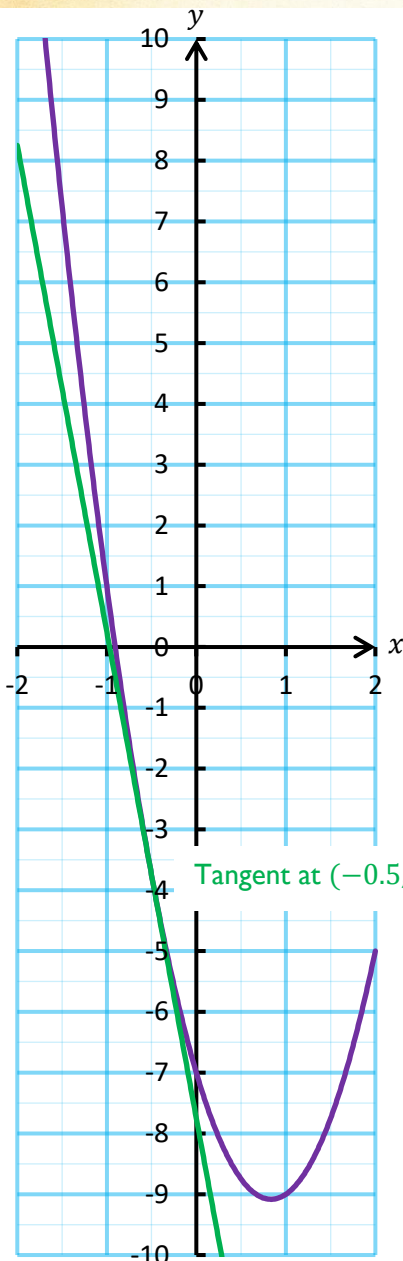




Quick

Differentiation



$$y = 3x^2 - 5x - 7$$

Name:

Background

What is the work?

Differentiating a function without having to go from first principles, and using this technique to find the equation of tangents, and stationary points.

What are the prerequisites?

GCSE Work: Solving equations.

Where does this lead to?

Additional Mathematics: Integrating quickly.

Applications: Finding velocity and acceleration; optimisation; designing efficient systems.

Theory

In the previous workbook, the technique of differentiating from first principles was introduced. Did you notice, in this work, the connection between each question and its answer? For example, we can differentiate from first principles to obtain the following answers.



Theory

y	$\frac{dy}{dx}$
$3x^2 + 8x$	$6x + 8$
$5x^2 - 9x$	$10x - 9$
$4x^2 + 11x + 3$	$8x + 11$
$7x^3$	$21x^2$
$26x + 18$	26

We can use the following rule to quickly differentiate a function without having to go from first principles:

If $y = ax^n$, then $\frac{dy}{dx} = nax^{n-1}$.

Example 1

y	a	n	$\frac{dy}{dx} = nax^{n-1}$
$6x^2$	6	2	$2 \times 6 \times x^1 = 12x$
$4x^3$	4	3	$3 \times 4 \times x^2 = 12x^2$
$-14x$	-14	1	$1 \times -14 \times x^0 = -14$
23	23	0	$0 \times 23 \times x^{-1} = 0$
$4x^{-6}$	4	-6	$-6 \times 4 \times x^{-7} = -24x^{-7}$
$-643x^{215}$	-643	215	$215 \times -643 \times x^{214} = -138245x^{214}$

Given a function that is a sum or difference of a series of terms, we can differentiate the terms individually to differentiate the function.

Example 2

y	$\frac{dy}{dx}$
$7x^4 + 8x^2 - 4$	$28x^3 + 16x$
$10x^5 + 8x^3 - 7x^{-3}$	$50x^4 + 24x^2 + 21x^{-4}$

Exercise 1

Complete the following table.

y	$\frac{dy}{dx}$
$12x^3$	
$5x$	
3	
$28x^2$	
$4x^5 + 2x^3 - 7x^2$	
$19x^3 - 3 + 4x^2 - 19x$	
$24x^{-3}$	
$8x^{100} + 11x^9$	

We can use the process of differentiation to find the equation of the **tangent** to a curve at a particular point.

Example 3

The curve for $y = 2x^3 - 3x$ is shown on the right.

Find the equation of the tangent to the curve at the point where $x = 1$.

