YSGOL CREUDDYN
DAWN DYSG DAIONI

The Mathematics Department

10

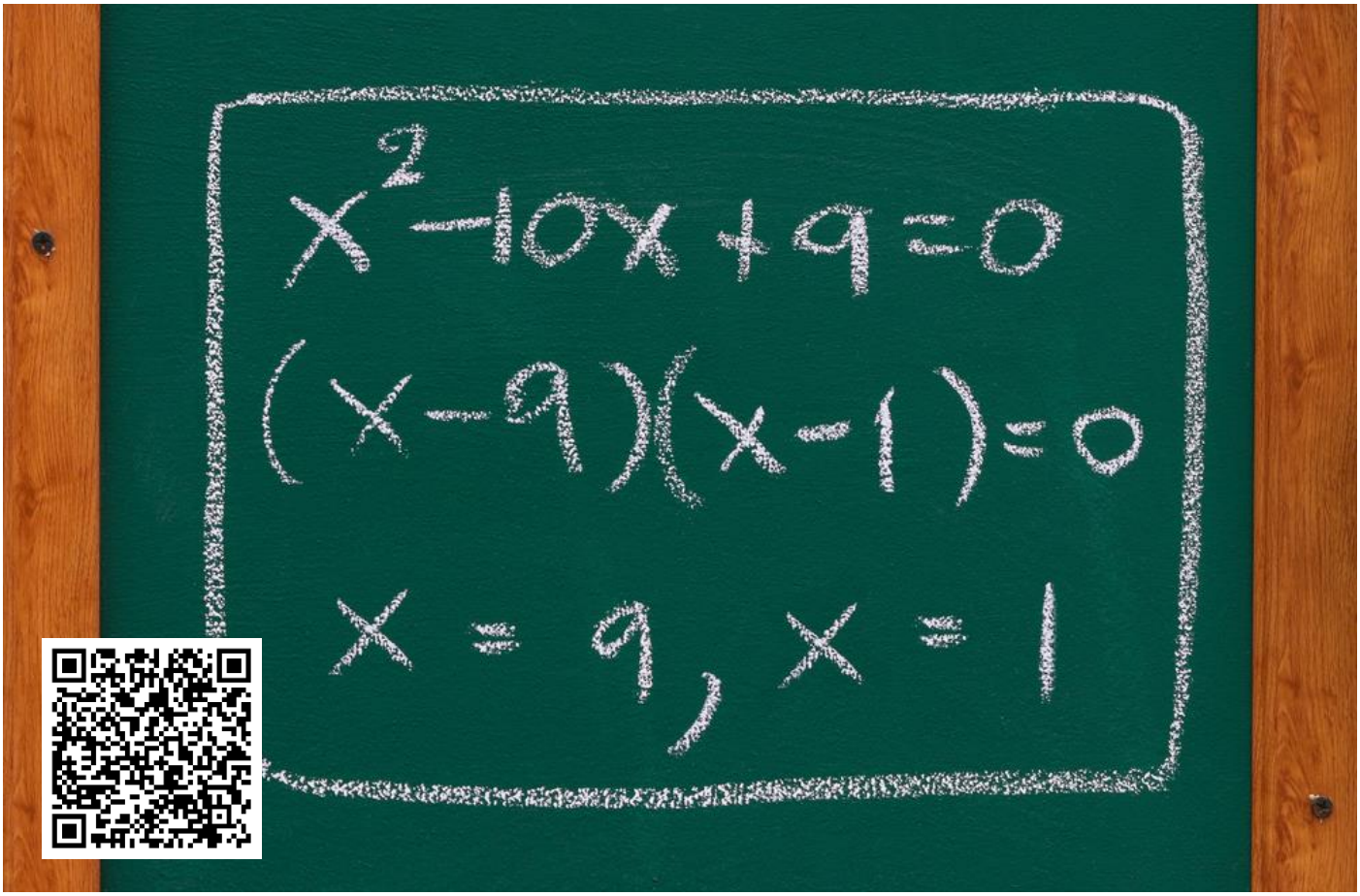
Developing Algebra 2

Higher Tier

Version 3.0
08/2025

Name:

Contents



Simple Factorising

Reversing Expansion

Like subtraction reverses addition, and division reverses multiplication, **factorising** reverses **expanding brackets,** a technique seen previously in the **Developing Algebra** workbook.

Exercise 1

Revision A

Expand the following algebraic expressions.

(a) $2(x+3)$ (b) $5(x+3)$ (c) $5(x-3)$

(d) $5(3-x)$ (e) $2(y+3)$ (f) $2(x+3+y)$

(g) $x(x+3)$ (h) $x(3+x)$ (i) $2x(x+3)$

(j) $5x(x+3)$ (k) $5x(2x+3)$ (l) $2x(5x-3)$

(m) $4(x-7)$ (n) $y(y+9)$ (o) $2z(z+4)$

(p) $7x(3x+4)$ (q) $-7x(3x+4)$ (r) $-7x(3x-4)$

Expanding gets rid of brackets, whereas factorising **reintroduces brackets.** Also, whilst expanding using multiplication sums, factorising uses **division** sums. To this intent, an important skill when factorising is to recognise the **highest common factor** of a set of numbers, which is the largest number that divides into the list of numbers.

Exercise 2

What is the highest common factor of the following numbers?

(a) 6 and 8 (b) 12 and 15 (c) 20 and 30 (d) 20 and 40 (e) 18 and 24

(f) 16 and 40 (g) 22 and 33 (h) 24 and 36 (i) 35 and 56 (j) 36 and 54

(k) 12, 14 and 16 (l) 12, 16 and 20 (m) 25, 35 and 45 (n) 30, 45 and 60 (o) 7, 11 and 13

In order to factorise an algebraic expression such as $12x+18$, we start by considering the highest common factor of the terms $12x$ and 18 in the expression.

Example

Factorise $12x+18$.

1. "What is the highest common factor of $12x$ and 18?" The answer is **6**, therefore we write 6 followed by a pair of brackets.

$$12x+18 = 6(\quad)$$

2. "6 multiplied by what gives $12x$?" 6 multiplied by $\mathbf{2x}$ is $12x$, therefore we write $2x$ inside the brackets.

$$12x+18 = 6(2x\quad)$$

3. "6 multiplied by what gives 18?" 6 multiplied by **3** gives 18, therefore we write $+3$ inside the brackets.

$$12x+18 = 6(2x+3)$$

Exercise 3

Factorise the following algebraic expressions.

(a) $4x + 6$	(b) $6 + 4x$	(c) $4x - 6$
(d) $6 - 4x$	(e) $4x + 8$	(f) $4z + 8$
(g) $6x + 8$	(h) $6x + 12$	(i) $12 - 6x$
(j) $9x + 21$	(k) $25x + 30$	(l) $18x + 30$
(m) $14x + 21$	(n) $28 + 14x$	(o) $30x - 40$
(p) $24y + 36$	(q) $60x + 80$	(r) $36x + 45$
(s) $36x + 54$	(t) $33y - 55$	(u) $33y - 66$
(v) $45 + 30x$	(w) $300x + 500$	(x) $80z - 240$
(y) $2x + 4y + 6$	(z) $5x + 10y + 20$	(α) $12x - 20y + 24$

Not all algebraic expressions can be factorised. For example, $5x + 7$ cannot be factorised since the highest common factor of $5x$ and 7 is 1. (We don't factorise $5x + 7$ as $1(5x + 7)$.)

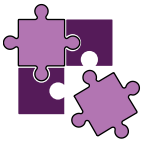
Exercise 4

Factorise all of the algebraic expressions that do factorise, and note which expressions do not factorise.

(a) $5x + 10$	(b) $5x + 11$	(c) $5x + 5$
(d) $16x$	(e) $16x + 2$	(f) $16x + 9$
(g) $8y - 12$	(h) $8y - 13$	(i) $8y - 14$

When factorising, it's not just numbers that can appear before the first bracket – we can include variables like x too.

Example



Factorise $6x^2 + 14x$.

1. "What is the highest common factor of $6x^2$ and $14x$?" The answer is $\mathbf{2x}$, therefore we write $2x$ followed by a pair of brackets.

$$6x^2 + 14x = 2x(\quad)$$

2. "$2x$ multiplied by what gives $6x^2$?" $2x$ multiplied by $\mathbf{3x}$ is $6x^2$, therefore we write $3x$ inside the brackets.

$$6x^2 + 14x = 2x(3x\quad)$$

3. "$2x$ multiplied by what gives $14x$?" $2x$ multiplied by $\mathbf{7}$ gives $14x$, therefore we write $+ 7$ inside the brackets.

$$6x^2 + 14x = 2x(3x + 7)$$

Exercise 5

What is the highest common factor of the following terms?

(a) $4x^2$ and $18x$	(b) $4x^2$ and $16x$	(c) $5x^2$ and $20x$	(d) $5x^2$ and $7x$	(e) $18x^2$ and $24x$
(f) x^2 and x	(g) x^3 and x^2	(h) $4x^3$ and $18x^2$	(i) $9x^4$ and $3x^2$	(j) $12x^4$ and $15x^3$

Exercise 6

Factorise the following algebraic expressions.

