



Linear

Programming



Name:

## Background

### What is the work?

Formulating and solving simple optimisation problems in two variables.

### What are the prerequisites?

**GCSE Work:** Co-ordinates; plotting straight lines and linear inequalities; shading regions of graphs.

### Where does this lead to?

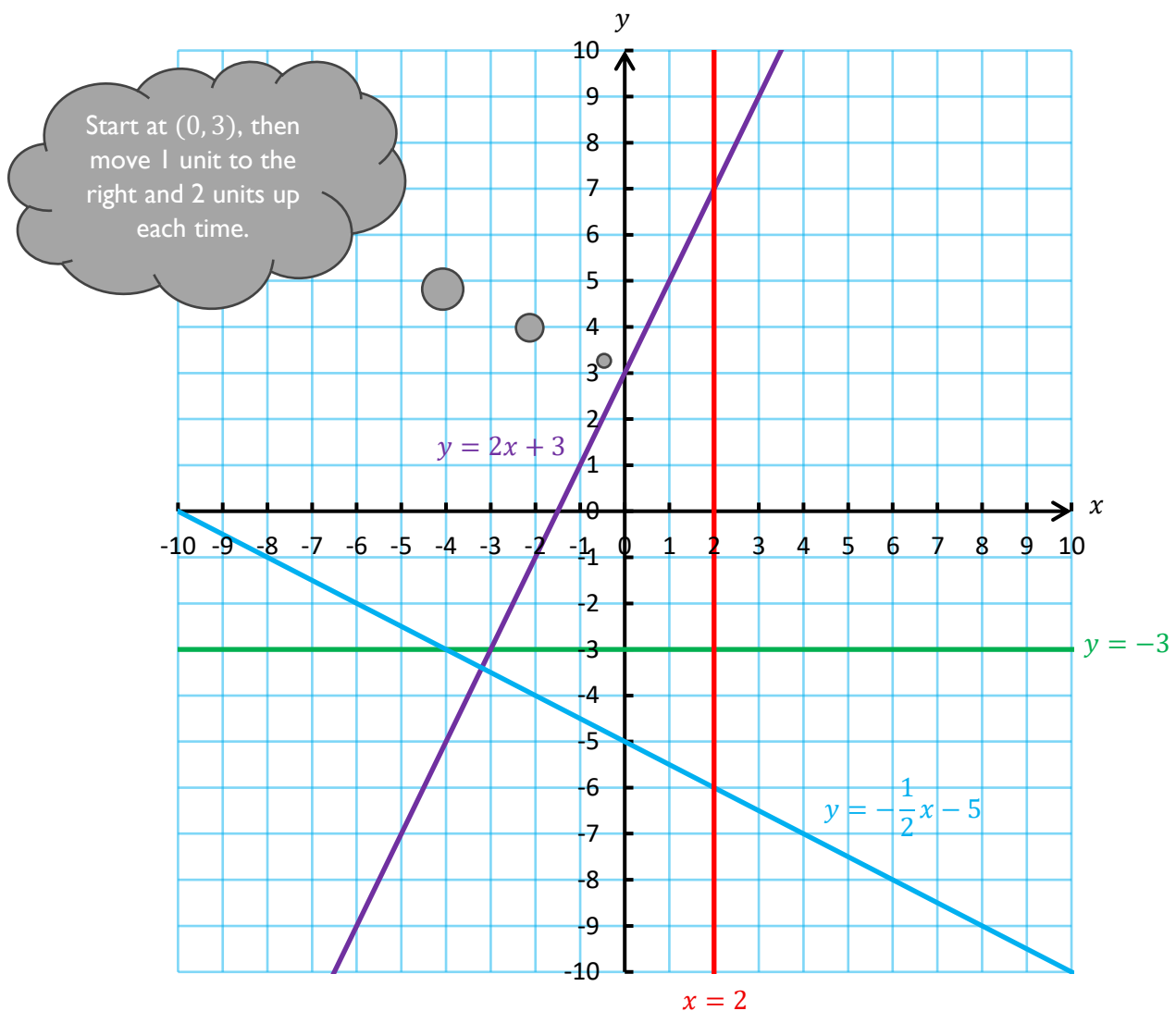
**Applications:** Linear programming can be used to reduce costs or increase profit by allocating limited resources such as time, labour and materials.

## Theory

### Straight Line Graphs

Plotting straight lines is part of the usual GCSE course.

#### Example 1



**Exercise 1**

On the graph paper below, plot the following lines.

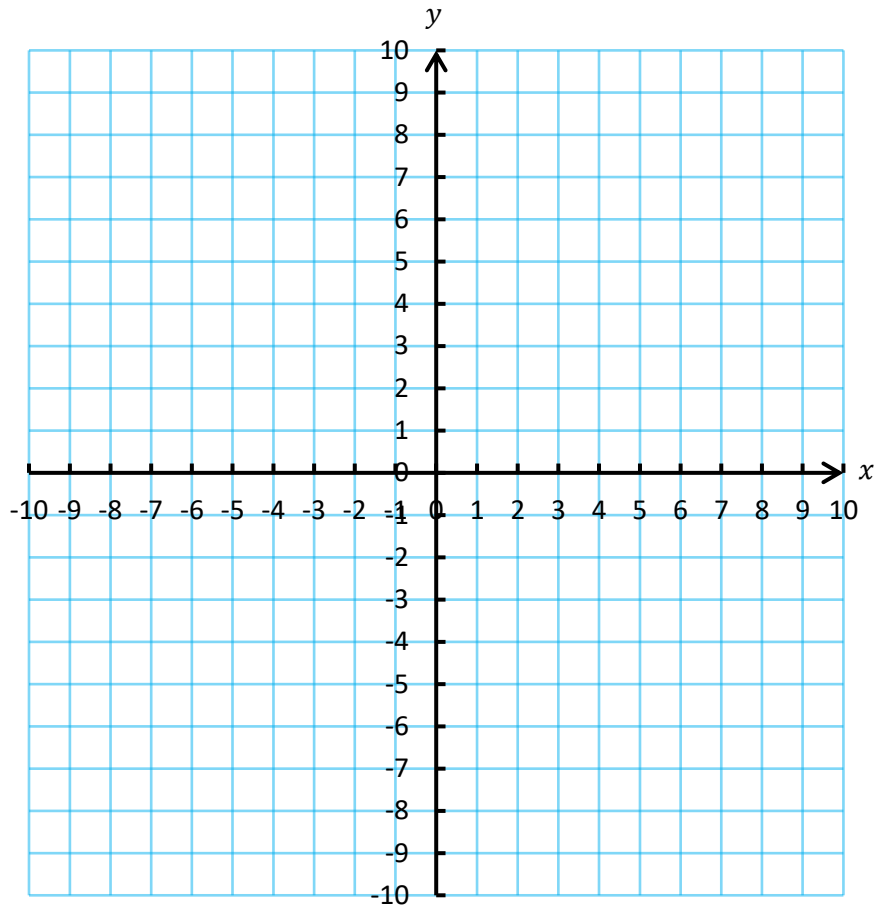
(a)  $y = 5$

(b)  $x = -3$

(c)  $y = 3x - 1$

(d)  $y = -2x + 7$

(e)  $y = \frac{1}{2}x$

**Example 2**

Plot a straight line for the equation  $2x + 3y - 6 = 0$ .

