obtain

$$\begin{aligned} \int\left[\frac{1}{x-1}+\frac{6}{x+2}-\frac{10}{(x+2)^2}\right]\mathrm{d}x &= \ln|x-1| + 6\ln|x+2| + \frac{10}{x+2}+k \\ &= \ln(|x-1|\,|x+2|^6) + \frac{10}{x+2}+k, \end{aligned}$$

on combining the logs.

<u>Summary</u>

The work in Examples 8.2, 8.3 may be summarised as follows. The lower case letters a, b, c, d, e, f, l, m, n, p, q, r, s, t are given constants and capital letters A, B, C, D, E and F denote constants to be determined, and k is an arbitrary constant of integration.

Case I

$$\int \left[\frac{1}{x-1} + \frac{6}{x+2} - \frac{10}{(x+2)^2} \right] \mathrm{d}x$$

$$= \int \frac{A}{ax+b} \mathrm{d}x \int \frac{B}{cx+d}$$

$$= \frac{A}{a} \ln|ax+b| + \frac{B}{c} \ln|cx+d| + k \, .$$

Case II

$$\int \frac{nx^2 + px + q}{(ax+b)(cx+d)^2} dx$$

$$= \int \frac{C}{ax+b} \mathrm{d}x + \int \frac{D}{cx+d} \mathrm{d}x + \int \frac{E}{(cx+d)^2} \mathrm{d}x$$

$$= \frac{C}{a} \ln|ax+b| + \frac{D}{c} \ln|cx+d| - \frac{E}{c(cx+d)} + k \, .$$

Exercises 8.2

1. Show that

$$\int \frac{3x+4}{(x-2)(x+3)} \mathrm{d}x = \ln|(x-2)^2(x+3)| + k.$$

2. Show that

$$\int \frac{4x+1}{(x+1)^2(x-2)} \mathrm{d}x = \ln\left|\left(\frac{x-2}{x+1}\right)\right| - \frac{1}{x+1} + k \, .$$

3. Integrate the following with respect to x.

(a) $\dfrac{x+1}{3x^2-x-2}$ (b) $\dfrac{5x+2}{x^2-4x+4}$ (c) $\dfrac{2x-5}{x^2-5x+6}$ ← **Do you need partial fractions here?**

(d) $\dfrac{1}{4x^2-9}$ (e) $\dfrac{1}{16-x^2}$ (f) $\dfrac{18-x}{12x^2-7x-12}$

(g) $\dfrac{1+5x+x^2}{x(x+1)^2}$ (h) $\dfrac{x+1}{(x-1)^2}$ (i) $\displaystyle\int \frac{1}{4x-x^2} \mathrm{d}x$

4. Find $\displaystyle\int \frac{x}{1-x^2} \mathrm{d}x$ by

(a) partial fractions (b) the substitution $u = 1 - x^2$.

8.2 Integration by parts

We have seen in Chapter 7 that under some circumstances products of functions may be integrated. Thus, for example,

$$\int x\cos(x^2)\,\mathrm{d}x = \frac{1}{2}(\sin x^2) + k.$$

substitution is $u = x^2$.

However, substitution does not assist when we attempt to find

$$\int x\cos x\,\mathrm{d}x$$

and a different approach is required.

The approach we adopt is known as integration by parts and is closely related to differentiation of a product.

Now $\qquad \dfrac{\mathrm{d}}{\mathrm{d}x}(uv) = v\dfrac{\mathrm{d}u}{\mathrm{d}x} + u\dfrac{\mathrm{d}v}{\mathrm{d}x}.$

See P2

Integrating, we obtain

$$\int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\mathrm{d}x = \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x$$

Integration and differentiation cancel each other.

so that $\qquad uv = \displaystyle\int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x$

$$\therefore \quad \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = uv - \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x. \qquad (1)$$

The result (1) is known as the integration by parts or the parts formula.

The essence of the integration by parts method is to replace an integral $\left(\int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x\right)$ by an integral that is easier to find $\left(\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x\right)$.

Example 8.4

Find $\int x\cos x\,\mathrm{d}x$.

Now the integrand is a product of two terms, namely x and $\cos x$.

The parts formula is

$$\int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = uv - \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x. \qquad (1)$$

Don't worry about the other possible choice :- $v = \cos x$, $\frac{\mathrm{d}u}{\mathrm{d}x} = x$ for the moment. We'll discuss this later.

Let $\qquad v = x, \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos x$

so that $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 1, \quad u = \sin x,$

ignoring the constant of integration when finding u from $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ by integration.

The substitution for u, v, $\frac{du}{dx}$, $\frac{dv}{dx}$ in (1) gives

$$\int x \cos x \, dx = (\sin x)x - \int \sin x \, . \, 1 \, dx$$

$$\begin{matrix} \uparrow & \uparrow & & \uparrow & \uparrow \\ v & \frac{du}{dx} & & u & v \end{matrix}$$

$$= x \sin x - (-\cos x) + k = x \sin x + \cos x + k.$$

Thus, $\int x \cos x \, dx$ was, in effect, replaced by the easier integral $\int \sin x \, dx$.

Sometimes, the integration by parts formula is used more than once in a question.

Example 8.5

Find $\int x^2 e^x \, dx$.

The parts formula is

$$\int v \frac{du}{dx} dx = uv - \int u \frac{dv}{dx} dx . \qquad (1)$$

Let $v = x^2, \quad \frac{du}{dx} = e^x$

so that $\frac{dv}{dx} = 2x, \quad u = e^x.$

Then the parts formula (1) gives

$$\int x^2 e^x \, dx = (e^x)x^2 - \int e^x \, . \, 2x \, dx$$

so that $\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx. \qquad (2)$

Now $\int x e^x \, dx$ is not immediately integrable because of the presence of the x term. Let's use integration by parts to work out $\int x e^x \, dx$.

Let $v = x, \quad \frac{du}{dx} = e^x$

so that $\frac{dv}{dx} = 1, \quad u = e^x.$

$$\int v \frac{du}{dx} dx = uv - \int u \frac{du}{dx} dx$$

Then

$$\int x e^x \, dx = e^x x - \int e^x \, . \, 1 \, dx$$

$$= e^x x - \int e^x \, dx$$

so that $\int x e^x \, dx = e^x x - e^x + k.$

Substitution from (2) and (3) into (1) gives

$$\int x^2 e^x \, dx = x^2 e^x - 2(e^x x - e^x) + k$$

$$= (x^2 - 2x + 2)e^x + k.$$

The integrals in examples 8.4 and 8.5 contained a power of x (x and x^2, respectively) and those powers of x were chosen as v. The choice for v was appropriate because u was easily found from $\frac{du}{dx}$ in each case.

These two examples can be summarised by the following rule for the choice of v and $\frac{du}{dx}$.

$\frac{du}{dx} = \cos x, u = \sin x$
$\frac{du}{dx} = e^x, \ u = e^x$

> When using integration by parts, if one of the functions involves a polynomial in x and the other function is easily integrated, let
>
> $$v = \text{the polynomial in } x,$$
> $$\frac{du}{dx} = \text{the other part of integrand.}$$

Rule I

Alternatively,

> When the other part is not easily integrated, we choose
>
> $$\frac{du}{dx} = \text{the polynomial in } x,$$
>
> and $v = $ the other part of integrand.

Rule II

Example 8.6

Find $\int x^2 \ln x \, dx$.

Here $\ln x$ is not easily integrated so Rule II applies.

Let $\quad v = \ln x, \quad \frac{du}{dx} = x^2,$

so that $\frac{dv}{dx} = \frac{1}{x}, \quad u = \frac{x^3}{3}.$

Then $\quad \int v\frac{du}{dx}dx = uv - \int u\frac{dv}{dx}dx$

gives
$$\int x^2 \ \ln x \ dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}.\frac{1}{x}dx$$
$$= \frac{x^3}{3}\ln x - \int \frac{x^2}{3}dx$$
$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + k.$$

We recall that $x^0 = 1$ so that Rule II may be applied when it is not apparent that a power of x is present.

Example 8.7

Find $\int \tan^{-1} x \, dx$.

Now $\quad \int \tan^{-1} dx = \int 1 . \tan^{-1} x \, dx.$

$x^0 = 1$

Rule II applies because $\tan^{-1} x$ is not easily integrated.

That's what the question is about!

Let $\quad v = \tan^{-1} x, \quad \dfrac{du}{dx} = 1,$

so that $\quad \dfrac{dv}{dx} = \dfrac{1}{1+x^2}, \quad u = x.$

Then
$$\int v \frac{du}{dx} dx = uv - \int u \frac{dv}{dx} dx$$

gives
$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$
$$= x \tan^{-1} x - \frac{1}{2} \ln|1+x^2| + k.$$

$\int \frac{g'(x)}{g(x)} dx = \ln|g(x)|$

Exercises 8.3

1. Use integration by parts to find the following indefinite integrals.

(a) $\int x \sin x \, dx$ (b) $\int x^2 \cos x \, dx$ (c) $\int (x+1)e^x \, dx$

(d) $\int x^3 \ln 2x \, dx$ (e) $\int x^2 e^{-x} \, dx$ (f) $\int \sin^{-1} x \, dx$

(g) $\int \ln x \, dx$ (h) $\int x \cos 2x \, dx$ (i) $\int (\pi - x) \sin 3x \, dx$

2. Show by means of a substitution that
$$\int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2},$$
omitting the constant of integration.

Use integration by parts to find
$$\int \frac{x}{\sqrt{1-x^2}} \sin^{-1} x \, dx.$$

3. Use integration by parts to show that
$$\int \frac{\ln(x+1)}{\sqrt{x+1}} dx = 2\sqrt{x+1}\left[\ln(x+1) - 2\right] + k \quad \text{assuming that } x > -1.$$

4. Use integration by parts to show that
$$\int \frac{\ln x}{(x+1)^2} dx = \frac{x}{x+1} \ln x - \ln(x+1) + k, \text{ assuming that } x > 0.$$

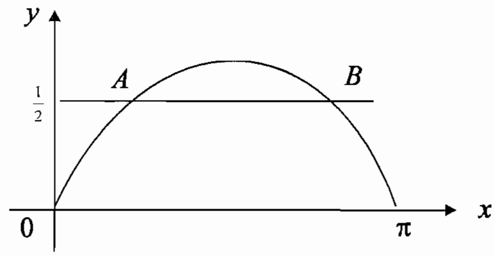
8.3 Definite integrals

We consider a number of examples. We recall that

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Example 8.8

Sketch the curve $y = \sin x$ and the line $y = \frac{1}{2}$ between $x = 0$ and $x = \pi$. For these values of x, find the area above the line and below the curve.



We require the shaded area.

The curve and the line intersect at the points A and B whose x coordinates satisfy

$$\sin x = \frac{1}{2} \quad \text{(equating values of } y\text{)}$$

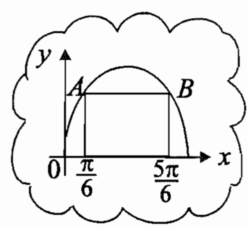
so $$x = \frac{\pi}{6}, \frac{5\pi}{6}.$$

The shaded area = area under the curve between the points A, B – area of shaded rectangle

$$= \int_{\pi/6}^{5\pi/6} \sin x \, dx - \frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$$

$$= \left[-\cos x\right]_{\pi/6}^{5\pi/6} - \frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3}.$$



Example 8.9

Find $\int_0^{\pi/2} x \cos 2x \, dx$

Use integration by parts with

$$v = x, \quad \frac{du}{dx} = \cos 2x$$

$$\frac{dv}{dx} = 1, \quad u = \frac{\sin 2x}{2}.$$

Rule I,
$$\int v \frac{du}{dx} dx = uv - \int u \frac{du}{dx} dx$$

Then $$\int_0^{\pi/2} x\cos 2x\,dx = \left[\left(\frac{\sin 2x}{2}\right)x\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\sin 2x}{2}dx$$

$$= \left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{\pi/2}$$

$$= \frac{\pi}{4}\sin\pi + \frac{\cos\pi}{4} - 0 - \frac{\cos 0}{4}$$

$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

As we have seen, integrals are sometimes evaluated by substitution of a new variable. It is sometimes troublesome to rewrite the result in terms of the original variable, (see Example 7.13). When finding definite integrals we may avoid the process of restoring the original variable by changing the limits to correspond to the new variable.

Example 8.10

Evaluate $\displaystyle\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx.$

We use the substitution

$$u = x^2 + 16$$

so that $$\frac{du}{dx} = 2x$$

and $$dx = \frac{1}{2x}\,du.$$

We also transform the limits.

When $x = 0$, $\quad u = x^2 + 16 = 0^2 + 16 = 16,$

$\qquad x = 3$, $\quad u = 3^2 + 16 = 9 + 16 = 25.$

Then the integral becomes

$$\int_{16}^{25}\frac{x}{\sqrt{u}}\frac{du}{2x} = \frac{1}{2}\int\frac{du}{\sqrt{u}}$$

$$= \frac{1}{2}\int_{16}^{25} u^{-\frac{1}{2}}\,du$$

$$= \left[u^{\frac{1}{2}}\right]_{16}^{25}$$

$$= 25^{\frac{1}{2}} - 16^{\frac{1}{2}} = 5 - 4 = 1.$$

Note the 'new' limits. Do not use 'old' limits with the new variable.

Example 8.11

Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ by means of the substitution of $x = 2\sin u$.

$$\begin{aligned}
\text{Now} \quad x &= 2\sin u \\
\text{so that} \quad \frac{dx}{du} &= 2\cos u \\
\text{or} \quad dx &= 2\cos u\,du. \\
\text{Also} \quad \sqrt{4-x^2} &= \sqrt{4-(2\sin u)^2} \\
&= \sqrt{4-4\sin^2 u} \\
&= \sqrt{4(1-\sin^2 u)} \\
&= \sqrt{4\cos^2 u}\,. \\
\therefore \quad \sqrt{4-x^2} &= 2\cos u.
\end{aligned}$$

$\sin^2 u + \cos^2 u = 1$

We also transform the limits.

$$\begin{aligned}
x = 0 \text{ gives} \quad 0 &= 2\sin u \\
\text{so that} \quad u &= \sin^{-1}\left(\frac{0}{2}\right) = 0. \\
x = 2 \text{ gives} \quad 2 &= 2\sin u \\
\text{so that} \quad \sin u &= 1, \\
u &= \sin^{-1}(1) = \frac{\pi}{2}.
\end{aligned}$$

$x = 2\sin u$

Then the integral becomes

$$\begin{aligned}
\int_0^{\pi/2} 2\cos u\, 2\cos u\,du &= 4\int_0^{\pi/2} \cos^2 u\,du \\
&= 4\int_0^{\pi/2} \frac{1+\cos 2u}{2}\,du \\
&= 2\int_0^{\pi/2} 1+\cos 2u\,du \\
&= 2\left[u + \frac{\sin 2u}{2}\right]_0^{\pi/2} \\
&= 2\left[\frac{\pi}{2} + \frac{\sin \pi}{2} - 0 - \frac{\sin 0}{2}\right] \\
&= \pi.
\end{aligned}$$

Substitute for dx, $\sqrt{4-x^2}$ and the limits

$\cos 2u = 2\cos^2 u - 1$

$\frac{\cos 2u + 1}{2} = \cos^2 u$

Example 8.12

Evaluate $\int_0^2 \frac{x(x^2+2)}{x^2+4}\mathrm{d}x$, by using the substitution $u = x^2 + 4$.

Now $\quad u = x^2 + 4$

so that $\quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

and $\quad \mathrm{d}x = \frac{1}{2x}\,\mathrm{d}u.$

We note that $x^2 + 2$ may be written in terms of u as follows:

$$x^2 + 2 = x^2 + 4 - 2 = u - 2.$$

We also transform the limits.

When $x = 0, \quad u = x^2 + 4 = 0^2 + 4 = 4,$

$x = 2, \quad u = 2^2 + 4 = 4 + 4 = 8.$

The integral becomes

$$\int_4^8 \frac{x(u-2)}{u}.\frac{1}{2x}\mathrm{d}x = \frac{1}{2}\int_4^8 \frac{u-2}{u}\mathrm{d}x$$

$$= \frac{1}{2}\left[u - 2\ln u\right]_4^8$$

$$= \frac{1}{2}[8 - 2\ln 8 - 4 + 2\ln 4]$$

$$= \frac{1}{2}\left[4 + \ln\frac{4^2}{8^2}\right] = \frac{1}{2}\left[4 + \ln\frac{16}{64}\right]$$

$$= 2 + \frac{1}{2}\ln\frac{1}{4} = 2 - \frac{1}{2}\ln 4$$

$$= 2 - \ln 2.$$

The final integral must contain u only.

Exercises 8.4

1. Evaluate the following integrals.

(a) $\int_0^4 \frac{\mathrm{d}x}{\sqrt{9-2x}}$ (b) $\int_0^1 xe^{-x^2}\mathrm{d}x$ (c) $\int_0^{\pi/2} \cos^2 x\,\mathrm{d}x$

(d) $\int_0^2 \frac{x}{\sqrt{25-4x^2}}\mathrm{d}x$ (e) $\int_0^{\pi/2} \sin^2 x\cos x\,\mathrm{d}x$

(f) $\int_0^2 \frac{x+1}{(x+2)^2}\mathrm{d}x$ **$u = x + 2$** (g) $\int_0^4 \frac{\mathrm{d}x}{\sqrt{16-x^2}}$ **$x = 4\sin u$ or $x = 4u$**

(h) $\int_0^{\pi/3} \sec^2 x\mathrm{d}x$ (i) $\int_0^5 \frac{\mathrm{d}x}{x^2+25}$ **$x = 5\tan u$ or $x = 5u$**

(j) $\int_0^{\pi/2} x^2\sin x\,\mathrm{d}x$ (k) $\int_1^2 x^4\ln x\,\mathrm{d}x$ (l) $\int_0^1 xe^{-2x}\mathrm{d}x$

2. Find the area between the curve $y = \cos x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.

3. Find the area between the curve $y = \cos^{-1} x$, the x-axis and the lines $x = 0$ and $x = 1$.

4. Find the area between the curve $y = \tan x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{4}$.

5. Find the area between the curve $y = x \sin x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.

6. Sketch the line $y = \frac{1}{2}$ and the curve $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{4}$. Evaluate the area in your sketch between the line $y = \frac{1}{2}$, the curve $y = \sin 2x$, and the line $x = \frac{\pi}{4}$.

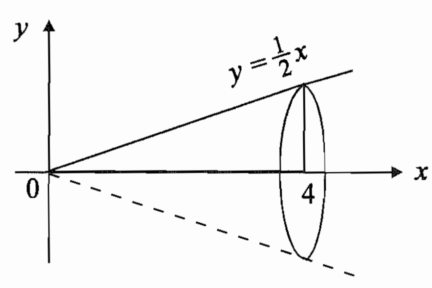
8.4 Volumes of solids of revolution

Definite integrals may be used to calculate volumes of surfaces of revolution. The method is illustrated by an example.

Example 8.13

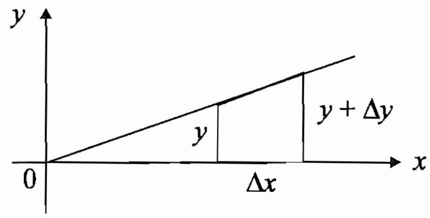
Find the volume of the solid generated by rotating about the x-axis the area under $y = \frac{1}{2}x$ from $x = 0$ to $x = 4$.

The area in question is shown shaded in the diagram. Rotation of the shaded area generates a cone.



We are asked to find the volume of this cone.

A typical element of area under $y = \frac{1}{2}x$ is shown below.



Rotation of this area about the x-axis generates a typical element of volume. This volume element is approximately a cylinder of thickness Δx, with one circular face of radius y, the other of radius $y + \Delta y$. Thus the volume generated lies between that of a cylinder of volume $\pi y^2 \Delta x$ and a cylinder of volume $\pi(y + \Delta y)^2 \Delta x$.

The size of the shaded element is exaggerated.

The sum of the volumes of the smaller (or bigger) cylinders is an approximation to the volume required. By making Δx sufficiently small, we can make this sum approach as close as we please to the volume of the solid of revolution.

Thus required volume = $\lim\limits_{\Delta x \to 0} \Sigma \pi y^2 \Delta x = \lim\limits_{\Delta x \to 0} \Sigma \pi (y + \Delta y)^2 \Delta x$.

This limit is $\int_{x=0}^{4} \pi y^2 \, dx$.

> In general, the volume of revolution between $x = a$, $x = b$ is $\int_a^b \pi y^2 d\,x$.

In the present case, $y = \frac{1}{2}x$, as given by the equation of the line.

Then the required volume is

$$\pi \int_0^4 \left(\frac{1}{2}x\right)^2 dx = \frac{\pi}{4}\int_0^4 x^2\, dx$$

$$= \left[\frac{\pi x^3}{12}\right]_0^4$$

$$= \frac{\pi . 4^3}{12} - 0 = \frac{16\pi}{3} \text{ units.}$$

Example 8.14

Find the volume of the solid generated by rotating about the x-axis

(a) the area under $y = x^2$ from $x = 1$ to $x = 2$,

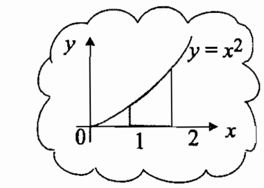
(b) the area under $y = \tan x$ from $x = 0$ to $x = \dfrac{\pi}{4}$.