(a) $3x^2 + 6x$ (b) $6x + 3x^2$ (c) $3x^2 - 6x$

(d) $x^2 + x$ (e) $x^3 + x$ (f) $x^3 + x^2$

(g) $4x^2 + 2x$ (h) $2x^3 - 6x$ (i) $2x^3 + 8x^2$

(j) $12x^2 + 4x$ (k) $4x - 12x^2$ (l) $10x^2 + 15x$

(m) $6x^4 + 9x^2$ (n) $24y^3 - 16$ (o) $21z^2 + 14z$

(p) $3x^2 + 5x$ (q) $7y - 11y^2$ (r) $4z^3 + 17z^2$

(s) $22x^2 + 33x^5$ (t) $12n - 4n^2$ (u) $2a + a^2$

(v) $x^6 + 6x$ (w) $x^6 + 6x^2$ (x) $6x^6 + 4x^4$

(y) $2x^3 + 4x^2 + 6x$ (z) $2x^2 + 4x + 6$ (α) $42x^4 - 30x^2 + 12x^3$

Exercise 7

What is the highest common factor of the following terms?

(a) x^2 and xy (b) $4x^2$ and $6xy$ (c) $12x^2y$ and $16xy$ (d) $5x^2y^2$ and $9y^2$ (e) $18x^3y$ and $12x^2y$

Example

$2x^2y + 4x = 2x(xy + 2)$ $30y^3z^2 - 24y^2z = 6y^2z(5yz - 4)$

Exercise 8

Factorise the following algebraic expressions.

(a) $xy + x$ (b) $4xy + 10y$ (c) $x^2y + 5xy$

(d) $2x^2y + 6xy$ (e) $10yz + 5yz^2$ (f) $12x^2y - 4x$

(g) $x^2 + xy^2$ (h) $16x^3z - 12z^2$ (i) $15x^4y + 25x^2y^2$

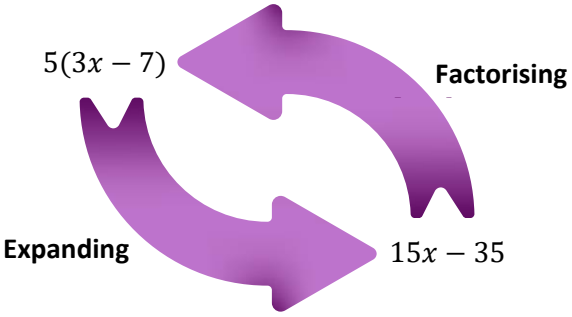
(j) $x^2yz + 4xyz$ (k) $8ab^2c^3 - 18abc^2$ (l) $26\pi x^2 + 65\pi x$

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Factorising Quadratic Expressions

As we saw in the previous chapter, factorising and expanding are two sides of the same coin.



Our aim in this chapter is to develop our understanding of factorising to be able to **factorise quadratic expressions.**

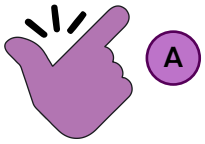
Preparation

To be able to factorise quadratic expressions, we must develop the following skill: given two numbers, for example 7 and 10, is it possible to find two numbers that **add to make 7** and **multiply to make 10**?

$$\square + \square = 7$$
$$\square \times \square = 10$$

In this case, the numbers we are looking for are **2 and 5**, since 2 + 5 = 7, and 2 × 5 = 10.

Exercise 9



Find a pair of numbers that satisfy the following calculations.

(a) $\square + \square = 10$, $\square \times \square = 24$
(b) $\square + \square = 11$, $\square \times \square = 24$
(c) $\square + \square = 14$, $\square \times \square = 24$
(d) $\square + \square = 25$, $\square \times \square = 24$

(e) $\square + \square = 9$, $\square \times \square = 18$
(f) $\square + \square = 11$, $\square \times \square = 18$
(g) $\square + \square = 8$, $\square \times \square = 16$
(h) $\square + \square = 17$, $\square \times \square = 16$

(i) $\square + \square = 13$, $\square \times \square = 30$
(j) $\square + \square = 17$, $\square \times \square = 30$
(k) $\square + \square = 8$, $\square \times \square = 12$
(l) $\square + \square = 7$, $\square \times \square = 12$

(m) $\square + \square = 15$, $\square \times \square = 14$
(n) $\square + \square = 11$, $\square \times \square = 28$
(o) $\square + \square = 5$, $\square \times \square = 4$
(p) $\square + \square = 2$, $\square \times \square = 1$

(q) $\square + \square = 13$, $\square \times \square = 42$
(r) $\square + \square = 23$, $\square \times \square = 42$
(s) $\square + \square = 17$, $\square \times \square = 42$
(t) $\square + \square = 43$, $\square \times \square = 42$

(u) $\square + \square = 32$, $\square \times \square = 60$
(v) $\square + \square = 16$, $\square \times \square = 60$
(w) $\square + \square = 19$, $\square \times \square = 60$
(x) $\square + \square = 17$, $\square \times \square = 60$

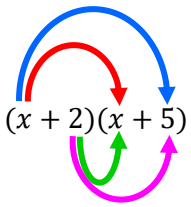
(y) $\square + \square = 16$, $\square \times \square = 55$
(z) $\square + \square = 24$, $\square \times \square = 80$
(α) $\square + \square = 29$, $\square \times \square = 100$
(β) $\square + \square = 36$, $\square \times \square = 99$

Challenge!

(a) $\square + \square = 27$, $\square \times \square = 72$
(b) $\square + \square = 19$, $\square \times \square = 84$
(c) $\square + \square = 22$, $\square \times \square = 96$
(d) $\square + \square = 39$, $\square \times \square = 360$

Factorising Quadratic Expressions of the form $x^2 + ax + b$

In the previous Developing Algebra workbook, we saw how to expand double brackets, for example $(x + 2)(x + 5)$, using the acronym **FOIL**, or by using a **multiplication grid**.



FIRST
OUTSIDE
INSIDE
LAST

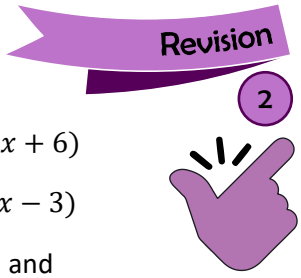
$= x^2 + 5x + 2x + 10$
$= x^2 + 7x + 10$

Question: $(x + 2)(x + 5)$

	x	2
x	x^2	$2x$
5	$5x$	10

Answer: $x^2 + 2x + 5x + 10$
$= x^2 + 7x + 10$

Exercise 10



Expand the following algebraic expressions.

(a) $(x + 3)(x + 5)$ (b) $(x + 2)(x + 8)$ (c) $(x - 2)(x + 6)$

(d) $(x - 4)(x - 5)$ (e) $(y + 1)(y + 3)$ (f) $(x + 9)(x - 3)$

Let us now look at the **reverse** process of factorising the quadratic expression $x^2 + 7x + 10$, and finishing with the double bracket $(x + 2)(x + 5)$.

Example

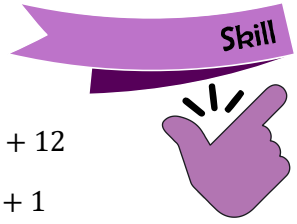
Factorise the quadratic expression $x^2 + 7x + 10$.

Answer: We need to consider the following question: 'Which two numbers add to make 7 (the coefficient of the x term) and multiply to make 10 (the constant)?'.

Add	Multiply
7	10

The answer is 2 and 5, since 2 + 5 = 7, and 2 × 5 = 10. **Therefore**, $x^2 + 7x + 10$ factorises to give $(x + 2)(x + 5)$.

Exercise 11



Factorise the following algebraic expressions.

(a) $x^2 + 6x + 8$ (b) $x^2 + 7x + 12$ (c) $x^2 + 8x + 12$

(d) $x^2 + 8x + 15$ (e) $x^2 + 16x + 15$ (f) $x^2 + 2x + 1$

(g) $x^2 + 11x + 18$ (h) $x^2 + 9x + 18$ (i) $x^2 + 19x + 18$

(j) $x^2 + 12x + 20$ (k) $x^2 + 9x + 20$ (l) $x^2 + 21x + 20$

(m) $x^2 + 18x + 32$ (n) $x^2 + 12x + 32$ (o) $x^2 + 33x + 32$

(p) $x^2 + 11x + 24$ (q) $x^2 + 25x + 24$ (r) $x^2 + 10x + 24$

(s) $x^2 + 14x + 33$ (t) $x^2 + 15x + 36$ (u) $x^2 + 16x + 39$

(v) $x^2 + 17x + 42$ (w) $x^2 + 26x + 48$ (x) $x^2 + 15x + 50$

(y) $x^2 + 16x + 60$ (z) $x^2 + 19x + 60$ (α) $x^2 + 23x + 60$

Exercise 12 2

Factorise the following algebraic expressions.

