



Quadratic

Functions

$$y = x^2 - 6x + 5$$

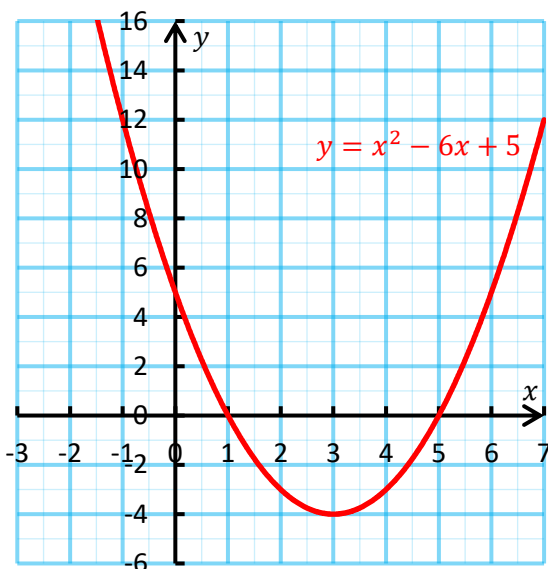
y-intercept (0, 5)

$$y = (x - 3)^2 - 4$$

Minimum point (3, -4)

$$y = (x - 1)(x - 5)$$

Roots $x = 1$, $x = 5$



Name: _____

Background

What is the work?

Consider the shape, solution and use of quadratic equations.

What are the prerequisites?

GCSE Work: Solving quadratic equations, graph plotting, solving simultaneous equations.

Where does this lead to?

Applications: Consider how a projectile (such as a javelin) moves; computer graphics; bridge design.

Theory

A **quadratic function** has the form $ax^2 + bx + c$, where a , b and c are constants with $a \neq 0$.

A **quadratic equation** has the form $ax^2 + bx + c = 0$.

We can plot a quadratic function on graph paper, and it always takes the form of a curve with a 'u' or 'n' shape.

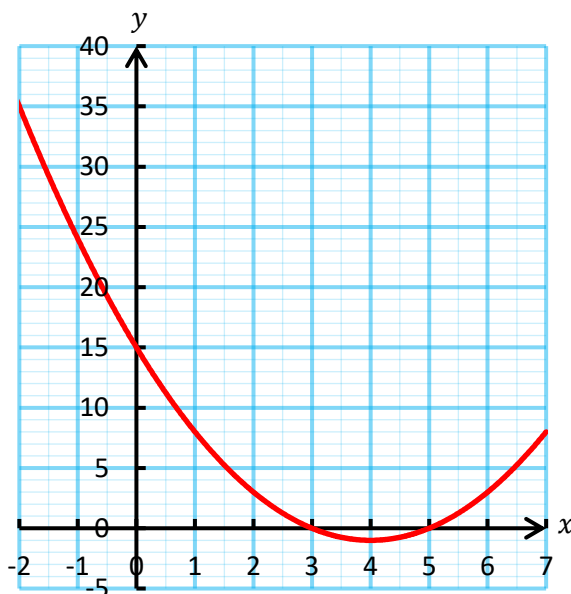
If $a > 0$, then the curve has a 'u' shape, and it has a minimum point.

If $a < 0$, then the curve has an 'n' shape, and it has a maximum point.

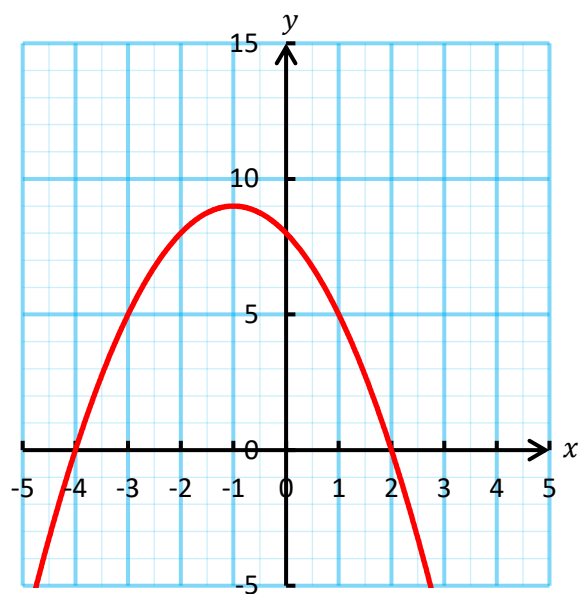
(If $a = 0$, then the function is a linear function, not a quadratic function.)