The formula to be used in both cases is

$$\text{Volume} = \int_a^b \pi y^2 \, dx,$$

where $y = f(x)$.

(a) The volume is

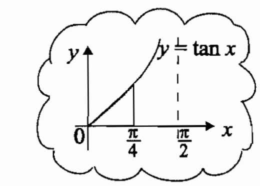


$$\pi \int_1^2 y^2 \, dx = \int_{x=1}^{x=2} (x^2)^2 \, dx = \pi \int_{x=1}^{x=2} x^4 \, dx$$

$$= \pi \left[\frac{x^5}{5} \right]_1^2$$

$$= \pi \left[\frac{32}{5} - \frac{1}{5} \right] = \frac{31\pi}{5}.$$

(b) The volume is



When you see $\tan^2 x$ think of $\sec^2 x$ because $1 + \tan^2 x = \sec^2 x$.

$$\pi \int_0^{\pi/4} y^2 \, dx = \pi \int_0^{\pi/4} (\tan x)^2 \, dx$$

$$= \pi \int_0^{\pi/4} \tan^2 x \, dx$$

$$= \pi \int_0^{\pi/4} (\sec^2 x - 1) dx$$

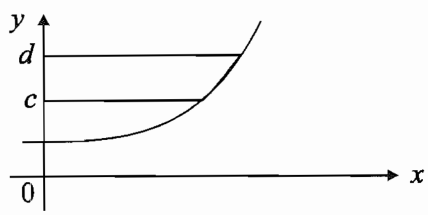
$$= \pi \left[\tan x - x \right]_0^{\pi/4}$$

$$= \pi \left[\tan \frac{\pi}{4} - \frac{\pi}{4} - 0 + 0 \right]$$

$$= \pi \left[1 - \frac{\pi}{4} \right] = \frac{\pi}{4}(4 - \pi).$$

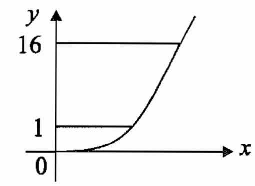
A similar formula applies if areas are rotated about the y-axis. In this case, the shaded area generates a volume $\pi \int_c^d x^2 \, dy$.

x and y have reversed their roles.



Example 8.15

Find the volume of the solid generated by rotating about the x-axis the area in the first quadrant enclosed by $y = x^4$, $y = 1$, $y = 16$ and the y-axis.



$y = x^4$, $x^2 = \sqrt{y}$

The volume is

$$\pi \int_1^{16} x^2\,dy = \pi \int_1^{16} \sqrt{y}\,dy$$

$$= \pi\left[\frac{2}{3}y^{\frac{3}{2}}\right]_1^{16}$$

$$= \frac{2\pi}{3}\left[16^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$= \frac{2\pi}{3}\left[4^3 - 1\right]$$

$$= \frac{2\pi}{3}\left[64 - 1\right]$$

$$= 42\pi.$$

Exercises 8.5

1. Find the volumes of the solids generated by rotating each of the areas bounded by the following curves and lines about the x-axis:

(a) $x + 2y - 4 = 0$, $x = 0$, $x = 4$, $y = 0$;

(b) $y = x^2 + 2$, $y = 0$, $x = 0$, $x = 1$;

(c) $y = \sqrt{x}$, $y = 0$, $x = 4$;

(d) $y = x(x - 2)$, $y = 0$, $x = 0$, $x = 2$;

(e) $y = \dfrac{1}{x}$, $y = 0$, $x = 1$, $x = 2$.

2. Find the volumes of the solids generated by rotating each of the areas bounded by the following curves and lines about the y-axis:-

(a) $y = x - 2$, $y = 0$, $y = 1$, $x = 0$;

(b) $x = \sqrt{y}$, $x = 0$, $y = 4$;

(c) $y = x^2 + 5$, $x = 0$, $y = 9$;

(d) $y^2 = x$, $x = 0$, $y = 2$.

3. The area enclosed by the curve $y = \cos x + \sin x$, the x-axis and the lines $x = 0$ and $x = \dfrac{\pi}{2}$ is rotated about the x-axis. Find the volume generated.

4. A cup is formed by rotating that part of the curve $y = x^2 - 4$ lying between (2, 0) and (3, 5) about the y-axis. Find the volume generated.

5. The area enclosed by the line $y = \dfrac{rx}{h}$, the x-axis, the lines $x = 0$, $x = h$ is rotated about the x-axis. Find the volume of the cone generated.

6. The equation of a circle centre the origin and radius r is $x^2 + y^2 = r^2$. Sketch the circle in the first quadrant ($x \geq 0$, $y \geq 0$) and shade the area enclosed by the curve and the x and y axes. Find the volume generated when the area is rotated about the x-axis. Deduce the volume of a sphere of radius r.

Chapter 9

Differential Equations

Mathematics is used in a variety of fields to develop theories: chemistry, economics, engineering, physics, sociology, for example. When mathematics is used in such a way we say that the situation has been modelled mathematically or we have set up a mathematical model of the situation.
A mathematical model may involve the rate of change of a quantity.

9.1 Rates of change

The following are examples of rates of change in various fields.

Velocity and Acceleration
If $v(t)$ is the velocity of a body at time t, its acceleration is the rate of change of velocity with time t and is therefore denoted by $\frac{dv}{dt}$.

Rates of Change of Functions
If $y = f(x)$ then $\frac{dy}{dx}$ is the rate of change of y with respect to x and is interpreted as the gradient of the curve $y = f(x)$ at a point.

Malthus Model of Population Growth
The economist Thomas Malthus developed a model involving the rate of change of a human population. If $N(t)$ is the population size at time t then the rate of change of $N(t)$ is $\frac{dN}{dt}$ or $N'(t)$.

There is an assumption here that $N(t)$ can take any non-negative value. In practice $N(t)$ may take non-negative integer values only.

Diffusion of Information
Here the spread of rumours or messages through a population is considered. If $P(t)$ is the proportion of a population that has received the message, the rate of change $\frac{dP}{dt}$ or $P'(t)$ is of interest.

Decay of Radioactive Material

Pure uranium decays very slowly into another substance due to the emission of alpha particles. The amount of uranium contained in a lump of matter will therefore decrease. If $m(t)$ is the amount of uranium then $\dfrac{dm}{dt}$ is the rate of change of the amount of uranium.

$\frac{dm}{dt} < 0$ because $m(t)$ decreases with time.

Rates of change in Economics

Rates of change of functions in Economics are often called the marginal value of the function. For example, if the cost of manufacturing x items is $C(x)$, the rate of change $\dfrac{dC}{dx}$ or $C'(x)$ is known as the marginal cost. The interpretation of $C'(x)$ is that it is the approximate cost of manufacturing an additional item when x items have been manufactured.

9.2 Differential equations

Differential equations are equations involving the derivatives or rates of change of functions. Typical examples are

$$\frac{dv}{dt} = 3t^2 + 4t + 6, \qquad \text{(i) (mechanics)}$$

$$\frac{dP}{dt} = \lambda P(1 - P), \qquad \text{(ii) (diffusion of rumours)}$$

and

$$\frac{d^2Q}{dt^2} + 3\frac{dQ}{dt} + 2Q = 2 \sin t. \qquad \text{(iii)}$$

electrical charge $Q(t)$ on a condenser

In equations (i), (ii) the highest derivative involved was the first $\left(\dfrac{dx}{dt} \text{ or } \dfrac{dP}{dt}\right)$.

In equation (iii) the highest derivative involved was the second $\left(\dfrac{d^2Q}{dt^2}\right)$.

The equations (i) and (ii) are known as first-order differential equations, equation (iii) being known as a second-order differential equation. We confine our consideration to first-order equations.

The first-order differential equations considered are of two types, namely

(a) directly integrable,

(b) variables separable.

The ideas are developed by means of examples.

(a) <u>Directly integrable</u>

These equations are of the form $\dfrac{dx}{dt} = f(t)$.

> **Variables other than x and t may be involved e.g. $\dfrac{dC}{dy} = f(y)$.**

Example 9.1

The gradient of a curve at the point (x, y) is $4x^3 + \dfrac{2}{x}$. Find y in terms of x.

Now the gradient is $\dfrac{dy}{dx}$ so that

$$\frac{dy}{dx} = 4x^3 + \frac{2}{x}.$$

To find y we require an expression which when differentiated with respect ot x gives $4x^3 + \dfrac{2}{x}$.

Then $$\frac{dy}{dx} = 4x^3 + \frac{2}{x}$$

may be rewritten as

$$y = \int \left(4x^3 + \frac{2}{x} \right) dx$$

giving $$y = x^4 + 2\ln|x| + k,$$

where k is a constant. (1)

(1) is known as the general solution of the differential equation, general because it involves an arbitrary constant k.

Two points arise out of Example 9.1. The first relates to the general method, namely that

if $\dfrac{dy}{dx} = f(x)$ then $y = \int f(x)dx$, or equivalent statements involving other variables.

(I)

The second point is that during the integration shown in (I) an unknown arbitrary constant of integration k (say) appears.

To assign a value to k, we require additional information.

Example 9.2

Find an expression for $y(x)$ given that

$$\frac{dy}{dx} = 2e^{-x} - 1$$

and that $y = 3$ when $x = 0$.

Applying (I) to

$$\frac{dy}{dx} = 2e^{-x} - 1,$$

we obtain $y = \int (2e^{-x} - 1)dx$

so that $$y = -2e^{-x} - x + k. \qquad (1)$$

Substitution of $x = 0$, $y = 3$ into (1) determines the value of k.

$$\therefore 3 = -2e^{-0} - 0 + k$$

$$\therefore k = 3 + 2e^{0} = 5.$$

Substitution of this value for k into (1) gives $y = -2e^{-x} - x + 5$.

Example 9.3

The cost of manufacturing x items is $c(x)$ pence. Assuming that the marginal cost $\frac{dc}{dx}$ is $\frac{3000}{(x+100)^2}$ and c = 50 when $x = 0$ (fixed cost), find an expression for $c(x)$. How much does it cost to manufacture 100 items?

Now $$\frac{dc}{dx} = \frac{3000}{(x+100)^2}$$

so that $$c = \int \frac{3000}{(x+100)^2} dx$$

$$\therefore c = \frac{-3000}{x+100} + k, \qquad (1)$$

where k is a constant.

Substitution of $x = 0$, $c = 50$ into (1) gives

$$50 = \frac{-3000}{100} + k$$

$$\therefore k = 80.$$

Substitution of the value of k into (1) gives

$$c = \frac{-3000}{x+100} + 80.$$

The cost of manufacturing 100 items is

$$c = \frac{-3000}{100+100} + 80 = 65 \text{ pence.}$$

Exercises 9.1

1. State whether the following differential equations are directly integrable. Find the general solutions of those equations which are directly integrable.

   (a) $\dfrac{dy}{dx} = 1 + x$ (b) $\dfrac{dy}{dx} = xy$ (c) $\dfrac{dx}{dt} = t \sin t$

   (d) $\dfrac{dx}{dt} = x \sin x$ (e) $\dfrac{dx}{dt} = x \sin t.$

2. Find the solution of

$$\frac{dx}{dt} = 4 \cos 2t,$$

   given that $x = 1$ when $t = 0$.

3. The gradient of a curve at a point (x, y) is $x\ln x$. Given that $y = 2$ when $x = 1$, find y in terms of x.

4. The cost of manufacturing x items is $c(x)$ pence. Assuming that the marginal cost $\dfrac{dc}{dx}$ is $3x^2 - 12x + 15$ pence and $c = 50$ when $x = 0$ (fixed costs) find an expression for $c(x)$. How much does it cost to manufacture 10 items?

5. Find the equations of the following curves:

   (a) A curve passing through the point $(0, -2)$ and $e^x \dfrac{dy}{dx} = 1$.

   (b) A curve passing through the point $\left(\dfrac{\pi}{2}, 0\right)$ and $\dfrac{dy}{dx} = \sin x - x$.

   (c) A curve passing through the point $(\pi, 1)$ whose gradient at the point (x,y) is $x\cos x$.

(b) Variables Separable

Here we consider equations of the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

or equivalently,

$$g(y)\frac{dy}{dx} = f(x),$$

where $f(x)$, $g(y)$ depend upon x and y only, respectively. The essence of such equations is that the x and y variables may be separated – hence the designation variables separable.

Some equations may involve other variables, of course.

Examples of such equations are

$$\frac{dy}{dx} = \frac{\sin x}{\cos y}, \qquad y\frac{dy}{dx} = x + 1, \qquad \frac{dy}{dx} = \frac{e^x}{1}.$$

$g(y) = 1$. This equation is also directly integrable.

Before solving such equations, we establish a result that will be required in the solution.

<u>Preliminary result</u>

Let's suppose y depends upon x and we differentiate $\sin y$ with <u>respect to x</u>.

Then $$\frac{d}{dx}(\sin y) = \frac{d}{dy}(\sin y)\,\frac{dy}{dx}$$

function of a function rule

so that $$\frac{d}{dx}(\sin y) = \cos y\,\frac{dy}{dx}. \quad (1)$$

Let's integrate (1) with respect to x.

Integration reverses differentiation

$$\therefore \quad \int \frac{d}{dx}(\sin y)\,dx = \int \cos y\,\frac{dy}{dx}\,dx$$

$$\therefore \quad \sin y = \int \cos y\,\frac{dy}{dx}\,dx.$$

Reversing this result, we write

No constant of integration is involved here

$$\int \cos y\,\frac{dy}{dx}\,dx = \sin y.$$

Since $$\int \cos y\ dy = \sin y,$$

we see that $$\int \cos y\,\frac{dy}{dx}\,dx = \int \cos y\ dy.$$

In effect, $\frac{dy}{dx}\,dx$ has been written as dy.

More generally,

$$\boxed{\int g(y)\frac{dy}{dx}\,dx = \int g(y)\,dy.} \quad \text{(II)}$$

Result (II) will enable us to solve equations of variable separable form.

Thus $$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

gives $$g(y)\frac{dy}{dx} = f(x).$$

Integrate both sides with respect to x.

$$\therefore \quad \int g(y)\frac{dy}{dx}\,dx = \int f(x)\,dx.$$

Using result (II), we have

$$\boxed{\int g(y)dy = \int f(x)\,dx.} \quad \text{(III)}$$

In words, result (III) states that:

(i) separate x and y,

(ii) integrate the resulting expressions with respect to x and y.

Example 9.4

Solve the differential equation

$$\frac{dy}{dx} = \frac{x}{y}$$

given that when $x = 1$, $y = 2$.

Using result (III), we obtain

$$\int y \, dy = \int x \, dx$$

so that

$$\frac{y^2}{2} = \frac{x^2}{2} + K, \qquad (1)$$

where K is an arbitrary constant of integration.

The variables are separable, $f(x) = x$, $g(y) = y$.

Whatever the form of a first order differential equation, the solution always contains one arbitrary constant.

[An aside: note that only one constant of integration is required although two integrations have been performed; because

$$\frac{y^2}{2} + C_1 = \frac{x^2}{2} + C_2$$

gives

$$\frac{y^2}{2} = \frac{x^2}{2} + C_2 - C_1$$

and $C_2 - C_1$ may be regarded as a single constant K.]

We multiply (1) throughout by 2 to clear the fractions

$$\therefore \quad y^2 = x^2 + K_1. \qquad (2)$$

$K_1 = 2K$. Such multiplication is not necessary but is convenient.

Now (2) contains an unknown constant.

Substitution of $x = 1, y = 2$ into (2) gives

$$2^2 = 1^2 + K_1$$

so that

$$K_1 = 2^2 - 1^2 = 3.$$

Substituting this value for K_1 into (2), we obtain

$$y^2 = x^2 + 3,$$

which is the required relation between x and y.

Example 9.5

Solve $\dfrac{dy}{dx} = \dfrac{y}{1+x}$, given that $y = 1$ when $x = 0$.

Result (III) gives

$$\int \frac{1}{y} \, dy = \int \frac{1}{1+x} \, dx.$$

$$\therefore \quad \ln|y| = \ln|1+x| + K$$

$$= \ln|1+x| + \ln A$$

Write $K = \ln A$ for convenience

so that

$$\ln|y| = \ln(A|1+x|),$$

on combining logs.

Then $\qquad e^{\ln|y|} = e^{\ln A|1+x|}$

or $\qquad |y| = A\,|\,1+x\,|. \qquad (1)$

Properties of logs

We assume $y > 0$ and $x + 1 > 0$ so that (1) may be written

$$y = A(x+1). \qquad (1)$$

Since $y = 1$ when $x = 0$,

$$1 = A(0+1).$$

giving $A = 1$.

Substituting this value for A in (1), we obtain

$$y = (x+1).$$

Variables-separable differential equations occur in many subjects.

Example 9.6

The amount of uranium $m(t)$ at time t contained in a lump of matter decreases with time. It is assumed that the rate of decrease of $m(t)$ at time t is proportional to the value of $m(t)$.
Write down a differential equation for $m(t)$.
At time $t = 0$, $m(t) = m_0$. If the amount of $m(t)$ is halved over a time period T, find an expression for $m(t)$ in terms of m_0 and t.

Now $m(t)$ decreases with time and the rate of decrease at time t is proportional to $m(t)$.

$$\therefore \qquad \frac{dm}{dt} = -km, \qquad (k > 0)$$

where k is an unknown constant of proportionality.

Separating variables, we obtain

The modulus is not used in ln $|m|$ because $m > 0$.

$$\int \frac{dm}{m} = \int -k\,dt.$$

$$\therefore \qquad \ln m(t) = -kt + C,$$

where C is a constant.

Then $\qquad e^{\ln m(t)} = e^{-kt+C}$

so that $\qquad m(t) = e^{-kt}\,.\,e^{C}.$

$\therefore \qquad m(t) = Ae^{-kt}. \qquad (1)$

$e^C = A$

This is the general solution. We are able to find the constant A because $m = m_0$ when $t = 0$.

Substitution of these values in (1) gives

$$m_0 = Ae^0$$

so that $\quad A = m_0.$

Substituting this value for A in (1), we obtain

$$m(t) = m_0 e^{-kt}. \qquad (2)$$

Up until now the proportionality constant k has not been specified. We have some additional information which we use as follows.

Now $\quad m = \frac{1}{2}m_0$ when $t = T.$

Substitution of these values in (2) gives

$$\frac{1}{2}m_0 = m_0 e^{-kT}$$

so that $\quad e^{-kT} = \frac{1}{2}.$

Take logs $\therefore \quad -kT = \ln\frac{1}{2}$

or $\quad kT = \ln 2.$

$\therefore \quad k = \frac{\ln 2}{T}.$

Substitution for k in (2) gives

$$m(t) = m_0 e^{-\frac{(\ln 2)t}{T}}$$

$$= m_0\left(e^{\ln 2}\right)^{-\frac{t}{T}} = m_0 2^{-\frac{t}{T}}$$

multiplication of indices. Notice the usefulness of $e^{\ln a} = a.$

$\therefore \quad m(t) = m_0\left(\frac{1}{2}\right)^{\frac{t}{T}}.$

Example 9.7

Let $P(t)$ be the proportion of a population at time t that has been infected with a disease. It is assumed that the rate of change of $P(t)$ is $rP(1-P)$, where r is a constant. Write down a differential equation satisfied by P and find the general solution of this equation.

Find $P(t)$ in terms of r and t if $P = 0.01$ when $t = 0$.

$$\frac{dP}{dt} = rP(1-P).$$

Variables separable type

Then $\quad \int \frac{dP}{P(1-P)} = \int r\,dt. \qquad (1)$

To find the left hand integral in (1) we use partial fractions.

two linear factors in denominator

Let $\quad \frac{1}{P(1-P)} \equiv \frac{A}{P} + \frac{B}{1-P}.$

$$\therefore \quad 1 \equiv A(1-P)+BP.$$

Let $P = 0$, $\therefore$ $1 = A(1-0)+B0$

so that $A = 1.$

Let $P = 1$, $\therefore$ $1 = A(0)+B$

so that $B = 1.$

$$\therefore \quad \frac{1}{P(1-P)} = \frac{1}{P}+\frac{1}{1-P}. \tag{2}$$

Then using (2) in (1), we obtain

$$\int\left[\frac{1}{P}+\frac{1}{1-P}\right]\mathrm{d}P = rt+K.$$

$$\therefore \quad \ln|P|-\ln|1-P| = rt+K.$$

$$\therefore \quad \ln\left|\frac{P}{1-P}\right| = rt+K.$$

$$\therefore \quad \mathrm{e}^{\ln\left|\frac{P}{1-P}\right|} = \mathrm{e}^{rt+K} = \mathrm{e}^{rt}\,.\,\mathrm{e}^{K} = A\mathrm{e}^{rt}.$$

$A = \mathrm{e}^K$

$$\therefore \quad \left|\frac{P}{1-P}\right| = A\mathrm{e}^{rt}. \tag{3}$$

Now $0 \le P \le 1$ so that

$$\left|\frac{P}{1-P}\right| = \frac{P}{1-P}.$$

Then (3) becomes

$$\frac{P}{1-P} = A\mathrm{e}^{rt}$$

$$\therefore \quad P = A(1-P)\mathrm{e}^{rt}$$

and

$$P(t) = \frac{A\,\mathrm{e}^{rt}}{1+A\,\mathrm{e}^{rt}} = \frac{A}{\mathrm{e}^{-rt}+A}. \tag{4}$$

This is the general solution.