(a) $x^2 + 10x + 25$ (b) $x^2 + 25 + 10x$ (c) $25 + 10x + x^2$

(d) $x^2 + 40 + 14x$ (e) $40 + 13x + x^2$ (f) $22x + 40 + x^2$

Example

Factorise the following algebraic expressions.

(a) $x^2 + 2x - 15$ (b) $x^2 - 2x - 15$ (c) $x^2 - 8x + 15$

Add	Multiply
2	–15

$\boxed{-3} + \boxed{5} = 2$

$\boxed{-3} \times \boxed{5} = -15$

$x^2 + 2x - 15 = (x - 3)(x + 5)$

Add	Multiply
–2	–15

$\boxed{-5} + \boxed{3} = -2$

$\boxed{-5} \times \boxed{3} = -15$

$x^2 - 2x - 15 = (x - 5)(x + 3)$

Add	Multiply
–8	15

$\boxed{-3} + \boxed{-5} = -8$

$\boxed{-3} \times \boxed{-5} = 15$

$x^2 - 8x + 15 = (x - 3)(x - 5)$

Exercise 13

Factorise the following algebraic expressions.

(a) $x^2 + 4x - 12$ (b) $x^2 - 4x - 12$ (c) $x^2 + x - 12$

(d) $x^2 - x - 12$ (e) $x^2 + 23x - 24$ (f) $x^2 - 23x - 24$

(g) $x^2 + 10x - 24$ (h) $x^2 - 10x - 24$ (i) $x^2 + 5x - 24$

(j) $x^2 - 5x - 24$ (k) $x^2 + 2x - 24$ (l) $x^2 - 2x - 24$

(m) $x^2 + 39x - 40$ (n) $x^2 - 39x - 40$ (o) $x^2 + 18x - 40$

(p) $x^2 - 18x - 40$ (q) $x^2 + 6x - 40$ (r) $x^2 - 6x - 40$

(s) $x^2 + 3x - 40$ (t) $x^2 - 3x - 40$ (u) $x^2 + 4x - 32$

(v) $x^2 - 4x - 32$ (w) $x^2 - 14x - 32$ (x) $x^2 + 14x - 32$

(y) $x^2 - 31x - 32$ (z) $x^2 + 31x - 32$ (α) $x^2 - 2x - 8$

Exercise 14

Factorise the following algebraic expressions.

(a) $x^2 - 7x + 12$ (b) $x^2 - 8x + 12$ (c) $x^2 - 13x + 12$

(d) $x^2 - 10x + 24$ (e) $x^2 - 11x + 24$ (f) $x^2 - 25x + 24$

(g) $x^2 - 14x + 40$ (h) $x^2 - 13x + 40$ (i) $x^2 - 22x + 40$

(j) $x^2 - 33x + 32$ (k) $x^2 - 12x + 32$ (l) $x^2 - 18x + 32$

(m) $x^2 - 5x + 6$ (n) $x^2 - 6x + 9$ (o) $x^2 - 11x + 18$

Challenge! 

Factorise the following algebraic expressions.

(a) $x^2 + 4x - 96$ (b) $x^2 - 5x - 84$ (c) $x^2 + x - 240$

Exercise 15 2

Factorise the following algebraic expressions.

(a) $x^2 + 8x + 16$ (b) $x^2 - 8x + 16$ (c) $x^2 + 10x + 16$

(d) $x^2 - 10x + 16$ (e) $x^2 + 6x - 16$ (f) $x^2 - 6x - 16$

(g) $x^2 + 17x + 16$ (h) $x^2 + 15x - 16$ (i) $x^2 - 17x + 16$

(j) $x^2 + 11x + 28$ (k) $x^2 + 16x + 28$ (l) $x^2 + 29x + 28$

(m) $x^2 - 11x + 28$ (n) $x^2 - 12x - 28$ (o) $x^2 + 27x - 28$

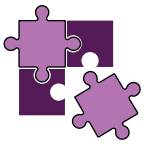
(p) $x^2 - 3x - 28$ (q) $x^2 - 16x + 28$ (r) $x^2 - 27x - 28$

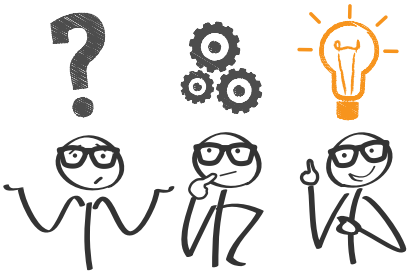
(s) $x^2 + 7x + 10$ (t) $x^2 + 11x + 10$ (u) $x^2 - 7x + 10$

(v) $x^2 + 9x - 10$ (w) $x^2 - 9x - 10$ (x) $x^2 - 11x + 10$

(y) $x^2 + x - 20$ (z) $x^2 - 8x - 20$ (α) $x^2 - 21x + 20$

Solving Quadratic Equations through Factorising





Example

Solve the quadratic equation $x^2 + 6x + 8 = 0$.

Step 1: Factorise.

Add	Multiply
6	8

$\boxed{2} + \boxed{4} = 6$

$\boxed{2} \times \boxed{4} = 8$

$x^2 + 6x + 8 = (x + 2)(x + 4)$

Step 2: Solve.

$x^2 + 6x + 8 = 0$

$(x + 2)(x + 4) = 0$

Either $x + 2 = 0$ or $x + 4 = 0$

$x = -2$ $x = -4$

Exercise 16

Solve the following quadratic equations.

(a) $x^2 + 5x + 6 = 0$ (b) $x^2 + 9x + 14 = 0$ (c) $x^2 + 12x + 27 = 0$

(d) $x^2 - 5x + 6 = 0$ (e) $x^2 - 9x + 14 = 0$ (f) $x^2 - 12x + 27 = 0$

(g) $x^2 + x - 6 = 0$ (h) $x^2 - 5x - 14 = 0$ (i) $x^2 + 6x - 27 = 0$

(j) $x^2 - x - 6 = 0$ (k) $x^2 + 5x - 14 = 0$ (l) $x^2 - 6x - 27 = 0$

(m) $x^2 + 7x + 6 = 0$ (n) $x^2 - 15x + 14 = 0$ (o) $x^2 - 26x - 27 = 0$

(p) $x^2 - 5x - 6 = 0$ (q) $x^2 + 13x - 14 = 0$ (r) $x^2 + 26x - 27 = 0$

Exercise 17

Applying

Wilf thinks of a number, x.

His sister says that if Wilf subtracts 6 from his number and then multiplies this new number with the number he first thought of, he will get an answer of –5.

Use this information to do the following.

(a) Form a quadratic equation of the form $x^2 + ax + b = 0$.

(b) Solve the equation to find the possible values of x.

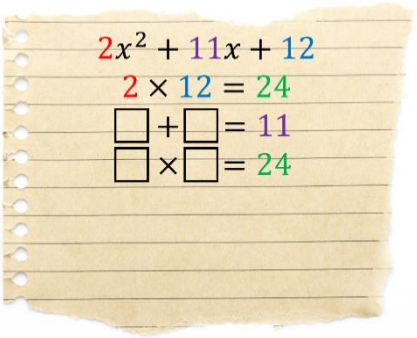
Factorising Quadratic Expressions of the form $ax^2 + bx + c$

Example

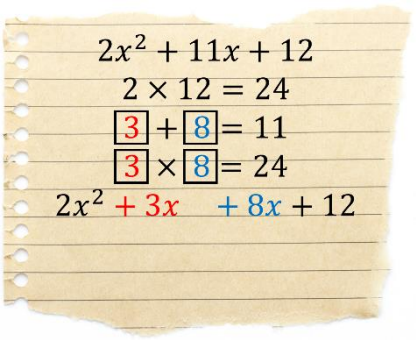
Factorise the quadratic expression $2x^2 + 11x + 12$.

Method 1: The Splitting Method

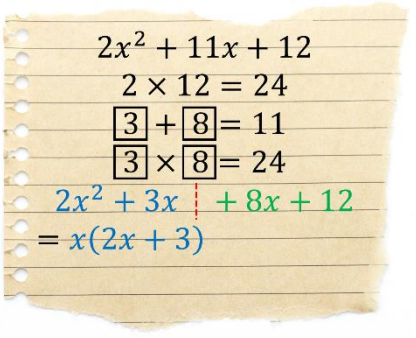
1. Multiply 2 (the coefficient of the x^2 term) by 12 (the constant) to obtain 24. We must look for a pair of numbers that add to give 11 (the coefficient of the x term) and multiply to give 24.



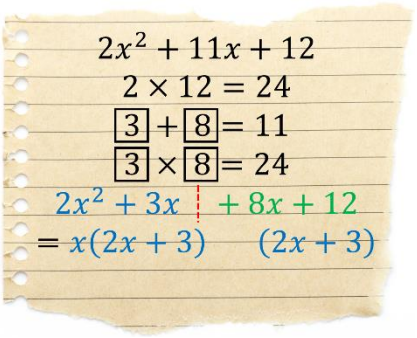
2. The numbers which work are 3 and 8. We rewrite the question by splitting the $11x$ to be $3x$ add $8x$.