Example 1

$$y = x^2 - 8x + 15$$



$$y = -x^2 - 2x + 8$$



From the functions and the graphs, we can see the following properties.

The y-intercept is (0, 15).

The roots are $x = 3$ and $x = 5$.

The minimum point is (4, -1).

The y-intercept is (0, 8).

The roots are $x = -4$ and $x = 2$.

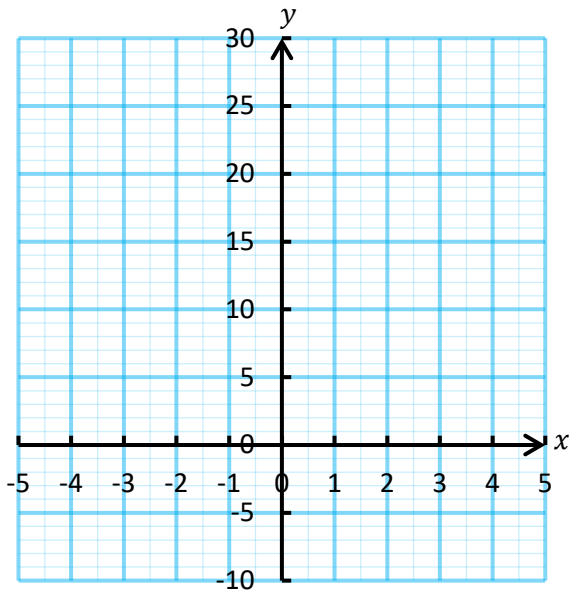
The maximum point is (-1, 9).

Exercise 1

Plot the following quadratic functions.

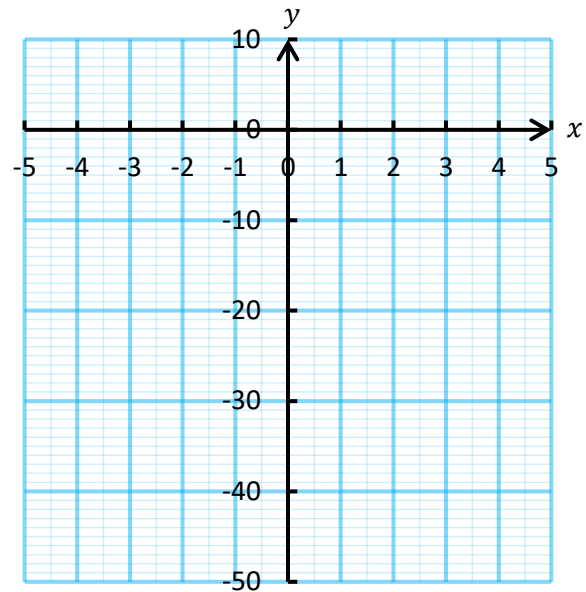
(a) $y = x^2 + 2x - 8$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											



(b) $y = -x^2 + 4x$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											



Complete the following sentences.

The y -intercept is

The roots are $x = \dots$ and $x = \dots$

The minimum point is

The y -intercept is

The roots are $x = \dots$ and $x = \dots$

The maximum point is

Exercise 2

Consider the quadratic function $y = x^2 + 4x - 5$.

(a) Will the graph for this function be 'u'-shaped, or 'n'-shaped?

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(b) Complete the following table.