[We note in passing that $\lim_{t\to\infty} P(t) = \dfrac{A}{0+A} = 1.$]

Now $P = 0.01$ when $t = 0$.

Substitution of these values in (4) gives

$$0.01 = \frac{A}{\mathrm{e}^0+A} = \frac{A}{A+1}.$$

$$\therefore \quad 0.01A+0.01 = A$$

so that

$$A = \frac{0.01}{1-0.01} = \frac{0.01}{0.99} = \frac{1}{99}.$$

Substitution of this value for A in (4) gives

$$P(t) = \frac{\frac{1}{99}}{\mathrm{e}^{-rt}+\frac{1}{99}} = \frac{1}{99\mathrm{e}^{-rt}+1}.$$

Exercises 9.2

1. Find the general solution of

$$\frac{dy}{dx} = 4x^3y^2.$$

2. Find y given that

$$\frac{dy}{dx} = e^{x+3y}$$

and that $y = 0$ when $x = 0$.

3. Find the general solution of

$$\sec x \frac{dy}{dx} + \sec y = 0.$$

4. Find the general solution of

$$\frac{dy}{dx} = \frac{2yx^2}{y^2+1}.$$

Do not attempt to write the solution in the form $y = f(x)$.

5. The height of a tank is 9 metres and it is completely full of water. At $t = 0$ the water begins to leak from a hole in the bottom of the tank. If the depth of the water at time t seconds is $x(t)$ metres and

$$100\frac{dx}{dt} = -\sqrt{x},$$

find the time (in minutes) required to empty the tank.

6. In a chemical reaction the amount $x(t)$ of one substance at time t is related to the rate of change of $x(t)$ by the differential equation

$$\frac{dx}{dt} = k(4-x)(8-x),$$

where k is a constant. If $x = 0$ when $t = 0$, express t in terms of x and k. Determine the value of k is it is known that $x = 3.6$ when $t = 2$.

7. A curve passes through a point (1, 4) and its slope at the point (x, y) is $\frac{y}{3+x}$ $(x > 0)$. Find the equation of the curve for $x > 0$.

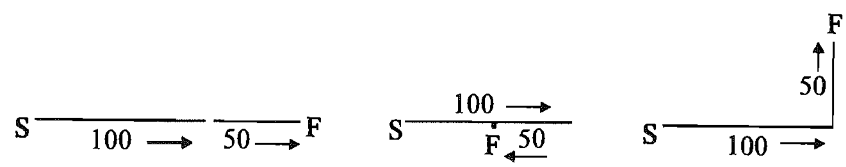
Chapter 10

Introduction to Vectors

In this chapter and the next we consider quantities which require both size (magnitude) and direction in their specification.

10.1 Vectors and scalars

Let's consider the following situation: I walk 100 metres, stop, then walk another 50 metres. How far am I from my starting point? A little thought would indicate that it's impossible to answer this question except to say that the answer lies between 50 and 150 metres. The diagram shows some of the possible situations that could arise; S is the starting point, F the finishing point.



Thus the question as posed is imprecise: we require more information concerning the directions in which I walk. In other words, to describe my final displacement SF we must specify both the directions and magnitudes of the two parts of my walk.

Quantities whose complete specifications involves both magnitude and direction are known as **vectors.**

They are represented in text by bold type, my replacement vector from S to F being denoted by **SF**, for example. When writing vectors you should underline and write quantities such as SF.

Other examples of vectors are velocity, acceleration and force (rugby players know the difference between being pushed forwards and backwards in scrums, for example).

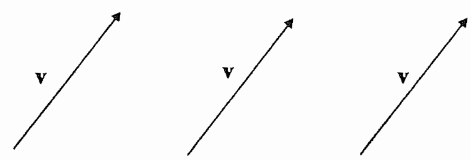
↑ 30 metres/second is different from ↓ 30 metres/second

Quantities whose complete specification involves only magnitude are called **scalars.** Examples of scalars are area, volume and speed.

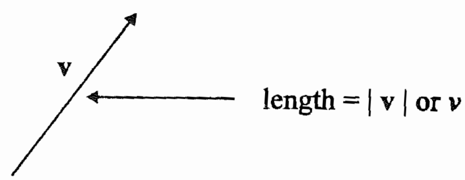
To make progress in dealing with vectors we require additional definitions and some rules of manipulation.

Definitions

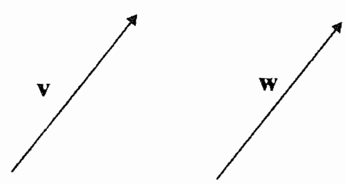
1. We represent a vector by an arrow, that is a straight line with an arrowhead on it. The tail of the vector is placed at some starting point and the direction of the arrow is chosen to be the same as the vector. Since this definition does not involve position (only magnitude and direction), we can draw an infinite number of lines to represent the vector **v**, three of which are shown.



2. The size or magnitude of a vector **v** is called its modulus and is the length of the line representing the vector. It is represented by | **v** | or *v*.



3. It follows from Definition 1 that two vectors are equal if they have the same magnitude and direction.



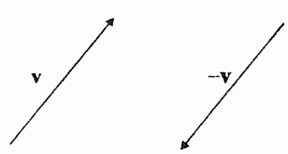
Then $\quad \mathbf{v} = \mathbf{w}$

because $\quad |\mathbf{v}| = |\mathbf{w}|$

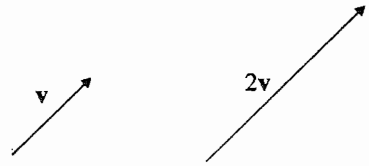
and the direction of **v** is the same as the direction of **w**.

or $v = w$

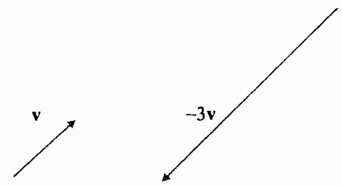
4. The vector which has the same modulus as *v* but the opposite direction is denoted by −**v**.



5. The vector *k***v**, where *k* is a real positive number, is a vector parallel to **v** of magnitude *k* | *v* | or *kv*. Thus 2**v** is parallel to **v** but is of magnitude 2 | **v** | or 2*v*.



6. The vector $-l\mathbf{v}$ (l real and positive) is a vector of magnitude $l\,|\,\mathbf{v}\,|$ with direction opposite that of **v**. Thus $-3\mathbf{v}$ has magnitude $3\,|\,\mathbf{v}\,|$ and direction opposite that of **v**.



7. Given two real numbers k and l and a vector **a**,

$$k(l\mathbf{a}) = (kl)\mathbf{a} = l(k\mathbf{a})$$

In particular, $3(2\mathbf{a}) = 6\mathbf{a} = 2(3\mathbf{a})$,

$$-4(5\mathbf{a}) = -20\mathbf{a} = 5\,(-4\mathbf{a}).$$

The vector does not have to be **v**.

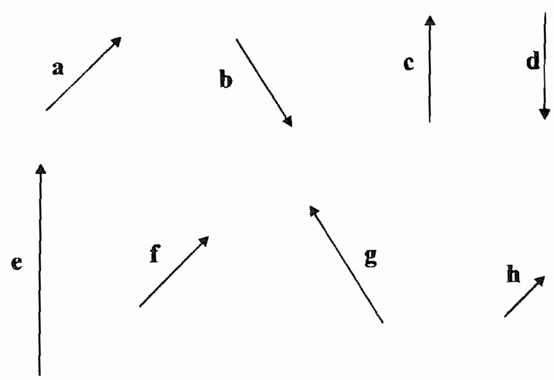
8. When a vector **b** is multiplied by 0 (zero) we obtain a vector of zero magnitude and no direction. We write this zero vector as

$$0\mathbf{v} = \mathbf{0}.$$

The zero vector is drawn as •.

Exercises 10.1

Select the correct options in relation to the following statements concerning the vectors shown.



Option	Yes	No

(i) Is $\mathbf{a} = \mathbf{b}$? (ii) $\mathbf{a} = \mathbf{0}$? (iii) $\mathbf{b} = \mathbf{g}$? (iv) $\mathbf{b} = -\mathbf{g}$?

(v) $\mathbf{b} = -l\mathbf{g}$ where $0 < 1 < 1$? (vi) $\mathbf{c} = -\mathbf{d}$ (vii) $|\,\mathbf{a}\,| = |\,\mathbf{e}\,|$?

(viii) $e = 3f$ (ix) $\mathbf{h} = \frac{1}{2}\mathbf{a}$?

10.2 Addition and Subtraction of vectors and rules of manipulation

To understand how vectors can be combined we first consider how displacements are combined.

displacements are directed distances

Let's suppose that I walk 30 metres to the east and then 40 metres to the north west, starting at S and finishing at F, and passing through the point A as shown.

Now the two displacements **SA** and **AF** have both magnitude and direction and may be regarded as vectors **a** and **b**, say. Representing the vectors as shown we see that the displacement **SA** or **a** followed by **AF** or **b** gives a resultant displacement **SF**.

Then we may regard the (displacement) vectors **a** and **b** as being added, and if

SF = **c** then

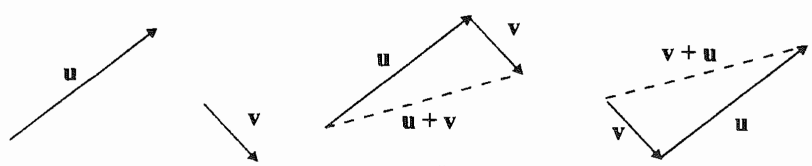
c = **a** + **b**.

This addition is different from the usual arithmetic +, of course.

This combining of displacements is an example of the general triangle law of addition of vectors.

<u>Triangular Law of Addition of Vectors</u>

Suppose we have two vectors **u** and **v** as shown. The vectors are added by placing the starting point of one vector (**v** say) next to the arrowhead of the other vector (**u**) and completing the triangle with the dotted line as shown. Since the vector **u** is drawn first (try it) the combination of vectors gives **u** + **v**.



(i) (ii)

If we draw **v** first and then place the starting position of **u** next to the arrowhead of **v** we obtain **v** + **u** as shown in (ii).

It is clear from (i) and (ii) that

u + **v** = **v** + **u**

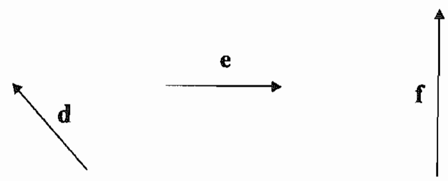
so that the order in which the vectors are combined (or 'added') is unimportant.

N.B. An alternative essentially equivalent method of adding vectors known as the parallelogram law of addition exists. We shall not introduce that method here.

Example 10.1

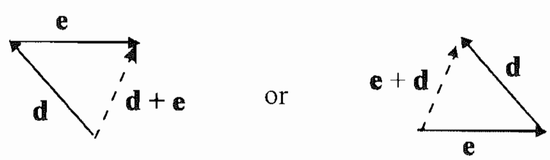
Use the triangular law of addition in the following.

(i) Add the two vectors **d** and **e**.

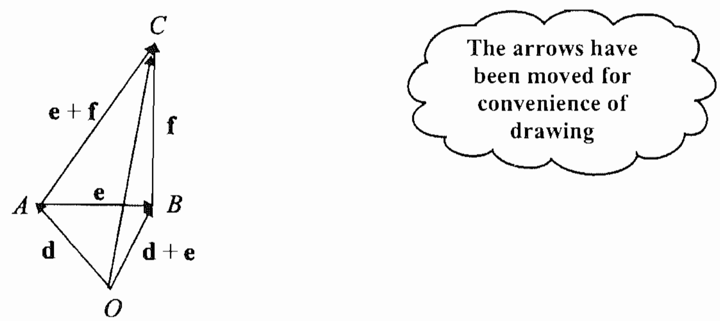


(ii) Show (**d** + **e**) + **f** = **d** + (**e** + **f**).

(i)



(ii)



Now (**d** + **e**) + **f** means add **d** and **e** first, then add **f** to the result.

Also **d** + (**e** + **f**) means add **e** and **f** first, and add this result to **d**.

In the diagram,

OA + **AB** = **OB**

and **OB** + **BC** = **OC**.

Also **AB** + **BC** = **AC**

and **OA** + **AC** = **OC**.

We see that (**d** + **e**) + **f** and **d** + (**e** + **f**) are both equal to **OC** so we conclude that

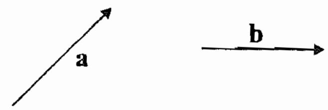
(**d** + **e**) + **f** = **d** + (**e** + **f**)

and both these sums are equal to

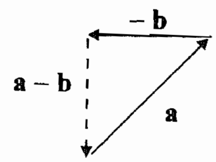
d + **e** + **f**.

<u>Subtraction of Vectors</u>

We recall that –**b** is a vector which has the same magnitude as **b** but is in the opposite direction.



We define **a** – **b** as **a** + (–**b**). Then to find **a** – **b** we reverse the direction of **b** and find **a** + (– **b**) as shown.

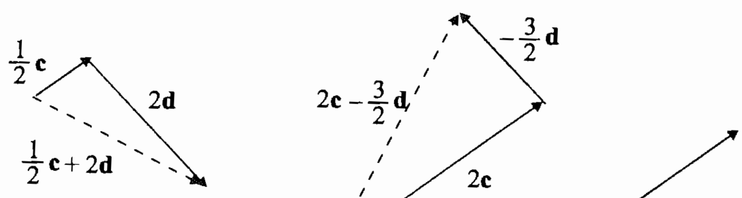


Example 10.2

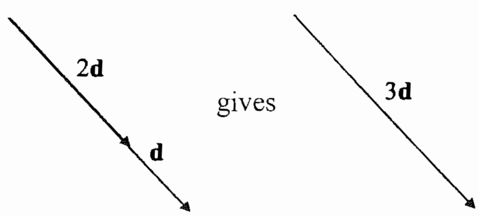
Given the vectors **c** and **d** as shown,

find

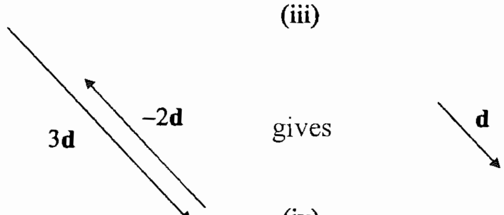
(i) $\frac{1}{2}\mathbf{c} + 2\mathbf{d}$ (ii) $2\mathbf{c} - \frac{3}{2}\mathbf{d}$ (iii) $2\mathbf{d} + \mathbf{d}$ (iv) $3\mathbf{d} - 2\mathbf{d}$



(i) **(ii)**



(iii)



(iv)

Parts (iii) and (iv) illustrate the result that if m and n are any real numbers and $\mathbf{v}$ is a vector then

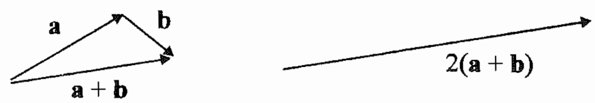
$$m\mathbf{v} + n\mathbf{v} = (m + n)\mathbf{v}$$

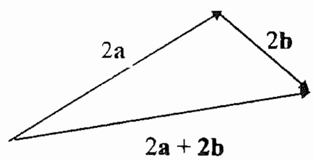
In (iii) $m = 2$, $n = 1$, in (iv) $m = 3$, $n = -2$ and $\mathbf{v} = \mathbf{d}$.

Example 10.3

Show that (i) $2(\mathbf{a} + \mathbf{b}) = 2\mathbf{a} + 2\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are shown.

(i)





From the diagrams it may be seen that $2(\mathbf{a} + \mathbf{b}) = 2\mathbf{a} + 2\mathbf{b}$.

Example 10.3 illustrates the general result that if m and are any real numbers, $\mathbf{a}$ and $\mathbf{b}$ are any vectors, then

$$m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}.$$

The addition and subtraction laws and the rules of manipulation given above enable us to simplify expressions and solve equations involving vectors, as in ordinary algebra.

Example 10.4

(i) Simplify $4(\mathbf{a} + \mathbf{b}) + 6(\mathbf{b} + \mathbf{c}) - 3(\mathbf{a} - \mathbf{b} + \mathbf{c})$.

(ii) Find the vector $\mathbf{x}$ given that

$$2\mathbf{a} + 6\mathbf{b} + 9\mathbf{x} = 8\mathbf{a} - 4\mathbf{x} - \mathbf{b}.$$

(i) $4(\mathbf{a} + \mathbf{b}) + 6(\mathbf{b} + \mathbf{c}) - 3(\mathbf{a} - \mathbf{b} + \mathbf{c}) = 4\mathbf{a} + 4\mathbf{b} + 6\mathbf{b} + 6\mathbf{c} - 3\mathbf{a} + 3\mathbf{b} - 3\mathbf{c}$

$$= \mathbf{a} + 13\mathbf{b} + 3\mathbf{c}.$$

(ii) $2\mathbf{a} + 6\mathbf{b} + 9\mathbf{x} = 8\mathbf{a} - 4\mathbf{x} - \mathbf{b}$.

We use the familiar rules of algebra.

$\therefore$ $\quad 9\mathbf{x} + 4\mathbf{x} = 8\mathbf{a} - \mathbf{b} - 2\mathbf{a} - 6\mathbf{b}$

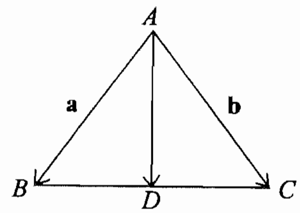
so that $\quad 13\mathbf{x} = 6\mathbf{a} - 7\mathbf{b}$

and $\quad \mathbf{x} = \frac{1}{13}(6\mathbf{a} - 7\mathbf{b})$.

We conclude this section by applying vectors in geometrical situations.

Example 10.5

The vectors **a** = **AB** and **b** = **AC** from two sides of the triangle *ABC*. *D* is the midpoint of the line *BC*. Find an expression for **DA** in terms of **a** and **b**.



Now $\mathbf{DA} = \mathbf{DB} - \mathbf{AB}$

$= \mathbf{DB} - \mathbf{a}$

(**DB** = **DA** + **AB** from triangular law)

Also $\mathbf{DB} = \frac{1}{2}\mathbf{CB} = \frac{1}{2}(\mathbf{a} - \mathbf{b})$

(**AC** = **CB** = **AB** so **CB** = **AB** – **A**)

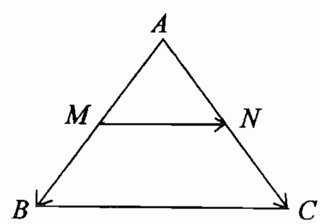
Then $\mathbf{DA} = \frac{1}{2}(\mathbf{a} - \mathbf{b}) - \mathbf{a}$

$= \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} - \mathbf{a}$

$= -\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Example 10.6

The vectors **a** = **AB** and **b** = **AC** form two sides of the triangle *ABC*. *M* and *N* are the midpoints of *AB* and *AC* respectively. Find expressions for **MN** and **BC**. Deduce that *MN* is parallel to *BC* and find the ratio $\frac{MN}{BC}$.



Now $\mathbf{BC} = \mathbf{AC} - \mathbf{AB}$

$= \mathbf{b} - \mathbf{a}$.

Also $\mathbf{MN} = \mathbf{AN} - \mathbf{AM}$

$= \frac{1}{2}\mathbf{AC} - \frac{1}{2}\mathbf{AB}$

$= \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$.

(*M*, *N* are midpoints)

Thus	$\mathbf{MN} = \frac{1}{2}\mathbf{BC}$	so that MN is parallel to BC
and	$MN = \frac{1}{2}BC$	
or	$\frac{MN}{BC} = \frac{1}{2}$.	

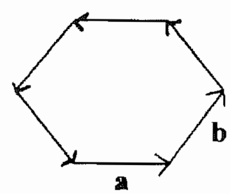
Exercises 10.2

1. By considering the diagram for the addition of two vectors, explain why

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|.$$

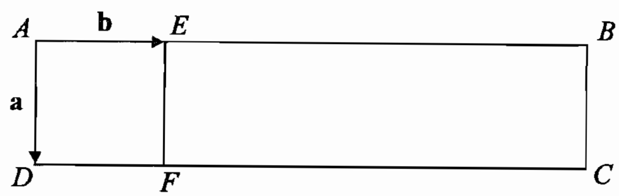
What is the relation between **a** and **b** when equality occurs?

2. **a** and **b** are vectors determined by the two adjacent sites of a regular hexagon.



Find the vectors determined by the other sides of the hexagon.