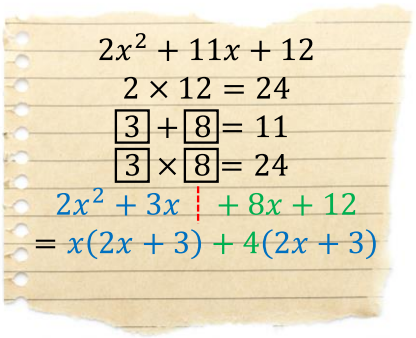
3. We split the four terms into two halves and factorise the first half.



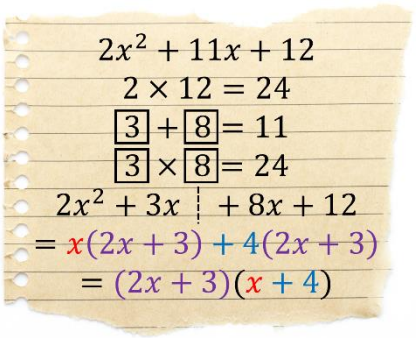
4. We copy the brackets, leaving a space between the old brackets and the new brackets.



5. $+ 4$ must appear in the space, since 4 multiplied by $2x$ is $8x$, and 4 multiplied by 3 is 12.

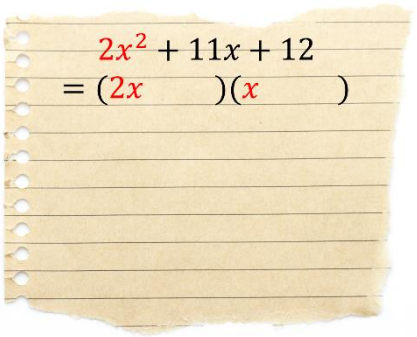


6. The expression has a common factor of $2x + 3$, therefore we factorise this out to leave the final answer.

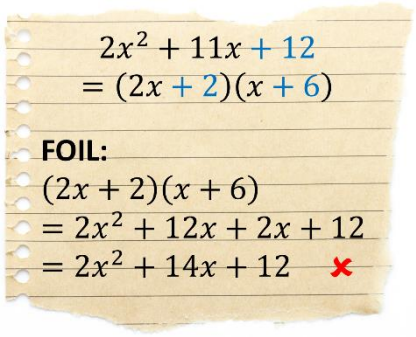


Method 2: The Detective Method

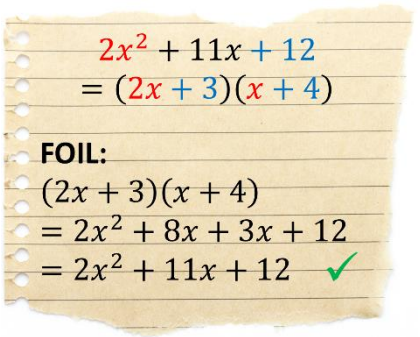
1. Write a pair of brackets. At the start of the brackets, write a pair of terms that multiply to give $2x^2$.



2. At the end of the brackets, write a pair of terms that multiply to give 12. Use (e.g.) FOIL (in your head or on paper) to check your answer.



3. If the answer is not correct, choose a different combination, repeating until you reach the correct answer.



Exercise 18

Skill 2

Factorise the following quadratic expressions.

(a) $2x^2 + 11x + 15$ (b) $2x^2 + 13x + 15$ (c) $2x^2 + 7x + 6$

(d) $3x^2 + 13x + 4$ (e) $3x^2 + 11x + 10$ (f) $3x^2 + 17x + 20$

(g) $4x^2 + 21x + 5$ (h) $4x^2 + 9x + 5$ (i) $4x^2 + 12x + 5$

(j) $5x^2 + 18x + 9$ (k) $5x^2 + 8x + 3$ (l) $6x^2 + 13x + 6$

(m) $2x^2 - x - 15$ (n) $3x^2 + x - 14$ (o) $5x^2 - 17x - 12$

(p) $3x^2 - 5x - 12$ (q) $4x^2 - 3x - 10$ (r) $2x^2 - 7x - 15$

(s) $4x^2 - 7x - 2$ (t) $3x^2 - 16x - 12$ (u) $4x^2 + 21x - 18$

(v) $3x^2 - 14x + 8$ (w) $5x^2 - 19x + 12$ (x) $3x^2 - 26x + 35$

(y) $2x^2 - 21x + 40$ (z) $2x^2 - 11x + 12$ (α) $4x^2 - 11x + 6$

Challenge!

Factorise the following quadratic expressions.

(a) $8x^2 - 2x - 15$ (b) $8x^2 - 19x + 6$ (c) $30x^2 - 42x + 12$

Example

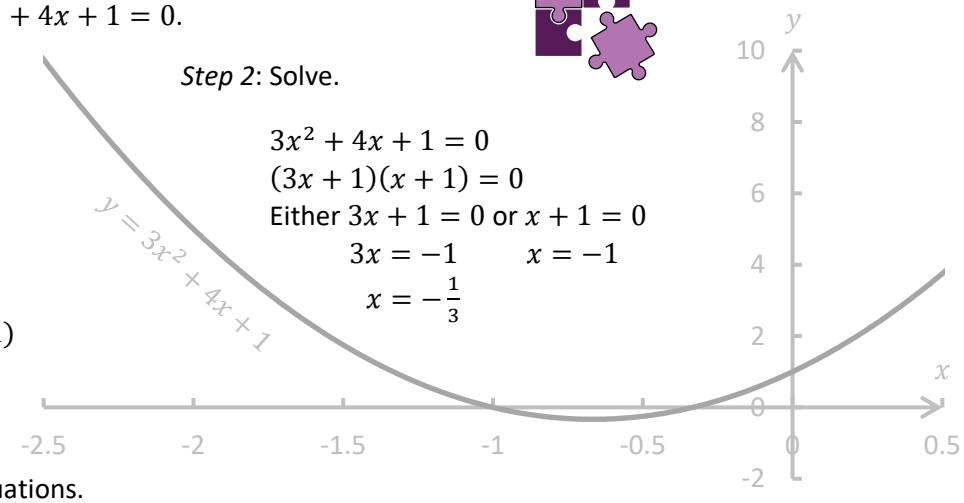
Solve the quadratic equation $3x^2 + 4x + 1 = 0$.

Step 1: Factorise.

$3 \times 1 = 3$.
$\boxed{1} + \boxed{3} = 4$
$\boxed{1} \times \boxed{3} = 3$
$3x^2 + 4x + 1$
$= 3x^2 + x + 3x + 1$
$= x(3x + 1) + 1(3x + 1)$
$= (3x + 1)(x + 1)$

Step 2: Solve.

$3x^2 + 4x + 1 = 0$
$(3x + 1)(x + 1) = 0$
Either $3x + 1 = 0$ or $x + 1 = 0$
$3x = -1$ $x = -1$
$x = -\frac{1}{3}$



Exercise 19

Solve the following quadratic equations.

(a) $2x^2 + 3x + 1 = 0$ (b) $2x^2 + 5x + 2 = 0$ (c) $2x^2 + 13x + 20 = 0$

(d) $2x^2 - 3x - 20 = 0$ (e) $2x^2 + 3x - 20 = 0$ (f) $2x^2 - 13x + 20 = 0$

(g) $3x^2 + 10x + 7 = 0$ (h) $3x^2 + 7x + 2 = 0$ (i) $3x^2 - 11x + 6 = 0$

(j) $2x^2 - 7x + 3 = 0$ (k) $2x^2 + 3x - 5 = 0$ (l) $2x^2 - 11x + 5 = 0$

(m) $4x^2 - 4x + 1 = 0$ (n) $4x^2 - 11x - 3 = 0$ (o) $5x^2 - 24x - 5 = 0$

(p) $6x^2 + x - 2 = 0$ (q) $6x^2 - 7x - 5 = 0$ (r) $15x^2 - 4x - 3 = 0$

Challenge!

Solve the following quadratic equations.

(a) $12x^2 + 28x - 5 = 0$ (b) $28x^2 + 15x + 2 = 0$ (c) $24x^2 - 2x - 15 = 0$

Example

Factorise the following quadratic expressions.

(a) $2x^2 + 20x + 42$

Answer: $2x^2 + 20x + 42$
$= 2(x^2 + 10x + 21)$
$= 2(x + 3)(x + 7)$

(b) $(x + 5)^2 + 8(x + 5)$

Answer: $(x + 5)^2 + 8(x + 5)$
$= (x + 5)((x + 5) + 8)$
$= (x + 5)(x + 13)$

(c) $2x^2 + 8x$

Answer: $2x^2 + 8x$
$= 2x(x + 4)$

Exercise 20 2

Factorise the following quadratic expressions.

