



Mathematics

Year 11

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01 Developing Algebra 3	(a) The Mathematics Department (b) Direct and Inverse Proportion. (c) Proportion Equations. (d) Quadratic Nth Term. (e) Inequalities. (f) Regions of Graphs.	4–6 weeks	5
02 Measuring Shapes 4	(a) Congruent Shapes. (b) Angles in Polygons. (c) Circle Theorems. (d) Transformations.	4 weeks	37
03 Developing Probability	(a) Relative Frequency. (b) Combined Events. (c) Tree Diagrams.	4 weeks	57
04 Developing Algebra 4	(a) Pre-Calculus. (b) Further Changing the Subject. (c) The Quadratic Formula. (d) Algebraic Fractions.	5–6 weeks	79
05 Measuring Shapes 5	(a) 3–D Trigonometry. (b) The Sine Rule, The Cosine Rule. (c) Area of a Triangle. (d) Trigonometric Graphs.	4 weeks	101
06 The End of Year 11	(a) Trial and Improvement. (b) AER, APR. (c) Mopping.	3 weeks	125



The Mathematics Department

11

Developing

Algebra 3

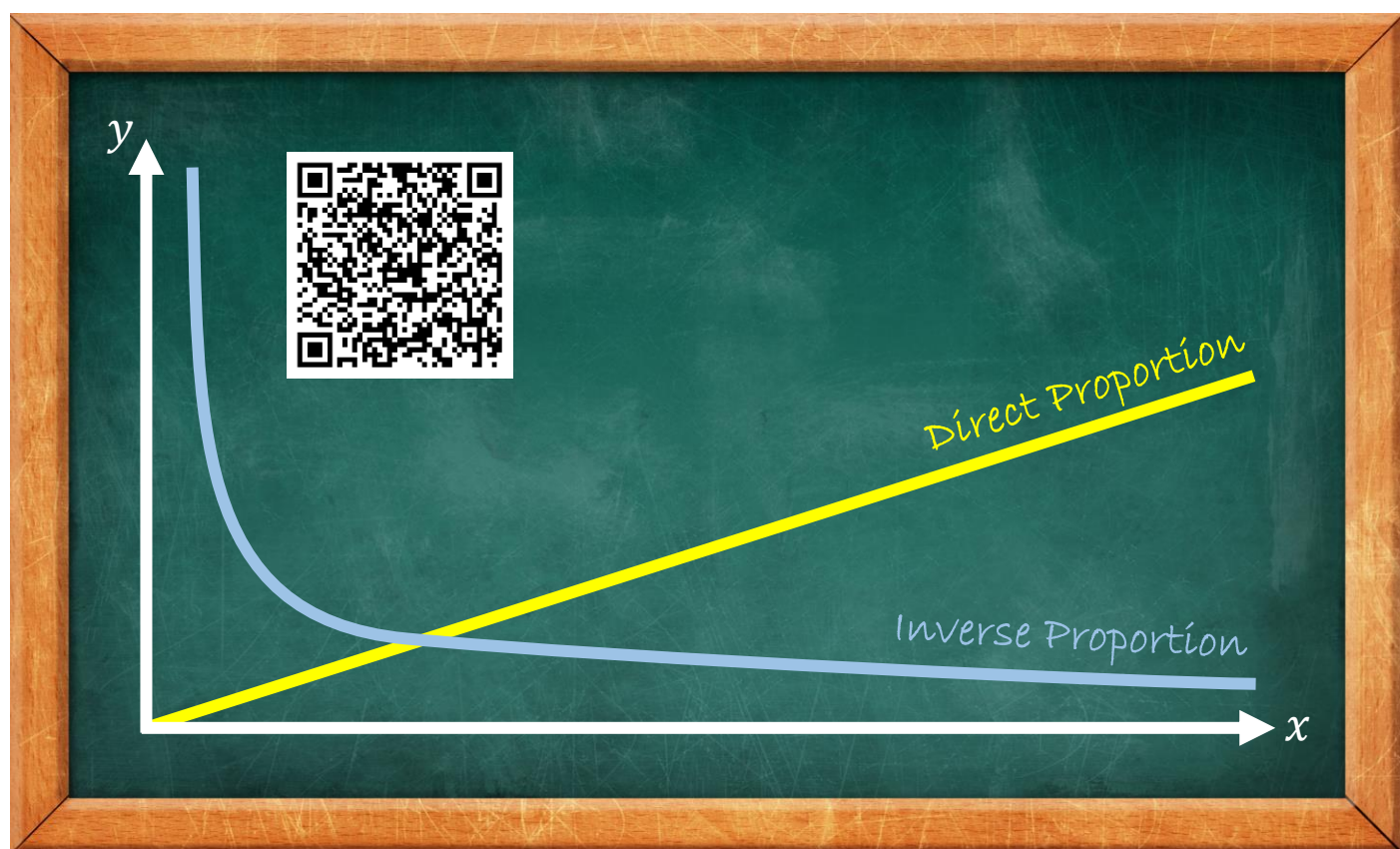
Higher Tier

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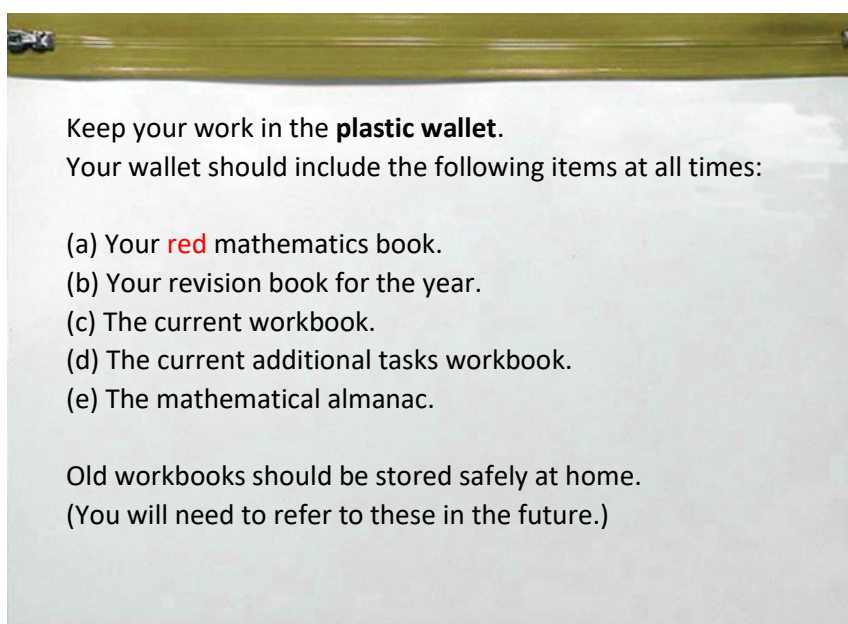
Chapter	Mathematics	Page Number
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Welcome back to year 11! Here is a reminder of the guidelines for looking after your work.

- At the start of every lesson, write “Classwork”, the date and a suitable title for the work. Each of these should be underlined.
- No blank pages should be left in the red mathematics book.