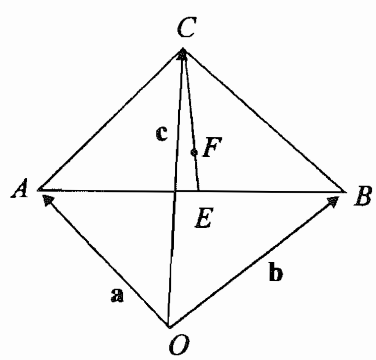
3.



The rectangle $ABCD$ is such that $AEFD$ is a square and $EB = 3AE$. If $\mathbf{AD} = \mathbf{a}$ and $\mathbf{AE} = \mathbf{b}$, find **EB, EC, DB, AC** and **FB** in terms of **a** and **b**.

4. Four points O, A, B and C are such that **OA, OB, OC** are **a**, **b** and **c** respectively.

E is the midpoint of AB and F is a point on EC such that EF is $\frac{1}{3}EC$.



Find expressions for the vectors **AB, AC, AE** and **EF.**

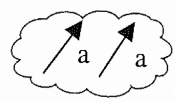
5. *O, A, B, C* and *D* are five points such that $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OC} = \mathbf{a} - 3\mathbf{b}$ and $\mathbf{OD} = 4\mathbf{a} + \mathbf{b}$. Find **AB, BC, AD** in terms of **a** and **b**.

6. Solve the following equations for the vector **x**.

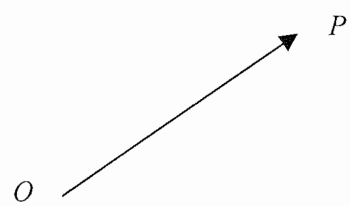
(i) $\mathbf{a} + 3\mathbf{b} - 2\mathbf{x} = 4\mathbf{b} - \mathbf{a} + 3\mathbf{x}$

(ii) $\mathbf{x} + 2\mathbf{a} - \mathbf{b} = -\frac{1}{3}\mathbf{x} + \mathbf{a} - \frac{2}{3}\mathbf{b}$

10.3 Position vectors



Up until now, the definition of a vector has not involved any specification concerning position: vectors with the same modulus and direction are equivalent, irrespective of their positions. Occasionally, it is appropriate to use a more localised definition and specify the starting position of a vector. In particular the position vector of a point *P* relative to an origin represents in magnitude and direction the line from *O* to *P*. Then **OP** is defined uniquely and cannot be represented by any other line parallel to *OP* and having the same length.



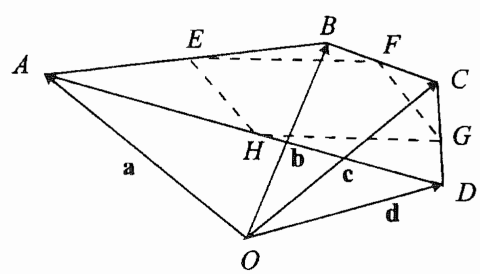
The usual rules of manipulation may be applied to position vectors.

Example 10.7

The points *A, B, C* and *D* have position vectors **a, b, c** and **d** relative to an origin *O*. The points *E, F, G* and *H* are the midpoints of *AB, BC, CD* and *AD*, respectively.

(a) Find the position vectors of *E, F, G* and *H*.

(b) Find the vectors **EH** and **FG** and deduce that *EFGH* is a parallelogram.



(a) The position vector of E is **OE** (not shown) and

E is midpoint of AB

$$\mathbf{OE} = \mathbf{OA} + \mathbf{AE}$$

so that $$\mathbf{OE} = \mathbf{a} + \frac{1}{2}\mathbf{AB} \qquad (1)$$

OB = OA + AB, *triangular law*

Now $$\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$$

so that $$\mathbf{AB} = \mathbf{b} - \mathbf{a}.$$

Substituting for **AB** in (1), we have

$$\mathbf{OE} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$$

We note in passing that the position vector of the midpoint of AB is $\frac{1}{2}$ (position vector of A + position vector of B), this being a particular case of a more general result.

Similarly, $$\mathbf{OF} = \frac{1}{2}(\mathbf{b} + \mathbf{c}), \quad \mathbf{OG} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$$

and $$\mathbf{OH} = \frac{1}{2}(\mathbf{a} + \mathbf{d}).$$

(b)
$$\begin{aligned}\mathbf{EH} &= \mathbf{OH} - \mathbf{OE}\\ &= \frac{1}{2}(\mathbf{a} + \mathbf{d}) - \frac{1}{2}(\mathbf{a} + \mathbf{b})\\ &= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{d} - \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b}\\ &= \frac{1}{2}(\mathbf{d} - \mathbf{b}).\end{aligned}$$

OH = OE + EH by triangular law.

Also
$$\begin{aligned}\mathbf{FG} &= \mathbf{OG} - \mathbf{OF}\\ &= \frac{1}{2}(\mathbf{c} + \mathbf{d}) = \frac{1}{2}(\mathbf{b} + \mathbf{c}) = \frac{1}{2}(\mathbf{d} - \mathbf{b})\end{aligned}$$

Then $$\mathbf{EH} = \mathbf{FG}$$

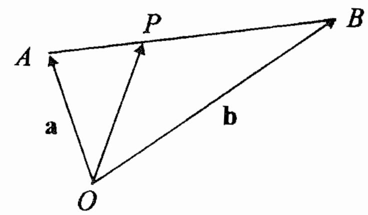
so that EH is parallel to FG

and $$EH = FG.$$

Thus, $EFGH$ is a parallelogram as it has a pair of sides which are parallel and of equal length.

In Example 10.7, we found the position vectors of the midpoints of various lines. It is often helpful to have available an expression for the positive vector of a point on a line joining two points whose position vectors are known.

To find the point which divides the join of two points in a given ratio.



Let's suppose that the points A and B have position vectors **a** and **b** and P is a point that divides AB such that $AP : PB = \lambda : \mu$. Then the position vector of P is given by

$$\mathbf{OP} = \mathbf{OA} + \mathbf{AP}$$

so that $$\mathbf{OP} = \mathbf{a} + \mathbf{AP} \qquad (1)$$

Now $$\frac{AP}{PB} = \frac{\lambda}{\mu}$$

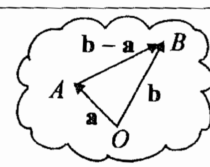
so that $$\frac{AP}{AB} = \frac{\lambda}{\lambda + \mu}$$

and $\therefore$ $$AP = \frac{\lambda}{\lambda + \mu} AB . \qquad (2)$$

> If you don't see how (2) is derived, read the following.
>
> $$\frac{AP}{PB} = \frac{\lambda}{\lambda + \mu} \text{ gives } AP = \frac{\lambda}{\mu} PB$$
>
> Then $$\frac{AP}{AB} = \frac{AP}{AP + PB} = \frac{\lambda}{\mu} PB \Big/ \frac{\lambda}{\mu} PB + PB$$
>
> $$= \frac{\lambda}{\mu} PB \Big/ \frac{\lambda PB + \mu PB}{\mu}$$
>
> $$= \frac{\lambda}{\mu} PB \Big/ \frac{PB}{\mu}(\lambda + \mu)$$
>
> $$= \frac{\lambda}{\lambda + \mu} .$$

(2) may be written in terms of vectors as

$$\mathbf{AP} = \frac{\lambda}{\lambda + \mu} \mathbf{AB} = \frac{\lambda}{\lambda + \mu} (\mathbf{b} - \mathbf{a}).$$



Substitution for **AP** in (1) then gives

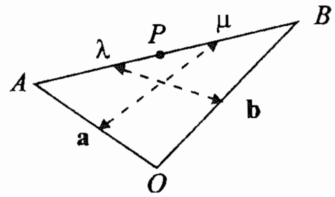
$$\mathbf{OP} = \mathbf{a} + \frac{\lambda}{\lambda + \mu}(\mathbf{b} - \mathbf{a})$$

$$= \frac{(\lambda + \mu)\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})}{\lambda + \mu}$$

$$= \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}.$$

Notes

1. Note that the pattern of the result and hence develop the ability to write down the result immediately.



$$\mathbf{OP} = \frac{\lambda \mathbf{b} + \mu \mathbf{a}}{\lambda + \mu}$$

2. You may have found the derivation of this result somewhat drawn out. You are advised to read the proof again and learn a compressed form of it. In particular, you should write down

$$\mathbf{AP} = \frac{\lambda}{\lambda + \mu}\mathbf{AB}$$

immediately.

Example 10.8

The points *A, B, C* and *D* have position vectors **a, b, c** and **a** + **b** − **c** relative to an origin *O*.

(a) Find the position vector of the point *E* on the line *AD* such that

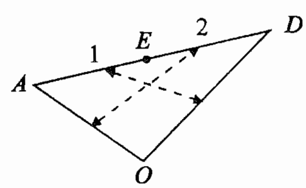
$AE : ED = 1 : 2.$

(b) Find the position vector of the point F on *AB* such that

$AF : FB = 1 : 2.$

(c) Show that *EF* is parallel to *DB*.

(a)



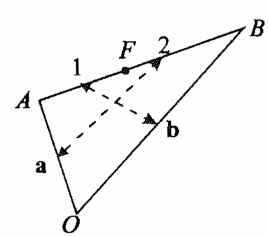
Using the result previously derived

$$\mathbf{OE} = \frac{1(\mathbf{OD}) + 2(\mathbf{OA})}{1+2} = \frac{1(\mathbf{a}+\mathbf{b}-\mathbf{c}) + 2(\mathbf{a})}{3}$$

$\lambda = 1,$ $\mu = 2$

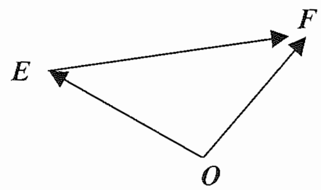
so that the position vector of E is $\frac{1}{3}(3\mathbf{a} + \mathbf{b} - \mathbf{c})$.

(b)



$$\mathbf{OF} = \frac{1(\mathbf{b}) + 2\mathbf{a}}{1+2} = \frac{1}{3}(2\mathbf{a} + \mathbf{b}).$$

(c)



$$\mathbf{EF} = \mathbf{OF} - \mathbf{OE}$$
$$= \frac{1}{3}(2\mathbf{a} + \mathbf{b}) - \frac{1}{3}(3\mathbf{a} + \mathbf{b} - \mathbf{c})$$
$$= \frac{1}{3}(2\mathbf{a} + \mathbf{b} - 3\mathbf{a} - \mathbf{b} + \mathbf{c})$$
$$= \frac{1}{3}(\mathbf{c} - \mathbf{a})$$

Similarly,
$$\mathbf{DB} = \mathbf{OB} - \mathbf{OD}$$
$$= \mathbf{b} - (\mathbf{a} + \mathbf{b} - \mathbf{c})$$
$$= \mathbf{b} - \mathbf{a} - \mathbf{b} + \mathbf{c}$$
$$= \mathbf{c} - \mathbf{a}$$

Thus
$$\mathbf{EF} = \frac{1}{3}\mathbf{BD}$$

so that EF is parallel to DB.

Exercises 10.3

1. A, B, C and D are points with position vectors **a**, **b**, **c** and $2\mathbf{a} - \mathbf{b} - \mathbf{c}$ respectively.

   (a) Find the position vector of the midpoint of BD.

   (b) Find the position vector of a point E on AD such that
   $$AE : ED = 3 : 2.$$

   (c) Find the position vector of the point E on AB produced such that
   $$AE = 3AB.$$

2. A, B and C have position vectors **a**, **b** and **c** relative to an origin O. The midpoints of OA, OB, CA and CB are D, E F and G, respectively. Show by finding their position vectors, that the midpoints of DG and EF coincide, i.e. are the same point.

3. The vertices A, B, C and D of a parallelogram $ABCD$ have position vectors **a**, **b**, **c** and **d**, respectively. Show that
   $$\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}.$$
   The points E, F lie on AC and AB, respectively where
   $$AE : EC = 1 : 3 \text{ and } AF : \text{FB} = 1 : 2.$$

   (a) Find the position vectors of E and F.

   (b) Show that D, E and F lie on a straight line.

4. The vertices A, B and C of a triangle have position vectors **a, b** and **c** respectively. The point D lies on BC such that

$$CD : DB = 2 : 1.$$

(a) Find the position vector of D.

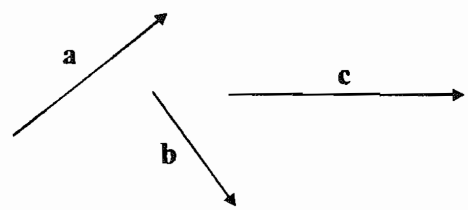
(b) Show that the point E with position vector

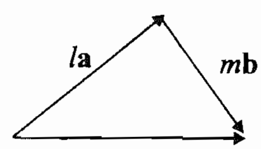
$$\frac{1}{7}\mathbf{a} + \frac{4}{7}\mathbf{b} + \frac{2}{7}\mathbf{c}$$

lies on AD and find $AE : ED$.

10.4 Two-dimensional and three-dimensional vectors

We consider two non-parallel vectors **a** and **b** (not necessarily localised) as shown. Then any vectors **c** as shown which lies in the plane containing **a** and **b** may be represented as a sum of linear multiples of **a** and **b**.





$\mathbf{c} = l\mathbf{a} + m\mathbf{b}$,
where l and m are real numbers.

A little thought and trial and error should convince you that this expressing of **c** in terms of **a** and **b** may be achieved in only one way.

> Any vector in the plane of non-parallel vectors of **a** and **b** may be represented uniquely as a sum of scalar multiples of **a** and **b**.

Example 10.9

Suppose **a** and **b** are two non-parallel vectors and $4\mathbf{a} + l\mathbf{b} = m\mathbf{a} - 2\mathbf{b}$, (1)
where l and m are real numbers.
The left hand and right hand sides of (1) are representation of a vector in terms of **a** and **b**. Since the representation is unique the coefficients of **a** and **b** on the left and right sides must be equal.

$\therefore \quad 4 = m,$

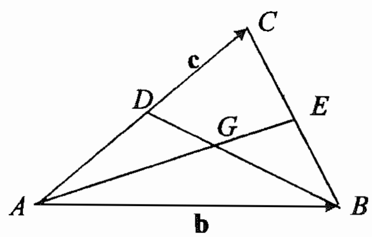
$\quad\; l = -2.$

The uniqueness of representation of a vector in terms of non-parallel vectors is very useful in problems.

This discussion has been drawn out. Once you've understood, the discussion may be shortened.

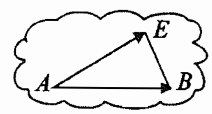
Example 10.10

The points D and E are the midpoints of the sides AC and BC respectively of triangle ABC. The lines AE and BD intersect at the point G. By first taking vectors $\mathbf{AB} = \mathbf{b}$ and $\mathbf{AC} = \mathbf{c}$, show that $DG = \frac{1}{3}DB$ and $EG = \frac{1}{3}EA$.



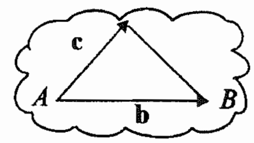
We take the vectors b and c as suggested. Our strategy is to find two expressions for the position vector of G (relative to A).

First, let's consider G as a point on AE. Then if $AG : AE = \lambda : 1$ the position vector of G relative to A is



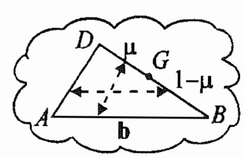
$$\begin{aligned}\mathbf{AG} &= \lambda\mathbf{AE}\\ &= \lambda(\mathbf{AB} + \mathbf{BE})\\ &= \lambda\mathbf{b} + \lambda\mathbf{BE}\\ &= \lambda\mathbf{b} + \lambda\frac{1}{2}\mathbf{BC}\\ &= \lambda\mathbf{b} + \frac{\lambda}{2}(\mathbf{c} - \mathbf{b})\end{aligned}$$

E is midpoint of *BC*



so that $$\mathbf{AG} = \frac{\lambda}{2}(\mathbf{b} + \mathbf{c}). \qquad (1)$$

Secondly, let's consider G as a point on BD and let $DG : GB = \mu : 1 - \mu$. Then position vector of G (on BD) is



$$\begin{aligned}\frac{\mu\mathbf{AB} + (1-\mu)\mathbf{AD}}{\mu + 1 - \mu} &= \mu\mathbf{AB} + (1-\mu)\mathbf{AD}\\ &= \mu\mathbf{b} + (1-\mu)\frac{1}{2}\mathbf{AC}\end{aligned}$$

$$\therefore \qquad \mathbf{AG} = \mu\mathbf{b} + \frac{1}{2}(1-\mu)\mathbf{c} \qquad (2)$$

D is midpoint of *AC*

Equating the two expressions for **AG** in(1) and (2), we obtain

$$\frac{\lambda}{2}\mathbf{b} + \frac{\lambda}{2}\mathbf{c} = \mu\mathbf{b} + \left(\frac{1}{2} - \frac{\mu}{2}\right)\mathbf{c}.$$

Since **b** and **c** are non-parallel vectors,

(**b**) $$\frac{\lambda}{2} = \mu, \qquad (3)$$

(**c**) $$\frac{\lambda}{2} = \frac{1}{2} - \frac{\mu}{2}. \qquad (4)$$

The argument given in Ex 10.9 has been shortened.

Solve (3) and (4) for λ and μ. Equating $\frac{\lambda}{2}$ from (3) and (4), we have

$$\mu = \frac{1}{2} - \frac{\mu}{2}.$$

$$\therefore \quad \frac{3\mu}{2} = \frac{1}{2}$$

$$\therefore \quad \mu = \frac{1}{3}.$$

Then from (3), $\lambda = \frac{2}{3}$.

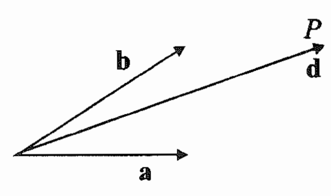
Thus $DG : DB = \mu : 1 = \frac{1}{3} : 1 = 1 : 3$

$$\therefore \; DG = \frac{1}{3}\,DB$$

and $EG : AE = 1 - \lambda : 1 = 1 - \frac{2}{3} : 1 = \frac{1}{3} : 1$ or $1 : 3$

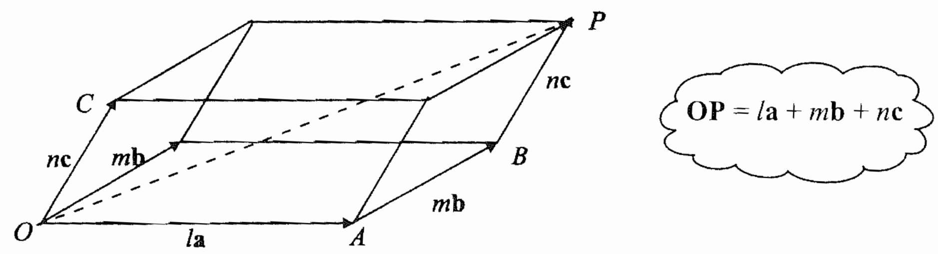
$$\therefore \; EG = \frac{1}{3}\,EA.$$

When a vector **d** is such that it does not lie in the plane of the vectors **a** and **b** it cannot be represented as a sum of multiples of **a** and **b.**



d is not contained in the plane containing **a** and **b** so that $\mathbf{d} \neq l\mathbf{a} + m\mathbf{b}$.

Crudely speaking, if we are confined to movements along the **a** and **b** directions we shall be confined to the plane containing **a** and **b.** To move out of this plane we require a movement parallel to a vector **c** (say) not contained in the plane of **a** and **b**.



It is convenient to consider the vectors as displacement vectors as shown. Then the displacement may be achieved by making

(i) a movement along **OA** ($l\mathbf{a}$),

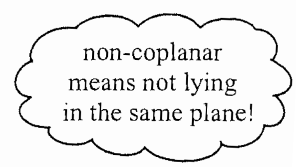
(ii) a movement along **AB** ($m\mathbf{b}$),

and (iii) a movement along **BP** ($n\mathbf{c}$).

Then $\mathbf{OP} = l\mathbf{a} + m\mathbf{b} + n\mathbf{c}$).

We ask you to accept that this representation is unique.

> Any vector **d** may be represented uniquely in terms of three non-parallel, non-coplanar vectors **a, b, c** as
>
> $\mathbf{d}$ = la + mb + nc,
>
> where l, m, n are real numbers.



The uniqueness of the representation is important.

Example 10.11

Given three non-parallel non-coplanar vectors **a, b** and **c** and

$$3\mathbf{a} + p\mathbf{b} + 4\mathbf{c} = q\mathbf{a} + 3\mathbf{b} + r\mathbf{c},$$

then by the arguments similar to those given in Example 10.9, we have

$$q = 3,$$
$$p = 3,$$
$$r = 4.$$

Example 10.12

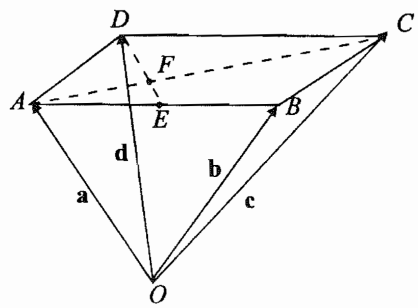
The points A, B, C, D have position vectors **a, b, c, d**, where **a, b, c**, are non-parallel, non-coplanar vectors. Given that $ABCD$ is a parallelogram, show that

$$\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}.$$

E is the midpoint of AB and the line DE intersects AC at F.