**The Hiding Method**

To find the value of  $y$  when  $x = 0$ , **hide** the term in  $x$  using your finger, and solve the equation that remains.

$$\begin{array}{lcl} 2x + 3y - 6 = 0 & \rightarrow & 3y - 6 = 0 \\ & & 3y = 6 \quad \text{[Add 6]} \\ & & y = 2 \quad \text{[Divide by 3]} \end{array}$$

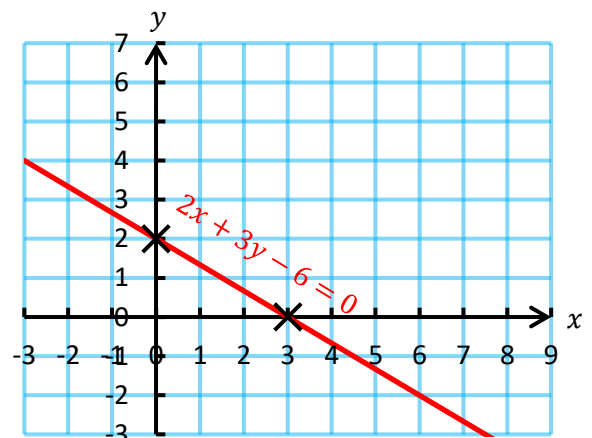
So, the line goes through the point  $(0, 2)$ .

To find the value of  $x$  when  $y = 0$ , **hide** the term in  $y$  with your finger, and solve the equation that remains.

$$\begin{array}{lcl} 2x + 3y - 6 = 0 & \rightarrow & 2x - 6 = 0 \\ & & 2x = 6 \quad \text{[Add 6]} \\ & & x = 3 \quad \text{[Divide by 2]} \end{array}$$

So, the line goes through the point  $(3, 0)$ .

To plot the line for the equation  $2x + 3y - 6 = 0$ , plot the two points  $(0, 2)$  and  $(3, 0)$  on graph paper before connecting them with a straight line.

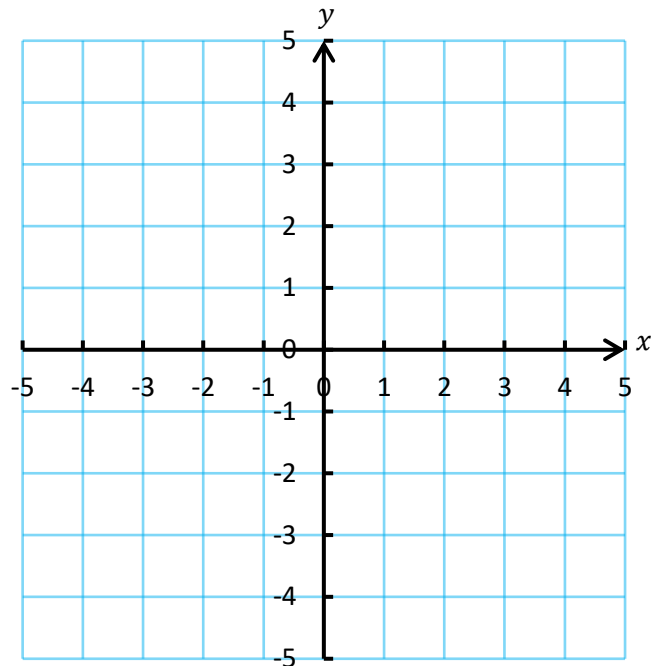
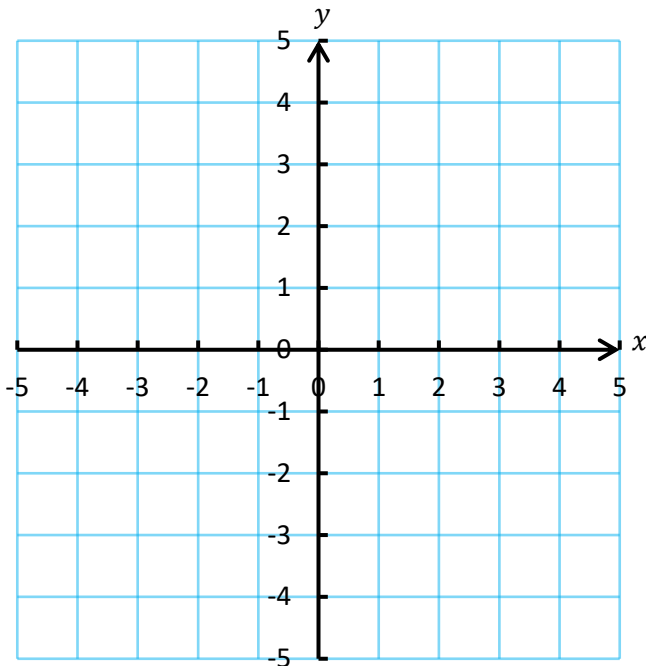


**Exercise 2**

Use the graph paper below to plot the following lines.

(a)  $2x + 4y - 8 = 0$

(b)  $3x - 2y - 6 = 0$

**Shading Regions**

It is possible to define a region on graph paper using a set of inequalities.

**Example 3**

Shade the region defined by the following inequalities.

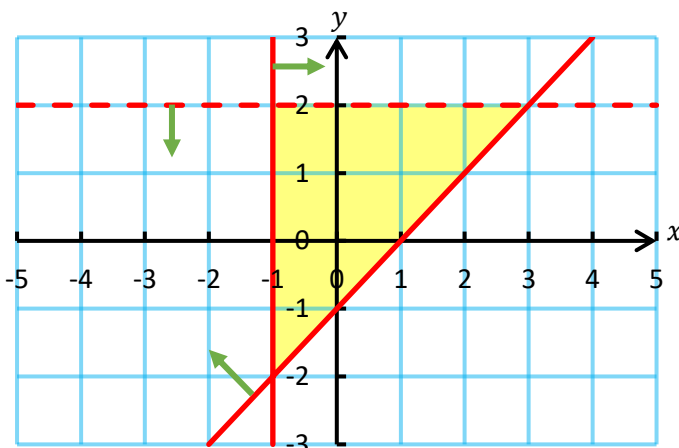
$$y < 2, \quad x \geq -1, \quad y \geq x - 1$$

Answer: **Step 1:** Plot the graphs of  $y = 2$ ,  $x = -1$ ,  $y = x - 1$ .

**Rule:** Use a solid line ( — ) for inequalities containing  $\geq$  or  $\leq$ ; and a dotted line ( - - - ) for inequalities containing  $>$  or  $<$ .

