



Additional

Mathematics

Contents

Unit	Workbooks	Page Number
1	(1) Algebraic Manipulation	5
	(2) Quadratic Functions	21
	(3) Arithmetic and Geometric Series	41
2	(1) Differentiating from First Principles	61
	(2) Quick Differentiation	75
	(3) Introducing Integration	91
3	(No workbooks)	
4	(No workbooks)	
5	(No workbooks)	
6	(1) Graphs and Networks	103
	(2) Introducing Algorithms	119
	(3) Critical Path Analysis	153
	(4) Linear Programming	179



Algebraic

Manipulation

$$\begin{aligned} & \frac{1}{2 + 3^{\frac{1}{2}}} \\ &= \frac{1}{2 + \sqrt{3}} \\ &= \frac{1}{(2 + \sqrt{3})} \times \frac{(2 - \sqrt{3})}{(2 - \sqrt{3})} \\ &= \frac{2 - \sqrt{3}}{4 - \cancel{2\sqrt{3}} + \cancel{2\sqrt{3}} - \sqrt{3} \times \sqrt{3}} \\ &= \frac{2 - \sqrt{3}}{4 - 3} \\ &= \frac{2 - \sqrt{3}}{1} \\ &= 2 - \sqrt{3} \end{aligned}$$

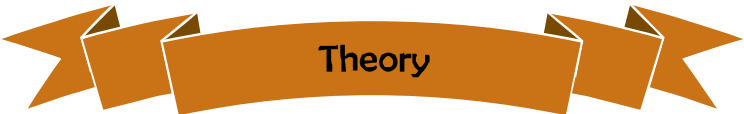
Name:



Background

What is the work?

Working with surds, rules of indices and mathematical proof.

What are the prerequisites?**GCSE Work:** Surds, rules of indices, identities.**Where does this lead to?****Applications:** Electrical engineering, digital signal processing.


Theory

Surds

The following rules develop the GCSE work on surds.

$$\sqrt{n} \times \sqrt{n} = n$$

$$\sqrt[n]{a} \times \sqrt[n]{m} = \sqrt[n]{am}$$

$$\sqrt[n]{a} \div \sqrt[n]{m} = \sqrt[n]{\frac{a}{m}} = \frac{\sqrt[n]{a}}{\sqrt[n]{m}}$$

Exercise I

Simplify the following.

(a) $\sqrt{3} \times \sqrt{3}$

(b) $\sqrt[3]{2} \times \sqrt[3]{4}$

(c) $\sqrt[4]{48} \div \sqrt[4]{3}$

(d) $\sqrt{54}$

(e) $(3 + \sqrt{5})(7 - \sqrt{5})$

(f) $\sqrt{150}$

(g) $(2\sqrt{7} - 3)(2\sqrt{7} + 3)$

(h) $(4\sqrt{3} + 1)^2 - 1$

Rationalising the denominator

Each fraction has an infinite number of equivalent fractions. For example, the fractions $\frac{2}{4}$, $\frac{3}{6}$ and $\frac{4}{8}$ are all equivalent to $\frac{1}{2}$. If we have a fraction whose denominator contains a surd, it is good practice to change it to an equivalent fraction whose denominator does **not** contain a surd. This way, working with the fraction will be more convenient.

The rules in this course for rationalising a denominator are

$$\frac{a}{b\sqrt{c}} = \frac{a \times \sqrt{c}}{b\sqrt{c} \times \sqrt{c}} = \frac{a\sqrt{c}}{bc}, \quad \frac{a}{d+e\sqrt{f}} = \left(\frac{a}{d+e\sqrt{f}} \right) \times \left(\frac{d-e\sqrt{f}}{d-e\sqrt{f}} \right)$$

Example 1

Rationalise the denominator in the following expressions. You must show all your working.

(a) $\frac{21}{\sqrt{7}}$

Answer: We must multiply by 1, in the form $\frac{\sqrt{7}}{\sqrt{7}}$:

$$\begin{aligned} \frac{21}{\sqrt{7}} &\times \frac{\sqrt{7}}{\sqrt{7}} \\ &= \frac{21\sqrt{7}}{\sqrt{7} \times \sqrt{7}} \\ &= \frac{21\sqrt{7}}{7} \\ &= 3\sqrt{7} \end{aligned}$$

(b) $\frac{5}{2+\sqrt{3}}$

Answer: We must multiply by 1, in the form $\frac{2-\sqrt{3}}{2-\sqrt{3}}$:

$$\begin{aligned} \frac{5}{(2+\sqrt{3})} &\times \frac{(2-\sqrt{3})}{(2-\sqrt{3})} \\ &= \frac{5(2-\sqrt{3})}{4-2\sqrt{3}+2\sqrt{3}-\sqrt{3} \times \sqrt{3}} \\ &= \frac{5(2-\sqrt{3})}{4-3} \\ &= \frac{5(2-\sqrt{3})}{1} \\ &= 5(2-\sqrt{3}) \end{aligned}$$

Exercise 2

Rationalise the denominator in the following expressions. You must show all your working.

(a) $\frac{18}{\sqrt{2}}$

.....

.....

.....

.....

.....

.....

(b) $\frac{24}{\sqrt{3}}$

.....

.....

.....

.....

.....

.....

(c) $\frac{2}{3+\sqrt{5}}$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(d) $\frac{1}{6-\sqrt{7}}$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(e) $\frac{11}{3\sqrt{5}}$

(f) $\frac{2}{6+\sqrt{3}}$

(g) $\frac{1}{12-\sqrt{11}}$

(h) $\frac{1}{8+\sqrt{5}}$

(i) $\frac{3}{5+\sqrt{2}}$

(j) $\frac{1}{2+3\sqrt{5}}$

Exercise 3

Simplify $(\sqrt{3} + 2)^2 - (\sqrt{3} - 2)^2$, leaving your answer in the form of a surd.

The area of a rectangle is 142 cm^2 . Its length is $(2 + 5\sqrt{3}) \text{ cm}$.

$$(2 + 5\sqrt{3}) \text{ cm}$$

Area = 142 cm²

The area of a trapezium is $(20 + 4\sqrt{5})$ cm².

The length of the parallel sides are $(1 + \sqrt{5})$ cm and $(6 + 2\sqrt{5})$ cm.

Calculate the perpendicular height of the trapezium.

Give your answer in the form $a + b\sqrt{5}$, where a and b are integers.

You must show all your working.

Rules of Indices

We will need the following rules of indices, from the GCSE course.

$$n^a \times n^b = n^{a+b}$$

$$n^a \div n^b = n^{a-b} = \frac{n^a}{n^b}$$

$$n^0 = 1$$

$$(n^a)^b = n^{ab}$$

$$(nm)^a = n^a m^a$$

$$\left(\frac{n}{m}\right)^a = \frac{n^a}{m^a}$$

$$n^{-a} = \frac{1}{n^a}$$

$$(\sqrt[b]{n})^a = n^{\frac{a}{b}} = \sqrt[b]{n^a}$$



GCSE
Playlist

Exercise 6

Simplify the following.

(a) $x^5 \times x^3$

(b) $\frac{y^{16}}{y^2}$

.....

.....

.....

.....

(c) w^0

(d) $(x^6)^4$

.....

.....

.....

.....

(e) $(x^3y)^5$

(f) $\left(\frac{3}{4}\right)^3$

.....

.....

.....

.....

(g) 4^{-2}

(h) $16^{\frac{1}{2}}$

.....

.....

.....

.....

(i) $16^{\frac{3}{4}}$

(j) $8^{-\frac{2}{3}}$

.....

.....

.....

.....

.....

.....

(k) $\frac{x^6 \times x^{-2}}{x^{-6}}$

(l) $\frac{2^{-1}}{\left(\frac{1}{4}\right)^{\frac{1}{2}}}$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

Exercise 7

Without using a calculator, find the value of the following.

(a) $\left(15^{\frac{1}{3}}\right)^6$

(b) $\left(12^{\frac{1}{2}}\right)^4$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(c) $\left(50^{\frac{1}{2}}\right)^4$

(d) $\left(49^{\frac{1}{2}}\right)^{-2}$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(e) $\left(5^{\frac{1}{4}}\right)^{-12}$

(f) $64^{-\frac{1}{2}} \times 36^{\frac{3}{2}}$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(g) $2^{-2} + \left(\frac{9}{16}\right)^{\frac{1}{2}}$

(h) $\left(16^{\frac{1}{16}}\right)^8 + \left(16^{\frac{1}{16}}\right)^0$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

Developing the Rules**Example 2**

Showing all your working, simplify the following.

$$(a) \frac{4x^{\frac{2}{5}} \times 3x^{\frac{17}{5}}}{x^{\frac{1}{2}}}$$

$$= \frac{12x^{\frac{2}{5} + \frac{17}{5}}}{x^{\frac{1}{2}}}$$

$$= \frac{12x^{\frac{19}{5}}}{x^{\frac{1}{2}}}$$

$$= \frac{12x^3}{x^{\frac{1}{2}}}$$

$$= 12x^{3 - \frac{1}{2}}$$

$$= 12x^{\frac{5}{2}}$$

$$(b) \frac{8x^{\frac{1}{9}} + x^{\frac{2}{9}}}{x^{\frac{2}{9}}}$$

$$= \frac{8x^{\frac{1}{9}}}{x^{\frac{2}{9}}} + \frac{x^{\frac{2}{9}}}{x^{\frac{2}{9}}}$$

[Split the fraction]

$$= 8x^{\frac{1}{9} - \frac{2}{9}} + 1$$

$$= 8x^{-\frac{1}{9}} + 1$$

Exercise 8

Showing all your working, simplify the following.

$$(a) \frac{6x^{\frac{13}{8}} \times 10x^{\frac{3}{8}}}{x^{\frac{1}{5}}}$$

$$(b) \frac{18x^{\frac{2}{5}} + 9x^{\frac{4}{5}}}{9x^{\frac{1}{5}}}$$

$$(c) \frac{5x^{\frac{5}{8}} \times 4x^{\frac{3}{8}}}{x^{\frac{2}{3}}}$$

$$(d) \frac{6x^{\frac{1}{4}} + 3x^{\frac{3}{4}}}{3x^{\frac{1}{4}}}$$

$$(e) \frac{3x^{-\frac{7}{4}} \times 2x^{\frac{17}{4}}}{x^{\frac{3}{2}}}$$

$$(f) \frac{28x^{\frac{1}{7}} + 7x^{\frac{2}{7}}}{7x^{\frac{1}{7}}}$$

$$(g) \frac{3x^{\frac{5}{4}} \times 4x^{\frac{7}{4}}}{x^{\frac{3}{2}}}$$

$$(h) \frac{12x^{\frac{1}{6}} + 4x^{\frac{2}{6}}}{4x^{\frac{1}{6}}}$$

$$(i) \left(x^{\frac{4}{5}} \times x^{\frac{1}{5}}\right)^3$$

$$(j) x^{-\frac{1}{8}} \left(5x^{\frac{1}{8}} + 7x^{\frac{5}{8}}\right)$$

$$(k) \left(x^{\frac{1}{4}} \times x^{\frac{3}{4}}\right)^5$$

$$(l) x^{-\frac{1}{4}} \left(2x^{\frac{1}{4}} + 5x^{\frac{3}{4}}\right)$$

$$(m) \left(x^{\frac{1}{8}} \times 2x^{\frac{3}{8}}\right)^{10}$$

$$(n) x^{-\frac{1}{5}} \left(7x^{\frac{3}{5}} - 6x^{-\frac{4}{5}}\right)$$

$$(o) \left(x \times x^{\frac{1}{4}}\right)^{\frac{1}{2}}$$

$$(p) \frac{x^{\frac{1}{8}} + 2x^{\frac{3}{8}} + 3x^{\frac{5}{8}}}{x^{\frac{1}{8}}}$$

Solving Equations**Example 3**

Solve the equation $8^x = \frac{2^{56}}{4^{10}}$.

Answer: We need to notice that every base in the question can be written as a power of 2.

$$(2^3)^x = \frac{2^{56}}{(2^2)^{10}}$$

$$2^{3x} = \frac{2^{56}}{2^{20}}$$

$$2^{3x} = 2^{56-20}$$

$$2^{3x} = 2^{36}$$

$$3x = 36$$

$$x = 12$$

[Multiply the exponents]

[With the same base of 2, the exponents must be equal]

Exercise 9

Solve the following equations.

(a) $8^x = \frac{2^{36}}{4^{12}}$

(b) $4^x = \frac{8^{21}}{2^5}$

(c) $9^x = \frac{3^{30}}{27^4}$

(d) $27^x = \frac{3^6}{9^{15}}$

A mathematical proof shows that a statement is true through a series of logical statements.

Prove that $\frac{3x}{2} - \frac{x-6}{5} + \frac{2x+3}{7} \equiv \frac{111x+114}{70}$.

Answer: We must start with the left hand side and change it, step-by-step, to the right hand side.

$$\begin{aligned}\text{Left hand side} &= \frac{3x}{2} - \frac{x-6}{5} + \frac{2x+3}{7} \\ &= \frac{3x}{2} \times \frac{35}{35} - \frac{x-6}{5} \times \frac{14}{14} + \frac{2x+3}{7} \times \frac{10}{10} \quad [\text{Form a common denominator}] \\ &= \frac{105x}{70} - \frac{14(x-6)}{70} + \frac{10(2x+3)}{70} \\ &= \frac{105x - 14(x-6) + 10(2x+3)}{70} \\ &= \frac{105x - 14x + 84 + 20x + 30}{70} \\ &= \frac{111x + 114}{70} \\ &= \text{Right hand side } \checkmark\end{aligned}$$

Prove that the following statements are true.

$$(a) \frac{x}{2} - \frac{x+3}{5} + \frac{x+5}{11} \equiv \frac{43x-16}{110}$$

$$(b) \frac{2x}{3} - \frac{x-7}{4} + \frac{3x+1}{5} \equiv \frac{61x+117}{60}$$

$$(c) \frac{2x}{7} - \frac{x-3}{2} + \frac{3x+2}{21} \equiv \frac{67-3x}{42}$$

$$(d) \frac{5x+6}{8} - \frac{2x-1}{4} + \frac{x-3}{3} \equiv \frac{11x}{24}$$

$$(e) \frac{2x}{5} + \frac{x-1}{6} + \frac{3x+5}{10} \equiv \frac{13x+5}{15}$$

$$(f) (x-1)(x+3)(x-2) \equiv x^3 - 7x + 6$$

Given that $a = v + u$, $b = v - u$ and $c = uv$, show that $a^2 - b^2 = 4c$.

n is an integer.
Show, **by factorising**, that $n^3 - n$ is the product of 3 consecutive numbers.
Hence find the value of n , given that $n^3 - n = 120$.

Pop Quiz: Algebraic Manipulation

(1) Simplify the following.

(a) $(2 + \sqrt{3})^2$

[2] (b) $\frac{7}{4-\sqrt{5}}$

[2]

/10

(c) $x^{-\frac{1}{6}}(3x^{\frac{1}{6}} + 7x^{\frac{5}{6}})$

[2] (d) $\frac{24x^{\frac{1}{9}} + 8x^{\frac{4}{9}}}{8x^{\frac{1}{9}}}$

[2]

(2) Prove that $\frac{x^2-9}{x^2+8x+15} \equiv \frac{x-3}{x+5}$.

[2]



Quadratic

Functions

$$y = x^2 - 6x + 5$$

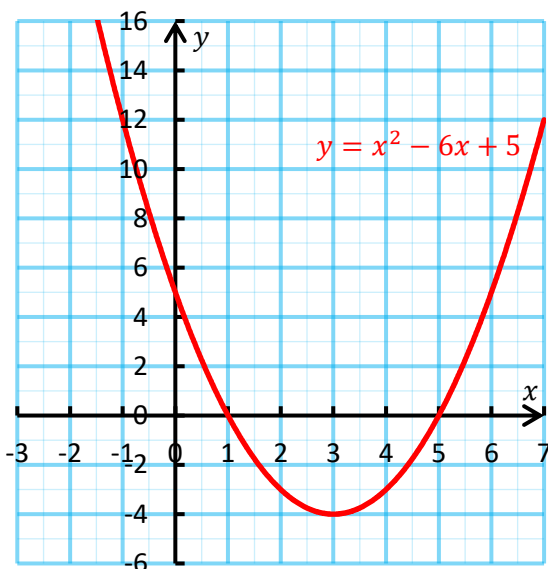
y-intercept (0, 5)

$$y = (x - 3)^2 - 4$$

Minimum point (3, -4)

$$y = (x - 1)(x - 5)$$

Roots $x = 1$, $x = 5$



Name: _____

Background

What is the work?

Consider the shape, solution and use of quadratic equations.

What are the prerequisites?

GCSE Work: Solving quadratic equations, graph plotting, solving simultaneous equations.

Where does this lead to?

Applications: Consider how a projectile (such as a javelin) moves; computer graphics; bridge design.

Theory

A **quadratic function** has the form $ax^2 + bx + c$, where a , b and c are constants with $a \neq 0$.

A **quadratic equation** has the form $ax^2 + bx + c = 0$.

We can plot a quadratic function on graph paper, and it always takes the form of a curve with a 'u' or 'n' shape.

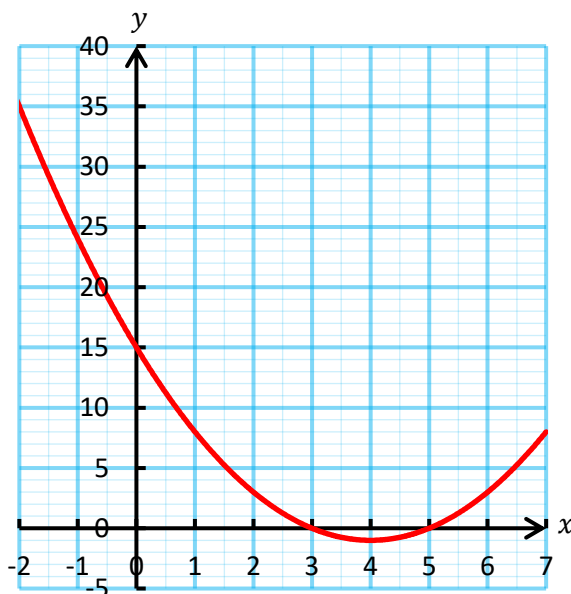
If $a > 0$, then the curve has a 'u' shape, and it has a minimum point.

If $a < 0$, then the curve has an 'n' shape, and it has a maximum point.

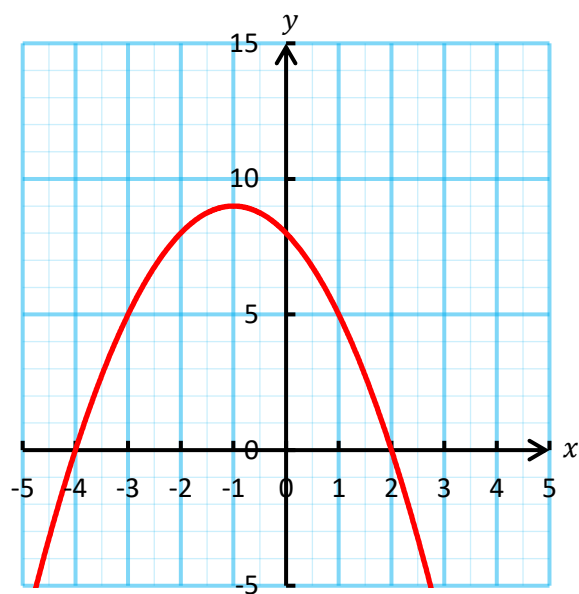
(If $a = 0$, then the function is a linear function, not a quadratic function.)

Example 1

$$y = x^2 - 8x + 15$$



$$y = -x^2 - 2x + 8$$



From the functions and the graphs, we can see the following properties.

The y-intercept is (0, 15).

The roots are $x = 3$ and $x = 5$.

The minimum point is (4, -1).

The y-intercept is (0, 8).

The roots are $x = -4$ and $x = 2$.

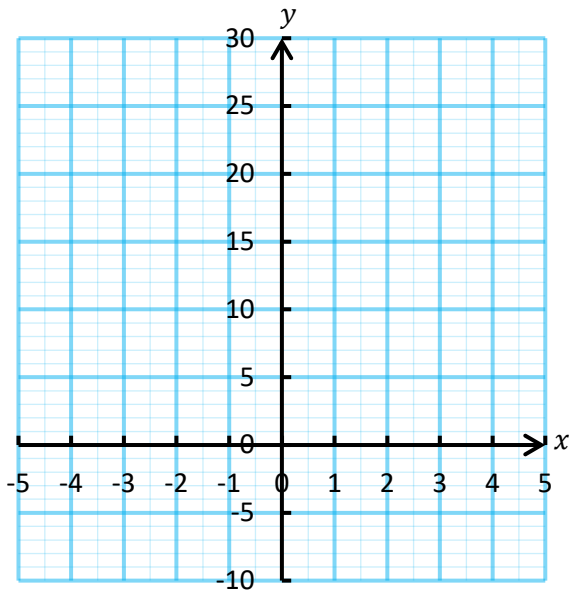
The maximum point is (-1, 9).

Exercise 1

Plot the following quadratic functions.

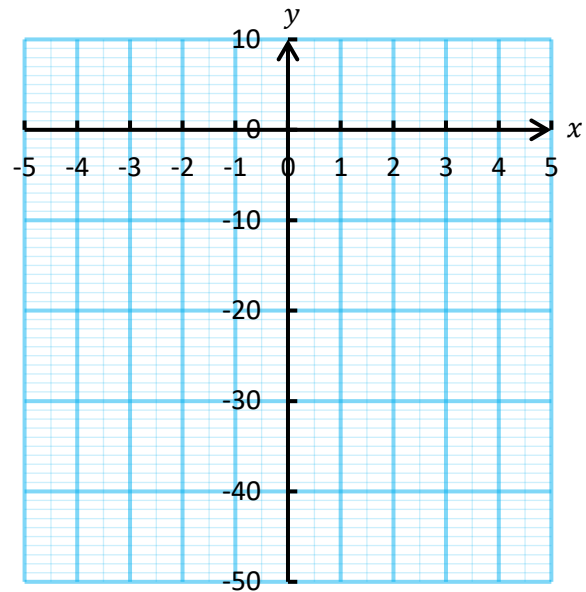
(a) $y = x^2 + 2x - 8$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											



(b) $y = -x^2 + 4x$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											



Complete the following sentences.

The y -intercept is

The roots are $x = \dots$ and $x = \dots$

The minimum point is

The y -intercept is

The roots are $x = \dots$ and $x = \dots$

The maximum point is

Exercise 2

Consider the quadratic function $y = x^2 + 4x - 5$.

(a) Will the graph for this function be 'u'-shaped, or 'n'-shaped?

.....

(b) Complete the following table.

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
y												

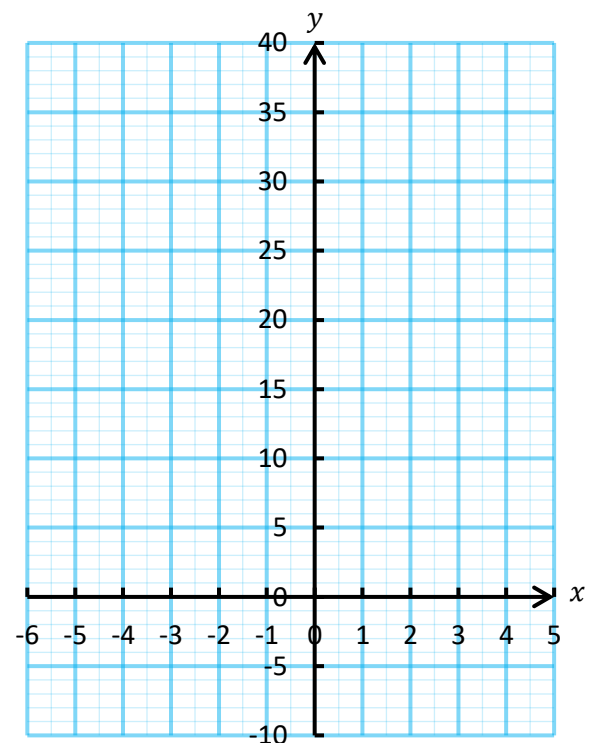
(c) Plot the graph of the function on the graph paper on the right.

(d) Complete the following sentences.

The y -intercept is

The roots are $x = \dots$ and $x = \dots$

The minimum point is



Factorising Quadratic Expressions

It is possible to factorise some quadratic expressions using GCSE methods.

Exercise 3

Factorise: (a) $x^2 + 6x + 8$

(b) $x^2 + 4x - 12$

.....

.....

.....

.....

.....

.....

Example 2

Factorise the quadratic expression $3x^2 - 7x - 20$.

The Splitting Method

$$3 \times -20 = -60$$

The two numbers that multiply to give -60 and add to give -7 are 5 and -12 .

$$\begin{aligned} 3x^2 - 7x - 20 &= 3x^2 + 5x - 12x - 20 \\ &= x(3x + 5) - 4(3x + 5) \\ &= (3x + 5)(x - 4) \end{aligned}$$

The Detective Method

$3x \times x = 3x^2$, so it is possible to factorise $3x^2 - 7x - 20$ in the form $(3x + ?)(x + ?)$.

By attempting pairs of numbers that multiply to give -20 , we see that $3x^2 - 7x - 20 = (3x + 5)(x - 4)$.

Exercise 4

Factorise: (a) $2x^2 + 13x + 15$

(b) $3x^2 - 35x - 12$

.....

.....

.....

.....

.....

.....

(c) $4x^2 + 9x + 5$

(d) $8x^2 - 14x + 6$

.....

.....

.....

.....

.....

.....

Factorising Expressions with Higher Indices

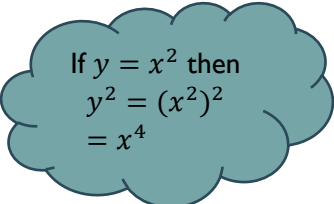
It is possible to use the above techniques to factorise **some** expressions with indices that are greater than 2.

Example 3

Factorise the quartic expression $2x^4 + 9x^2 + 4$.

We can use the substitution $y = x^2$ to change the expression to be a quadratic expression:

$$\begin{aligned} 2x^4 + 9x^2 + 4 &= 2y^2 + 9y + 4 \\ &= 2y^2 + 8y + y + 4 \\ &= 2y(y + 4) + 1(y + 4) \\ &= (y + 4)(2y + 1) \\ &= (x^2 + 4)(2x^2 + 1) \end{aligned}$$



If $y = x^2$ then
 $y^2 = (x^2)^2$
 $= x^4$

Exercise 5

Use an appropriate substitution to factorise the following expressions.

(a) $2x^4 + 7x^2 + 6$

.....

.....

.....

.....

.....

(b) $5x^4 + 18x^2 + 9$

.....

.....

.....

.....

.....

(c) $3x^6 + 13x^3 + 4$

.....

.....

.....

.....

.....

(d) $4x^8 + 21x^4 - 18$

.....

.....

.....

.....

.....

(e) $2x^{24} - 21x^{12} + 40$

.....

.....

.....

.....

.....

(f) $x^{100} + 3x^{50} + 2$

.....

.....

.....

.....

.....

Solving Equations

We can use the above factorising techniques to solve some equations.

Example 4

(a) Solve the equation $x^2 + 2x - 8 = 0$

$$(x + 4)(x - 2) = 0$$

Either $x + 4 = 0$ or $x - 2 = 0$

$$x = -4 \quad \text{or} \quad x = 2$$

(c) Solve the quartic equation

$$x^4 - 13x^2 + 36 = 0$$

We can use the substitution $y = x^2$ to change the expression to be a quadratic expression:

$$y^2 - 13y + 36 = 0$$

$$(y - 4)(y - 9) = 0$$

$$(x^2 - 4)(x^2 - 9) = 0$$

Either $x^2 - 4 = 0$ or $x^2 - 9 = 0$

$$x^2 = 4 \quad \text{or} \quad x^2 = 9$$

$$x = \pm\sqrt{4} \quad \text{or} \quad x = \pm\sqrt{9}$$

Either $x = 2$, $x = -2$, $x = 3$ or $x = -3$.

(b) Solve the equation $2x^2 + 3x - 20 = 0$

$$2 \times -20 = -40$$

$$2x^2 + 8x - 5x - 20 = 0$$

$$2x(x + 4) - 5(x + 4) = 0$$

$$(x + 4)(2x - 5) = 0$$

Either $x + 4 = 0$ or $2x - 5 = 0$

$$x = -4 \quad \text{or} \quad x = \frac{5}{2}$$

Exercise 6

Solve the following equations by factorising.

(a) $x^2 + 12x + 32 = 0$

.....

.....

.....

.....

.....

(b) $2x^2 - 13x + 20 = 0$

.....

.....

.....

.....

.....

(c) $x^4 - 17x^2 + 16 = 0$

.....

.....

.....

.....

.....

.....

.....

.....

.....

(d) $x^6 - 7x^3 - 8 = 0$

.....

.....

.....

.....

.....

.....

.....

.....

.....

Completing the Square

It is not possible to solve every quadratic equation by using the above factorising techniques.

For these equations, we can try to **complete the square** to solve the equation.**Example 5**Solve the equation $x^2 + 6x + 7 = 0$ by completing the square.

$$x^2 + 6x + 7 = 0$$

$$(x + 3)^2 - 3^2 + 7 = 0 \quad \text{[Half of 6 is 3]}$$

$$(x + 3)^2 - 9 + 7 = 0$$

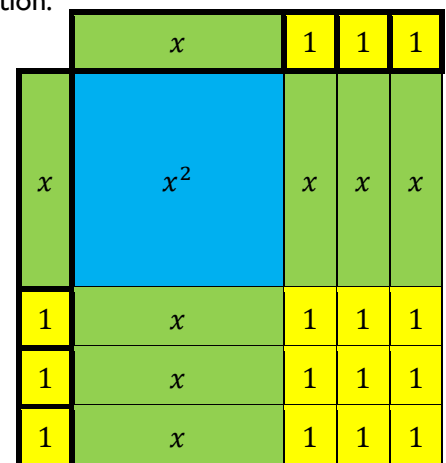
$$(x + 3)^2 - 2 = 0$$

$$(x + 3)^2 = 2$$

$$x + 3 = \pm\sqrt{2}$$

$$x = -3 \pm \sqrt{2}$$

$$\text{Either } x = -3 + \sqrt{2} \text{ or } x = -3 - \sqrt{2}$$

The *algebra tiles* on the right shows what is happening during the solution.The diagram shows the sum $(x + 3)^2 = x^2 + 6x + 9$. Comparing with $x^2 + 6x + 7$, we see that we agree with the x^2 and the $6x$, but not with the constants. We must **subtract 9** (or 3^2) and then **add 7** to obtain what we need:

$$(x + 3)^2 = x^2 + 6x + 9$$

$$(x + 3)^2 - 9 = x^2 + 6x + 9 - 9$$

$$(x + 3)^2 - 9 = x^2 + 6x$$

$$(x + 3)^2 - 9 + 7 = x^2 + 6x + 7$$

$$(x + 3)^2 - 2 = x^2 + 6x + 7$$

Exercise 7

Solve the following equations by completing the square.

(a) $x^2 + 10x + 3 = 0$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(b) $x^2 - 8x + 9 = 0$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(c) $x^2 + 6x + 7 = 0$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(d) $x^2 - 14x - 1 = 0$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(e) $x^2 + 12x + 1 = 0$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(f) $x^2 + x - 2 = 0$

.....

.....

.....

.....

.....

.....

.....

.....

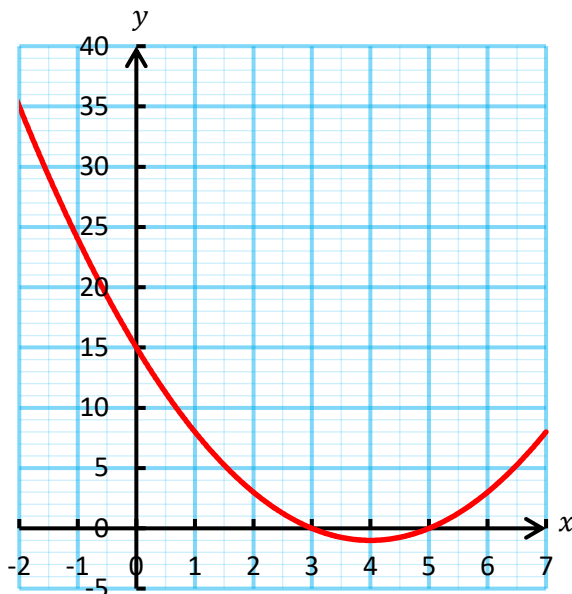
.....

.....

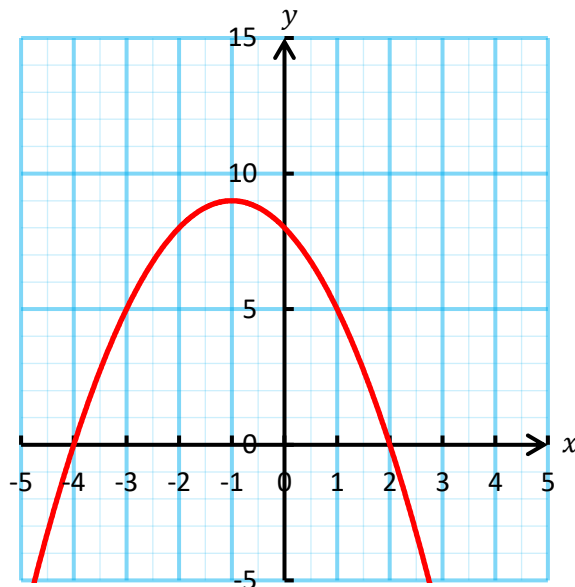
Finding a minimum point or maximum point by completing the square

Let us return to the quadratic functions we first saw in Example 1.

$$y = x^2 - 8x + 15$$



$$y = -x^2 - 2x + 8$$



By completing the square, it is possible to find the roots and the minimum / maximum point.

Finding the roots:

$$\begin{aligned} x^2 - 8x + 15 &= 0 \\ (x - 4)^2 - (-4)^2 + 15 &= 0 \\ (x - 4)^2 - 16 + 15 &= 0 \\ (x - 4)^2 - 1 &= 0 \\ (x - 4)^2 &= 1 \\ x - 4 &= \pm\sqrt{1} \\ x - 4 &= \pm 1 \\ x &= 4 \pm 1 \\ \text{Either } x &= 4 + 1 \text{ or } x = 4 - 1 \\ x &= 5 \quad \text{or } x = 3 \end{aligned}$$

Minimum / maximum point:

By completing the square, we can write $y = x^2 - 8x + 15$ as $y = (x - 4)^2 - 1$.
No matter the value of x , we have $(x - 4)^2 \geq 0$.
So, the minimum value of the function is -1 .
This happens when the bracket $x - 4 = 0$
 $x = 4$
So, the minimum point is $(4, -1)$.

$$\begin{aligned} -x^2 - 2x + 8 &= 0 \\ x^2 + 2x - 8 &= 0 && [\text{Multiply by } -1] \\ (x + 1)^2 - 1^2 - 8 &= 0 \\ (x + 1)^2 - 1 - 8 &= 0 \\ (x + 1)^2 - 9 &= 0 \\ (x + 1)^2 &= 9 \\ x + 1 &= \pm\sqrt{9} \\ x + 1 &= \pm 3 \\ x &= -1 \pm 3 \\ \text{Either } x &= -1 + 3 \text{ or } x = -1 - 3 \\ x &= 2 \quad \text{or } x = -4 \end{aligned}$$

By completing the square, we can write $y = -x^2 - 2x + 8$ as $y = -(x + 1)^2 + 9$.
So, we can write $y = -x^2 - 2x + 8$ as $y = -(x + 1)^2 + 9$, or $y = 9 - (x + 1)^2$.
No matter the value of x , we have $(x + 1)^2 \geq 0$.
So, the maximum value of the function is 9 .
This happens when the bracket $x + 1 = 0$
 $x = -1$
So, the maximum point is $(-1, 9)$.

Use your calculations from Exercise 7 to help you find the minimum point for the following functions.

(b) $x^2 - 8x + 9 = 0$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(d) $x^2 - 14x - 1 = 0$

[illegible]

.....

.....

.....

.....

.....

(f) $x^2 + x - 2 = 0$

[illegible]

.....

.....

.....

.....

.....

Find the roots and the minimum / maximum point for the following functions.

(b) $y = -x^2 - 2x + 3$

The Discriminant

Consider the graphs shown below, plotting three quadratic functions.

$$y = 2x^2 - 16x + 30$$

$$y = 2x^2 - 16x + 32$$

$$y = 2x^2 - 16x + 34$$



The first graph, for $y = 2x^2 - 16x + 30$, crosses the x -axis twice, at the points (3, 0) and (5, 0). We say that the function has **two real roots**.

The second graph, for $y = 2x^2 - 16x + 32$, meets the x -axis at one point, namely (4, 0). We say that the function has **one repeated real root**.

The third graph, for $y = 2x^2 - 16x + 34$, does not cross the x -axis at all. We say that the function has **no real roots**.

We could find the above roots by using the quadratic formula.

$$y = 2x^2 - 16x + 30$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2, b = -16, c = 30$$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4 \times 2 \times 30}}{2 \times 2}$$

$$x = \frac{16 \pm \sqrt{256 - 240}}{4}$$

$$x = \frac{16 \pm \sqrt{16}}{4}$$

$$x = \frac{16 \pm 4}{4}$$

$$\text{Either } x = \frac{16+4}{4} \text{ or } x = \frac{16-4}{4}$$

$$\text{Either } x = 5 \text{ or } x = 3$$

$$y = 2x^2 - 16x + 32$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2, b = -16, c = 32$$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4 \times 2 \times 32}}{2 \times 2}$$

$$x = \frac{16 \pm \sqrt{256 - 256}}{4}$$

$$x = \frac{16 \pm \sqrt{0}}{4}$$

$$x = \frac{16 \pm 0}{4}$$

$$\text{Either } x = \frac{16+0}{4} \text{ or } x = \frac{16-0}{4}$$

$$\text{Either } x = 4 \text{ or } x = 4$$

$$y = 2x^2 - 16x + 34$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2, b = -16, c = 34$$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4 \times 2 \times 34}}{2 \times 2}$$

$$x = \frac{16 \pm \sqrt{256 - 272}}{4}$$

$$x = \frac{16 \pm \sqrt{-16}}{4}$$

'Math Error' due to $\sqrt{-16}$

The part of the quadratic formula that influences the number of real roots obtained is the expression $b^2 - 4ac$, which appears within the square root. This expression is known as the **discriminant**.

If $b^2 - 4ac > 0$ then we have two real roots.

If $b^2 - 4ac = 0$ then we have one repeated real root.

If $b^2 - 4ac < 0$ then we do not have any real roots.

Exercise 10

How many real roots do the following quadratic equations have?

(a) $9x^2 + 12x + 4 = 0$

.....

.....

.....

.....

.....

(b) $9x^2 + 12x - 4 = 0$

.....

.....

.....

.....

.....

(c) $9x^2 + 4x + 12 = 0$

.....

.....

.....

.....

.....

(d) $x^2 + 5x + 2 = 0$

.....

.....

.....

.....

.....

(e) $x^2 + 2x + 2 = 0$

.....

.....

.....

.....

.....

(f) $-x^2 + 6x - 9 = 0$

.....

.....

.....

.....

.....

Exercise 11

Consider the quadratic equation $2x^2 + 8x + k = 0$. For which values of k does the equation have

(a) two real roots?

.....

.....

.....

.....

.....

(b) one repeated real root?

.....

.....

.....

.....

.....

(c) no real roots?

.....

.....

.....

.....

.....

Exercise 12

Consider the quadratic equation $2x^2 + kx + 8 = 0$. For which values of k does the equation have

(a) two real roots?

.....

.....

.....

.....

.....

.....

.....

.....

.....

(b) one repeated real root?

.....

.....

.....

.....

.....

.....

.....

.....

.....

(c) no real roots?

.....

.....

.....

.....

.....

.....

.....

.....

.....

Pop Quiz: Quadratic Functions I

/16

(I) Consider the quadratic function $y = x^2 + 8x - 20$.

(a) Will the graph of this function be 'u'-shaped, or 'n'-shaped?

[1]

(b) Complete the following table.

[2]

x	-12	-10	-8	-6	-4	-2	0	2	4	6	8	10	12
y	28	0		-32	-36	-32	-20		28	64	108	160	220

(c) Plot the graph of the function on the graph paper on the right.

[1]

(d) Complete the following sentences.

The y -intercept is The roots are $x = \dots$ and $x = \dots$

The minimum point is

[3]

(e) By factorising, solve the equation

[2]

$x^2 + 8x - 20 = 0$.

.....

.....

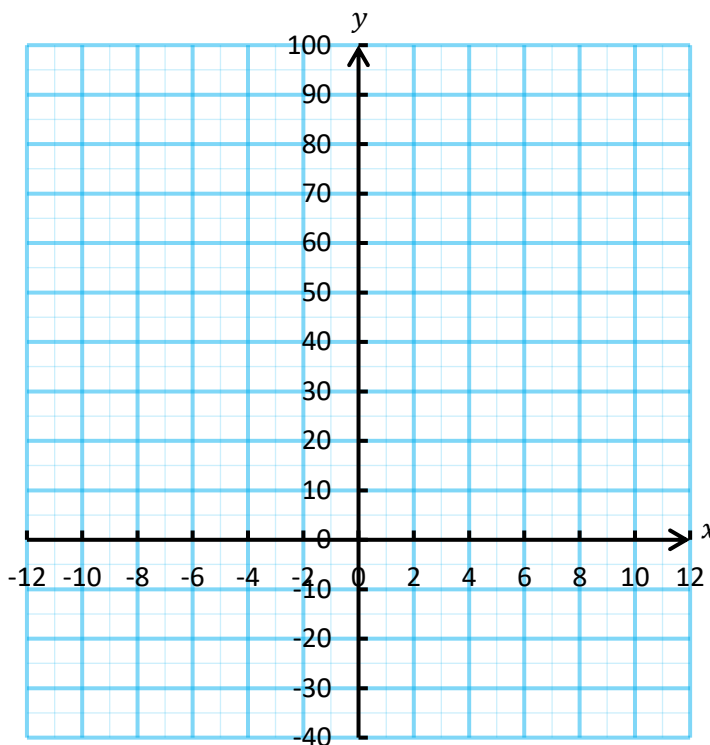
.....

.....

.....

.....

.....



(f) By completing the square, solve the equation

[2]

$x^2 + 8x - 20 = 0$.

.....

.....

.....

.....

.....

.....

.....

(g) Use your answer to part (f) to write down the minimum point of $y = x^2 + 8x - 20$.

[1]

.....

.....

.....

.....

.....

.....

.....

(h) Solve the equation $x^2 + 8x - 20 = 0$.

[2]

(i) Use the discriminant to determine the number of real roots of the equation $x^2 + 8x - 20 = 0$.

[2]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

(b) $y = 5x^2 - 8x - 6$, $y = 10x + 2$

(c) $y = x^2 - 7x + 12$, $x + y = 4$

(d) $2x + y = 13$, $x^2 + xy - 30 = 0$

(e) $y = 4x^2 + 2x - 3$, $10x - 2y + 3 = 0$

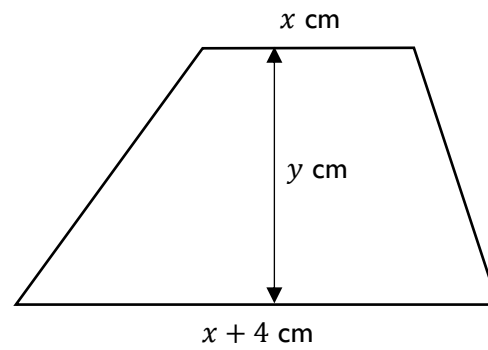
Exercise 14

The area of the larger square is 30 cm^2 more than the area of the smaller square.

You must use an algebraic method, **not** trial and improvement.

A square and rectangle are such that the length of a side of the square is equal to the length of the **shorter** side of the rectangle. The total area of the square and the rectangle is 198 cm^2 and the total of their perimeters is 80 cm . Calculate the dimensions of the rectangle.

Megan knows that $x = y + 5$. She also knows that the sum of x and y is equal to the product of x and y . Using an algebraic method, show that x and y are not integers.

Exercise 17

The diagram is not drawn to scale

The area of the trapezium in the diagram is 28 cm^2 .

The area of a rectangle of length $(x + y) \text{ cm}$ and width $y \text{ cm}$ is 43 cm^2 .

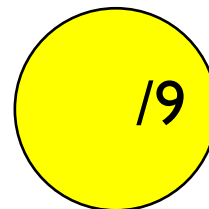
(a) Show that $xy = 28 - 2y$ and $xy = 43 - y^2$.

(b) Hence, write a quadratic equation in terms of y and calculate the length of the parallel sides of the trapezium. You must use an algebraic method, **not** trial and improvement.

Pop Quiz: Quadratic Functions 2

(1) Find the coordinates of the points where the following functions intersect.

[4]



$$y = x^2 - 4x + 6, \quad y = -x + 16$$

(2) Five times one number is equal to twice a second number. The sum of the two numbers is 0.7.
Find the two numbers.

[5]



Unit 1, Workbook 3

11

Arithmetic

and Geometric

Series



Name:

Background

What is the work?

Given an arithmetic series where we add the same number to obtain the next number, or a geometric series where we multiply the same number to obtain the next number, how can we find the n th term, or the sum of the first n terms?

What are the prerequisites?

GCSE Work: Solving equations, n th term, simultaneous equations.

Where does this lead to?

Applications:

- Finance: Borrowing and investing money.
- Studying how populations change.
- Managing time in large projects.
- Expansion rate of the universe.
- Radioactive decay.
- Compound interest.

Theory

Arithmetic Series

A sequence of numbers is an **arithmetic sequence** if we add the same constant to obtain the next number in the sequence. For example, the sequence 5, 8, 11, 14, 17, ... is an arithmetic sequence where the first term is 5 and the common difference is 3.

Terminology

a	First term of the sequence
d	The common difference
l	Last term of the sequence, if one exists
t_n	The n th term of the sequence

For an arithmetic sequence with first term a and common difference d :

First term:	$t_1 = a$
Second term:	$t_2 = a + d$
Third term:	$t_3 = a + 2d$
Fourth term:	$t_4 = a + 3d$

Nth term: $t_n = a + (n - 1)d$

For all integers n ,

$$d = t_{n+1} - t_n$$

Exercise 1

For the arithmetic sequence 9, 14, 19, 24, 29, ..., write down

- (a) a
 (c) t_7

- (b) d
 (d) t_n

For the arithmetic sequence 5, 8, 11, 14, 17, ...

$$S_6 = 5 + 8 + 11 + 14 + 17 + 20 = 75$$

$$S_n = \frac{1}{2}n(a + l).$$

$$S_n = \frac{1}{2}n(2a + (n - 1)d).$$

For the arithmetic sequence 9, 11, 13, 15, 17, ..., calculate S_5 .

Method 1: using $S_n = \frac{1}{2}n(a + l)$.

$$S_n = \frac{1}{2} \times 5 \times 26$$

$$S_n = 65$$

Method 2: using $S_n = \frac{1}{2}n(2a + (n - 1)d)$.

$$S_n = \frac{1}{2} \times 5 \times (18 + 4 \times 2)$$

$$S_n = \frac{1}{2} \times 5 \times 26$$

$$S_n = 65$$

For the arithmetic sequence 5, 9, 13, 17, 21, ..., calculate S_4 .

For the arithmetic sequence 11, 14, 17, 20, 23, ..., calculate S_6 .

Example 2

The seventh term of an arithmetic series is twice the third term. The sum of the first ten terms is 195.

- (a) Find the common difference of the series.
- (b) Find the sum of the first sixty terms.

Answer: (a) To begin, we must change the clues at the start of the question to be equations.

Clue 1:

$$t_7 = 2t_3$$

$$a + 6d = 2(a + 2d)$$

$$a + 6d = 2a + 4d$$

$$2d = a$$

Equation 1

Clue 2:

$$S_{10} = 195$$

$$\frac{1}{2} \times 10 \times (2a + (10 - 1)d) = 195$$

$$5(2a + 9d) = 195$$

$$2a + 9d = 39$$

Equation 2

Substituting for a from Equation [1] into Equation [2]:

$$2(2d) + 9d = 39$$

$$4d + 9d = 39$$

$$13d = 39$$

$$d = 3$$

- (b) Substituting back into Equation **1**: $2 \times 3 = a$
 $a = 6$

Using $S_n = \frac{1}{2}n(2a + (n - 1)d)$:

$$S_{60} = \frac{1}{2} \times 60 \times (2 \times 6 + (60 - 1) \times 3)$$

$$S_{60} = 30(12 + 59 \times 3)$$

$$S_{60} = 30(12 + 177)$$

$$S_{60} = 30 \times 189$$

$$S_{60} = 5670$$

$$\begin{array}{r} 189 \\ \times 3 \\ \hline 567 \\ \hline 22 \end{array}$$

Exercise 4

In an arithmetic series, the eighth term is twice the third term. The twentieth term of the series is 11. Find the common difference and the first term of the series.

Exercise 5

The third term of an arithmetic series is four times the sixth term of the series. The sum of the first twenty terms of the series is 350. Find the first term and the common difference of the series.

Exericse 6

The sum of the first ten terms of an arithmetic series is 320. The sum of the twelfth and sixteenth terms of the series is 166. Find the first term and the common difference of the series.

(a) The eighth term of an arithmetic series is 28. The sum of the first twenty terms of the series is 710. Find the first term and common difference of the arithmetic series.

The sum of the first eight terms of an arithmetic series is 124 and the sum of the first twenty terms is 910.

- (a) Find the first term and common difference of the series.
- (b) The n th term of the series is equal to 183. Find the value of n .

(a) The eighth term of an arithmetic series is 46. The sum of the first nine terms of the series is 225. Find the first term and common difference of the series.

$$3 + 7 + 11 + 15 + \dots$$

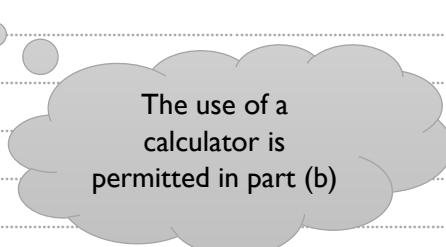
(a) The sum of the third, fourth and tenth terms of an arithmetic series is 79. The sum of the sixth and seventh terms of the series is 61. Find the first term and common difference of the series.

$$15 + 13 + 11 + 9 + \dots$$

Exercise 11

Geraint opens a savings account. He deposits £10 in the first month. In each subsequent month, the amount he deposits is 20 pence greater than the amount he deposited in the previous month.

- (a) Find the amount that Geraint deposits into the savings account in the 12th month.
- (b) Determine the number of months it takes for the total amount in the savings account to reach £954.



