Unit 2, Workbook 3

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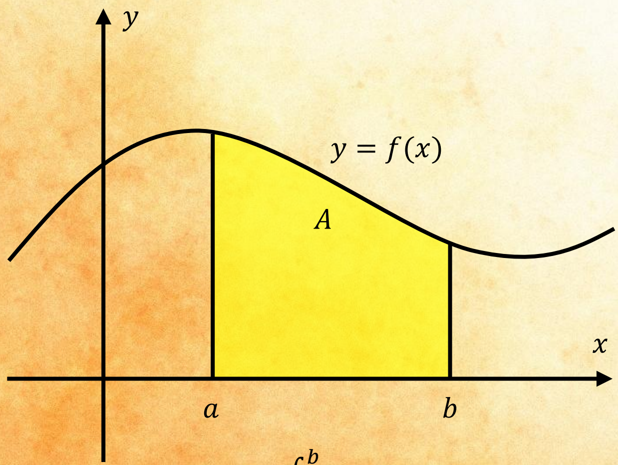
Introducing Integration



$$A = \int_a^b f(x)\,dx$$
$$A = [F(x)]_a^b$$
$$A = F(b) - F(a)$$

Name:

Background

What is the work?	Introducing the process of integration as the inverse of differentiation, and using this to find the area beneath curves.
What are the prerequisites?	**Additional Mathematics:** Differentiation.
Where does this lead to?	**Mathematics A Level:** Further integration. **Applications:** Finding areas; volumes; mass; location; cumulative distribution functions in probability.

Theory

Indefinite Integration

Theory

Integration is the process of reversing differentiation.

Exercise 1

Complete the following table.

y	$\frac{dy}{dx}$
$5x^2+9x+3$	
$5x^2+9x+4$	
$5x^2+9x-3$	
$5x^2+9x+\frac{3}{4}$	
$5x^2+9x+\pi$	

In completing the above exercise, you will see the problem facing us when attempting to integrate: whilst differentiation always gives a unique answer, this will not be the case for integration. Above, every function y in the first column differentiates to give the same answer. This leads to the question: if we are integrating $10x+9$, what should the answer be? To deal with this problem, we introduce a **constant of integration**. Different textbooks use different letters for the constant of integration: sometimes c, sometimes k, sometimes something else. This workbook will use c as the constant of integration.

When integrating $10x+9$, we say that the answer is $5x^2+9x+c$, where c represents any number, because any expression of the form $5x^2+9x+c$ would differentiate to give $10x+9$. The formal notation for this process is

$$\int 10x+9\,dx = 5x^2+9x+c$$

where dx denotes we are integrating with respect to the variable x, and $\int$ is the symbol for integration.

The rule for integrating a term of the form ax^n is the following:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

In words, we add one to the power n, and then divide by the new power $n+1$.

Example 1

Question	Answer
$\int 28x^3\,dx$	$\frac{28x^4}{4} + c = 7x^4 + c$
$\int 35x^4 - 6x + 2\,dx$	$\frac{35x^5}{5} - \frac{6x^2}{2} + 2x + c = 7x^5 - 3x^2 + 2x + c$

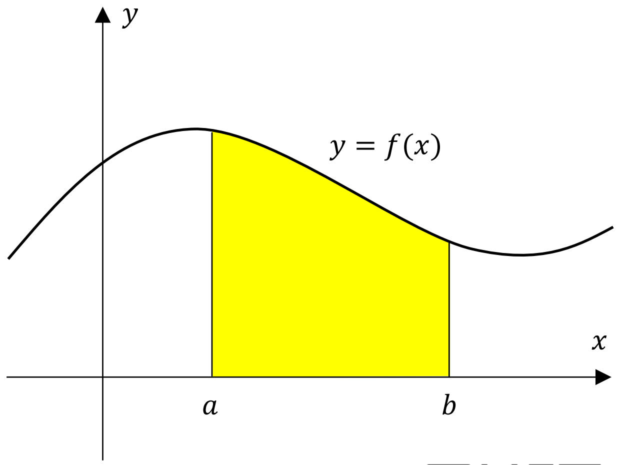
Exercise 2

Complete the following table.