**Step 2:** Note, using an arrow, a region corresponding to each of the three lines.

**Step 3:** Shade the region that is satisfied by all of the arrows / inequalities.



For lines that aren't vertical or horizontal, **substitute a point that doesn't lie on the line to decide which way the arrow should point.** For example, considering  $y \geq x - 1$ , substitute the point  $(0, 0)$ :

$$0 \geq 0 - 1$$

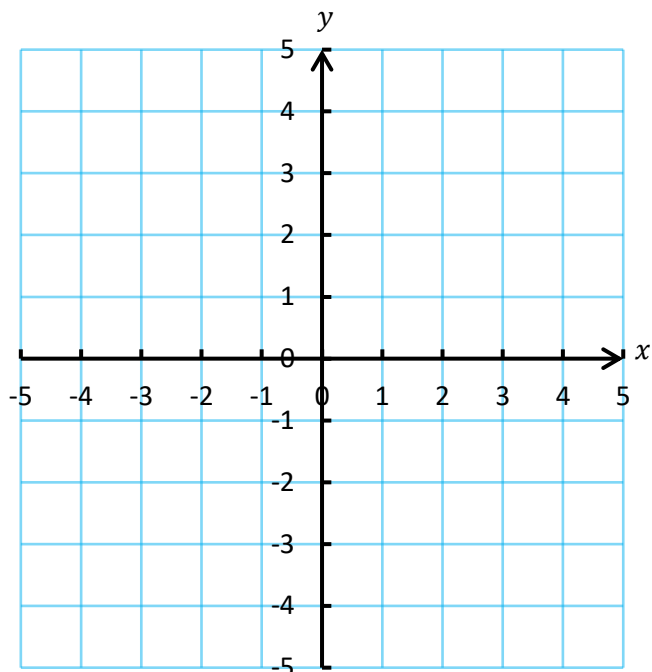
$$0 \geq -1$$

This inequality is **true**, so the arrow should point **towards** the point  $(0, 0)$ .

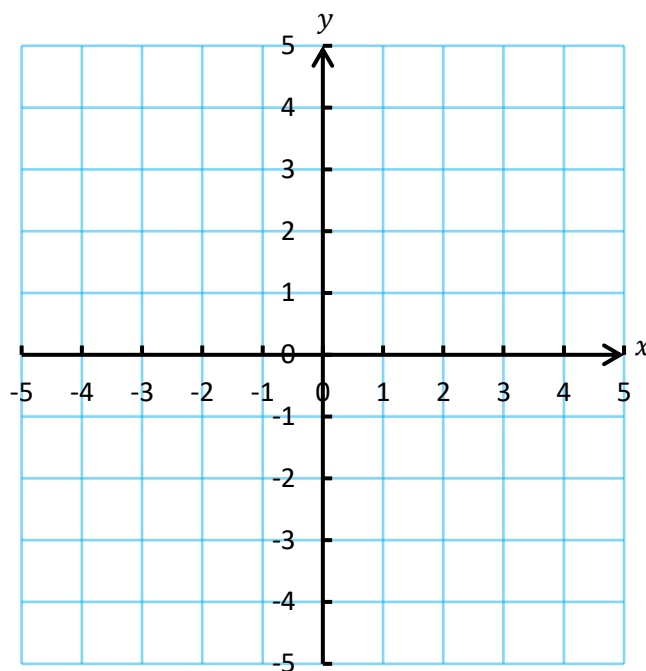
**Exercise 3**

Shade the region that is defined by the following sets of inequalities.

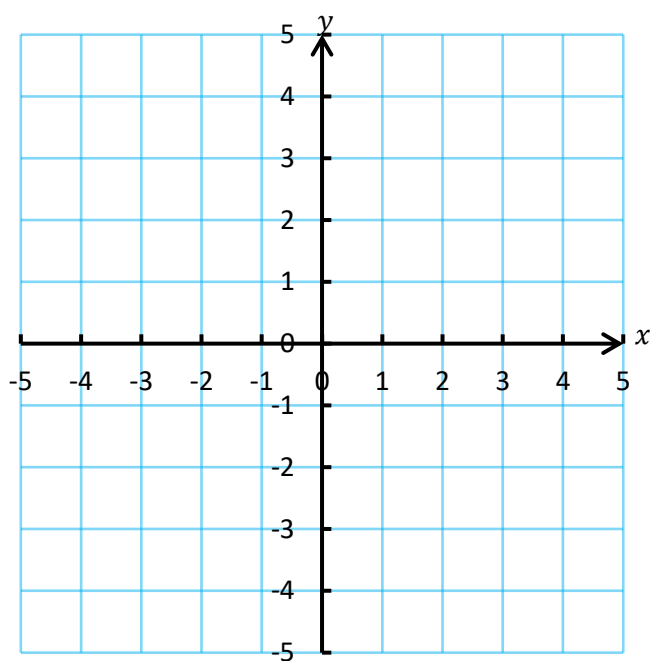
(a)  $y < 3$ ,  $x < 2$ ,  $y \geq -1$ ,  $x \geq -3$



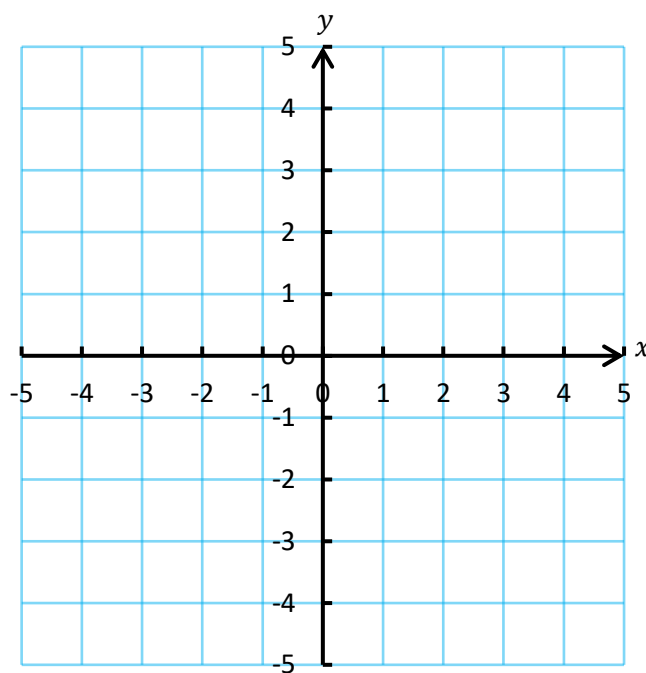
(b)  $y \leq 4$ ,  $x \geq -1$ ,  $y \geq 2x - 1$



(c)  $y > 0$ ,  $x \geq 1$ ,  $x + 3y - 3 \leq 0$



(d)  $y > -2$ ,  $y < x + 2$ ,  $y < -2x + 4$



## Linear Programming

### Example 4

Awel has bought 100 lemons and 30 kg of sugar to sell products at her stall at the local fair.

Awel sells two products:

- Lemonade, which uses 1 lemon and 25 g of sugar for each cup sold;
- Lemon Curd, which uses 2 lemons and 125 g of sugar for each jar sold.

Awel makes £1 profit selling a cup of lemonade, and £3 profit selling a jar of lemon curd. How much lemonade and lemon curd should Awel make to maximise her profit?

*Answer:* To begin, we need to represent the problem as a **linear programming problem**.