The use of a
calculator is
permitted in part (b)

First term

 d
$$S_n$$
$$a$$

[1]

$$t_{10} = a + 9d$$

$$t_{10} = a + 10d$$

$$t_{10} = a + 11d$$

(a) Find the first term and common difference of this arithmetic series.

[6]

Geometric Series

A sequence of numbers is a **geometric sequence** if we multiply by the same constant to obtain the next number in the sequence. For example, the sequence 3, 6, 12, 24, 48, ... is a geometric sequence where the first term is 3 and the common ratio is 2.

Terminology

a	First term of the sequence
r	The common ratio
l	Last term of the sequence, if one exists
t_n	The n th term of the sequence

For a geometric series with first term a and common ratio r :

First term:	$t_1 = a$
Second term:	$t_2 = ar$
Third term:	$t_3 = ar^2$
Fourth term:	$t_4 = ar^3$

Nth term: $t_n = ar^{n-1}$

For all integers n ,

$$r = \frac{t_{n+1}}{t_n}$$

Exercise 12

For the geometric series 2, 6, 18, 54, 162, ..., write down

- (a) a (b) r
 (c) t_7 (d) t_n

It is possible to consider the sum of the first n terms in a geometric series, written as S_n .

For the geometric series 3, 6, 12, 24, 48, ...

$$S_3 = 3 + 6 + 12 = 21$$

$$S_6 = 3 + 6 + 12 + 24 + 48 + 96 = 189$$

The general formula for calculating S_n is

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Example 3

For the geometric series 5, 10, 20, 40, 80, ..., calculate S_{10} .

Answer: In this series, we have $a = 5$ and $r = 2$. The sum of the first ten terms is

$$S_{10} = \frac{5(1 - 2^{10})}{1 - 2}$$

$$S_{10} = \frac{5(1 - 1024)}{-1}$$

$$S_{10} = -5 \times -1023$$

$$S_{10} = 5115$$

$$\begin{array}{r} 1023 \\ \times 5 \\ \hline 5115 \\ \hline \end{array}$$

2, 4, 8, 16,
32, 64, 128,
256, 512,
1024

Exercise 13

For the geometric series 5, 10, 20, 40, 80, ..., calculate S_6 .

Exercise 14

For the geometric series 9, -18, 36, -72, 144, ..., calculate S_5 .

Example 4

The fourth term of a geometric series is 2 and the seventh term is 54.

(a) Find the common ratio of the series.

(b) Find the sum of the first three terms of the series, giving your answer as a fraction.

Answer: (a) To begin, we must change the clues at the start of the question to be equations.

Clue 1:

$$t_4 = 2$$

$$ar^3 = 2$$

Equation [1]

Clue 2:

$$t_7 = 54$$

$$ar^6 = 54$$

Next, we must look for a connection between the above equations.

$$ar^3(r^3) = 54$$

Equation [2]

Substituting for ar^3 from Equation [1] into Equation [2]:

$$2r^3 = 54$$

$$r^3 = 27$$

$$r = \sqrt[3]{27}$$

$$r = 3$$

(b) To use the formula for S_n , we must find the value of a by substituting for r back into Equation [1]:

$$a(3^3) = 2$$

$$27a = 2$$

$$a = \frac{2}{27}$$

We can now find the value of S_3 :

$$S_3 = \frac{\frac{2}{27}(1-3^3)}{1-3}$$

$$S_3 = \frac{\frac{2}{27}(1-27)}{-2}$$

$$S_3 = \frac{\frac{2}{27}(-26)}{-2}$$

$$S_3 = \frac{2}{27} \times 13$$

$$S_3 = \frac{26}{27}$$

The ninth and tenth terms of a geometric series are 36 and 108, respectively. Find the common ratio of the series.

The fourth term of an arithmetic series is -108 and the seventh term is 4 . Find the common ratio of the series.

The second term of a geometric series is 6 and the fifth term is 384.

The eighth and ninth terms of a geometric series are 576 and 2304, respectively.

A geometric series has first term a and common ratio r . The sum of the second and third terms of the series is -216 . The sum of the fifth and sixth terms of the series is 8 . Prove that $r = -\frac{1}{3}$.

The sum of the first and third terms of a geometric series is 20. The sum of the fourth and sixth terms of the series is 540. Prove that $r = 3$, where r denotes the common ratio of the series.

A geometric series has first term a and common ratio r . The sum of the first and second terms of the series is 72. The sum of the first and third terms of the series is 120.

$$3r^2 - 5r - 2 = 0.$$

Exercise 22

A rich businessman makes one donation per year to a certain charity. He starts by donating £100 in the first year. In each subsequent year, the value of the donation is 1.2 times the value of the previous year's donation. Find the value of the businessman's donation in the 12th year. Give your answer correct to the nearest pound.

The use of a calculator
is permitted in this
question.

Exercise 23

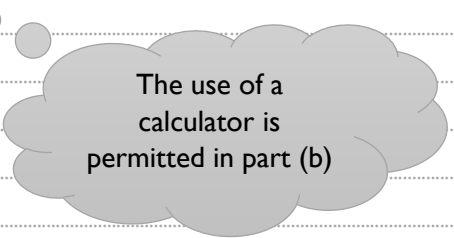
A tree is 80 cm in height when it is planted. In the first year, the tree grows in height by 32 cm. In each subsequent year, the tree grows in height by 90% of the growth of the previous year. Find the height of the tree 10 years after it was planted.

The use of a calculator
is permitted in this
question.

Exercise 24

The sum of the first and second terms of a geometric series is equal to twice the sum of the second and third terms of the series.

- (a) Given that the common ratio of the series is positive, find the value of the common ratio.
- (b) Given that $a = 6$, find, correct to two decimal places, the sum of the first eight terms of the series.



[3]

$$r$$
$$S_n$$
$$a$$

[1]

$$t_{10} = ar^{11}$$

[7]



Differentiating

from First

Principles



Isaac Newton



Gottfried Wilhelm Leibniz

In the 18th century a huge row developed between Leibniz and Newton regarding who was the first person to develop the idea of differentiation.

Name:

Background

What is the work?

Finding the gradient of a curve using algebraic techniques.

What are the prerequisites?

GCSE Work: Expanding brackets; simplifying; finding the gradient of a curve.

Where does this lead to?

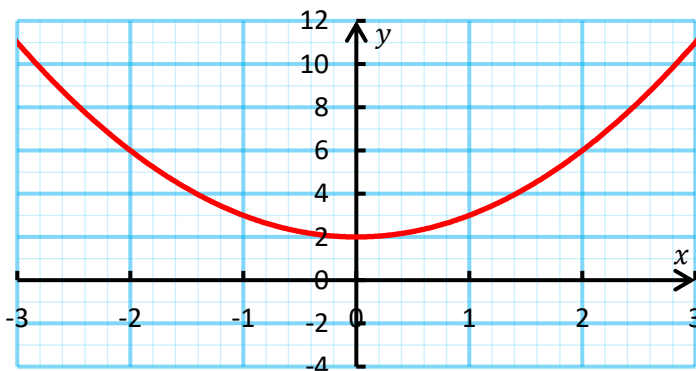
Additional Mathematics: Differentiating and integrating quickly.

Applications: Finding velocity and acceleration; optimisation; designing efficient systems.

Theory

Consider the function $y = x^2 + 2$. It is possible to draw a graph of this function by forming a table of values and plotting them on graph paper.

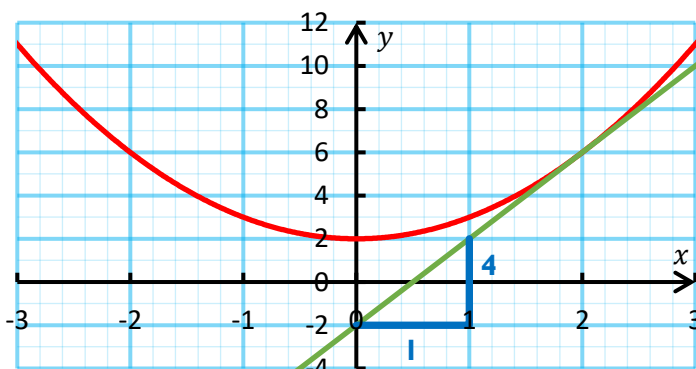
x	-3	-2	-1	0	1	2	3
y	11	6	3	2	3	6	11



The gradient of a graph can be very useful. For example, the gradient of a displacement-time graph gives the velocity.

We see that the **gradient** of the curve (how steep the curve is) changes as we move along the curve. If $x = 0$, then the gradient is zero. If x is negative, then the gradient is negative, and if x is positive, then the gradient is positive.

What is the gradient of the graph if $x = 2$? One way of finding this would be to draw a tangent to the graph at the point (2, 6) and find the gradient of the tangent by drawing a triangle.

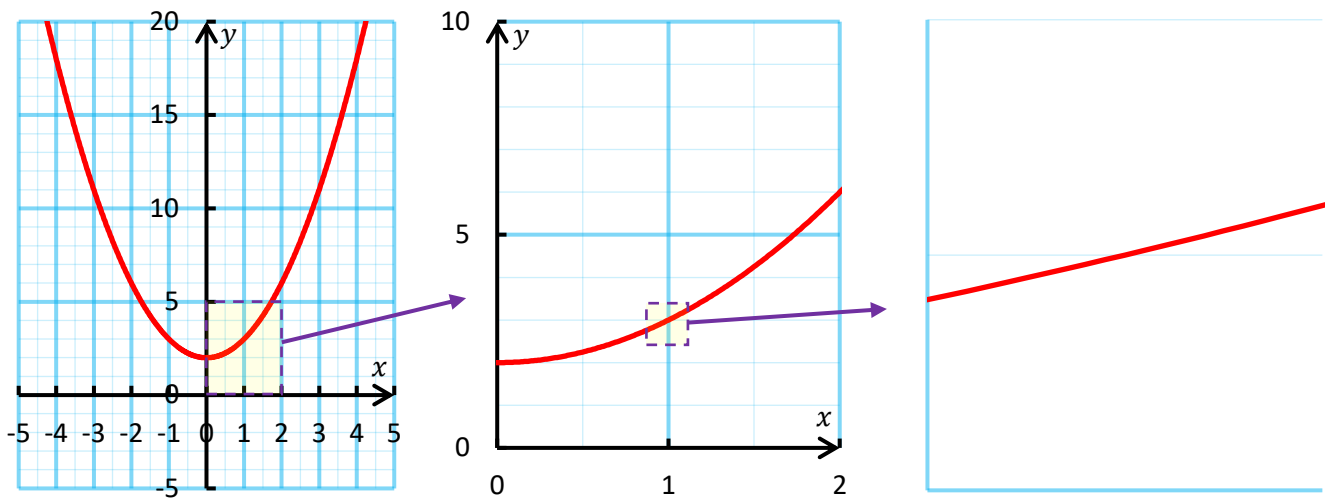


We see from the diagram on the left that the gradient of the graph when $x = 2$ is 4.

Differentiating is a method for finding the gradient of a curve at a particular point **without** having to draw a tangent and forming a triangle. We use special notation to represent this gradient: if y represents any function, then $\frac{dy}{dx}$ or y' represents the gradient of the function.

Differentiating from first principles

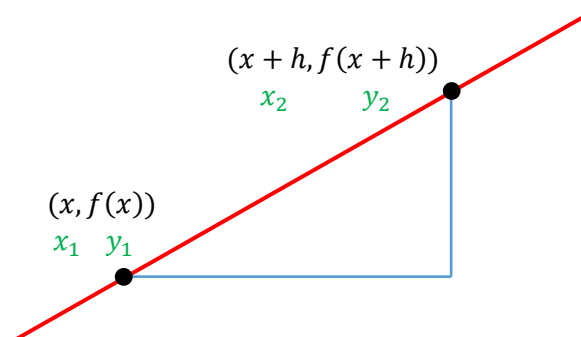
Consider again the function $y = x^2 + 2$ from the previous page. If we zoom in enough on the graph, then it will look like a straight line. (Try this yourself on the website www.desmos.com/calculator.)



This is true for any part of the graph, so to find the gradient of the graph at any point,

- Zoom in to the point – enough so that you see a straight line;
- Find the gradient of the straight line by adding a right-angled triangle to the line.

Let $(x, f(x))$ represent the co-ordinate of the point we are zooming in to. (In the above example, we are zooming in on the point $(1, 3)$, where $f(1) = 1^2 + 2 = 3$.) Consider that we have zoomed in enough so that we see the following **straight line**.



Theory

Let h represent a very small distance along the x -axis. We can move right from the point $(x, f(x))$ to reach a distance $x + h$ along the x -axis. The point on the function corresponding to $x + h$ is $(x + h, f(x + h))$.

Considering that h is small enough so that the function behaves like a straight line between x and $x + h$, we can use the fraction $\frac{y_2 - y_1}{x_2 - x_1}$ to calculate the gradient of the straight line.

$$\begin{aligned} \frac{y_2 - y_1}{x_2 - x_1} &= \frac{f(x + h) - f(x)}{x + h - x} \\ &= \frac{f(x + h) - f(x)}{h} \end{aligned}$$

The problem now is to ensure that h is small enough to give a straight line each time. To make sure of this, we use the idea of a **limit** so that h can be as close to zero as we want. So, the gradient $\frac{dy}{dx}$ of the function at the point $(x, f(x))$ is

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Note: Some textbooks and worksheets use δx instead of h to represent the small distance along the x -axis.

Example 1

From first principles, find $\frac{dy}{dx}$ if $y = x^2 + 2$.

We need to construct $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ step-by-step.

We have $f(x) = x^2 + 2$

So

$$\begin{aligned} f(x+h) &= (x+h)^2 + 2 \\ f(x+h) &= (x+h)(x+h) + 2 \\ f(x+h) &= x^2 + xh + xh + h^2 + 2 \\ f(x+h) &= x^2 + 2xh + h^2 + 2 \end{aligned}$$

We can use FOIL to expand the brackets.

Now

$$\begin{aligned} f(x+h) - f(x) &= x^2 + 2xh + h^2 + 2 - (x^2 + 2) \\ f(x+h) - f(x) &= \cancel{x^2} + 2xh + h^2 + \cancel{2} - \cancel{x^2} - \cancel{2} \\ f(x+h) - f(x) &= 2xh + h^2 \end{aligned}$$

The x^2 cancels the $-x^2$ and $+2$ cancels -2 .

Next

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{2xh+h^2}{h} \\ \frac{f(x+h)-f(x)}{h} &= 2x + h \end{aligned}$$

Finally

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} 2x + h \\ \frac{dy}{dx} &= 2x \end{aligned}$$

After simplifying as much as we can, we now use the limit by substituting $h = 0$.

Example 2

From first principles, find $\frac{dy}{dx}$ if $y = x^2 + 4x$.

We need to construct $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ step-by-step.

We have $f(x) = x^2 + 4x$

So

$$\begin{aligned} f(x+h) &= (x+h)^2 + 4(x+h) \\ f(x+h) &= (x+h)(x+h) + 4(x+h) \\ f(x+h) &= x^2 + xh + xh + h^2 + 4x + 4h \\ f(x+h) &= x^2 + 2xh + h^2 + 4x + 4h \end{aligned}$$

Now

$$\begin{aligned} f(x+h) - f(x) &= x^2 + 2xh + h^2 + 4x + 4h - (x^2 + 4x) \\ f(x+h) - f(x) &= \cancel{x^2} + 2xh + h^2 + \cancel{4x} + 4h - \cancel{x^2} - \cancel{4x} \\ f(x+h) - f(x) &= 2xh + h^2 + 4h \end{aligned}$$

Next

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{2xh+h^2+4h}{h} \\ \frac{f(x+h)-f(x)}{h} &= 2x + h + 4 \end{aligned}$$

Finally

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} 2x + h + 4 \\ \frac{dy}{dx} &= 2x + 4 \end{aligned}$$

From first principles, find $\frac{dy}{dx}$ if $y = 2x^2 - 6x + 9$.

We need to construct $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ step-by-step.

We have $f(x) = 2x^2 - 6x + 9$

So

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 6(x+h) + 9 \\ f(x+h) &= 2(x+h)(x+h) - 6(x+h) + 9 \\ f(x+h) &= 2(x^2 + xh + xh + h^2) - 6x - 6h + 9 \\ f(x+h) &= 2x^2 + 4xh + 2h^2 - 6x - 6h + 9 \end{aligned}$$

Now $f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 - 6x - 6h + 9 - (2x^2 - 6x + 9)$
 $f(x+h) - f(x) = \cancel{2x^2} + 4xh + 2h^2 - \cancel{6x} - 6h + \cancel{9} - \cancel{2x^2} + \cancel{6x} - \cancel{9}$
 $f(x+h) - f(x) = 4xh + 2h^2 - 6h$

Next $\frac{f(x+h)-f(x)}{h} = \frac{4xh+2h^2-6h}{h}$
 $\frac{f(x+h)-f(x)}{h} = 4x + 2h - 6$

Finally

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} 4x + 2h - 6 \\ \frac{dy}{dx} &= 4x - 6\end{aligned}$$

From first principles, find $\frac{dy}{dx}$ if $y = x^2 + 5$.

Given that $y = x^2 - 3$, find $\frac{dy}{dx}$ from first principles.

Given that $y = x^2 + 7x$, find $\frac{dy}{dx}$ from first principles.

Given that $y = x^2 - 4x$, find $\frac{dy}{dx}$ from first principles.

Given that $y = 3x^2 + 4$, find $\frac{dy}{dx}$ from first principles.

Given that $y = 5x^2 - 2$, find $\frac{dy}{dx}$ from first principles.

Given that $y = 11x^2 + 5x$, find $\frac{dy}{dx}$ from first principles.

Given that $y = 7x^2 - 3x$, find $\frac{dy}{dx}$ from first principles.

Given that $y = 8x^2 + 4x - 3$, find $\frac{dy}{dx}$ from first principles.

Pop Quiz: Differentiating from First Principles

(1) Which one of the following is correct?

[1]

/6

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)+f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{x}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x-h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{x \rightarrow 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(h)}{h}$$

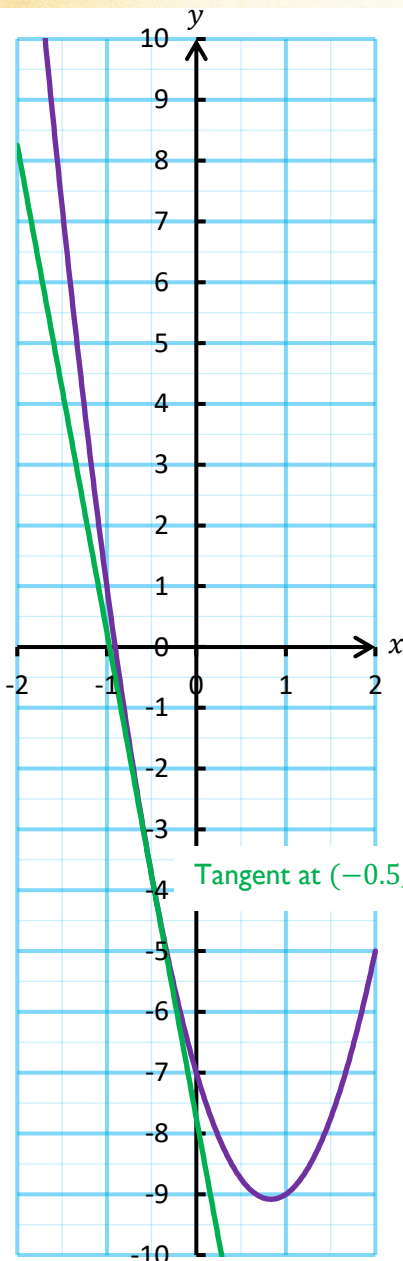
(2) Given that $y = 3x^2 + 14x$, find $\frac{dy}{dx}$ from first principles.

[5]



Quick

Differentiation



Tangent at $(-0.5, -3.75)$

$$y = 3x^2 - 5x - 7$$

Name:

Background

What is the work?

Differentiating a function without having to go from first principles, and using this technique to find the equation of tangents, and stationary points.

What are the prerequisites?

GCSE Work: Solving equations.

Where does this lead to?

Additional Mathematics: Integrating quickly.

Applications: Finding velocity and acceleration; optimisation; designing efficient systems.

Theory

In the previous workbook, the technique of differentiating from first principles was introduced. Did you notice, in this work, the connection between each question and its answer? For example, we can differentiate from first principles to obtain the following answers.



Theory

y	$\frac{dy}{dx}$
$3x^2 + 8x$	$6x + 8$
$5x^2 - 9x$	$10x - 9$
$4x^2 + 11x + 3$	$8x + 11$
$7x^3$	$21x^2$
$26x + 18$	26

We can use the following rule to quickly differentiate a function without having to go from first principles:

If $y = ax^n$, then $\frac{dy}{dx} = nax^{n-1}$.

Example 1

y	a	n	$\frac{dy}{dx} = nax^{n-1}$
$6x^2$	6	2	$2 \times 6 \times x^1 = 12x$
$4x^3$	4	3	$3 \times 4 \times x^2 = 12x^2$
$-14x$	-14	1	$1 \times -14 \times x^0 = -14$
23	23	0	$0 \times 23 \times x^{-1} = 0$
$4x^{-6}$	4	-6	$-6 \times 4 \times x^{-7} = -24x^{-7}$
$-643x^{215}$	-643	215	$215 \times -643 \times x^{214} = -138245x^{214}$

Given a function that is a sum or difference of a series of terms, we can differentiate the terms individually to differentiate the function.

Example 2

y	$\frac{dy}{dx}$
$7x^4 + 8x^2 - 4$	$28x^3 + 16x$
$10x^5 + 8x^3 - 7x^{-3}$	$50x^4 + 24x^2 + 21x^{-4}$

Exercise 1

Complete the following table.

y	$\frac{dy}{dx}$
$12x^3$	
$5x$	
3	
$28x^2$	
$4x^5 + 2x^3 - 7x^2$	
$19x^3 - 3 + 4x^2 - 19x$	
$24x^{-3}$	
$8x^{100} + 11x^9$	

We can use the process of differentiation to find the equation of the **tangent** to a curve at a particular point.

Example 3

The curve for $y = 2x^3 - 3x$ is shown on the right.

Find the equation of the tangent to the curve at the point where $x = 1$.

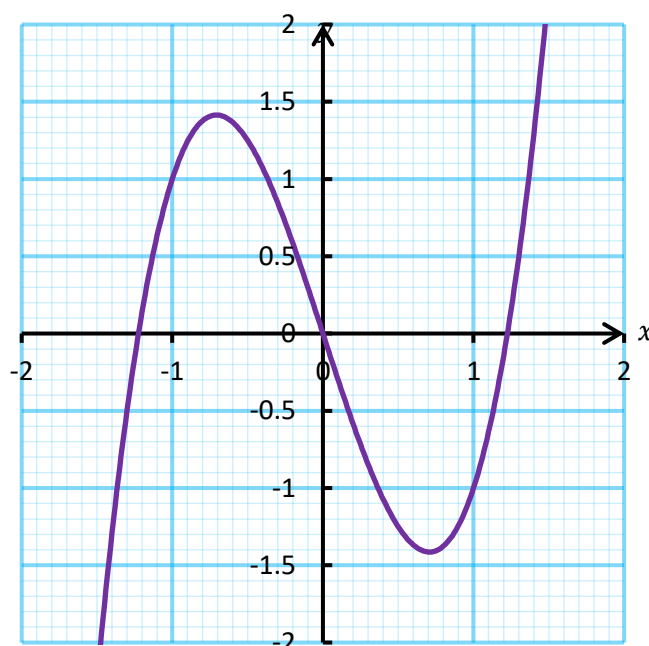
Answer: If $y = 2x^3 - 3x$, then $\frac{dy}{dx} = 6x^2 - 3$.

When $x = 1$, we have $y = 2(1^3) - 3(1) = -1$, and $\frac{dy}{dx} = 6(1^2) - 3 = 3$. So, the tangent that goes through the point $(1, -1)$ has gradient 3.

The equation $y - y_1 = m(x - x_1)$ gives the equation of a straight line with gradient m going through the point (x_1, y_1) .

So, the equation of the tangent at $(1, -1)$ is

$$\begin{aligned} y - (-1) &= 3(x - 1) \\ y + 1 &= 3x - 3 \\ y &= 3x - 4 \end{aligned}$$

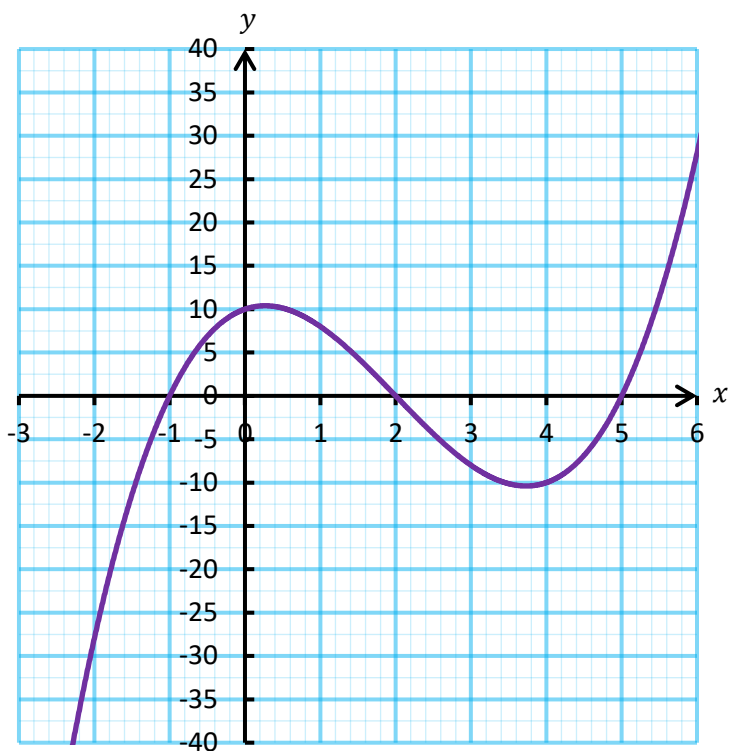


Exercise 2

The curve for $y = x^3 - 6x^2 + 3x + 10$ is shown on the right.

(a) Find the equation of the tangent to the curve at the point where $x = -1$.

(b) Find the equation of the tangent to the curve at the point where $x = 3$.

**Exercise 3**

The equation of the curve C is $y = x^2 + 6x + 8$. The point P has coordinates $(-2, 0)$ and lies on C . Find the equation of the tangent to C at P .

The equation of the curve C is $y = 3x^2 - 4x - 2$.

- (a) The point P has coordinates $(1, -3)$ and lies on C . Find the equation of the tangent to C at P .
- (b) The point Q has coordinates $(-1, 5)$ and lies on C . Find the equation of the tangent to C at Q .
- (c) Find the equation of the tangent to the curve C at the point where $x = 3$.

The equation of the curve C is $y = 2x^3 - 5x^2 + 8$.

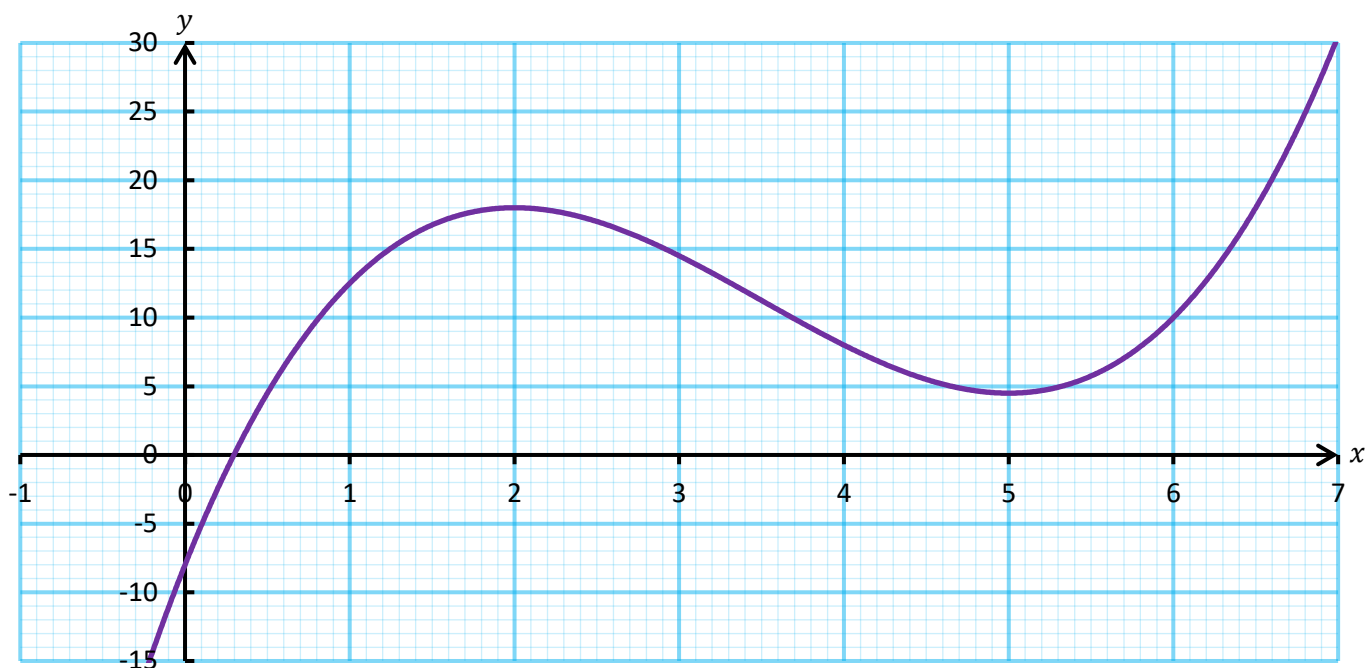
- (a) Find $\frac{dy}{dx}$.
- (b) The point P has coordinates $(2, 4)$ and lies on C . Find the equation of the tangent to C at P .
- (c) The point Q has coordinates $(-2, -28)$ and lies on C . Find the equation of the tangent to C at Q .
- (d) Find the equation of the tangent to the curve C at the point where $x = 0$.

The equation of the curve C is $y = 2x^2 - 10x + 7$.

The equation of the curve C is $y = 7 + 13x - 2x^2$.

The Second Derivative

Consider the graph below which shows the function $y = x^3 - 10.5x^2 + 30x - 8$.



We can differentiate the function to give $\frac{dy}{dx} = 3x^2 - 21x + 30$. This gives the **gradient** of the purple curve at a particular point. If we differentiate again, we form the **second derivative** $\frac{d^2y}{dx^2}$, which in this case is $\frac{d^2y}{dx^2} = 6x - 21$. The second derivative will be helpful later on in this workbook in finding maximum and minimum points.

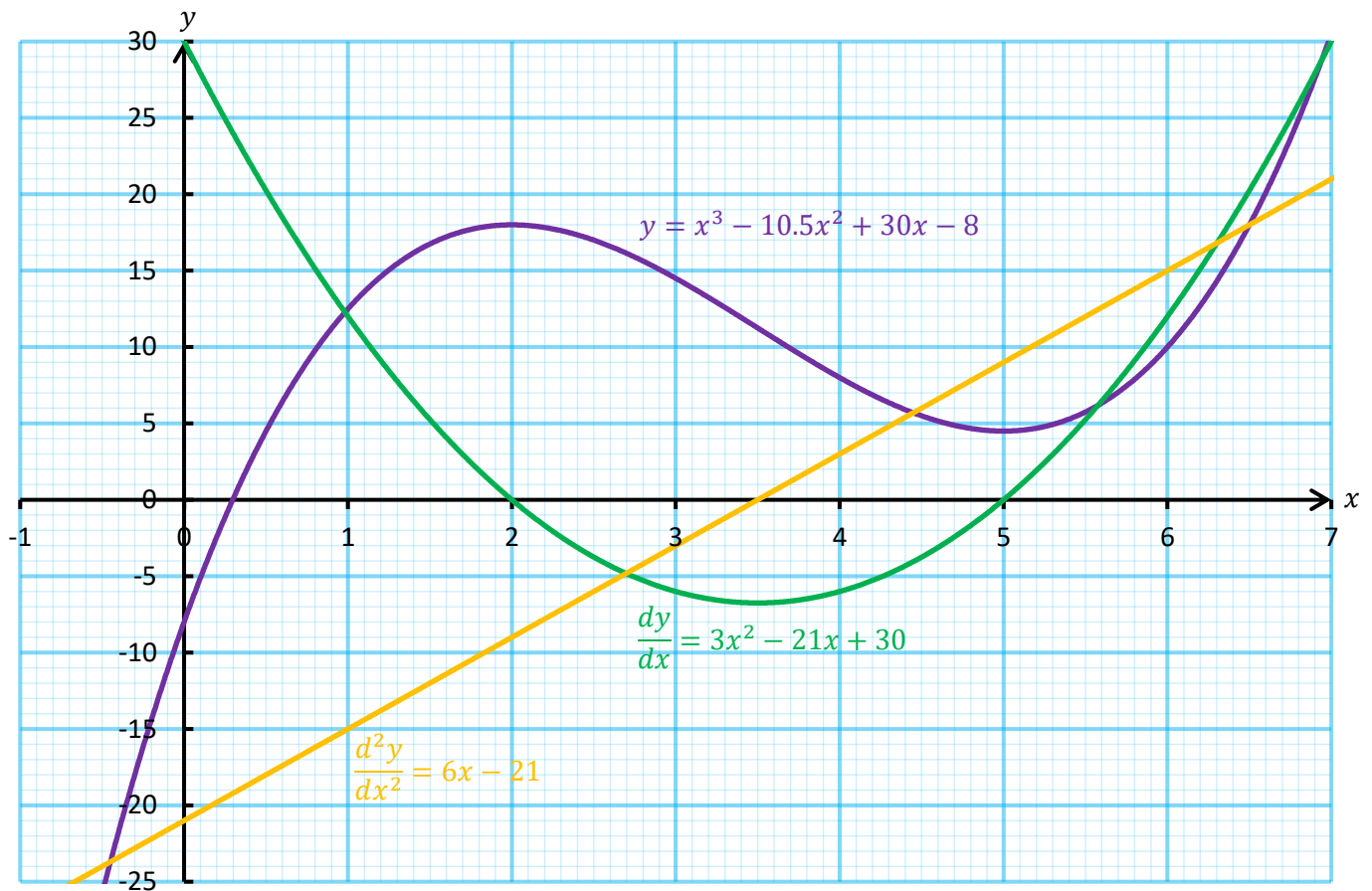
Exercise 8

Complete the following table.

y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
$4x^3$		
$10x^5$		
$8x^2$		
$18x$		
$9x^4 + 2x$		
$-3x^3 + 10x^2$		
$x^3 + 5x^2 - 4x + 5$		
$6x^7 - 2x^8 + 10x^{10}$		
$4x^{-2}$		
$20x^5 + 10x^{-5}$		

Stationary Points

Let us consider again the example from the previous page, where $y = x^3 - 10.5x^2 + 30x - 8$, $\frac{dy}{dx} = 3x^2 - 21x + 30$, and $\frac{d^2y}{dx^2} = 6x - 21$. The graph below shows these three functions, and the connections between them.



Any point where $\frac{dy}{dx} = 0$ is called a **stationary** point. At these points, the tangent to the curve is horizontal, and the curve does not increase or decrease. For the curve $y = x^3 - 10.5x^2 + 30x - 8$, we see from the above graph that there are two stationary points, where $x = 2$ and $x = 5$. For the first stationary point, where $x = 2$,

- the **purple curve** reaches a local maximum;
- the value of the **green curve** is zero;
- the value of the **orange line** is negative.

For the second stationary point, where $x = 5$,

- the **purple curve** reaches a local minimum;
- the value of the **green curve** is zero;
- the value of the **orange line** is positive.

We can use the above information to form a strategy for finding the coordinates and the nature of a curve's stationary points:

- Solve $\frac{dy}{dx} = 0$ to find the x -coordinate of each stationary point.
- Substitute back into y to find the y -coordinates.
- Substitute into $\frac{d^2y}{dx^2}$ to find the nature of each stationary point. If $\frac{d^2y}{dx^2} > 0$, then the stationary point is a local minimum point. If $\frac{d^2y}{dx^2} < 0$, then the stationary point is a local maximum point¹.

¹ We will deal with the case $\frac{d^2y}{dx^2} = 0$ in the mathematics A Level course.

Find the coordinates and the nature of the stationary points for the curve $y = x^3 - 10.5x^2 + 30x - 8$.

Solve $\frac{dy}{dx} = 0$:

$$3x^2 - 21x + 30 = 0$$

$$3 \times 30 = 90$$

$$3x^2 - 15x - 6x + 30 = 0$$

$$3x(x - 5) - 6(x - 5) = 0$$

$$(x - 5)(3x - 6) = 0$$

Either $x - 5 = 0$ or $3x - 6 = 0$

$x = 5$

$$3x = 6$$

$x = 2$

$$-15 \times -6 = 90$$

$$-|5 + -6| = -21$$

Substitute back into y : If $x = 5$ then $y = 5^3 - 10.5 \times 5^2 + 30 \times 5 - 8$
 $y = 4.5$

If $x = 2$ then $y = 2^3 - 10.5 \times 2^2 + 30 \times 2 - 8$
 $y = 18$

Differentiate again: $\frac{d^2y}{dx^2} = 6x - 21$.

Substitute into $\frac{d^2y}{dx^2}$:

If $x = 5$ then $\frac{d^2y}{dx^2} = 6 \times 5 - 21$

$$\frac{d^2y}{dx^2} = 9$$

$\frac{d^2y}{dx^2}$ is positive so $(5, 4.5)$ is a local minimum point.

If $x = 2$ then $\frac{d^2y}{dx^2} = 6 \times 2 - 21$

$$\frac{d^2y}{dx^2} = -9$$

$\frac{d^2y}{dx^2}$ is negative so $(2, 18)$ is a local maximum point.

Find the coordinates and the nature of the stationary points for the curve $y = x^3 - 3x^2 - 9x + 2$.

Exercise 10

Find the coordinates and nature of the stationary points for the curve $y = x^3 - x^2 - x + 2$.

Find the coordinates and nature of the stationary points for the curve $y = \frac{1}{3}x^3 + x^2 - 15x$.

Find the coordinates and nature of the stationary points for the curve $y = 2x^3 - 6x^2 - 7$.

Exercise 13

Find the coordinates and nature of the stationary points for the curve $y = 5x^3 + 15x^2 - 8$.

117

[8]

y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
$6x^4$		
$2x^3 - 16x$		
$5x^{-3}$		
$6x + 3x^2 - 8x^3$		

[5]

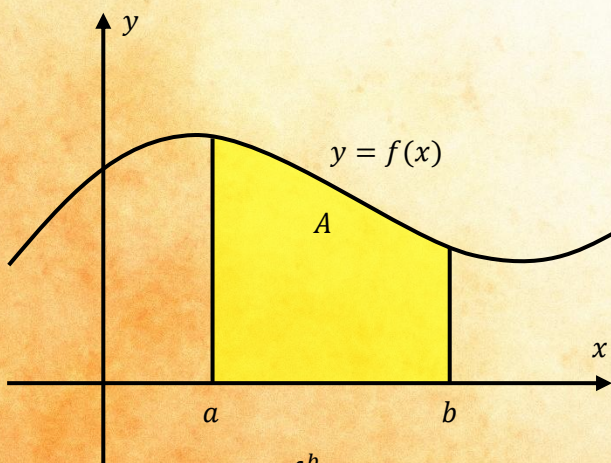
[5]

[4]



Introducing

Integration



$$A = \int_a^b f(x) dx$$

$$A = [F(x)]_a^b$$

$$A = F(b) - F(a)$$

Name:

Background

What is the work?

Introducing the process of integration as the inverse of differentiation, and using this to find the area beneath curves.

What are the prerequisites?

Additional Mathematics: Differentiation.

Where does this lead to?

Mathematics A Level: Further integration.

Applications: Finding areas; volumes; mass; location; cumulative distribution functions in probability.

Theory

Indefinite Integration

Integration is the process of reversing differentiation.

Exercise I

Complete the following table.



Theory

y	$\frac{dy}{dx}$
$5x^2 + 9x + 3$	
$5x^2 + 9x + 4$	
$5x^2 + 9x - 3$	
$5x^2 + 9x + \frac{3}{4}$	
$5x^2 + 9x + \pi$	

In completing the above exercise, you will see the problem facing us when attempting to integrate: whilst differentiation always gives a unique answer, this will not be the case for integration. Above, every function y in the first column differentiates to give the same answer. This leads to the question: if we are integrating $10x + 9$, what should the answer be? To deal with this problem, we introduce a **constant of integration**. Different textbooks use different letters for the constant of integration: sometimes c , sometimes k , sometimes something else. This workbook will use c as the constant of integration.

When integrating $10x + 9$, we say that the answer is $5x^2 + 9x + c$, where c represents any number, because any expression of the form $5x^2 + 9x + c$ would differentiate to give $10x + 9$. The formal notation for this process is

$$\int 10x + 9 \, dx = 5x^2 + 9x + c$$

where dx denotes we are integrating with respect to the variable x , and $\int$ is the symbol for integration.

The rule for integrating a term of the form ax^n is the following:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

In words, we add one to the power n , and then divide by the new power $n + 1$.

Example 1

Question	Answer
$\int 28x^3 dx$	$\frac{28x^4}{4} + c = 7x^4 + c$
$\int 35x^4 - 6x + 2 dx$	$\frac{35x^5}{5} - \frac{6x^2}{2} + 2x + c = 7x^5 - 3x^2 + 2x + c$

Exercise 2

Complete the following table.

Question	Answer
$\int 15x^2 dx$	
$\int 24x^3 dx$	
$\int 10x dx$	
$\int 4 dx$	
$\int 8x^{-3} dx$	
$\int 9x^2 + 6x + 3 dx$	
$\int 10x^4 - 4x^3 dx$	
$\int 2 + x^{-2} dx$	
$\int 0 dx$	
$\int 3 - 16x^{-5} dx$	
$\int 4x^2 dx$	
$\int x^2 + x^4 + x^6 dx$	
$\int 5x^3 - 2x + 9 dx$	
$\int 6x^2 + 10 - 2x^{-2} dx$	
$\int \frac{2}{3}x^2 dx$	
$\int 2(4 + x) dx$	
$\int x(2x - 3) dx$	
$\int 20x^4 - 12x^3 + 6x^2 - 4x + 7 dx$	

Definite Integration

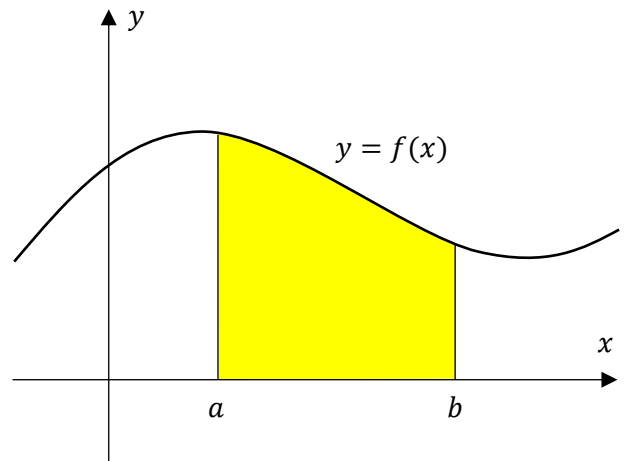
It is possible to use integration to find the area between a curve and the x -axis.

Let us consider a general curve of the form $y = f(x)$.

The **yellow area** on the right is bounded by the curve, the x -axis and the lines $x = a$, $x = b$.

To find the size of the area, we integrate between a and b using the notation

$$\int_a^b f(x) dx$$



We say that a is the **lower limit** and b is the **upper limit**.

The **Fundamental Theorem of Calculus** tells us how to calculate the above integral:

If $\int f(x) dx = F(x) + c$, then

$$\begin{aligned} \int_a^b f(x) dx &= [F(x) + c]_a^b \\ &= [F(b) + c] - [F(a) + c] \\ &= F(b) - F(a) \end{aligned}$$



Theory

Notice that the constants of integration cancel out, so we can write

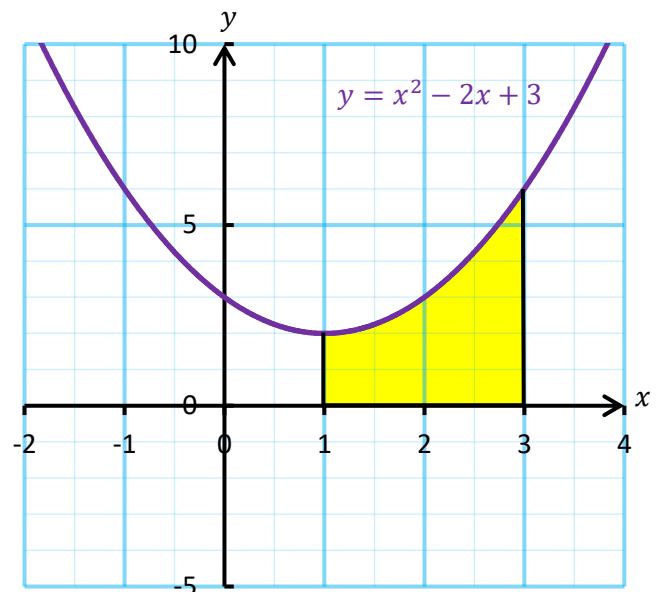
$$\begin{aligned} \int_a^b f(x) dx &= [F(x)]_a^b \\ &= F(b) - F(a) \end{aligned}$$

Example 2

Find the area underneath the curve $y = x^2 - 2x + 3$ between $x = 1$ and $x = 3$.

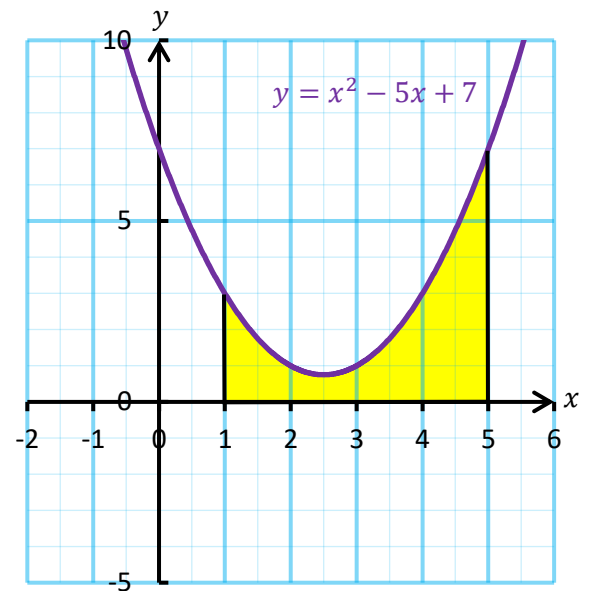
Answer: We need to find the value of

$$\begin{aligned} &\int_1^3 x^2 - 2x + 3 dx \\ &= \left[\frac{x^3}{3} - \frac{2x^2}{2} + 3x \right]_1^3 \\ &= \left[\frac{x^3}{3} - x^2 + 3x \right]_1^3 \\ &= \left[\frac{3^3}{3} - 3^2 + 3 \times 3 \right] - \left[\frac{1^3}{3} - 1^2 + 3 \times 1 \right] \\ &= [9 - 9 + 9] - \left[\frac{1}{3} - 1 + 3 \right] \\ &= 9 - \frac{7}{3} \\ &= \frac{20}{3} \text{ square units} \end{aligned}$$

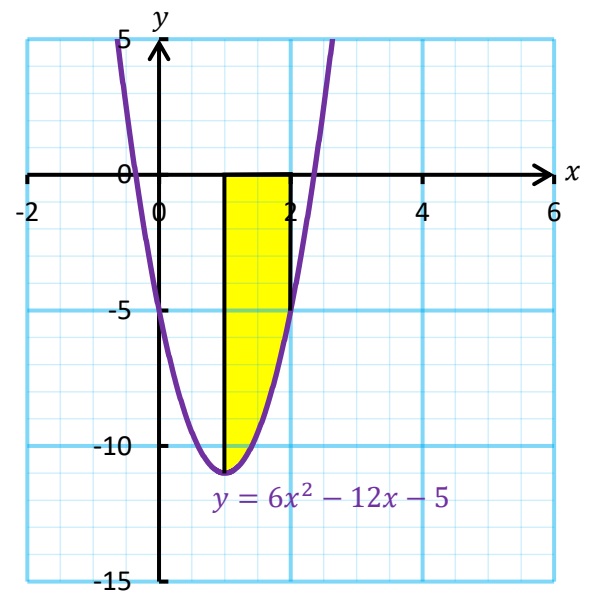


Exercise 3

Find the area underneath the curve
 $y = x^2 - 5x + 7$ between $x = 1$ and $x = 5$.

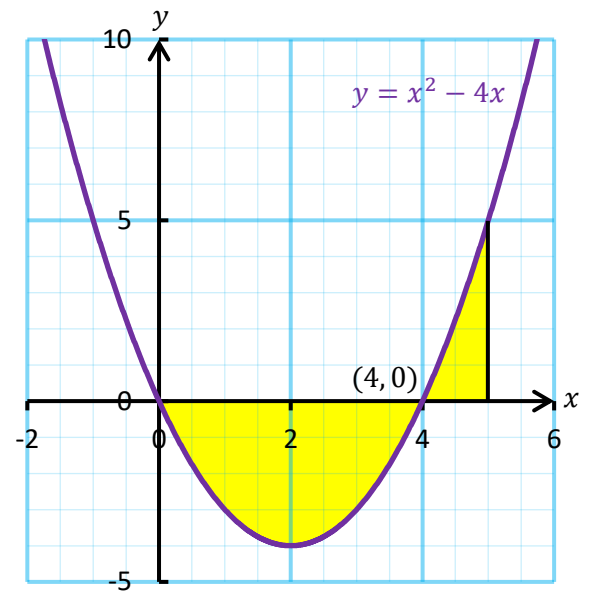
**Exercise 4**

Find the area above the curve
 $y = 6x^2 - 12x - 5$ between $x = 1$ and $x = 2$.



Exercise 5

Find the total area between the curve $y = x^2 - 4x$ and the x -axis between $x = 0$ and $x = 5$.