Question	Answer
$\int 15x^2\,dx$	
$\int 24x^3\,dx$	
$\int 10x\,dx$	
$\int 4\,dx$	
$\int 8x^{-3}\,dx$	
$\int 9x^2 + 6x + 3\,dx$	
$\int 10x^4 - 4x^3\,dx$	
$\int 2 + x^{-2}\,dx$	
$\int 0\,dx$	
$\int 3 - 16x^{-5}\,dx$	
$\int 4x^2\,dx$	
$\int x^2 + x^4 + x^6\,dx$	
$\int 5x^3 - 2x + 9\,dx$	
$\int 6x^2 + 10 - 2x^{-2}\,dx$	
$\int \frac{2}{3}x^2\,dx$	
$\int 2(4 + x)\,dx$	
$\int x(2x - 3)\,dx$	
$\int 20x^4 - 12x^3 + 6x^2 - 4x + 7\,dx$	

Definite Integration

It is possible to use integration to find the area between a curve and the x–axis.



Let us consider a general curve of the form $y = f(x)$.
The yellow area on the right is bounded by the curve, the x–axis and the lines $x = a$, $x = b$.
To find the size of the area, we integrate between a and b using the notation

$$\int_a^b f(x)\, dx$$

We say that a is the **lower limit** and b is the **upper limit**.

Theory

The **Fundamental Theorem of Calculus** tells us how to calculate the above integral:

If $\int f(x)\, dx = F(x) + c$, then

$$\begin{aligned} &\int_a^b f(x)\, dx \\ &= [F(x) + c]_a^b \\ &= [F(b) + c] - [F(a) + c] \\ &= F(b) - F(a) \end{aligned}$$

Notice that the constants of integration cancel out, so we can write

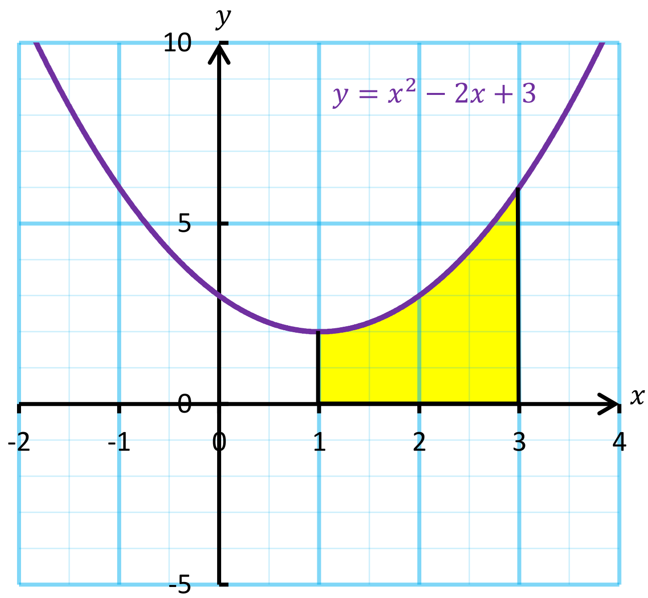
$$\begin{aligned} &\int_a^b f(x)\, dx \\ &= [F(x)]_a^b \\ &= F(b) - F(a) \end{aligned}$$

Example 2

Find the area underneath the curve $y = x^2 - 2x + 3$ between $x = 1$ and $x = 3$.

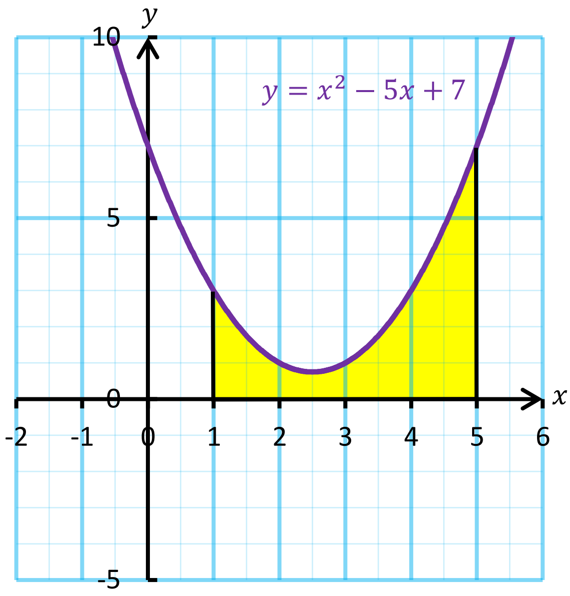
Answer: We need to find the value of

$$\begin{aligned} &\int_1^3 x^2 - 2x + 3\, dx \\ &= \left[\frac{x^3}{3} - \frac{2x^2}{2} + 3x\right]_1^3 \\ &= \left[\frac{x^3}{3} - x^2 + 3x\right]_1^3 \\ &= \left[\frac{3^3}{3} - 3^2 + 3 \times 3\right] - \left[\frac{1^3}{3} - 1^2 + 3 \times 1\right] \\ &= [9 - 9 + 9] - \left[\frac{1}{3} - 1 + 3\right] \\ &= 9 - \frac{7}{3} \\ &= \frac{20}{3} \text{ square units} \end{aligned}$$