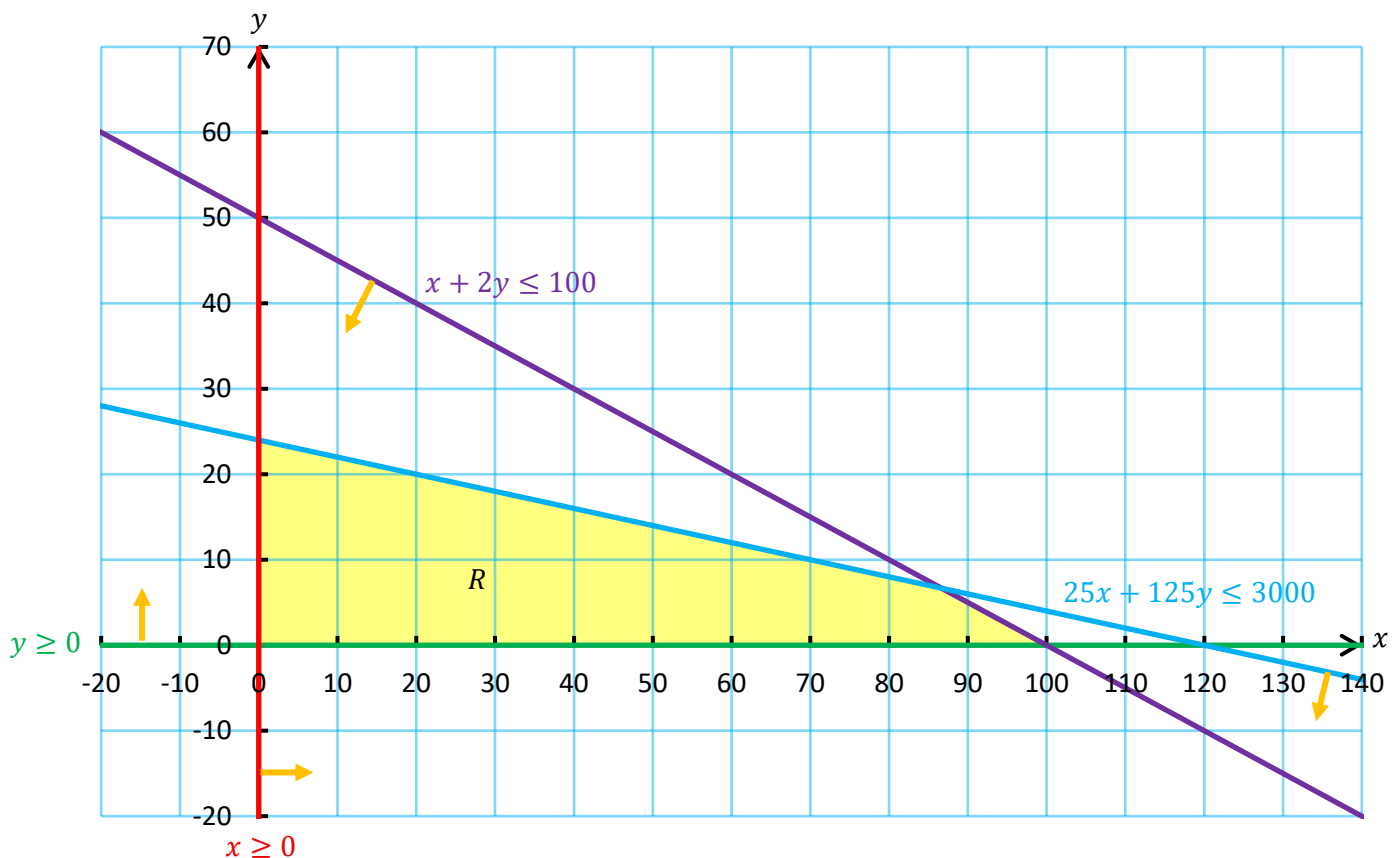
Let  $x$  represent how many cups of lemonade Awel produces, and let  $y$  represent how many jars of lemon curd Awel produces. These are the **decision variables** for the problem. Considering the ingredients, we can write the **constraints**

$$\begin{aligned} x + 2y &\leq 100 && \text{(lemons)} \\ 25x + 125y &\leq 3000 && \text{(sugar, in grams)} \end{aligned}$$

Also, we must have  $x \geq 0$ ,  $y \geq 0$  (it is not sensible for  $x$  and  $y$  to be negative).

The total profit is  $\text{£}1 \times x + \text{£}3 \times y$ , so  $E = x + 3y$  is the **objective function** that must be maximised.

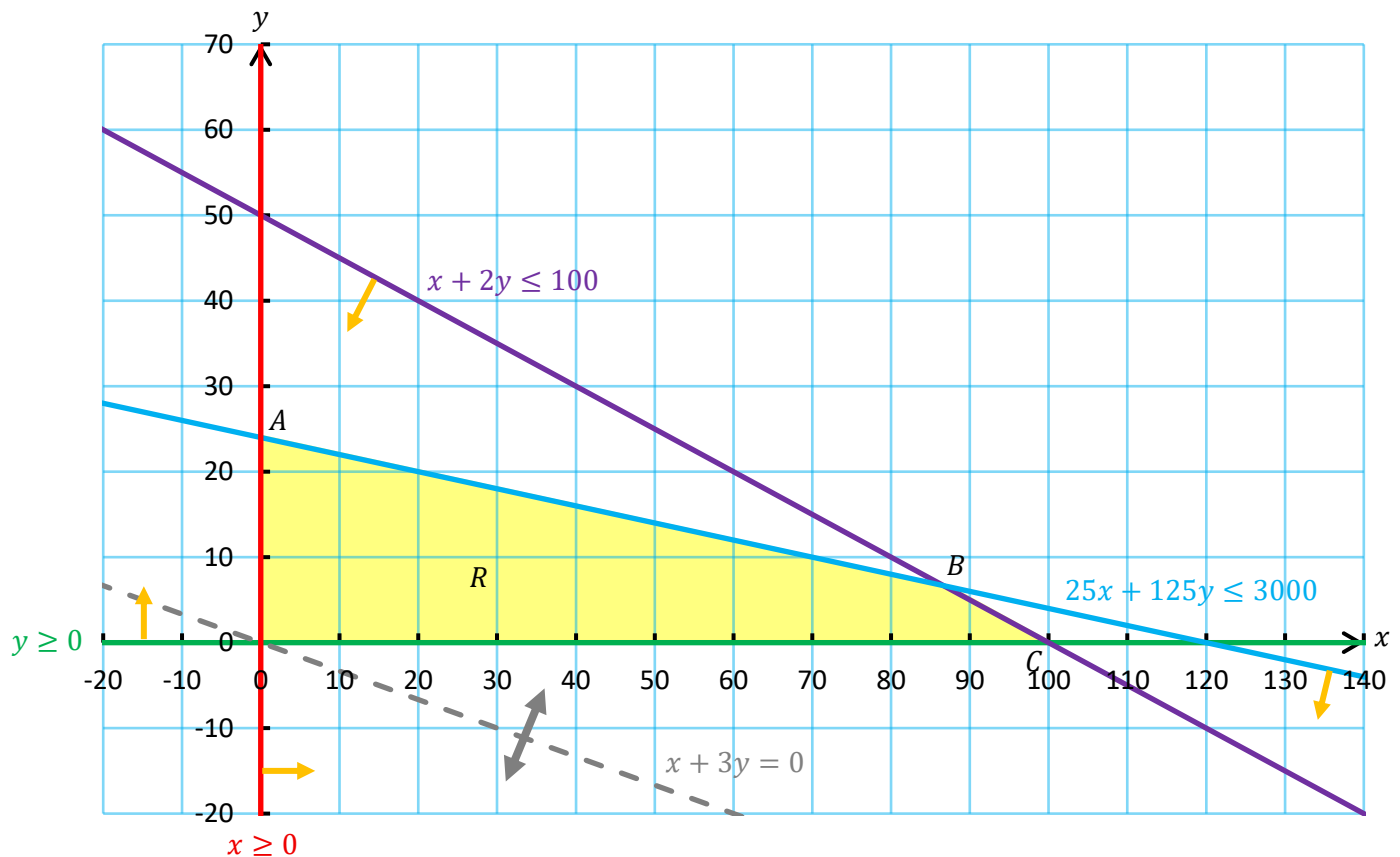
We can draw the above inequalities on the graph below. The yellow region  $R$  shows the **feasible region**.