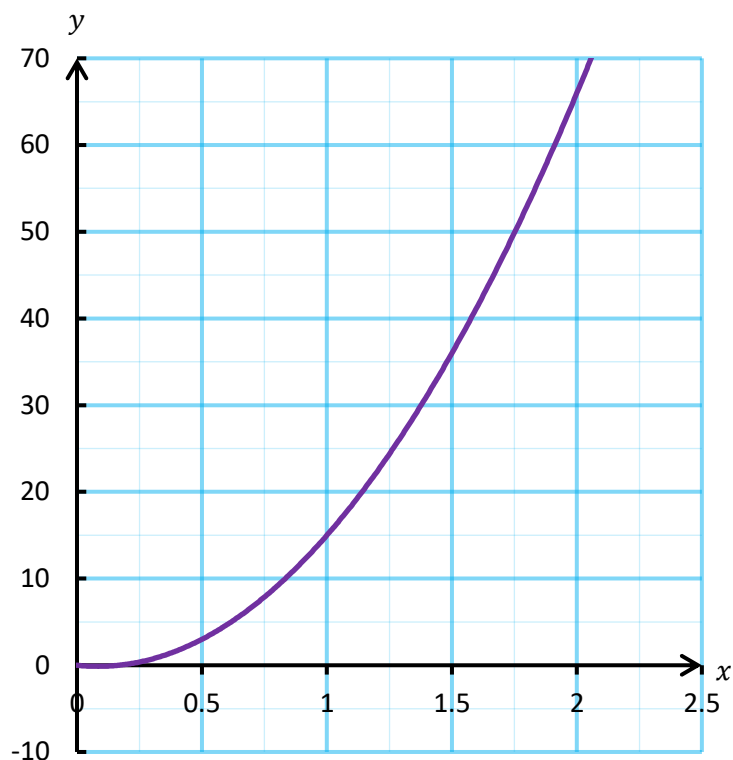
Exercise 6

The diagram on the right shows a sketch of the curve $y = 18x^2 - 3x$.

Calculate the area of the region defined by:

- the curve $y = 18x^2 - 3x$,
- the line $x = 1$,
- the line $x = 2$,
- the x -axis.

You must show all your working.



Exercise 7

The diagram on the right shows a sketch of the curve $y = -x^2 + 5x + 14$.

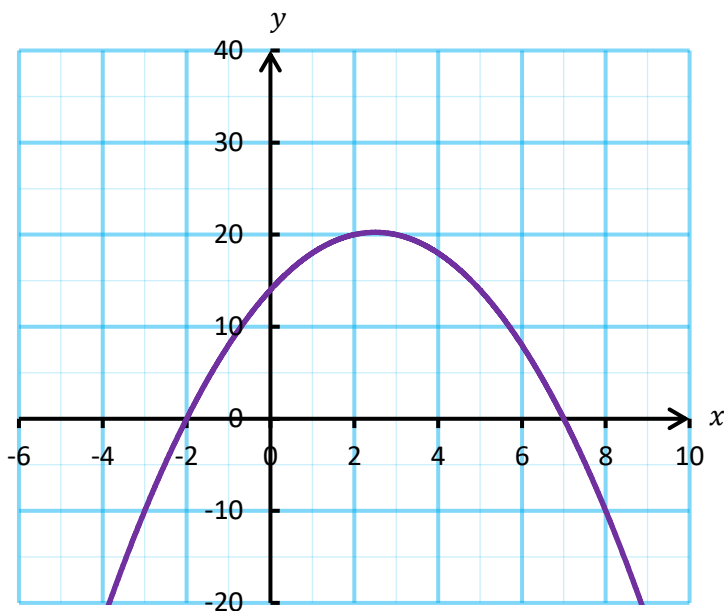
Calculate the area of the region defined by the following inequalities.

$$y \leq -x^2 + 5x + 14$$

$$x \geq 6$$

$$y \geq 0$$

You must show all your working.



Given the following facts, find the values of a , b , c and d .

- $y = ax^3 + bx^2 + cx + d$
- $\frac{dy}{dx} = 27x^2 + 8x + 13$
- The value of d is equal to $\frac{1}{6}$ of the coefficient of x in $\frac{d^2y}{dx^2}$.

$a =$ $b =$ $c =$ $d =$

Given the following facts, find the values of a , b , c and d .

$a =$ $b =$ $c =$ $d =$

Given the following facts, find the values of a , b and c .

- $y = ax^3 + bx^2 + c$
- $\frac{dy}{dx} = 3ax^2 + 24x$
- $\frac{d^2y}{dx^2} = 60x + 24$
- The point $(2, 150)$ lies on $y = ax^3 + bx^2 + c$.

$C =$

Pop Quiz: Introducing Integration

(1) Find $\int 21x^6 - 3x^2 - x^{-2} + 6 \, dx$.

[5]

/15

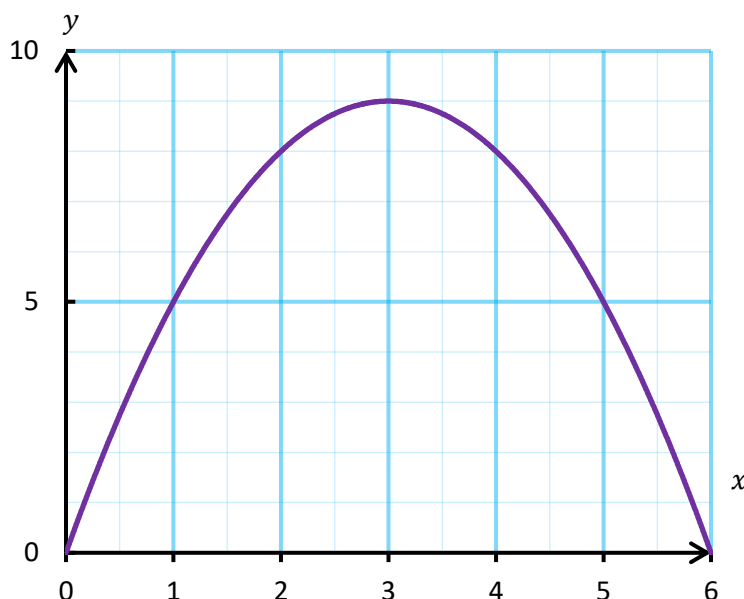
(2) Showing all your working, evaluate $\int_2^5 6x^2 + 4x \, dx$.

[5]

(3) The diagram shows the curve $y = 6x - x^2$.

Showing all your working, calculate the area of the region bounded by the curve $y = 6x - x^2$ and the x -axis.

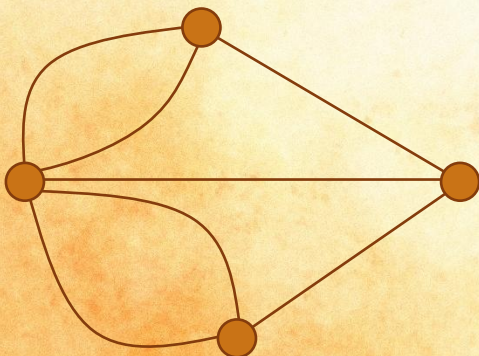
[5]





Graphs and

Networks



Name:

Background

What is the work?

Working with graphs and networks, like the Welsh railway network. Recognising Eulerian and semi-Eulerian graphs.

What are the prerequisites?

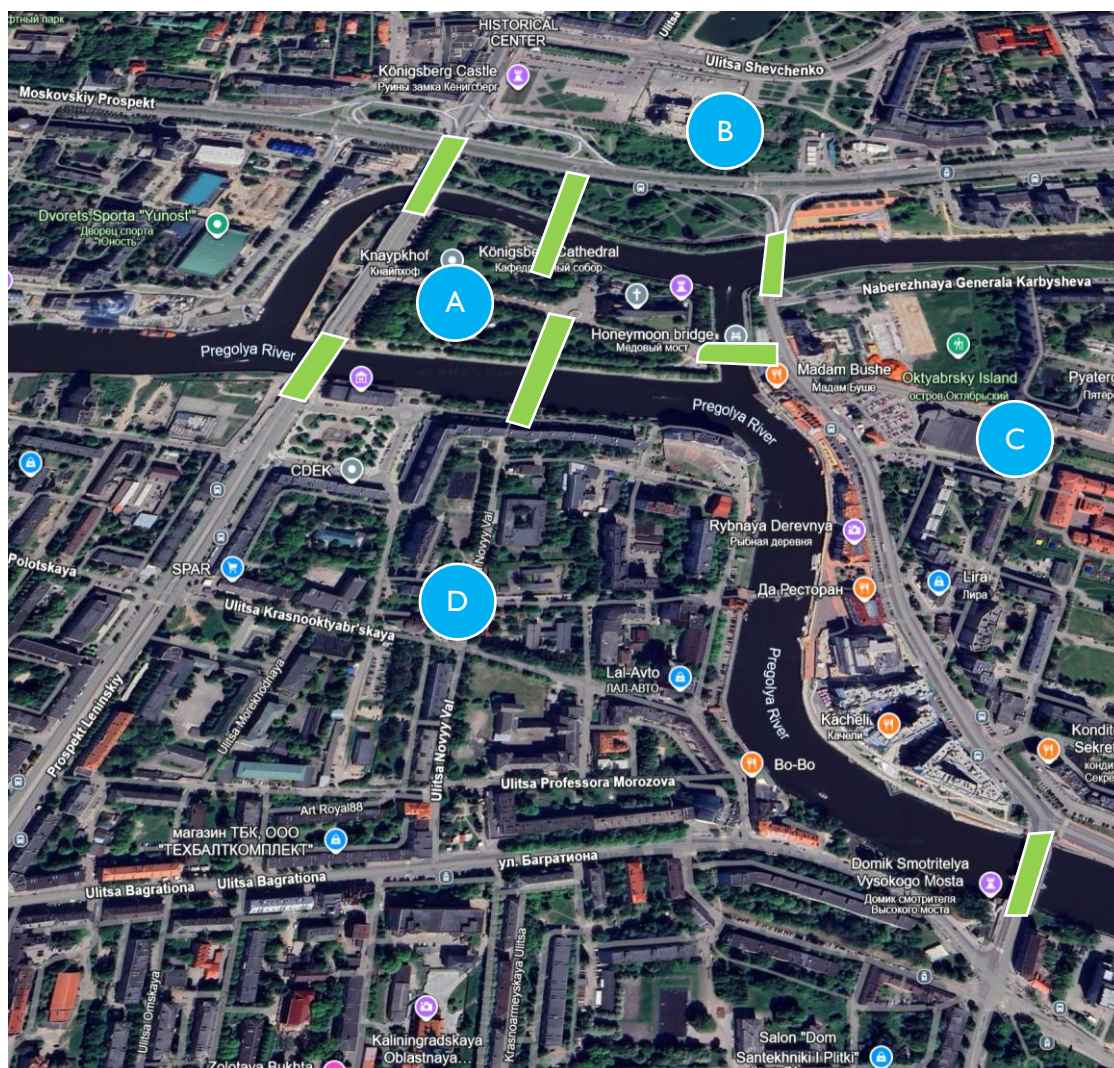
GCSE Work: Even numbers and odd numbers.

Where does this lead to?

Applications: Tracking how disease spreads; analysing social networks; recommendation systems (for example for films or products).

Theory

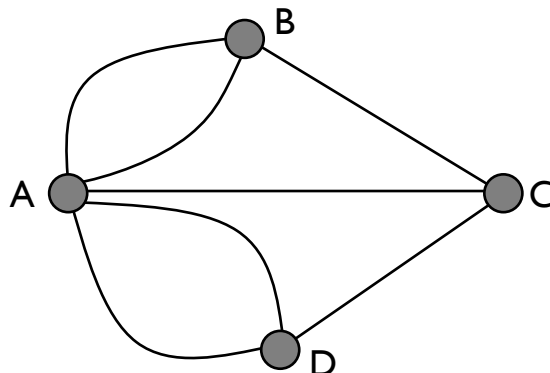
In the 18th century, there were 7 bridges in the middle of Königsberg, Prussia (now Kaliningrad, Russia). The residents of the city tried to devise a walking route crossing each bridge exactly once.



In 1736, **Leonhard Euler** showed that such a walking route was not possible, using the theory of **graphs**, the subject of this workbook.

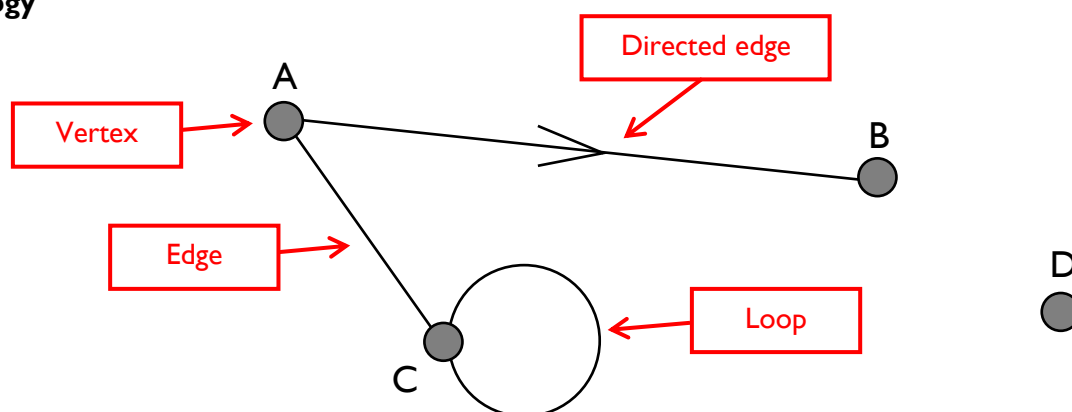
- He used **edges** to represent each bridge.
- He used **vertices** to represent each piece of land.

The following **graph** represents the situation.



The number of edges (bridges) coming out of each vertex is an **odd number**, so it was possible for Euler to show that there was not a walking route crossing each bridge exactly once.

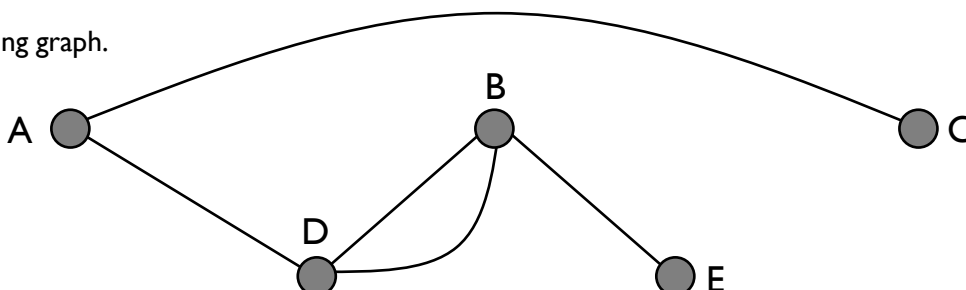
Terminology



Vertex	A	B	C	D
Degree of the vertex	2	1	3	0

Exercise 1

Study the following graph.



(a) How many vertices are in the graph? _____ (b) How many edges are in the graph? _____

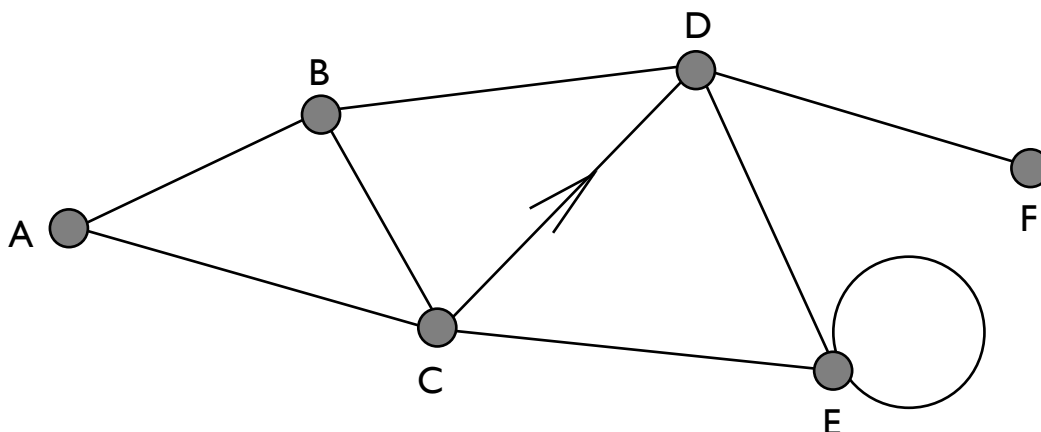
(c) Complete the following table.

Vertex	A	B	C	D	E
Degree of the vertex					

Walks

- A **walk** is a route through a graph along edges from one vertex to the next.
- An **open walk** is a walk where the start vertex is not the same as the end vertex.
- A **closed walk** is a walk where the start vertex is the same as the end vertex.
- A **trail** is a walk in which no edge is visited more than once.
- A **path** is a walk in which no vertex is visited more than once.
- A **cycle** is a closed walk where no vertex (apart from the start vertex) is visited more than once.

Example



Walk	Open?	Closed?	Trail?	Path?	Cycle?
$A \rightarrow B \rightarrow C$	✓	✗	✓	✓	✗
$A \rightarrow B \rightarrow C \rightarrow A$	✗	✓	✓	✗	✓
$A \rightarrow B \rightarrow C \rightarrow A \rightarrow B$	✓	✗	✗	✗	✗

$A \rightarrow B \rightarrow C \rightarrow D$ is a walk in the above graph.

$A \rightarrow B \rightarrow D \rightarrow C$ is not a walk in the above graph. (Why?)

Exercise 2

For the above graph,

(a) Is $E \rightarrow E \rightarrow C \rightarrow D$ a walk? _____

(b) Is $F \rightarrow D \rightarrow C \rightarrow B$ a walk? _____

(c) Complete the following table.

Walk	Open?	Closed?	Trail?	Path?	Cycle?
$B \rightarrow D \rightarrow F$					
$C \rightarrow D \rightarrow E \rightarrow E$					
$C \rightarrow D \rightarrow E \rightarrow C$					
$C \rightarrow E \rightarrow C \rightarrow E \rightarrow C$					

(d) Complete the following table.

Vertex	A	B	C	D	E	F
Degree of the vertex						

Exercise 3

Draw a graph with four vertices A, B, C, D so that the degree of each vertex is as noted below.

(a)

Vertex	A	B	C	D
Degree	0	0	0	0

(b)

Vertex	A	B	C	D
Degree	1	1	0	0

--	--

(c)

Vertex	A	B	C	D
Degree	2	1	1	0

(d)

Vertex	A	B	C	D
Degree	3	2	1	2

--	--

(e)

Vertex	A	B	C	D
Degree	3	3	3	3

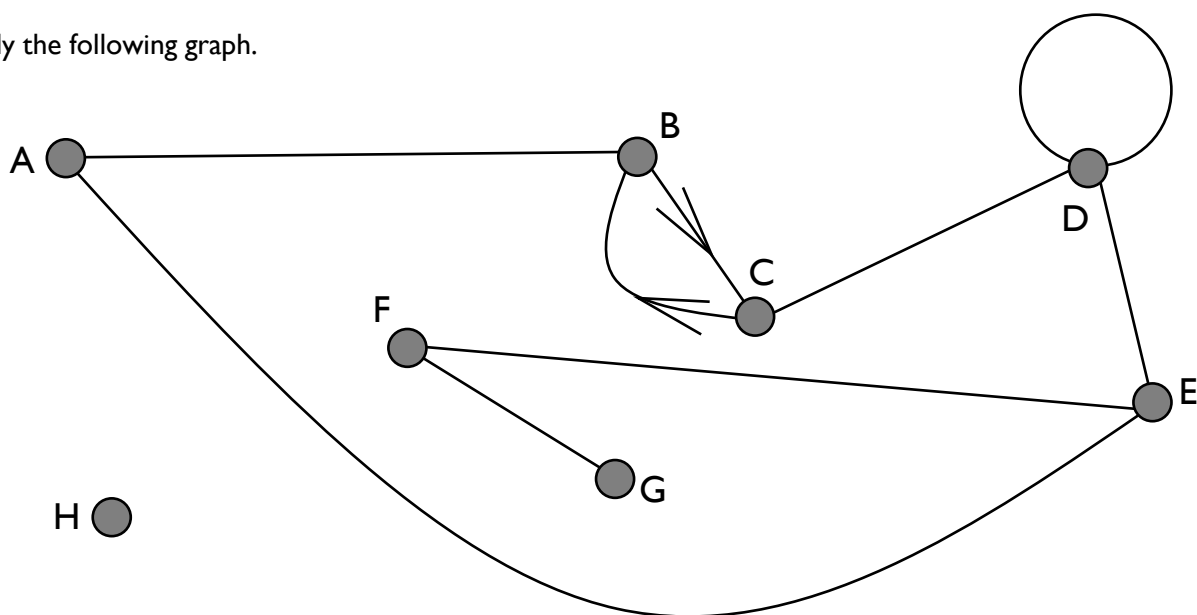
(f)

Vertex	A	B	C	D
Degree	5	3	3	3

--	--

Exercise 4

Study the following graph.



(a) Complete the following table.

Vertex	A	B	C	D	E	F	G	H
Degree								

(b) Which vertex has a loop? _____

(c) How many directed edges are in the graph? _____

(d) List 4 different walks that go from A to D.

(i) _____

(ii) _____

(iii) _____

(iv) _____

(e) Complete the following table.

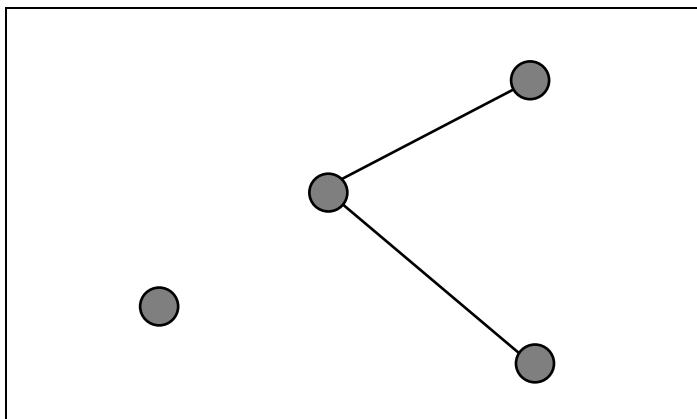
Walk	Open?	Closed?	Trail?	Path?	Cycle?
$E \rightarrow F \rightarrow G$					
$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$					
$A \rightarrow B \rightarrow A$					
$C \rightarrow D \rightarrow E$					
$C \rightarrow D \rightarrow D \rightarrow E$					
$F \rightarrow E \rightarrow D \rightarrow C \rightarrow B \rightarrow A \rightarrow E \rightarrow F$					
$B \rightarrow C \rightarrow B$					
$C \rightarrow B \rightarrow C \rightarrow D \rightarrow D \rightarrow E$					

Graph Types

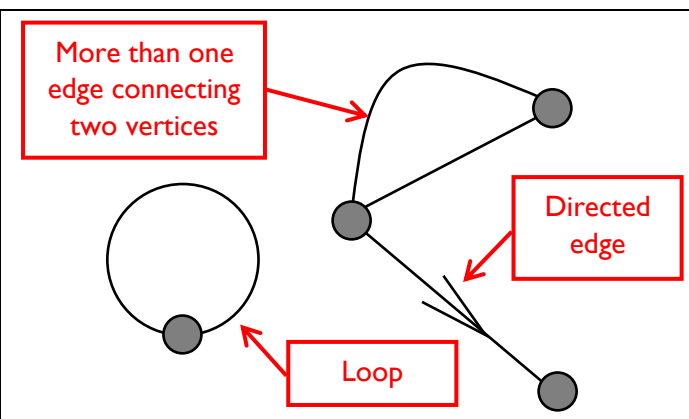
- A graph is a **simple graph** if it contains
 - no loops;
 - no directed edges;
 - only one edge connecting any pair of vertices.
- A graph is a **connected graph** if every pair of vertices is connected by a walk.
- A graph is a **complete graph** if every vertex is connected to every other vertex by a unique edge.

Example

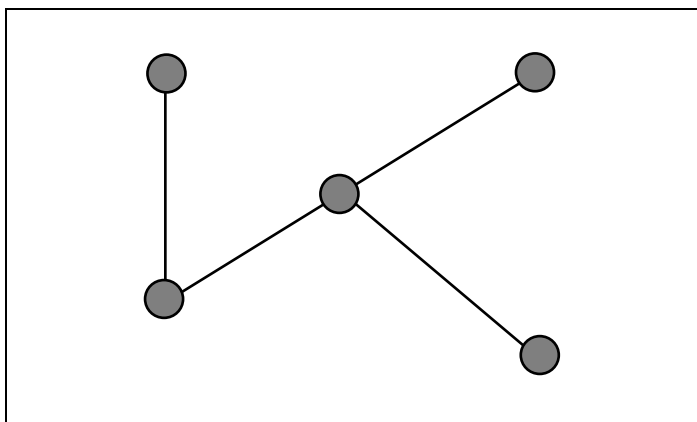
A simple graph



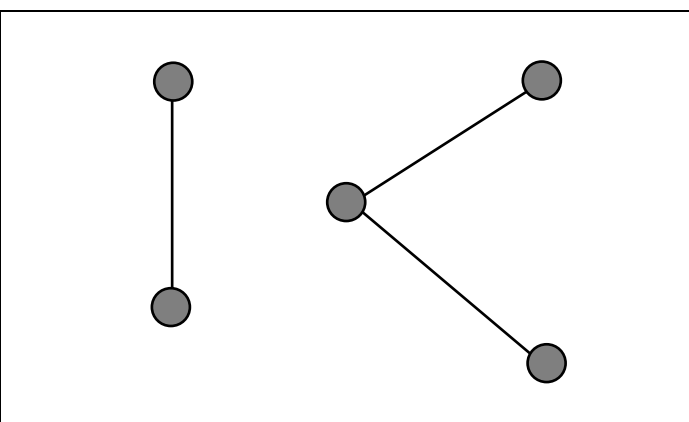
Not a simple graph



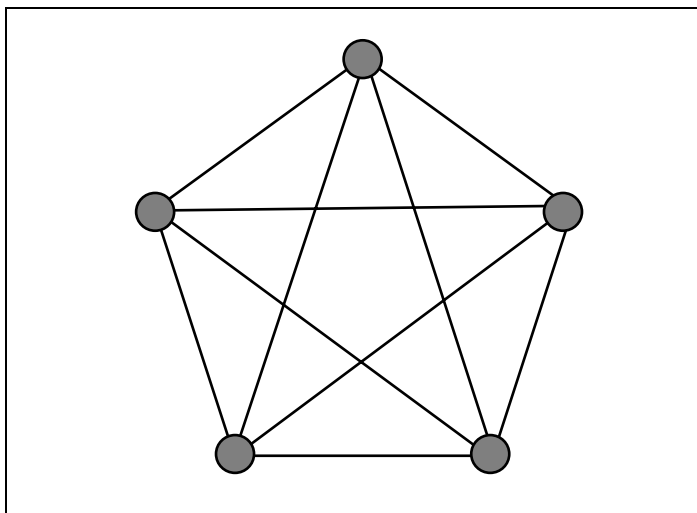
A connected graph



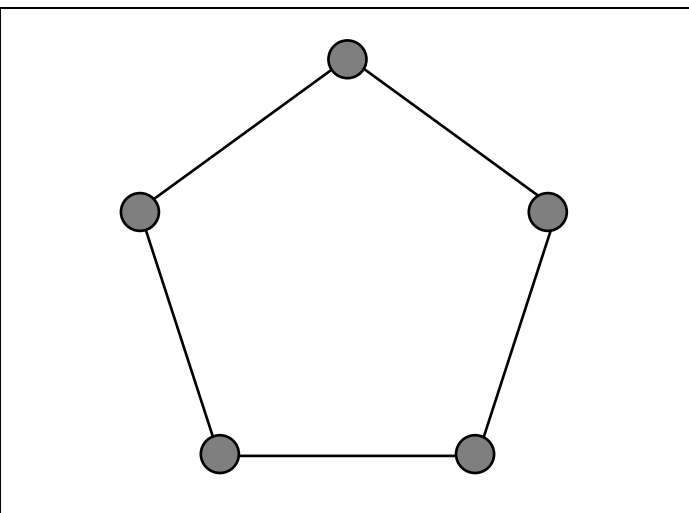
Not a connected graph



A complete graph



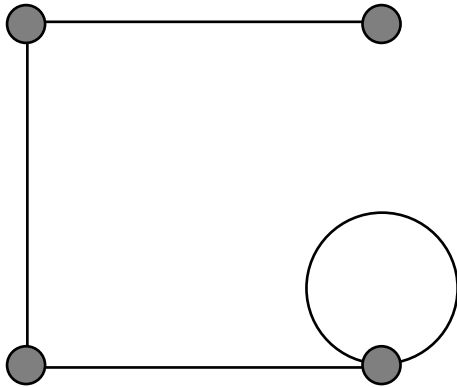
Not a complete graph



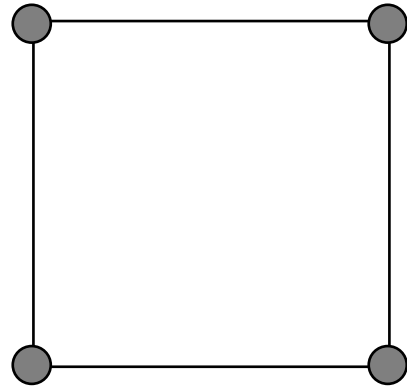
Example 5

Four graphs are shown below.

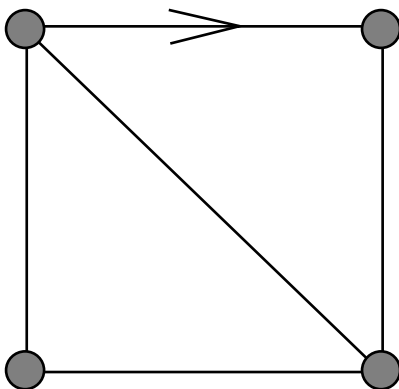
Graph 1



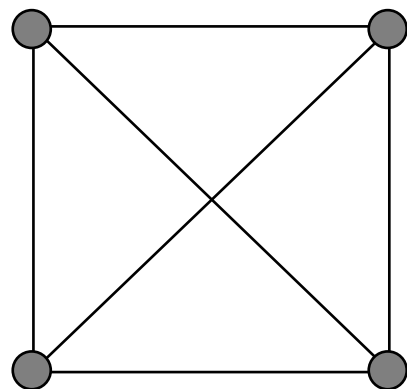
Graph 2



Graph 3



Graph 4



Tick the boxes, as appropriate.

Graph	Simple Graph?	Connected Graph?	Complete Graph?
1			
2			
3			
4			

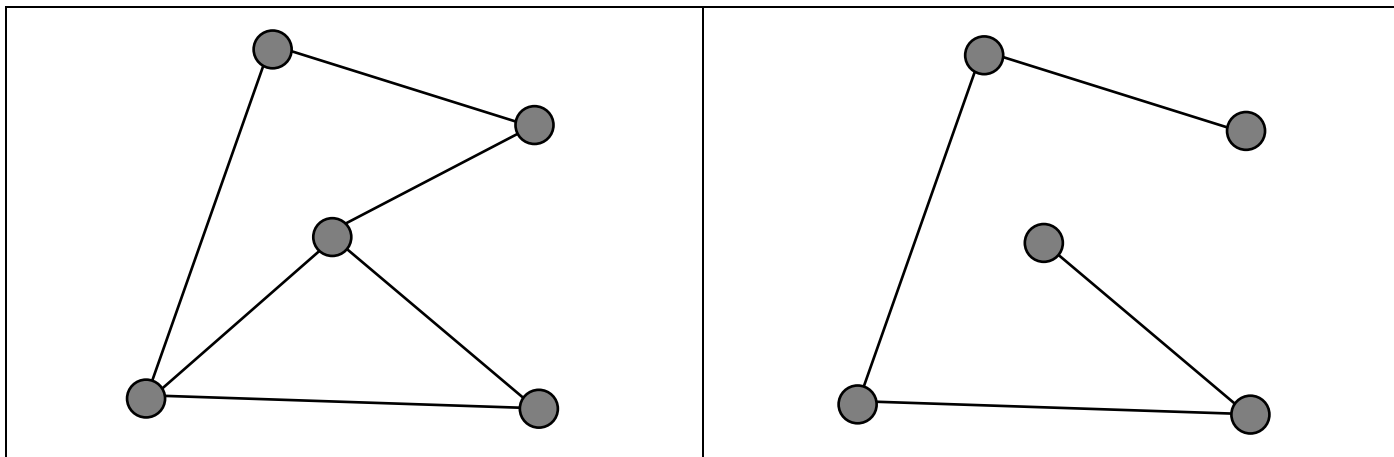
Trees

- A graph is a **tree** if it is a connected graph with no cycles.
- A **spanning tree** is a subgraph of a graph G that includes all vertices and is also a tree.

Example

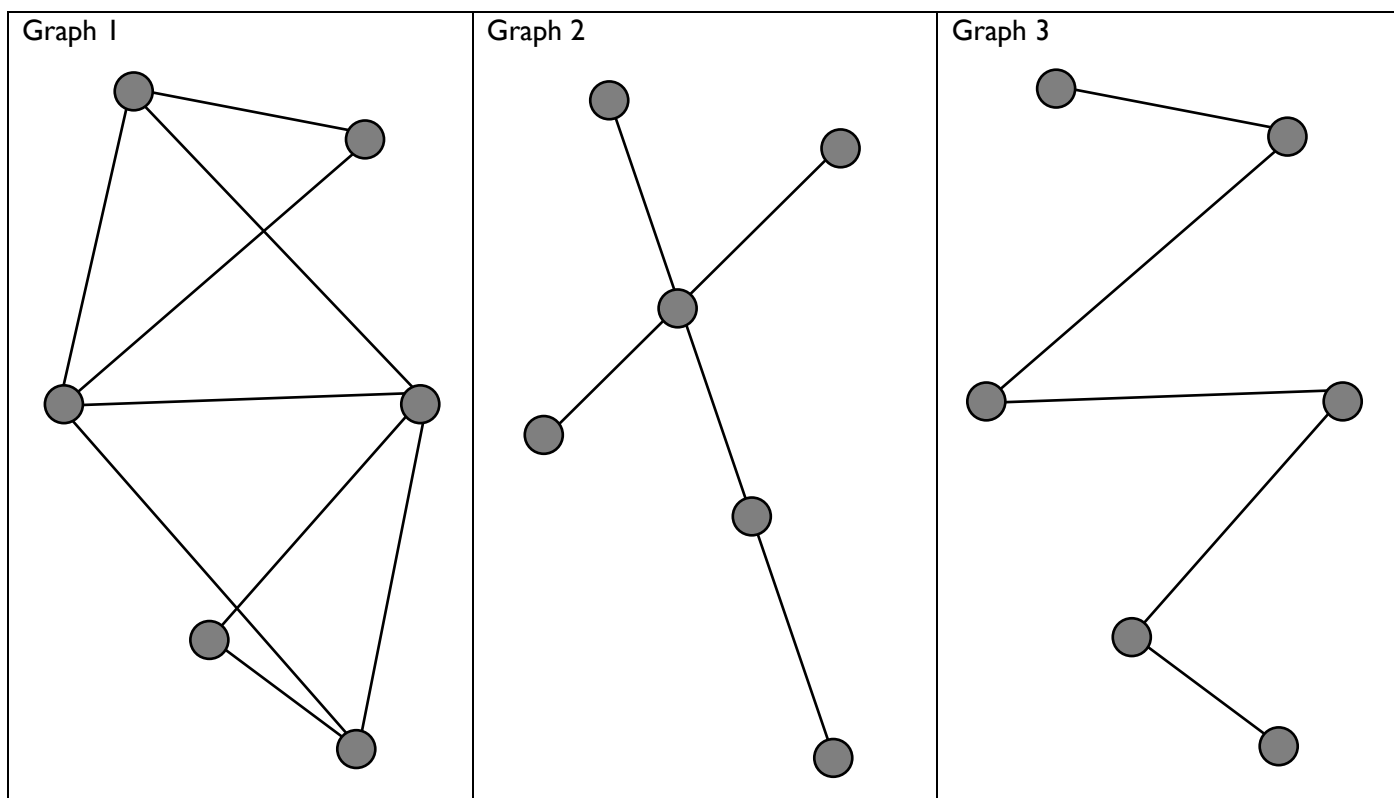
The graph below is not a tree...

...but the subgraph below is a tree.



Exercise 6

Three graphs are shown below.



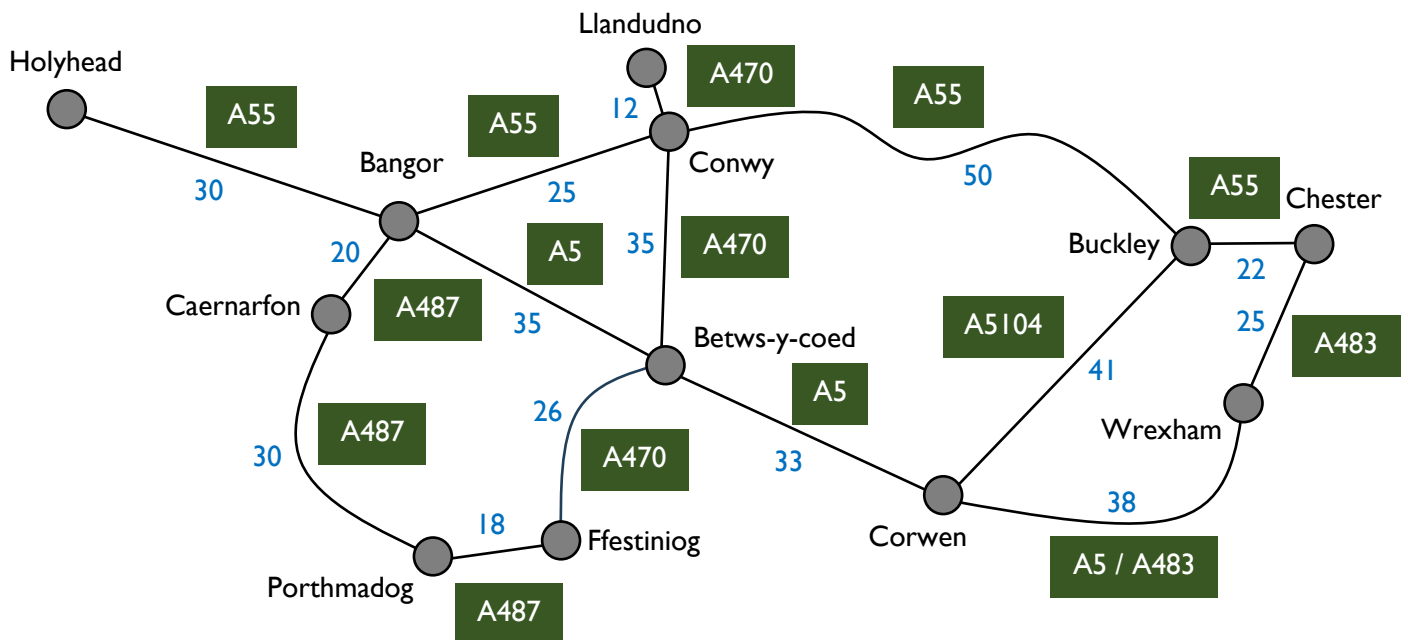
- (a) Which graphs are trees? _____
- (b) Which graph is a spanning tree of graph 1? _____
- (c) How many edges must be removed from graph 1 to change it to be a tree? _____

Weighted graphs

A graph is a **weighted graph** if each edge has a numerical value.

Exercise 7

Here is part of North Wales' road network, with the usual journey time (in minutes) shown on each edge.



(a) What is the journey time between

(i) Bangor and Conwy? _____ (ii) Caernarfon and Porthmadog? _____

(iii) Betws-y-coed and Corwen? _____ (iv) Wrexham and Chester? _____

(b) What would be the journey time of the following journeys?

(i) Bangor → Conwy → Betws-y-coed _____

(ii) Ffestiniog → Betws-y-coed → Corwen → Wrexham _____

(iii) Chester → Buckley → Conwy → Bangor _____

(c) What would be the **minimum** journey time between the following places?

(i) Porthmadog and Conwy _____

(ii) Holyhead and Wrexham _____

(iii) Ffestiniog and Bangor _____

(d) Is the above graph a

(i) Simple graph? _____ (ii) Connected graph? _____

(iii) Complete graph? _____ (iv) Tree? _____

(e) Which two vertices in the graph have degree one? _____

(f) How many vertices in the graph have degree four? _____

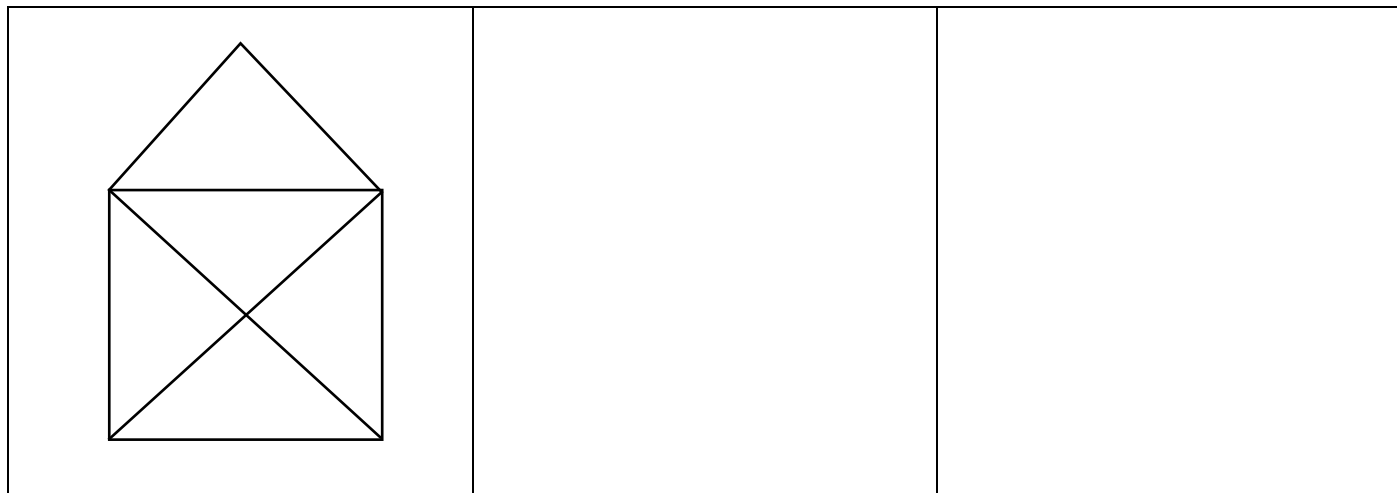
(g) What is the cycle with the least journey time? _____

Eulerian graphs

- A graph is an **Eulerian graph** if it is a connected graph where every vertex has an even degree.
- A graph is a **semi-Eulerian graph** if it is a connected graph where precisely two vertices have odd degree.

Exercise 8

Try to recreate the diagram below without taking your pencil from the paper, and without repeating any line.



It is possible to draw a graph without taking your pencil from the paper, and without repeating an edge, if the graph is Eulerian or semi-Eulerian.

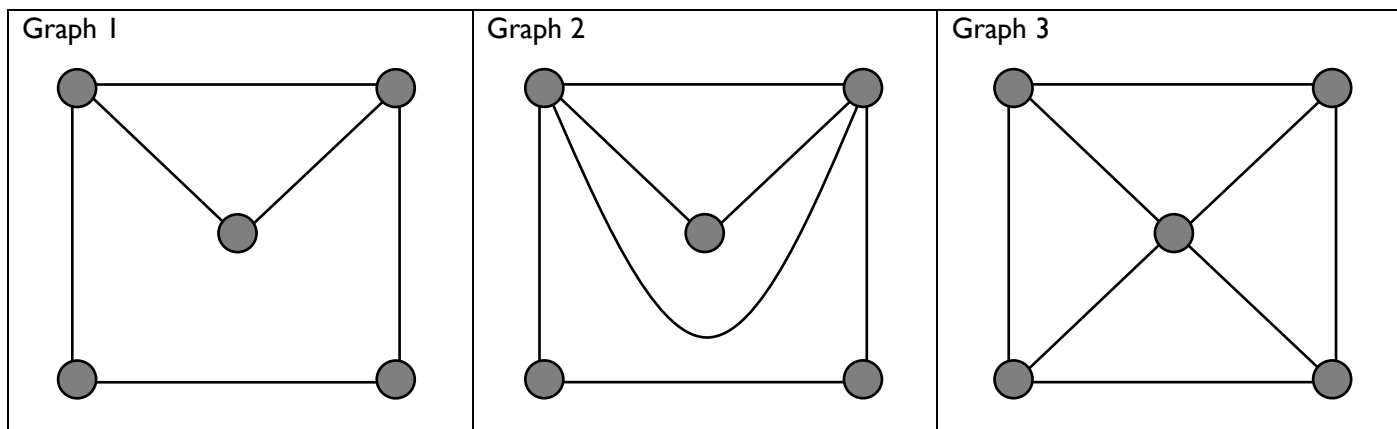
Exercise 9

(a) Is the graph in Exercise 8 Eulerian? _____

(b) Is the graph in Exercise 8 semi-Eulerian? _____

Exercise 10

Three graphs are shown below.

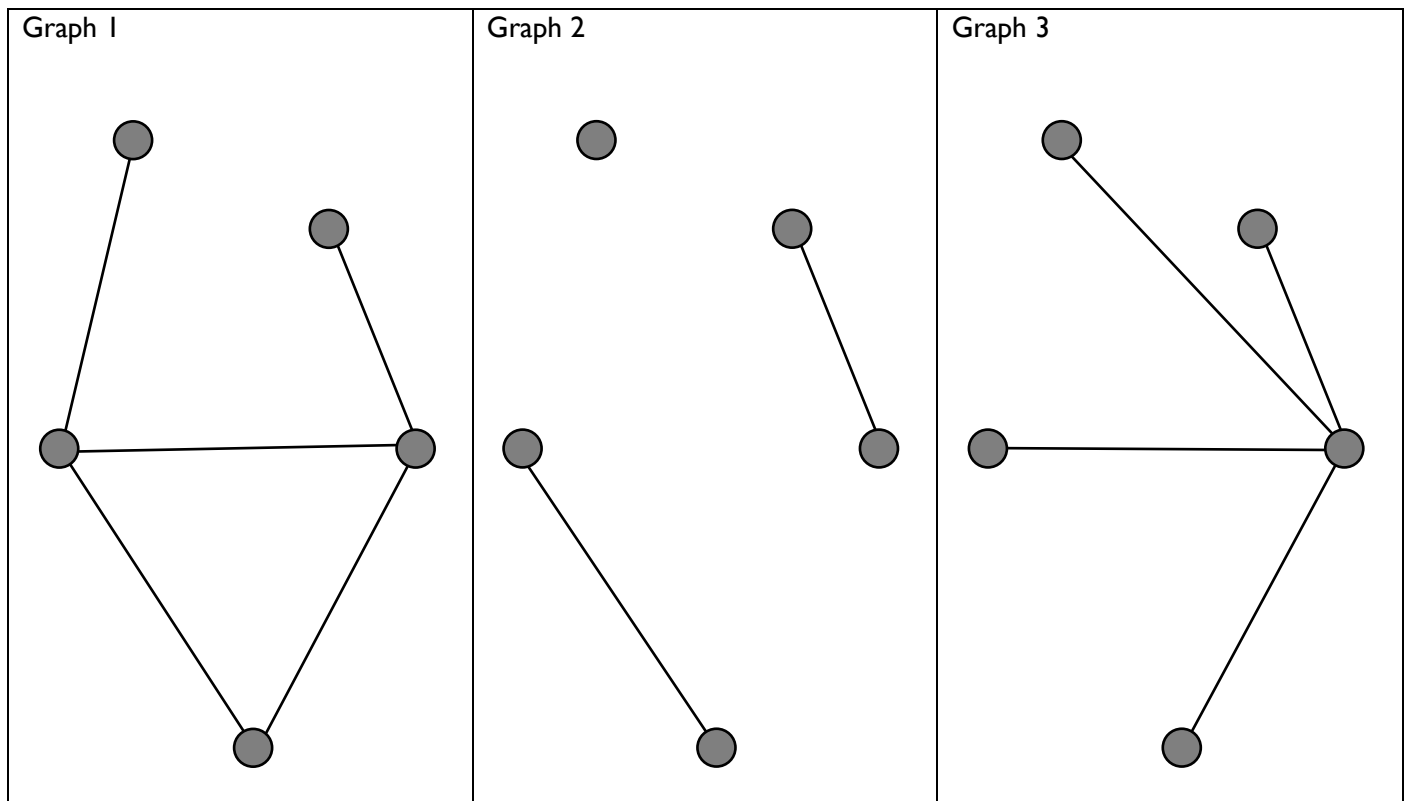


Tick the boxes, as appropriate.

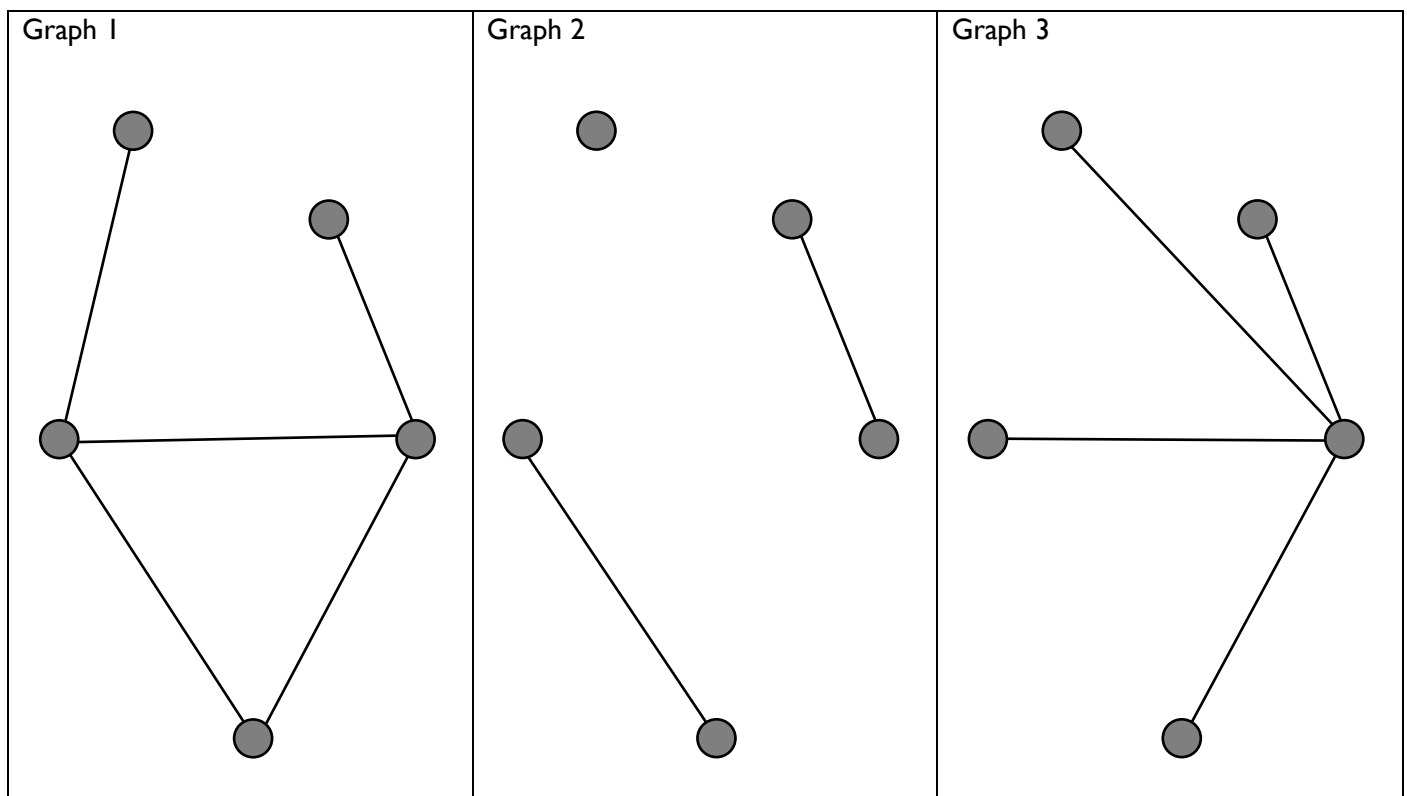
Graph	Eulerian graph?	Semi-Eulerian graph?
1		
2		
3		

Exercise 11

Add the least number of edges to the graphs below to turn them into semi-Eulerian graphs.

