



Differentiating

from First

Principles



Isaac Newton



Gottfried Wilhelm Leibniz

In the 18th century a huge row developed between Leibniz and Newton regarding who was the first person to develop the idea of differentiation.

Name:

Background

What is the work?

Finding the gradient of a curve using algebraic techniques.

What are the prerequisites?

GCSE Work: Expanding brackets; simplifying; finding the gradient of a curve.

Where does this lead to?

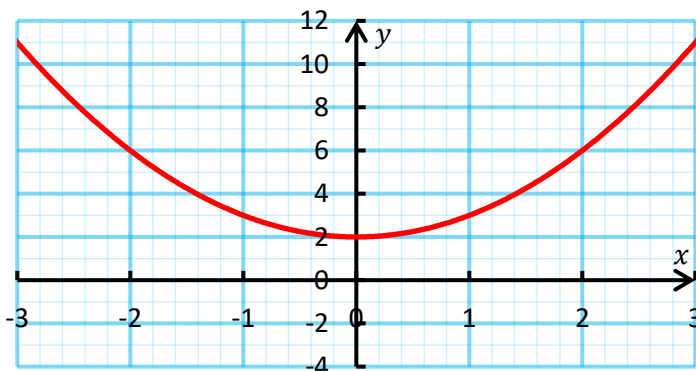
Additional Mathematics: Differentiating and integrating quickly.

Applications: Finding velocity and acceleration; optimisation; designing efficient systems.

Theory

Consider the function $y = x^2 + 2$. It is possible to draw a graph of this function by forming a table of values and plotting them on graph paper.

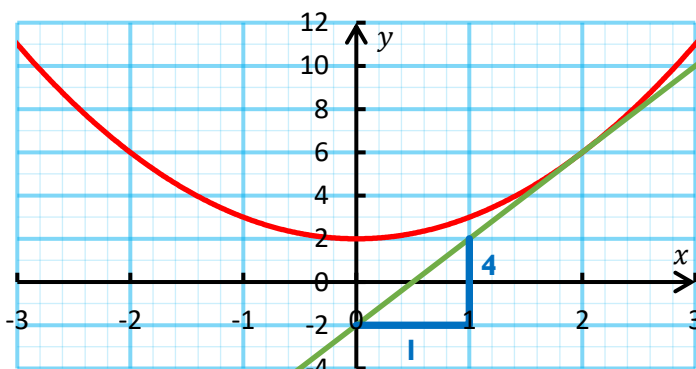
x	-3	-2	-1	0	1	2	3
y	11	6	3	2	3	6	11



The gradient of a graph can be very useful. For example, the gradient of a displacement-time graph gives the velocity.

We see that the **gradient** of the curve (how steep the curve is) changes as we move along the curve. If $x = 0$, then the gradient is zero. If x is negative, then the gradient is negative, and if x is positive, then the gradient is positive.

What is the gradient of the graph if $x = 2$? One way of finding this would be to draw a tangent to the graph at the point $(2, 6)$ and find the gradient of the tangent by drawing a triangle.