To find the **optimal solution**, we must imagine the line  $E = x + 3y$  moving along the graph.

This line is parallel to the line  $x + 3y = 0$ , which is shown on the graph below.

(An interactive GeoGebra resource to show this: <https://www.geogebra.org/classic/varzdkn9>)



To maximise  $E$ , the line  $E = x + 3y$  must be as far as possible from the origin.

So, the optimal solution lies on one of the corners of the feasible region (at  $A$ ,  $B$  or  $C$ ).

At  $A$ ,  $x = 0$  in the equation  $25x + 125y = 3000$ .

So, we have  
 $125y = 3000$   
 $y = 24$ .

If  $x = 0$ ,  $y = 24$ , the profit is  
 $E = 0 + 3 \times 24$   
 $E = £72$

At  $B$ , the lines  $x + 2y = 100$  and  $25x + 125y = 3000$  intersect.

Multiplying the first equation by 25:  
 $25x + 50y = 2500$

Subtracting this equation from the second equation:  
 $75y = 500$   
 $y = 6\frac{2}{3}$

Substituting back:

$$x + 2\left(6\frac{2}{3}\right) = 100$$

$$x = 100 - 13\frac{1}{3}$$

$$x = 86\frac{2}{3}$$

If  $x = 86\frac{2}{3}$ ,  $y = 6\frac{2}{3}$ , the profit is

$$E = 86\frac{2}{3} + 3 \times 6\frac{2}{3}$$

$$E = £106.67, \text{ to the nearest penny.}$$

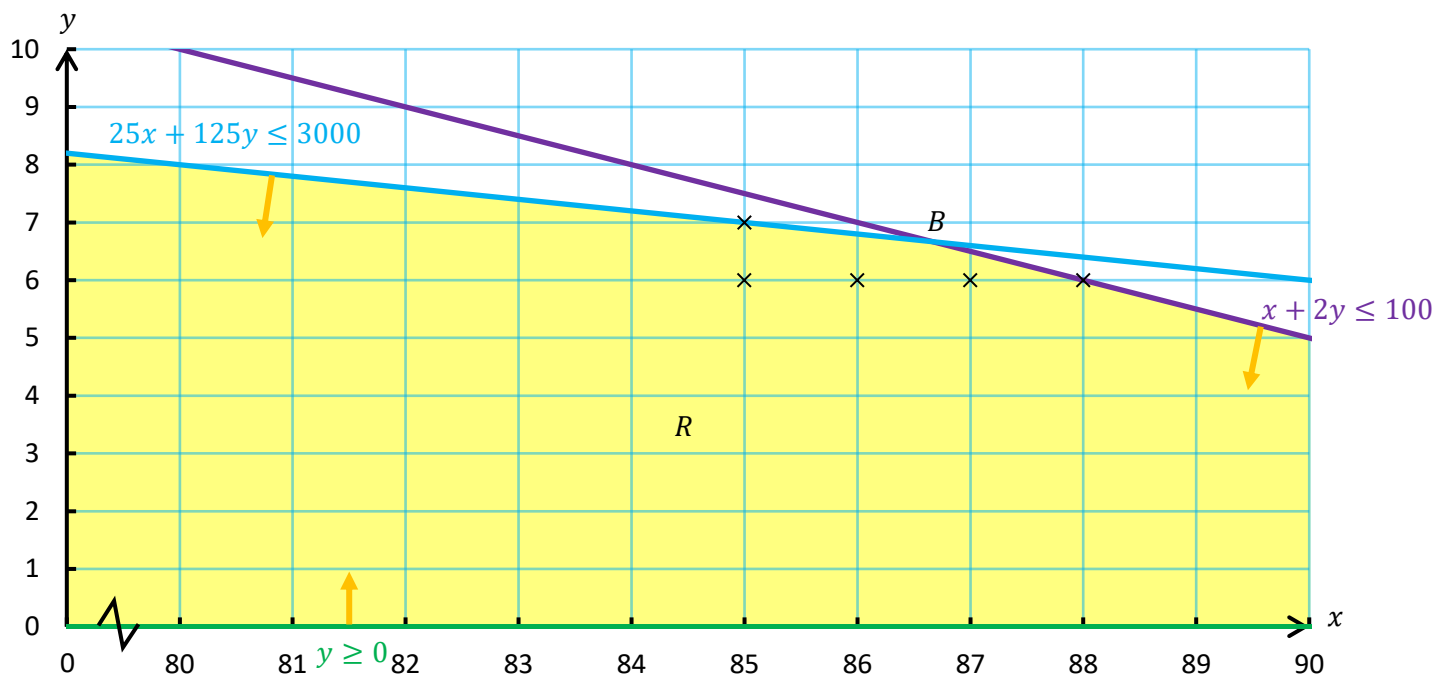
At  $C$ ,  $y = 0$  in the equation  $x + 2y = 100$ .

So, we have  
 $x = 100$ .

If  $x = 100$ ,  $y = 0$ , the profit is  
 $E = 100 + 3 \times 0$   
 $E = £100$

We see that the largest profit is found at the point  $B$ . However,  $x$  and  $y$  are not whole numbers at this point, so we need to inspect nearby points with whole number co-ordinates.

Zooming into the graph around the point  $B$ , we see the following situation.



The points with whole number co-ordinates that are in  $R$  and are close to  $B$  are  $(85, 7)$ ,  $(85, 6)$ ,  $(86, 6)$ ,  $(87, 6)$  and  $(88, 6)$ .

Considering the objective function  $E = x + 3y$ ,  $E$  will be at its largest at either  $(85, 7)$  or  $(88, 6)$ .

If  $x = 85$ ,  $y = 7$ , the profit is

$$E = 85 + 3 \times 7$$

$$E = £106$$

If  $x = 88$ ,  $y = 6$ , the profit is

$$E = 88 + 3 \times 6$$

$$E = £106$$

So, for this problem, there are two optimal solutions: Awel should either produce **85 lemonades and 7 lemon curds**, or **88 lemondades and 6 lemon curds**. By doing this, Awel's profit will be £106.