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
y												

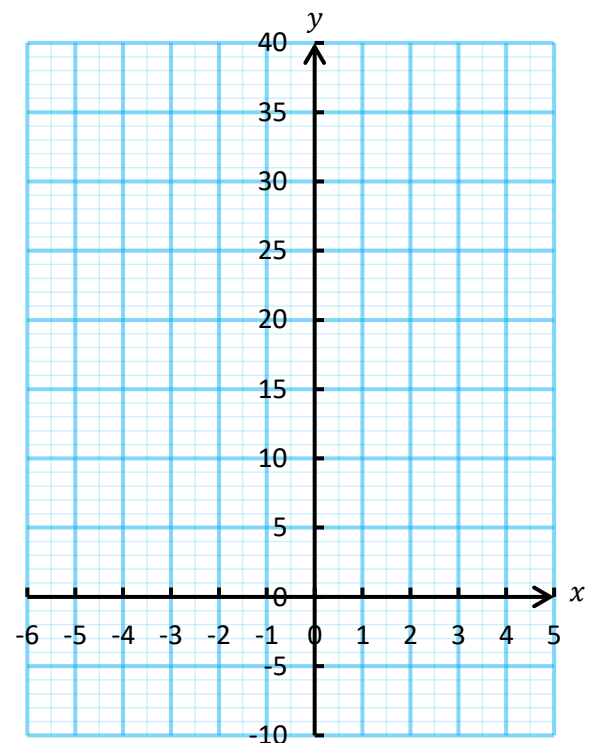
(c) Plot the graph of the function on the graph paper on the right.

(d) Complete the following sentences.

The y -intercept is

The roots are $x = \dots$ and $x = \dots$

The minimum point is



Factorising Quadratic Expressions

It is possible to factorise some quadratic expressions using GCSE methods.

Exercise 3

Factorise: (a) $x^2 + 6x + 8$

(b) $x^2 + 4x - 12$

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Example 2

Factorise the quadratic expression $3x^2 - 7x - 20$.

The Splitting Method

$$3 \times -20 = -60$$

The two numbers that multiply to give -60 and add to give -7 are 5 and -12 .

$$\begin{aligned} 3x^2 - 7x - 20 &= 3x^2 + 5x - 12x - 20 \\ &= x(3x + 5) - 4(3x + 5) \\ &= (3x + 5)(x - 4) \end{aligned}$$

The Detective Method

$3x \times x = 3x^2$, so it is possible to factorise $3x^2 - 7x - 20$ in the form $(3x + ?)(x + ?)$.

By attempting pairs of numbers that multiply to give -20 , we see that $3x^2 - 7x - 20 = (3x + 5)(x - 4)$.

Exercise 4

Factorise: (a) $2x^2 + 13x + 15$

(b) $3x^2 - 35x - 12$

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(c) $4x^2 + 9x + 5$

(d) $8x^2 - 14x + 6$

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Factorising Expressions with Higher Indices

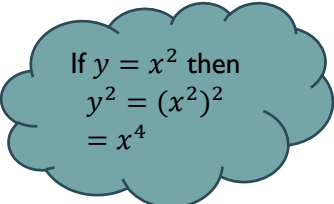
It is possible to use the above techniques to factorise **some** expressions with indices that are greater than 2.

Example 3

Factorise the quartic expression $2x^4 + 9x^2 + 4$.

We can use the substitution $y = x^2$ to change the expression to be a quadratic expression:

$$\begin{aligned} 2x^4 + 9x^2 + 4 &= 2y^2 + 9y + 4 \\ &= 2y^2 + 8y + y + 4 \\ &= 2y(y + 4) + 1(y + 4) \\ &= (y + 4)(2y + 1) \\ &= (x^2 + 4)(2x^2 + 1) \end{aligned}$$



If $y = x^2$ then
 $y^2 = (x^2)^2$
 $= x^4$

Exercise 5

Use an appropriate substitution to factorise the following expressions.

(a) $2x^4 + 7x^2 + 6$

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(b) $5x^4 + 18x^2 + 9$

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(c) $3x^6 + 13x^3 + 4$

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(d) $4x^8 + 21x^4 - 18$

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(e) $2x^{24} - 21x^{12} + 40$

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(f) $x^{100} + 3x^{50} + 2$

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Solving Equations

We can use the above factorising techniques to solve some equations.