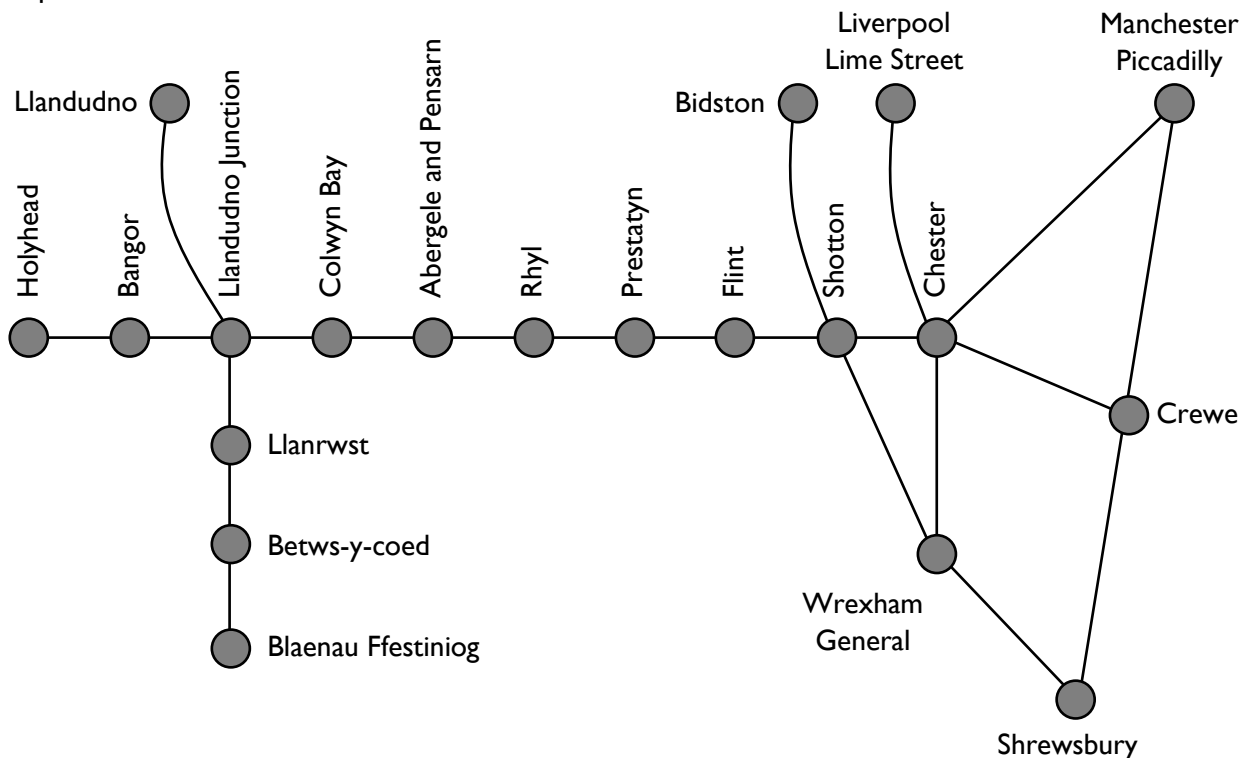
**Exercise 12**

Add the least number of edges to the graphs below to turn them into Eulerian graphs.



Exercise 13

Here is part of North Wales' rail network.



(a) What is the degree of the following vertices?

Holyhead _____

Llandudno Junction _____

Chester _____

Crewe _____

Shrewsbury _____

Abergele and Pensarn _____

(b) Are there any loops in the network? _____

(c) Are the following journeys open walks or closed walks?

Holyhead → Bangor → Llandudno Junction → Bangor

Chester → Crewe → Shrewsbury → Wrexham General → Chester

Liverpool Lime Street → Chester → Shotton → Bidston

Manchester Piccadilly → Crewe → Chester → Manchester Piccadilly

(d) Are the following journeys paths?

Bangor → Llandudno Junction → Llanrwst → Betws-y-coed

Chester → Wrexham General → Shotton → Chester

(e) Are the following journeys trails?

Chester → Wrexham General → Shotton → Chester

Chester → Wrexham General → Shotton → Chester → Wrexham General

(f) Is the above network a

Simple graph? _____

Connected graph? _____

Complete graph? _____

Tree? _____

Eulerian graph? _____

Exercise 14

(a) Draw a simple connected graph with 6 vertices and 9 edges.

(b) Draw a simple connected graph with 6 vertices and 9 edges, where the degree of one vertex is 5 and the degree of all other vertices is odd.

(c) Draw a simple connected graph with 6 vertices and 9 edges, which is also an Eulerian graph.

WJEC Terminology

Term	Definition
Edge	An edge is a line segment that connects two vertices in a graph.
Directed edge	An edge that has an orientation and can only be traversed in one direction. Also known as an arc.
Vertex/node	A vertex is a point on a graph that may or may not be connected to other vertices by edges. <i>Note:</i> vertices are also known as points.
Vertex degree	The number of edges that are incident to the vertex. A vertex is even/odd if the degree is even/odd. <i>Note:</i> vertex degree is also known as valency, or the “order” of a vertex.
Loop	An edge that starts and finishes at the same vertex.
Graph/network	A graph G consists of vertices that are connected by edges. <i>Note:</i> graphs are also known as networks in certain contexts.
Weighted graph/network	A graph/network where each edge has a numerical value (weight) assigned to it.
Subgraph	A graph G' whose vertices and edges belong to G .
Simple graph	A graph in which there are no loops, no directions on the edges, and where there is at most one edge connecting any pair of vertices.
Walk	A route through a graph along edges from one vertex to the next.
Connected graph	A graph in which every vertex is connected by a walk.
Complete graph	A graph in which every vertex is connected to every other vertex by a unique edge.
Open walk	A walk where the start vertex is not the same as the end vertex.
Closed walk	A walk where the start vertex is the same as the end vertex.
Trail	A walk in which no edge is visited more than once.
Path	A walk in which no vertex is visited more than once.
Cycle	A walk in which the end vertex is the same as the start vertex and no other vertex is visited more than once.
Tree	A connected graph with no cycles.
Spanning tree	A subgraph of G that includes all vertices and is also a tree.
Minimum spanning tree	A spanning tree such that the total weight of its edges is as small as possible. <i>Note:</i> a minimum spanning tree is also known as a minimum connector.
Eulerian graph	A connected graph where every vertex has an even degree.
Semi-Eulerian graph	A connected graph where precisely two vertices have an odd degree.

Pop Quiz: Graphs and Networks

(1) Study the graph on the right.

Pair the following.

A

B

C

D

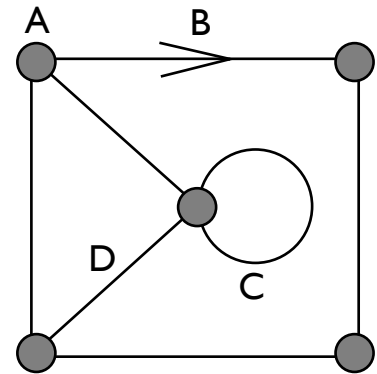
Edge

Vertex

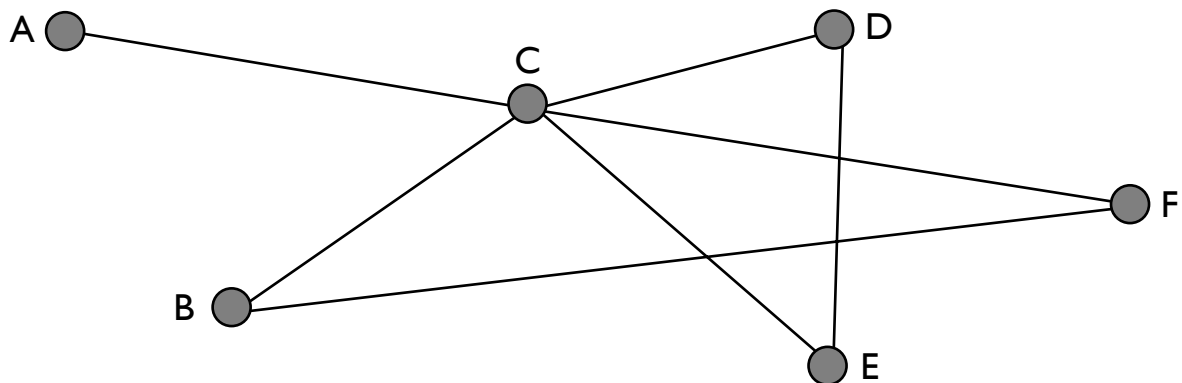
Directed Edge

Loop

[4]



(2) Study the following graph.



(a) What is the degree of vertex C? _____

- (b) Is the graph a
- (i) simple graph? _____
 - (ii) connected graph? _____
 - (iii) Eulerian graph? _____
 - (iv) semi-Eulerian graph? _____
 - (v) complete graph? _____
 - (vi) tree? _____

(c) Is the walk $C \rightarrow D \rightarrow E \rightarrow C \rightarrow B \rightarrow F \rightarrow C$

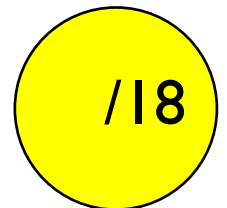
- (i) an open walk? _____
- (ii) a closed walk? _____
- (iii) a path? _____
- (iv) a trail? _____

(d) Add one edge to the graph to change it to an Eulerian graph.

[12]

(3) Draw a simple semi-Eulerian graph with 5 vertices and 7 edges.

[2]



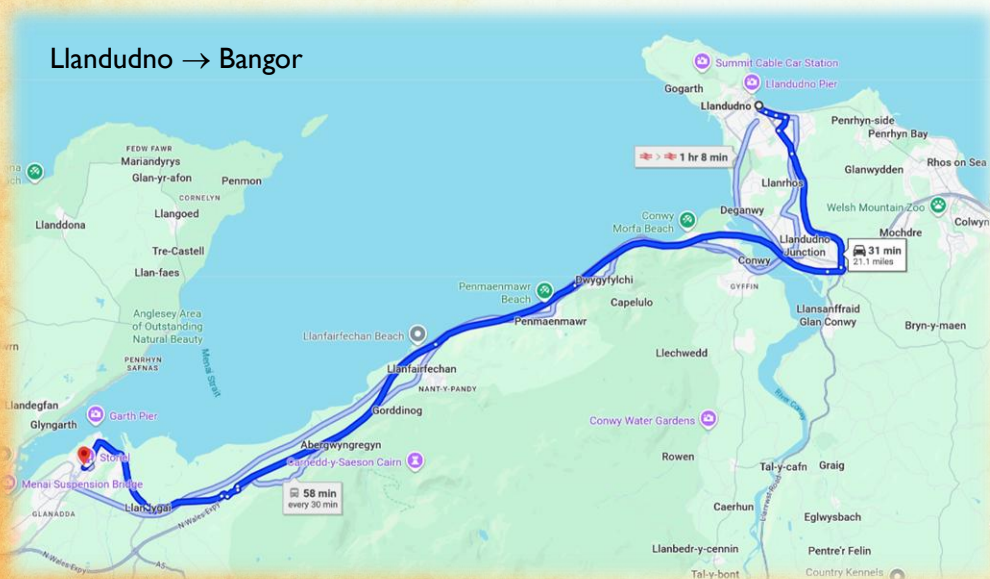


Unit 6, Workbook 2

11

Introducing

Algorithms



Name:

Background

What is the work?

Understanding how algorithms can be represented by a flowchart or a set of instructions. The Prim, Kruskal and Dijkstra algorithms. Algorithms for sorting data.

What are the prerequisites?

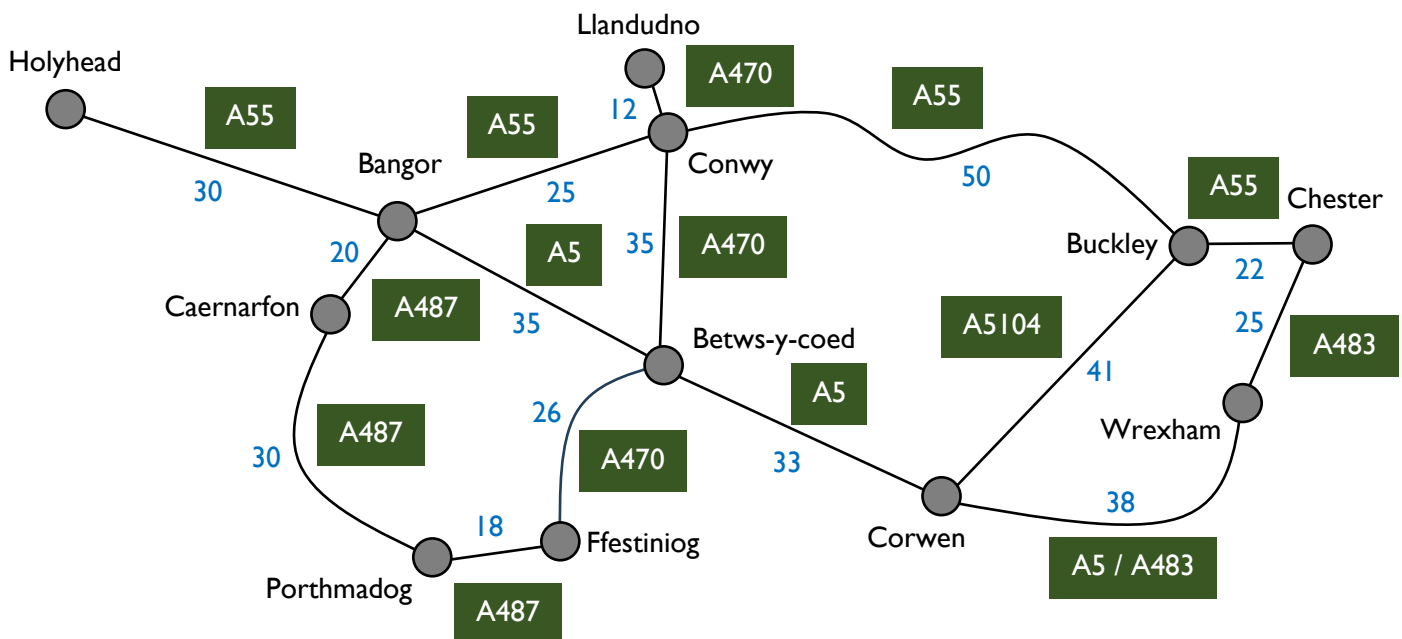
GCSE Work: Sorting numbers from least to greatest.

Where does this lead to?

Applications: Minimising the cost of cables in an electrical network; the travelling salesman problem; optimising computing networks.

Theory

In the previous workbook, we introduced the concept of a graph, which uses vertices and edges. For example, the following weighted graph shows part of the **road network** of North Wales, with the **usual travelling time** (in minutes) shown on each edge.



The above graph is not a tree, as there are a number of circuits in the graph, for example Bangor → Conwy → Betws-y-coed → Bangor. By removing some of the roads from the graph, we can form a spanning tree, but what is the **minimal spanning tree**, namely the spanning tree with the least total weight?

There are a number of algorithms for finding a minimal spanning tree. In this workbook, we will consider two such algorithms, namely **Kruskal's algorithm** and **Prim's algorithm**.

Kruskal's AlgorithmOn the
exam paper

- Sort all the edges into ascending order of weight.
- Select the edge of least weight to start the tree.
- Consider the next edge of least weight. If it would form a cycle with those edges already selected, reject it. If it does not form a cycle, add it to the tree. If there is a choice of equal edges, choose one of them at random and consider each in turn.
- Repeat the third step until all vertices are connected.

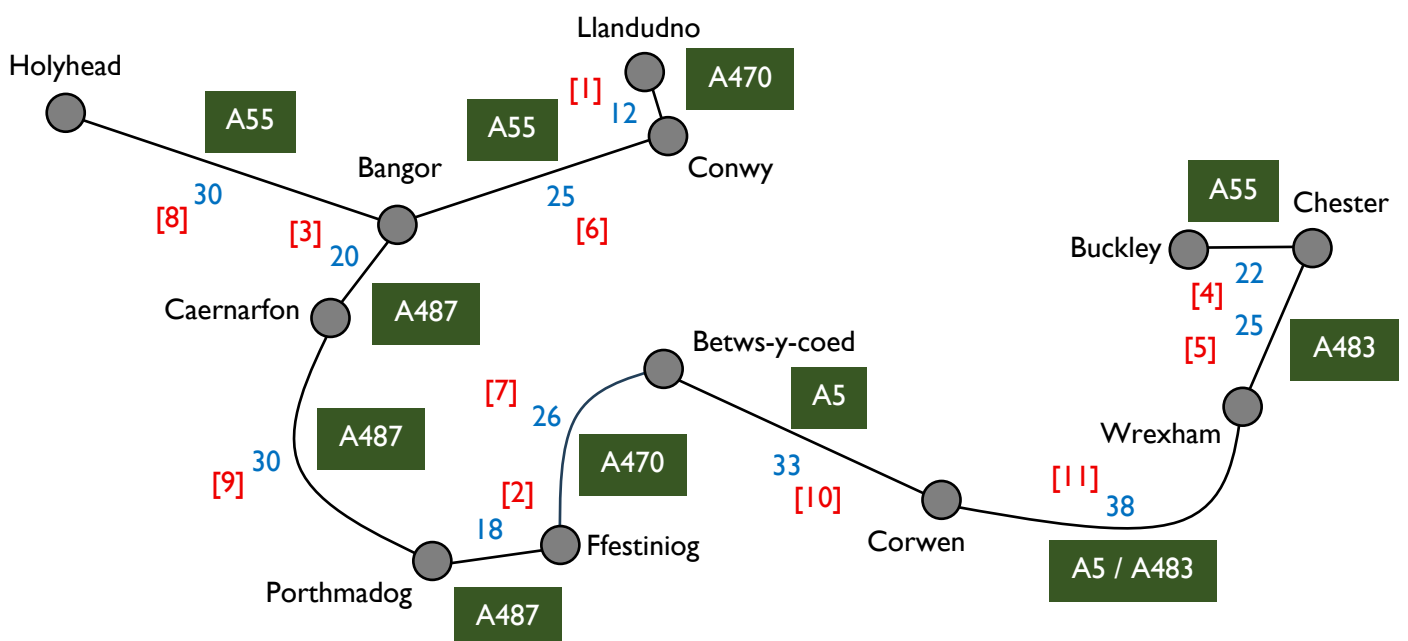
Example 1

Let us apply Kruskal's algorithm to the network shown on the previous page.

Sorting all the edges, we can form the following table.

Edge	Weight
Llandudno → Conwy	12
Porthmadog → Ffestiniog	18
Caernarfon → Bangor	20
Buckley → Chester	22
Bangor → Conwy	25
Wrexham → Chester	25
Ffestiniog → Betws-y-coed	26
Holyhead → Bangor	30
Caernarfon → Porthmadog	30
Betws-y-coed → Corwen	33
Bangor → Betws-y-coed	35
Conwy → Betws-y-coed	35
Corwen → Wrexham	38
Corwen → Buckley	41
Conwy → Buckley	50

Following the algorithm, we create the following spanning tree. (The **bracketed numbers** show the order of adding the edges.)



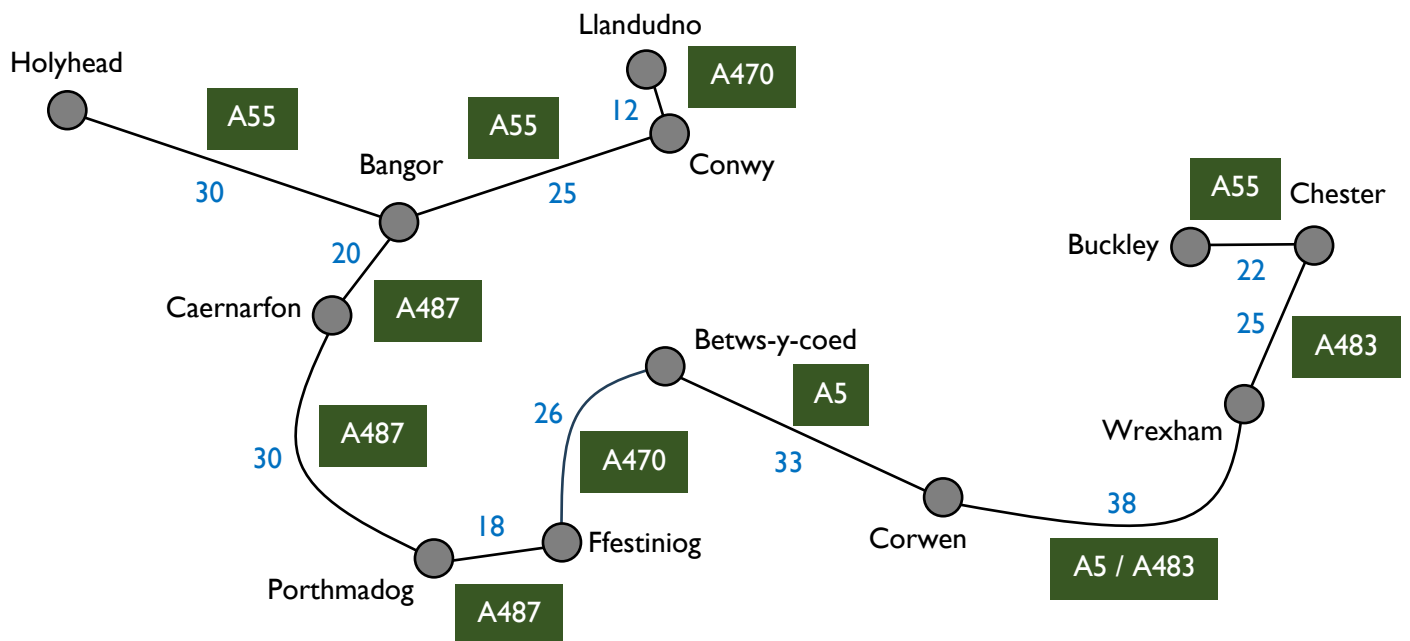
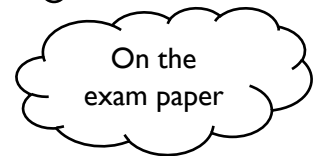
Prim's Algorithm

- Choose any vertex to start the tree.
- Select an edge of least weight that joins a vertex already in the tree to a vertex not yet in the tree. If there is a choice of edges of equal weight, choose any of them.
- Repeat the second step until all the vertices are connected.

Example 2

Let us apply Prim's algorithm to the network shown on page 2.

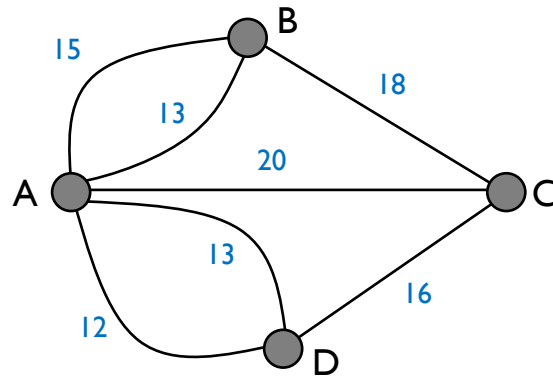
- We choose Betws-y-coed as the start vertex.
- We add the following edges:
 - Betws-y-coed → Ffestiniog (26)
 - Ffestiniog → Porthmadog (18)
 - Porthmadog → Caernarfon (30)
 - Caernarfon → Bangor (20)
 - Bangor → Conwy (25)
 - Conwy → Llandudno (12)
 - Bangor → Holyhead (30)
 - Betws-y-coed → Corwen (33)
 - Corwen → Wrexham (38)
 - Wrexham → Chester (25)
 - Chester → Buckley (22)
 - Buckley → Wrexham (25)



Notice, in examples 1 and 2, that Kruskal and Prim's algorithms both give the same minimal spanning tree. This will not happen with every graph however – and it is possible that an individual algorithm will give different spanning trees, due to the choices involved in the algorithms.

Exercise I

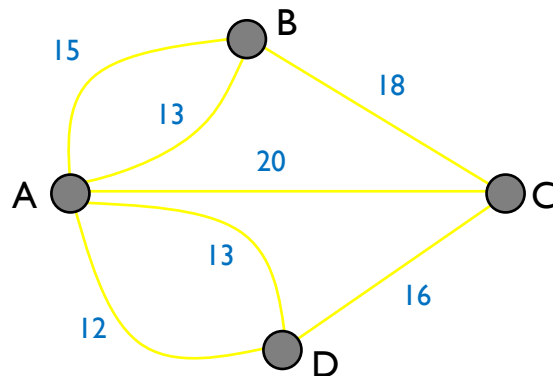
Consider the following network that connects four vertices.



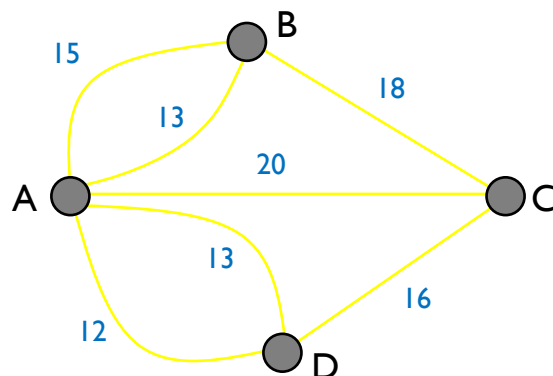
(a) Complete the following table, to order all the edges. (The first one has been done for you.)

Edge	Weight
A → D	12

(b) Use Kruskal's algorithm to choose a minimal spanning tree for the network.

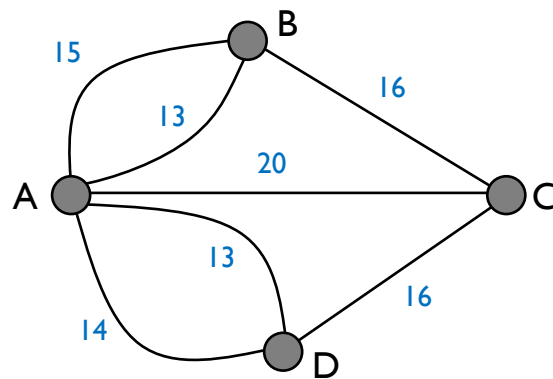


(c) Use Prim's algorithm to choose a minimal spanning tree for the network. Start from vertex A.



Exercise 2

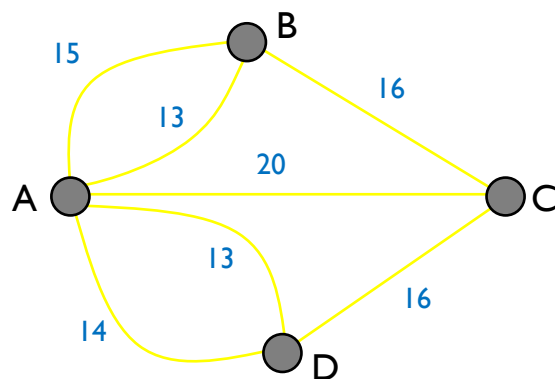
Consider the following network that connects four vertices.



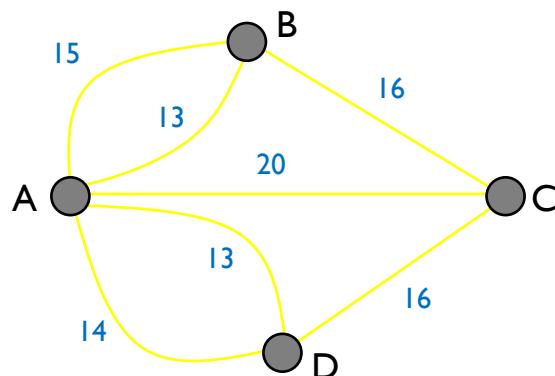
(a) Complete the following table, to order all the edges.

Edge	Weight

(b) Use Kruskal's algorithm to choose a minimal spanning tree for the network.

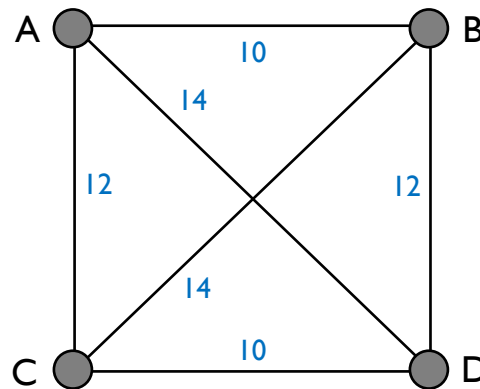


(c) Use Prim's algorithm to choose a minimal spanning tree for the network. Start from vertex C.



Exercise 3

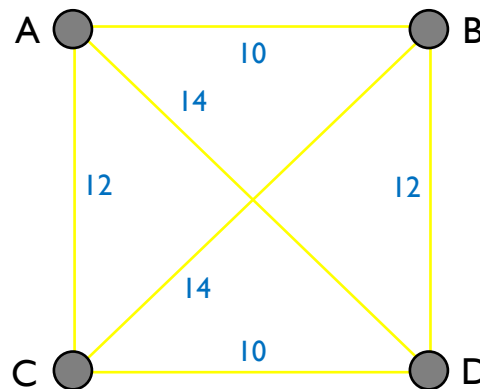
Consider the following network that connects four vertices.



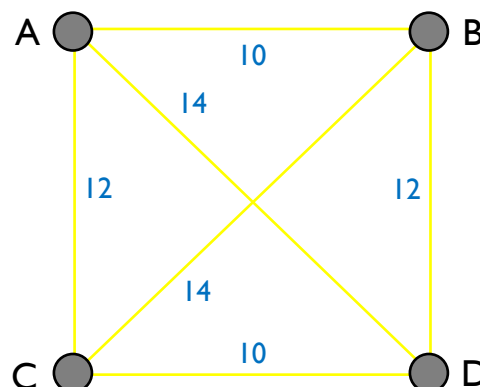
(a) Complete the following table, to order all the edges.

Edge	Weight

(b) Use Kruskal's algorithm to choose a minimal spanning tree for the network.

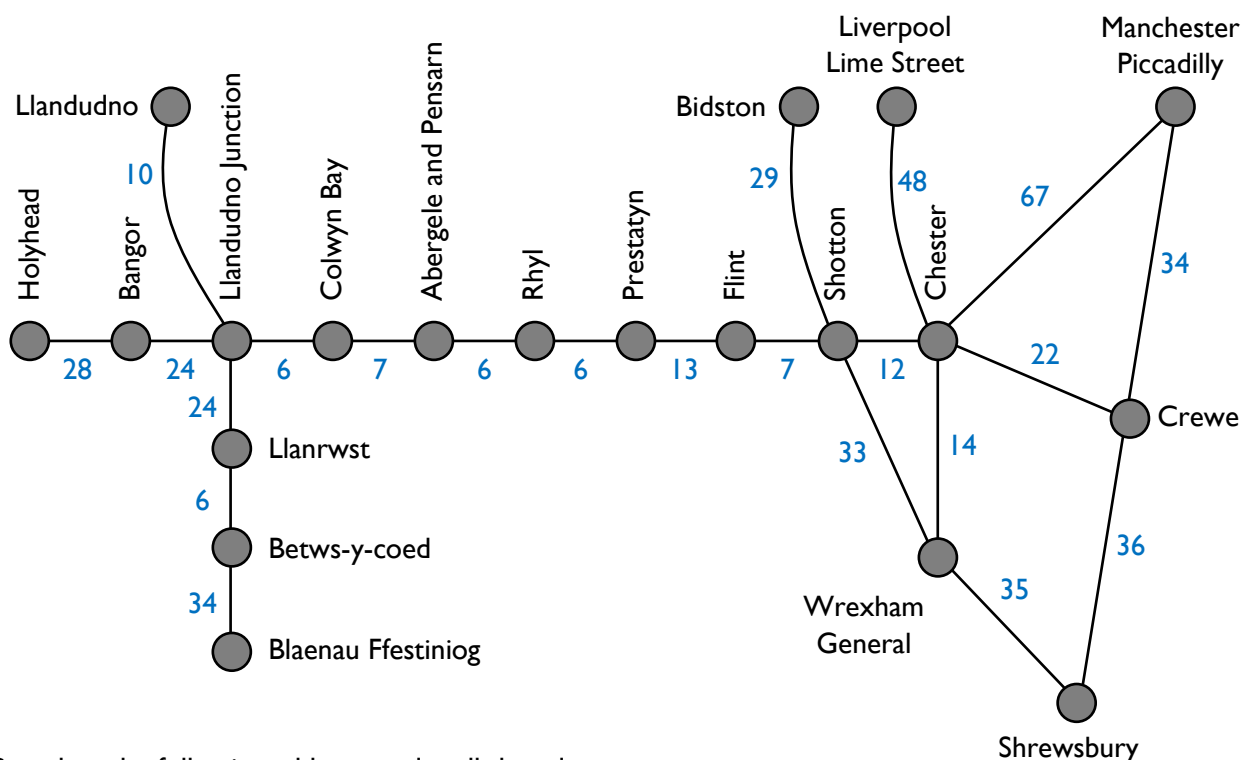


(c) Use Prim's algorithm to choose a minimal spanning tree for the network. Start from vertex B.



Exercise 4

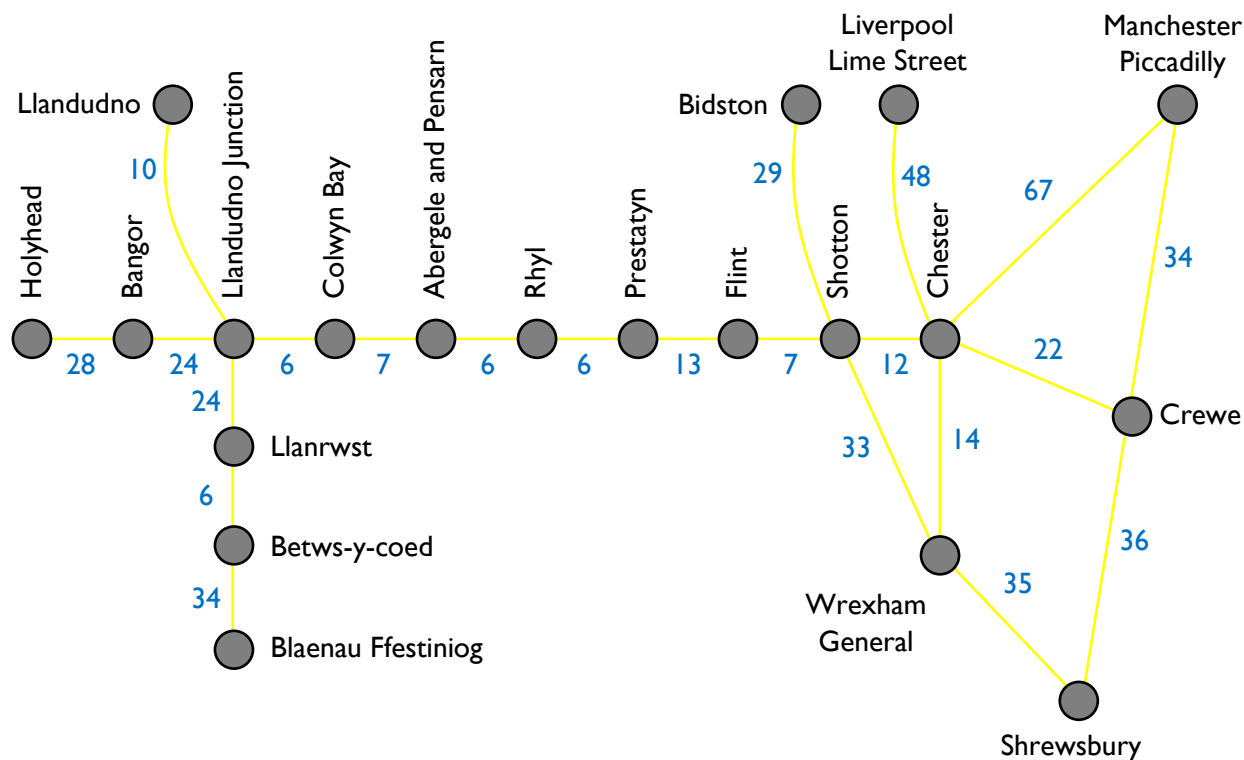
Here is part of North Wales' rail network, with the usual travelling time (in minutes) shown on each edge.



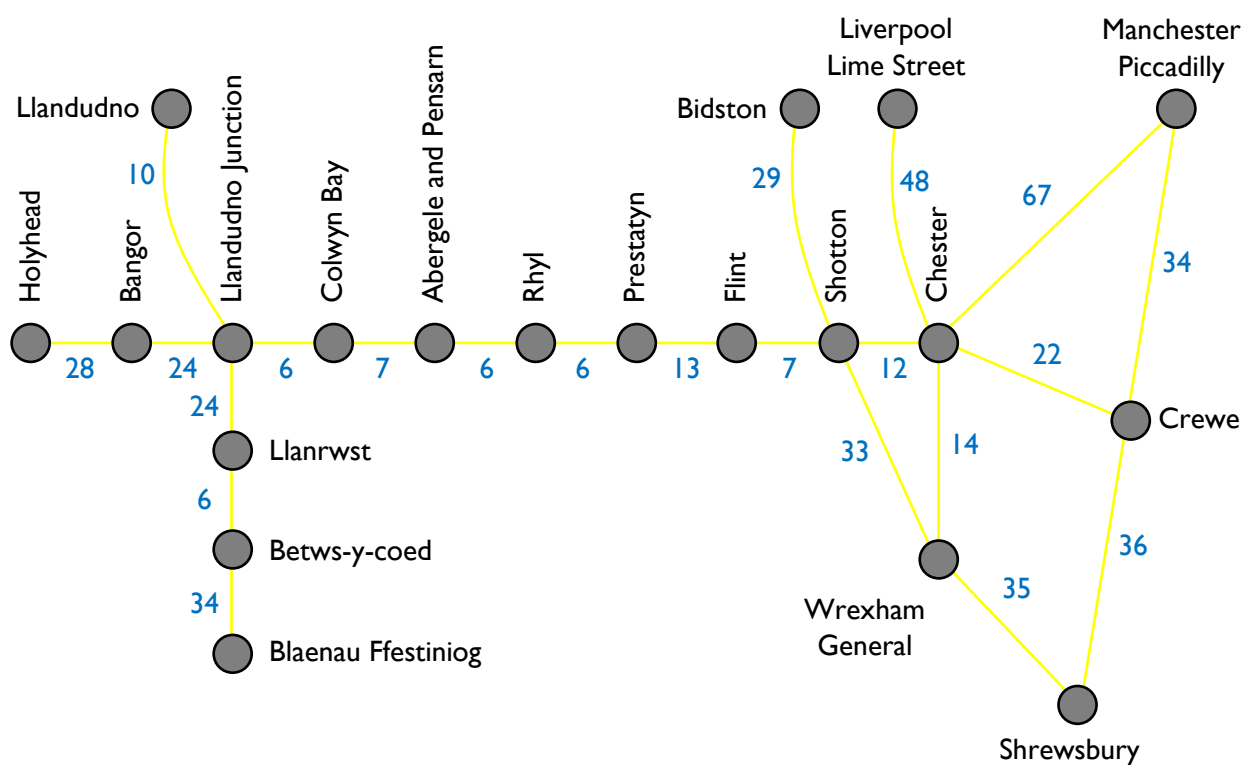
(a) Complete the following table, to order all the edges.

Edge	Weight
Llandudno Junction → Colwyn Bay	6
Abergele and Pensarn → Rhyl	6
Rhyl → Prestatyn	6
	6
Colwyn Bay → Abergele and Pensarn	7
Flint → Shotton	7
Llandudno Junction → Llandudno	
Shotton → Chester	12
Prestatyn → Flint	13
	14
Chester → Crewe	22
Bangor → Llandudno Junction	24
Llandudno Junction → Llanrwst	24
Holyhead → Bangor	
Shotton → Bidston	29
	33
Betws-y-coed → Blaenau Ffestiniog	34
Crewe → Manchester Piccadilly	34
Wrexham General → Shrewsbury	
Crewe → Shrewsbury	36
	48
Chester → Manchester Piccadilly	

(b) Use Kruskal's algorithm to find a minimal spanning tree for the North Wales rail network.

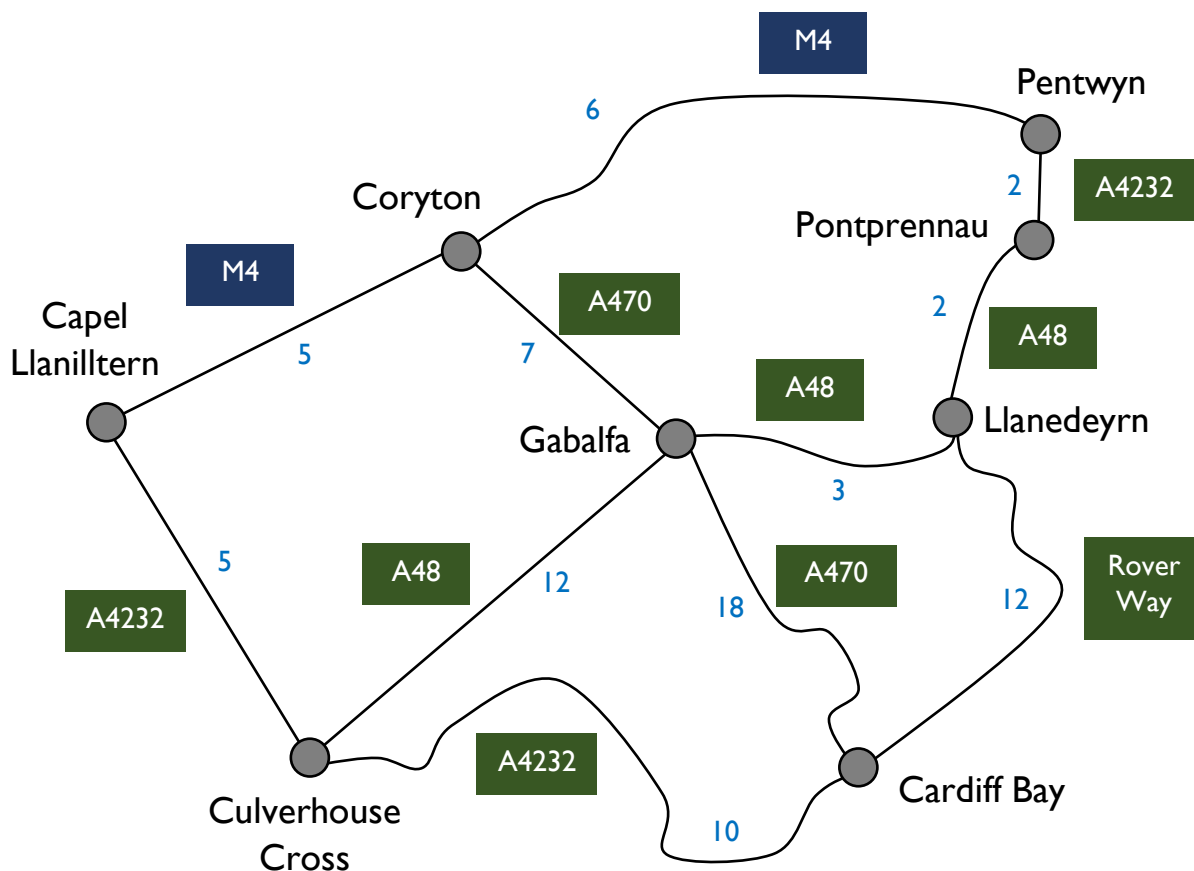


(c) Use Prim's algorithm to find a minimal spanning tree for the North Wales rail network.



Exercise 5

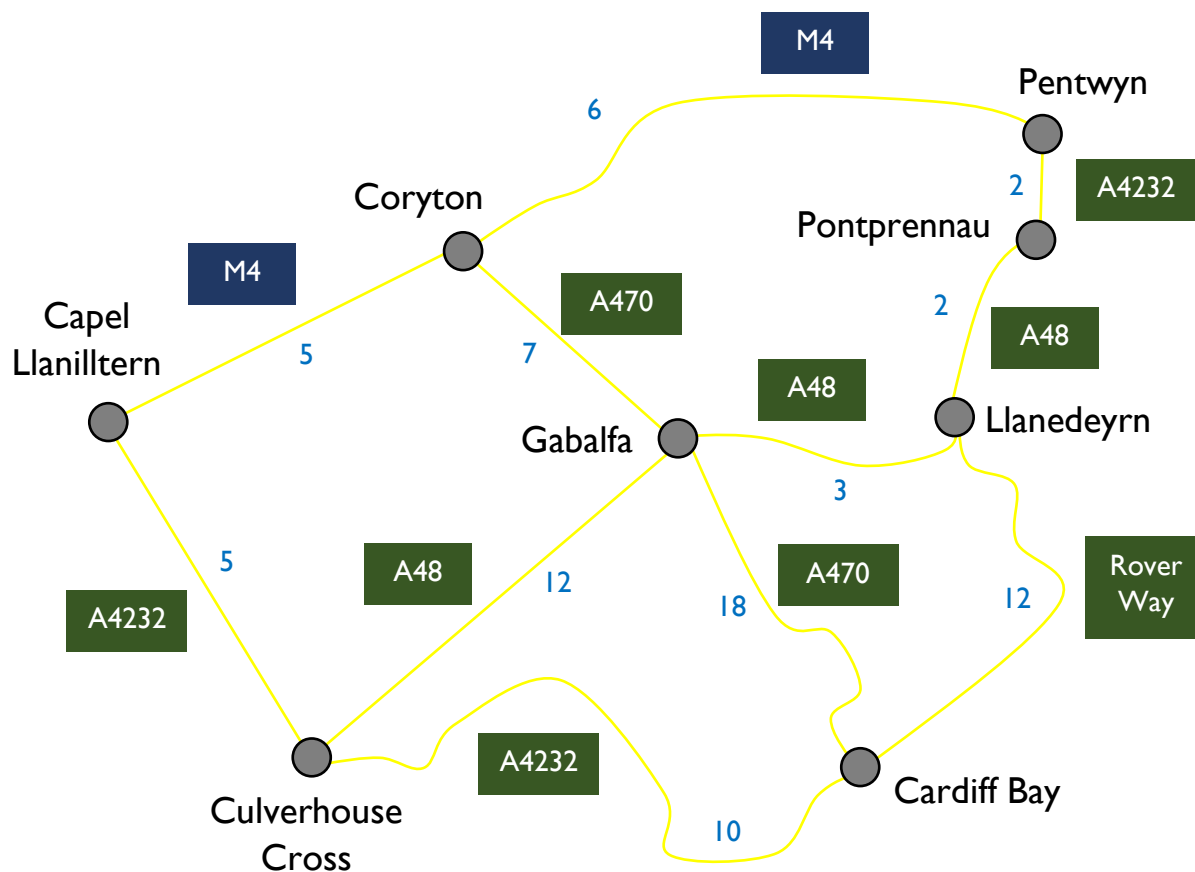
Here is part of Cardiff's road network, with the **usual travelling time** (in minutes, at a quiet time!) shown on each edge.



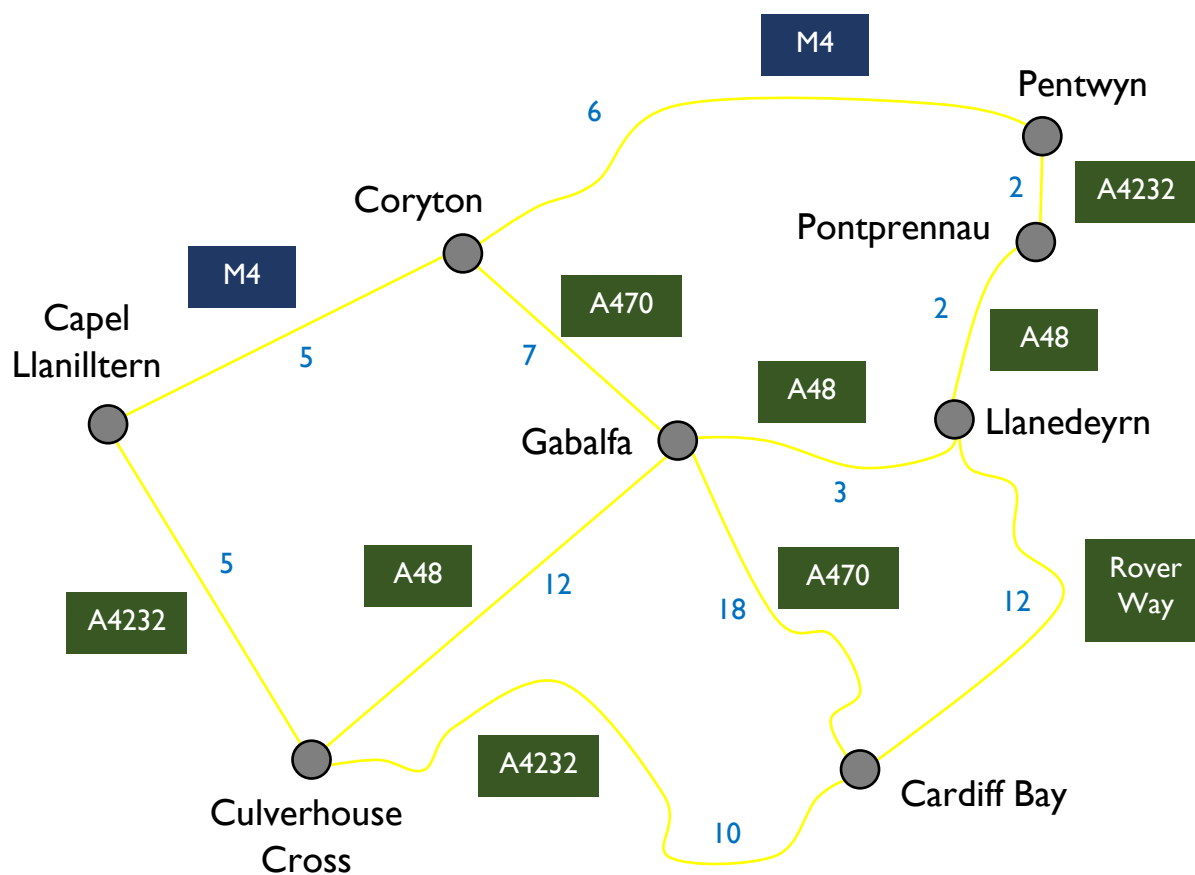
(a) Complete the following table, to order all the edges.