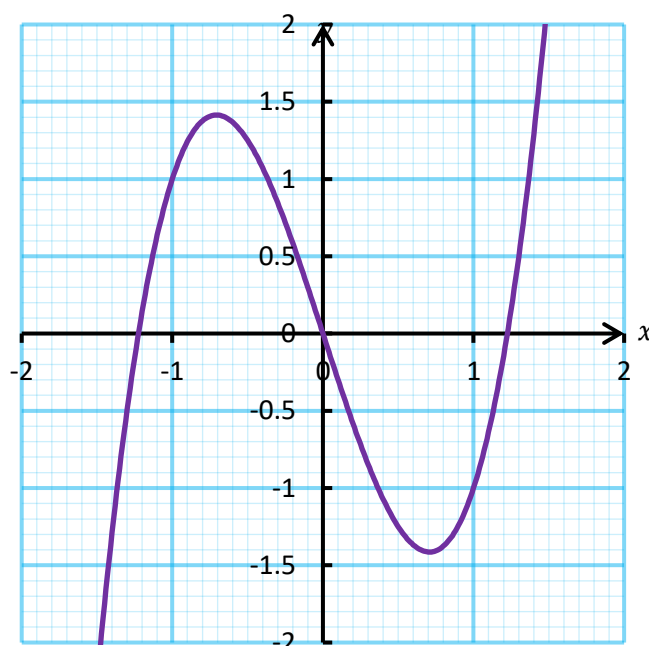
Answer: If $y = 2x^3 - 3x$, then $\frac{dy}{dx} = 6x^2 - 3$.

When $x = 1$, we have $y = 2(1^3) - 3(1) = -1$, and $\frac{dy}{dx} = 6(1^2) - 3 = 3$. So, the tangent that goes through the point $(1, -1)$ has gradient 3.

The equation $y - y_1 = m(x - x_1)$ gives the equation of a straight line with gradient m going through the point (x_1, y_1) .

So, the equation of the tangent at $(1, -1)$ is

$$\begin{aligned} y - (-1) &= 3(x - 1) \\ y + 1 &= 3x - 3 \\ y &= 3x - 4 \end{aligned}$$

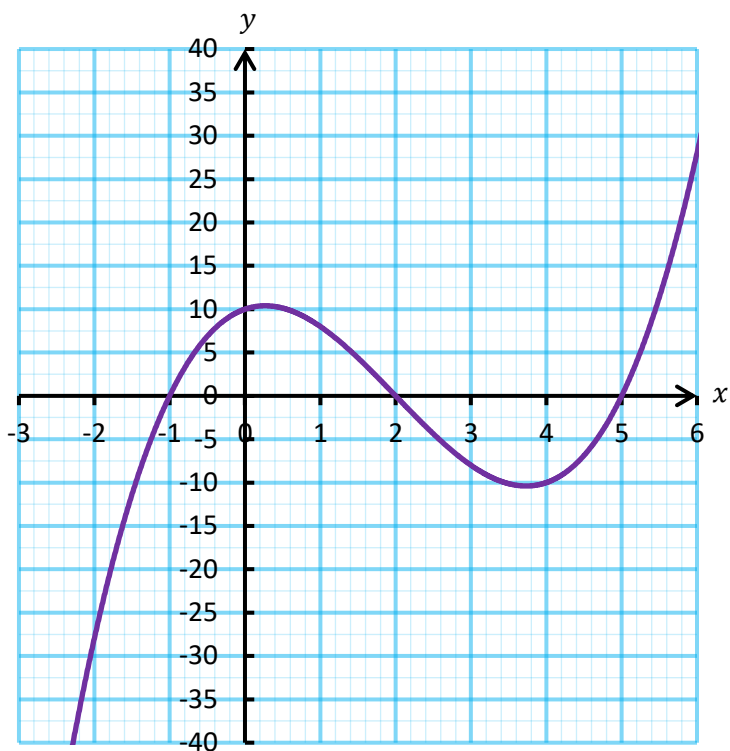


Exercise 2

The curve for $y = x^3 - 6x^2 + 3x + 10$ is shown on the right.

(a) Find the equation of the tangent to the curve at the point where $x = -1$.

(b) Find the equation of the tangent to the curve at the point where $x = 3$.