Example 4

(a) Solve the equation $x^2 + 2x - 8 = 0$

$$(x + 4)(x - 2) = 0$$

Either $x + 4 = 0$ or $x - 2 = 0$

$$x = -4 \quad \text{or} \quad x = 2$$

(c) Solve the quartic equation

$$x^4 - 13x^2 + 36 = 0$$

We can use the substitution $y = x^2$ to change the expression to be a quadratic expression:

$$y^2 - 13y + 36 = 0$$

$$(y - 4)(y - 9) = 0$$

$$(x^2 - 4)(x^2 - 9) = 0$$

Either $x^2 - 4 = 0$ or $x^2 - 9 = 0$

$$x^2 = 4 \quad \text{or} \quad x^2 = 9$$

$$x = \pm\sqrt{4} \quad \text{or} \quad x = \pm\sqrt{9}$$

Either $x = 2$, $x = -2$, $x = 3$ or $x = -3$.

(b) Solve the equation $2x^2 + 3x - 20 = 0$

$$2 \times -20 = -40$$

$$2x^2 + 8x - 5x - 20 = 0$$

$$2x(x + 4) - 5(x + 4) = 0$$

$$(x + 4)(2x - 5) = 0$$

Either $x + 4 = 0$ or $2x - 5 = 0$

$$x = -4 \quad \text{or} \quad x = \frac{5}{2}$$

Exercise 6

Solve the following equations by factorising.

(a) $x^2 + 12x + 32 = 0$

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(b) $2x^2 - 13x + 20 = 0$

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(c) $x^4 - 17x^2 + 16 = 0$

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(d) $x^6 - 7x^3 - 8 = 0$

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Completing the Square

It is not possible to solve every quadratic equation by using the above factorising techniques.

For these equations, we can try to **complete the square** to solve the equation.**Example 5**Solve the equation $x^2 + 6x + 7 = 0$ by completing the square.

$$x^2 + 6x + 7 = 0$$

$$(x + 3)^2 - 3^2 + 7 = 0 \quad \text{[Half of 6 is 3]}$$

$$(x + 3)^2 - 9 + 7 = 0$$

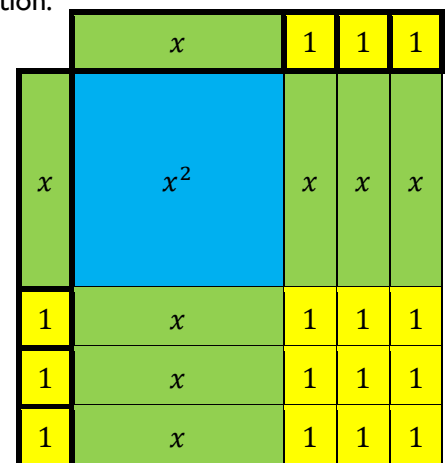
$$(x + 3)^2 - 2 = 0$$

$$(x + 3)^2 = 2$$

$$x + 3 = \pm\sqrt{2}$$

$$x = -3 \pm \sqrt{2}$$

$$\text{Either } x = -3 + \sqrt{2} \text{ or } x = -3 - \sqrt{2}$$

The *algebra tiles* on the right shows what is happening during the solution.The diagram shows the sum $(x + 3)^2 = x^2 + 6x + 9$. Comparing with $x^2 + 6x + 7$, we see that we agree with the x^2 and the $6x$, but not with the constants. We must **subtract 9** (or 3^2) and then **add 7** to obtain what we need:

$$(x + 3)^2 = x^2 + 6x + 9$$

$$(x + 3)^2 - 9 = x^2 + 6x + 9 - 9$$

$$(x + 3)^2 - 9 = x^2 + 6x$$

$$(x + 3)^2 - 9 + 7 = x^2 + 6x + 7$$

$$(x + 3)^2 - 2 = x^2 + 6x + 7$$

Exercise 7

Solve the following equations by completing the square.