Edge	Weight
Pentwyn → Pontprennau	2
	2
Gabalfa → Llanedeyrn	
Capel Llanilltern → Coryton	5
	5
Coryton → Pentwyn	6
Culverhouse Cross → Cardiff Bay	10
Culverhouse Cross → Gabalfa	
Llanedeyrn → Cardiff Bay	12
	18

(b) Use Kruskal's algorithm to find a minimal spanning tree for Cardiff's road network.



(c) Use Prim's algorithm to find a minimal spanning tree for Cardiff's road network. Start at Culverhouse Cross.



Dijkstra's Algorithm

Dijkstra's algorithm is used to find the minimal path between two vertices in a graph.

Label the start vertex with a **final** label, 0.

Record a working value at each vertex that is connected to the vertex that has just received its final label.

- Working value is final label plus the weight of the edge.
- If there is already a working value, it is only replaced if the new value is smaller.
- Once a vertex has a final value, it is no longer revisited, and its working values are no longer considered.

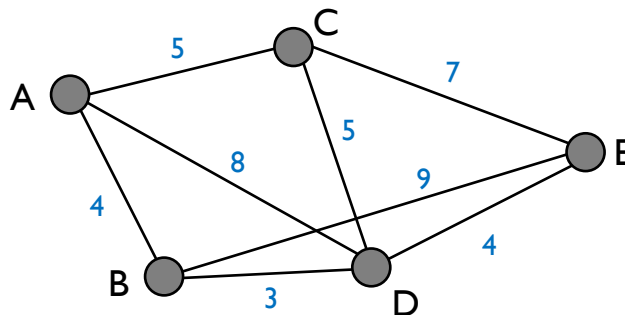
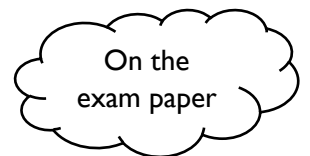
Look at the working values at all vertices without final labels. Select the smallest working value. This now becomes the final label at that vertex. If two vertices have the same working value, choose either of them first.

Repeat the second and third steps until the destination vertex receives its final label.

To find the shortest path, trace back from the destination to the start.

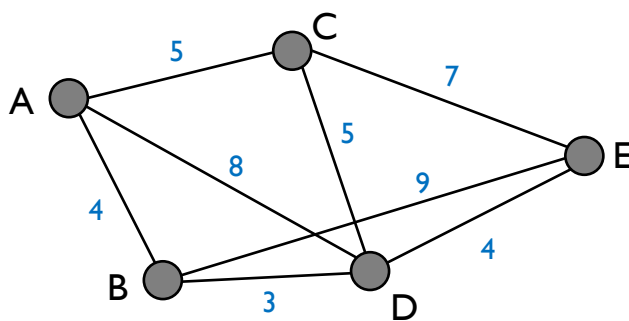
Example 3

Consider the network below that connects five vertices.



Use Dijkstra's algorithm to find the shortest path from A to E.

Answer: To begin, form tables of labels as shown below on the right.



Key

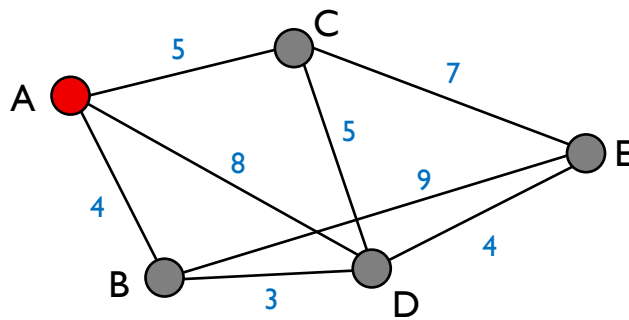
Vertex	Final Label
	Working Value

A	0	C		E	
B		D			

We start from A so its final label becomes 0.

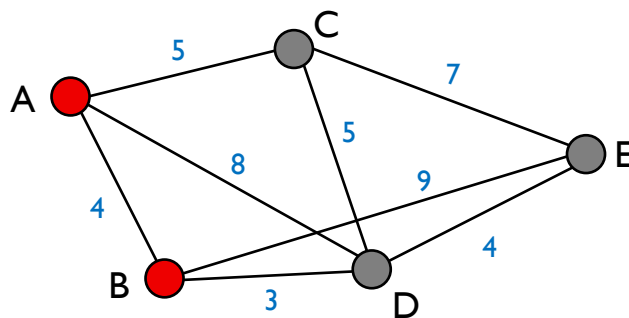
All the other vertices start with empty labels.

The first vertex to be considered is A, the start of the path. We need to consider all vertices that can be reached from A (using exactly one edge).



A	0	C		E	
			5		
B		D			
	4		8		

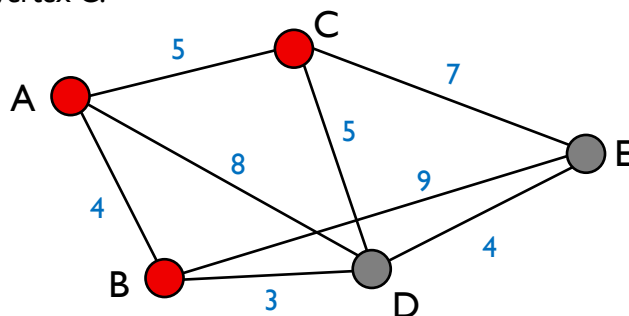
The next vertex to be considered is the one with the least working value that has not yet been considered. This is vertex B.



A	0	C		E	
			5		13
B	4	D			
	4		8, 7		

Notice that the working value of D changes from 8 to 7, as $A \rightarrow B \rightarrow D$ is shorter than $A \rightarrow D$.

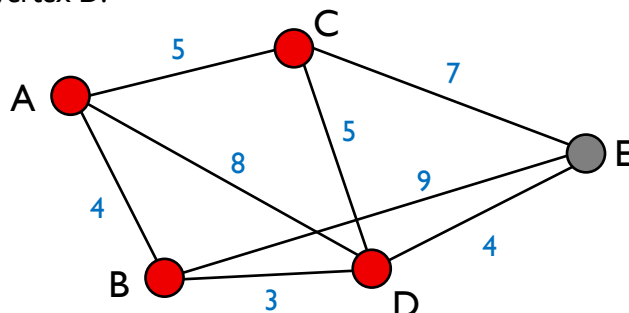
The next vertex to be considered is the one with the least working value that has not yet been considered. This is vertex C.



A	0	C	5	E	
			5		13, 12
B	4	D			
	4		8, 7		

This time, the working value of E changes from 13 to 12, as $A \rightarrow C \rightarrow E$ is shorter than $A \rightarrow B \rightarrow E$.

The next vertex to be considered is the one with the least working value that has not yet been considered. This is vertex D.



A	0	C	5	E	
			5		13, 12, 11
B	4	D	7		
	4		8, 7		

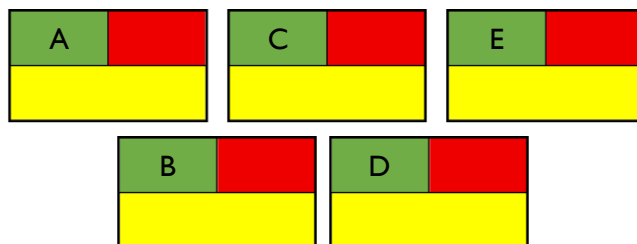
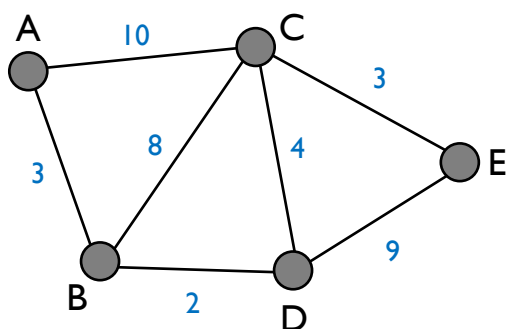
Again, the working value of E needs to change, as $A \rightarrow B \rightarrow D \rightarrow E$ is shorter than $A \rightarrow C \rightarrow E$.

Now, only the destination vertex remains, E. Its final label becomes 11, and we can work back through the tables to find the shortest path from A to E, $A \rightarrow B \rightarrow D \rightarrow E$, with weight 11.

E	11
	13, 12, 11

Exercise 6

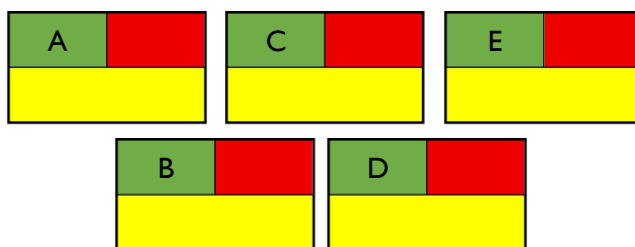
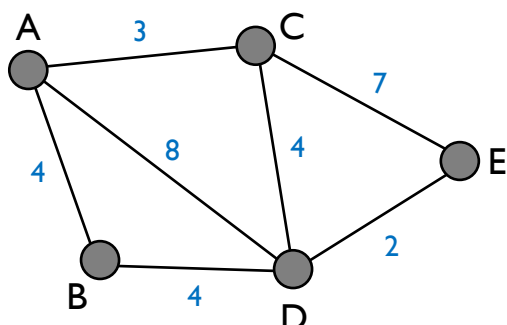
Use Dijkstra's algorithm to find the shortest path from A to E in the network below.



The shortest path is _____ with weight ____

Exercise 7

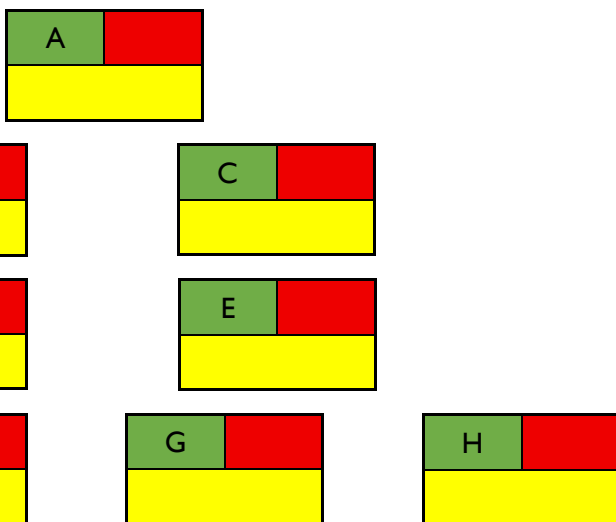
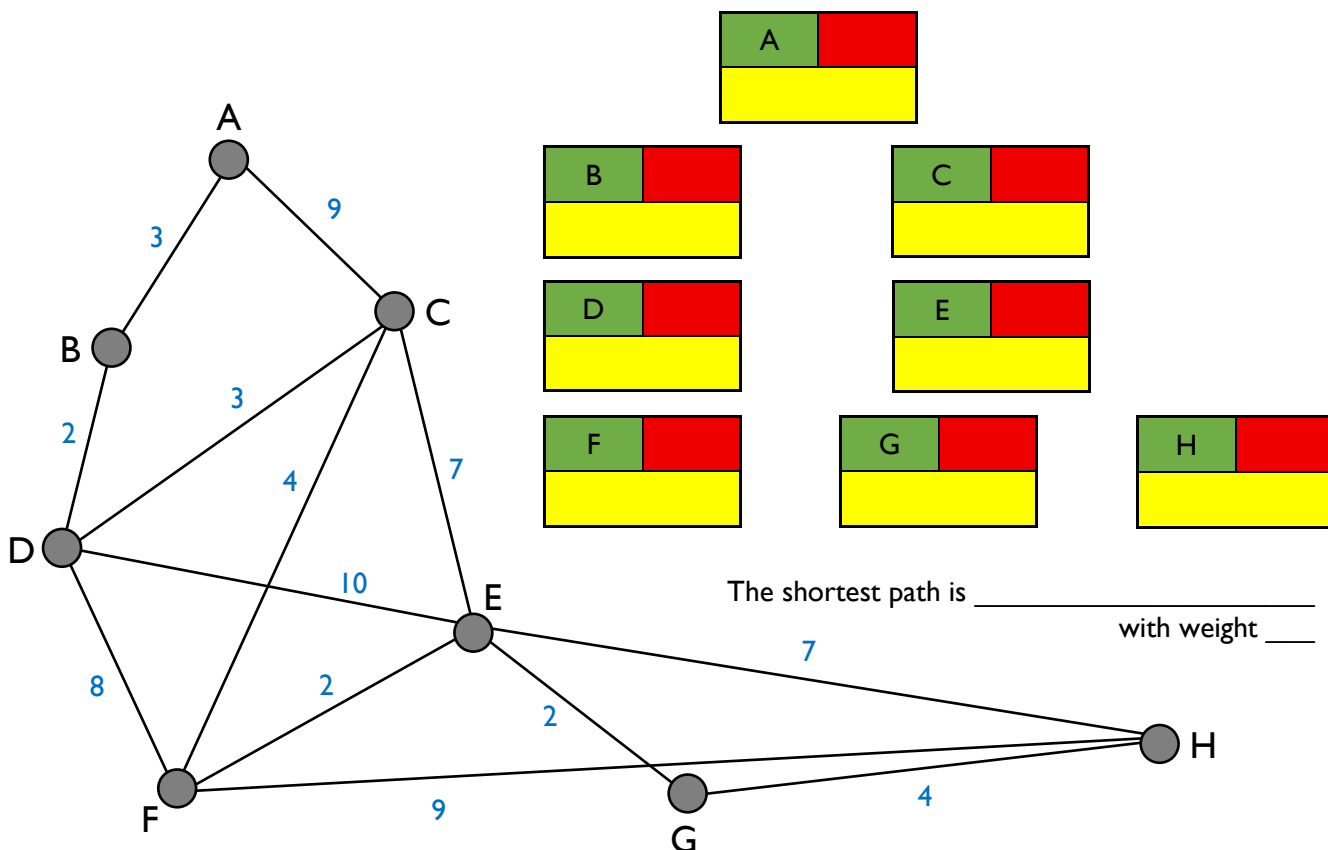
Use Dijkstra's algorithm to find the shortest path from A to E in the network below.



The shortest path is _____ with weight ____

Exercise 8

Use Dijkstra's algorithm to find the shortest path from A to H in the network below.

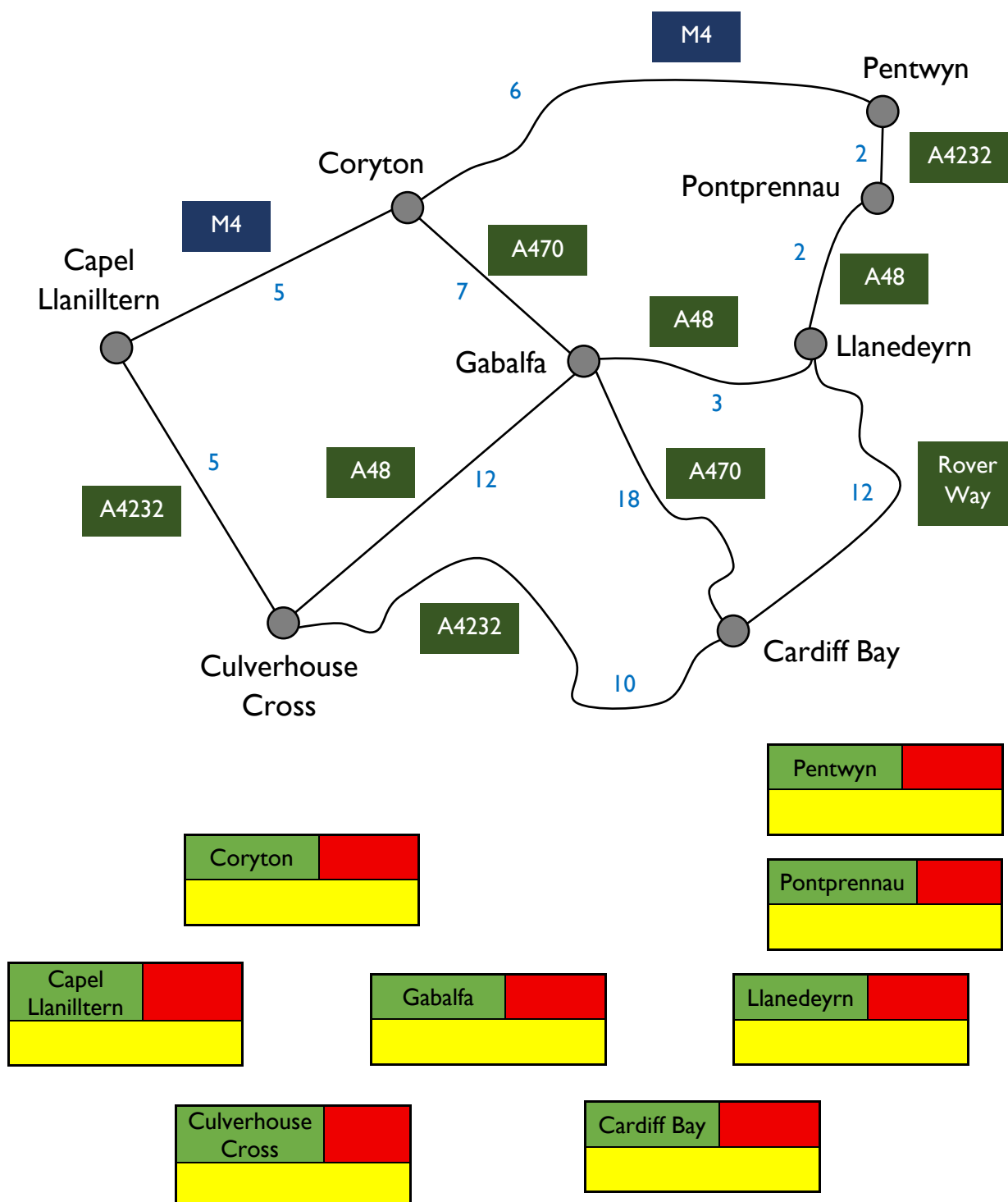


The shortest path is _____ with weight ____

Exercise 9

Here is part of Cardiff's road network, with the usual travelling time (in minutes, at a quiet time!) shown on each edge.

(a) Use Dijkstra's algorithm to find the shortest path from Cardiff Bay to Coryton.



The shortest path is _____ with weight ____

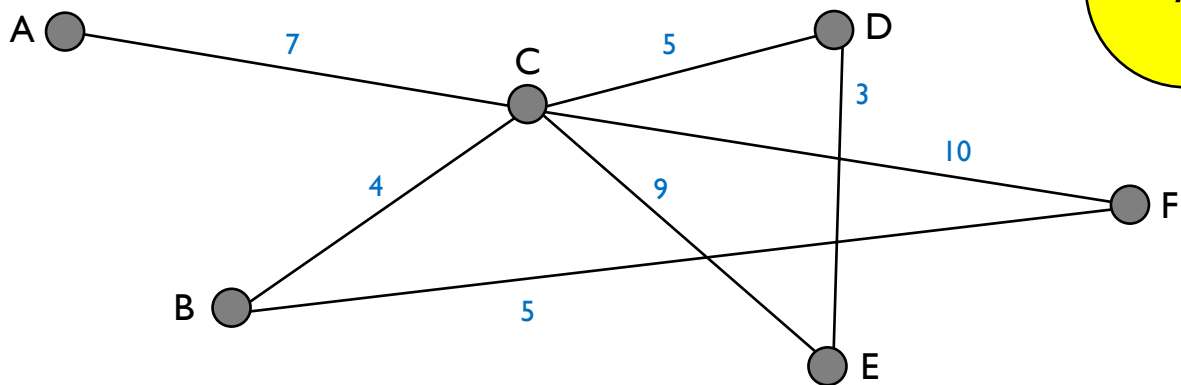
(b) An accident on the road means that the road between Cardiff Bay and Culverhouse Cross is closed. What is now the shortest path between Cardiff Bay and Coryton?

The shortest path is _____ with weight ____

Pop Quiz: Kruskal, Prim and Dijkstra

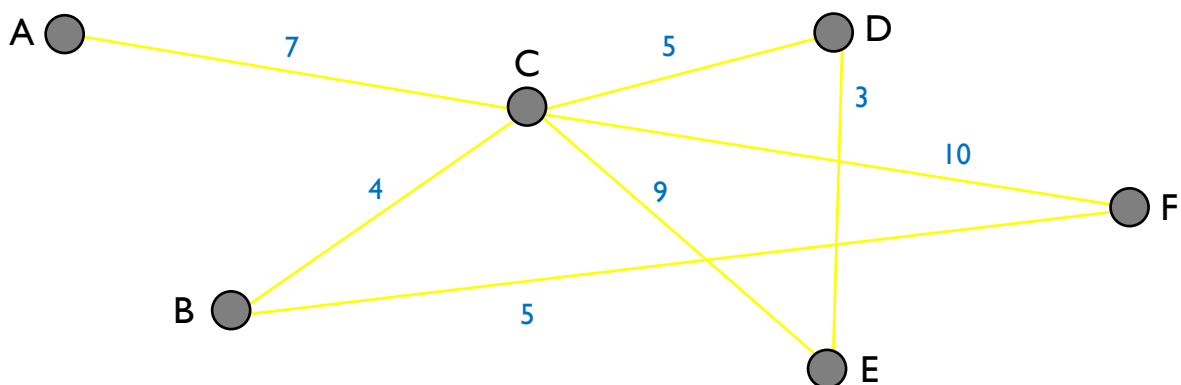
/18

(I) Consider the network below.



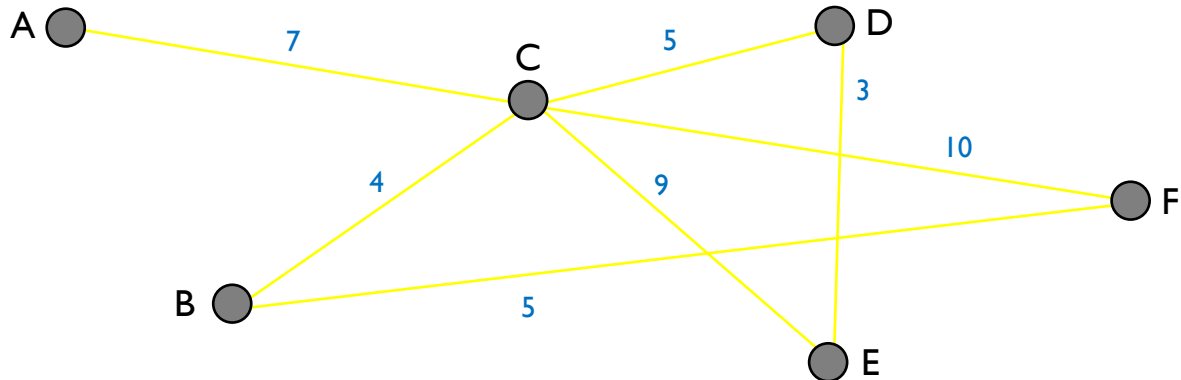
(a) Apply Kruskal's algorithm to find a minimal spanning tree for the network.

[5]



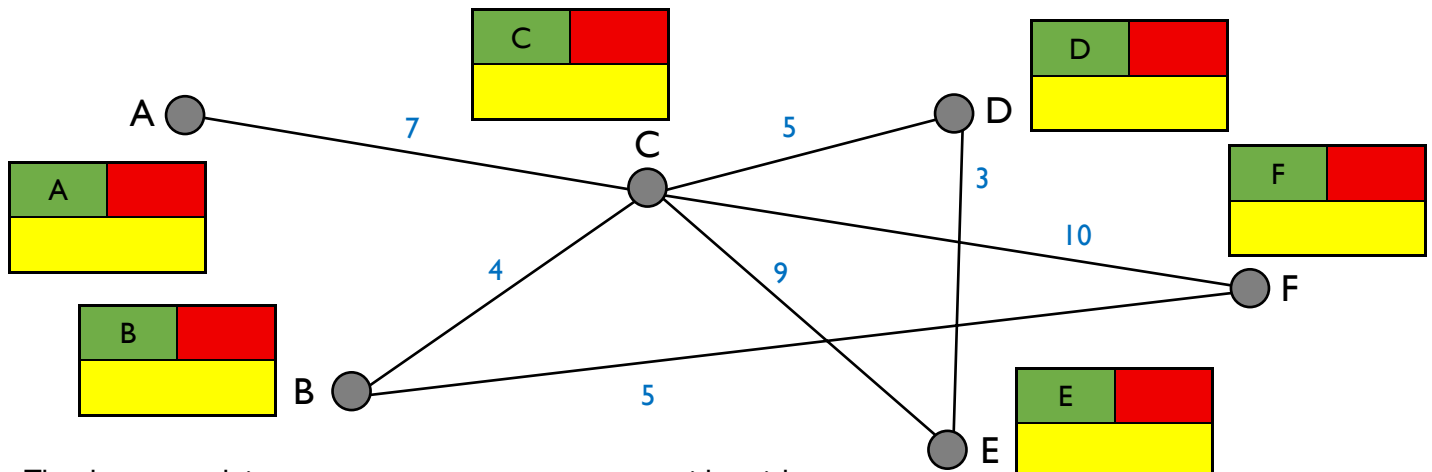
(b) Apply Prim's algorithm to find a minimal spanning tree for the network. Start at A.

[5]



(c) Apply Dijkstra's algorithm to find the shortest path between A and E.

[8]



The shortest path is _____ with weight _____

Sorting Algorithms

Look at the following set of numbers.

24, 43, 75, 8, 19, 87, 48, 29, 70, 36

How would you go about sorting the numbers from least to greatest?

Computers use different algorithms to sort numbers or letters.

We will study two such algorithms in this workbook.

Bubble Sort

Start at the beginning of the working list and move from left to right comparing adjacent items. If they are in order, leave them, if they are not, swap them.

When you get to the end of the working list, the last item will be in its final position. This item is no longer in the working list.

If you made some swaps on the last pass, repeat the first instruction.

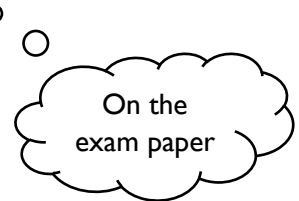
When a pass is completed without any swaps, every item is in its final position and the list is in order, stop.

Example 4

Here is how the bubble sort algorithm would sort the numbers at the top of this page, from least to greatest.

The green cells show pairs of numbers that have been swapped.

The red cells show the numbers that are no longer in the working list.



24	43	75	8	19	87	48	29	70	36
24	43	8	75	19	87	48	29	70	36
24	43	8	19	75	87	48	29	70	36
24	43	8	19	75	48	87	29	70	36
24	43	8	19	75	48	29	87	70	36
24	43	8	19	75	48	29	70	87	36
24	43	8	19	75	48	29	70	36	87
24	8	43	19	75	48	29	70	36	87
24	8	19	43	75	48	29	70	36	87
24	8	19	43	48	75	29	70	36	87
24	8	19	43	48	29	75	70	36	87
24	8	19	43	48	29	70	75	36	87
24	8	19	43	48	29	70	36	75	87
8	24	19	43	48	29	70	36	75	87
8	19	24	43	48	29	70	36	75	87
8	19	24	43	29	48	70	36	75	87
8	19	24	43	29	48	36	70	75	87
8	19	24	29	43	48	36	70	75	87
8	19	24	29	43	36	48	70	75	87
8	19	24	29	43	36	48	70	75	87

Exercise 10

Use the bubble sort algorithm to sort the following numbers, from least to greatest.

Use a new row to show when a pair of numbers have been swapped.

24	37	5	16	42	30	25

Exercise 11

Use the bubble sort algorithm to sort the following numbers, from greatest to least.

Use a new row to show when a pair of numbers have been swapped.

24	37	5	16	42	30	25

Exercise 12

Bill wishes to use the bubble sort algorithm to sort four numbers, from least to greatest.

--	--	--	--

What will be the (a) minimum; (b) maximum number of swaps that need to take place?

.....

.....

Exercise 13

Use the bubble sort algorithm to sort the following letters, in alphabetical order. (“*Lexicographic order*”).
Use a new row to show when a pair of letters have been swapped.

P	O	L	Y	G	O	N

Exercise 14

Use the bubble sort algorithm to sort the following letters, in **reverse** alphabetical order. (“*Antilexicographic order*”).
Use a new row to show when a pair of letters have been swapped.

E	U	C	L	I	D

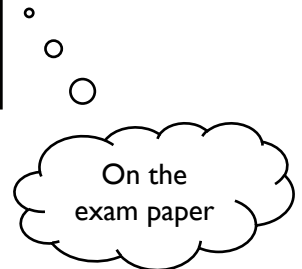
Exercise 15

Use the bubble sort algorithm to sort the following colours, in rainbow order (starting with **red**).
Use a new row to show when a pair of colours have been swapped.

Blue	Red	Yellow	Orange	Green	Violet	Indigo

Quick Sort

Choose the item at the midpoint of the list to be the first pivot.
 If the list has an even number of items, the pivot should be to the right of the middle.
 Write down all the items that are less than the pivot, keeping their order, in a sub list.
 Write down the pivot.
 Write down the remaining items (greater than the pivot, keeping their order) in a sub list.
 Repeat for each sub list of size greater than one.
 When all items have been chosen as pivots or any subgroups only contain one item, stop.

**Example 5**

Use the quick sort algorithm to sort the following numbers, from least to greatest.

53, 24, 45, 38, 4, 18, 63

Answer: The first pivot is 38, the number in the middle. (We use a circle to show the pivot.)

53, 24, 45, 38, 4, 18, 63

We write the numbers that are less than 38 **to the left of 38** (in their original order), and we write the numbers that are greater than 38 **to the right of 38** (again, in their original order). (We use a square to show that we have processed the pivot.)

24, 4, 18, 38, 53, 45, 63

The pivots of the red and blue sub lists are 4 and 45, respectively.

24, 4, 18, 38, 53, 45, 63

Next, we consider the **red** sub list. There are no red numbers less than 4, so we write the pivot 4, followed by the **two numbers greater than 4**. The pivot of these two numbers is 18, as this is on the right.

4, 24, 18, 38, 53, 45, 63

Next, we consider the **purple** sub list. There are no purple numbers less than 18, so we write the pivot 18, followed by the number greater than 18.

4, 18, 24, 38, 53, 45, 63

Next, we consider the **blue** sub list. There are no blue numbers less than 45, so we write the pivot 45, followed by the **two numbers greater than 45**. The pivot of these two numbers is 63, as this is on the right.

4, 18, 24, 38, 45, 53, 63

To finish, we consider the **green** sub list. We write the number less than the pivot 63, and then we write the pivot. This finishes the sort, as there are no remaining sub lists of size greater than one.

4, 18, 24, 38, 45, 53, 63

Example 6

Use the quick sort algorithm to sort the following numbers, from least to greatest.

29, 8, 43, 85, 32, 24, 14, 48

Answer: The first pivot is 32, as this is to the right of the two numbers in the middle.

29, 8, 43, 85, 32, 24, 14, 48

We write the numbers that are less than 32 **to the left of 32** (in their original order), and we write the numbers that are greater than 32 **to the right of 32** (again, in their original order).

29, 8, 24, 14, 32, 43, 85, 48

The pivots of the red and blue sub lists are 24 and 85, respectively.

29, 8, 24, 14, 32, 43, 85, 48

Next, we consider the **red** sub list. We write the numbers that are less than 24 **to the left of 24** (in their original order), and we write the number greater than 24 to the right of 24. The pivot of 8 and 14 is 14, as this is on the right.

8, 14, 24, 29, 32, 43, 85, 48

Next, we consider the **purple** sub list. We write the number less than 14 on the left, and then we write the pivot 14.

8, 14, 24, 29, 32, 43, 85, 48

Next, we consider the **blue** sub list. We write the numbers that are less than 85 **to the left of 85** (in their original order), and then we write the pivot 85. The pivot of 43 and 48 is 48, as this is on the right.

8, 14, 24, 29, 32, 43, 48, 85

To finish, we consider the **green** sub list. We write the number less than the pivot 48, and then we write the pivot. This finishes the sort, as there are no remaining sub lists of size greater than one.

8, 14, 24, 29, 32, 43, 48, 85

Exercise 16

Use the quick sort algorithm to sort the following numbers, from least to greatest.

32, 65, 94, 34, 54, 72, 5

Exercise 17

Use the quick sort algorithm to sort the following numbers, from least to greatest.

8, 2, 6, 4, 9, 3

Exercise 18

Use the quick sort algorithm to sort the following numbers, from least to greatest.

37, 81, 29, 94, 23, 31, 55, 70, 19, 34, 42, 48

Exercise 19

Use the quick sort algorithm to sort the following numbers, from greatest to least.

37, 81, 29, 94, 23, 31, 55, 70, 19, 34, 42, 48

Exercise 20

Use the quick sort algorithm to sort the following letters, in alphabetical order.

C O S I N E

Exercise 21

Use the quick sort algorithm to sort the following letters, in alphabetical order.

A L A N T U R I N G

Exercise 22

Use the quick sort algorithm to sort the following letters, in **reverse** alphabetical order.

A L A N T U R I N G

Exercise 23

Use the quick sort algorithm to sort the following colours, in rainbow colour (starting with **red**).

Blue	Red	Yellow	Orange	Green	Violet	Indigo
------	-----	--------	--------	-------	--------	--------

Pop Quiz: Sorting Algorithms

(1) Use the bubble sort algorithm to sort the following letters, in alphabetical order.
Use a new row to show when a pair of letters have been swapped.

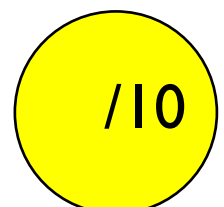
[5]

F	O	U	R	I	E	R

(2) Use the quick sort algorithm to sort the following letters, in alphabetical order.

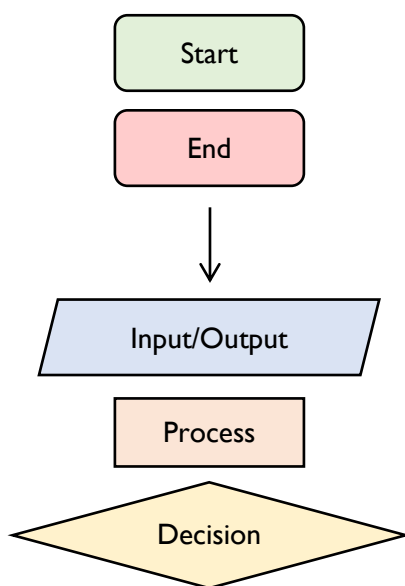
[5]

F O U R I E R



Flow Charts

It is possible to use a flow chart to represent a specific algorithm. Here are some common flow chart symbols.



The start of a flow chart.

The end of a flow chart.

An arrow that moves from one part of a flow chart to another.

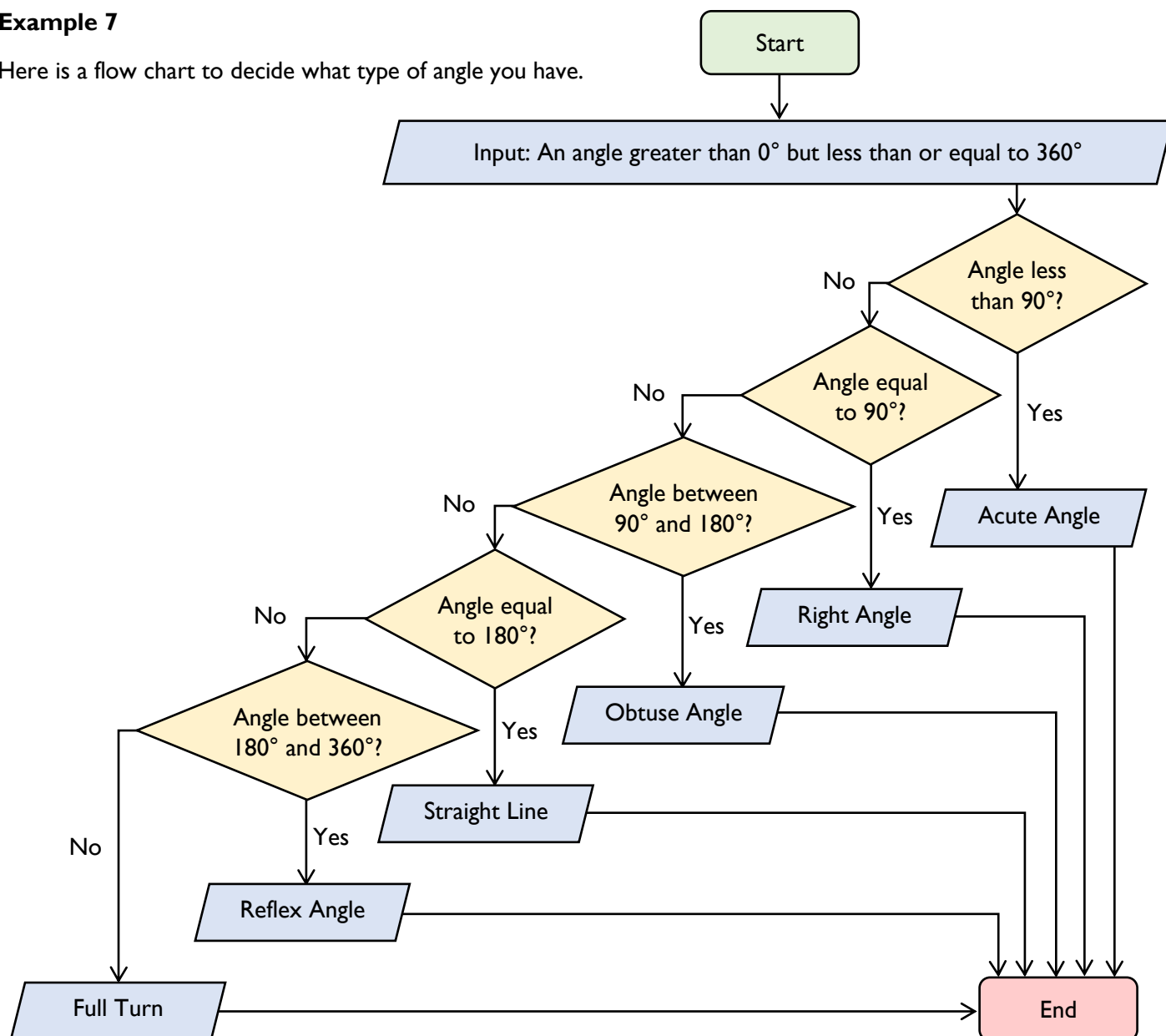
Something that goes into the algorithm, or comes out.

An instruction or command.

Something that provides a “yes” or “no” answer.

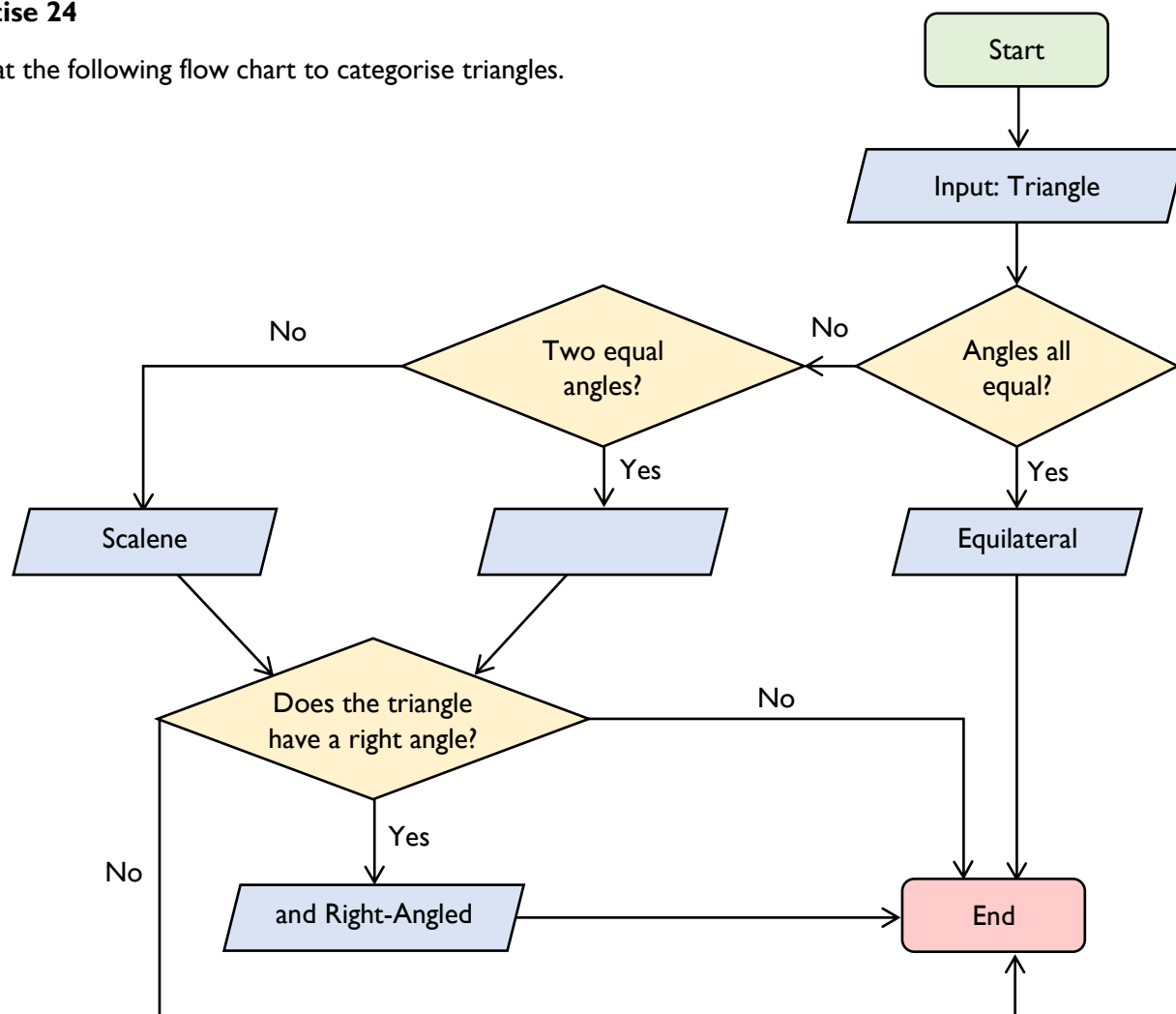
Example 7

Here is a flow chart to decide what type of angle you have.



Exercise 24

Look at the following flow chart to categorise triangles.

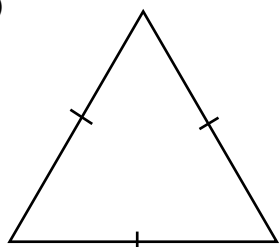


(a) Complete the empty output box.

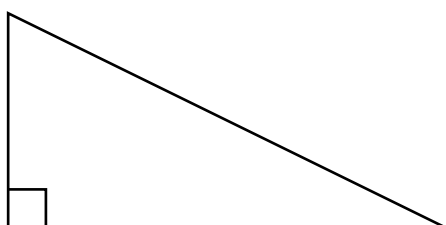
(b) Why does an equilateral triangle not need a check to see whether it is a right-angled triangle?

(c) What would be the output of the algorithm for the following triangles?

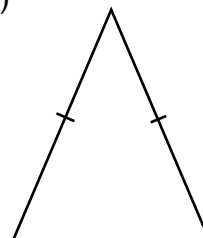
(i)



(ii)



(iii)

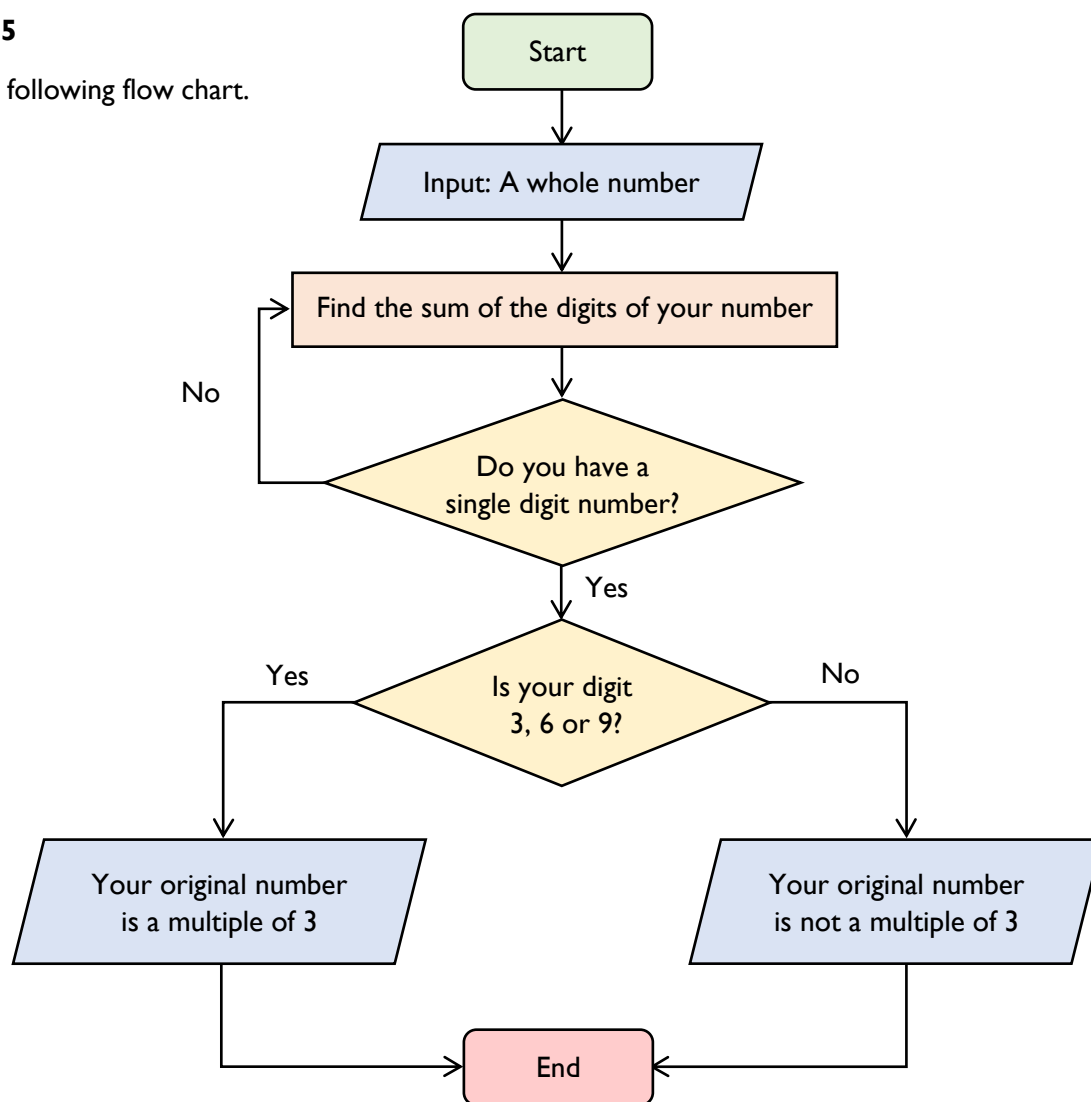


(iv) A triangle with angles 30° , 80° , 70°

(v) A triangle with angles 45° , 90° , 45°

Exercise 25

Look at the following flow chart.



(a) What is the output of the algorithm if the input number is...

(i) 387

(ii) 872

(iii) 3,489

(iv) 9,238,238

(b) What happens if the input is 0? How could you change the algorithm to deal with this problem?

.....

.....

.....

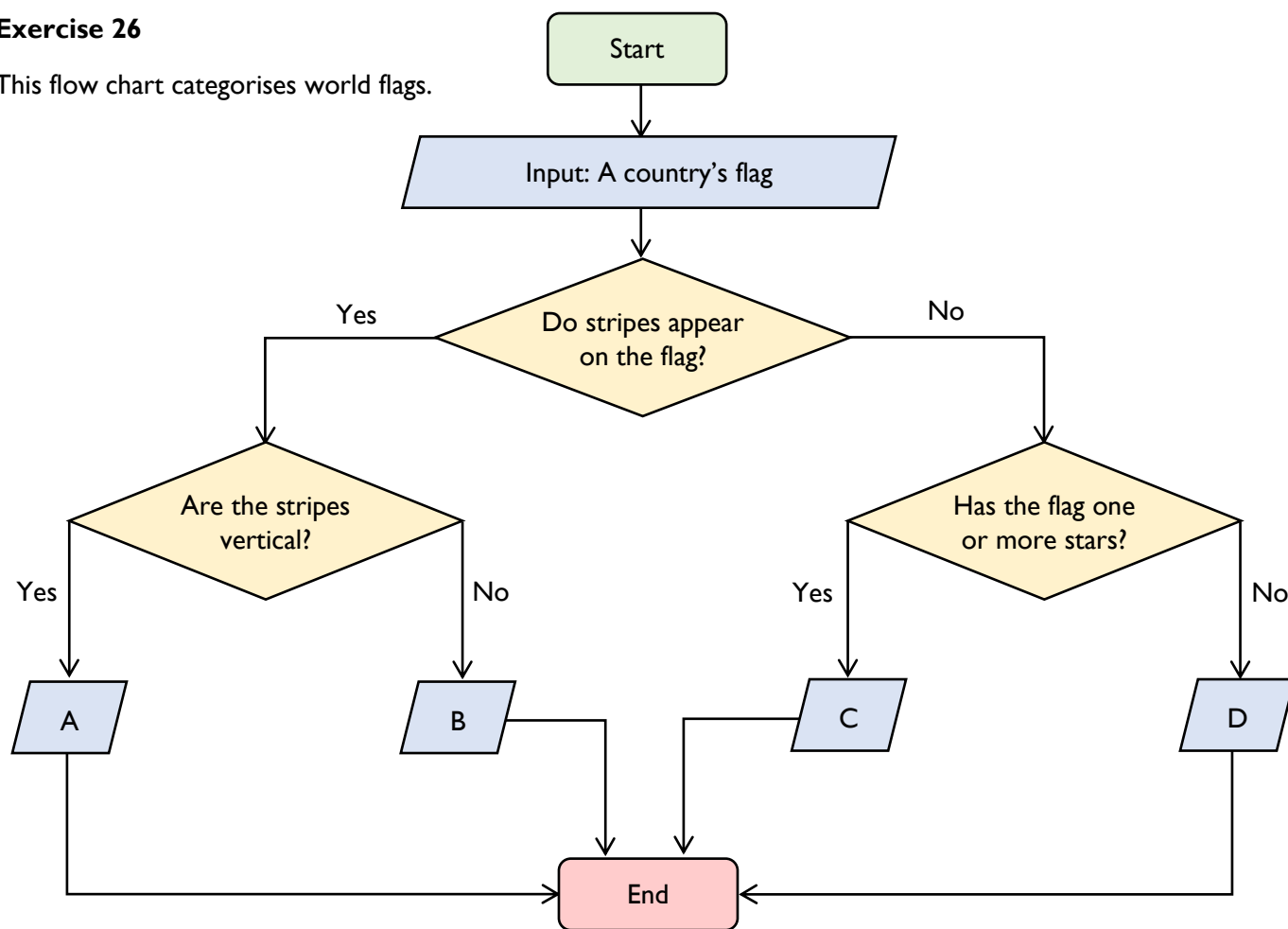
(c) What changes would be needed to draw a flow chart to check whether a positive integer is a multiple of 9?