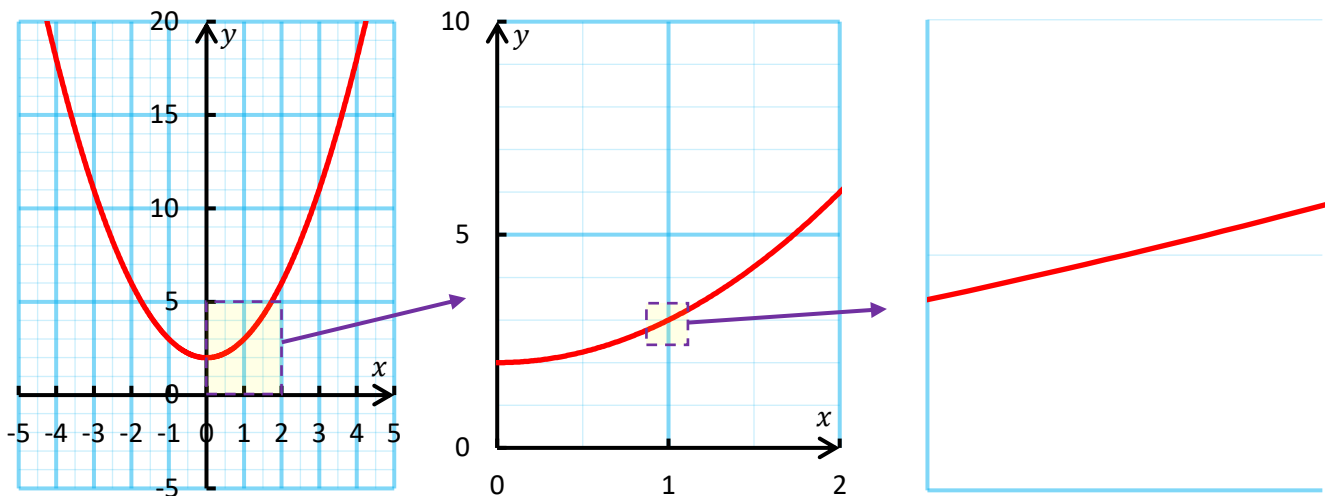


We see from the diagram on the left that the gradient of the graph when $x = 2$ is 4.

Differentiating is a method for finding the gradient of a curve at a particular point **without** having to draw a tangent and forming a triangle. We use special notation to represent this gradient: if y represents any function, then $\frac{dy}{dx}$ or y' represents the gradient of the function.

Differentiating from first principles

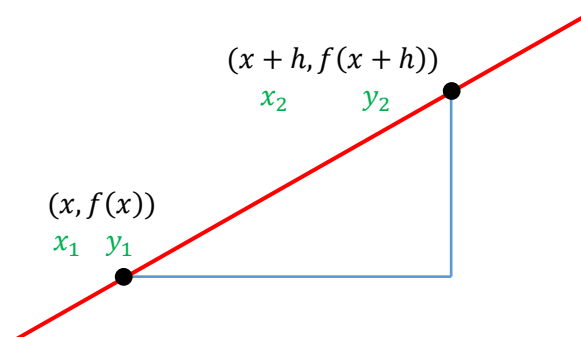
Consider again the function $y = x^2 + 2$ from the previous page. If we zoom in enough on the graph, then it will look like a straight line. (Try this yourself on the website www.desmos.com/calculator.)



This is true for any part of the graph, so to find the gradient of the graph at any point,

- Zoom in to the point – enough so that you see a straight line;
- Find the gradient of the straight line by adding a right-angled triangle to the line.

Let $(x, f(x))$ represent the co-ordinate of the point we are zooming in to. (In the above example, we are zooming in on the point $(1, 3)$, where $f(1) = 1^2 + 2 = 3$.) Consider that we have zoomed in enough so that we see the following **straight line**.



Theory

Let h represent a very small distance along the x -axis. We can move right from the point $(x, f(x))$ to reach a distance $x + h$ along the x -axis. The point on the function corresponding to $x + h$ is $(x + h, f(x + h))$.

Considering that h is small enough so that the function behaves like a straight line between x and $x + h$, we can use the fraction $\frac{y_2 - y_1}{x_2 - x_1}$ to calculate the gradient of the straight line.

$$\begin{aligned} \frac{y_2 - y_1}{x_2 - x_1} &= \frac{f(x + h) - f(x)}{x + h - x} \\ &= \frac{f(x + h) - f(x)}{h} \end{aligned}$$

The problem now is to ensure that h is small enough to give a straight line each time. To make sure of this, we use the idea of a **limit** so that h can be as close to zero as we want. So, the gradient $\frac{dy}{dx}$ of the function at the point $(x, f(x))$ is

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Note: Some textbooks and worksheets use δx instead of h to represent the small distance along the x -axis.

Example 1

From first principles, find $\frac{dy}{dx}$ if $y = x^2 + 2$.

We need to construct $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ step-by-step.

We have $f(x) = x^2 + 2$

So

$$\begin{aligned} f(x+h) &= (x+h)^2 + 2 \\ f(x+h) &= (x+h)(x+h) + 2 \\ f(x+h) &= x^2 + xh + xh + h^2 + 2 \\ f(x+h) &= x^2 + 2xh + h^2 + 2 \end{aligned}$$

We can use FOIL to expand the brackets.