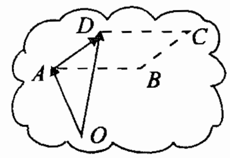
Show that $AF : AC = 1 : 3$

and $DF : DE = 2 : 3$.



The position vector of D is given by

$$\begin{aligned} \mathbf{OD} &= \mathbf{OA} + \mathbf{AD} \\ &= \mathbf{OA} + \mathbf{BC} \\ &= \mathbf{a} + \mathbf{OC} - \mathbf{OB} \\ &= \mathbf{a} + \mathbf{c} - \mathbf{b}. \end{aligned}$$



The strategy for the remainder of this question is to find two expressions for **OF** and equate them.

Let $\quad AF : AC = \lambda : 1,$

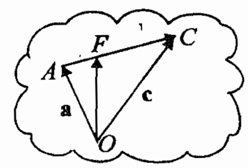
$\qquad DF : DE = \mu : 1.$

The problem is to find λ and μ.

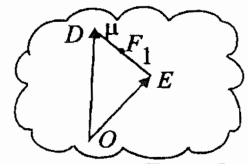
Since F lies on AC,

$$\begin{aligned} \mathbf{OF} &= \mathbf{OA} + \mathbf{AF} \\ &= \mathbf{OA} + \mathbf{AC} \\ &= \mathbf{a} + \lambda\,(\mathbf{c} - \mathbf{a}) \end{aligned}$$

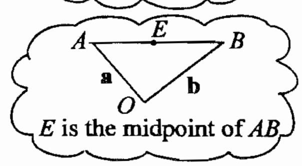
so that $\quad \mathbf{OF} = (1 - \lambda)\,\mathbf{a} + \mathbf{c}. \qquad (1)$



Similarly, since F lies on **DE**,



$$\begin{aligned} \mathbf{OF} &= \mathbf{OD} + \mathbf{DF} \\ &= \mathbf{d} + \mu\mathbf{DE} \\ &= \mathbf{d} + \mu\,(\mathbf{OE} - \mathbf{OD}) \\ &= \mathbf{d} + \mu\mathbf{OE} - \mu\mathbf{OD} \\ &= \mathbf{d}\,(1 - \mu) + \mu\,\frac{1}{2}(\mathbf{a} + \mathbf{b}) \\ &= (\mathbf{a} + \mathbf{c} - \mathbf{b})(1 - \mu) + \frac{\mu}{2}\mathbf{a} + \frac{\mu}{2}\mathbf{b} \end{aligned}$$



so that $\quad \mathbf{OF} = \mathbf{a}\left(1 - \frac{\mu}{2}\right) + \left(\frac{3\mu}{2} - 1\right)\mathbf{b} + (1 - \mu)\mathbf{c} \qquad (2)$

Equate expressions in (1) and (2) for **OF.**

$$\therefore (1 - \lambda)\mathbf{a} + \lambda\mathbf{c} = \mathbf{a}\left(1 - \frac{\mu}{2}\right) + \left(\frac{3\mu}{2} - 1\right)\mathbf{b} + (1 - \mu)\mathbf{c} \qquad (3)$$

Since **a, b** and **c** are non-parallel, non-coplanar vectors:

$$1-\lambda = 1-\frac{\mu}{2}, \qquad (4) \qquad \textbf{(a)}$$

$$0 = \frac{3\mu}{2}-1, \qquad (5) \qquad \textbf{(b)}$$

$$\lambda = 1-\mu. \qquad (6) \qquad \textbf{(c)}$$

(5) gives $\mu = \frac{2}{3}$ and $\lambda = \frac{1}{3}$, $\mu = \frac{2}{3}$ satisfies (4) and (6).

Then $\quad AF : AC = \lambda : 1 = \frac{1}{3} : 1$

and $\quad DF : DE = \mu : 1 = \frac{2}{3} : 1$

or $\quad AF : AC = 1 : 3$ and $DF : DE = 2 : 3$.

Exercises 10.4

1. Four points *A, B, C, D* have position vectors **a, b, c, a + b – c** relative to an origin, respectively. The lines *AB* and *CD* intersect at E. You may assume that **a, b, c** are non-parallel coplanar vectors.

(i) Given that $AE : AB = \lambda : 1$ and $CE : CD = \mu : 1$, write down two expressions for **OE**.

(ii) Find **OE**.

2. In the parallelogram *ABCD* the points *A, B, C, D* have position vectors **a, b, c, a + c – b**, respectively, where **a, b, c** are non-parallel non coplanar vectors. *M* is the midpoint of *BC* and *N* is the midpoint of *CD*.
The lines *AM* and *BN* intersect at the point *P*. Given that $AP : AM = \lambda : 1$ and $BP : BN = \mu : 1$, find

(a) two expressions for the position vector of *P*,

(b) the position vector of *P*.

3. The points *A, B, C* and *D* have position vectors **a, b, c** and 2**a** – 2**b** + **c** respectively, where **a, b** and **c** are non-parallel non-coplanar vectors. The lines *AC* and *BD* intersect at *P*. Show that *P* has position vector

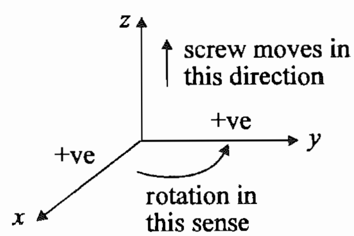
$$\frac{2}{3}\mathbf{a}+\frac{1}{3}\mathbf{c}.$$

4. The points *A, B* and *C* have position vectors **a, b** and –2**a** + **b** relative to an origin *O*, where **a** and **b** are non-parallel vectors. *L* is the midpoint of *OA*. Find the position vector of the point where *BL* intersects *AC*.

10.5 Cartesian components of vectors

In Section 10.4, we showed that any vector could be represented in terms of three non-parallel, non-coplanar vectors **a, b** and **c**. A particular choice of the vectors **a**, **b** and **c** simplifies calculations.

Let's introduce a three-dimensional Cartesian frame of reference consisting of a fixed point *O*, the origin and three mutually perpendicular axes *Ox, Oy* and *Oz*.



The arrangements of axes for a right-handed set of axes is that if we screw along the positive z-axis, the thumb rotates from the positive *x*-axis to the positive *y*-axis.

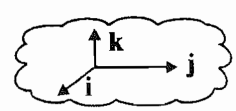
Then unit vectors **i, j, k** are taken parallel to *Ox, Oy* and *Oz* axes respectively.

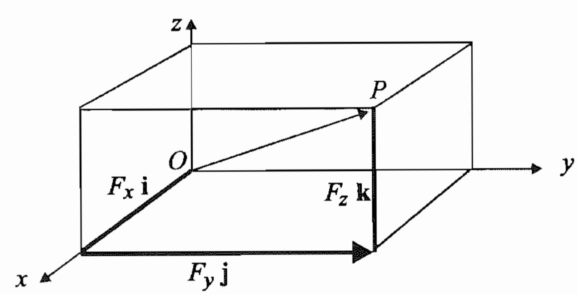
Each of magnitude 1, of course

Then a vector *F* may be written in the form

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k},$$

where $F_x\mathbf{i}$, $F_y\mathbf{j}$, $F_z\mathbf{k}$ are its components parallel to the *Ox, Oy, Oz* axes.





The vector **F** may be a line *OP* where *P* has coordinates (F_x, F_y, F_z) relative to *O*.

Note that the vector **F** may not be localised. In that case the directions of the axes are important but their location in space is not important.

For localised vectors, the location of the axes is important.

The unit vectors **i, j, k** are non-parallel and non-coplanar; and therefore the representation of a vector in terms of them is unique.

Example 10.13

Given $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

then $a = 3,\ b = -1,\ c = 4.$

Modulus of a vector

The modulus of the vector

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_x\mathbf{k}$$

is represented by the length of the line OP where P is the point (F_x, F_y, F_z). Then

$$OP = \sqrt{F_x^{\ 2} + F_y^{\ 2} + F_z^{\ 2}}$$

and thus $|\ F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}| = OP = \sqrt{F_x^{\ 2} + F_y^{\ 2} + F_z^{\ 2}}$.

A double application of Pythagoras' Theorem gives this result.

Example 10.14

The modulus of 3i – 2j – 4k is $\sqrt{3^2 + (-2)^2 + (-4)^2}$

$$= \sqrt{9 + 4 + 16} = \sqrt{29}\ .$$

Note that –2, –4 are squared and give 4 and 16.

The formula for modulus may be applied to find the modulus of the algebraic sum of a number of vectors.

Example 10.15

Given
$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k},$$
$$\mathbf{b} = 6\mathbf{i} - 2\mathbf{j} + \mathbf{k},$$
$$\mathbf{c} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k},$$

find $|\ 2\mathbf{a} - 3\mathbf{b} + \mathbf{c}|$.

Note this first step

We first find $2\mathbf{a} - 3\mathbf{b} + \mathbf{c}$.

Then $2\mathbf{a} - 3\mathbf{b} + \mathbf{c} = 2(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) - 3(6\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$= (6 - 18 + 2)\mathbf{i} + (4 + 6 + 1)\mathbf{j} + (-4 - 3 - 3)\mathbf{k}$

$= -10\mathbf{i} + 11\mathbf{j} - 10\mathbf{k}.$

add the **i, j, k** parts separately

Then $|2a - 3b + c| = \sqrt{(-10)^2 + 11^2 + (-10)^2} = \sqrt{321}.$

Given a vector in Cartesian form it is straightforward to find a vector of any given magnitude parallel to the given vector.

Example 10.16

Given a vector $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$, find

(i) a unit vector parallel to **a**,

(ii) a vector of magnitude 14 parallel to **a**.

(i) Now $|\mathbf{a}| = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}.$

Then a unit vector parallel to **a** is $k\mathbf{a}$ where $k|\mathbf{a}| = 1$.

$$\therefore \quad k = \frac{1}{|\mathbf{a}|}$$

$\therefore$ a unit vector parallel to **a** is

$$\frac{a}{|\mathbf{a}|} = \frac{1}{\sqrt{29}}(3i + 4j + 2k)$$

$$\text{or } \frac{3}{\sqrt{29}}\mathbf{i} + \frac{4}{\sqrt{29}}\mathbf{j} + \frac{2}{\sqrt{29}}\mathbf{k}.$$

N.B. It's worthwhile remembering that $\frac{a}{|a|}$ is a unit vector parallel to a.

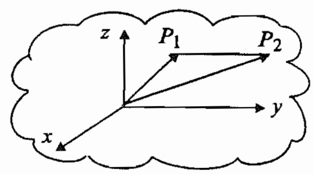
(ii) A vector parallel to **a** of magnitude 14 is $k\mathbf{a}$ where $k|\mathbf{a}| = 14$.

$$\therefore \quad k = \frac{14}{|\mathbf{a}|}$$

Then the required vector is

$$\frac{14}{|\mathbf{a}|} = \frac{14}{\sqrt{29}}(3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$

$$\text{or } \frac{42}{\sqrt{29}}\mathbf{i} + \frac{56}{\sqrt{29}}\mathbf{j} + \frac{28}{\sqrt{29}}\mathbf{k}.$$



The distance between two points in three-dimensional space may be considered as the modulus of a vector. The points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ may be regarded as being at the arrow heads of two position vectors OP_1, OP_2 so that

$$\mathbf{OP}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k},$$

$$\mathbf{OP}_2 = x_2 i + y_2\mathbf{j} + z_2\mathbf{k},$$

and $\mathbf{P}_1\mathbf{P}_1 = \mathbf{OP}_2 - \mathbf{OP}_1 = (x_1 - x_2)\mathbf{i} + (y_1 - y_2)\mathbf{j} + (z_1 - z_2)\mathbf{k}.$

Then $|\mathbf{P}_1\mathbf{P}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$

This formula may be used immediately to find the distance between two points.

Example 10.17

Find the distance between the points (1, 2, −3) and (−2, 1, −4)

Then distance $= \sqrt{(1-(-2))^2 + (2-1)^2 + (-3-(-4))^2}$

$= \sqrt{3^2 + 1^2 + 1^2} = \sqrt{11}.$

Note that $(x_1 - x_2)^2 = (x_2 - x_1)^2$, etc

Exercises 10.5

1. Find the resultant vectors and their magnitudes in (i) and (ii), given that

$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = -3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \quad \mathbf{c} = \mathbf{i} + \mathbf{k}.$

(i) $\mathbf{a} + \mathbf{b} + \mathbf{c}$

(ii) $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$.

2. Find unit vectors parallel to

(i) $\mathbf{i} + \mathbf{j} + \mathbf{k}$

(ii) $2\mathbf{i} + \mathbf{j}$

(iii) $5\mathbf{i}$.

3. Given that

$2a\mathbf{i} + 3(a + b)\mathbf{j} + 2(a - b + c)\mathbf{k} = (2 + a)\mathbf{i} + 4b\mathbf{j} + (3c - 2)\mathbf{k},$

find the values of *a, b* and *c*.

4. Find the distances between the following pairs of points:

(0, 1, −1), (2, −1, −2); (0, −1, 1), (−3, −4, −5).

Chapter 11

More on Vectors

In the first part of this chapter vector methods are used to find the equations of straight lines. Later, a method of combining two vectors to produce a scalar is considered.

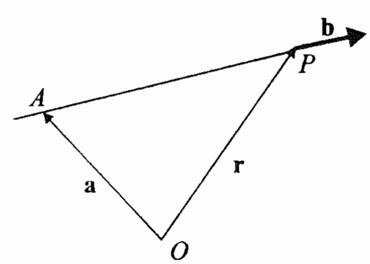
11.1 The vector equation of a straight line

A line in three-dimensional space may be specified in one of two ways, namely

(i) where the line is specified as passing through a given point in a given direction,

or (ii) where the line is specified as passing through two given points.

(i) **Line of known direction passing through a given point**



Here the line passes through the point *A* having position vector **a**, and is parallel to the vector **b**.

Let's find the position vector of a general point *P* on the line.

Now $\mathbf{r} = \mathbf{OA} + \mathbf{AP}$,

or $\mathbf{r} = \mathbf{a} + \mathbf{AP}$. (1)

The vector **AP** is parallel to the vector **b** and is therefore of the form $\lambda\mathbf{b}$, where λ is some real number. Then the equation(1) may be converted into

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}. \qquad (2)$$

(2) is called the vector equation of the line; as λ varies in (2) various points on the line will be obtained.

The equation (2) may be expressed in Cartesian form.

Then if

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$
$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

(2) may be written

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + \lambda(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

so that $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (a_1 + \lambda b_1)\mathbf{i} + (a_2 + \lambda b_2)\mathbf{j} + (a_3 + \lambda b_3)\mathbf{k}$.

Then considering the terms in **i, j, k,** we obtain

$$\left.\begin{aligned} x &= a_1 + \lambda b_1, \\ y &= a_2 + \lambda b_2, \\ \text{and} \quad z &= a_3\ \lambda b_3. \end{aligned}\right\} \qquad (3)$$

(3) are called the parametric equations of the line. When λ is eliminated from each of the three equations in (3), we obtain

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}. \qquad (= \lambda)$$

Example 11.1

Find the vector form and Cartesian form of the equation passing through (1, 2, –4) which is parallel to the vector 4i – **2j** + 3**k**.

The point (1, 2, –4) has position vector **i** + 2**j** – 4**k**.

Then the vector equation of the line is

$$\begin{aligned} \mathbf{r} &= \mathbf{i} + 2\mathrm{j} - 4\mathbf{k} + \lambda(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &= (1 + 4\lambda)\mathbf{i} + (2 - 2\lambda)\mathbf{j} + (-4 + 3\lambda)\mathbf{k}. \end{aligned}$$

The Cartesian forms of the equations of the straight line are

$$x = 1 + 4\lambda$$
$$y = 2 - 2\lambda$$

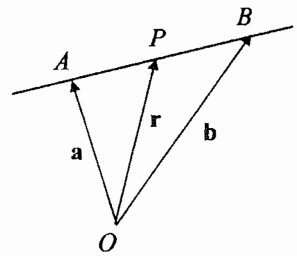
and

$$z = -4 + 3\lambda$$

or

$$\frac{x-1}{4} = \frac{y-2}{-2} = \frac{z+4}{3}.$$

(ii) **Line passing through two known points**



Here the line passes through the points A and B having position vectors **a** and **b**, respectively. Let's find an expression for the position vector **r** of some general point P on the line AB.

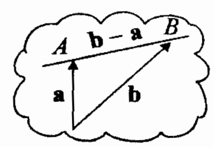
Then $$\mathbf{r} = \mathbf{OA} + \mathbf{AP}$$

or $$\mathbf{r} = \mathbf{a} + \mathbf{AP}. \qquad (4)$$

Now **AP** is parallel to the vector **AB** so that

$$\mathbf{AP} = \lambda\mathbf{AB},$$

where λ is some real number.



Also $$\mathbf{AB} = \mathbf{b} - \mathbf{a}$$

so that $$\mathbf{AP} = \lambda(\mathbf{b} - \mathbf{a}). \qquad (5)$$

Substitution from (5) into (4) gives

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) \qquad (6)$$

(6) is the equation of the straight line; various points on the line are produced as λ varies. The Cartesian form of the equation is easily found from (6). Then if

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

and $$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

are substituted into (5), we have

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + \lambda(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}).$$

Consideration of the terms in **i, j** and **k** then give

$$\left.\begin{aligned} x &= a_1 + \lambda(b_1 - a_1), \\ y &= a_2 + \lambda(b_2 - a_2), \\ \text{and} \quad z &= a_3 + \lambda(b_3 - a_3). \end{aligned}\right\} \qquad (7)$$

Example 11.2

Find the vector form of the equation of the straight line passing through the points (1, –1, –2) and (–1, 1, –3).

The position vectors are

$$\mathbf{a} = \mathbf{i} - \mathbf{j} - 2\mathbf{k},$$
$$\mathbf{b} = -\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

so that the vector form of the line is

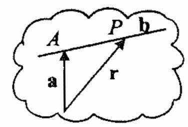
$$\mathbf{r} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda[-\mathbf{i} + \mathbf{j} - 3\mathbf{k} - (\mathbf{i} - \mathbf{j} - 2\mathbf{k})]$$

or $$x\mathbf{i} + y\mathbf{i} + z\mathbf{k} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda[-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}].$$

It is useful to look again at the vector equations of the straight lines.

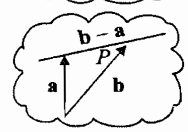
Case (i)

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}.$$



Case (ii)

$$\mathbf{r} = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a}).$$



In the case (i), the direction of the line is parallel to the vector **b**; in case (ii) the direction of the line is parallel to the vector **b** – **a**.

In both cases (i) and (ii) the direction of the line is given by the terms in λ.

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$,
$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$.

Example 11.3

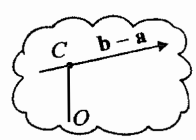
(a) Find the equation of the line which is parallel to the line passing through the points *A* (4, –3, 11) and *B* (2, 1, 5) and which passes through the point *C* (–1, 1, 3).

(b) Find a unit vector parallel to the straight line found in (a).

(a) The position vectors **a** and **b** are

$$\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + 11\mathbf{k},$$
$$\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}.$$

Then $$\mathbf{b} - \mathbf{a} = -2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}.$$



The equation of the straight line is

$$\mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(-2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}). \quad (1)$$

(b) The direction of the straight line in (a) is given by the term in λ, i.e. by the vector $-2\mathbf{i}+4\mathbf{j}-6\mathbf{k}$. The magnitude of this vector is

$$\sqrt{(-2)^2+4^2+(-6)^2}=\sqrt{56}.$$

Thus, the unit vector parallel to the given line is

$$\frac{1}{\sqrt{56}}(-2\mathbf{i}+4\mathbf{j}-6\mathbf{k})$$

which reduces to $\quad \dfrac{1}{\sqrt{14}}(-\mathbf{i}+2\mathbf{j}-3\mathbf{k}).$

$$\sqrt{56}=\sqrt{4\times14}=2\sqrt{14}$$

When the equations of two lines are known their point of intersection is easily found (if it exists).

Example 11.4

Given the vector equations of two lines as

$$\mathbf{r} = \mathbf{i}+\mathbf{j}+\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j}-\mathbf{k}),$$
$$\mathbf{r} = 2\mathbf{i}-\mathbf{j}+2\mathbf{k}+\mu(3\mathbf{i}+\mathbf{j}-18\mathbf{k}),$$

find their point of intersection.

Note that different parameters λ and μ must be used to define the lines.

The position vector of the point of intersection must satisfy the equations of both lines. Then equating the expressions for **r**, we have

$$\mathbf{i}+\mathbf{j}+\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j}-\mathbf{k}) = 2\mathbf{i}-\mathbf{j}+2\mathbf{k}+\mu(3\mathbf{i}+\mathbf{j}-18\mathbf{k})$$

Then matching the terms in **i, j, k,** we have

$$1+2\lambda = 2+3\mu, \quad (1)$$
$$1-3\lambda = -1+\mu \quad (2)$$
$$1-\lambda = 2-18\mu \quad (3)$$

Only two equations are required to find two values, of course!