.....

.....

Exercise 26

This flow chart categorises world flags.



(a) What would be the output of the flow chart for the following flags?



(b) Name another country whose flag would give the output

- (i) A (ii) B (iii) C (iv) D

(c) Do all flags containing a star receive the output C?

.....

.....

A list of instructions

It is possible to write an algorithm using a set of instructions.

Example 8

Here is an algorithm (a recipe!) for making Yorkshire Puddings.

- Turn the oven on at a temperature of 230°C / fan 210°C / gas 8.
- Pour some sunflower oil into the holes of a Yorkshire pudding tin that holds 12 puddings. Place the tin into the oven to heat up.
- To make the batter, put 140 g of plain flour into a bowl. Mix in 4 eggs until the mixture becomes smooth.
- Gradually, add 200 ml of milk, beating the mixture until the lumps have disappeared. Add salt and pepper to taste.
- Pour the batter into a jug, before removing the hot tin from the oven. Carefully pour the batter equally into the holes.
- Put the tin back into the oven to bake, for 20-25 minutes (or until the puddings have risen and turned brown). Do not open the oven door during baking.

**Exercise 27**

Here is an algorithm for finding the prime numbers between 1 and 100.

- Cross out the number 1.
- Repeat the following instructions until all numbers have a circle or cross on them.
 - Look for the least number left without a circle or a cross. Circle this number.
 - Cross all multiples of this number.

(a) Use the algorithm to find the prime numbers between 1 and 100, using the grid on the right to help.

(b) How many one digit prime numbers exist?

.....

(c) How many prime numbers are there between 1 and 100?

.....

(d) How many even prime numbers exist?

.....

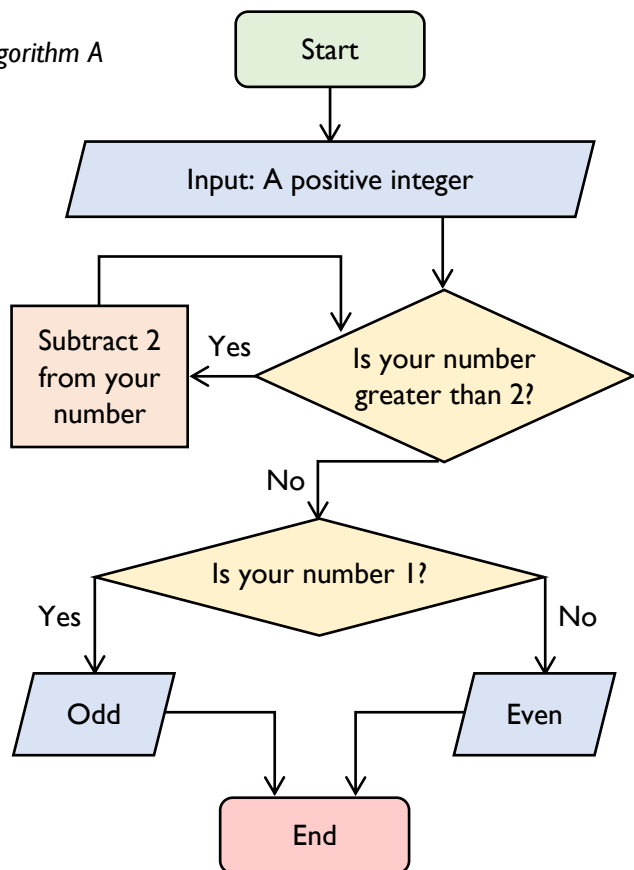
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

This algorithm is known as
“the sieve of Eratosthenes”.

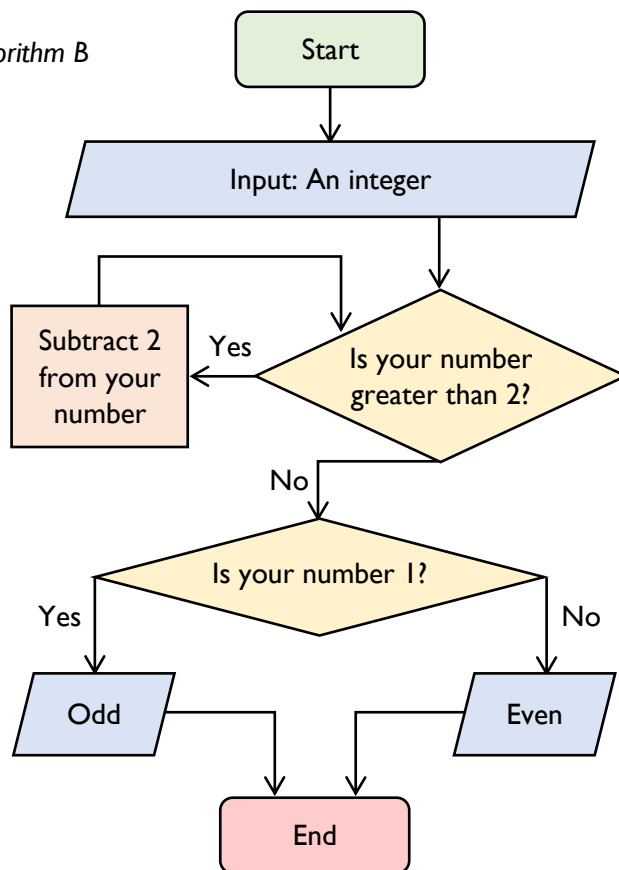
Types of Algorithms**Exercise 28**

Consider the following four algorithms.

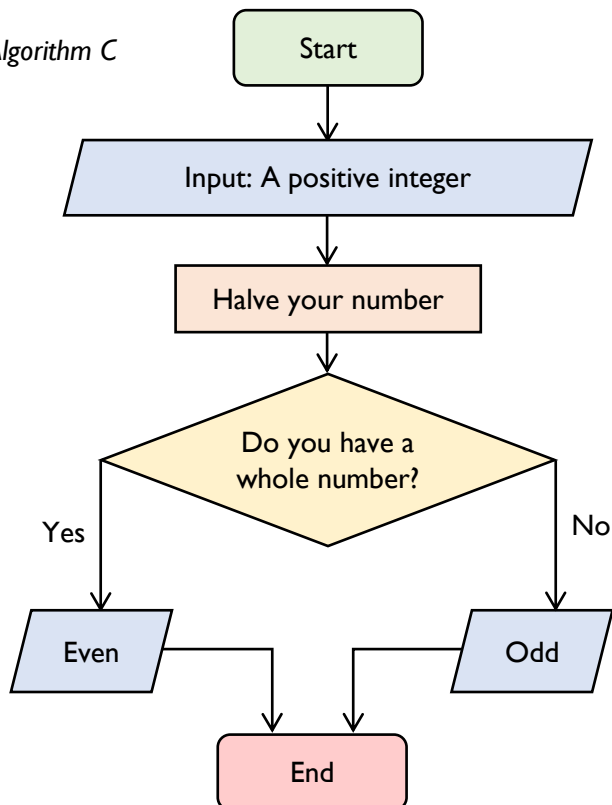
Algorithm A



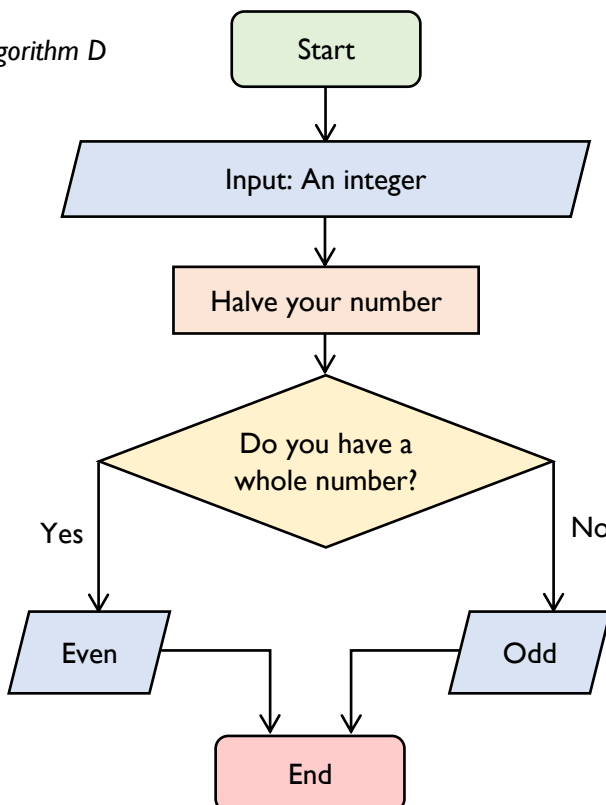
Algorithm B



Algorithm C



Algorithm D



(a) Tick the following boxes to show which algorithms allow you to use the following numbers.

Algorithm	Allows 5?	Allows 0?	Allows -5?
A			
B			
C			
D			

(b) What is the output of the algorithms for the following inputs?
(If an output is not possible, write "No output".)

Algorithm	5	6	0	-4	-7
A					
B					
C					
D					

(c) The algorithms have been designed to categorise numbers as odd numbers or even numbers.
Which algorithms do this correctly – for every possible input?

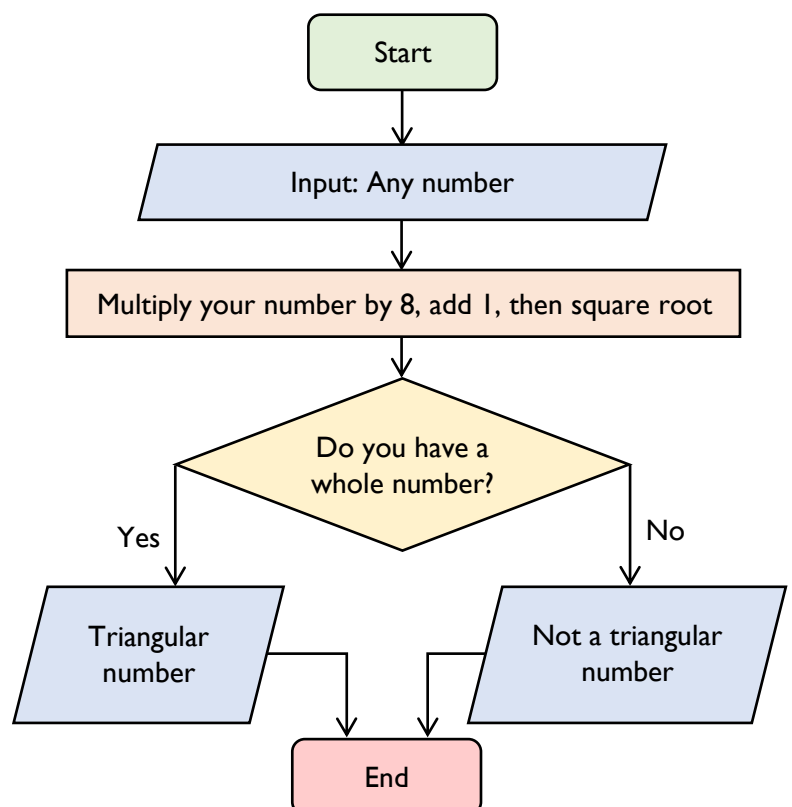
WJEC Definitions

Term	Definition
Finiteness	The algorithm terminates after a finite number of steps.
Definiteness	Each step in the algorithm is precisely defined and unambiguous.
Correctness	The algorithm solves the given problem without errors or mistakes.

Example 9

Look at the algorithm on the right.

- The algorithm is finite, because we always reach the end of the algorithm, regardless of what the input is.
- The algorithm is definite, because it is not possible to misinterpret the instructions.
- The algorithm is not correct, as it would say that the number 1.875 (for example) is a triangular number, which isn't true. By changing the input to "Any positive whole number", the algorithm would then be correct.



Example 10

Look again at the recipe for Yorkshire puddings on page 29.

It would be possible to argue that the algorithm isn't definite because it is not clear what type of eggs need to be used.

- Should the eggs be medium sized, or large?
- From which animal should the eggs come from?
- Should the eggs be used straight from the fridge, or allowed to reach room temperature first?

Exercise 29

Write down another reason why the recipe for Yorkshire puddings isn't a definite algorithm.

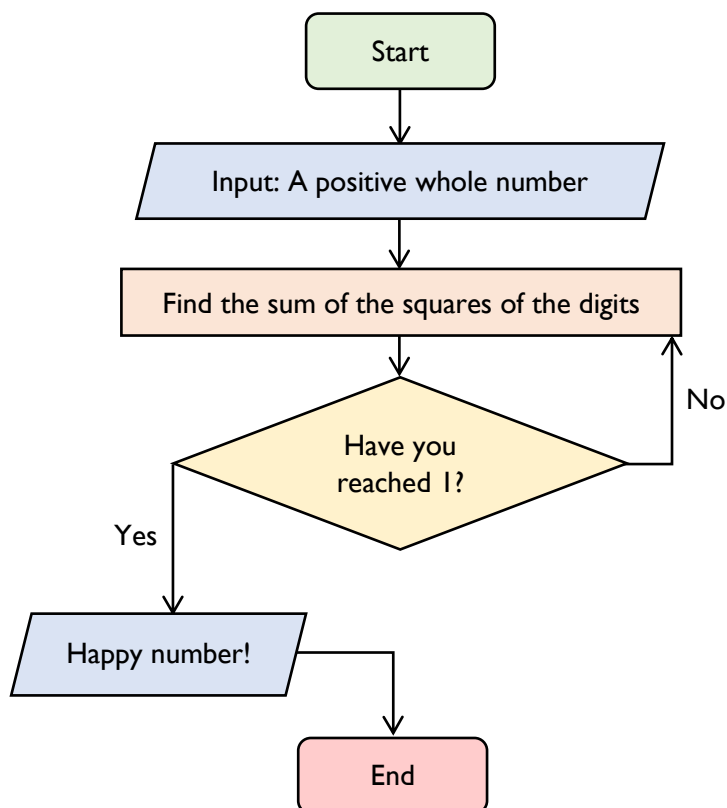
.....

.....

Exercise 30

A number is a **happy number** if it is possible to reach 1 in the sequence of numbers formed by finding the sum of the squares of the digits of the current number. For example, 13 is a happy number, because $1^2 + 3^2 = 10$, and $1^2 + 0^2 = 1$.

Here is an algorithm to try to find whether a number is happy, or not.



(a) What happens if the input to the algorithm is 49?

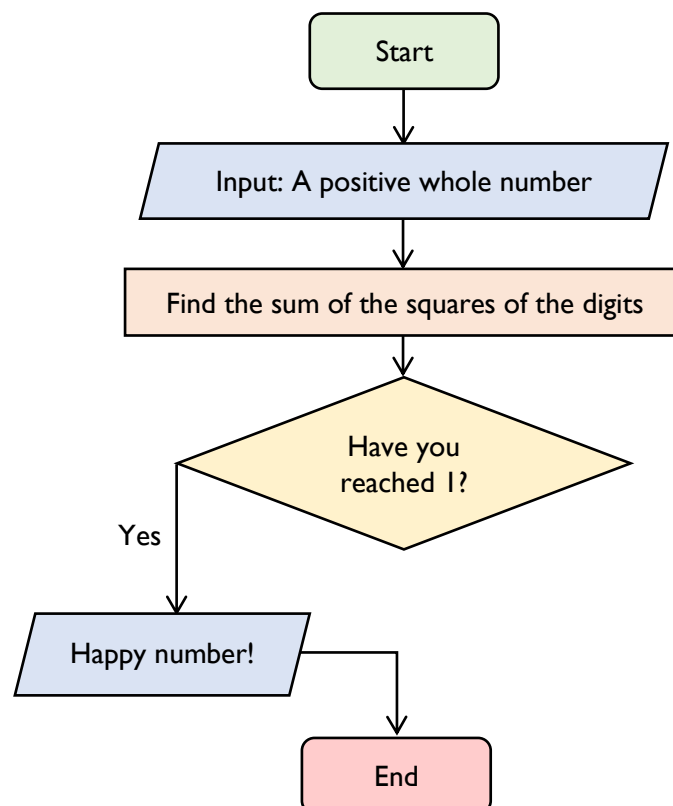
(b) What happens if the input to the algorithm is 4?

(c) After completing part (b), explain why the algorithm is not a finite algorithm.

.....

.....

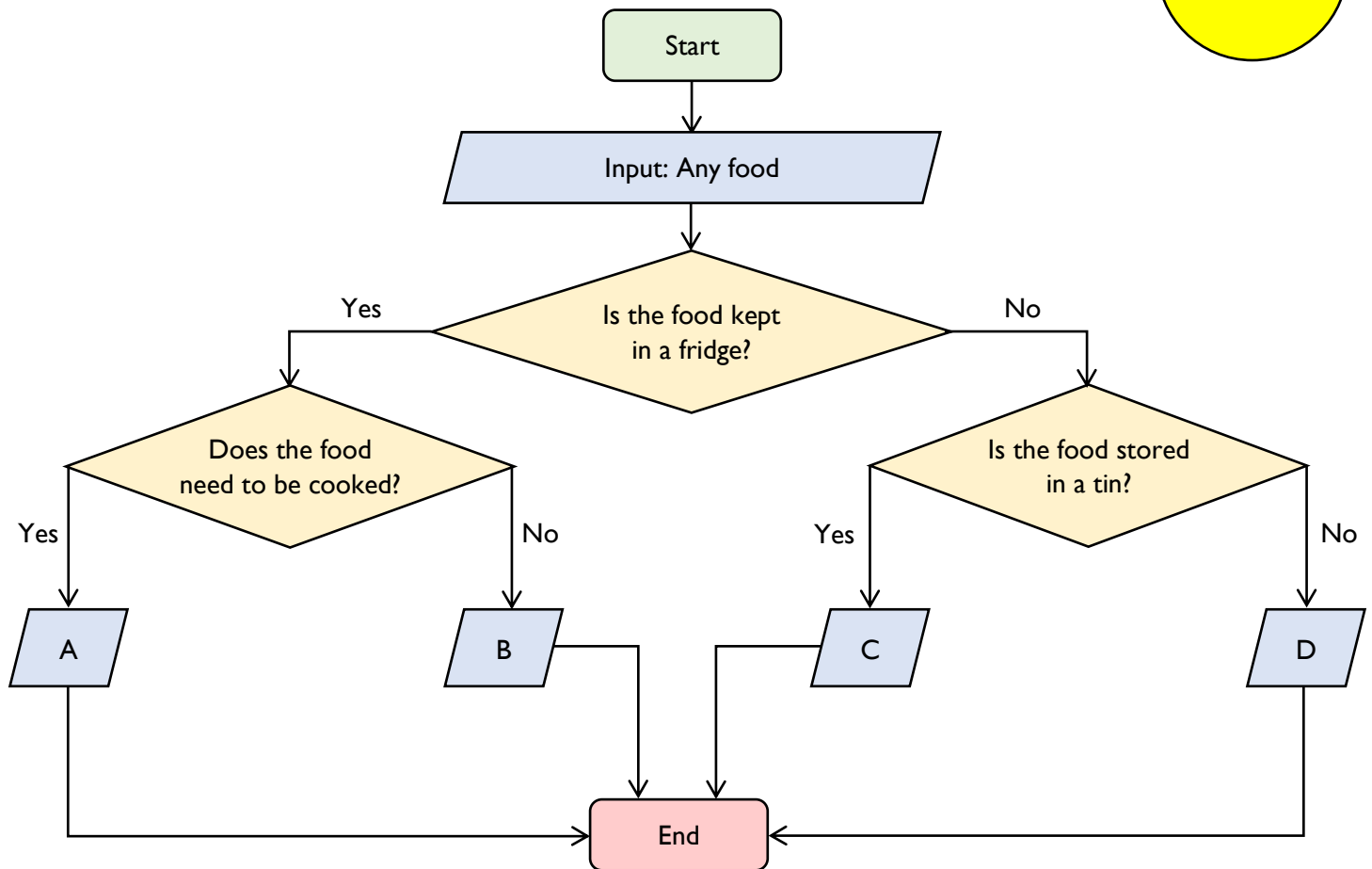
(d) Modify the algorithm so that it becomes a finite algorithm.



Pop Quiz: Flow Charts; Types of Algorithms

/12

(1) The following flow chart sorts food commonly kept in a kitchen.



(a) What would be the output of the algorithm for the following foods?

[4]

Tin of soup



Packet of ham



Packet of crisps



Minced beef



Coco Pops



--	--	--	--	--

(b) Give an example of a different food that would receive the following outputs.

[4]

A	B	C	D
---	---	---	---

(c) Give an example of a food that could receive more than one output.

[1]

.....

(d) After answering part (c), is the algorithm definite or not? [1]

(e) Is the algorithm a finite algorithm? [1]

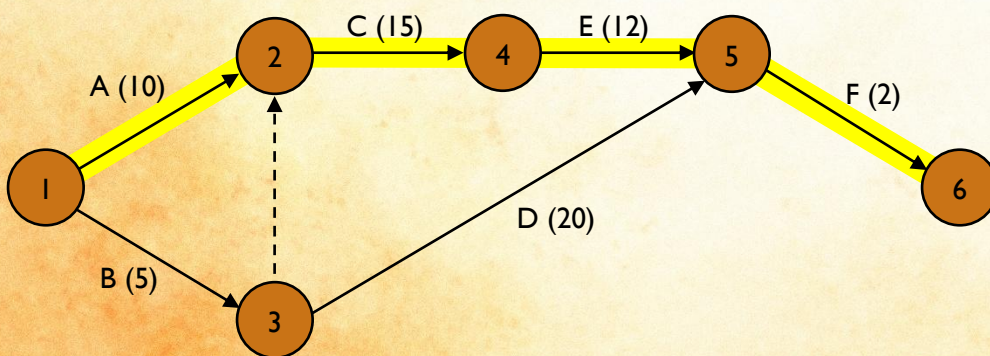
(f) Explain what is meant by a 'correct' algorithm.

..... [1]



Critical Path

Analysis



Name: _____

Background

What is the work?

Working with activities on arc networks, precedence tables and dummies. Using these to find possible critical paths. Interpreting Gantt charts.

What are the prerequisites?

GCSE Work: Ordering numbers; addition and subtraction.

Where does this lead to?

Applications: Timetabling activities in various projects in areas such as construction, engineering and information technology.

Theory

Here is a recipe for making “Jerk BBQ Cauli” by the company Simply Cook:

JERK BBQ CAULI

serves 2

Ingredients

- 1 small cauliflower
finely cut into small florets
leaves reserved (optional)
- 200ml tin coconut milk
- 120g basmati rice
- 210g tin kidney beans
don't drain!

Storecupboard

- olive oil

Let's Cook! Heat the oven to 200C / 180C fan / gas 6.

1. Prep the cauliflower

- Toss the **cauliflower** and its **leaves** (optional) with some oil and put on a baking tray in the oven for 15 mins.
- Mix the **Jerk Seasoning** and the **Jerk Paste** together in a bowl with a drop of oil and keep aside.

2. Meanwhile, make the rice

- Bring 100ml water to the boil in a lidded, non-stick saucepan, then mix in the **Rice Stock**.
- Stir in the **coconut milk**, then add the **rice** and the **kidney beans** along with their tin liquid to the pan. Mix well, gently bring to the boil, then cover, turn the heat down to low, and let it simmer for 15 mins. Then turn off the heat and keep covered until ready to serve.

3. Season the cauliflower

- Take the cauliflower out of the oven and tip the Jerk marinade mixture onto the florets. Mix well to fully coat them in the spices, then return to the oven for 10 more mins, (keeping the optional crisp cauli leaves aside).
- If you're having **sweetcorn** alongside, either toast it in a dry frying pan until lightly charred or just season it with pepper and some olive oil.

Serve! Serve the roasted cauliflower over your rice and peas with sweetcorn on the side (if having).

Make it your own!

Extras

- 200g tin sweetcorn

Swaps

- Cauliflower for quorn pieces

Make it milder

- Go easy on the Jerk marinade
- Serve with coleslaw

Tip

“Rice and peas” is the traditional name for this Caribbean rice dish. Kidney beans are traditional, but you could use gungo peas, black beans or black-eyed beans.

Take me shopping

serves 2

- ☐ 1 small cauliflower
- ☐ 200ml tin coconut milk
- ☐ 120g basmati rice
- ☐ 210g tin kidney beans

Extras

- ☐ 200g tin sweetcorn

JERK BBQ CAULI

Ingredients: 1 small cauliflower, 200ml tin coconut milk, 120g basmati rice, 210g tin kidney beans, 200g tin sweetcorn, olive oil, Jerk Seasoning, Jerk Paste.

Instructions: Heat the oven to 200C / 180C fan / gas 6. Toss the cauliflower and its leaves (optional) with some oil and put on a baking tray in the oven for 15 mins. Mix the Jerk Seasoning and the Jerk Paste together in a bowl with a drop of oil and keep aside. Bring 100ml water to the boil in a lidded, non-stick saucepan, then mix in the Rice Stock. Stir in the coconut milk, then add the rice and the kidney beans along with their tin liquid to the pan. Mix well, gently bring to the boil, then cover, turn the heat down to low, and let it simmer for 15 mins. Then turn off the heat and keep covered until ready to serve. Take the cauliflower out of the oven and tip the Jerk marinade mixture onto the florets. Mix well to fully coat them in the spices, then return to the oven for 10 more mins, (keeping the optional crisp cauli leaves aside). If you're having sweetcorn alongside, either toast it in a dry frying pan until lightly charred or just season it with pepper and some olive oil. Serve the roasted cauliflower over your rice and peas with sweetcorn on the side (if having).

Dr Evans cooking the recipe! →



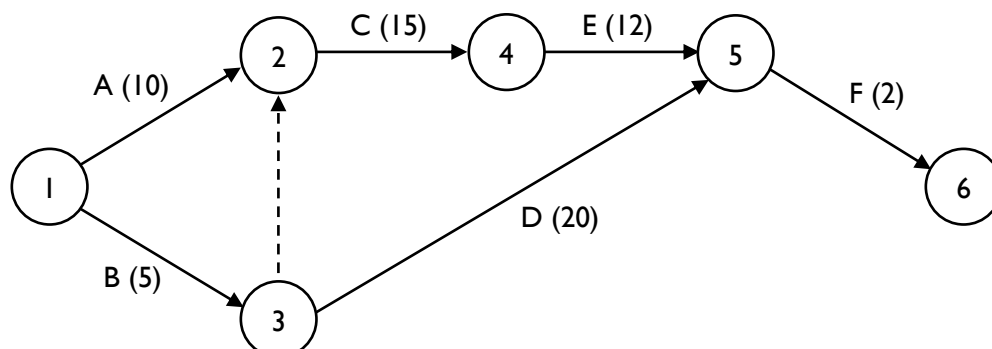
It would be possible to make the recipe by completing the following **activities**.

Activity		Time (in minutes)
A	Heat the oven to 200°C / fan 180°C / gas 6.	10
B	Collect and prepare the ingredients (cut the cauliflower).	5
C	Prepare the cauliflower (coat in oil, put in the oven), mix the paste.	15
D	Make the rice.	20
E	Finish cooking the cauliflower.	12
F	Serve.	2

The following **precedence table** shows which activities must be completed before others can start.

The activity	A, B	must finish before activity	C	can start.
	B		D	
	C		E	
	D, E		F	

The following **arc network** represents the project. Each **arc** (or arrow) represents an activity, with the label showing the letter of the activity and its completion time in brackets. The **nodes** (circles) show the start and end of activities and are considered to be **events**.



We can represent the above network using the following table.

Activity	Start node	End node	Completion time (in minutes)
A	1	2	10
B	1	3	5
C	2	4	15
D	3	5	20
E	4	5	12
F	5	6	2

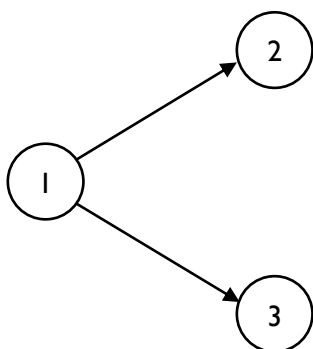
Notice that a **dummy activity** is included in the arc network, going from node 3 to node 2. This shows that activity B must be completed before activity C can start, and is shown using a dotted line.

Exercise 1

The following table defines the activities in a small project.

Activity	Start node	End node	Completion time (in minutes)
A	1	2	5
B	1	3	10
C	2	4	10
D	3	5	15
E	4	6	15
F	5	6	10
G	6	7	5

(a) Complete the arc network for the project.



(b) Complete the precedence table for the project.

The activity	A	must finish before activity	C	can start.

(c) Complete the following table to show the **earliest** time each activity can start.

Activity	Earliest starting time (in minutes)
A	0
B	
C	5
D	
E	
F	
G	

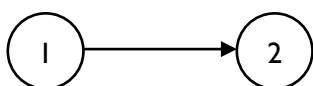
(d) What is the earliest time the entire project can finish?

Exercise 2

The following table defines the activities in a small project.

Activity	Start node	End node	Completion time (in minutes)
A	1	2	5
B	2	3	8
C	2	4	4
D	3	5	7
E	4	5	14
F	5	6	5

(a) Complete the arc network for the project.



(b) Complete the precedence table for the project.

The activity	A	must finish before activity	B, C	can start.

(c) Complete the following table to show the **latest** time each activity can start.

Activity	Latest starting time (in minutes)
A	0
B	8
C	
D	
E	
F	

(d) What is the earliest time the entire project can finish?

.....

Exercise 3

Here is a recipe for cooking a baked potato with tuna mayo.

Activity		Time (in minutes)
A	Prick the potato with a fork before covering with kitchen roll.	1
B	Cook the potato in a microwave.	10
C	Drain a tin of tuna. In a small bowl, mix the tuna with 2–4 tablespoons of mayonnaise (to taste), and black pepper.	5
D	Cut the potato before adding the tuna mayo. Enjoy!	1

The following precedence table shows which activities must be completed before others can start.

The activity	A	must finish before activity	B, C	can start.
	B, C		D	

(a) Draw an arc network for the project.

(b) Complete the following table to show the **earliest** time each activity can start.

Activity	Earliest starting time (in minutes)
A	
B	
C	
D	

(c) Complete the following table to show the **latest** time each activity can start.

Activity	Latest starting time (in minutes)
A	
B	
C	
D	

(d) How much time does the cook spend doing nothing during the project?

.....

(e) What is the earliest time the entire project can finish?

.....

Exercise 4

You have a large tin of baked beans, two slices of bread, and access to a toaster, Pyrex jug, microwave, small saucepan and stove.



- (a) Write a recipe for making beans on toast.
(You do not have to use all the rows in the table below.)

Activity		Time (in minutes)

- (b) Complete a precedence table for your project. (Again, not all rows need to be used.)

The activity		must finish before activity		can start.

- (c) Draw an arc network for the project.

- (d) Complete the following table to define the activities.

Activity	Start node	End node	Completion time (in minutes)

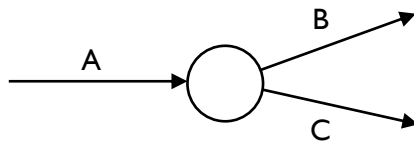
- (e) What is the earliest time the entire project can finish?

.....

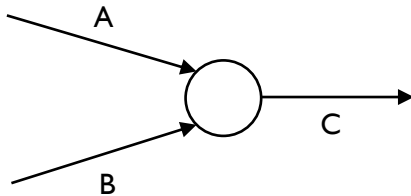
Arc Networks

To create an arc network, the key question is “which activities can start now?”.

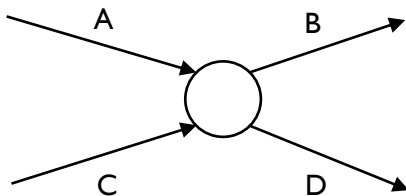
The following situations can appear in an arc network.



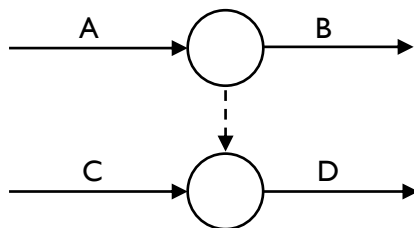
A must finish before B or C can start.



A and B must finish before C can start.



A and C must finish before B or D can start.



A must finish before B can start.

A and C must finish before D can start.

The last situation uses a dummy activity (the dotted line). Dummy activities are useful to show precedence relationships which are otherwise hard to express using the project's activities.

Note the difference between the final two situations above. In the first, C must finish before B can start. In the second, there is no connection between C finishing and B starting.

Early and late event times

In the previous exercises, you calculated the earliest and latest start time for activities. We can also consider when events can start. Formally,

- An **early event time** is the earliest time of arrival at an event allowing for the completion of all preceding activities.
- A **late event time** is the latest time that an event can be left without extending the time needed for the project.

Float

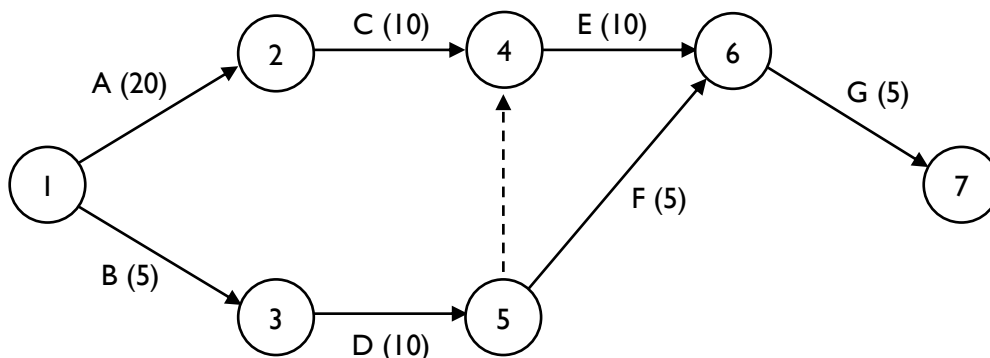
Consider an activity going from event i to event j .

The **float** of the activity (or the *slack* of the activity) is

$$\text{Late event time for event } j - \text{early event time for event } i - \text{length of the activity}$$

Example 1

The following arc network represents a small project, where the times shown are in minutes.



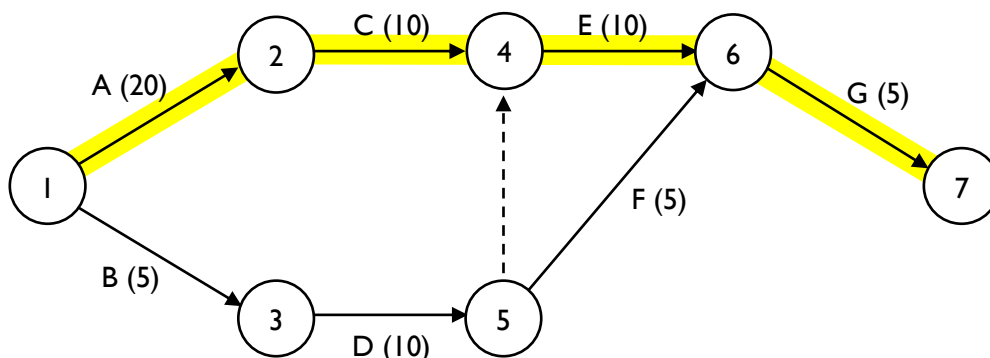
The table below shows information about the seven activities in the project.

Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in minutes)	Float
A	1	2	20	0	20	0
B	1	3	20	0	5	15
C	2	4	30	20	10	0
D	3	5	30	5	10	15
E	4	6	40	30	10	0
F	5	6	40	15	5	20
G	6	7	45	40	5	0

Critical activities

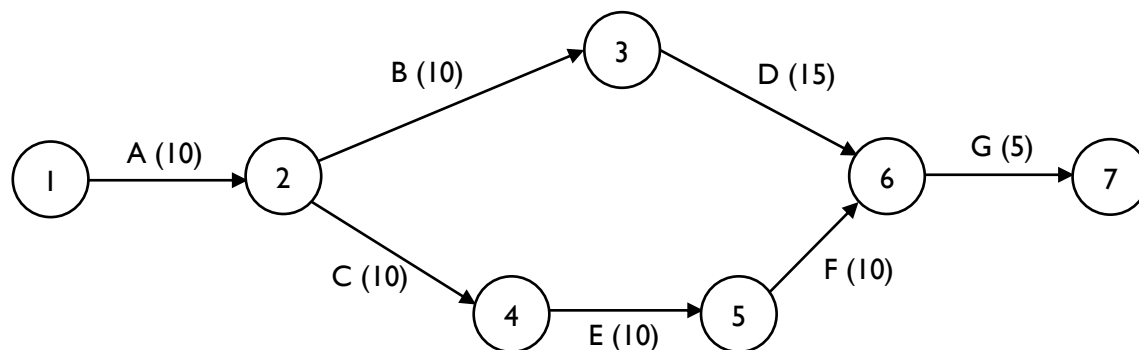
Any activity with a float of zero is considered to be a **critical activity**. An increase in the length of a critical activity leads to a corresponding increase in the length of the entire project.

A **critical path** in a network is a path from the start node to the end node that uses only critical activities. For the network in Example 1, here (in yellow) is the critical path:



Exercise 5

The following arc network represents a small project, where the times shown are in minutes.



(a) Complete the precedence table for the project.

The activity	A	must finish before activity	B, C	can start.

(b) Complete the following table.

Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in minutes)	Float
A						
B						
C						
D						
E						
F						
G						

(c) Draw the critical path on the network.

(d) What is the earliest time the entire project can finish?

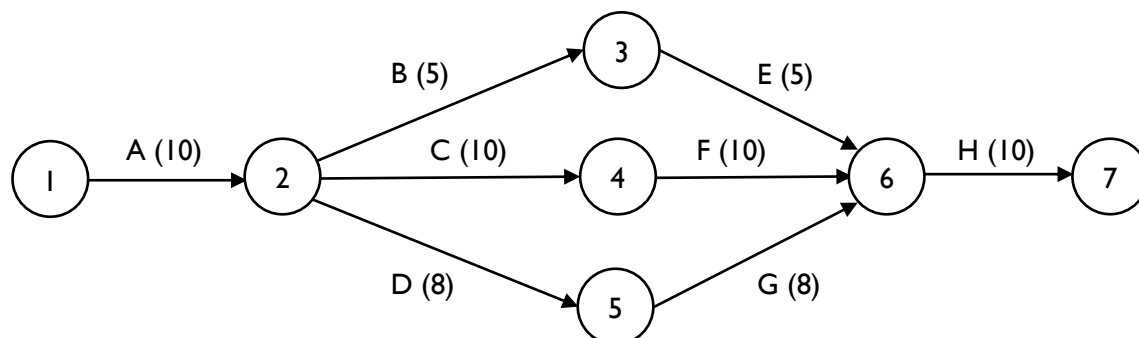
.....

(e) In reality, activity E took 8 minutes, and activity D took 17 minutes. After how many minutes did the project finish?

.....

Exercise 6

The following arc network represents a small project, where the times shown are in minutes.



(a) Complete the precedence table for the project.

The activity	A	must finish before activity	B, C, D	can start.

(b) Complete the following table.

Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in minutes)	Float
A						
B						
C						
D						
E						
F						
G						
H						

(c) Draw the critical path on the network.

(d) What is the earliest time the entire project can finish?

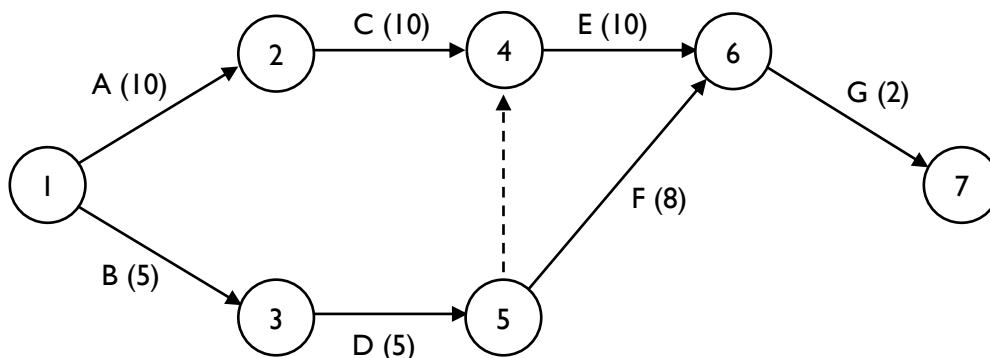
.....

(e) In reality, activity B took 25 minutes, and activity H took 15 minutes. After how many minutes did the project finish?

.....

Exercise 7

The following arc network represents a small project, where the times shown are in hours.



(a) Complete the precedence table for the project.

The activity		must finish before activity		can start.

(b) Complete the following table.

Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in hours)	Float
A						
B						
C						
D						
E						
F						
G						

(c) Draw the critical path on the network.

(d) What is the earliest time the entire project can finish?

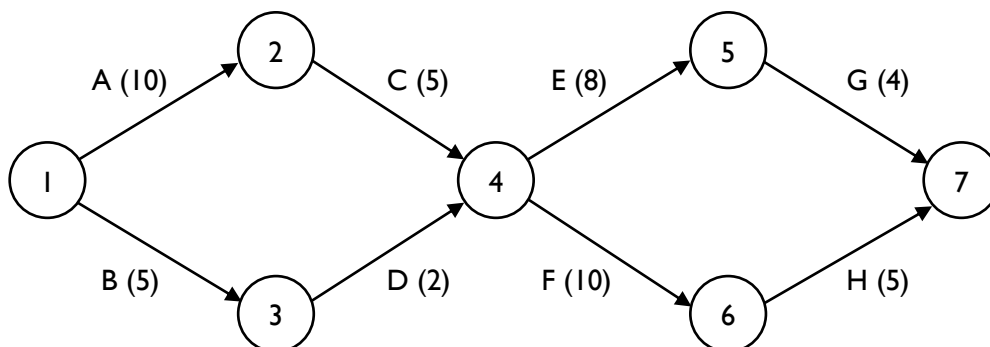
.....

(e) In reality, activity B took 10 hours, and activity F took 10 hours. What is the new float for activity D?

.....

Exercise 8

The following arc network represents a small project, where the times shown are in hours.



(a) Complete the precedence table for the project.

The activity		must finish before activity		can start.

(b) Complete the following table.

Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in hours)	Float
A						
B						
C						
D						
E						
F						
G						
H						

(c) Draw the critical path on the network.

(d) What is the earliest time the entire project can finish?

.....

(e) In reality, **every** activity took 2 hours more than expected. After how many hours did the project finish?

.....

Exercise 9

Here is a recipe for cooking pasta with broccoli.

Activity		Time (in minutes)
A	Put 2 saucepans full of salted water to boil on the stove.	1
B	Put 200 g of pasta of your choice into one of the saucepans to cook.	15
C	Cut the broccoli into florets. Put the florets into the other saucepan.	12
D	Grate cheese of your choice to put on top of the pasta.	2
E	Drain the pasta and broccoli before mixing them. Add the cheese before serving!	2

The following precedence table shows which activities must finish before others can begin.

The activity	A	must finish before activity	B, C, D	can start.
	B, C, D		E	

(a) Draw an arc network for the project.

(b) Complete the following table.

Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in minutes)	Float
A						
B						
C						
D						
E						

(c) Draw the critical path on the network.

(d) What is the earliest time the entire project can finish?

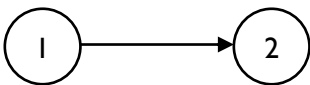
.....

Exercise 10

The following table defines the activities in a small project.

Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in minutes)	Float
A	1	2			10	
B	2	3			5	
C	2	4			10	
D	3	5			10	
E	4	6			15	
F	5	7			5	
G	6	7			5	
H	7	8			10	

(a) Complete the arc network for the project.



(b) Complete the precedence table for the project.

The activity		must finish before activity		can start.

(c) Complete the table at the top of the page.

(d) Draw the critical path on the network.

(e) What is the earliest time the entire project can finish?

.....

(f) In reality, activity B took 10 minutes, and activity C took 12 minutes. What is the new float for activity F?

.....

Gantt Charts

A **Gantt chart** shows when different activities run (or can run) in a project.

Example 2

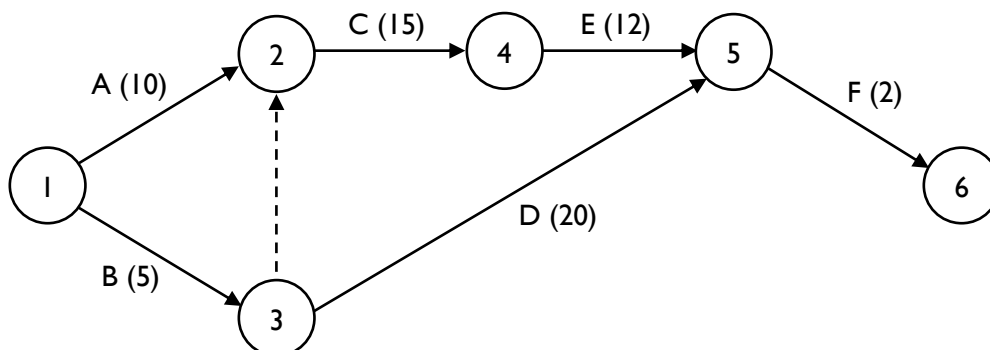
Let us consider again the example of cooking the 'Jerk BBQ Cauli' from earlier in the workbook.

Activity		Time (in minutes)
A	Heat the oven to 200°C / fan 180°C / gas 6.	10
B	Collect and prepare the ingredients (cut the cauliflower).	5
C	Prepare the cauliflower (coat in oil, put in the oven), mix the paste.	15
D	Make the rice.	20
E	Finish cooking the cauliflower.	12
F	Serve.	2

Here is the **precedence table**:

The activity	A, B	must finish before activity	C	can start.
	B		D	
	C		E	
	D, E		F	

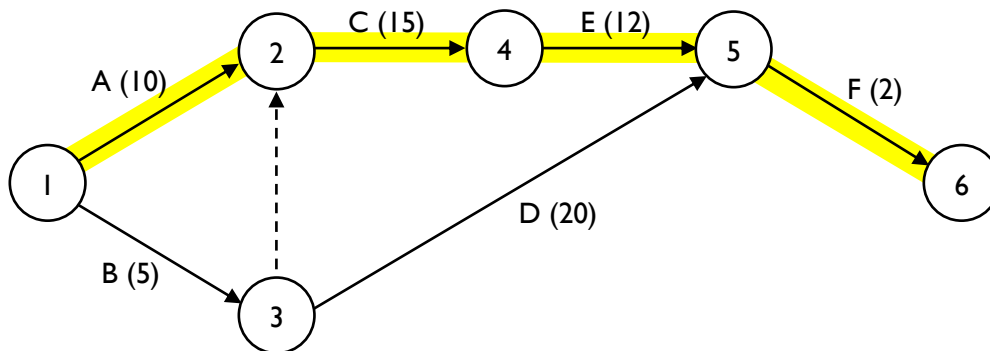
Here is the **arc network**:



Here is the **information table**:

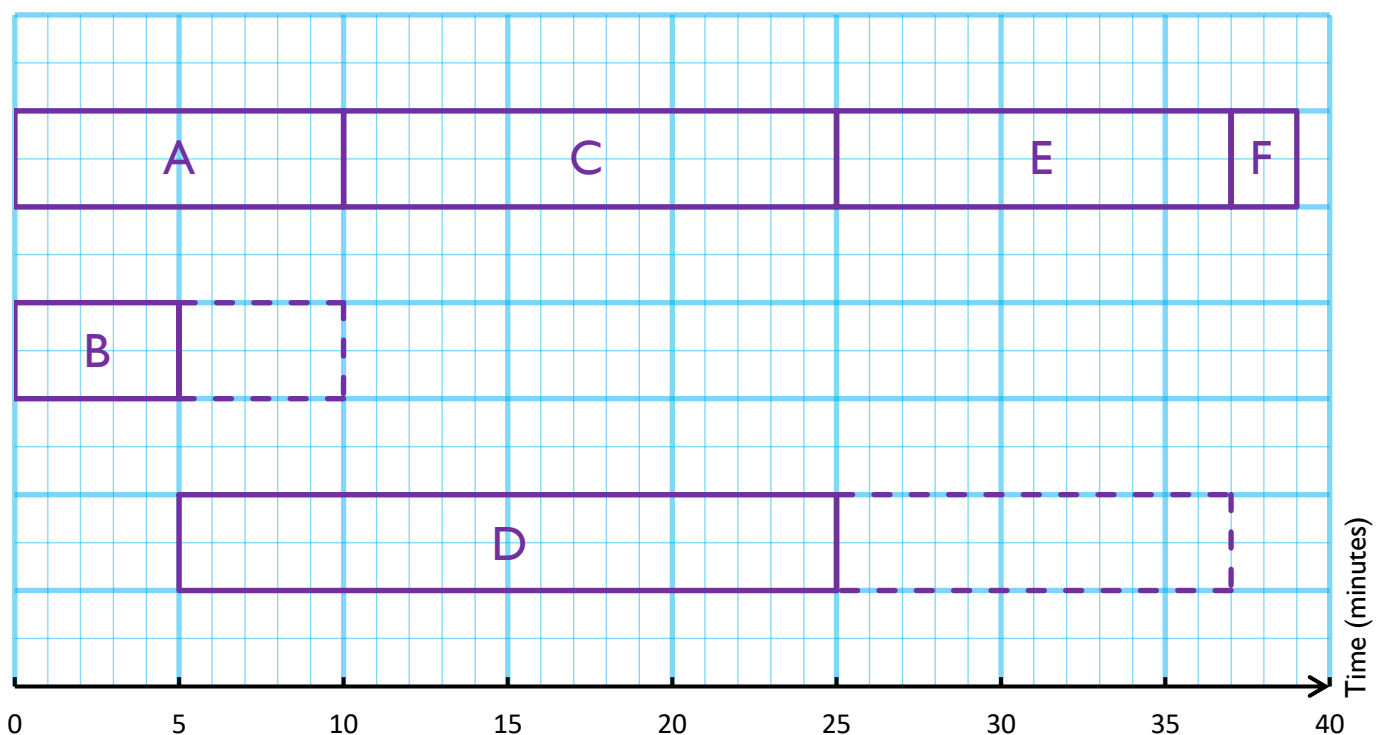
Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in minutes)	Float
A	1	2	10	0	10	0
B	1	3	10	0	5	5
C	2	4	25	10	15	0
D	3	5	37	5	20	12
E	4	5	37	25	12	0
F	5	6	39	37	2	0

From the information table, we see that activities A, C, E and F are critical activities.
So, the path $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$ is a critical path.



