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General Certificate of Education Advanced Level/Advanced Subsidiary	Tystysgrif Addysg Gyffredinol Safon Uwch/Uwch Gyfrannol

MATHEMATICS

FORMULA BOOKLET

Issued 2004

Pure Mathematics

Mensuration

Surface area of sphere = $4\pi r^2$

Area of curved surface of cone = $\pi r \times$ slant height

Arithmetic Series

$$u_n = a + (n-1)d$$

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\,[2a + (n-1)d]$$

Geometric Series

$$u_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |\,r\,| < 1$$

Summations

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Binomial Series

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

$$\text{where } \binom{n}{r} = {}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \quad (|x| < 1,\ n \in \mathbb{R})$$

Logarithms and exponentials

$$\mathrm{e}^{x\ln a} = a^x$$

Complex Numbers

$$\{r(\cos\theta + \mathrm{i}\sin\theta)\}^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$$

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

The roots of $z^n = 1$ are given by $z = \mathrm{e}^{\frac{2\pi k\,\mathrm{i}}{n}}$, for $k = 0, 1, 2, \ldots, n-1$

Maclaurin's and Taylor's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^r}{r!} f^{(r)}(a) + \ldots$$

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!} f''(a) + \ldots + \frac{x^r}{r!} f^{(r)}(a) + \ldots$$

$$\mathrm{e}^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1} \frac{x^r}{r} + \ldots \qquad (-1 < x \le 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{for all } x$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 \le x \le 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{for all } x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{for all } x$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 < x < 1)$$

Hyperbolic Functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^{-1} x = \ln\left\{x + \sqrt{x^2 - 1}\right\} \qquad (x \ge 1)$$

$$\sinh^{-1} x = \ln\left\{x + \sqrt{x^2 + 1}\right\}$$

$$\tanh^{-1} x = \tfrac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$$

Coordinate Geometry

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$y^2=4ax$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$xy=c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$(ct, \frac{c}{t})$
Eccentricity	$e<1$ $b^2=a^2(1-e^2)$	$e=1$	$e>1$ $b^2=a^2(e^2-1)$	$e=\sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x=\pm\frac{a}{e}$	$x=-a$	$x=\pm\frac{a}{e}$	$x+y=\pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a}=\pm\frac{y}{b}$	$x=0, y=0$

Trigonometric Identities

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$

$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}\qquad(A\pm B\neq(k+\tfrac{1}{2})\pi)$$

For $t=\tan\frac{1}{2}A$: $\sin A=\frac{2t}{1+t^2}$, $\cos A=\frac{1-t^2}{1+t^2}$

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Vectors

The resolute of **a** in the direction of **b** is $\frac{\mathbf{a.b}}{|\mathbf{b}|}$

The point dividing AB in the ratio $\lambda:\mu$ is $\frac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu}$

Matrix transformations

Anticlockwise rotation through θ about O: $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

Reflection in the line $y=(\tan\theta)x$: $\begin{pmatrix}\cos2\theta & \sin2\theta\\ \sin2\theta & -\cos2\theta\end{pmatrix}$

Differentiation

Function	**Derivative**
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$

Integration (+ *constant*; $a > 0$ *where relevant*)

Function	**Integral**
$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert = \ln\lvert\tan(\frac{1}{2}x)\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert = \ln\lvert\tan(\frac{1}{2}x + \frac{1}{4}\pi)\rvert$
$\sec^2 x$	$\tan x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\cosh x$
$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\dfrac{1}{a^2 + x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right) = \ln\left\{x + \sqrt{x^2 - a^2}\right\} \quad (x > a)$
$\dfrac{1}{\sqrt{a^2 + x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right) = \ln\left\{x + \sqrt{x^2 + a^2}\right\}$
$\dfrac{1}{a^2 - x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a + x}{a - x}\right\rvert = \dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\dfrac{1}{x^2 - a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x - a}{x + a}\right\rvert$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

Area of a sector

$$A = \tfrac{1}{2}\int r^2 \, \mathrm{d}\theta \quad \text{(polar coordinates)}$$

Arc length

$$s = \int \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x \quad \text{(cartesian coordinates)}$$

$$s = \int \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t \quad \text{(parametric form)}$$

Surface area of revolution

$$S_x = 2\pi\int y \, \mathrm{d}s = 2\pi \int y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t$$

$$S_y = 2\pi\int x \, \mathrm{d}s = 2\pi \int x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t$$

Numerical Mathematics

Numerical integration

The trapezium rule: $\int_a^b y \, \mathrm{d}x \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

Simpson's Rule: $\int_a^b y \, \mathrm{d}x \approx \tfrac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\}$,

where $h = \dfrac{b-a}{n}$ and n is even

Numerical Solution of Equations

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Mechanics

Motion in a circle

Transverse velocity: $v = r\dot{\theta} = \omega r$

Radial acceleration: $-r\dot{\theta}^2 = -\dfrac{v^2}{r} = -\omega^2 r$

Centres of Mass of Uniform Bodies

Triangular lamina: $\tfrac{2}{3}$ along median from vertex

Probability & Statistics

Probability

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B \mid A)$$

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(B \mid A)\,\mathrm{P}(A)}{\mathrm{P}(B \mid A)\,\mathrm{P}(A) + \mathrm{P}(B \mid A')\,\mathrm{P}(A')}$$