If the points intersect the <u>two</u> parameters λ and μ must satisfy the <u>three equations</u> (1), (2), (3).

From (1) and(2),

$$2\lambda-3\mu = 1 \quad (1)$$
$$3\lambda+\mu = 2 \quad (2)$$

These may be solved in the usual way to give

$$\lambda=\frac{7}{11}, \qquad \mu=\frac{1}{11}. \qquad (4)$$

(1) + 3 × (2), for example

Substitution of the values of λ and μ given in (4) into (3) shows that these values satisfy (3):

$$1-\lambda \quad = \quad 2-18\mu$$
$$\downarrow \qquad\qquad \downarrow$$
$$1-\frac{7}{11}=\frac{4}{11}, \quad 2-\frac{18}{11}=\frac{4}{11}.$$

Thus $\lambda = \frac{7}{11}, \mu = \frac{1}{11}$, satisfy (1), (2) and (3) which demonstrates that the lines intersect. The position vector of the point may be found by either substituting $\lambda = \frac{7}{11}$ in $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} - \mathbf{k})$,

or substituting $\mu = \frac{1}{11}$ in $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu\,(3\mathbf{i} + \mathbf{j} - 18\mathbf{k})$.

The first gives
$$\begin{aligned}\mathbf{r} &= \mathbf{i} + \mathbf{j} + \mathbf{k} + \frac{7}{11}(2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) \\ &= \frac{25}{11}\mathbf{i} - \frac{10}{11}\mathbf{j} + \frac{4}{11}\mathbf{k}.\end{aligned}$$

You are asked to check that the same position vector is obtained after substitution of $\mu = \frac{1}{11}$ in the second case.

Example 11.5

Show that the lines given by the vector equations
$$\begin{aligned}\mathbf{r} &= 2\mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}), \\ \mathbf{r} &= 3\mathbf{i} - \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})\end{aligned}$$
do not intersect.

We assume that the point of intersection exists, i.e. the lines intersect.
Equating the expressions for **r**, we have
$$2\mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 3\mathbf{i} - \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$
so that
$$\begin{aligned}2 + \lambda &= 3 + 2\mu, &&(1)\\ 1 + \lambda &= -1 + 3\mu, &&(2)\\ -1 + \lambda &= 1 - \mu. &&(3)\end{aligned}$$

The values λ and μ will satisfy (1), (2), (3) if the lines intersect.

From (1) and (2),
$$\begin{aligned}\lambda - 2\mu &= 1, &&(1)\\ \lambda - 3\mu &= -2. &&(2)\end{aligned}$$

Subtraction of (2) from (1) gives
$$\mu = 3$$
and substitution in (1) or (2) gives
$$\lambda = 7.$$

Substitution of these values in (3) leads to
$$\underset{\text{left hand side}}{-1 + 7} \neq \underset{\text{right hand side}}{1 - 3}$$

so that the derived values of λ and μ do not satisfy the third equation. In other words, there are no values of λ and μ satisfying the three equations (1), (2), (3). We conclude that the lines do not intersect.

The point of intersection of two lines may be found even if the unit vectors **i, j, k** are not used.

Example 11.6

Find the position vector of the point of intersection of the lines given by

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} \text{ and } \mathbf{r} = 2\mathbf{a} + \mathbf{b} + \mu(\mathbf{b} - \mathbf{a})$$

where a and b are non-parallel vectors.

At the point of intersection,

$$\mathbf{a} + \lambda\mathbf{b} = 2\mathbf{a} + \mathbf{b} + \mu(\mathbf{b} - 1)$$

$$\therefore \quad \mathbf{a} + \lambda\mathbf{b} = (2 - \mu)\mathbf{a} + (1 + \mu)\mathbf{b}.$$

Since **a** and **b** are non-parallel vectors,

$$1 = 2 - \mu, \qquad (1)$$

$$\lambda = 1 + \mu. \qquad (2)$$

a terms,
b terms.

(1) gives immediately $\mu = 1$ and substitution into (2) gives $\lambda = 2$.

The point of intersection is

$$\mathbf{a} + \lambda\mathbf{b} = \mathbf{a} + 2\mathbf{b}.$$

or use $\mu = 1$ in the second equation

Example 11.7

a, b and **c** are three non-parallel, non-coplanar vectors. Show that the lines

$$\mathbf{r} = 2\mathbf{a} + \lambda(\mathbf{a} + \mathbf{b} + \mathbf{c}),$$

$$\mathbf{r} = \mathbf{a} + 2\mathbf{b} + \mu\mathbf{c}$$

do not intersect.

We attempt to find the point of intersection by equating the position vectors.

$$\therefore \quad 2\mathbf{a} + \lambda(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a} + 2\mathbf{b} + \mu\mathbf{c} \qquad (1)$$

Equating coefficients of the non-parallel, non-coplanar vectors, we have

$$2 + \lambda = 1, \qquad (2)$$

$$\lambda = 2, \qquad (3)$$

$$\lambda = \mu. \qquad (4)$$

From (2), $\lambda = -1$ which is contradicted by $\lambda = 2$ from (3).

Thus we cannot find values of and μ satisfying (2), (3), (4). We conclude that the lines do not intersect.

Exercises 11.1

1. Find the vector forms of the lines which pass through the given point and which are parallel to the given direction.

(a) (2, 1, 1); direction $\mathbf{i} - \mathbf{j} - \mathbf{k}$ (b) (1, -1, 0); direction $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

(c) (0, 1, 0); direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$ (d) (0,0,0); direction $3\mathbf{i} + \mathbf{j} + \mathbf{k}$.

2. Find the vector equations of the lines passing through the points A and B.

(a) $A(1, 1, -3)$ $B(2, 0, 3)$

(b) $A(1, 1, 1)$ $B(3, 1, -2)$

(C) $A(0, 0, 0)$ $B(5, -3, 1)$

(d) $A(2, 1, 0)$ $B(-2, 3, 1)$

3. State whether the following pairs of lines are parallel.

(a) $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k})$

$\mathbf{r} = \mathbf{i} - \mathbf{j} + \mu\left(\frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{3}{2}\right)\mathbf{k}$

(b) $\mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + (4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$

$\mathbf{r} = \mathbf{i} + \mu\left(\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}\right)$

4. Find the vector equation of a line passing through the point $\mathbf{i} + \mathbf{j} + \mathbf{k}$ which is parallel to the straight line given by $\mathbf{r} = \mathbf{i} + \lambda(2\mathbf{i} - \mathbf{j})$.

5. Determine whether the following pairs of lines intersect.

(a) $\mathbf{r} = 6\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})$

$\mathbf{r} = \mathbf{i} + \mu(\mathbf{i} - \mathbf{j} - \mathbf{k})$

(b) $\mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$

$\mathbf{r} = + \mu\mathbf{k}$

(c) $\mathbf{r} = \mathbf{a} + \mathbf{b} - 2\mathbf{c} + \lambda(\mathbf{a} + \mathbf{b} + \mathbf{c})$,

$\mathbf{r} = \mathbf{b} + \mu(2\mathbf{a} - \mathbf{b} + \mathbf{c})$,

where **a, b, c** are non-parallel, non-coplanar vectors.

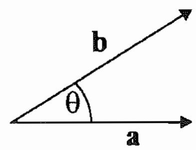
11.2 The scalar product of two vectors

Up until now we have defined essentially two algebraic operations, namely addition (or subtraction) of vectors and multiplication of a vector by a real number (or scalar).

Other operations involving vectors exist which are important in Applied Mathematics. One such operation is the scalar product or dot product. We shall use the name scalar product in the following.

Scalar Product

Suppose two vectors **a** and **b** are as shown, where θ is the angle between them.



The scalar product of **a** and **b** is defined as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta.$$

The following points are emphasised in relation to this definition.

(a) Care should be taken to ensure that the dot between the vectors **a** and **b** is made clear.

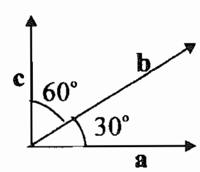
(b) The scalar product is referred to as **a** dot **b**, hence the alternative name the dot product.

(c) The product **a . b** is a scalar so that, as the name suggests, the scalar product combines two vectors to produce a scalar.

(d) The scalar product depends upon the magnitudes of **a** and **b** and the cosine of the angle between them.

Example 11.8

The coplanar vectors **a**, **b** and **c** have magnitudes 3, 4, 5 respectively and are as shown.



Find **a . b, b . c, c . a, a . a, b . b** and **c . c.**

Now $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 5$ (given).

the angle between **a** and **b** is 30°

Then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos 30°$

$$= 3 \times 4 \times \frac{\sqrt{3}}{2}$$

$$= 6\sqrt{3}.$$

$$\mathbf{b} \cdot \mathbf{c} = 4 \times 5 \times \frac{1}{2}.$$

$$= 10.$$

$$\mathbf{c} \cdot \mathbf{a} = |\mathbf{c}|\,|\mathbf{a}| \cos 90°$$

$$= 5 \times 3 \times 0 = 0.$$

Now $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}| \cos$ (angle between **a** and **a**)

$= 3 \times 3 \times \cos 0$

$= 9;$

similarly, $\mathbf{b} \cdot \mathbf{b} = |\mathbf{b}||\mathbf{b}| \cos 0$

$= 4 \times 4 \times 1$

$= 16;$

and $\mathbf{c} \cdot \mathbf{c} = |\mathbf{c}||\mathbf{c}| \cos 0$

$= 5 \times 5 \times 1$

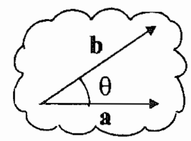
$= 25.$

Example 11.9

Two vectors **a** and **b** are such that

$\mathbf{a} \cdot \mathbf{b} = 6$ and $|\mathbf{a}| = 5$, $|\mathbf{b}| = 3$.

Find the angle between the vectors **a** and **b**.



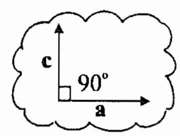
Now $\mathbf{a} \cdot \mathbf{b} = |a||b| \cos \theta$.

$\therefore \quad 6 = 5 \times 3 \cos \theta$

so that $\cos \theta = \frac{6}{15} = \frac{2}{5}$.

$\therefore \quad \theta = \cos^{-1}\left(\frac{2}{5}\right) = 66.4°$

In Example 11.8, we saw that



$\mathbf{c} \cdot \mathbf{a} = 0$

because the vectors c and a are perpendicular to each other and

$\cos 90° = 0.$

The above illustrates the following rule for testing whether two vectors are perpendicular.

<u>Test for Perpendicularity</u>

If $|\mathbf{a}| \neq 0$, $|\mathbf{b}| \neq 0$ and $\mathbf{a} \cdot \mathbf{b} = 0$ then **a** and **b** are perpendicular vectors.

The converse of this text is:

If $\mathbf{a} \cdot \mathbf{b} = 0$, then $|\mathbf{a}| = 0$, $|\mathbf{b}| = 0$ or the vectors are perpendicular vectors.

Example 11.8 also showed that

$$\mathbf{a} \cdot \mathbf{a} = |\,\mathbf{a}\,|^2,$$
$$\mathbf{b} \cdot \mathbf{b} = |\,\mathbf{b}\,|^2,$$
$$\mathbf{c} \cdot \mathbf{c} = |\,\mathbf{b}\,|^2,$$

9
16
25

The general rule is:

> The scalar product of a vector with itself is the square of the magnitude of the vector.

Example 11.10

The unit vectors **i, j, k** are naturally perpendicular.
Find **i . i, i . j, i . k, j . j, j . k, k . k**.

These results are important and are therefore displayed.

> Now $\mathbf{i} \cdot \mathbf{i} = |\,\mathbf{i}\,|^2 = 1^2 = 1,$
> since **i** is a unit vector.
> Similarly, $\mathbf{j} \cdot \mathbf{j} = 1,$
> $\mathbf{k} \cdot \mathbf{k} = 1.$

> Now $\mathbf{i} \cdot \mathbf{j} = 1 \times 1 \ \cos 90° = 0.$
> Similarly, $\mathbf{j} \cdot \mathbf{k} = 0,$
> $\mathbf{i} \cdot \mathbf{k} = 0.$

There are a number of properties of the scalar product which are important.

The Commutative Law

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

↓

or $$|\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta = |\,\mathbf{b}\,|\,|\,\mathbf{a}\,| \cos\theta.$$

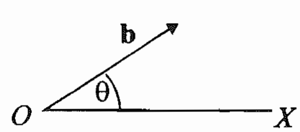
The order is unimportant.

The Distributive Law

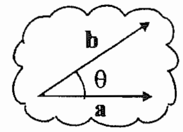
For three vectors,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

Before proving the result, we define the projection of a vector onto a line. If the vector **b** makes an angle θ with a direction *OX* then
$|\,\mathbf{b}\,| \cos\theta$ is the projection of the vector **b** onto *OX*.

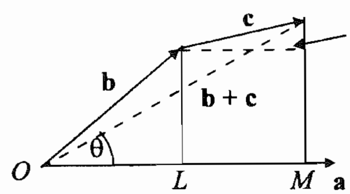


Also if the vector a is directed along OX, then



$$
\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\cos\theta \\
&= |\mathbf{a}| \times (|\mathbf{b}|\cos\theta) \\
&= |\mathbf{a}| \times \text{projection of } \mathbf{b} \text{ onto } \mathbf{a} \\
\text{or} \quad \mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\cos\theta \\
&= |\mathbf{b}|\,|\mathbf{a}|\cos\theta \\
&= |\mathbf{b}| \times (|\mathbf{a}|\cos\theta) \\
&= |\mathbf{b}| \times \text{projection of } \mathbf{a} \text{ onto } \mathbf{b}.
\end{aligned}
$$

We prove $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ by considering the projection of the various vectors shown in the diagram.

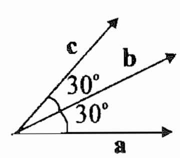


This proof is non-examinable.

$$
\begin{aligned}
\text{Note that} \quad OL &= \text{projection of } \mathbf{c} \text{ onto } \mathbf{a}, \\
LM &= \text{projection of } \mathbf{b} \text{ onto } \mathbf{a}, \\
OM &= \text{projection of } \mathbf{b}+\mathbf{c} \text{ onto } \mathbf{a}. \\
\text{Then } \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= |\mathbf{a}| \times (\text{projection of } \mathbf{b}+\mathbf{c} \text{ onto } \mathbf{a}) \\
&= |\mathbf{a}| \times OM \\
&= |\mathbf{a}|\,(OL + LM) \\
&= |\mathbf{a}|\,(OL) + |\mathbf{a}|\,(LM) \\
&= |\mathbf{a}|\,(\text{projection of } \mathbf{b} \text{ onto } \mathbf{a}) + |\mathbf{a}|\,(\text{projection of } \mathbf{c} \text{ onto a}) \\
&= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}. \\
\text{Thus } \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}.
\end{aligned}
$$

Example 11.11

Given coplanar vectors **a, b** and **c** as shown with $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 2$, find $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$.



$$
\begin{aligned}
\text{Now} \quad \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} \\
&= 3 \times 4 \times \cos 30^\circ + 3 \times 2 \times \cos 60^\circ \\
&= 3 \times 4 \times \frac{\sqrt{3}}{2} + 3 \times 2 \times \frac{1}{2} \\
&= 6\sqrt{3} + 3.
\end{aligned}
$$

Scaling Law

It can be shown (but we shall not do so here) that

$$l\mathbf{a} \,.\, m\mathbf{b} \quad = \quad lm\,(\mathbf{a} \,.\, \mathbf{b}),$$

where l, m are two real numbers.

Example 11.12

Evaluate the following scalar products.

(a) $6\mathbf{i} \,.\, 3\mathbf{j}$ (b) $2\mathbf{i} \,.\, -4\mathbf{i}$

(c) $a_1\mathbf{i}.(b_1\mathbf{i} + b_2\mathbf{j})$ (d) $(a_1\mathbf{i} + a_2\mathbf{j}) \,.\, (b_1\mathbf{i} + b_2\mathbf{j})$

(a) $6\mathbf{i} \,.\, 3\mathbf{j} = 6 \times 3(\mathbf{i} \,.\, \mathbf{j}) = 18 \times 0 = 0.$

Scaling law

(b) $2\mathbf{i} \,.\, -4\mathbf{i} = 2 \times -4(\mathbf{i} \,.\, \mathbf{i}) = -8 \times 1 = -8$

(c) $a_1\mathbf{i} \,.(b_1\mathbf{i} + b_2\mathbf{j}) = a_1\mathbf{i}.b_1\mathbf{i} + a_1\mathbf{i}.b_2\mathbf{j} = a_1b_1\mathbf{i}.\mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} = a_1b_1.$

Distributive law

(d) $(a_1\mathbf{i} + a_2\mathbf{j}) \,.\, (b_1\mathbf{i} + b_2\mathbf{j})$

$$= (a_1\mathbf{i} + a_2\mathbf{j}) \,.\, b_1\mathbf{i} + (a_1\mathbf{i} + a_2\mathbf{j}) \,.\, b_2\mathbf{j}$$

Distributive law

$$= a_1\mathbf{i} \,.\, b_1\mathbf{i} + a_2\mathbf{j} \,.\, b_1\mathbf{i} + a_1\mathbf{i}.b_2\mathbf{j} + a_2\mathbf{j} \,.\, b_2\mathbf{j}$$

$$= a_1b_1\mathbf{i} \,.\, \mathbf{i} + a_2b_1\mathbf{j}.\, \mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} + a_2b_2\mathbf{j} \,.\, \mathbf{j}$$

Distributive and Commutative laws

so that $(a_1\mathbf{i} + a_2\mathbf{j}) \,.\, (b_1\mathbf{i} + b_2\mathbf{j}) = a_1b_1 + a_2b_2.$

The result in 13.12(d) may be generalised, namely that

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \,.\, (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = a_1b_1 + a_2b_2 + a_3b_3$$

= coefficients of **i** multiplied
+ coefficients of **j** multiplied
+ coefficients of **k** multiplied.

Example 11.13

(a) $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \,.\, (3\mathbf{i} + \mathbf{j} - 6\mathbf{k}) = 1 \times 3 + 1 \times 1 + (-1) \times (-6)$

$= 3 + 1 + 6 = 10.$

Coeffs of **i** multiplied
+ coeffs of **j** multiplied
+ coeffs of **k** multiplied

(b) $(\mathbf{i} + \mathbf{j} - 4\mathbf{k}) \,.\, (2\mathbf{i} - 3\mathbf{j}) = 1 \times 2 + 1 \times (-3) + (-4) \times 0$

$= 2 - 3 + 0 = -1.$

There is no term in **k** in second bracket

The negative sign shows that the scalar product is negative which means that $\cos\theta < 0$, i.e. the angle between the vectors is obtuse.

$\mathbf{a} \,.\, \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ and $|\mathbf{a}| > 0$, $|\mathbf{b}| > 0$.

When vectors are expressed in terms of **i, j, k** the angle between the vectors is easily found.

Example 11.14

Find the angle between the vectors

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$$

and

$$\mathbf{b} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}.$$

Now $\quad \mathbf{a} \,.\, \mathbf{b} = |\,\mathbf{a}\,||\,\mathbf{b}\,| \cos\theta$

so that $\quad \cos\theta = \dfrac{|\,\mathbf{a.b}\,|}{|\,\mathbf{a}\,||\,\mathbf{b}\,|}$

We find

$$\begin{aligned} |\,\mathbf{a} \,.\, \mathbf{b}\,| &= (\mathbf{i} + \mathbf{j} - \mathbf{k}) \,.\, (\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) \\ &= 1 \times 1 + 1 \times (-2) + (-1) \times (-3) \\ &= 1 - 1 + 3 = 2. \end{aligned}$$

Also

$$\begin{aligned} |\,\mathbf{a}\,| &= \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}, \\ |\,\mathbf{b}\,| &= \sqrt{1^2 + (-2)^2 + (-3)^2} = \sqrt{14}. \end{aligned}$$

Then $\quad \cos\theta = \dfrac{2}{\sqrt{3}\sqrt{14}} = \dfrac{2}{\sqrt{42}}$

so that $\quad \theta = 72.02°$

Example 11.15

Find the angle between the lines given by the vector equations

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}),$$

$$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \mu(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}).$$

The direction of the lines are given by the terms involving λ and μ i.e. by the vectors

$$\mathbf{i} - 3\mathbf{j} - 2\mathbf{k},\ \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

We find the angle between these vectors

Now $\quad \cos\theta = \dfrac{|\,\mathbf{a.b}\,|}{|\,\mathbf{a}\,||\,\mathbf{b}\,|}.$

Then for the two vectors

$$\begin{aligned} \cos\theta &= \frac{(\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}).(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})}{|\,\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}\,||\,\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\,|} \\ &= \frac{1 \times 1 + (-3) \times (-2) + (-2) \times (2)}{\sqrt{1^2 + (-3)^2 + 2^2}\sqrt{1^2 + (-2)^2 + 2^2}} \\ &= \frac{1 + 6 - 4}{\sqrt{14}\sqrt{9}} = \frac{3}{\sqrt[3]{14}} = \frac{1}{\sqrt{14}}. \end{aligned}$$

$\therefore \quad \theta = 74.50°$

The scalar product enables us to determine a vector perpendicular to two given vectors.