(a) $2x^2 + 22x + 56$

(b) $(x + 3)^2 + 7(x + 3)$

(c) $2x^2 + 20x$

(d) $3x^2 + 18x + 24$

(e) $(x - 5)^2 + 8(x - 5)$

(f) $3x^2 - 12x$

(g) $4x^2 + 12x - 40$

(h) $(x - 2)^2 - 4(x - 2)$

(i) $4x^2 - 18x$

(j) $4x^2 + 26x + 30$

(k) $7(x + 4)^2 + 3(x + 4)$

(l) $5x^2 + 45x$

The Difference of Two Squares

An expression of the form $a^2 - b^2$ factorises in a special way.

$$\boldsymbol{a^2 - b^2 = (a + b)(a - b)}$$

Example

Factorise the following expressions.

(a) $x^2 - 9$

Answer: $x^2 - 9$
$= (x + 3)(x - 3)$

(b) $4x^2 - 49$

Answer: $4x^2 - 49$
$= (2x + 7)(2x - 7)$

(c) $27x^2 - 75y^2$

Answer: $27x^2 - 75y^2$
$= 3(9x^2 - 25y^2)$
$= 3(3x + 5y)(3x - 5y)$

Exercise 21 2

Factorise the following expressions.

(a) $x^2 - 4$

(b) $x^2 - 16$

(c) $x^2 - 1$

(d) $x^2 - 144$

(e) $y^2 - 100$

(f) $z^2 - 36$

(g) $4x^2 - 25$

(h) $9x^2 - 4$

(i) $49x^2 - 81$

(j) $64x^2 - 1$

(k) $100y^2 - 9$

(l) $16z^2 - 121$

(m) $2x^2 - 18$

(n) $2x^2 - 50$

(o) $3x^2 - 48$

(p) $8x^2 - 18$

(q) $6x^2 - 24$

(r) $5x^2 - 125$

(s) $x^2 - y^2$

(t) $4x^2 - z^2$

(u) $x^2y^2 - 1$

(v) $16x^2 - \pi^2$

(w) $8x^2 - 72z^2$

(x) $4x^2z^2 - 36y^2$

(y) $x^4 - 4$

(z) $9y^4 - 16$

(α) $32z^6 - 128y^2$

Exercise 22 2

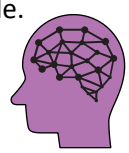
Solve the following quadratic equations.

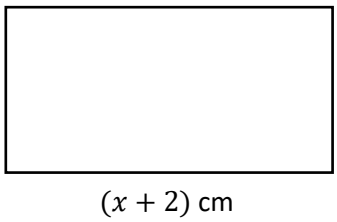
(a) $x^2 - 25 = 0$ (b) $y^2 - 64 = 0$ (c) $z^2 - 169 = 0$

(d) $4x^2 - 49 = 0$ (e) $9x^2 - 1 = 0$ (f) $4x^2 - 16 = 0$

Exercise 23

The area of the rectangle on the right is 45 cm². Use the difference of two squares method to calculate the height and width of the rectangle.





Exercise 24 (Revision)

Solve the following quadratic equations.

(a) $x^2 + 15x + 44 = 0$ (b) $x^2 - 15x + 44 = 0$ (c) $x^2 + 7x - 44 = 0$

(d) $4x^2 + 14x = 0$ (e) $4x^2 - 14x = 0$ (f) $14x - 4x^2 = 0$

(g) $2x^2 + 13x + 21 = 0$ (h) $2x^2 - 13x + 21 = 0$ (i) $2x^2 + x - 21 = 0$

(j) $x^2 - 36 = 0$ (k) $9x^2 - 100 = 0$ (l) $4x^2 - 36 = 0$

Challenge!

The picture below shows Menai Bridge. We can model the cable between the two towers using the quadratic equation $y = \frac{43}{7744}x^2 - \frac{43}{44}x$. Given that the origin is at the highest point of one of the towers, solve the equation $\frac{43}{7744}x^2 - \frac{43}{44}x = 0$ to calculate the horizontal distance (in metres) between the top of the two towers.

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Simultaneous Equations

Our aim in this chapter is to solve problems similar to the one below.

"Deiniol buys 2 *fish* and 3 *chips* in the local *fish & chips* shop, and he pays £8. Awel buys 4 *fish* and 2 *chips* in the same shop, and pays £12. What is the cost of 1 *fish* and 1 *chips* in the shop?"

By using the variable f to represent the cost of 1 *fish*, and c to represent the cost of 1 *chips*, we can write the following equations to represent the problem.

$2f + 3c = 8$
$4f + 2c = 12$

To solve the above equations, which are **simultaneous equations,** we must develop a number of **algebraic techniques** for our algebraic *toolbox* …

Multiplying Equations

Example

Multiply the equation $3x + 2y = 5$ by 4.

Answer: We multiply every term in the equation by four to obtain the equation $12x + 8y = 20$.

Exercise 25

Skill 2

Multiply the following equations by the numbers in the boxes.

(a) $2x + 6y = 4$	$\times 2$	(b) $3x + 4y = 5$	$\times 2$	(c) $7x + 2y = 6$	$\times 2$
(d) $4x + 3y = 7$	$\times 3$	(e) $8x + 11y = 3$	$\times 4$	(f) $6x + 3y = 11$	$\times 5$
(g) $2x - 5y = 3$	$\times 2$	(h) $-5x + 2y = 4$	$\times 2$	(i) $3x - 8y = -7$	$\times 2$
(j) $x + 7y = 3$	$\times 6$	(k) $8x + y = 9$	$\times 7$	(l) $3x - 4y = 1$	$\times 8$
(m) $3x + 6y = 7$	$\times -2$	(n) $5x - 2y = 10$	$\times -3$	(o) $-3x + 2y = -5$	$\times -4$

Subtracting Equations

Example

$$\begin{array}{rr} & 5x + 8y = 16 \\ - & 2x + 3y = 7 \\ \hline & 3x + 5y = 9 \end{array} \qquad \begin{array}{rr} & 7x + 9y = 25 \\ - & 3x + 9y = 4 \\ \hline & 4x \quad\;\; = 21 \end{array} \qquad \begin{array}{rr} & 5x + 6y = 10 \\ - & 5x - 2y = 3 \\ \hline & 8y = 7 \end{array}$$

Exercise 26

Subtract the second equation from the first equation.

(a) $7x + 8y = 20$ $3x + 4y = 12$	(b) $9x + 5y = 13$ $3x + 2y = 5$	(c) $15x + 9y = 14$ $12x + 4y = 9$
(d) $6x + 2y = 31$ $2x + 7y = 8$	(e) $8x + 3y = 15$ $2x + \;\; y = 10$	(f) $19x - 8y = 8$ $4x + 4y = 3$

Exercise 26 (continued)

(g) $8x + 7y = 15$
$2x + 7y = 6$

(h) $11x + 4y = 27$
$11x + 2y = 3$

(i) $20x + 5y = -8$
$18x - 5y = 4$

(j) $-4x + 8y = 18$
$4x + 2y = 4$

(k) $6x - 4y = 8$
$2x - 4y = 10$

(l) $8x - 4y = 15$
$8x - 6y = 2$

(m) $x + 18y = 12$
$x + 17y = 2$

(n) $18x - 2y = -5$
$11x - 9y = 3$

(o) $-4x + 10y = 8$
$-4x - 2y = -3$

Multiplying Equations to Obtain Equal Coefficients

The **coefficient** of an algebraic term is the **number** which is part of the term. For example, the coefficient of $16x$ is 16.

Example

Consider the following simultaneous equations.

$3x + 10y = 16$
$4x + 5y = 13$

By multiplying the first equation by 4, and the second equation by 3, we can ensure that the x **coefficients** are equal.

$3x + 10y = 16 \xrightarrow{\times 4} 12x + 40y = 64$
$4x + 5y = 13 \xrightarrow{\times 3} 12x + 15y = 39$

On the other hand, by leaving the first equation as it is, and multiplying the second equation by 2, we can ensure that the y **coefficients** are equal.

$3x + 10y = 16$ $\quad\quad$ $3x + 10y = 16$
$4x + 5y = 13 \xrightarrow{\times 2}$ $8x + 10y = 26$

Try to use the **smallest possible** numbers that work.

Exercise 27

Which numbers do we need to multiply the following equations by to obtain equal x coefficients?

(a) $2x + 4y = 4$
$3x + 5y = 7$

(b) $2x + 4y = 4$
$4x + 12y = 7$

(c) $2x + 4y = 4$
$5x + 8y = 7$

(d) $5x + 7y = 6$
$2x + 2y = 8$

(e) $5x + 7y = 6$
$6x + 3y = 8$

(f) $5x + 7y = 6$
$15x + 14y = 8$

(g) $4x + 8y = 12$
$5x + 4y = 5$

(h) $4x + 8y = 12$
$6x + 3y = 5$

(i) $4x + 8y = 12$
$12x + 12y = 5$

(j) $24x + 32y = 20$
$8x + 8y = 20$

(k) $24x + 32y = 20$
$12x + 64y = 20$

(l) $24x + 32y = 20$
$16x + 16y = 20$

Exercise 28

Which numbers do we need to multiply the equations from Exercise 27 by to obtain equal y coefficients?