Here is the Gantt chart for the project.

Gantt chart for cooking 'Jerk BBQ Cauliflower'



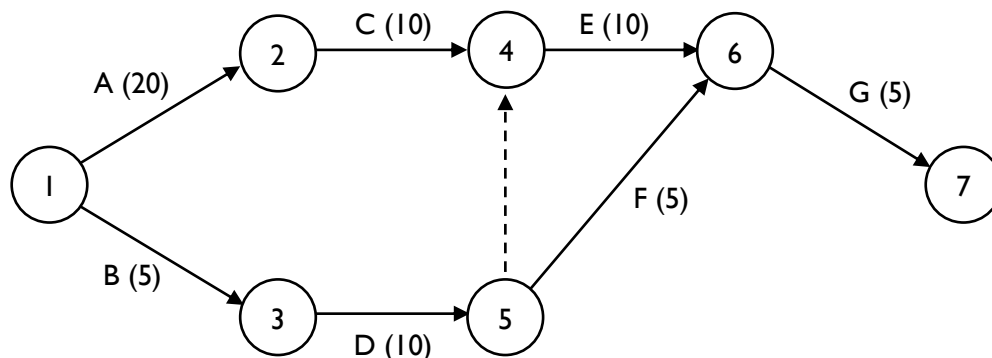
The first row of the chart shows the activities that appear in the critical path. The length of this row shows the length of the entire project, 39 minutes.

The remaining rows of the chart show the activities that are not critical. The dotted lines show that these activities can start later. For example, activity D can run earliest between 5 minutes and 25 minutes. However, it could also run between 10 minutes and 30 minutes, or (at the latest) between 17 minutes and 37 minutes.

It is possible to use a Gantt chart so see which activities run at a particular time. For example, at time 4 minutes above, activity A must be running, and *perhaps* activity B is running. At time 20 minutes, activities C and D must be running (regardless of when activity D starts).

Exercise 11

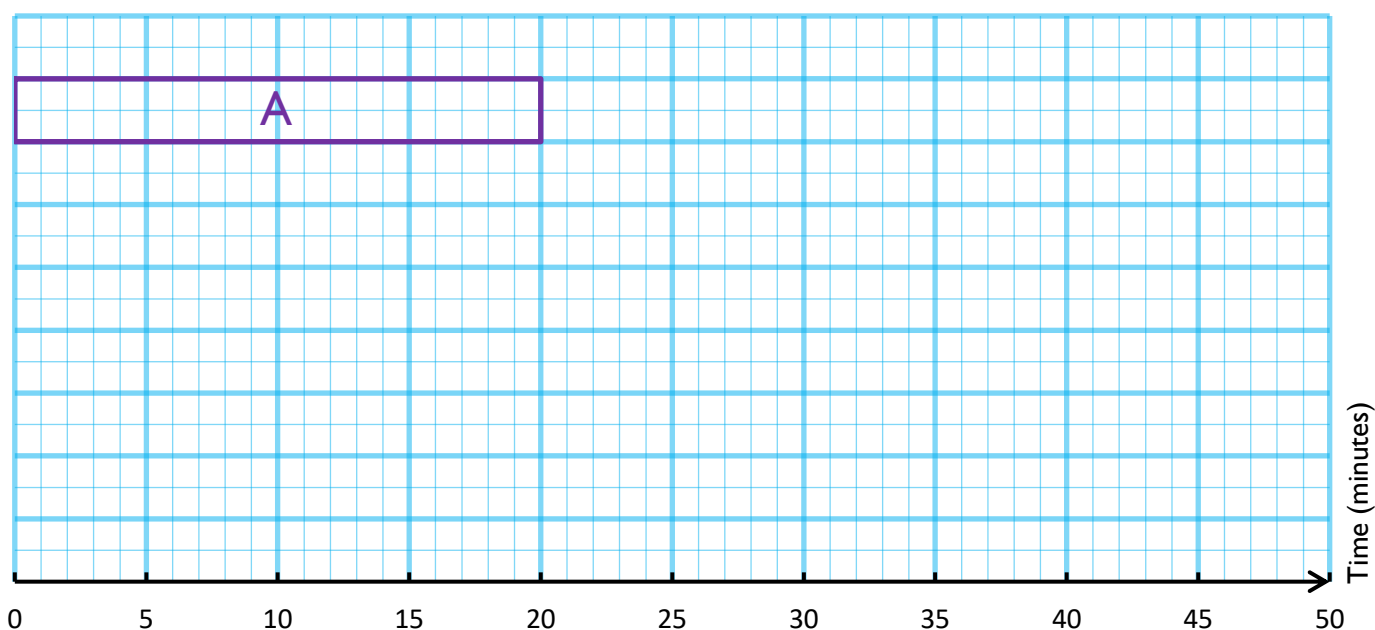
The following arc network represents a small project, where the times shown are in minutes.



The following table shows information about the seven activities in the project.

Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in minutes)	Float
A	1	2	20	0	20	0
B	1	3	20	0	5	15
C	2	4	30	20	10	0
D	3	5	30	5	10	15
E	4	6	40	30	10	0
F	5	6	40	15	5	20
G	6	7	45	40	5	0

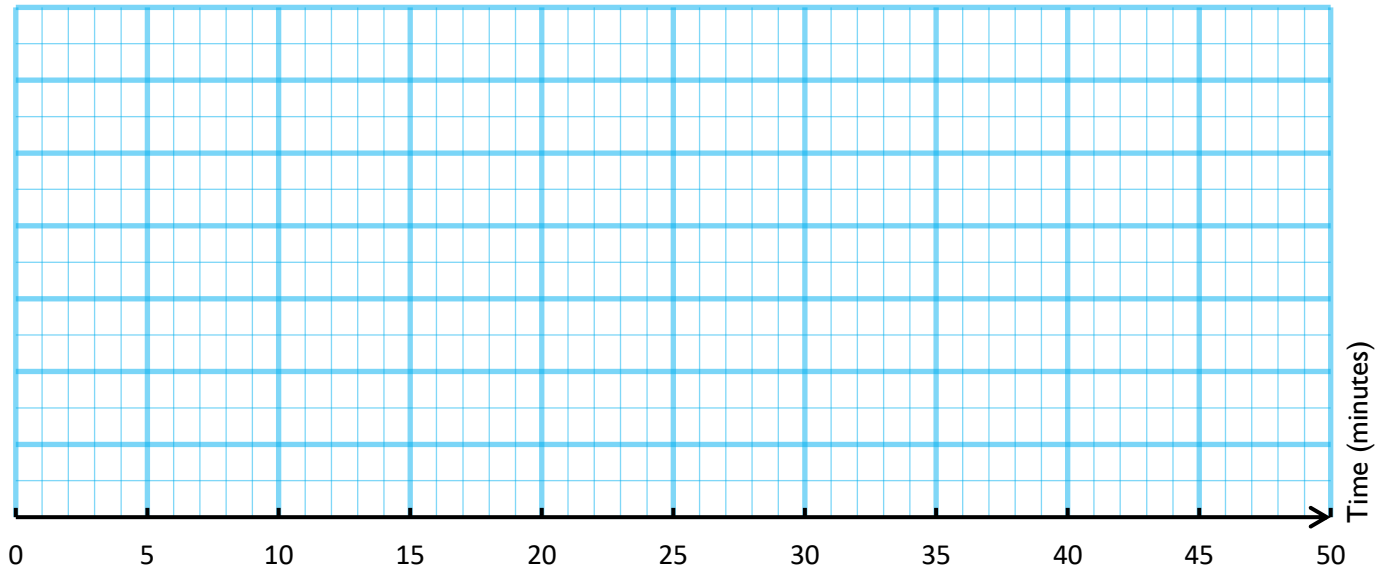
Draw a Gantt chart for the project. (Activity A has been done for you.)

Gantt chart for Exercise 11

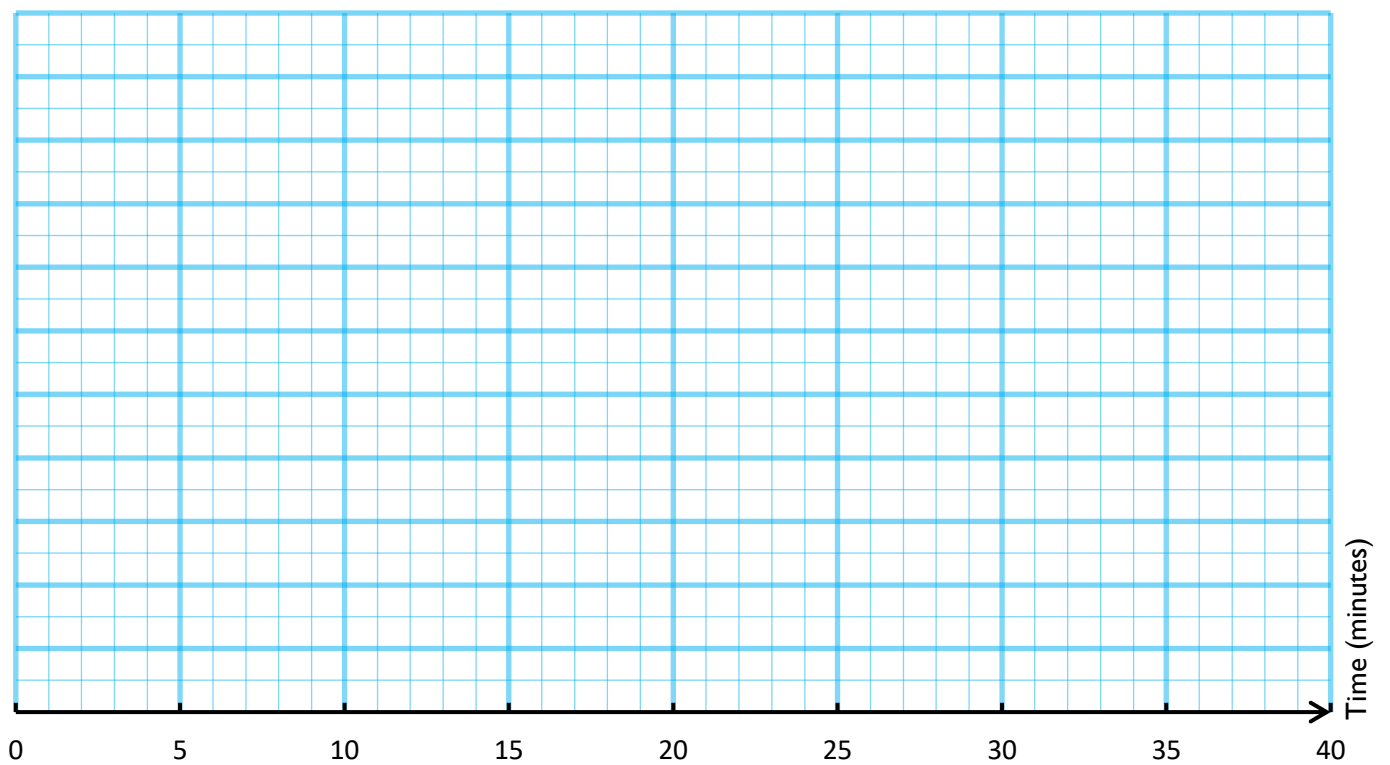
Exercise 12

Draw Gantt charts for the projects in...

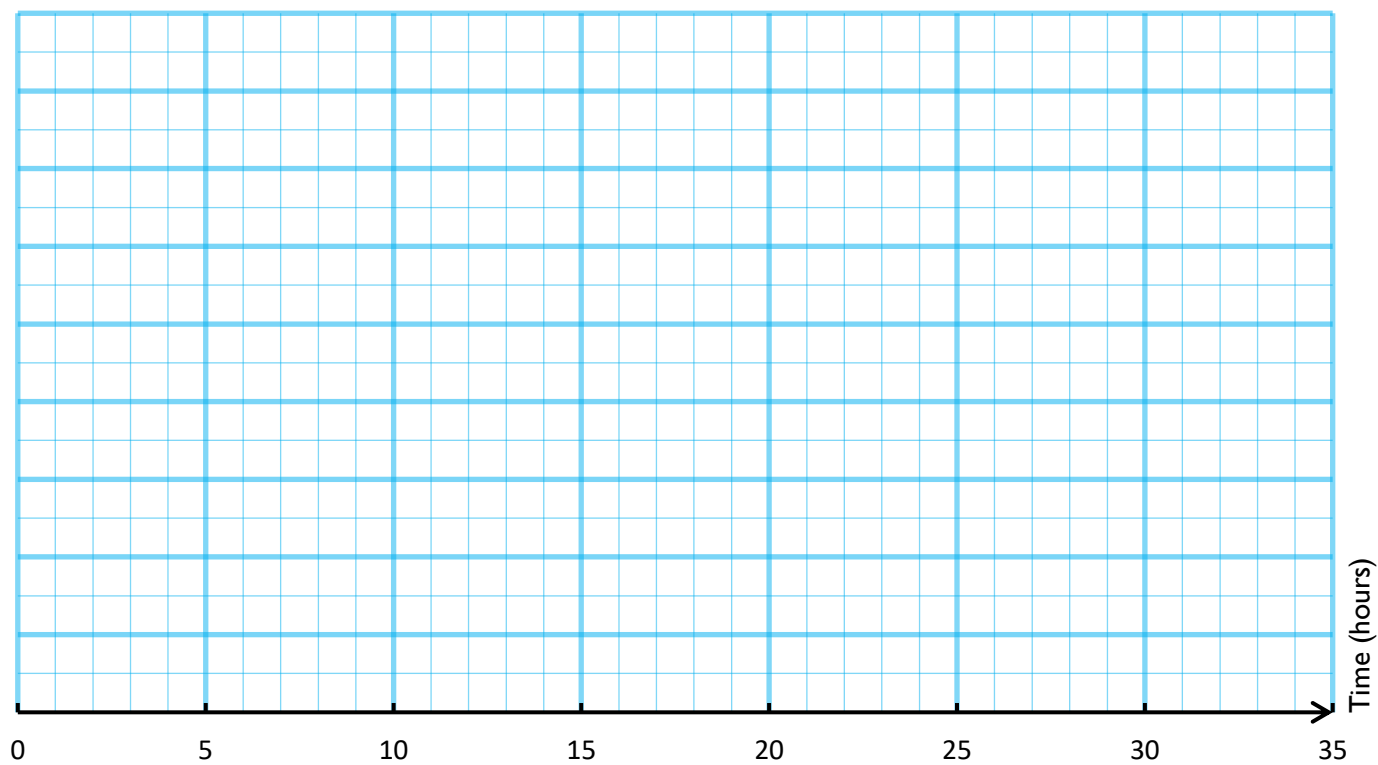
(a) Exercise 5

Gantt chart for Exercise 5

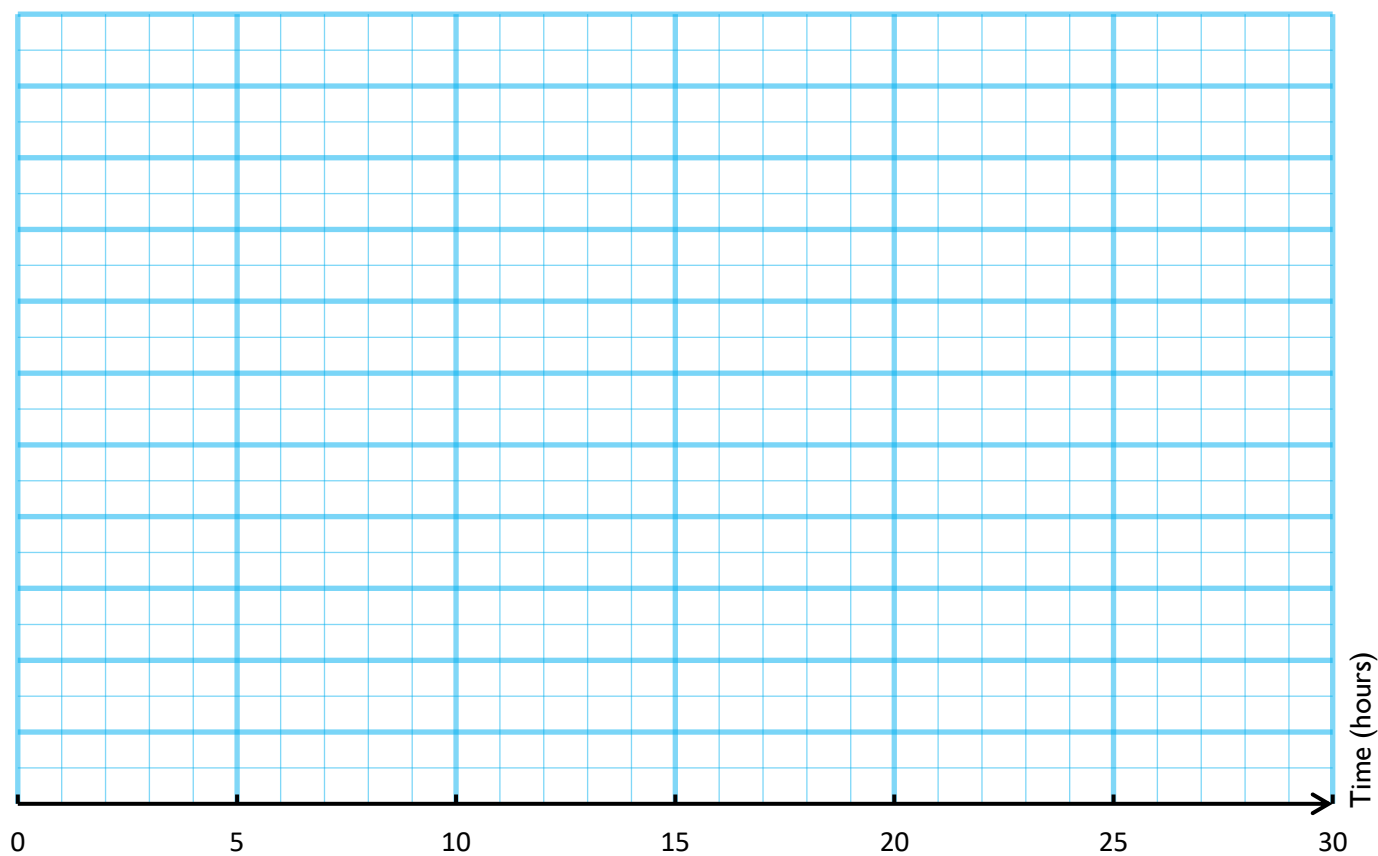
(b) Exercise 6

Gantt chart for Exercise 6

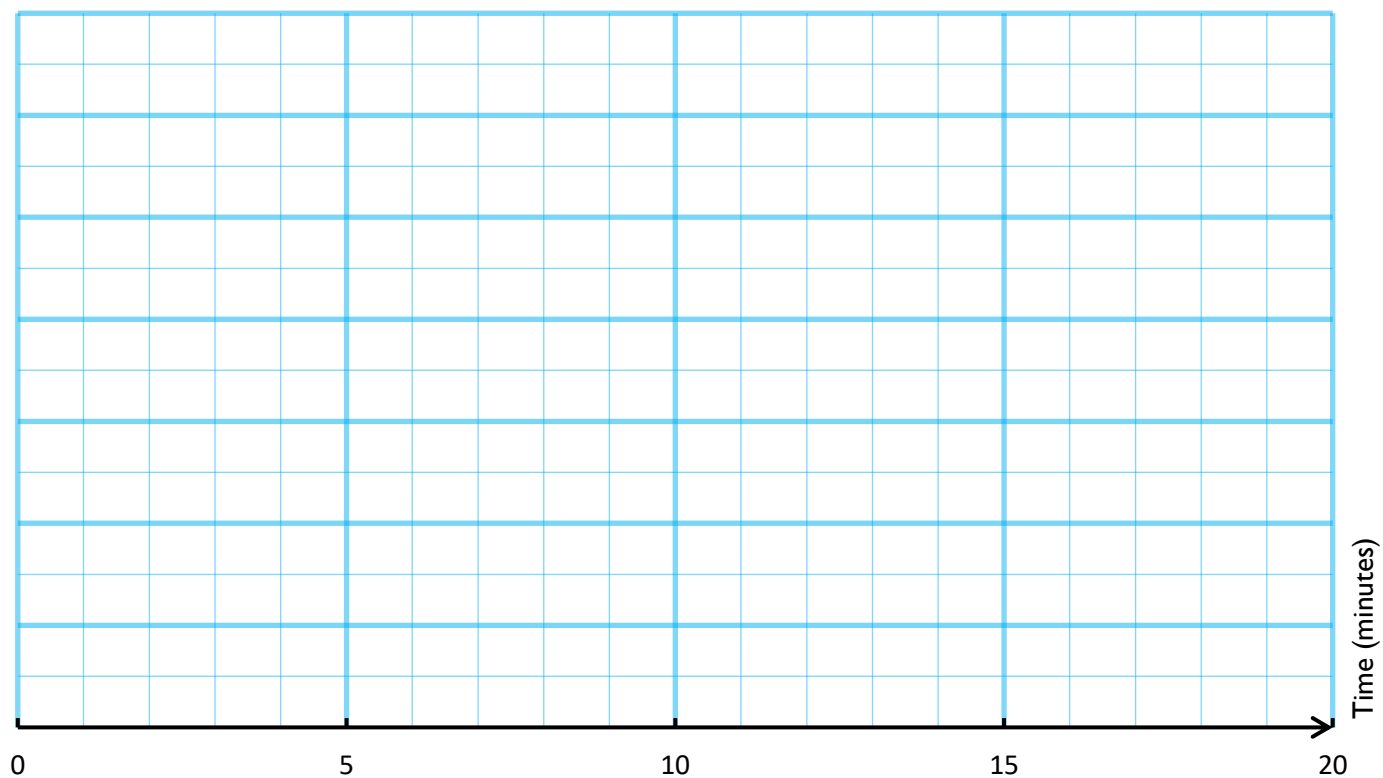
(c) Exercise 7

Gantt chart for Exercise 7

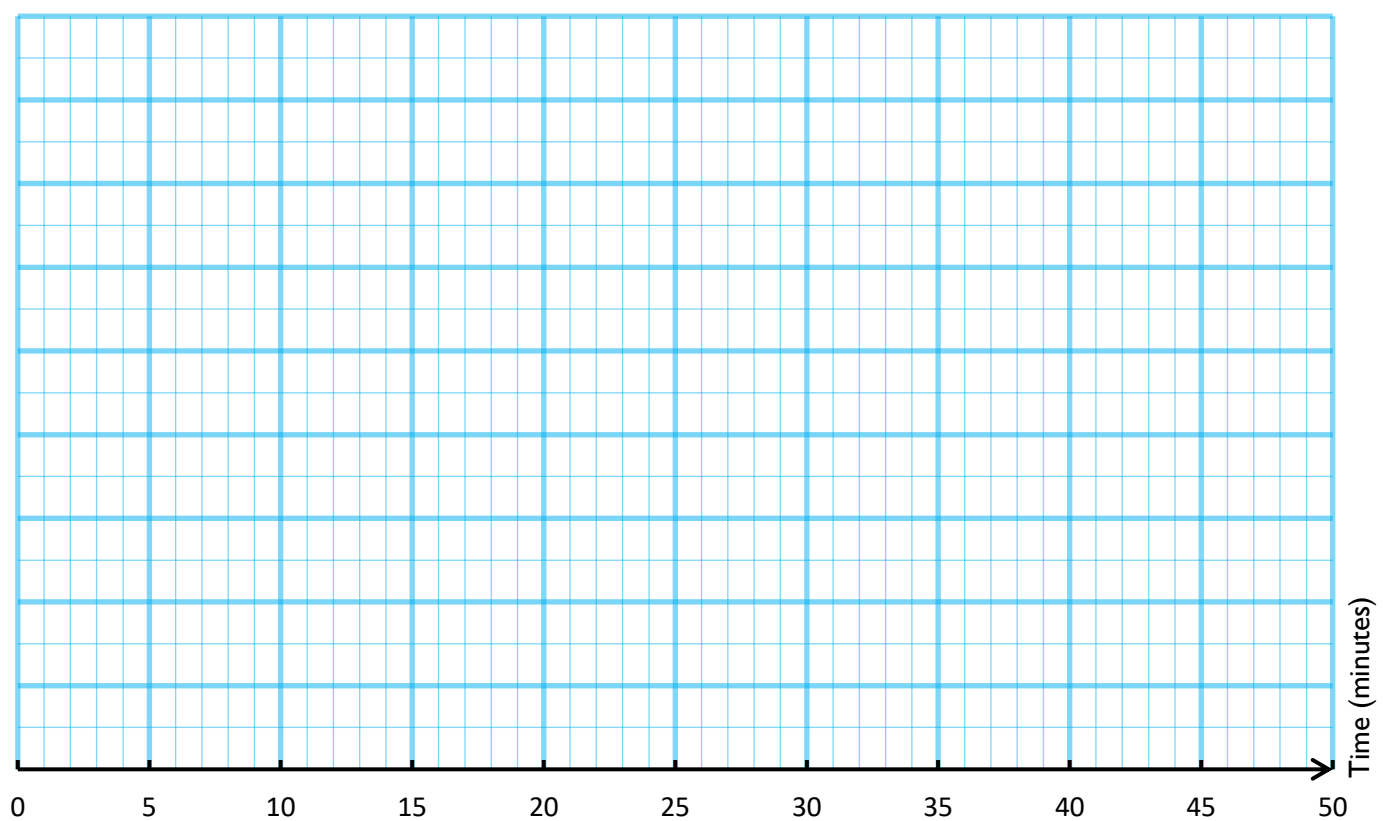
(d) Exercise 8

Gantt chart for Exercise 8

(e) Exercise 9

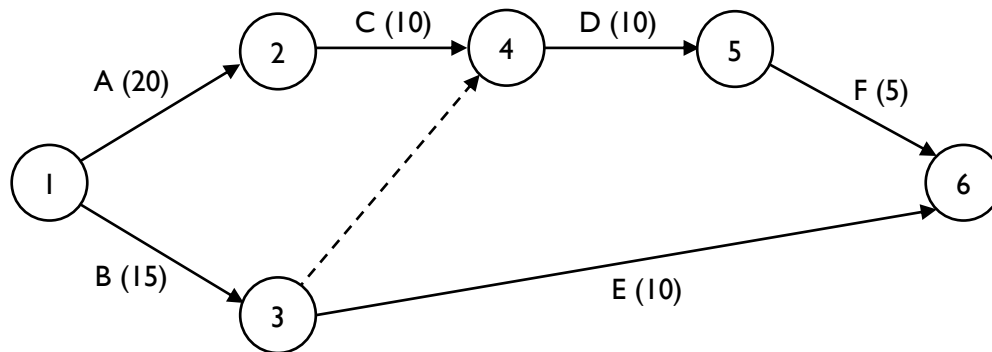
Gantt chart for Exercise 9

(f) Exercise 10

Gantt chart for Exercise 10

Exercise 13

The following arc network represents a small project, where the times shown are in minutes.

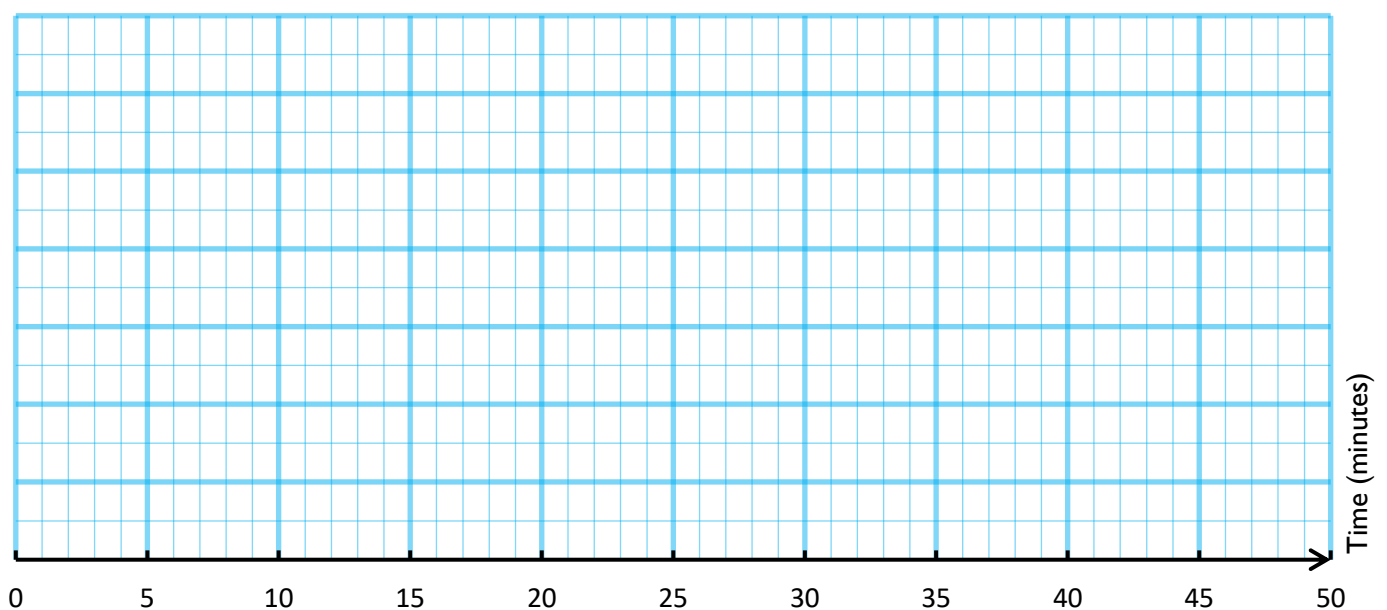


(a) Complete the following table.

Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in minutes)	Float
A						
B						
C						
D						
E						
F						

(b) Draw the critical path on the network.

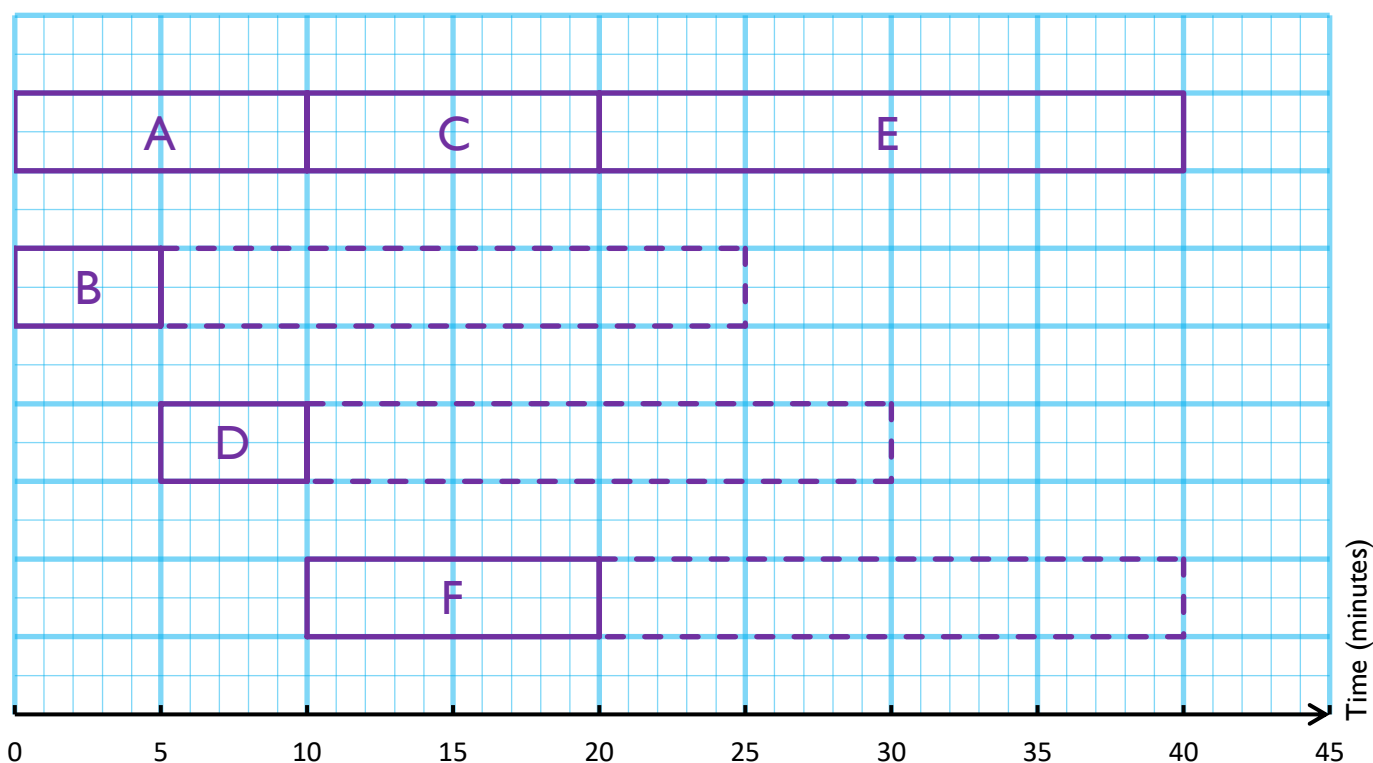
(c) Draw a Gantt chart for the project.

Gantt chart for Exercise 13

Exercise 14

Look at the following Gantt chart for a small project.

Gantt chart for Exercise 14



(a) What is the length of the entire project, in minutes?

.....

(b) Which activities are critical?

.....

(c) How many activities of length 10 minutes are in the project?

.....

(d) Which activities could be running at time 3 minutes?

.....

(e) Gwion says 'the chart shows that activities C, B, D and E could all be running at time 15 minutes'.
Is Gwion correct?

.....

.....

.....

(f) What is the total float for activity B?

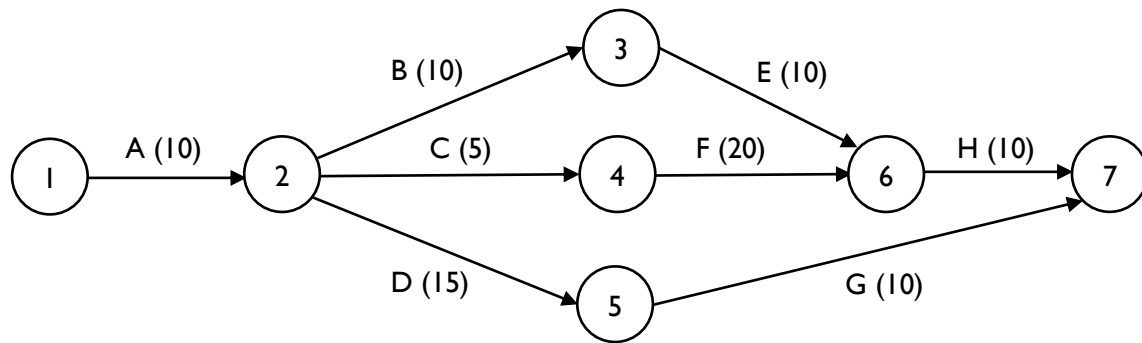
.....

(g) Is it possible for activities A and F to run at the same time?

.....

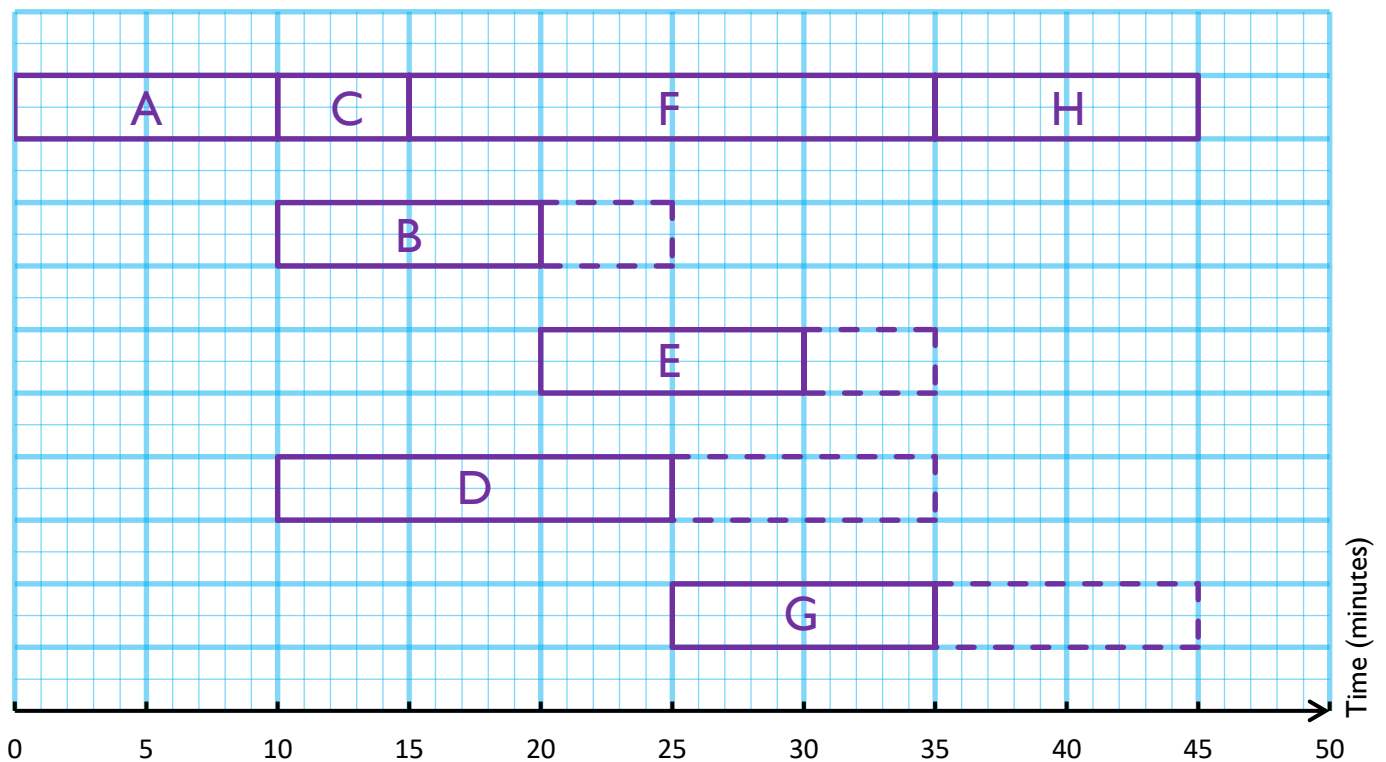
Exercise 15

The following arc network represents a small project, where the times shown are in minutes.



Here is the Gantt chart for the project.

Gantt chart for Exercise 15



(a) What is the length of the entire project, in minutes?

.....

(b) Which activities are critical?

.....

(c) What is the total float for activity B?

.....

(d) Is it possible for activities B and E to run at the same time?

.....

(e) Is it possible for activities D and E to **not** run at the same time?

.....

(f) Which activities are definitely running at time 22 minutes?

.....

(g) Fill in the table using ticks (✓), as appropriate.

Which activity is running?	Exactly one activity is running	Exactly two activities are running	Exactly three activities are running	Exactly four activities are running
A				
C				
F				
H				

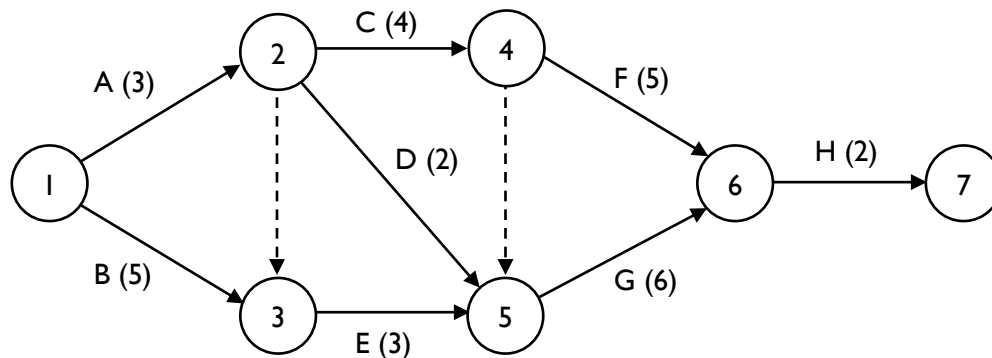
WJEC Terminology

Term	Definition
Precedence / dependence table	A table that shows the activities that must be completed before others are started.
Activity on arc network	Activities are represented by arcs, and nodes represent the start and end of activities.
Duration	The length of time it takes to complete an activity.
Critical activity	Any activity in which an increase in its duration would result in a corresponding increase in the duration of the whole project.
Critical path	A path from the start/source vertex/node to the end/sink vertex/node that entirely follows critical activities.
Early event time	The earliest time of arrival at the event allowing for the completion of all preceding activities.
Late event time	The latest time that an event can be left without extending the time needed for the project.
Total float	The amount of time that its start may be delayed without affecting the duration of the project.
Gantt (cascade) chart	A graphical way to represent the range of possible start and finish times for all activities on a single diagram.

Pop Quiz: Critical Path Analysis

/17

(1) The following arc network represents a project, where the times shown are in weeks.



(a) Complete the following table.

[8]

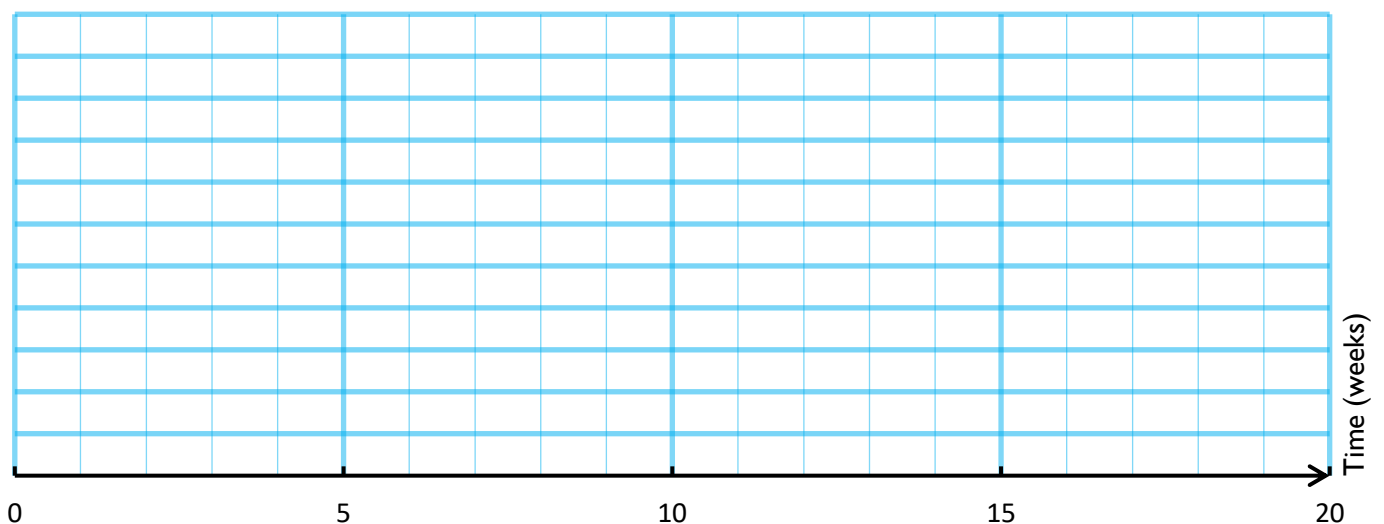
Activity	Start node	End node	Late event time for the end node	Early event time for the start node	Completion time (in weeks)	Float
A						
B						
C						
D						
E						
F						
G						
H						

(b) Draw the critical path on the network.

[1]

(c) Draw the Gantt chart for the project.

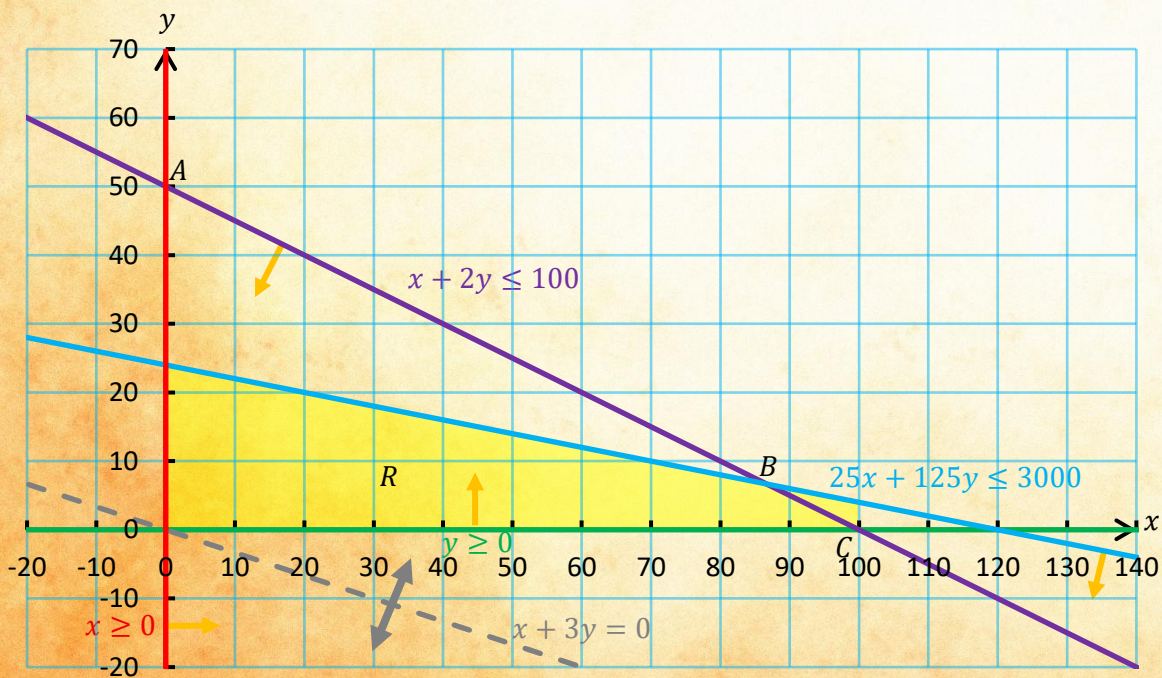
[8]

Gantt chart for the Pop Quiz



Linear

Programming



Name:

Background

What is the work?

Formulating and solving simple optimisation problems in two variables.

What are the prerequisites?

GCSE Work: Co-ordinates; plotting straight lines and linear inequalities; shading regions of graphs.

Where does this lead to?

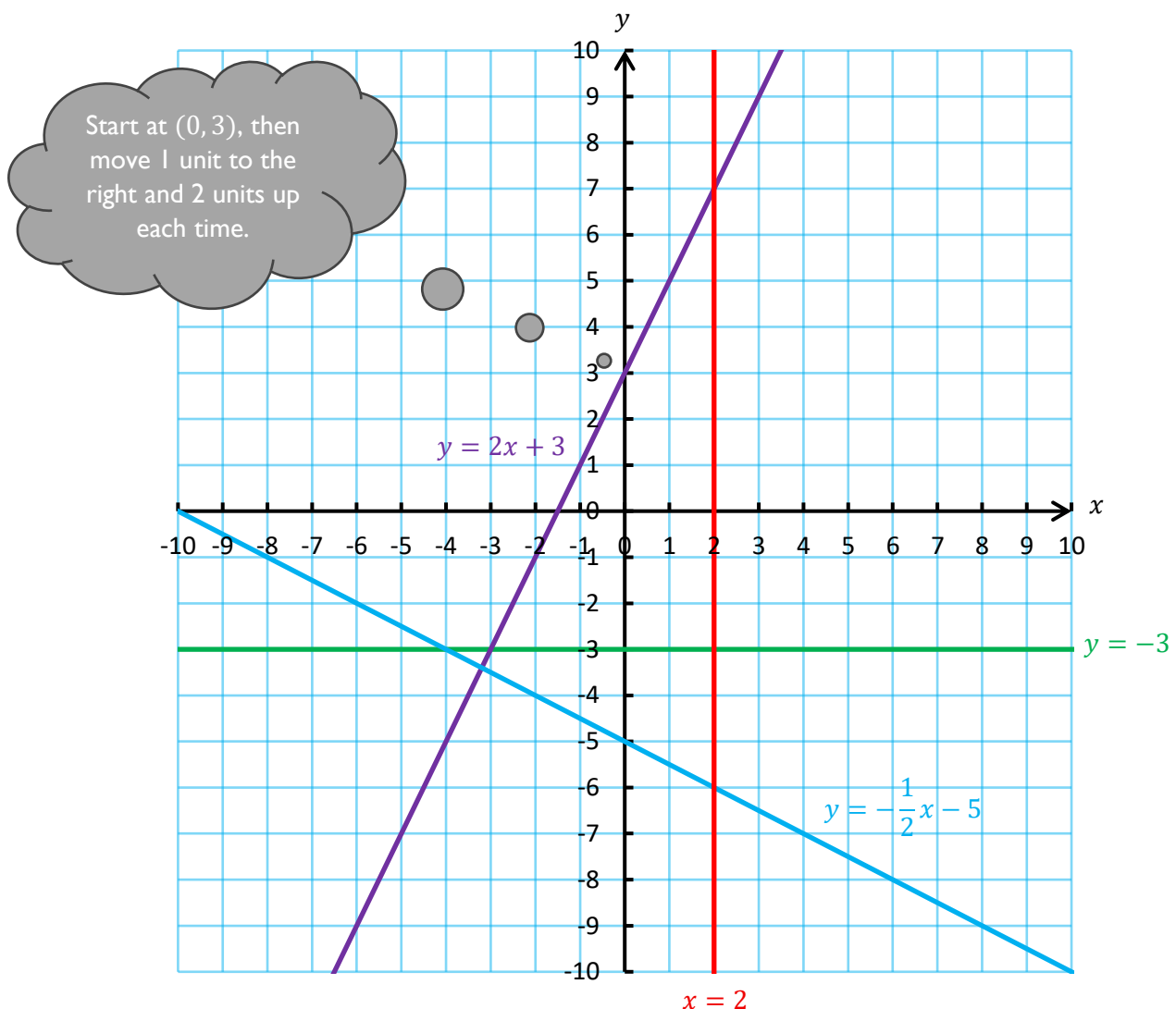
Applications: Linear programming can be used to reduce costs or increase profit by allocating limited resources such as time, labour and materials.