- Draw diagrams using a pencil and, where appropriate, using the correct equipment, e.g. ruler, protractor, compass.
- Show your method fully.
- Remember to include relevant units in your answers, e.g. cm, £, ml.



Equipment

- Black, **red** and **blue** biros.
- HB Pencil.
- Ruler (a 30 cm one is better).
- Eraser.
- Protractor.
- Compass.
- Scientific calculator (Casio fx-83GTCW).
- Highlighter.

The Revision Book

This will form the basis for your GCSE revision.

- Complete at least 4 pages in your revision book for each unit of work.
- You should include material you will need in the future for remembering the work quickly. This can include notes on the work; examples; important facts; and revision posters.

Workbooks

You will receive 1 copy of the workbook and 1 copy of the additional tasks workbook at the start of each new unit of work. (If you lose the workbook, a new one will cost 50p.)

A Welsh copy of the workbook, and additional supporting materials, can be found on the department's website, www.mathemateg.com



Content of the workbooks

When you see a QR code (like the one on the left), scan it using your mobile device in order to reach a Welsh YouTube video hosted on the following channel.

www.youtube.com/adolygumathemateg

The numbers in circles, for example **3**, show which GCSE unit the work appears in.

Circle	1	2	3	12	13	23	A
Units	1	2	3	1 and 2	1 and 3	2 and 3	1, 2 and 3

All the workbooks contain a variety of exercises, labelled as follows.



Exercises on new topics.