Solving Linear Equations

The final part of the jigsaw is to be able to solve linear equations like the following ones.

Exercise 29

Solve the following equations.

(a) $4x = 8$

(b) $4x = 32$

(c) $7x = 35$

(d) $4x = 2$

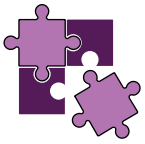
(e) $3x = 5$

(f) $8x = 7$

Solving Simultaneous Equations

We now have enough tools in our algebraic *toolbox* to return to the *fish & chips* problem from the beginning of this chapter.

"Deiniol buys 2 *fish* and 3 *chips* in the local *fish & chips* shop, and he pays £8. Awel buys 4 *fish* and 2 *chips* in the same shop, and pays £12. What is the cost of 1 *fish* and 1 *chips* in the shop?"



Step 1: Change the word problem into a pair of equations.

$2f + 3c = 8$
$4f + 2c = 12$

Step 2: Multiply the first equation by 2 so that the f coefficients are equal.

$$\begin{array}{l} 2f + 3c = 8 \\ 4f + 2c = 12 \end{array} \xrightarrow{\times 2} \begin{array}{l} 4f + 6c = 16 \\ 4f + 2c = 12 \end{array}$$

Step 3: Subtract the second equation from the first equation.

$$\begin{array}{l} 2f + 3c = 8 \\ 4f + 2c = 12 \end{array} \xrightarrow{\times 2} \begin{array}{rl} & 4f + 6c = 16 \\ - & 4f + 2c = 12 \\ \hline & 4c = 4 \end{array}$$

Step 4: Solve the equation $4c = 4$ to obtain $c = 1$.

Conclusion: The cost of 1 *chips* in the shop is £1.

To find the value of f (and thus the cost of 1 *fish*), we can use any of the following methods.

Method A: Repeat steps 2–4 above, but this time making sure that the c coefficients are equal.

$$\begin{array}{l} 2f + 3c = 8 \\ 4f + 2c = 12 \end{array} \begin{array}{l} \xrightarrow{\times 2} \\ \xrightarrow{\times 3} \end{array} \begin{array}{rl} & 4f + 6c = 16 \\ - & 12f + 6c = 36 \\ \hline - & 8f = -20 \end{array}$$

$$f = \frac{-20}{-8}$$
$$f = \frac{5}{2}$$

Method B: Substitute $c = 1$ into one of the original equations.

Substitute $c = 1$ into the equation $2f + 3c = 8$:

$2f + 3 \times 1 = 8$
$2f + 3 = 8$
$2f = 5$
$f = \frac{5}{2}$

Conclusion: The cost of 1 *fish* is £2.50.

Note: You can check that the solutions are correct by substituting the values $c = 1$, $f = 2.5$ into the left-hand side of any of the original equations.

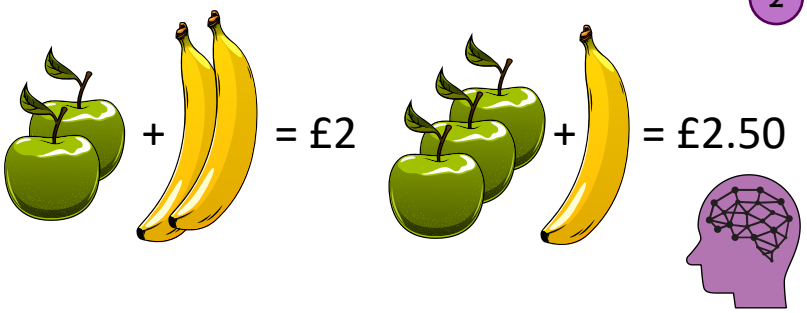
$2f + 3c$
$= 2 \times 2.5 + 3 \times 1$
$= 5 + 3$
$= £8$ ✓

$4f + 2c$
$= 4 \times 2.5 + 2 \times 1$
$= 10 + 2$
$= £12$ ✓

Applying 2

Exercise 30

Two apples and two bananas costs £2.
Three apples and one banana costs £2.50.
Find the cost of one apple and one banana.



Exercise 31

Skill 2

Solve the following simultaneous equations.

(a) $3x + 4y = 18$
$2x + 2y = 10$

(b) $2x + 3y = 9$
$4x + y = 13$

(c) $2x + 4y = 16$
$2x + 3y = 14$

(d) $5x - 2y = 6$
$2x + 2y = 8$

(e) $6x + 3y = 18$
$-2x + 2y = 6$

(f) $-2x + y = -2$
$4x - 3y = 0$

(g) $2x + 3y = 3$
$2x - y = 7$

(h) $3x + 2y = 7$
$3x - y = -8$

(i) $2x + 3y = 14$
$3x + 2y = 16$

(j) $-3x + 2y = 0$
$3x - 4y = 6$

(k) $x + 5y = 9$
$2x + 3y = 11$

(l) $5x - 2y = 19$
$3x + y = 18$

(m) $4x + y = 8$
$7x + 3y = 9$

(n) $2x - 3y = 8$
$x + 2y = -10$

(o) $2x + 6y = 34$
$4x - 2y = 5$

(p) $2x + 3y = 10$
$5x - 6y = 16$

(q) $2x + 3y = 0$
$8x + 9y = -1$

(r) $7x + 8y = 19$
$3x - 2y = -19$

(s) $3x + 4y = 15$
$x - 6y = -6$

(t) $3x - 4y = 14$
$5x - 8y = 30$

(u) $3x + 5y = 21$
$4x + 3y = 17$

(v) $3x - 2y = 17$
$2x + 7y = 3$

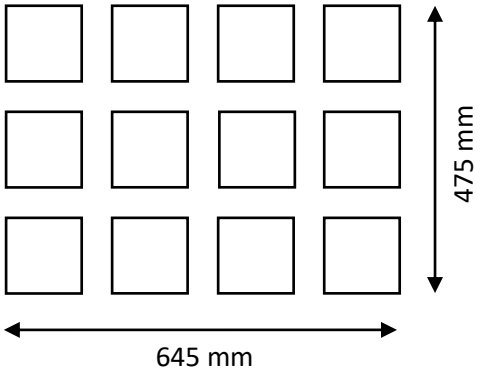
(w) $5x - 2y = 26$
$3x - 5y = 27$

(x) $2x + 4y = 5$
$5x + 7y = 8$

Exercise 32

(a) Aled buys 2 Cornish pasties and 3 sausage rolls in a shop, and pays £7. Ceinwen buys 4 Cornish pasties and 1 sausage roll in the same shop, and pays £9. What is the cost of 1 Cornish pasty and 1 sausage roll from the shop?

(b) A rectangle shape is made by using 12 square tiles with equal spaces between them. The total length of the rectangle is 645 mm and the total width of the rectangle is 475 mm. Find the dimensions of the tiles and the width of the space in mm.



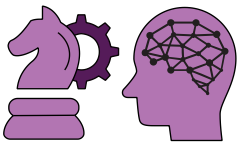
(c) Glyn employs two people, Ben and Ceri. Ben and Ceri are paid at different hourly rates. Glyn has recorded how many hours both Ben and Ceri have worked on Monday and Tuesday. He has also noted the total he paid in wages.

Day	Number of hours worked		Total wages (£)
	Ben	Ceri	
Monday	6	5	116
Tuesday	4	8	138

Applying

Use an algebraic method to calculate how much Ben and Ceri are paid per hour.

(d) Ysgol Trefswm organised a concert to raise money for a charity.
All of the 120 tickets were sold for a total of £1,210.
The price of an adult ticket was £12.
The price of a child ticket was £7.
How many adult tickets and how many child tickets were sold?



Extension 2

Exercise 33: Old, Older, Oldest

Use the clues on the cards to solve the problem.

The information for these people has been shuffled. To begin you must form simultaneous equations and solve them. Every answer is a whole number.

(1) 18 is the sum of Arwyn's age and Bedwyr's age.

(2) 47 is the sum of Rhodri's age and Sali's age.

Your job is to find everyone's age and list them from oldest to youngest. There are two sets of twins.

(3) 38 is the sum of three times Gwen's age add two times Heledd's age.

(4) Three times Wil's age subtract four times Tesni's age is 33.

(5) Three times Ynyr's age add Penri's age is 27.

(6) Arwyn's age subtract Bedwyr's age is 8.

(7) Two times Wil's age add three times Tesni's age is 107.

(8) Three times Iolo's age add four times John's age is 147.

(9) 20 is two times Cai's age subtract Dewi's age.

(10) 78 is two times Fred's age subtract Elfyn's age.

(11) Four times Kate's age add three times Lora's age is 40.