There is another method of doing this, using another type of product.

Example 11.16

Given that $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$,
find **a** unit vector **c** perpendicular to both **a** and **b**.

Let $\mathbf{c} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,
where x, y and z are constants to be determined.
Since **c** is perpendicular to **a**,

$$\mathbf{c.a.} = 0$$

$$\therefore (x\mathbf{i} + y\mathbf{i} + z\mathbf{k}).(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0$$

so that $3x + 2y - z = 0. \qquad (1)$

Similarly $\mathbf{c} \,.\, \mathbf{b} = 0$
gives

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).(\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$$

so that $x - y + z = 0. \qquad (2)$

We solve equation (1) and (2) for any two unknowns in terms of the third. Adding (1) and (2), we obtain

$$4x + y = 0$$

so that $y = -4x$

Substitute in (2) for y

$$\therefore \quad x - (-4x) + z = 0$$

so that $z = -5x$.

Then writing y, z in terms of x in the vector **c**, we obtain

$$\mathbf{c} = x\mathbf{i} - 4x\mathbf{j} - 5x\mathbf{k}$$

so that $\mathbf{c} = x(\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}) \qquad (4)$

Then taking any non-zero value of x in (4), we obtain a vector perpendicular to both **a** and **c**. In fact, the unit vector is required. The unit vector is

$$\frac{\mathbf{c}}{|\mathbf{c}|} = \frac{x(\mathbf{i} - 4\mathbf{j} - 5\mathbf{k})}{\sqrt{x^2 + (-4x)^2 + (-5x)^2}}$$

$$= \frac{x(\mathbf{i} - 4\mathbf{j} - 5\mathbf{k})}{\sqrt{42x^2}}$$

$$= \frac{x}{x}\frac{(\mathbf{i} - 4\mathbf{j} - 5\mathbf{k})}{\sqrt{42}}$$

$$= \frac{(\mathbf{i} - 4\mathbf{j} - 5\mathbf{k})}{\sqrt{42}}.$$

Exercises 11.2

1. Find the scalar product of **a** and **b** given that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and the angle between the vectors is 150°.

2. Show that the vectors $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $-\mathbf{i} - \mathbf{j} - \mathbf{k}$ are perpendicular.

3. Find the projection of the vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ in the direction of the vector $4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

4. Find the angles between the vector $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and the x-axis, y-axis and the z-axis.
Hint: **i, j, k** are vectors parallel to the axes.

5. Find a unit vector perpendicular to the vectors $-2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

6. Find the angles between the pairs of lines given by the following equations.

(a) $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$; $\mathbf{r} = \mathbf{i} - \mathbf{j} + \mu(2\mathbf{i} - \mathbf{j})$

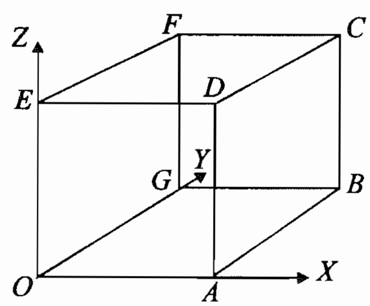
(b) $\mathbf{r} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j})$; $\mathbf{r} = 4\mathbf{i} + \mu(3\mathbf{i} - 4\mathbf{j} + \mathbf{k})$

7. Find the angles between the pairs of lines passing through the given points.

(a) (1, 2, 3), (-2, 1, 2); (-1, 2, -3), (0, 2, -1)

(b) (1, 0, 1), (0, 1, 1); (1, 0, 0), (-1, 1, 1)

8.



The figure shows a cube of side a. Taking axes OX, OY, OZ as shown and the associated unit vectors **i, j** and **k**:

(a) write down the position vectors of A, B, C, D, E and F,

(b) express the vectors BD, DF and GE in terms of **i, j** and **k**,

(c) find the angles between the lines DF and GE,

(d) find the angles between BD and OC.

Revision Paper 1

1. Write down the expansion of $(1-3x)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^3.

Hence obtain the expansion of $\dfrac{1-4x}{\sqrt{1-3x}}$ in ascending powers of x up to and including the term in x^3. State the range of x for which the expansion is valid.

2. (a) Express

$$\frac{x+4}{(x+1)(x-2)^2} \quad \text{in terms of partial fractions.}$$

(b) Determine the equation of the normal to the curve

$$y = \frac{x+4}{(x+1)(x-2)^2}$$

at the point (0, 1).

3. Find the values of x in the range 0° to 360° in the following:

(a) $\cos 2x - \cos x + 1 = 0$,

(b) $\tan^2 x - 3\sec x = 3$.

4. Find $\dfrac{dy}{dx}$ in the following cases.

(a) $y = x\sin^{-1} x + \sqrt{1-x^2}$.

(b) $x^2 + 3xy - y^2 = 3$.

5. (a) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \tan^2\left(\frac{\pi}{6} + x\right) dx$.

(b) Use integration by parts to find

$$\int_1^2 \ln x \, dx.$$

(c) Using the substitution $u^2 = 1 + \sin^2\theta$, or otherwise, evaluate

$$\int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta\sqrt{1+\sin^2\theta}\, d\theta.$$

6. A liquid is heated in such a way that its temperature x°C at time t seconds satisfies the differential equation

$$\frac{dx}{dt} = \alpha(100 - x),$$

where α is a positive constant.

The temperature at $t = 0$ is 20°C.

(a) Show that $x = 100 - 80e^{-2t}$.

(b) Given that $x = 80$ when $t = 300$, find the value of α.

7. The points A, B, C, D and E have position vectors $\mathbf{i} + 11\mathbf{j}$, $2\mathbf{i} + 8\mathbf{j}$, $-\mathbf{i} + 7\mathbf{j}$, $-2\mathbf{i} + 8\mathbf{j}$, and $-4\mathbf{i} + 6\mathbf{j}$, respectively. The lines AB and DC intersect at F.

(a) Show that the vector equation of the line AB is

$$\mathbf{r} = (1 + \lambda)\mathbf{i} + (11 - 3\lambda)\mathbf{j},$$

where λ is a parameter, and find the vector equation of the line DC.

(b) Show that the position vector of F is

$$\mathbf{i} + 11\mathbf{j}.$$

(c) Find the angle between FD and EA.

Revision Paper 2

1. Express $\dfrac{1-x-x^2}{(1+2x)(1+x)^2}$ as the sum of three partial fractions.

   Hence or otherwise, expand this expression in ascending powers of x up to and including the term in x^3.
   State the range of values of x for which the expansion is valid.

2. (a) By first writing $\cos 3x = \cos(2x + x)$, show that
   $$\cos 3x = 4\cos^3 x - 3\cos x.$$
   (b) Find all values of x between 0° and 360° satisfying
   $$\cos 3x + 2\cos x = 0.$$

3. The point $P\left(t, \dfrac{1}{t}\right)$ lies on the curve C given by $xy = 1$.

   (a) Show that the equation of the tangent to C at the point P is
   $$t^2y + x - 2t = 0$$
   The tangent at P meets the x and y axes at A and B. Show that the area of triangle AOB is independent of t.

   (b) $S\left(s, \dfrac{1}{s}\right)$ is another point on C. Find the coordinates of U, the point of intersection of the tangents at S and T.
   If V is the midpoint of PQ, show that the origin O and the points U and V lie on a straight line.

4. The region bounded by the curve $y = \tan\left(\dfrac{x}{2}\right)$, the lines $x = 0$, $x = \dfrac{\pi}{4}$ and the x-axis is rotated about the x-axis through four right angles. Find the volume of the solid generated.

5. (a) Use integration by parts to find $\int xe^{-4x}\,dx$.

   (b) Use the substitution $x = \tan u$ to show that
   $$\int_0^{\sqrt{3}} \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \frac{\sqrt{3}}{2}.$$

6. The slope of a curve at a point (x, y) is equal to $2y^2x$.

(a) Write down a differential equation satisfied by y.

(b) Given that the curve passes through the point (1, 1), find y as a function of x.

7. The vector equations of two lines are

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda\,(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

and

$$\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + a\mathbf{k} + \mu\,(\mathbf{i} + 2\mathbf{j} + \mathbf{k}),$$

where a is a constant.

(a) Given that the two lines intersect, find a and the point of intersection of the lines.

(b) find the angle between the lines, giving your answer correct to the nearest degree.

Revision Paper 3

1. Write down and simplify the binomial expansion of $\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$ up to and including the term in x^3. State the range of x for which the expansion is valid. By putting $x = -1$, use your expansion to obtain an approximation to $\sqrt{3}$, giving your answer as a ratio of two integers.

2. Use partial fractions to find

$$\int \frac{2x^2+7x+3}{(x-1)(x+1)^2}\,dx\,.$$

3. Find $\frac{dy}{dx}$ in the following:

(a) $y = (\tan^{-1}x)^2$

(b) $y^3 = x\,(x + 2y)$

(c) $x = \sec t,\ \ y = \tan t + 2.$

4. Solve the following equations for values of x from 0° to 360° inclusive.

(a) $6 \sin x - 2 \operatorname{cosec} x = 1.$

(b) $\sin x = 6 \sin 2x.$

5. Rewrite $\sqrt{3}\cos\theta - \sin\theta$ in the form $R \cos (\theta + \alpha)$, giving the values of R and α.

(a) Find the values of θ between 0° and 360° satisfying

$$\sqrt{3}\cos\theta - \sin\theta = 1.$$

(b) Find the maximum and minimum values of

$$\frac{3}{\sqrt{3}\cos\theta - \sin\theta + 4}$$

6. (a) By using the substitution $x = 3\sin\theta$, or otherwise, evaluate

$$\int_0^3 \frac{x^2}{\sqrt{9-x^2}}\,dx$$

(b) (i) Given that $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, show that if $y = \sin^{-1}x$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 = x^2}}.$$

(ii) Given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\sqrt{1 - x^2}}$$

and $y = 1$ when $x = \frac{1}{2}$, find y in terms of x.

7. The vertices *A, B* of the triangle *OAB* have position vectors **a, b** relative to *O*. *C* and D are the midpoints of *OA* and *AB* respectively.

(a) Show that the position vector of *M*, the midpoint of *CD*, is $\frac{1}{2}\mathbf{a} + \frac{1}{4}\mathbf{b}$.

(b) Write down, in terms of *a* and *b* and λ, the position vector of the point which divides *BM* in the ratio λ : 1 – λ. Hence find the position vector of the point of intersection of *BM* and *OD*.

Revision Paper 4

1. Find the expansion of

$$\frac{1-2x}{\sqrt{1+2x}}$$

in ascending powers of x up to the term in x^2. State the range of values of x for which the expansion is valid.

Use your expansion to find an approximate non-zero root of the equation

$$\frac{1-2x}{\sqrt{1+2x}} = 1 - 2 \cdot 99x + x^2 .$$

2. (a) Express $\dfrac{2x-6}{(x-1)(x+2)}$ in terms of partial fractions.

(b) Find the points on the curve

$$y = \frac{3x-6}{(x-1)(x+2)}$$

at which the tangents are parallel to the x-axis.

3. Solve the following equations for values of x between 0° and 360°.

(a) $\cot^2 x = 2 \operatorname{cosec} x + 2$.

(b) $\tan 2x = 4 \tan x$.

4. A curve is given in terms of the parameter t by the equations

$$x = a \sin^2 t, \quad y = a \cos^3 t, \quad 0 < t < \frac{\pi}{2}$$

where a is a positive constant.

(a) Find and simplify an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}t}$.

(b) The normal to the curve at the point where $t = \dfrac{\pi}{3}$ cuts the x and y-axes at the points M and N, respectively. Find the area of triangle OMN.

5. (a) Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$ and by first finding $\dfrac{\mathrm{d}x}{\mathrm{d}y}$, show that if

$$y = \ln (x + a)$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x+a}.$$

(b) Find the equation of the tangent at the point (1, 2) for the curve given by

$$y^4 + xy = 12 + 6x^4.$$

6. The region bounded by the curve $y = \sin 2x$, the lines $x = 0$, $x = \frac{\pi}{2}$ and the x-axis is rotated about the x-axis through four right-angles. Find the volume of the solid generated.

7. A population is growing in such a way that, at time t years, the rate at which the population is increasing is proportional to the size, x, of that population at that time. Initially the size of the population is 4.

(a) Write down a differential equation describing the rate of growth of x.

(b) Show that $x = 4e^{kt}$, where k is a positive constant.

(c) After 6 years the population size is 100.

(i) show that $k = \frac{1}{6}\ln 25$.

(ii) Estimate the size of the population after 10 years.

Revision Paper 5

1. (a) Express $\dfrac{2x+1}{(2x+3)^2}$ in terms of partial fractions.

(b) Find $\displaystyle\int_{-1}^{2} \frac{2x+1}{(2x+3)^2}\,\mathrm{d}x$.

2. (a) Find all values of θ between 0° and 360° satisfying the equation

$$3\cos\theta + 2\sin\theta = 1.$$

(b) Given that $\tan 2x = -1$, show that $\tan x = 1 \pm \sqrt{2}$. Deduce the values of $\tan 67\tfrac{1}{2}°$ and $\tan 157\tfrac{1}{2}°$ in surd form.

3. (a) A curve has parametric equations $x = 3\sec t$, $y = 5\tan t$. Find the coordinates of the point on the curve at which the tangent to the curve is parallel to the line $y = 2x$.

(b) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in the following cases:

(i) $y = \sin^{-1}(3x)$

(ii) $y^3 + xy^2 + x^4 = 3$.

4. (a) Given that the derivative of $\tan x$ is $\sec^2 x$ show that if

$$y = \tan^{-1} x$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}.$$

(b) Use the substitution $u = \cos x$ to evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x.$$

(c) Find $\int (2x+1)e^{-x}\,\mathrm{d}x$.

5. Given that

$$\cos^2 x \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y$$

and $y = 1$ when $x = 0$, find y in terms of x.

6. Given that

$$(1 + ax)^n = 1 + 6x + 6x^2 + \dots ,$$

find the values of a and n.

7. The points A and B have position vectors, relative to an origin O, $2\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$ and $4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ respectively.

(a) Show that OA and OB are perpendicular.

(b) Find the vector equation for the line AB.

(c) Show that AB intersects the line given by

$$\mathbf{r} = (3 - \lambda)\,\mathbf{i} + (-1 + \lambda)\mathbf{j} + (2 - 3\lambda)\mathbf{k}$$

and find the position vector of the point of intersection.

Revision Paper 6

1. If x is so large that $\frac{1}{x^3}$ and higher powers of $\frac{1}{x}$ can be neglected, show that

$$\frac{x+1}{(x+2)^2} = \frac{1}{x} - \frac{3}{x^2}.$$

2. Express $f(x) = \dfrac{3x-1}{(x+2)(2x-1)}$ in terms of partial fractions.

Find $f'(x)$.

3. (a) Show that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$.

(b) Using the result derived in (a), or otherwise, find all the values of x between 0° and 180° satisfying

$$\tan x + \cot x = 8.$$

4. $P\ (a\cos^3 t,\ a\sin^3 t)$ is a point on the curve C given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(a) Show that the equation of the tangent to C at the point P is given by

$$y + x\tan t = a\sin t.$$

(b) The tangent at P meets the x and y axes at A and B, respectively. Find the area of triangle AOB in terms of t and show that the value of this area lies between 0 and $\frac{1}{4}a^2$.

5. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in the following:

(a) $y = \ln(\sec x + \tan x)$, expressing your answer in its simplest form;

(b) $y = \cos^{-1}(x-1)$,

(c) $x^3 + \sin y + xy = 1$.

6. (a) (i) Use the substitution $u = 1 - x^2$ to evaluate

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

(ii) Evaluate

$$\int_0^1 \sin^{-1} x\,dx$$

(b) Evaluate

$$\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$$

7. In the rectangle $OABC$, $OA = \mathbf{a}$ and $OC = \mathbf{c}$. R is a point on AB such that $AR : RB = 1 : 2$ and S is a point on BC such that $BS : SC = 3 : 1$. AS meets OR at P.

(a) Find an expression for OP in terms of $\mathbf{a}$ and $\mathbf{c}$, and λ where

$$OP : PR = \lambda : 1 - \lambda.$$

(b) Show that $OP : PR = 4 : 1$.

(c) Find also the value of the ratio $AP : PS$.

ANSWERS

Chapter 1

Exercise 1.1

$-1,\ 1$

Exercises 1.2

1. (a) $1 - 2x - 2x^2 - 4x^3;$ $|x| < \frac{1}{4}$

(b) $1 - 4x + 12x^2 - 32x^3;$ $|x| < \frac{1}{2}$

(c) $1 + 3x + 9x^2 + 27x^3;$ $|x| < \frac{1}{3}$

(d) $\sqrt{3}\left(1+\frac{x}{6}-\frac{x^2}{72}+\frac{x^3}{432}\right);$ $|x| < 3$

(e) $\frac{1}{4}-\frac{x}{4}+\frac{3}{16}x^2-\frac{1}{8}x^3;$ $|x| < 2$

(f) $\frac{1}{3}+\frac{2x}{27}+\frac{2}{81}x^2+\frac{20}{2187}x^3$ $|x| < \frac{9}{4}$

2. (a) $1-\frac{x}{2}-\frac{5}{8}x^2;$ $|x| < 1$

(b) $2-2x+\frac{21}{4}x^2;$ $|x| < \frac{1}{3}$

(c) $1+5x+\frac{27x^2}{2};$ $|x| < \frac{1}{2}$

(d) $1 + 3x + 7x^2;$ $|x| < \frac{1}{2}$

3. $x+\frac{1}{2x^3}-\frac{1}{8x^7}+\frac{1}{16x^{11}};$ $|x| > 1$

4. $1 + 2x;$ 3.60

5. $1-3x+\frac{7x^2}{2};$ $\frac{1}{35}\approx 0.029$

6. $5,-\frac{2}{5}$

7. $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{8}x^2$ with $x=-\frac{1}{100}, \sqrt{0.99}\approx 0.9950$

8. $1+\frac{x}{18}-\frac{x^2}{648};$ $|x| < 9.$

Chapter 2

Exercises 2.1

1. (a) $\dfrac{2x^3-3x^2+x+2}{x^2+2x+1} = 2x-7+\dfrac{13x+9}{x^2+2x+1}$

(b) $\dfrac{x^4-3x^2+2x+1}{x^3+3x^2+2} = x-3+\dfrac{6x^2+7}{x^3+3x^2+2}$

(c) $\dfrac{2x^3-3x^2+x+1}{x^3+x^2-x-2} = 2+\dfrac{5+3x-5x^2}{x^3+x^2-x-2}$

(d) $\dfrac{12x^4-4x^3+12x^2-7}{4x^3-3x+2} = 3x-1+\dfrac{21x^2-9x-5}{4x^3-3x+2}$

Exercise 2.2

$A=\dfrac{9}{2}$, $B=-\dfrac{17}{2}$

Exercises 2.3

1. (i) $\dfrac{1}{5(x+2)}+\dfrac{9}{5(x-3)}$ (ii) $\dfrac{2}{x+3}-\dfrac{1}{x+5}$

(iii) $\dfrac{3}{4(x-2)}+\dfrac{5}{4(x+2)}$ (iv) $\dfrac{7}{5(x+2)}+\dfrac{1}{5(2x-1)}$

(v) $-\dfrac{2}{x-1}-\dfrac{2}{(x-1)^2}+\dfrac{2}{x-2}$ (vi) $\dfrac{3}{1-3x}-\dfrac{2}{1-2x}$

(vii) $\dfrac{6}{x+1}-\dfrac{1}{x+2}-\dfrac{15}{(x+2)^2}$ (viii) $\dfrac{3}{x+1}-\dfrac{2}{(x+1)^2}$

2. $a=3, b=-1, c=7$; $\dfrac{2}{x-1}-\dfrac{3}{x+2}$

3. $\dfrac{1}{x}-\dfrac{2}{3x-1}+\dfrac{6}{(3x-1)^2}$; $-\dfrac{1}{x^2}+\dfrac{6}{(3x-1)^2}-\dfrac{36}{(3x-1)^3}$

4. $a=4, b=2, c=26$; $\dfrac{32}{5(x-3)}-\dfrac{22}{5(x+2)}$; $-\dfrac{32}{5(x-3)^2}+\dfrac{22}{5(x+2)^2}$

5. (a) $\dfrac{2}{7(1+2x)}+\dfrac{1}{7(3-x)}$; (b) $\dfrac{1}{3}-\dfrac{5x}{9}+\dfrac{31}{27}x^2, |x|<\dfrac{1}{2}$

6. $-\dfrac{1}{2(2-x)}+\dfrac{1}{(2-x)^2}+\dfrac{1}{2(4+x)}$; $\dfrac{1}{8}+\dfrac{3}{32}x+\dfrac{17}{128}x^2, |x|<2.$

Chapter 3

Exercises 3.1

1. (a) 45°, 63.4° (b) 30°, 150° (c) 45°, 60°, 120°.