Answering a question in context, or solving a problem.



A more difficult question.



Revision of material from previous workbooks.



There are evaluation boxes at the end of each chapter to revise the completed work.

Key words	Corrections	I am happy with...	I need to revise...
Write down the new or important mathematical terms from the chapter.	What do you need to remember when completing this type of work in the future?	Write down the topics you had success with.	Write down the topics you need to look at again.

Curriculum for Wales Proficiencies



Conceptual understanding



Communication using symbols



Strategic competence



Logical reasoning



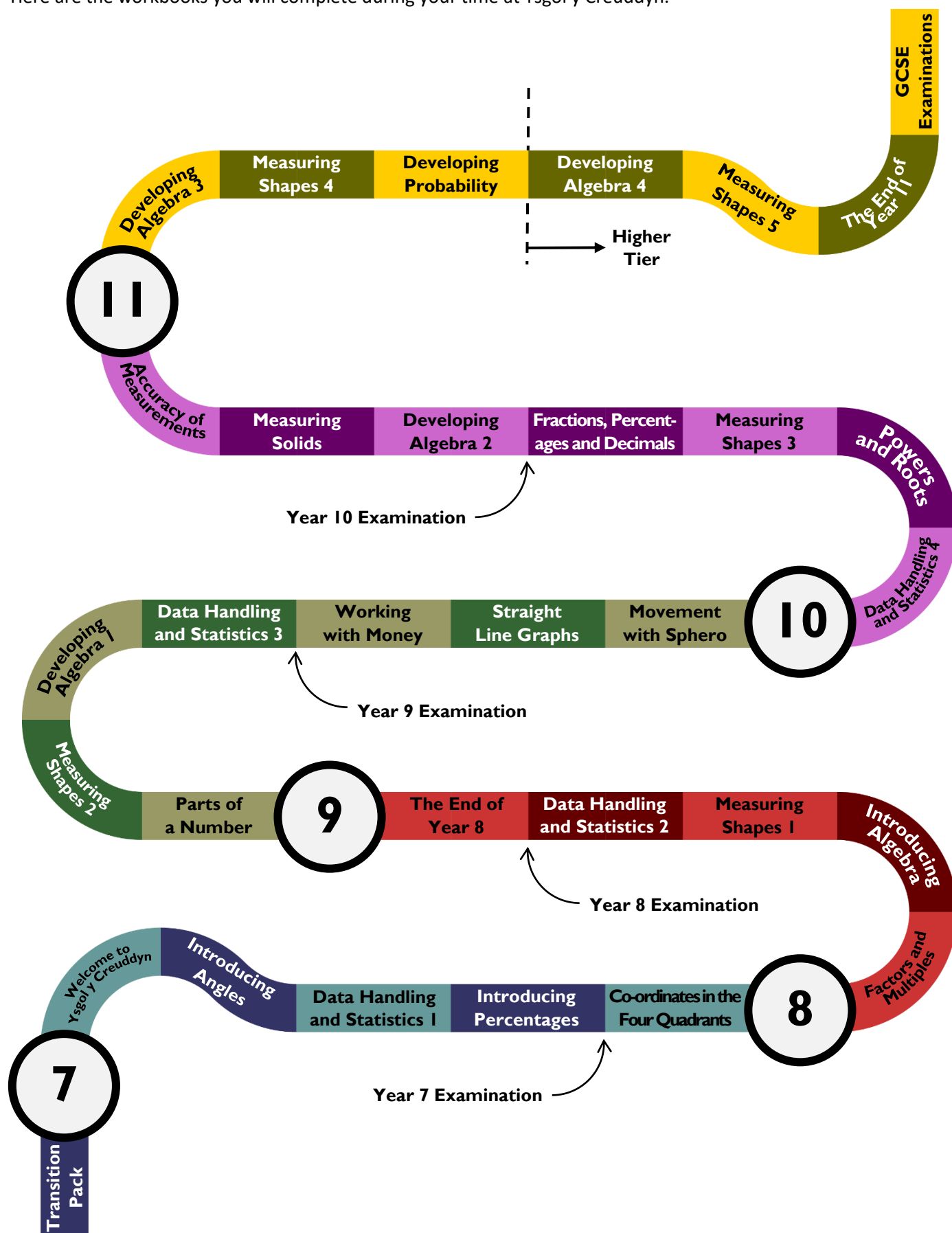
Fluency

Supporting Materials:

- Diagnostic Questions
 - A quiz for each workbook on the website www.diagnosticquestions.com.
- Reflection Sheet
 - An opportunity to assess your understanding of a workbook, and to see the test question order.
- Old WJEC examination questions; worksheets; investigations; puzzles.
 - Available for some topics.