Theory

Straight Line Graphs

Plotting straight lines is part of the usual GCSE course.

Example 1



Exercise 1

On the graph paper below, plot the following lines.

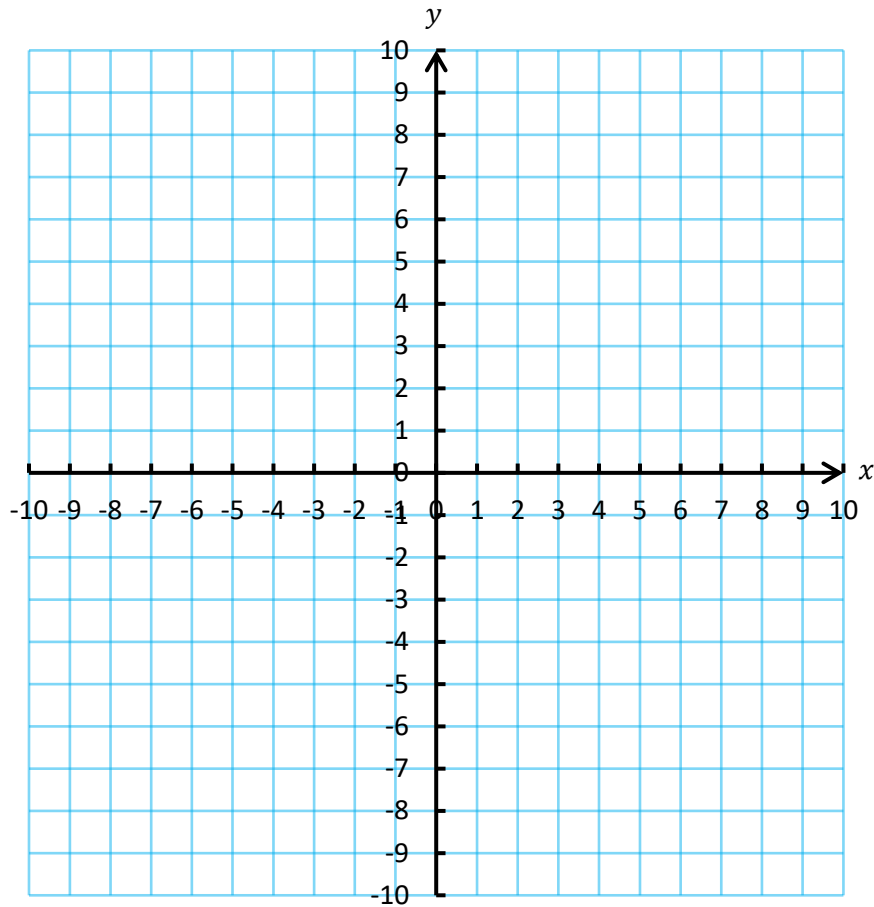
(a) $y = 5$

(b) $x = -3$

(c) $y = 3x - 1$

(d) $y = -2x + 7$

(e) $y = \frac{1}{2}x$

**Example 2**

Plot a straight line for the equation $2x + 3y - 6 = 0$.

The Hiding Method

To find the value of y when $x = 0$, **hide** the term in x using your finger, and solve the equation that remains.

$$\begin{array}{lcl} 2x + 3y - 6 = 0 & \rightarrow & 3y - 6 = 0 \\ & & 3y = 6 \quad \text{[Add 6]} \\ & & y = 2 \quad \text{[Divide by 3]} \end{array}$$

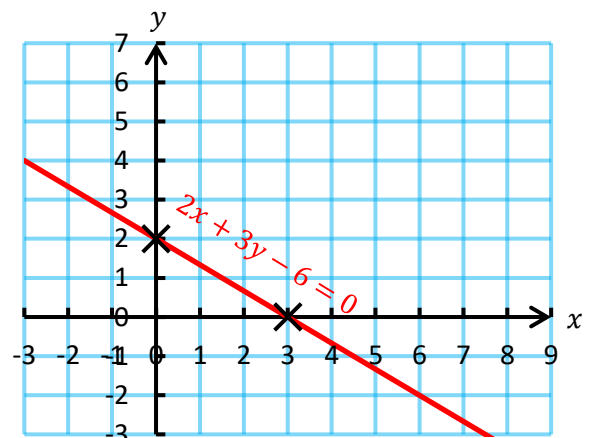
So, the line goes through the point $(0, 2)$.

To find the value of x when $y = 0$, **hide** the term in y with your finger, and solve the equation that remains.

$$\begin{array}{lcl} 2x + 3y - 6 = 0 & \rightarrow & 2x - 6 = 0 \\ & & 2x = 6 \quad \text{[Add 6]} \\ & & x = 3 \quad \text{[Divide by 2]} \end{array}$$

So, the line goes through the point $(3, 0)$.

To plot the line for the equation $2x + 3y - 6 = 0$, plot the two points $(0, 2)$ and $(3, 0)$ on graph paper before connecting them with a straight line.

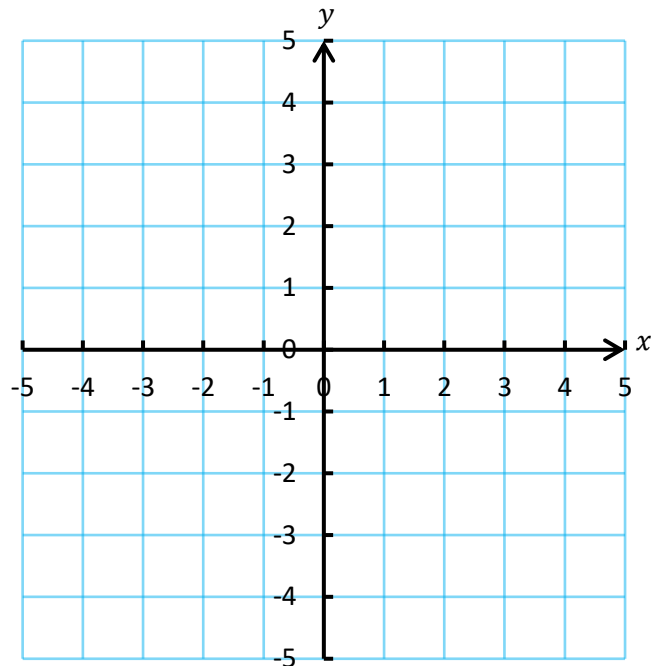
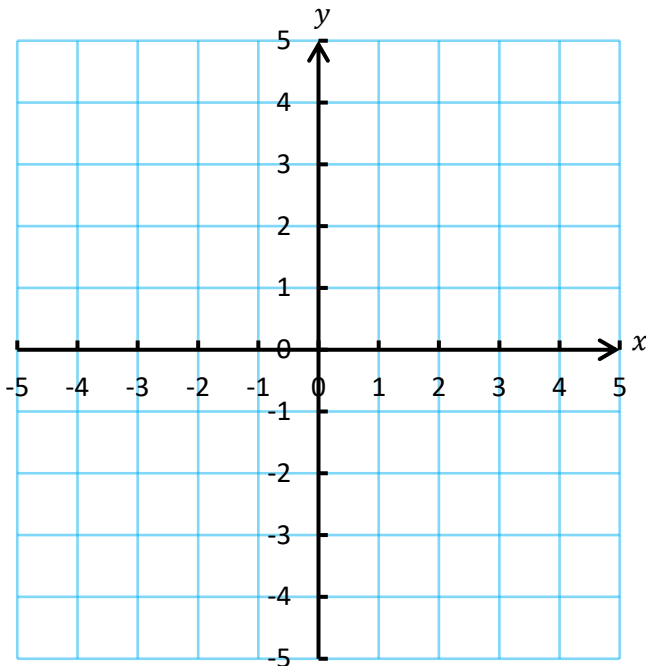


Exercise 2

Use the graph paper below to plot the following lines.

(a) $2x + 4y - 8 = 0$

(b) $3x - 2y - 6 = 0$

**Shading Regions**

It is possible to define a region on graph paper using a set of inequalities.

Example 3

Shade the region defined by the following inequalities.

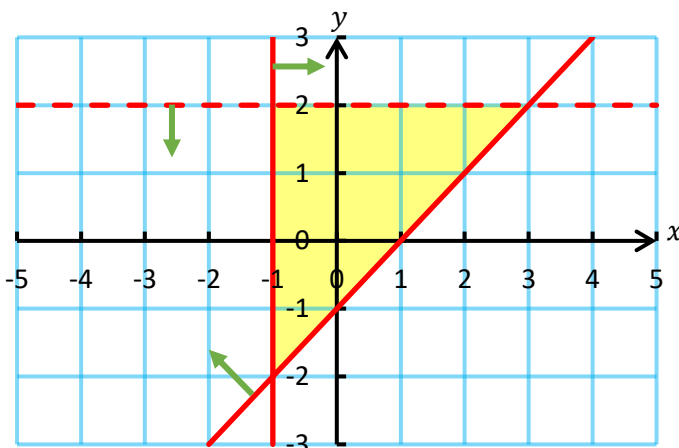
$$y < 2, \quad x \geq -1, \quad y \geq x - 1$$

Answer: **Step 1:** Plot the graphs of $y = 2$, $x = -1$, $y = x - 1$.

Rule: Use a solid line (—) for inequalities containing $\geq$ or $\leq$; and a dotted line (- - -) for inequalities containing $>$ or $<$.

Step 2: Note, using an arrow, a region corresponding to each of the three lines.

Step 3: Shade the region that is satisfied by all of the arrows / inequalities.



For lines that aren't vertical or horizontal, substitute a point that doesn't lie on the line to decide which way the arrow should point. For example, considering $y \geq x - 1$, substitute the point $(0, 0)$:

$$0 \geq 0 - 1$$

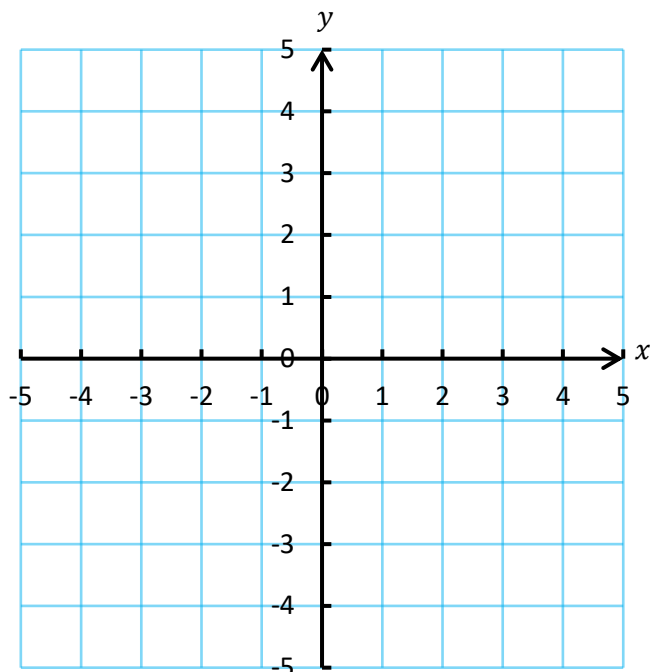
$$0 \geq -1$$

This inequality is **true**, so the arrow should point **towards** the point $(0, 0)$.

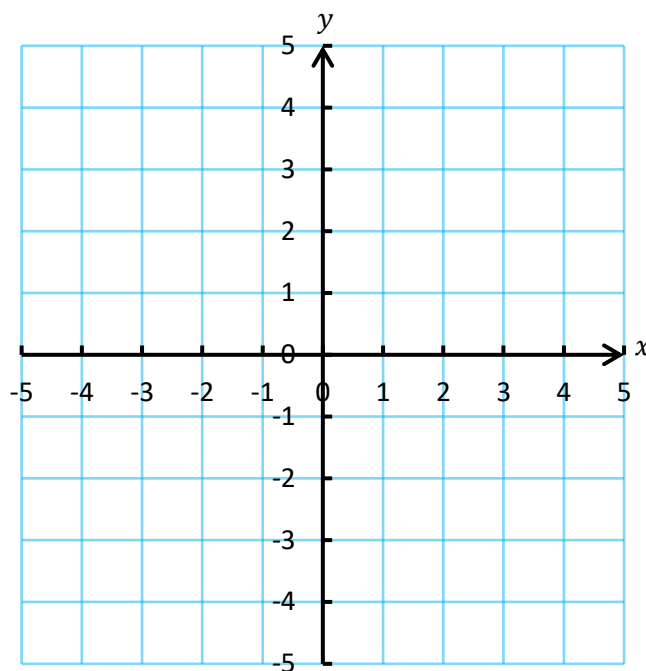
Exercise 3

Shade the region that is defined by the following sets of inequalities.

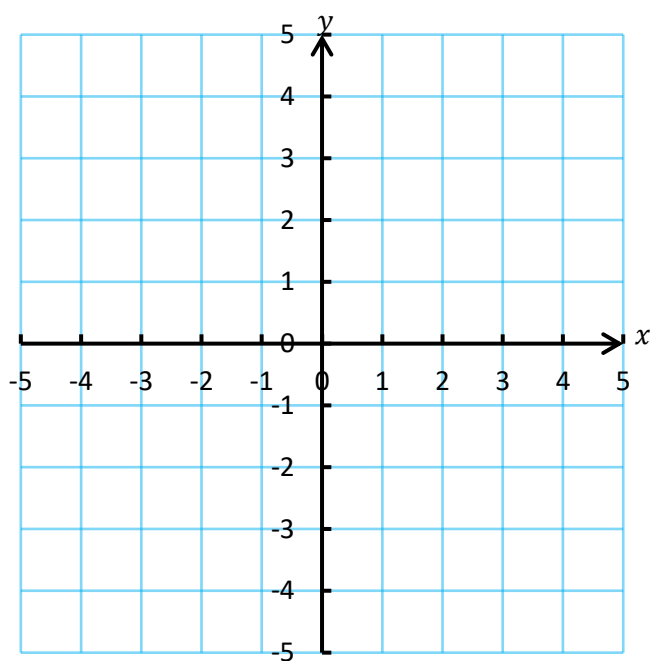
(a) $y < 3$, $x < 2$, $y \geq -1$, $x \geq -3$



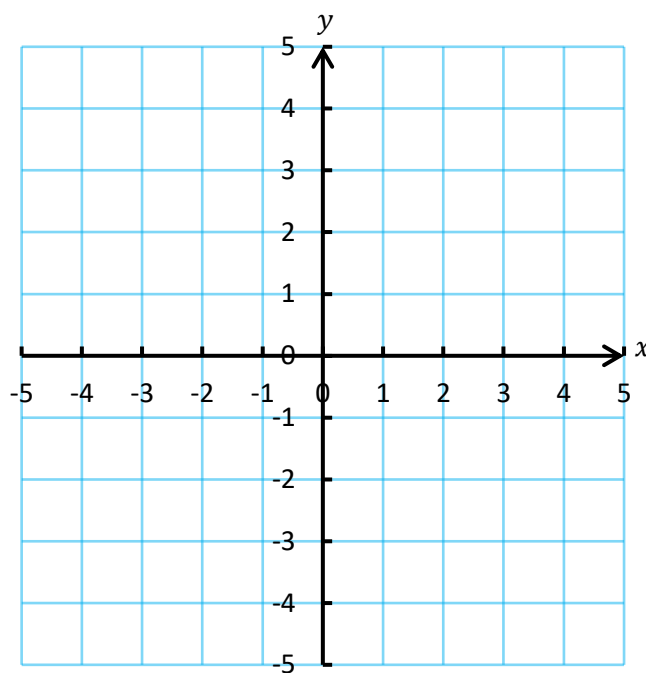
(b) $y \leq 4$, $x \geq -1$, $y \geq 2x - 1$



(c) $y > 0$, $x \geq 1$, $x + 3y - 3 \leq 0$



(d) $y > -2$, $y < x + 2$, $y < -2x + 4$



Linear Programming

Example 4

Awel has bought 100 lemons and 30 kg of sugar to sell products at her stall at the local fair.

Awel sells two products:

- Lemonade, which uses 1 lemon and 25 g of sugar for each cup sold;
- Lemon Curd, which uses 2 lemons and 125 g of sugar for each jar sold.

Awel makes £1 profit selling a cup of lemonade, and £3 profit selling a jar of lemon curd. How much lemonade and lemon curd should Awel make to maximise her profit?

Answer: To begin, we need to represent the problem as a **linear programming problem**.

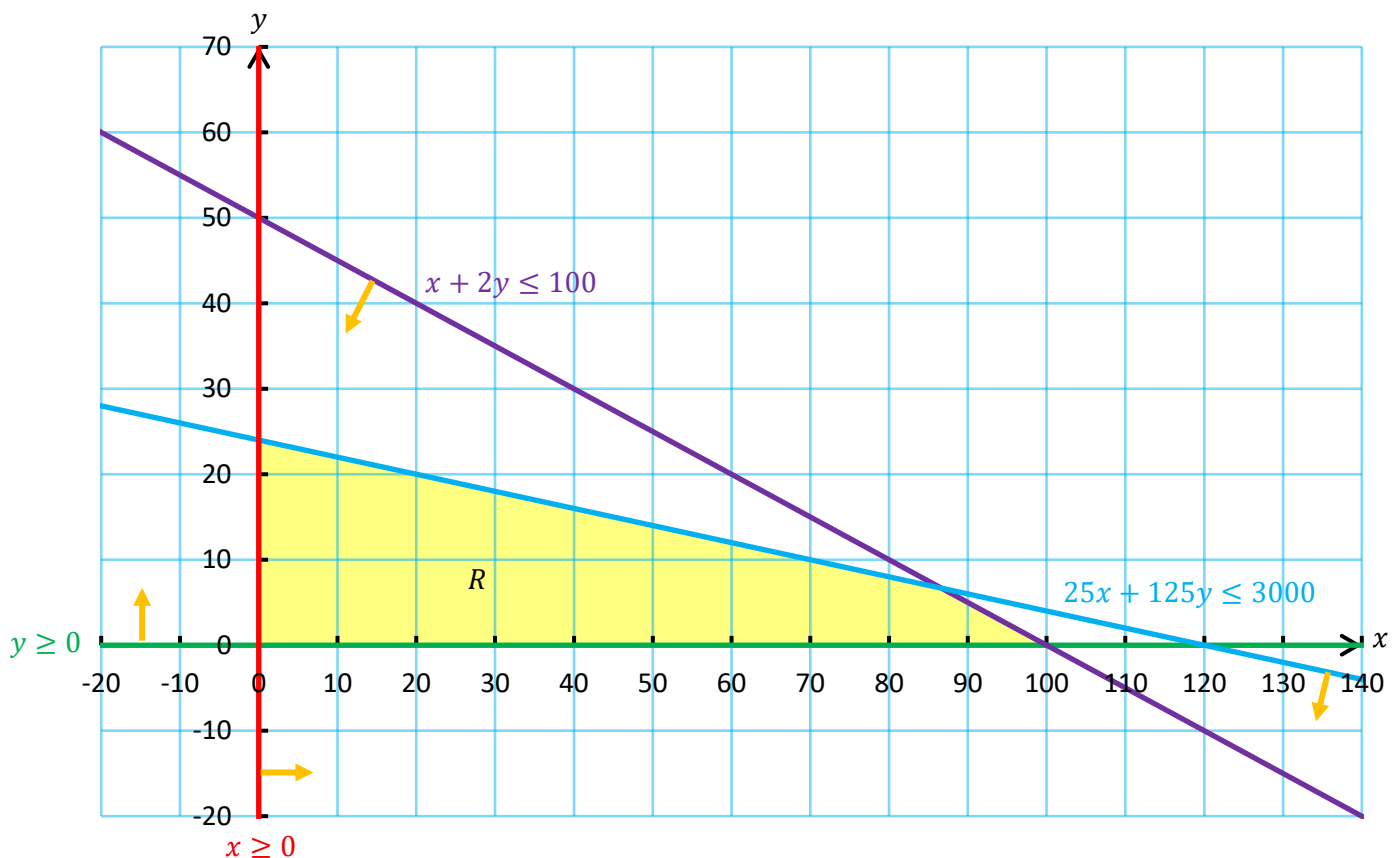
Let x represent how many cups of lemonade Awel produces, and let y represent how many jars of lemon curd Awel produces. These are the **decision variables** for the problem. Considering the ingredients, we can write the **constraints**

$$\begin{aligned} x + 2y &\leq 100 && \text{(lemons)} \\ 25x + 125y &\leq 3000 && \text{(sugar, in grams)} \end{aligned}$$

Also, we must have $x \geq 0$, $y \geq 0$ (it is not sensible for x and y to be negative).

The total profit is $\text{£}1 \times x + \text{£}3 \times y$, so $E = x + 3y$ is the **objective function** that must be maximised.

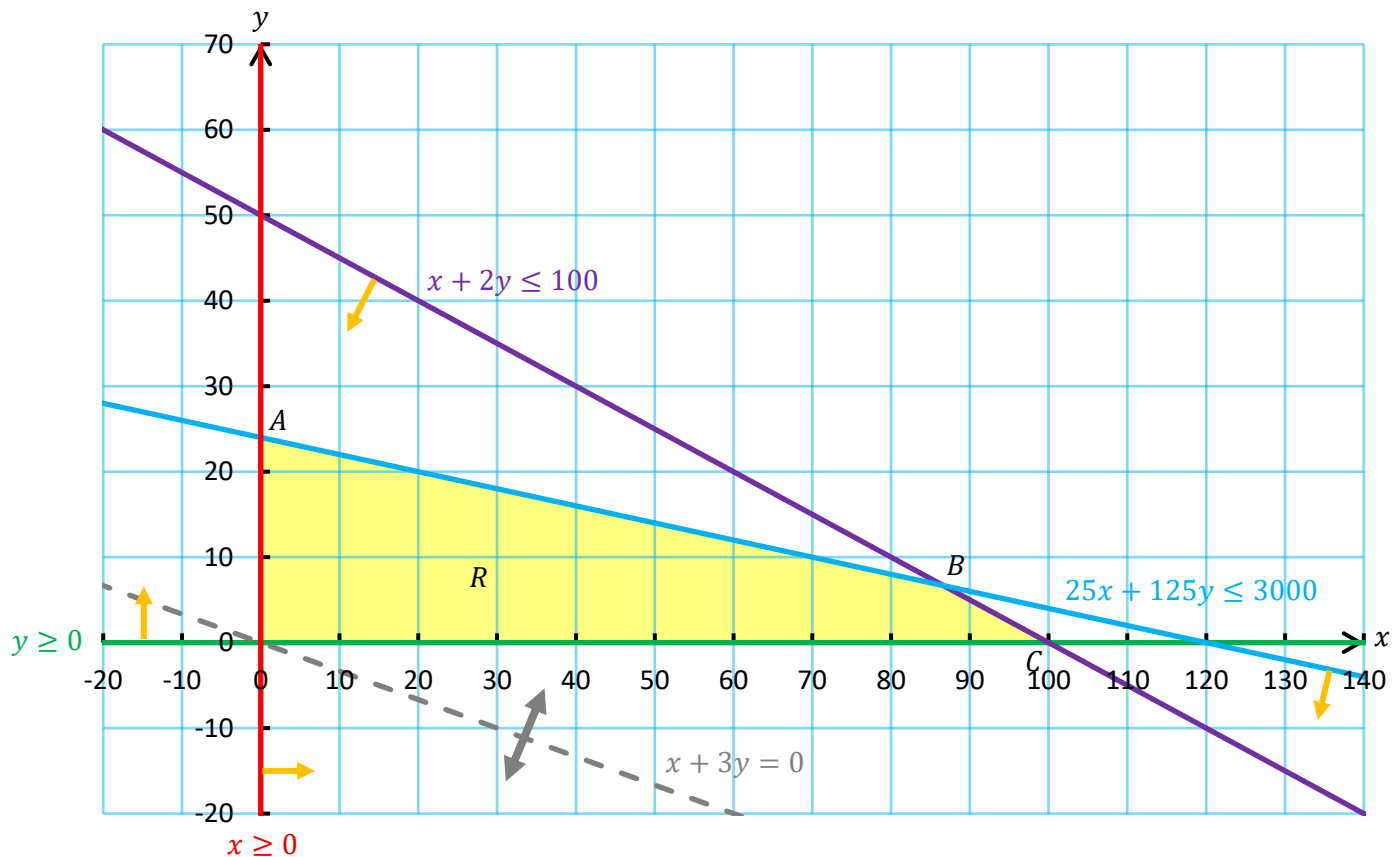
We can draw the above inequalities on the graph below. The yellow region R shows the **feasible region**.



To find the **optimal solution**, we must imagine the line $E = x + 3y$ moving along the graph.

This line is parallel to the line $x + 3y = 0$, which is shown on the graph below.

(An interactive GeoGebra resource to show this: <https://www.geogebra.org/classic/varzdkn9>)



To maximise E , the line $E = x + 3y$ must be as far as possible from the origin.

So, the optimal solution lies on one of the corners of the feasible region (at A , B or C).

At A , $x = 0$ in the equation $25x + 125y = 3000$.

So, we have
 $125y = 3000$
 $y = 24$.

If $x = 0$, $y = 24$, the profit is
 $E = 0 + 3 \times 24$
 $E = £72$

At B , the lines $x + 2y = 100$ and $25x + 125y = 3000$ intersect.

Multiplying the first equation by 25:
 $25x + 50y = 2500$

Subtracting this equation from the second equation:
 $75y = 500$
 $y = 6\frac{2}{3}$

Substituting back:

$$x + 2\left(6\frac{2}{3}\right) = 100$$

$$x = 100 - 13\frac{1}{3}$$

$$x = 86\frac{2}{3}$$

If $x = 86\frac{2}{3}$, $y = 6\frac{2}{3}$, the profit is

$$E = 86\frac{2}{3} + 3 \times 6\frac{2}{3}$$

$$E = £106.67, \text{ to the nearest penny.}$$

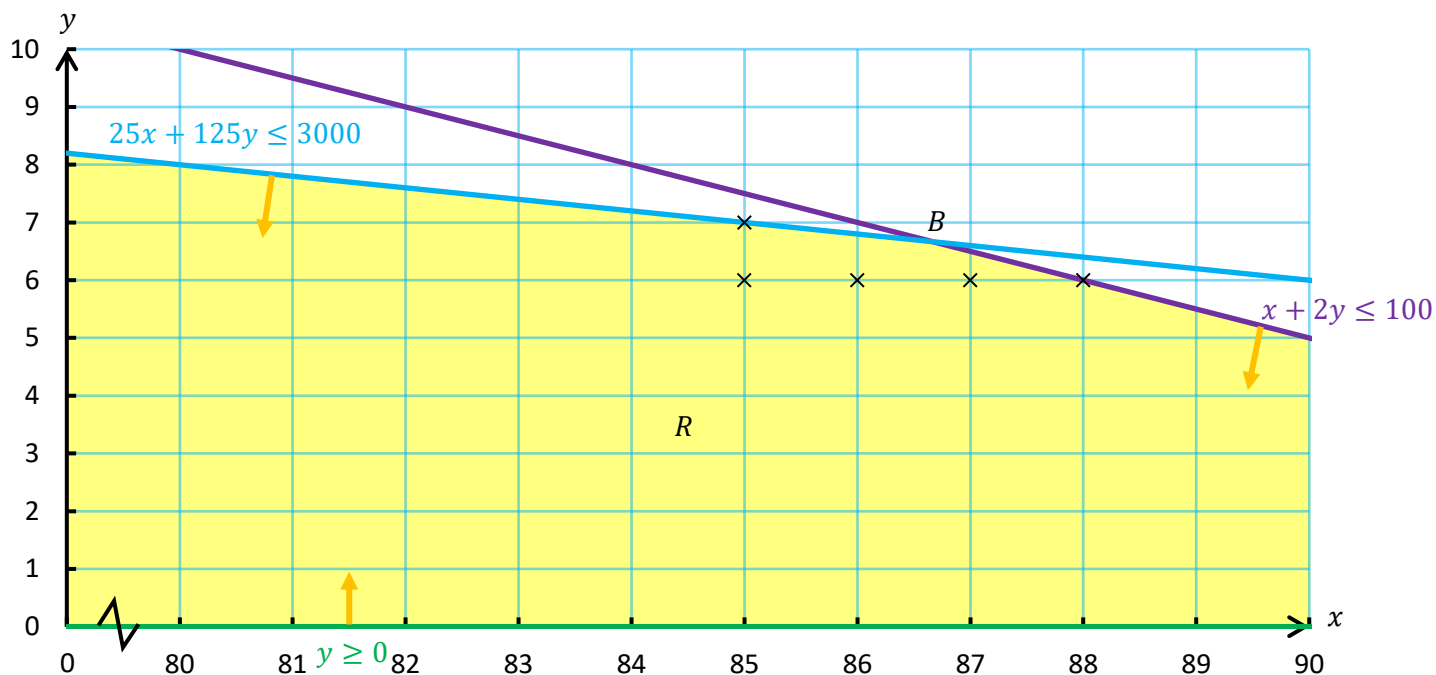
At C , $y = 0$ in the equation $x + 2y = 100$.

So, we have
 $x = 100$.

If $x = 100$, $y = 0$, the profit is
 $E = 100 + 3 \times 0$
 $E = £100$

We see that the largest profit is found at the point B . However, x and y are not whole numbers at this point, so we need to inspect nearby points with whole number co-ordinates.

Zooming into the graph around the point B , we see the following situation.



The points with whole number co-ordinates that are in R and are close to B are $(85, 7)$, $(85, 6)$, $(86, 6)$, $(87, 6)$ and $(88, 6)$.

Considering the objective function $E = x + 3y$, E will be at its largest at either $(85, 7)$ or $(88, 6)$.

If $x = 85$, $y = 7$, the profit is

$$E = 85 + 3 \times 7$$

$$E = £106$$

If $x = 88$, $y = 6$, the profit is

$$E = 88 + 3 \times 6$$

$$E = £106$$

So, for this problem, there are two optimal solutions: Awel should either produce **85 lemonades and 7 lemon curds**, or **88 lemondades and 6 lemon curds**. By doing this, Awel's profit will be £106.

WJEC Terminology

Term	Definition
Decision variable	A variable associated with linear programming problems. A solution is formed by assigning appropriate values to all decision variables.
Constraint	A restriction on the values of the decision variables.
Objective function	The function that defines the criteria for evaluating the solution.
Feasible solution	Values for the decision variables that satisfy each of the constraints in the linear programming problem.
Feasible region	The region that contains all the feasible solutions.
Optimal solution	A feasible solution that has the optimum objective function value.

Exercise 4

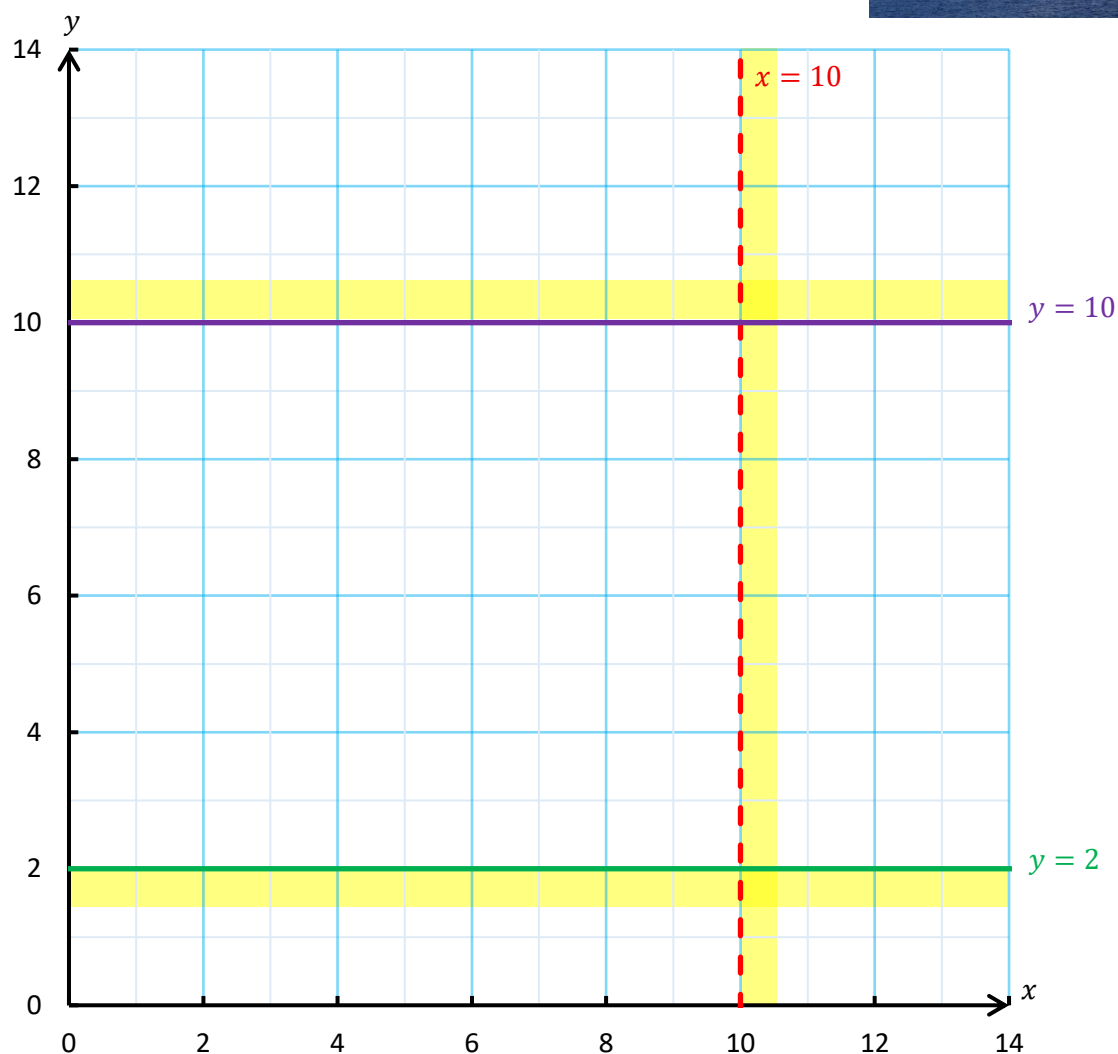
The captain of a pleasure boat charges passengers to take a lake trip on her boat.

The number of children is represented by x , and the number of adults by y .

Two of the constraints that control how many passengers can be transported are

$$x < 10 \text{ and } 2 \leq y \leq 10$$

These are shown on the graph below, where the rejected regions are shaded yellow.



(a) Explain why the line $x = 10$ is shown as a dotted line.

.....

.....

(b) Use the constraints to write statements that describe the number of children and the number of adults that can be transported on the boat.

.....

.....

.....

.....

For each journey, the tickets cost £2 per child and £3 per adult. The total tickets sold must be at least £24 for the boat to be able to sail.

It is not permitted for double the number of children on the boat to be greater than the number of adults.

(c) Use this information to write two inequalities.

.....

.....

(d) Add two lines to the graph on the previous page to show your inequalities.
Hence, label the feasible region with the letter R .

(e) Use your graph to decide how many children and how many adults go on the trip if the captain takes

(i) the least number of passengers,

.....

.....

(ii) the greatest number of passengers.

.....

.....

(f) Use your graph to find the maximum profit the captain can make each time the boat sails.

.....

.....

.....

.....

.....

.....

.....

.....

.....

Exercise 5

Eifion is arranging two types of sporting event, 'Cracking Cricket' and 'Mastering Mountaineering'. He must decide how many times each event should be held.

Let x represent how many times the event 'Cracking Cricket' is held.

Let y represent how many times the event 'Mastering Mountaineering' is held.

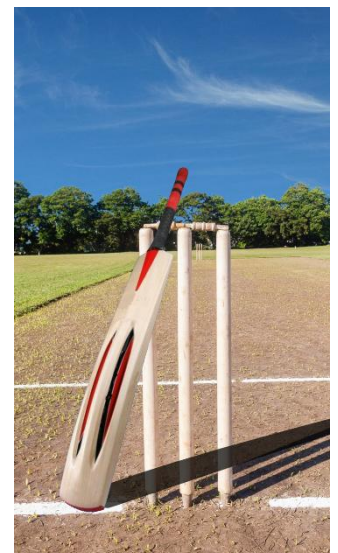
Two of the constraints are

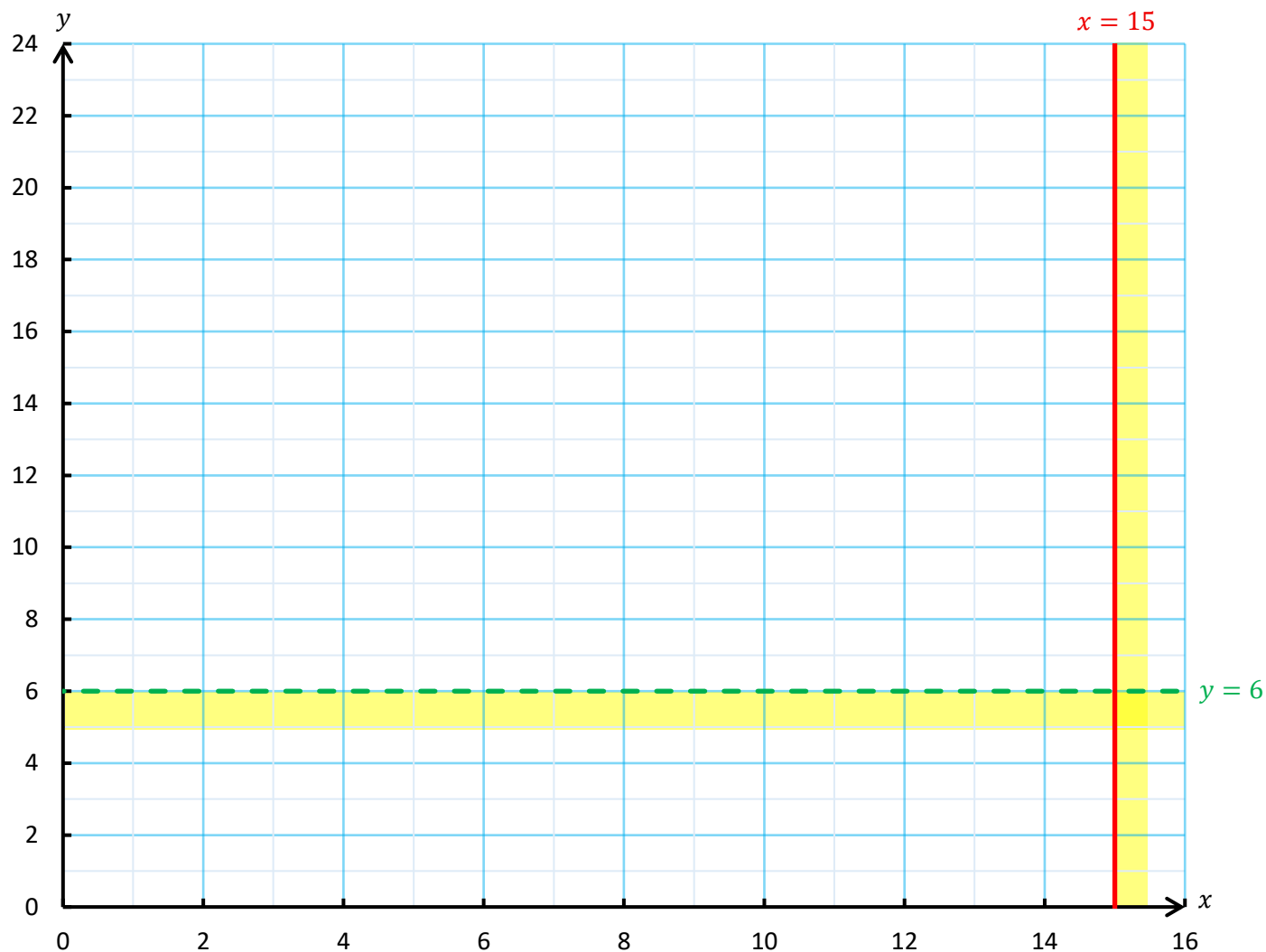
$$x \leq 15$$

and

$$y > 6$$

There are shown on the graph on the next page, where the rejected regions are shaded yellow.





Two more constraints are

$$3x \leq 2y \quad \text{and} \quad 5x + 4y \leq 80$$

(a) Add lines to the above graph to represent the two new constraints.

Hence, label the feasible region with the letter R .

Each 'Cracking Cricket' event costs £500 to run.

Each 'Mastering Mountaineering' event costs £800 to run.

Eifion wants to minimise the total costs.

(b) Write an objective function for the costs, C , in terms of x and y .

.....

.....

(c) Use your graph to decide how many times each event should be held, and what the total costs are.

.....

.....

.....

.....

.....

[illegible]

Rhys sells bouquets of flowers at his local market. He sells two types of bouquets: roses and tulips.

Each bouquet of roses costs £7 to make and each bouquet of tulips costs £5 to make.

.....

$$y \leq 20 \quad \text{and} \quad y \leq 4x$$

.....

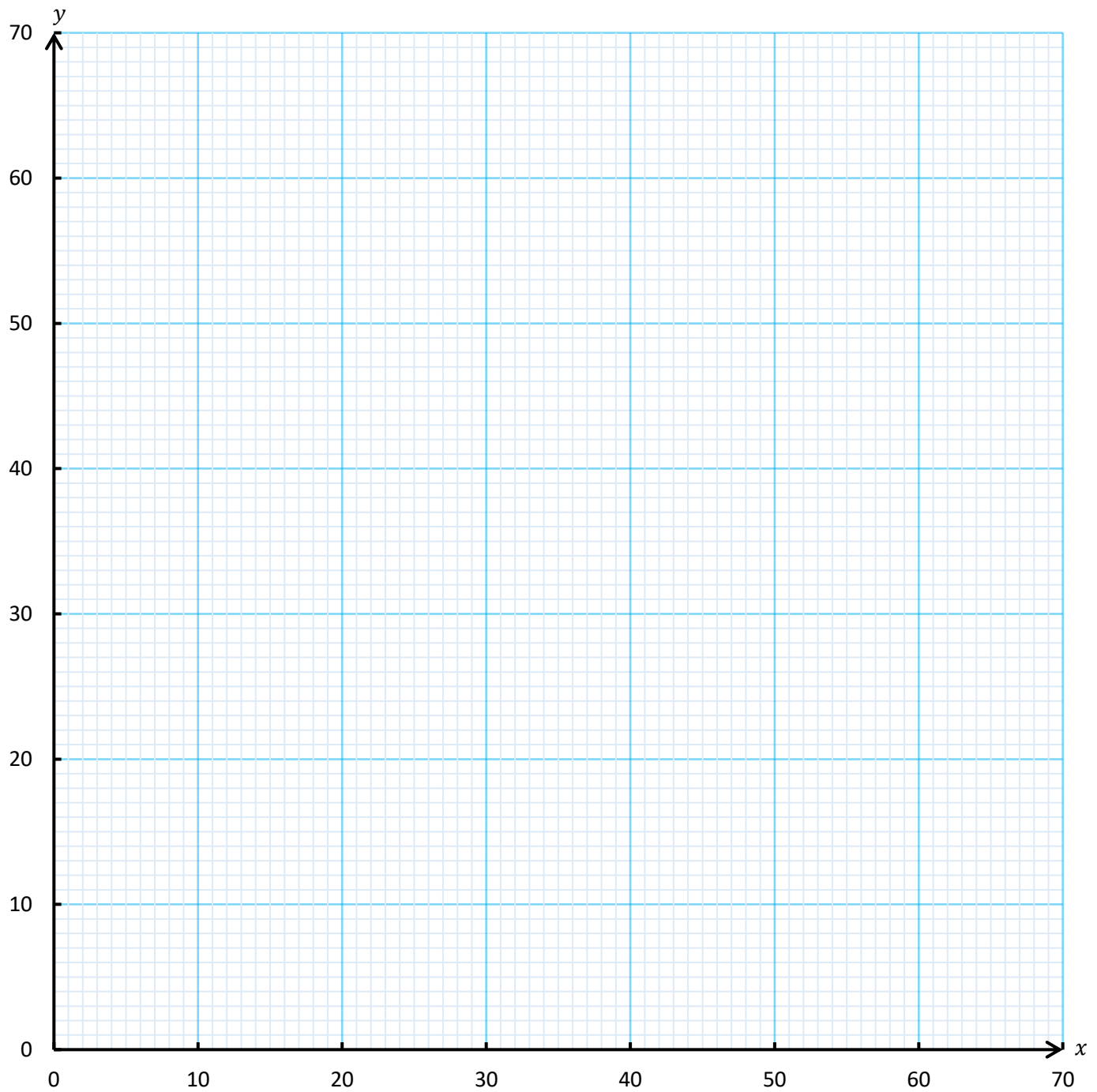
.....

.....

.....



(c) On the graph below, show all the constraints, as well as $x \geq 0$, $y \geq 0$.
Hence, label the feasible region with the letter R .



Rhys makes £2 profit on each bouquet of roses, and £3 profit on each bouquet of tuplis.
Rhys wishes to maximise his profit, £ P .

(d) Write down the objective function.

.....
.....

(e) Use your graph to find the optimal solution, noting clearly Rhys' optimal profit.

[illegible]

Exercise 7

Megan makes two types of party bag, which sell at £1 each.

Type A contains 2 balloons and 6 sweets.

Type B contains 3 balloons and 5 sweets.

Megan has a total of 20 balloons and 30 sweets, and wishes to make as much profit as possible.

(a) Letting a represent the number of party bags of Type A, and b represent the number of party bags of Type B, write a set of inequalities to represent the constraints of the problem.

.....

.....

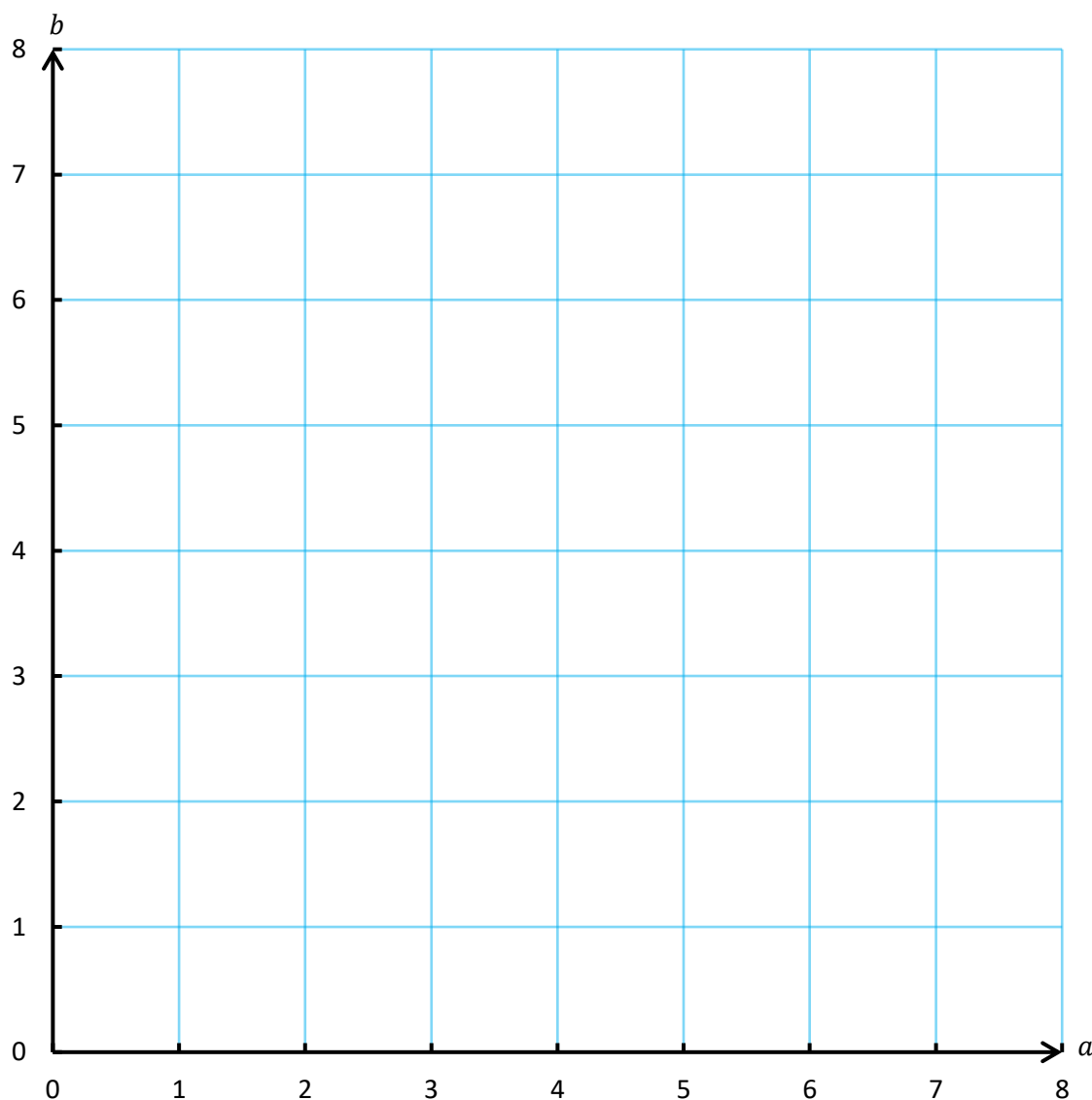
.....

.....

(b) Write down the objective function for the profit, £ P .

.....

(c) On the graph below, show all the constraints. Hence, label the feasible region with the letter R .



(d) Your feasible region should be a quadrilateral. Find the co-ordinates of each corner of this quadrilateral.

This image shows a full page of primary-ruled paper. It features approximately 28 horizontal dashed lines spaced evenly down the page, providing a guide for handwriting practice. The paper is otherwise blank, with no margins, text, or other markings.

(e) Find the optimal solution, clearly noting (i) the optimal profit, and (ii) how many bags of Type A and Type B Megan should produce.

[illegible]

Pop Quiz: Linear Programming

/9

(I) Class IIC have decided to sell apples and bananas during each break time this week, in order to raise money for charity. The profit on selling each apple is 20p, and the profit on selling each banana is 15p. They have done some market research which give them the following constraints.

- They will sell at most 800 pieces of fruit during the week.
- They will sell at least twice as many apples as bananas.
- They will sell between 50 and 150 bananas.

(a) Write the above constraints as a set of inequalities, letting a represent the number of apples sold, and b the number of bananas sold. [3]

.....

.....

.....

(b) Assuming they sell all their fruit, write down an objective function for the profit P . [1]

.....

(c) On the graph below, show all the constraints, as well as $a \geq 0$. Hence, label the feasible region with the letter R . [5]