2. 70.5°, 289.5°

3. (a) 45°, 225°, 63.4°, 243,4°

(b) 30°, 150°.

(c) 16.6°, 163.4°, 23.6°, 156.4°.

4. 39.2°, 140.8°, 30°, 150°.

5. 153.4°, 333.4°, 116.6°, 296.6°, 71.6°, 251.6°.

6. 30°, 330°.

7. $\frac{1}{2}\tan 2x - x + C.$

8. (a) $-\frac{1}{2}\cot 2x + C$ (b) $-\frac{1}{2}\cot 2x - x + C.$

9. $-\operatorname{cosec} x \cot x$

10. $\sec x \tan x.$

Exercises 3.2

1. -0.253^c, 1.772^c, -0.124^c (^{c} indicates radian measure)

Chapter 4

Exercise 4.1

$$\cos(A+B) = \frac{OR - NR}{OP} = \frac{OR}{OP} - \frac{NR}{OP} = \frac{OR}{OQ}\frac{OQ}{OP} - \frac{SQ}{OP}$$

$$= \cos A \cos B - \frac{SQ}{PQ}\frac{PQ}{OP}$$

$$= \cos A \cos B - \sin A \sin B.$$

Exercises 4.2

1. (i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\cos(\theta + 3\phi)$

(iv) $\sin A$ (v) $\cos B$ (vi) $\sin A$

(vii) 1

2. (i) $\frac{24}{25}, \frac{7}{25}$ (ii) $-\frac{4}{3}$ (iii) $-2 \pm \sqrt{7}$

(iv) $\text{Sin } 135° = \frac{1}{\sqrt{2}}$ (v) 105°

3.	$2 \sin A \cos B$; 45°, 135°						
4.	$\frac{2\tan\theta}{1-\tan^2\theta}$						
5.	(i)	66.2, 246.2	(ii)	67.5°, 247.5°			
	(iii)	0, 180, 360°	(iv)	50.7°, 230.7°			

Exercises 4.3

0, 1, 0.

Exercises 4.4

1.	(i)	sin 24°	(ii)	tan 30°	(iii)	cos 48°
	(iv)	$\sin x$	(v)	cos 40°	(vi)	$\tan x$
	(vii)	cos 32°	(viii)	$\cos\theta$	(ix)	$\cos 6\theta$
	(x)	$\tan 8x$				
2.	$-\frac{4}{3}$					
3.	(i)	$-\frac{24}{25},-\frac{7}{25}$	(ii)	$-\frac{120}{169},-\frac{119}{169}$	(iii)	$\frac{-\sqrt{3}}{2},-\frac{1}{2}$
4.	(i)	0°, 30°, 150°, 180°, 210°, 330, 360°				
	(ii)	14.5°, 90°, 155.5°, 270°				
5.	90°, 228.6°, 311.4°					
6.	(i)	0°, 30°, 150°, 180°, 210°, 330°, 360°				
	(ii)	18.4°, 161.6°, 198.4°, 341.6°.				

Exercises 4.6

1.	(a)	13, 67.4°	(b)	$\sqrt{10}$, 71.6°		
	(c)	$\sqrt{2}$, 45°	(d)	5, 53.1°		
2.	(a)	2, –2	(b)	5, –5	(c)	$\sqrt{10},\ -\sqrt{10}$
	(d)	$\frac{1}{3-\sqrt{2}},\ \frac{1}{3+\sqrt{2}}$	(e)	$\frac{1}{4},\frac{1}{10}$	(f)	25, 0
3.	(a)	0°, 120°, 360°	(b)	29.5°, 256.7°		
	(c)	10°, 124.8°	(d)	7.9°, 231.5°		
	(e)	26.2°, 110.2°	(f)	120.8, 173.2, 300.8		
4.	$1 + \cos 2\theta + 3 \sin 2\theta$; 0, 71.6°, 180°					

Chapter 5

Exercises 5.1

1. $\dfrac{1}{5x^{4/5}}$ **2.** $\dfrac{1}{2(x-1)^{\frac{1}{2}}}$ **3.** $\dfrac{1}{4x^{\frac{1}{2}}(x^{\frac{1}{2}}-1)^{\frac{1}{2}}}$

Exercise 5.2

$\boxed{\tan y} = \boxed{x}$; $\boxed{\sec^2 y}$; $\boxed{\dfrac{1}{\sec^2 y}}$; $\boxed{\dfrac{1}{1+\tan^2 y}}$

Exercise 5.3

1. (i) $\dfrac{1}{\sqrt{a^2-x^2}}$ (ii) $-\dfrac{1}{\sqrt{a^2-x^2}}$ (iii) $\dfrac{a}{x^2+a^2}$

(iv) $\dfrac{2x}{\sqrt{1-x^4}}$ (v) $-\dfrac{2x}{x^4+1}$ (vi) $2x\sin^{-1}(1-x)-\dfrac{x^{\frac{3}{2}}}{\sqrt{2-x}}$

(vii) $\dfrac{1}{2\sqrt{x-x^2}}$ (viii) -1 (ix) $\dfrac{-x}{\sqrt{1-x^2}}\cos^{-1}x-1$

(x) $\dfrac{1}{2(1+x^2)\sqrt{\tan^{-1}x}}$ (xi) $\dfrac{-(x+1)}{\sqrt{1-x^2}}$

(xii) $-\dfrac{1}{x^2+1}$ (xiii) $\dfrac{1-2x\tan^{-1}x}{(1+x^2)^2}$

(xiv) $\dfrac{2\sin^{-1}x}{\sqrt{1-x^2}}$ (xv) $-\dfrac{1}{(\tan^{-1}x)^2(1+x^2)}$

Exercises 5.4

1. (i) $-\dfrac{x}{y}$ (ii) $\dfrac{2}{y^3}$ (iii) $-\dfrac{1}{4y}$

(iv) $-\sqrt{\dfrac{y}{x}}$ (v) $\dfrac{-(2x+3y)}{3x+2y}$ (vi) $-\dfrac{x^2}{y^2}$

(vii) $\dfrac{-y}{x+3y^2}$ (viii) $\dfrac{y\sin x}{\cos x+2y}$ (ix) $-\dfrac{2y}{3x}$

(x) $\dfrac{-(2xy+y^2)}{x^2+2xy}$ (xi) $\dfrac{2+2y-2x}{2y-2x+3}$

2. $-\dfrac{1}{3}, 3$

3. -1

4. $1, -1$

Exercises 5.5

1. (i) $-\cot\theta$ (ii) $-\frac{1}{t^2}$ (iii) $\frac{1}{3t}$

(iv) cosec t (v) $\frac{1}{t}$ (vi) $-\frac{5t^{\frac{3}{2}}}{6(1+t^2)}$

(vii) $-\tan t$ (viii) $2t - t^2$ (ix) $\frac{t}{t+1}$

2. (i) ± 1 (ii) $\frac{\pi}{2}$ (iii) 1

3. (i) $-\frac{1}{(t+1)^2}$; $(t+1)^2$ (ii) $\frac{4t}{3}$; $-\frac{3}{4t}$

(iii) $-\frac{1}{2\sin t}$; $2\sin t$ (iv) $-\frac{3}{2}\cot t$; $\frac{2}{3}\tan t$

Exercises 5.6

1. $\frac{32}{27a}$ **2.** -2 **3.** $-\frac{3}{4}\operatorname{cosec}^3 t$

4. $\cos^3 t$ **5.** $\frac{2t-2}{t+3}$

6. $\left(\frac{\pi}{6}-\frac{1}{2}, \frac{\pi}{6}+\sqrt{3}\right)$ maximum; $\left(\frac{5\pi}{6}-\frac{1}{2}, \frac{5\pi}{6}-\sqrt{3}\right)$ minimum

Chapter 6

Exercises 6.1

1. $\left(-\frac{9}{2}, -\frac{4}{3}\right)$ **2.** $(5\sin t)\, y + (3\cos t)\, x - 15 = 0$

5. $y + px - 2ap - ap^3 = 0$; 2 **6.** $\frac{1}{4}a^2$

Chapter 7

Exercises 7.1

The constant of integration is omitted in each case.

(a) $\frac{(2x+1)^3}{6}$ (b) $-\frac{1}{3(3x+5)}$

(c) $-\frac{1}{9}\cos(9x+1)$ (d) $\frac{1}{2}\tan(2x+9)$

(e) $-\frac{1}{3}$ cosec $(3x+1)$ (f) $\frac{2}{9}(3x-7)^{\frac{3}{2}}$

(g) $\frac{1}{2}\ln|2x+1|$ (h) $\frac{2}{9}\sqrt{9x+2}$

Exercises 7.2

The constant of integration is omitted in each case.

(a) $\frac{1}{15}(x^3+1)^5$ (b) $\frac{1}{4}\ln|x^4+2|$

(c) $-\frac{1}{2(x^2+3)}$ (d) $\frac{1}{3}\tan(x^3+1)$

(e) $\frac{1}{3}\sqrt{1+3x^2}$ (f) $-\cos(x^2+x+5)$

(g) $\frac{1}{4}(1+e^x)^4$ (h) $-\frac{2}{3}(1-\sin x)^{\frac{3}{2}}$

(i) $\frac{1}{3}(x^2+4x+1)^{\frac{3}{2}}$ (j) $\frac{2}{3}\ln|1+x^{\frac{3}{2}}|$

(k) $\ln|x^2+x-3|$ (l) $\frac{1}{8}(\sin(2x)+4)^4$

(m) $\frac{1}{2}(\ln x)^2$ (n) $\ln|1-\cos x|$

(o) $\ln(e^x+x)$ (p) $\frac{\tan^3 x}{3}$

(q) $\frac{\sin^4 x}{4}$ (r) $-\frac{1}{2}\cot^2 x$

Exercises 7.3

The constant of integration is omitted in each case.

1. (a) $-\frac{1}{3(x^3+1)}$ (b) $\sin^{-1}\left(\frac{x}{2}\right)$

(c) $-\sqrt{1-x^2}$ (d) $-\sqrt{1-x^2}$

(e) $\frac{1}{6}(2x+1)^{\frac{3}{2}}-\frac{1}{2}\sqrt{2x+1}$ (f) $\sqrt{1+2\sin x}$

(g) $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$ (h) $\frac{1}{2}(x^2+4)-3\ln|x^2+4|$

(i) $\frac{1}{3}(x^2+1)^{\frac{3}{2}}$ (j) $\frac{1}{2}\sin^{-1}(2x)$

(k) $\sin^{-1}(e^x)$ (l) $-\ln|\cos x|$

(m) $\frac{1}{2}\sqrt{x^2+1}$ (n) $\frac{1}{2}\sqrt{x^2+1}$

(o) $\frac{\sin^5 x}{5}$ (p) $\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)+\frac{x\sqrt{25-x^2}}{2}$

2. (a) $-\frac{1}{2}\sqrt{4-x^2}$ (b) $\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+\frac{x\sqrt{9-x^2}}{2}$

(c) $\frac{2}{3}(1+\sin x)^{\frac{3}{2}}$ (d) $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$

(e) $\frac{1}{3}\tan^{-1}(3x)$ (f) $-\frac{\cos^6 x}{6}$

(g) $\frac{\tan^2 x}{2}$ (h) $\sqrt{9+x^2}$

3. (a) $\frac{10}{3}(x-2)^{\frac{3}{2}}+20\sqrt{x-2}$ (b) $\frac{1}{2}\sin^{-1}(x^2)$

(c) $\frac{(x-1)^8}{8}+\frac{(x-1)^7}{7}$ (d) $\frac{(x+2)^7}{7}-\frac{(x+2)^6}{2}$

(e) $\frac{1}{2}\sin^{-1}x-\frac{x\sqrt{1-x^2}}{2}$ (f) $-\frac{\sqrt{4-x^2}}{4x}$

Chapter 8

Exercises 8.1

The constants of integration are omitted

(a) $-\frac{1}{2}\ln|3-2x|$ (b) $-\frac{3}{2(4x-7)}$

(c) $\frac{2}{3}\ln|3x+2|-\frac{5}{3(3x+2)}$

Exercises 8.2

The constant of integration is omitted in each case

3. (a) $-\frac{1}{15}\ln|3x+2|+\frac{2}{5}\ln|x-1|$

(b) $5\ln|x-2|-\frac{12}{x-2}$

(c) $\ln|x-2|+\ln|x-3|$

(d) $\frac{1}{12}\ln|2x-3|-\frac{1}{12}\ln|2x+3|$

(e) $\frac{1}{8}\ln|4+x|-\frac{1}{8}\ln|4-x|$

(f) $\frac{2}{3}\ln|3x-4|-\frac{3}{4}\ln|4x+3|$

(g) $\ln|x|-\frac{3}{x+1}$

(h) $\ln|x-1|-\frac{2}{x-1}$

(i) $\frac{1}{4}\ln|x|-\frac{1}{4}\ln|4-x|$

4. $-\frac{1}{2}\ln|1-x^2|$

Exercises 8.3

The constant of integration is omitted in each case

1. (a) $-x\cos x+\sin x$ (b) $(x^2-2)\sin x+2x\cos x$

(c) xe^x (d) $\frac{x^4}{4}\ln(2x)-\frac{x^4}{16}$

(e) $-(x^2+2x+2)e^{-x}$ (f) $x\sin^{-1}x+\sqrt{1-x^2}$

(g) $x\ln x-x$ (h) $\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x$

(i) $-\frac{1}{3}(\pi-x)\cos 3x-\frac{1}{9}\sin 3x$

2. $-\frac{1}{2}\sqrt{1-x^2}\sin^{-1}x+\frac{1}{2}x$

Exercises 8.4

1. (a) 2 (b) $\frac{1}{2}(1-e^{-1})$ (c) $\frac{\pi}{4}$

(d) $\frac{1}{2}$ (e) $\frac{1}{3}$ (f) $\ln 2-\frac{1}{4}$

(g) $\frac{\pi}{2}$ (h) $\sqrt{3}$ (i) $\frac{\pi}{20}$

(j) $\pi-2$ (k) $\frac{32}{5}\ln 2-\frac{31}{25}$ (l) $\frac{1}{4}(1-3e^{-2})$

2. 1 **3.** 1 **4.** $\frac{1}{2}\ln 2$

5. 1 **6.** $\frac{3}{\sqrt{4}}-\frac{\pi}{12}$

Exercises 8.5

1. (a) $\frac{16\pi}{3}$ (b) $\frac{73\pi}{15}$ (c) 8π

(d) $\frac{16\pi}{15}$ (e) $\frac{\pi}{2}$

2. (a) $\frac{19}{3}\pi$ (b) 8π (c) 8π

(d) $\frac{32\pi}{5}$

3. $\pi\left(\frac{\pi}{2}+1\right)$ **4.** $\frac{65\pi}{2}$ **5.** $\frac{1}{3}\pi r^2 h$

6. $\frac{2}{3}\pi r^3, \frac{4}{3}\pi r^3$

Chapter 9

Exercises 9.1

1. Only (a) and (c) are directly integrable

(a) $y = x + \frac{x^2}{2} + k$ (c) $x = -t\cos t + \sin t + k$

2. $x = 2\sin 2t + 1$

3. $y = \frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{9}{4}$

4. $c = x^3 - 6x^2 + 15x + 50$, £6

5. (a) $y = -e^{-x} - 1$ (b) $y = -\cos x - \frac{x^2}{2} + \frac{\pi^2}{8}$ (c) $y = x\sin x + \cos x + 2$

Exercises 9.2

1. $y = \frac{-1}{x^4 + C}$

2. $y = -\frac{1}{3}\ln(4 - 3e^x)$

3. $y = \sin^{-1}(A - \sin x)$

4. $\frac{y^2}{4} + \frac{1}{2}\ln(y) = \frac{2x^3}{3} + A$

5. 10 minutes

6. $t = \frac{1}{4k}\ln\left(\frac{8-x}{8-2x}\right), \frac{1}{8}\ln(5.5)$

7. $y = 3 + x$

Chapter 10

Exercises 10.1

(i)	No	(ii)	No	(iii)	No	(iv)	No	(v)	Yes	(vi)	Yes
(vii)	No	(viii)	No	(ix)	Yes						

Exercises 10.2

1. Sum of lengths of two sides of a triangle is greater than the length of the third side. Equality occurs when a and b are parallel.

2. The sides of hexagon are given by the vectors **a, b, b – a, –a, –b, a – b**.

3. **EB** = 3**b**, **EC** = **a** + 3**b**, **DB** = 4**b** – **a**, **AC** = 4**b** + **a**, **FB** = 3**b** – **a**.

4. **AB** = **b** – **a**, **AC** = **c** – **a**, **AE** = $\frac{1}{2}$(**b** – **a**), **EF** = $\frac{1}{3}$**c** – $\frac{1}{6}$**a** – $\frac{1}{6}$ **b**.

5. **AB** = **b** – **a**, **BC** = **a** – 4**b**, **AD** = 3 **a** + **b**.

6. (i) $\frac{1}{5}(2\mathbf{a} - \mathbf{b})$ (ii) $\frac{1}{4}(\mathbf{b} - 3\mathbf{a})$

Exercises 10.3

1. (a) $\frac{1}{2}(2\mathbf{a} - \mathbf{c})$ (b) $\frac{1}{5}(8\mathbf{a} - 3\mathbf{b} - 3\mathbf{c})$ (c) $3\mathbf{b} - 2\mathbf{a}$

2. $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$

3. (a) $\frac{2\mathbf{b} + \mathbf{c}}{3}$ (b) 6 : 1

Exercises 10.4

1. (i) $(1 - \lambda)\,\mathbf{a} + \lambda\mathbf{b},\ (1 - \mu)\mathbf{c} + \mu(\mathbf{a} + \mathbf{b} - \mathbf{c})$

(ii) $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

2. (a) $(1 - \lambda)\,\mathbf{a} + \frac{\lambda}{2}\mathbf{b} + \frac{\lambda}{2}\mathbf{c},\quad \frac{\mu\mathbf{a}}{2} + \left(1 - \frac{3\mu}{2}\right)\mathbf{b} + \mu\mathbf{c}$

(b) $\frac{1}{5}\mathbf{a} + \frac{2}{5}\mathbf{b} + \frac{2}{5}\mathbf{c}$

4. $\frac{2}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}$

Exercises 10.5

1. (i) $-\mathbf{j} - \mathbf{k};\ \sqrt{2}$ (ii) $11\mathbf{i} - 7\mathbf{j} + 10\mathbf{k};\ \sqrt{270}$

2. (i) $\frac{i}{\sqrt{3}} + \frac{j}{\sqrt{3}} + \frac{k}{\sqrt{3}}$ (ii) $\frac{2}{\sqrt{5}}i + \frac{1}{5}j$ (iii) $\mathbf{i}$

3. $a = 2,\ b = 6,\ c = -6$

4. $3;\ \sqrt{54}$

Chapter 11

Exercises 11.1

1. (a) $\mathbf{r} = (2 + \lambda)\mathbf{i} + (1 - \lambda)\,\mathbf{j} + (1 - \lambda)\,\mathbf{k}$

(b) $\mathbf{r} = (1 + 3\lambda)\mathbf{i} + (-1 + 2\lambda)\mathbf{j} + \lambda\mathbf{k}$

(c) $\mathbf{r} = \lambda\mathbf{i} + (1 + \lambda)\mathbf{j} + \lambda\mathbf{k}$

(d) $\mathbf{r} = 3\lambda\mathbf{i} + \lambda\mathbf{j} + \lambda\mathbf{k}$

2. (a) $\mathbf{r} = (1 + \lambda)\mathbf{i} + (1 - \lambda)\mathbf{j} + (-3 + 6\lambda)\mathbf{k}$

(b) $\mathbf{r} = (1 + 2\lambda)\mathbf{i} + \mathbf{j} + (1 - 3\lambda)\mathbf{k}$

(c) $\mathbf{r} = 5\lambda\mathbf{i} - 3\lambda\mathbf{j} + \lambda\mathbf{k}$

(d) $\mathbf{r} = (2 - 4\lambda)\mathbf{i} + (1 + 2\lambda)\mathbf{j} + \lambda\mathbf{k}$

3. (a) parallel (b) not parallel

4. $\mathbf{r} = (\mathbf{i} + 2\mu)\mathbf{i} + \mathbf{j}(1 - \mu) + \mathbf{k}$

5. (a) Yes, at $\mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ (b) No (c) No

Exercises 11.2

1. $-5\sqrt{3}$

3. $\frac{12}{\sqrt{29}}$

4. 72.5°, 107.5°, 25.2°

5. $\frac{1}{\sqrt{13}}(3\mathrm{i}+2\mathrm{j})$

6. (a) 56.7° (b) 116.0°

7. (a) 132.4° (b) 30°

8. (a) $a\mathbf{i}$, $a\mathbf{i}+a\mathbf{j}$, $a\mathbf{i}+a\mathbf{j}+a\mathbf{k}$, $a\mathbf{i}+a\mathbf{k}$, $a\mathbf{j}$, $a\mathbf{j}+a\mathbf{k}$

(b) $a\mathbf{k}-a\mathbf{j}$, $-a\mathbf{i}+a\mathbf{j}$, $-a\mathbf{j}+a\mathbf{k}$

(c) 120°

(d) 90°

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