Now

$$\begin{aligned} f(x+h) - f(x) &= x^2 + 2xh + h^2 + 2 - (x^2 + 2) \\ f(x+h) - f(x) &= \cancel{x^2} + 2xh + h^2 + \cancel{2} - \cancel{x^2} - \cancel{2} \\ f(x+h) - f(x) &= 2xh + h^2 \end{aligned}$$

The x^2 cancels the $-x^2$ and $+2$ cancels -2 .

Next

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{2xh+h^2}{h} \\ \frac{f(x+h)-f(x)}{h} &= 2x + h \end{aligned}$$

Finally

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} 2x + h \\ \frac{dy}{dx} &= 2x \end{aligned}$$

After simplifying as much as we can, we now use the limit by substituting $h = 0$.

Example 2

From first principles, find $\frac{dy}{dx}$ if $y = x^2 + 4x$.

We need to construct $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ step-by-step.

We have $f(x) = x^2 + 4x$

So

$$\begin{aligned} f(x+h) &= (x+h)^2 + 4(x+h) \\ f(x+h) &= (x+h)(x+h) + 4(x+h) \\ f(x+h) &= x^2 + xh + xh + h^2 + 4x + 4h \\ f(x+h) &= x^2 + 2xh + h^2 + 4x + 4h \end{aligned}$$

Now

$$\begin{aligned} f(x+h) - f(x) &= x^2 + 2xh + h^2 + 4x + 4h - (x^2 + 4x) \\ f(x+h) - f(x) &= \cancel{x^2} + 2xh + h^2 + \cancel{4x} + 4h - \cancel{x^2} - \cancel{4x} \\ f(x+h) - f(x) &= 2xh + h^2 + 4h \end{aligned}$$

Next

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{2xh+h^2+4h}{h} \\ \frac{f(x+h)-f(x)}{h} &= 2x + h + 4 \end{aligned}$$

Finally

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} 2x + h + 4 \\ \frac{dy}{dx} &= 2x + 4 \end{aligned}$$

From first principles, find $\frac{dy}{dx}$ if $y = 2x^2 - 6x + 9$.

We need to construct $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ step-by-step.

We have $f(x) = 2x^2 - 6x + 9$

So

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 6(x+h) + 9 \\ f(x+h) &= 2(x+h)(x+h) - 6(x+h) + 9 \\ f(x+h) &= 2(x^2 + xh + xh + h^2) - 6x - 6h + 9 \\ f(x+h) &= 2x^2 + 4xh + 2h^2 - 6x - 6h + 9 \end{aligned}$$

Now $f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 - 6x - 6h + 9 - (2x^2 - 6x + 9)$
 $f(x+h) - f(x) = \cancel{2x^2} + 4xh + 2h^2 - \cancel{6x} - 6h + \cancel{9} - \cancel{2x^2} + \cancel{6x} - \cancel{9}$
 $f(x+h) - f(x) = 4xh + 2h^2 - 6h$

Next $\frac{f(x+h)-f(x)}{h} = \frac{4xh+2h^2-6h}{h}$
 $\frac{f(x+h)-f(x)}{h} = 4x + 2h - 6$

Finally

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} 4x + 2h - 6 \\ \frac{dy}{dx} &= 4x - 6\end{aligned}$$

From first principles, find $\frac{dy}{dx}$ if $y = x^2 + 5$.

Given that $y = x^2 - 3$, find $\frac{dy}{dx}$ from first principles.

Given that $y = x^2 + 7x$, find $\frac{dy}{dx}$ from first principles.

Given that $y = x^2 - 4x$, find $\frac{dy}{dx}$ from first principles.

Given that $y = 3x^2 + 4$, find $\frac{dy}{dx}$ from first principles.

Given that $y = 5x^2 - 2$, find $\frac{dy}{dx}$ from first principles.

Given that $y = 11x^2 + 5x$, find $\frac{dy}{dx}$ from first principles.

Given that $y = 7x^2 - 3x$, find $\frac{dy}{dx}$ from first principles.

Given that $y = 8x^2 + 4x - 3$, find $\frac{dy}{dx}$ from first principles.

[1]

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(h)}{h}$$

[5]