(12) The sum of Cai's age and Dewi's age is 73.

(13) 91 is two times Mali's age add Nia's age.

(14) 43 is five times Kate's age add two times Lora's age.

(15) Fred is older than Elfyn by thirteen years.

(16) By doubling Ynyr's age and adding Penri's age you get 25.

(17) Two times Rhodri's age add Sali's age is 76.

(18) Seven times Iolo's age subtract three times John's age is 10.

(19) Double of Gwen's age add three times Heledd's age gives 42.

(20) Mali is eight years older than Nia.

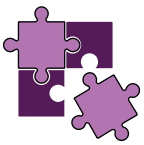
Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Changing the Subject

The purpose of **changing the subject** is to re-arrange formulae so that a particular **variable** appears on its own on the left-hand side of the formula. For example, consider the formula $p = 3w + d$, a formula to calculate the number of points (p) a football team has given how many games they have won (w) and how many drawn games (d) they have had. We can re-arrange the formula to give the number of games won by a football team.

$p = 3w + d$
$3w + d = p$ [Swap sides]
$3w = p - d$ [Subtract d from both sides]
$w = \frac{p-d}{3}$ [Divide both sides by 3]



After re-arranging the formula as shown above, we say that w is the **subject** of the formula.

There are a number of 'movements' we can perform to help re-arrange a formula to give a specific subject. Here are some of the most common movements.

Add a number to both sides of the formula

E.g. $y - 3 = x$
$y = x + 3$ [Add 3 to both sides]

Subtract a number from both sides of the formula

E.g. $y + 7 = x$
$y = x - 7$ [Subtract 7 from both sides]

Multiply both sides of the formula by a number

E.g. $\frac{y}{2} = 5x$
$y = 10x$ [Multiply both sides by 2]

Divide both sides of the formula by a number

E.g. $4y = x - 3$
$y = \frac{x-3}{4}$ [Divide both sides by 4]

Square both sides of the formula

E.g. $\sqrt{y} = 4x + 5$
$y = (4x + 5)^2$ [Square both sides]

Take the square root of both sides of the formula

E.g. $y^2 = 2x - 9$
$y = \sqrt{2x - 9}$ [Square root both sides]

Swap sides

E.g. $5x + 3 = y$
$y = 5x + 3$ [Swap sides]

Expand brackets

E.g. $4(y + 2) = 5x$
$4y + 8 = 5x$ [Expand brackets]

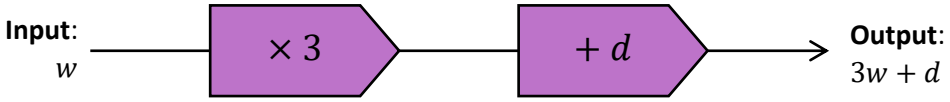
Let us reconsider the example at the top of this page. The question in the example can be set as follows.

Make w the subject of the formula $p = 3w + d$.

The aim in this question is to re-arrange the formula to leave only the variable w on the left-hand side of the formula. As the variable w initially appears on the right-hand side of the formula, it makes sense to start by swapping the sides of the formula, so that w appears on the left-hand side of the formula.

$p = 3w + d$
$3w + d = p$ [Swap sides]

There are several ways to proceed. You can think of the formula as an equation, and 'solve' to leave w on its own. Or you can think of how to calculate the left-hand side of the formula, if you start with the variable w.



By working backwards through the function machine, we can see the steps required to proceed, namely subtracting d from both sides of the formula, and then dividing both sides by 3.

Exercise 34

(a) Make e the subject of the formula $p = 2e + c$.

(b) Make c the subject of the formula $p = 2e + c$.

(c) Make c the subject of the formula $y = mx + c$.

(d) Make m the subject of the formula $y = mx + c$.

(e) Make p the subject of the formula $c = p - 3t$.

(f) Make t the subject of the formula $c = p - 3t$.

(g) Make p the subject of the formula $A = p(q + r)$.

(h) Make q the subject of the formula $A = p(q + r)$.

(i) Make t the subject of the formula $F = \frac{m+4n}{t}$.

(j) Make m the subject of the formula $F = \frac{m+4n}{t}$.

(k) Make n the subject of the formula $F = \frac{m+4n}{t}$.

(l) Make r the subject of the formula $A = \pi r^2$.

(m) Make R the subject of the formula $I = \frac{PRT}{100}$.

(n) Make s the subject of the formula $A = \frac{su}{2}$.

(o) Make u the subject of the formula $C = \frac{1}{3}\pi r^2 u$.

(p) Make r the subject of the formula $C = \frac{1}{3}\pi r^2 u$.

(q) Make u the subject of the formula $A = \frac{1}{2}(a + b)u$.

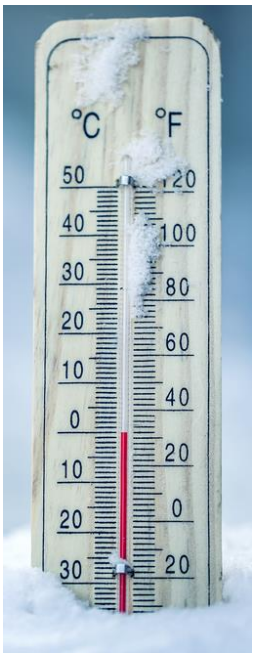
(r) Make b the subject of the formula $A = \frac{1}{2}(a + b)u$.

Exercise 35

The following formula was used by festival planners to calculate the parking fee for mini buses.

Parking Fee = Number of Passengers × 30p + £5

(a) What was the parking fee for a mini bus with 12 passengers?
(b) The parking fee for another mini bus was £7.40. How many passengers were on this mini bus?



Exercise 36

To change from degrees Celsius (°C) to degrees Fahrenheit (°F), you can use the following formula.

$$F = \frac{9}{5}(C + 40) - 40$$

(a) The temperature is 60°C. What is this in °F?
(b) Re-arrange the formula to find C in terms of F.

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Expression, Equation, Formula, Identity

In algebra, what's the difference between expressions, equations, formulae and identities?

Expression

An expression is a collection of terms (e.g. $5x$ or 7) and operators (e.g. $+$ or $\times$).
$4x + 2$ and $\sqrt{6y - 4z}$ are examples of expressions.
There are no equals signs ($=$) in expressions.

Equation

An equation notes that two terms or expressions are equal. Two sides of an equation are separated by an equals sign ($=$). Sometimes, it is possible to solve an equation to find the value of a variable.

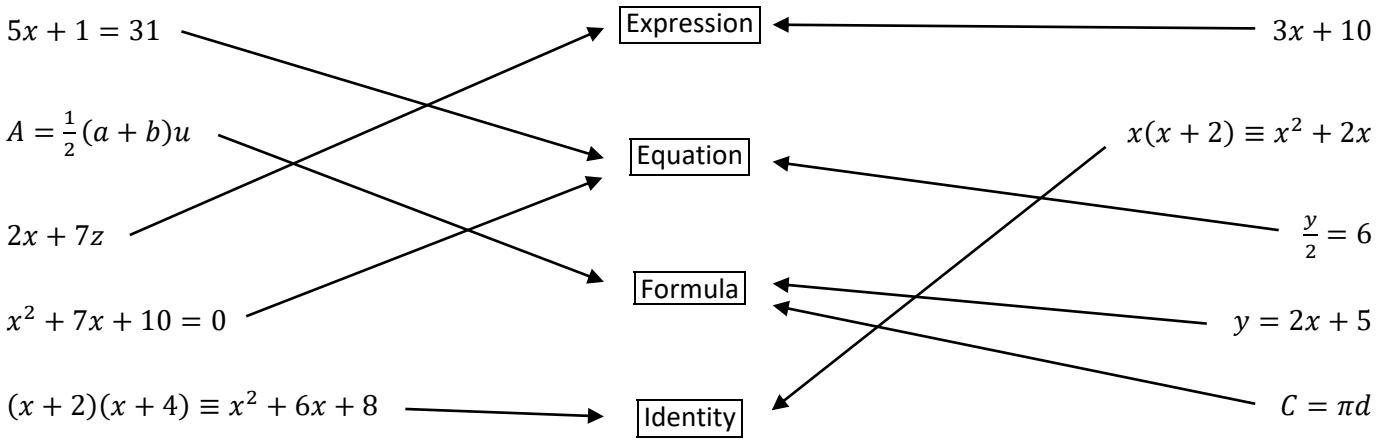
Formula

A formula is a special type of equation which shows the connection between different variables.
$P = 2a + 2b$ is an example of a formula, one which is used to calculate the perimeter of a rectangle with length a and width b.

Identity

An identity is an equation which is always true, no matter what the values of the variables are.
$2(x + 4) \equiv 2x + 8$ is an example of an identity. Two sides of an identity are separated by an equivalence sign ($\equiv$).