Bayes' Theorem: $\mathrm{P}(A_j \mid B) = \dfrac{\mathrm{P}(A_j)\,\mathrm{P}(B \mid A_j)}{\Sigma\,\mathrm{P}(A_i)\,\mathrm{P}(B \mid A_i)}$

Discrete distributions

For a discrete random variable X taking values x_i with probabilities p_i

Expectation (mean): $\mathrm{E}(X) = \mu = \Sigma x_i p_i$

Variance: $\mathrm{Var}(X) = \sigma^2 = \Sigma(x_i - \mu)^2 p_i = \Sigma x_i^2 p_i - \mu^2$

For a function $\mathrm{g}(X)$: $\mathrm{E}(\mathrm{g}(X)) = \Sigma\,\mathrm{g}(x_i)p_i$

Standard discrete distributions:

Distribution of X	$\mathrm{P}(X = x)$	Mean	Variance
Binomial $\mathrm{B}(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Poisson $\mathrm{Po}(\lambda)$	$\mathrm{e}^{-\lambda} \dfrac{\lambda^x}{x!}$	λ	λ

Continuous distributions

For a continuous random variable X having probability density function f

Expectation (mean): $\mathrm{E}(X) = \mu = \int x f(x)\,\mathrm{d}x$

Variance: $\mathrm{Var}(X) = \sigma^2 = \int (x-\mu)^2\,\mathrm{f}(x)\,\mathrm{d}x = \int x^2 f(x)\,\mathrm{d}x - \mu^2$

For a function $\mathrm{g}(X)$: $\mathrm{E}(\mathrm{g}(X)) = \int \mathrm{g}(x) f(x)\,\mathrm{d}x$

Cumulative distribution function: $\mathrm{F}(x) = \mathrm{P}(X \le x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$

Standard continuous distributions:

Distribution of X	P.D.F.	Mean	Variance
Uniform (Rectangular) on $[a, b]$ $\mathrm{U}[a,b]$	$\dfrac{1}{b-a}$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$
Normal $\mathrm{N}(\mu, \sigma^2)$	$\dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2

Expectation algebra

For independent random variables X and Y

$$\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y)\,, \quad \mathrm{Var}(aX \pm bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y)$$

Sampling distributions

For a random sample $X_1, X_2, \ldots, X_n$ of n independent observations from a distribution having mean μ and variance σ^2

$\overline{X}$ is an unbiased estimator of μ, with $\mathrm{Var}(\overline{X}) = \dfrac{\sigma^2}{n}$

S^2 is an unbiased estimator of σ^2, where $S^2 = \dfrac{\Sigma(X_i - \overline{X})^2}{n-1}$

For a random sample of n observations from $\mathrm{N}(\mu, \sigma^2)$

$$\frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim \mathrm{N}(0,1)$$

$$\frac{\overline{\mathrm{X}} - \mu}{S / \sqrt{n}} \sim t_{(n-1)}$$

If X is the observed number of successes in n independent Bernoulli trials in each of which the probability of success is p, and $Y = \dfrac{X}{n}$, then

$$\mathrm{E}(Y) = p \quad \text{and} \quad \mathrm{Var}(Y) = \frac{p(1-p)}{n}$$

For a random sample of n_x observations from $\mathrm{N}(\mu_x, \sigma_x^2)$ and, independently, a random sample of n_y observations from $\mathrm{N}(\mu_y, \sigma_y^2)$

$$\frac{(\overline{X} - \overline{Y}) - (\mu_x - \mu_y)}{\sqrt{\dfrac{\sigma_x^2}{n_x} + \dfrac{\sigma_y^2}{n_y}}} \sim \mathrm{N}(0,1)$$

Method of Least Squares

If $y_i = \alpha + \beta\, x_i + e_i, i = 1, 2, ..., n,$

then the least squares estimates of β and α respectively, are

$$b = S_{xy} / S_{xx,}\ a = \bar{y} - b\bar{x},$$

where $S_{xy} \equiv \sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - (\sum x_i)(\sum y_i) / n$

$$S_{xx} \equiv \sum (x_i - \bar{x})^2 = \sum x_i^2 - (\sum x_i)^2 / n$$

If $e_1, .e_2, ..., e_n$ are independent and each is distributed as N(0, σ^2), then

(i) a is an observation from $\mathrm{N}\left\{\alpha, \dfrac{\sigma^2 \sum x_i^2}{nS_{xx}}\right\}$,

(ii) b is an observation from $\mathrm{N}\left\{\beta, \dfrac{\sigma^2}{S_{xx}}\right\}$

(iii) $\hat{y}_0 = a + bx_0$, the least squares estimate of $y_0 = \alpha + \beta x_0$,

is an observation from $\mathrm{N}\left\{y_0, \sigma^2\left[\dfrac{1}{n} + \dfrac{(x_0 - \bar{x})^2}{S_{xx}}\right]\right\}$