(a) $x^2 + 10x + 3 = 0$

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(b) $x^2 - 8x + 9 = 0$

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(c) $x^2 + 6x + 7 = 0$

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(d) $x^2 - 14x - 1 = 0$

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(e) $x^2 + 12x + 1 = 0$

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(f) $x^2 + x - 2 = 0$

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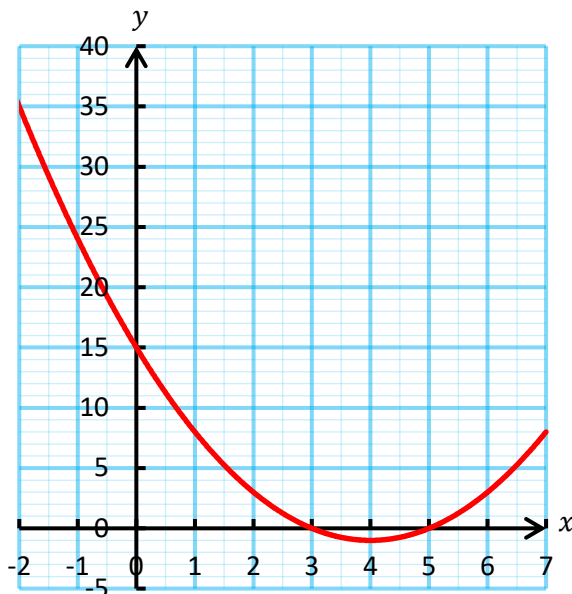
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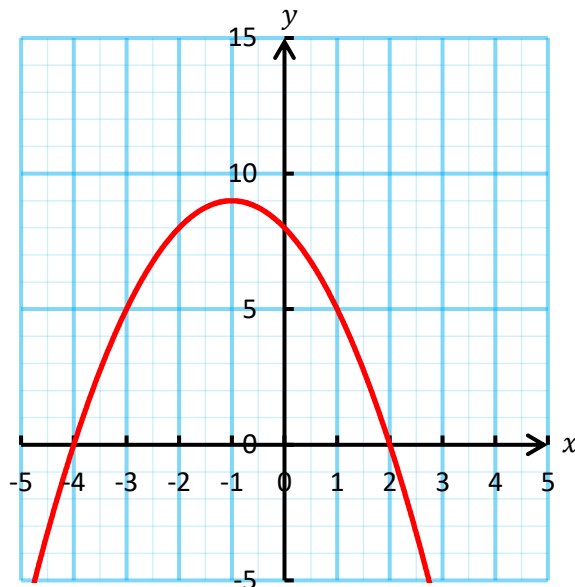
Finding a minimum point or maximum point by completing the square

Let us return to the quadratic functions we first saw in Example 1.

$$y = x^2 - 8x + 15$$



$$y = -x^2 - 2x + 8$$



By completing the square, it is possible to find the roots and the minimum / maximum point.

Finding the roots:

$$\begin{aligned} x^2 - 8x + 15 &= 0 \\ (x - 4)^2 - (-4)^2 + 15 &= 0 \\ (x - 4)^2 - 16 + 15 &= 0 \\ (x - 4)^2 - 1 &= 0 \\ (x - 4)^2 &= 1 \\ x - 4 &= \pm\sqrt{1} \\ x - 4 &= \pm 1 \\ x &= 4 \pm 1 \\ \text{Either } x &= 4 + 1 \text{ or } x = 4 - 1 \\ x &= 5 \quad \text{or } x = 3 \end{aligned}$$

Minimum / maximum point:

By completing the square, we can write $y = x^2 - 8x + 15$ as $y = (x - 4)^2 - 1$.
No matter the value of x , we have $(x - 4)^2 \geq 0$.
So, the minimum value of the function is -1 .
This happens when the bracket $x - 4 = 0$
 $x = 4$
So, the minimum point is $(4, -1)$.

$$\begin{aligned} -x^2 - 2x + 8 &= 0 \\ x^2 + 2x - 8 &= 0 && [\text{Multiply by } -1] \\ (x + 1)^2 - 1^2 - 8 &= 0 \\ (x + 1)^2 - 1 - 8 &= 0 \\ (x + 1)^2 - 9 &= 0 \\ (x + 1)^2 &= 9 \\ x + 1 &= \pm\sqrt{9} \\ x + 1 &= \pm 3 \\ x &= -1 \pm 3 \\ \text{Either } x &= -1 + 3 \text{ or } x = -1 - 3 \\ x &= 2 \quad \text{or } x = -4 \end{aligned}$$