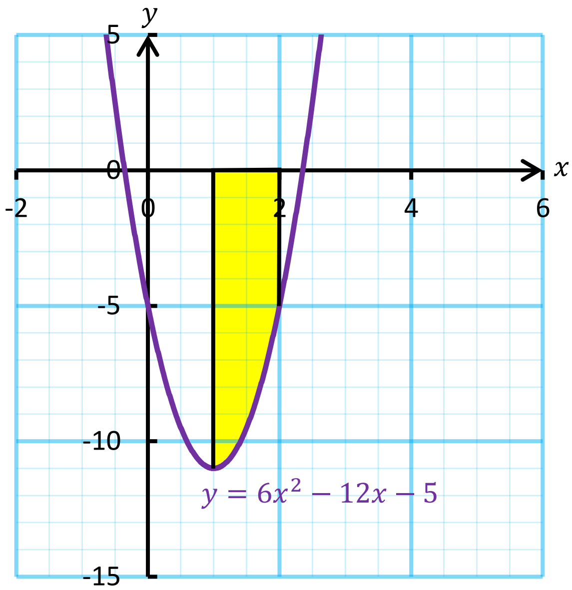
Exercise 3

Find the area underneath the curve
$y = x^2 - 5x + 7$ between $x = 1$ and $x = 5$.



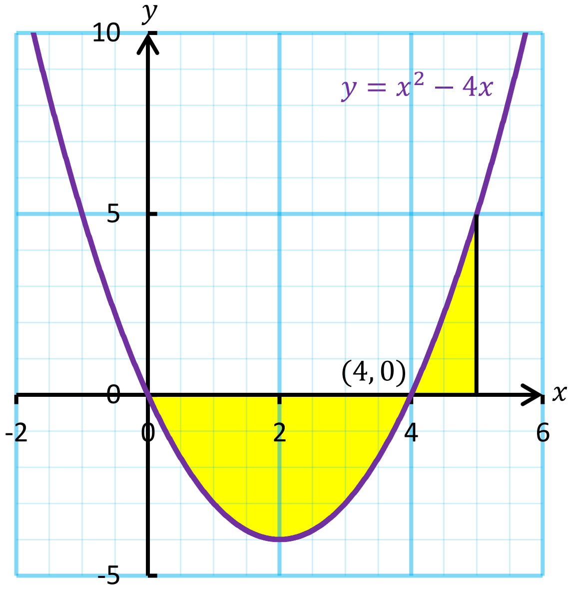
Exercise 4

Find the area above the curve
$y = 6x^2 - 12x - 5$ between $x = 1$ and $x = 2$.



Exercise 5

Find the total area between the curve
$y = x^2 - 4x$ and the x–axis between $x = 0$ and $x = 5$.