**Exercise 3**

The equation of the curve C is $y = x^2 + 6x + 8$. The point P has coordinates $(-2, 0)$ and lies on C . Find the equation of the tangent to C at P .

The equation of the curve C is $y = 3x^2 - 4x - 2$.

- (a) The point P has coordinates $(1, -3)$ and lies on C . Find the equation of the tangent to C at P .
- (b) The point Q has coordinates $(-1, 5)$ and lies on C . Find the equation of the tangent to C at Q .
- (c) Find the equation of the tangent to the curve C at the point where $x = 3$.

The equation of the curve C is $y = 2x^3 - 5x^2 + 8$.

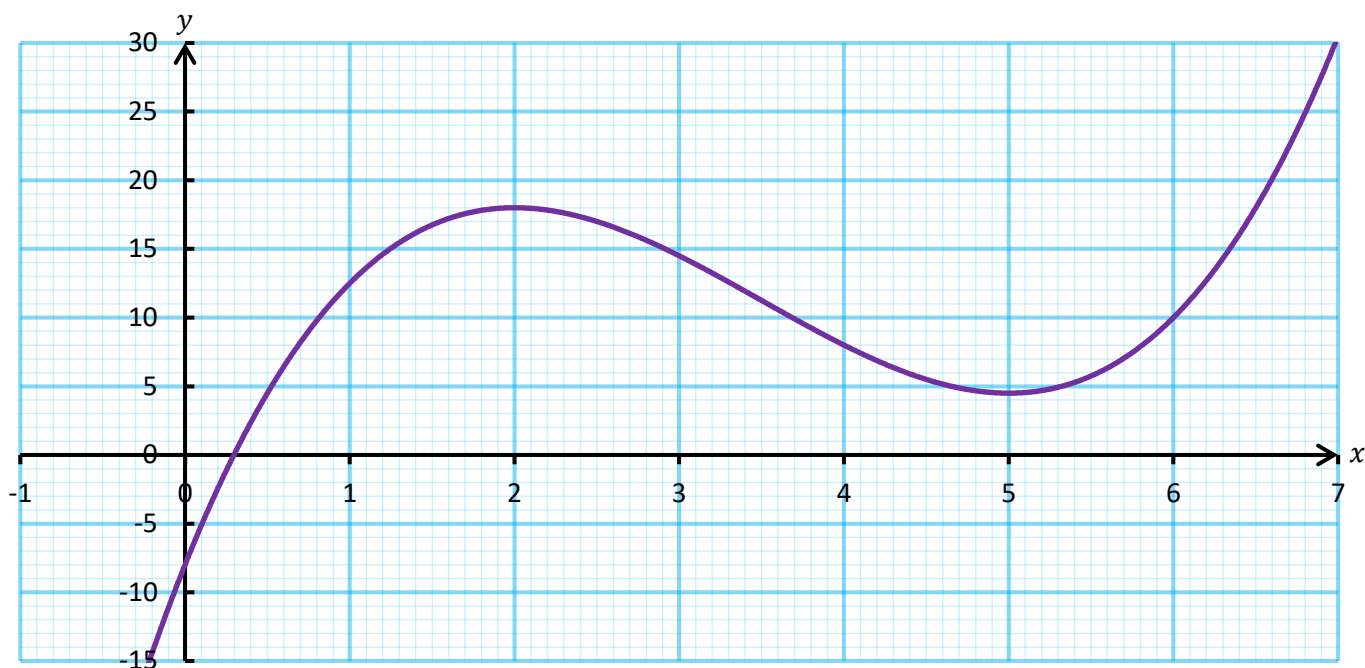
- (a) Find $\frac{dy}{dx}$.
- (b) The point P has coordinates $(2, 4)$ and lies on C . Find the equation of the tangent to C at P .
- (c) The point Q has coordinates $(-2, -28)$ and lies on C . Find the equation of the tangent to C at Q .
- (d) Find the equation of the tangent to the curve C at the point where $x = 0$.

The equation of the curve C is $y = 2x^2 - 10x + 7$.

The equation of the curve C is $y = 7 + 13x - 2x^2$.

The Second Derivative

Consider the graph below which shows the function $y = x^3 - 10.5x^2 + 30x - 8$.