### WJEC Terminology

| Term               | Definition                                                                                                                              |
|--------------------|-----------------------------------------------------------------------------------------------------------------------------------------|
| Decision variable  | A variable associated with linear programming problems. A solution is formed by assigning appropriate values to all decision variables. |
| Constraint         | A restriction on the values of the decision variables.                                                                                  |
| Objective function | The function that defines the criteria for evaluating the solution.                                                                     |
| Feasible solution  | Values for the decision variables that satisfy each of the constraints in the linear programming problem.                               |
| Feasible region    | The region that contains all the feasible solutions.                                                                                    |
| Optimal solution   | A feasible solution that has the optimum objective function value.                                                                      |

**Exercise 4**

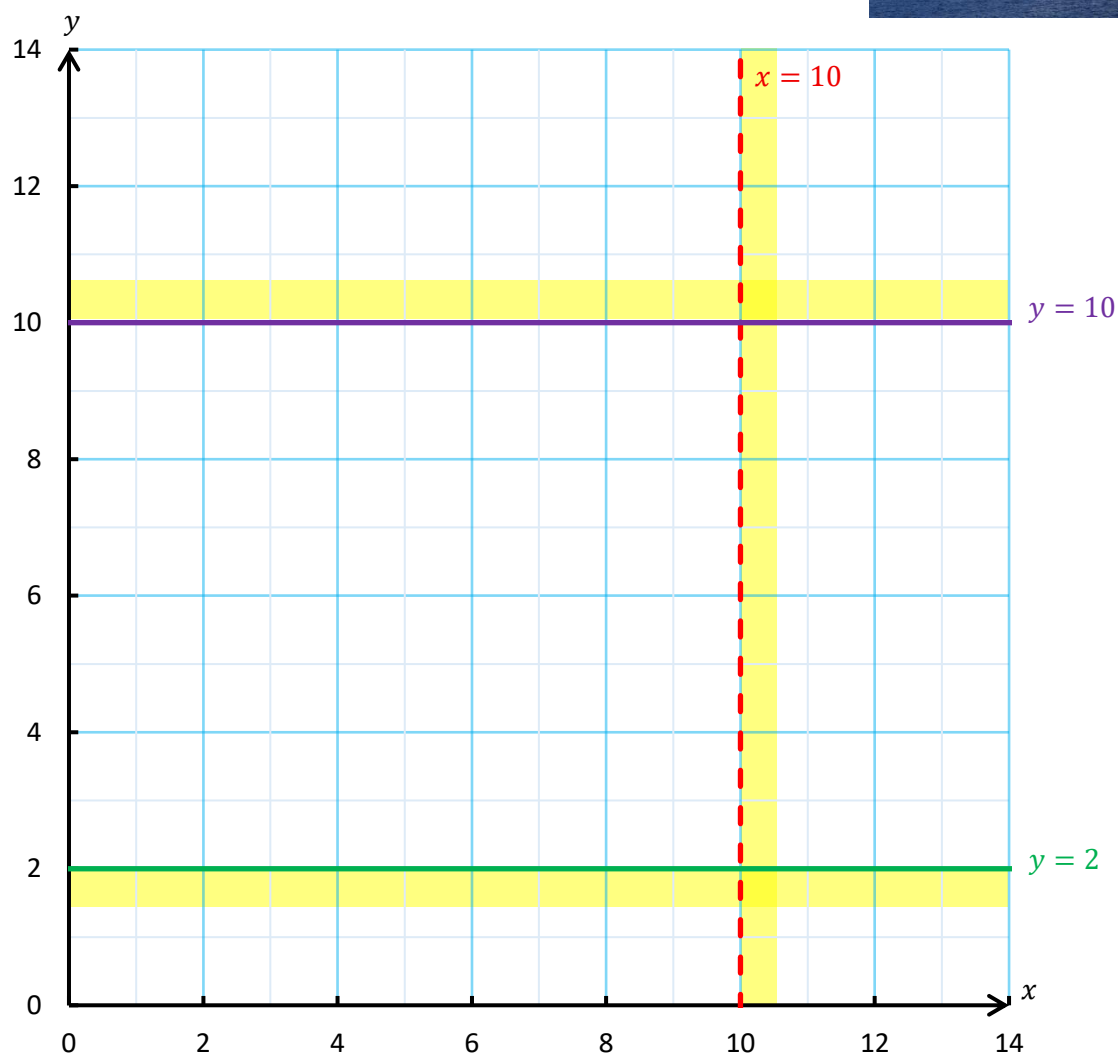
The captain of a pleasure boat charges passengers to take a lake trip on her boat.

The number of children is represented by  $x$ , and the number of adults by  $y$ .

Two of the constraints that control how many passengers can be transported are

$$x < 10 \text{ and } 2 \leq y \leq 10$$

These are shown on the graph below, where the rejected regions are shaded yellow.



(a) Explain why the line  $x = 10$  is shown as a dotted line.

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(b) Use the constraints to write statements that describe the number of children and the number of adults that can be transported on the boat.

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For each journey, the tickets cost £2 per child and £3 per adult. The total tickets sold must be at least £24 for the boat to be able to sail.

It is not permitted for double the number of children on the boat to be greater than the number of adults.

(c) Use this information to write two inequalities.

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(d) Add two lines to the graph on the previous page to show your inequalities.  
Hence, label the feasible region with the letter  $R$ .

(e) Use your graph to decide how many children and how many adults go on the trip if the captain takes

(i) the least number of passengers,

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(ii) the greatest number of passengers.

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(f) Use your graph to find the maximum profit the captain can make each time the boat sails.

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### Exercise 5

Eifion is arranging two types of sporting event, 'Cracking Cricket' and 'Mastering Mountaineering'. He must decide how many times each event should be held.

Let  $x$  represent how many times the event 'Cracking Cricket' is held.

Let  $y$  represent how many times the event 'Mastering Mountaineering' is held.