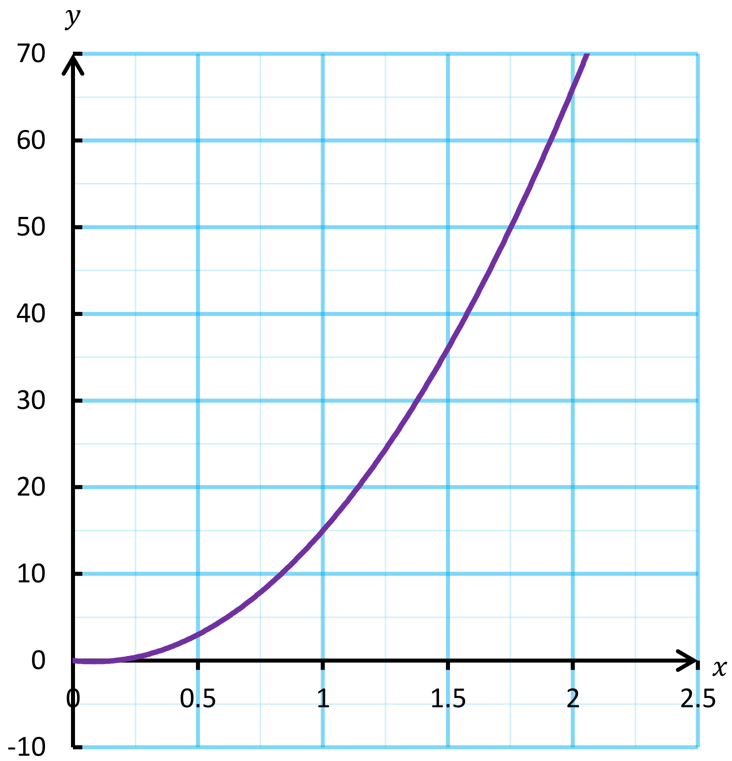
Exercise 6

The diagram on the right shows a sketch of the curve $y = 18x^2 - 3x$.

Calculate the area of the region defined by:

- the curve $y = 18x^2 - 3x$,
- the line $x = 1$,
- the line $x = 2$,
- the x–axis.

You must show all your working.



Exercise 7

The diagram on the right shows a sketch of the curve $y = -x^2 + 5x + 14$.

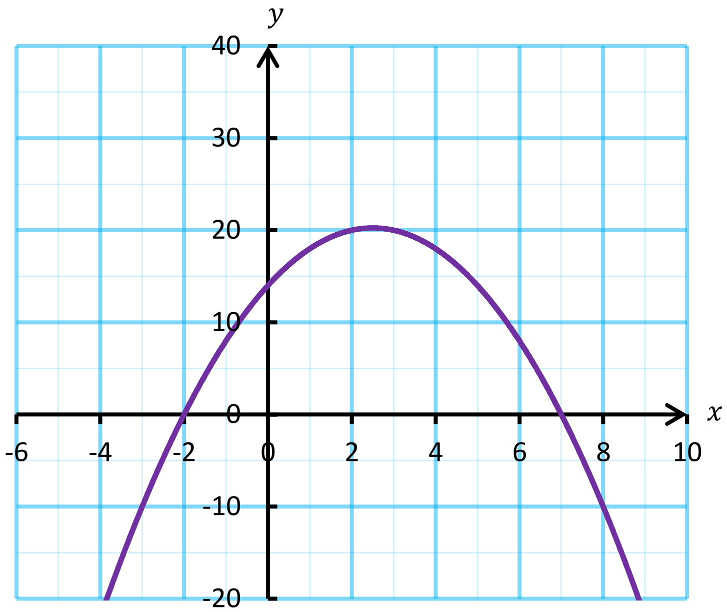
Calculate the area of the region defined by the following inequalities.

$y \leq -x^2 + 5x + 14$

$x \geq 6$

$y \geq 0$

You must show all your working.



Exercise 8

Given the following facts, find the values of a, b, c and d.

- $y = ax^3 + bx^2 + cx + d$
- $\frac{dy}{dx} = 27x^2 + 8x + 13$
- The value of d is equal to $\frac{1}{6}$ of the coefficient of x in $\frac{d^2y}{dx^2}$.

$a =$ .... $b =$ .... $c =$ .... $d =$ ....

Exercise 9

Given the following facts, find the values of a, b, c and d.

- $y = ax^3 + bx^2 + cx + d$
- $\frac{dy}{dx} = 12x^2 + 4x + 1$
- When $x = 1$, we have $y = 10$.

$a =$ .... $b =$ .... $c =$ .... $d =$ ....

Exercise 10

Given the following facts, find the values of a, b and c.

- $y = ax^3 + bx^2 + c$
- $\frac{dy}{dx} = 3ax^2 + 24x$
- $\frac{d^2y}{dx^2} = 60x + 24$
- The point $(2, 150)$ lies on $y = ax^3 + bx^2 + c$.

$a =$ .... $b =$ .... $c =$ ....

Pop Quiz: Introducing Integration

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(1) Find $\int 21x^6 - 3x^2 - x^{-2} + 6 \ dx$. [5]

(2) Showing all your working, evaluate $\int_2^5 6x^2 + 4x \ dx$. [5]

(3) The diagram shows the curve $y = 6x - x^2$.

Showing all your working, calculate the area of the region bounded by the curve $y = 6x - x^2$ and the x–axis. [5]