By completing the square, we can write $y = -x^2 - 2x + 8$ as $y = -(x + 1)^2 + 9$.
So, we can write $y = -x^2 - 2x + 8$ as $y = -(x + 1)^2 + 9$, or $y = 9 - (x + 1)^2$.
No matter the value of x , we have $(x + 1)^2 \geq 0$.
So, the maximum value of the function is 9 .
This happens when the bracket $x + 1 = 0$
 $x = -1$
So, the maximum point is $(-1, 9)$.

Use your calculations from Exercise 7 to help you find the minimum point for the following functions.

(b) $x^2 - 8x + 9 = 0$

(d) $x^2 - 14x - 1 = 0$

(f) $x^2 + x - 2 = 0$

Find the roots and the minimum / maximum point for the following functions.

(b) $y = -x^2 - 2x + 3$

The Discriminant

Consider the graphs shown below, plotting three quadratic functions.

$$y = 2x^2 - 16x + 30$$

$$y = 2x^2 - 16x + 32$$

$$y = 2x^2 - 16x + 34$$



The first graph, for $y = 2x^2 - 16x + 30$, crosses the x -axis twice, at the points (3, 0) and (5, 0). We say that the function has **two real roots**.

The second graph, for $y = 2x^2 - 16x + 32$, meets the x -axis at one point, namely (4, 0). We say that the function has **one repeated real root**.

The third graph, for $y = 2x^2 - 16x + 34$, does not cross the x -axis at all. We say that the function has **no real roots**.

We could find the above roots by using the quadratic formula.

$$y = 2x^2 - 16x + 30$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2, b = -16, c = 30$$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4 \times 2 \times 30}}{2 \times 2}$$

$$x = \frac{16 \pm \sqrt{256 - 240}}{4}$$

$$x = \frac{16 \pm \sqrt{16}}{4}$$

$$x = \frac{16 \pm 4}{4}$$

$$\text{Either } x = \frac{16+4}{4} \text{ or } x = \frac{16-4}{4}$$

$$\text{Either } x = 5 \text{ or } x = 3$$

$$y = 2x^2 - 16x + 32$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2, b = -16, c = 32$$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4 \times 2 \times 32}}{2 \times 2}$$

$$x = \frac{16 \pm \sqrt{256 - 256}}{4}$$

$$x = \frac{16 \pm \sqrt{0}}{4}$$

$$x = \frac{16 \pm 0}{4}$$

$$\text{Either } x = \frac{16+0}{4} \text{ or } x = \frac{16-0}{4}$$

$$\text{Either } x = 4 \text{ or } x = 4$$

$$y = 2x^2 - 16x + 34$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2, b = -16, c = 34$$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4 \times 2 \times 34}}{2 \times 2}$$

$$x = \frac{16 \pm \sqrt{256 - 272}}{4}$$

$$x = \frac{16 \pm \sqrt{-16}}{4}$$

'Math Error' due to $\sqrt{-16}$

The part of the quadratic formula that influences the number of real roots obtained is the expression $b^2 - 4ac$, which appears within the square root. This expression is known as the **discriminant**.

If $b^2 - 4ac > 0$ then we have two real roots.

If $b^2 - 4ac = 0$ then we have one repeated real root.

If $b^2 - 4ac < 0$ then we do not have any real roots.

Exercise 10

How many real roots do the following quadratic equations have?

(a) $9x^2 + 12x + 4 = 0$

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(b) $9x^2 + 12x - 4 = 0$

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(c) $9x^2 + 4x + 12 = 0$

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(d) $x^2 + 5x + 2 = 0$

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(e) $x^2 + 2x + 2 = 0$

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(f) $-x^2 + 6x - 9 = 0$

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Exercise 11

Consider the quadratic equation $2x^2 + 8x + k = 0$. For which values of k does the equation have

(a) two real roots?

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(b) one repeated real root?