We can differentiate the function to give $\frac{dy}{dx} = 3x^2 - 21x + 30$. This gives the **gradient** of the purple curve at a particular point. If we differentiate again, we form the **second derivative** $\frac{d^2y}{dx^2}$, which in this case is $\frac{d^2y}{dx^2} = 6x - 21$. The second derivative will be helpful later on in this workbook in finding maximum and minimum points.

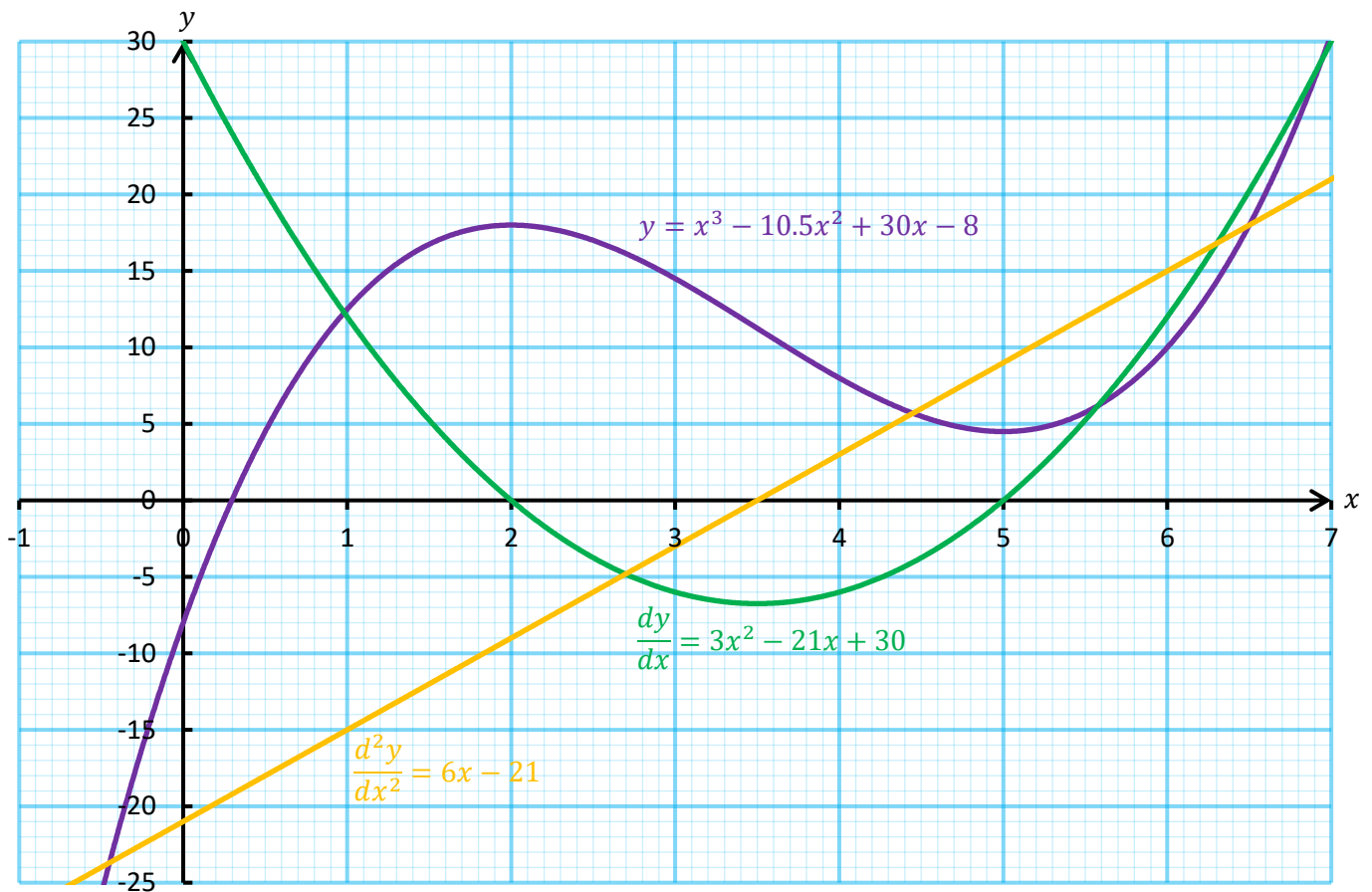
Exercise 8

Complete the following table.

y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
$4x^3$		
$10x^5$		
$8x^2$		
$18x$		
$9x^4 + 2x$		
$-3x^3 + 10x^2$		
$x^3 + 5x^2 - 4x + 5$		
$6x^7 - 2x^8 + 10x^{10}$		
$4x^{-2}$		
$20x^5 + 10x^{-5}$		

Stationary Points

Let us consider again the example from the previous page, where $y = x^3 - 10.5x^2 + 30x - 8$, $\frac{dy}{dx} = 3x^2 - 21x + 30$, and $\frac{d^2y}{dx^2} = 6x - 21$. The graph below shows these three functions, and the connections between them.