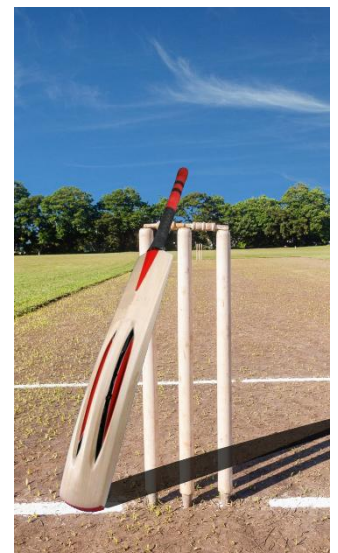
Two of the constraints are

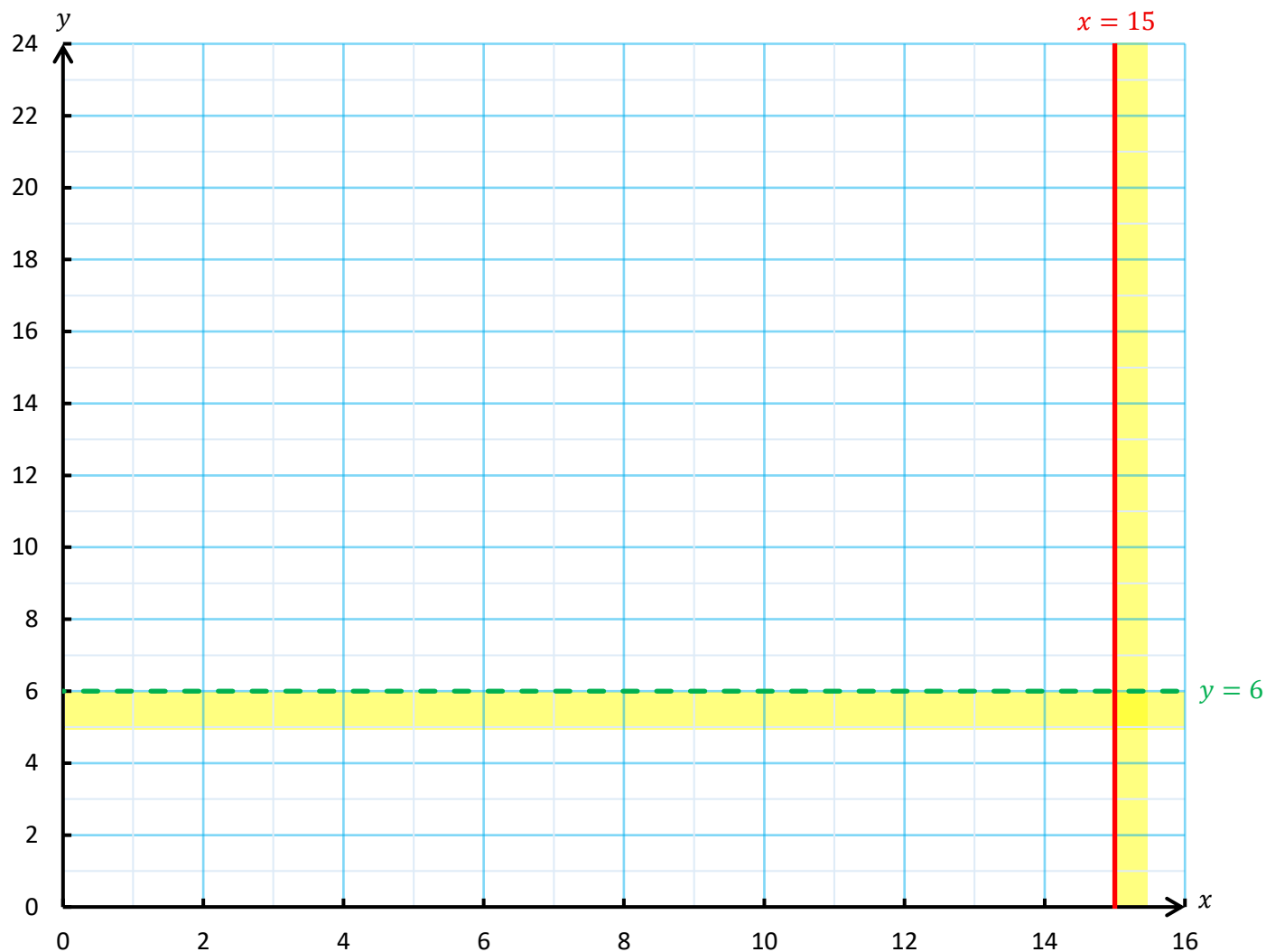
$$x \leq 15$$

and

$$y > 6$$

There are shown on the graph on the next page, where the rejected regions are shaded yellow.





Two more constraints are

$$3x \leq 2y \quad \text{and} \quad 5x + 4y \leq 80$$

(a) Add lines to the above graph to represent the two new constraints.

Hence, label the feasible region with the letter  $R$ .

Each 'Cracking Cricket' event costs £500 to run.

Each 'Mastering Mountaineering' event costs £800 to run.

Eifion wants to minimise the total costs.

(b) Write an objective function for the costs,  $C$ , in terms of  $x$  and  $y$ .

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(c) Use your graph to decide how many times each event should be held, and what the total costs are.

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[illegible]

Rhys sells bouquets of flowers at his local market. He sells two types of bouquets: roses and tulips.

Each bouquet of roses costs £7 to make and each bouquet of tulips costs £5 to make.

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$$y \leq 20 \quad \text{and} \quad y \leq 4x$$

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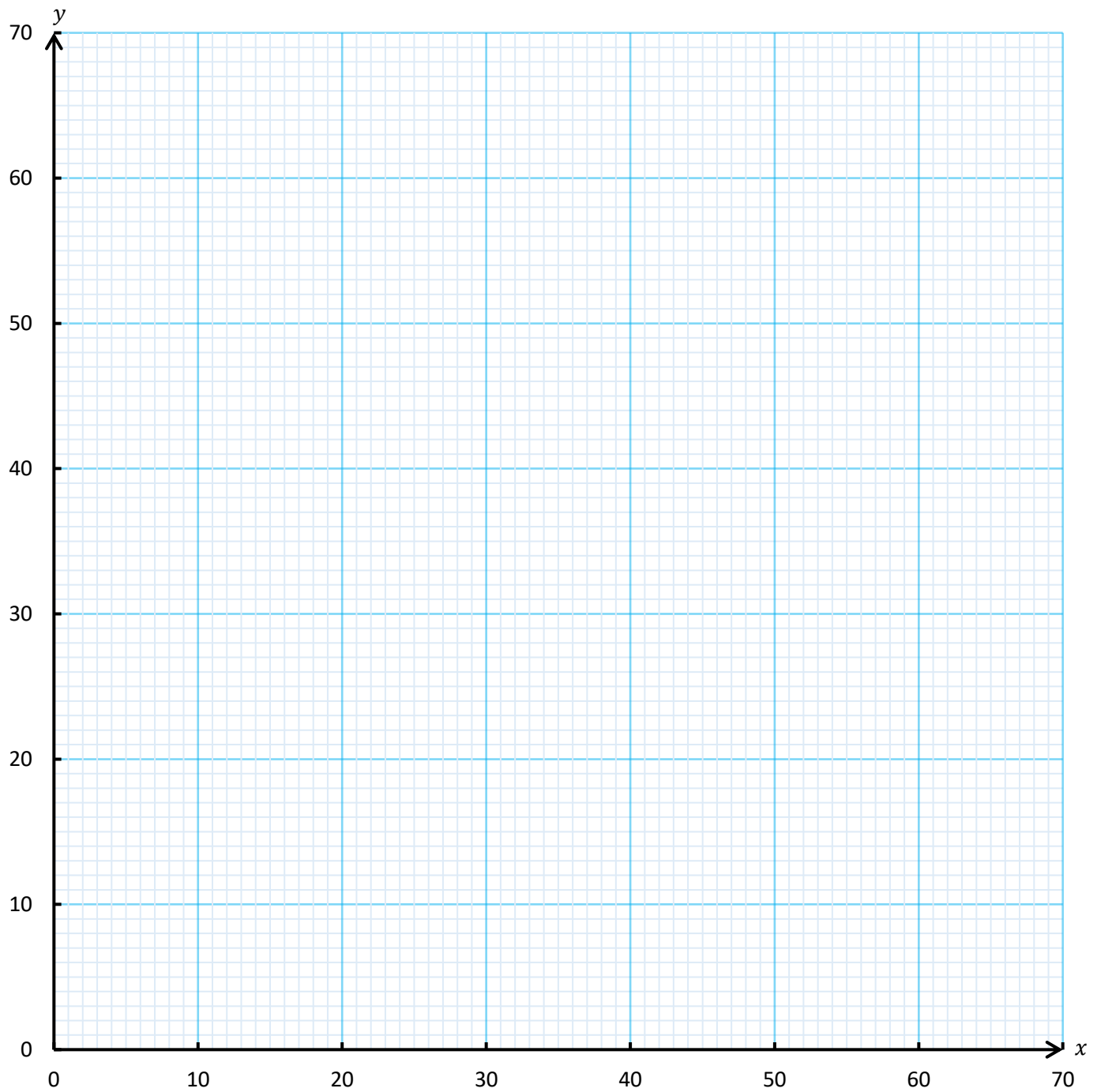
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(c) On the graph below, show all the constraints, as well as  $x \geq 0$ ,  $y \geq 0$ .  
Hence, label the feasible region with the letter  $R$ .