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(c) no real roots?

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Exercise 12

Consider the quadratic equation $2x^2 + kx + 8 = 0$. For which values of k does the equation have

(a) two real roots?

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(b) one repeated real root?

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(c) no real roots?

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Pop Quiz: Quadratic Functions I

/16

(I) Consider the quadratic function $y = x^2 + 8x - 20$.

(a) Will the graph of this function be 'u'-shaped, or 'n'-shaped?

[1]

(b) Complete the following table.

[2]

x	-12	-10	-8	-6	-4	-2	0	2	4	6	8	10	12
y	28	0		-32	-36	-32	-20		28	64	108	160	220

(c) Plot the graph of the function on the graph paper on the right.

[1]

(d) Complete the following sentences.

The y -intercept is The roots are $x = \dots$ and $x = \dots$

The minimum point is

[3]

(e) By factorising, solve the equation

[2]

$x^2 + 8x - 20 = 0$.

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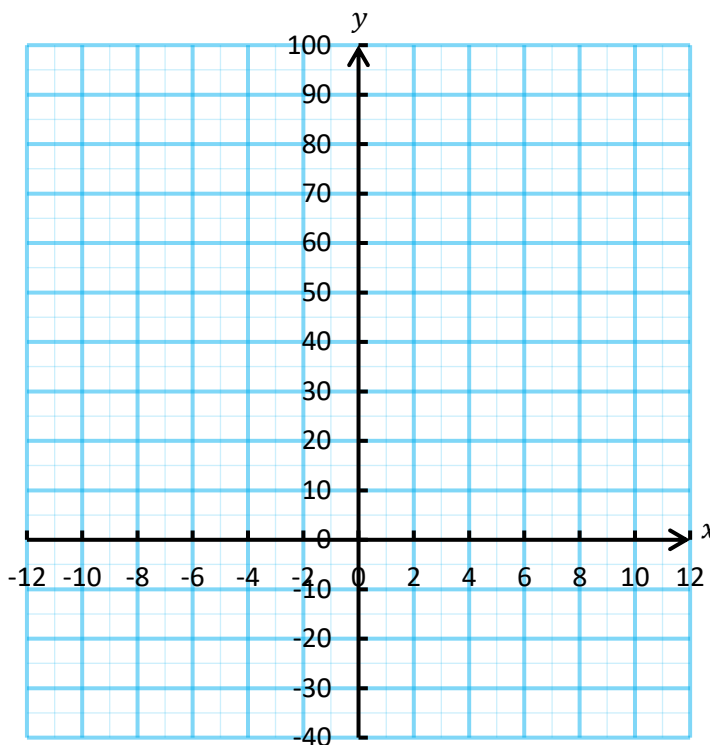
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(f) By completing the square, solve the equation

[2]

$x^2 + 8x - 20 = 0$.

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(g) Use your answer to part (f) to write down the minimum point of $y = x^2 + 8x - 20$.

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(h) Solve the equation $x^2 + 8x - 20 = 0$.

[2]

(i) Use the discriminant to determine the number of real roots of the equation $x^2 + 8x - 20 = 0$.

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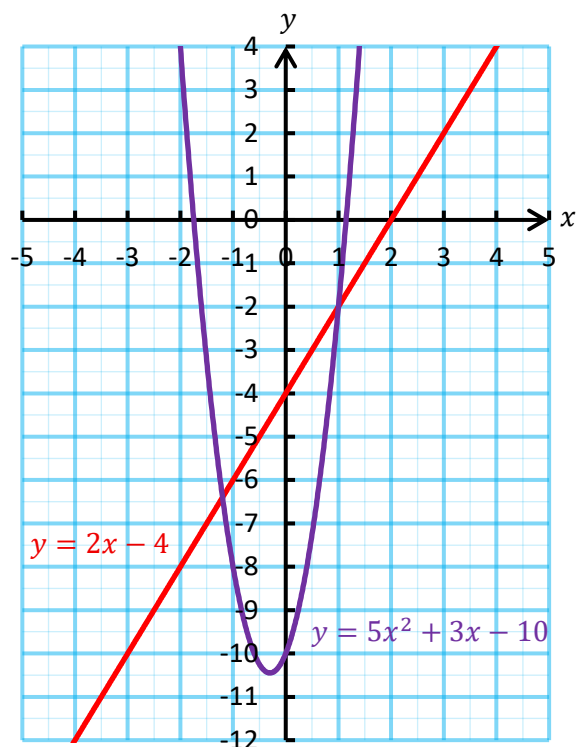
The graph on the right shows the quadratic function $y = 5x^2 + 3x - 10$ and the linear function $y = 2x - 4$.