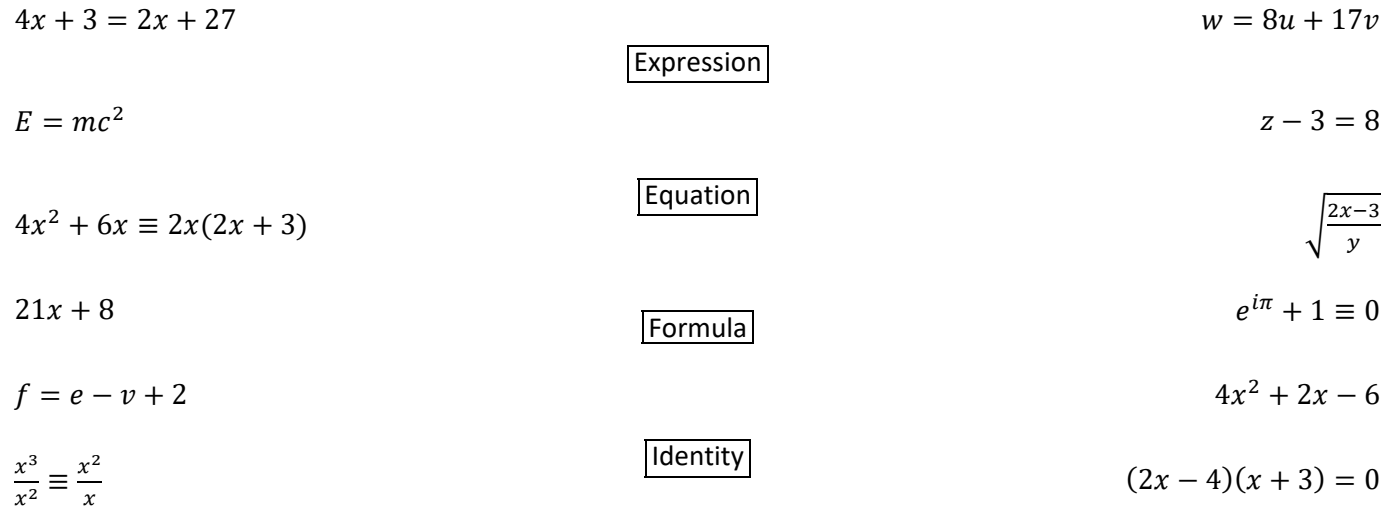
Example



Exercise 37

Skill A

Add arrows pointing to the correct descriptions.



Proving Identities

To prove an identity such as $(x+6)(x-2)-x(x+3) \equiv x-12$, we must use algebraic steps to change the left-hand side to be the right-hand side.

Left-hand side $= (x+6)(x-2)-x(x+3)$ [Expand brackets]
$= x^2-2x+6x-12-(x^2+3x)$ [Collect like terms]
$= x^2+4x-12-x^2-3x$ [Simplify]
$= x-12$
$=$ Right-hand side ✓

23

Exercise 38

Prove the following identities.

(a) $4(x+2) \equiv 4x+8$

(b) $2(x+4)+5(x+8) \equiv 7x+48$

(c) $(x+8)(x-3) \equiv x^2+5x-24$

(d) $6(x+8)-2(x-4) \equiv 4(x+14)$

(e) $(x+5)(x+2)+(x+8)(x+8) \equiv 2x^2+23x+74$

(f) $(x+6)(x-2)-(x+8)(x+2) \equiv -6x-28$

(g) $(y+4)(y-7)+3y(y-1) \equiv 4y^2-6y-28$

(h) $(2x+1)(x+2)-2x(x+4) \equiv -3x+2$

Exercise 39

Three of the following identities are incorrect. Which ones?

(a) $3(x-4) \equiv 3x-12$

(b) $(x+4)(x-2) \equiv x^2+2x-8$

(c) $(x+3)^2 \equiv x^2+9$

(d) $7(x+3)+2(x-2) \equiv 9x+17$

(e) $4(x+8)-2(x+8) \equiv 2(x+8)$

(f) $(x+2)(x-2) \equiv x^2+4$

(g) $5(y-2)-2(y-3) \equiv 3y-4$

(h) $\frac{x^2+6x+8}{x^2+5x+6} \equiv \frac{x+4}{x+3}$

(i) $(x+3)(x-3)-(x-4)(x+4) \equiv 7$

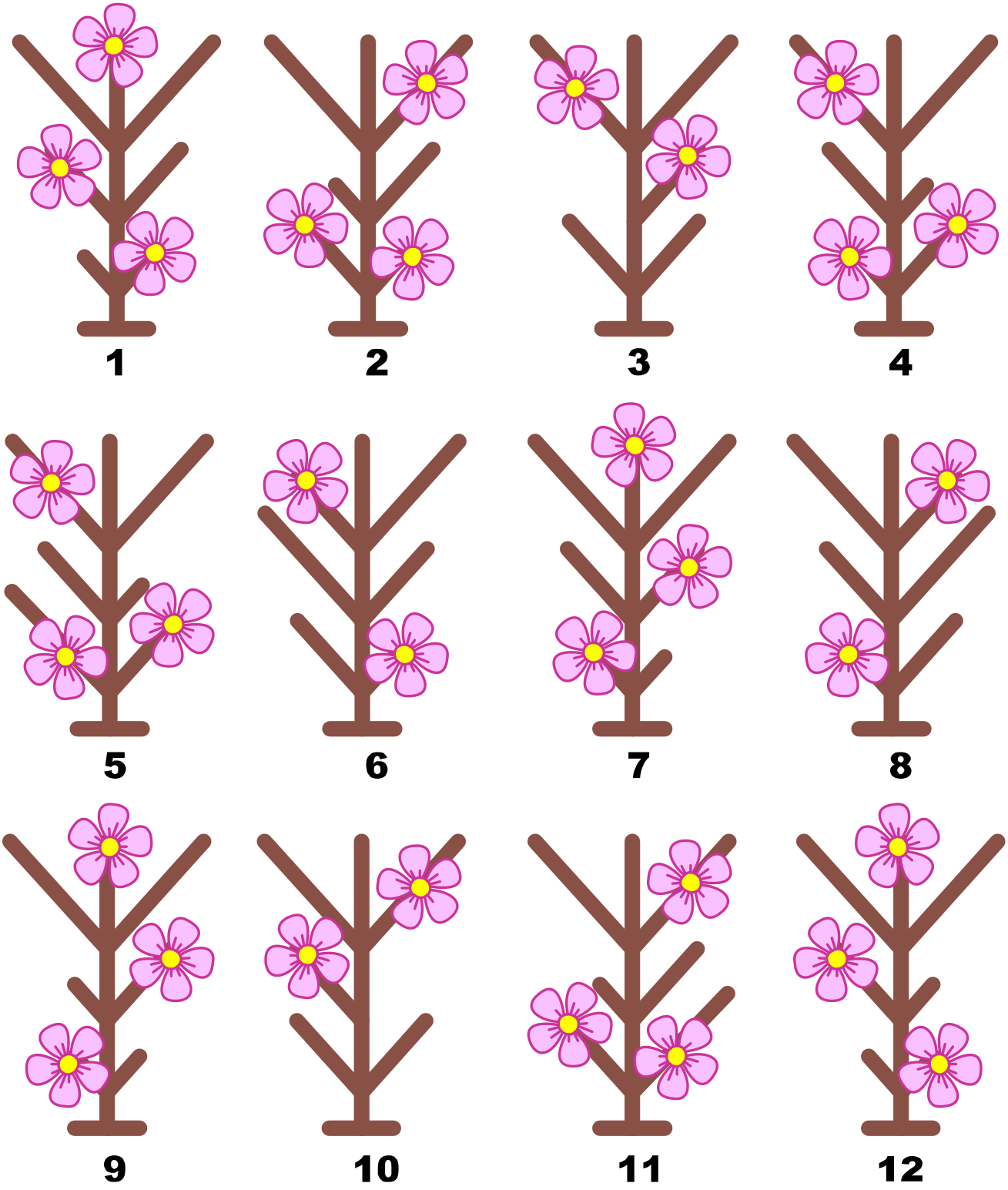
(j) $4(x+2)+(x-4)(x+7) \equiv x^2-20$

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Puzzle

Can you find the mirrored copy of each picture?



Developing Algebra 2

Reflection Sheet

Name: .....

Percentage in the test: .....

	I know this. 	I need to revise this. 	Question in the test	Correct in the test?
I can **factorise** simple expressions such as $8x + 12$ or $15x^2 - 10$.			1	
I can **factorise quadratic expressions** of the form $x^2 + ax + b$.			1	
I can **factorise quadratic expressions** of the form $ax^2 + bx + c$.			2, 3	
I can factorise quadratic expressions of the form $a^2 - b^2$ (a **difference of two squares**).			1	
I can **solve quadratic equations** through factorisation.			2, 3	
I can **solve simultaneous equations**.			4, 5	
I can **re-arrange a formula** in order to make a specific variable (e.g. x) the **subject** of the formula.			6, 7	
I can recognise **expressions, equations, formulae** and **identities**.			8	
I can **prove identities**.			9	