Rhys makes £2 profit on each bouquet of roses, and £3 profit on each bouquet of tuplis.  
Rhys wishes to maximise his profit, £ $P$ .

(d) Write down the objective function.

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(e) Use your graph to find the optimal solution, noting clearly Rhys' optimal profit.

This image shows a single page of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

**Exercise 7**

Megan makes two types of party bag, which sell at £1 each.

Type A contains 2 balloons and 6 sweets.

Type B contains 3 balloons and 5 sweets.

Megan has a total of 20 balloons and 30 sweets, and wishes to make as much profit as possible.

(a) Letting  $a$  represent the number of party bags of Type A, and  $b$  represent the number of party bags of Type B, write a set of inequalities to represent the constraints of the problem.

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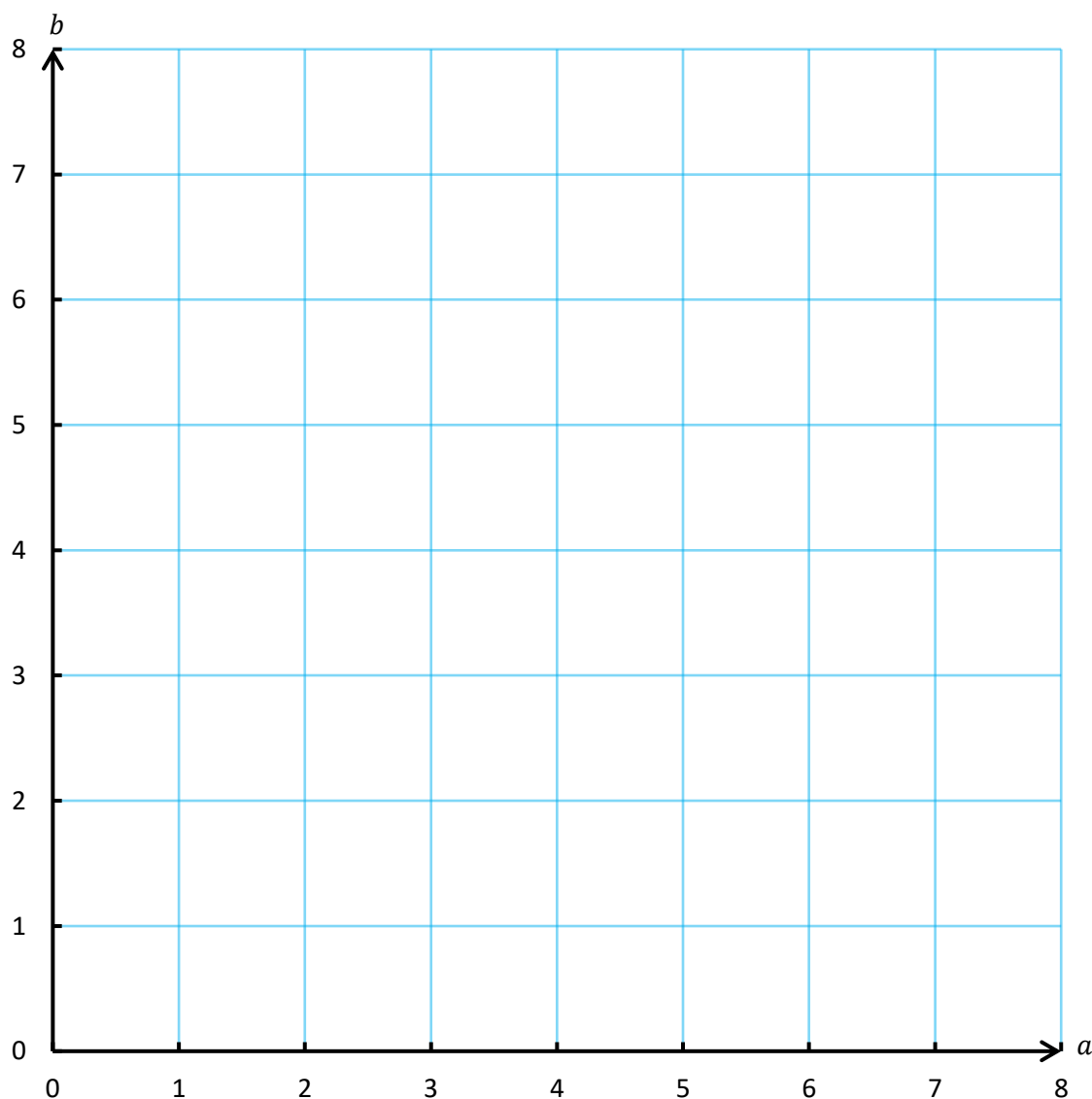
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(b) Write down the objective function for the profit, £ $P$ .

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(c) On the graph below, show all the constraints. Hence, label the feasible region with the letter  $R$ .



(d) Your feasible region should be a quadrilateral. Find the co-ordinates of each corner of this quadrilateral.

This image shows a full page of primary-ruled paper. It consists of numerous horizontal dashed lines spaced evenly apart, providing a guide for handwriting practice. The lines are black and extend across the entire width of the page. There are no margins, text, or other markings present.

(e) Find the optimal solution, clearly noting (i) the optimal profit, and (ii) how many bags of Type A and Type B Megan should produce.

[illegible]

**Pop Quiz: Linear Programming**

/9

(I) Class IIC have decided to sell apples and bananas during each break time this week, in order to raise money for charity. The profit on selling each apple is 20p, and the profit on selling each banana is 15p. They have done some market research which give them the following constraints.

- They will sell at most 800 pieces of fruit during the week.
- They will sell at least twice as many apples as bananas.
- They will sell between 50 and 150 bananas.

(a) Write the above constraints as a set of inequalities, letting  $a$  represent the number of apples sold, and  $b$  the number of bananas sold. [3]

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(b) Assuming they sell all their fruit, write down an objective function for the profit  $P$ . [1]

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(c) On the graph below, show all the constraints, as well as  $a \geq 0$ . Hence, label the feasible region with the letter  $R$ . [5]