To answer this question, we must consider where the two functions agree, and so we must solve the equation

$$\begin{array}{l} 5x^2 + 3x - 10 = 2x - 4 \\ 5x^2 + x - 10 = -4 \quad \text{[Subtract } 2x\text{]} \\ 5x^2 + x - 6 = 0 \quad \text{[Add 4]} \\ 5 \times -6 = -30 \\ 5x^2 - 5x + 6x - 6 = 0 \\ 5x(x - 1) + 6(x - 1) = 0 \\ (x - 1)(5x + 6) = 0 \end{array}$$

$$\begin{array}{rcl} x = 1 & 5x = -6 & \\ & x = -\frac{6}{5} & \end{array}$$

Either $y = 2 \times 1 - 4$ or $y = 2 \times -\frac{6}{5} - 4$
 $y = -2$ $y = -\frac{32}{5}$



So, the curve and line intersect at the points $(1, -2)$ and $(-\frac{6}{5}, -\frac{32}{5})$.

Find the coordinates of the points where the following functions intersect.

(a) $y = x^2 - 5x + 13$, $y = 2x + 1$

(b) $y = 5x^2 - 8x - 6$, $y = 10x + 2$

(c) $y = x^2 - 7x + 12$, $x + y = 4$

(d) $2x + y = 13$, $x^2 + xy - 30 = 0$

(e) $y = 4x^2 + 2x - 3$, $10x - 2y + 3 = 0$

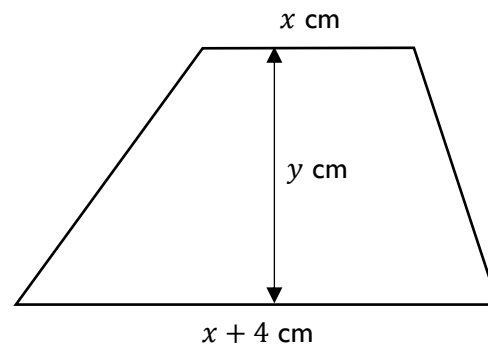
Exercise 14

The area of the larger square is 30 cm^2 more than the area of the smaller square.

You must use an algebraic method, **not** trial and improvement.

A square and rectangle are such that the length of a side of the square is equal to the length of the **shorter** side of the rectangle. The total area of the square and the rectangle is 198 cm^2 and the total of their perimeters is 80 cm . Calculate the dimensions of the rectangle.

Megan knows that $x = y + 5$. She also knows that the sum of x and y is equal to the product of x and y . Using an algebraic method, show that x and y are not integers.

Exercise 17

The diagram is not drawn to scale

The area of the trapezium in the diagram is 28 cm^2 .

The area of a rectangle of length $(x + y) \text{ cm}$ and width $y \text{ cm}$ is 43 cm^2 .

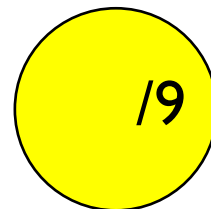
(a) Show that $xy = 28 - 2y$ and $xy = 43 - y^2$.

(b) Hence, write a quadratic equation in terms of y and calculate the length of the parallel sides of the trapezium. You must use an algebraic method, **not** trial and improvement.

Pop Quiz: Quadratic Functions 2

(1) Find the coordinates of the points where the following functions intersect.

[4]



$$y = x^2 - 4x + 6, \quad y = -x + 16$$

(2) Five times one number is equal to twice a second number. The sum of the two numbers is 0.7.
Find the two numbers.

[5]