Any point where $\frac{dy}{dx} = 0$ is called a **stationary** point. At these points, the tangent to the curve is horizontal, and the curve does not increase or decrease. For the curve $y = x^3 - 10.5x^2 + 30x - 8$, we see from the above graph that there are two stationary points, where $x = 2$ and $x = 5$. For the first stationary point, where $x = 2$,

- the **purple curve** reaches a local maximum;
- the value of the **green curve** is zero;
- the value of the **orange line** is negative.

For the second stationary point, where $x = 5$,

- the **purple curve** reaches a local minimum;
- the value of the **green curve** is zero;
- the value of the **orange line** is positive.

We can use the above information to form a strategy for finding the coordinates and the nature of a curve's stationary points:

- Solve $\frac{dy}{dx} = 0$ to find the x -coordinate of each stationary point.
- Substitute back into y to find the y -coordinates.
- Substitute into $\frac{d^2y}{dx^2}$ to find the nature of each stationary point. If $\frac{d^2y}{dx^2} > 0$, then the stationary point is a local minimum point. If $\frac{d^2y}{dx^2} < 0$, then the stationary point is a local maximum point¹.

¹ We will deal with the case $\frac{d^2y}{dx^2} = 0$ in the mathematics A Level course.

Find the coordinates and the nature of the stationary points for the curve $y = x^3 - 10.5x^2 + 30x - 8$.

$$-15 \times -6 = 90$$
$$-|5 + -6| = -2|$$

Solve $\frac{dy}{dx} = 0$:

$$3x^2 - 21x + 30 = 0$$
$$3 \times 30 = 90$$
$$3x^2 - 15x - 6x + 30 = 0$$
$$3x(x - 5) - 6(x - 5) = 0$$
$$(x - 5)(3x - 6) = 0$$

Either $x - 5 = 0$ or $3x - 6 = 0$

$$x = 5 \qquad 3x = 6$$
$$\qquad \qquad x = 2$$

Substitute back into y : If $x = 5$ then $y = 5^3 - 10.5 \times 5^2 + 30 \times 5 - 8$
 $y = 4.5$
 If $x = 2$ then $y = 2^3 - 10.5 \times 2^2 + 30 \times 2 - 8$
 $y = 18$

Differentiate again: $\frac{d^2y}{dx^2} = 6x - 21$.

Substitute into $\frac{d^2y}{dx^2}$:

If $x = 5$ then $\frac{d^2y}{dx^2} = 6 \times 5 - 21$
 $\frac{d^2y}{dx^2} = 9$
 $\frac{d^2y}{dx^2}$ is positive so $(5, 4.5)$ is a local minimum point.

If $x = 2$ then $\frac{d^2y}{dx^2} = 6 \times 2 - 21$
 $\frac{d^2y}{dx^2} = -9$
 $\frac{d^2y}{dx^2}$ is negative so $(2, 18)$ is a local maximum point.

Find the coordinates and the nature of the stationary points for the curve $y = x^3 - 3x^2 - 9x + 2$.

Exercise 10

Find the coordinates and nature of the stationary points for the curve $y = x^3 - x^2 - x + 2$.

Exercise 11

Find the coordinates and nature of the stationary points for the curve $y = \frac{1}{3}x^3 + x^2 - 15x$.

Find the coordinates and nature of the stationary points for the curve $y = 2x^3 - 6x^2 - 7$.

Exercise 13

Find the coordinates and nature of the stationary points for the curve $y = 5x^3 + 15x^2 - 8$.

[8]

y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
$6x^4$		
$2x^3 - 16x$		
$5x^{-3}$		
$6x + 3x^2 - 8x^3$		

[5]

[5]

[4]