Mathematics Learning Journey for Ysgol y Creuddyn

Here are the workbooks you will complete during your time at Ysgol y Creuddyn.



Proportion



Two measures are in **proportion to each other** if there is a **connection** between the measures. For example, the more pieces of paper there are in a pile of paper, the higher the pile will be. We say that the height of the pile of paper and the number of pages in the pile are in proportion to each other.



The **type** of proportion depends on the type of connection between the measures.

Direct Proportion	Inverse Proportion
As one measure increases, the other measure also increases.	As one measure increases, the other measure decreases.

Example

- (a) The distance a car travels is in direct proportion to the amount of petrol it uses.
- (b) The average speed of a car on a specific journey is in inverse proportion to the time the car takes to make the journey.



Exercise 1

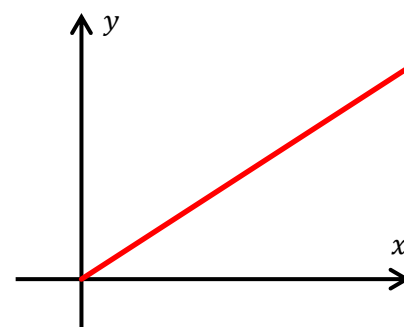
Note which type of proportion (direct proportion or inverse proportion) the following questions describe.

- (a) The height of a pile of paper and the number of pages in the pile.
- (b) The length of a piece of string and the mass of the string.
- (c) The time taken to build a wall and the number of workers used to build the wall.
- (d) The number of tins of soup bought and the total cost of the tins.
- (e) The time taken to empty a water tank and the number of water pumps used to empty the tank.
- (f) The number of pages in a book and the time taken to read the book.
- (g) The distance a car travels in half an hour and the average speed of the car.
- (h) The age of a car and the monetary value of the car (during the first decade after the initial purchase).
- (i) The mass of a bar of gold and the monetary value of the bar.



Direct Proportion

With direct proportion, when one measure increases (e.g. miles travelled, x), another measure must also increase (e.g. amount of petrol used, y). We can write this relationship as $y \propto x$. The symbol $\propto$ means "in proportion to". The graph on the right illustrates direct proportion. The gradient of the line (the multiplier of the proportion, k) can be any positive value.



Example

A digger can dig a 560 m long ditch in 21 days. How much time would it take to dig a 240 m long ditch?

Answer: To begin, we set out the information in a table, ensuring that the quantity we want to find appears in the bottom-right of the table.

Length of ditch	Time
560 m	21 days
240 m	?



By multiplying or dividing, we need to find a sum (or a series of sums) that changes 560 m to be 240 m.

Unitary Method (always works)

Starting with 560, divide by 560 to reach 1, then multiply by 240 to reach 240. Use the **same** sums with the time.

Length of ditch	Time
$\div 560$ 560 m	21 days $\div 560$
1 m	$\frac{21}{560}$ days
$\times 240$ 240 m	9 days $\times 240$

Inspection Method

Starting with 560, divide by 7 to reach 80, then multiply by 3 to reach 240. Use the **same** sums with the time.

Length of ditch	Time
$\div 7$ 560 m	21 days $\div 7$
80 m	3 days
$\times 3$ 240 m	9 days $\times 3$

Exercise 2

- A train travels 165 metres in 3 seconds. How far would it travel in 8 seconds?
- An aeroplane travels 216 miles in 27 minutes. How far would it travel in 12 minutes?
- £50 is worth \$90. How much is £175 worth?
- A 7-metre ladder has 28 steps. How many steps would a similar 5-metre ladder have?
- The mass of a 27 metre long piece of string is 351 grams. What would be the mass of 15 metres of the same type of string?
- A rabbit can burrow a 4 metre long tunnel in 26 hours. How long would the rabbit take to burrow a 7 metre long tunnel?
- A landscape gardener can paint 15 fence panels in 6 hours. How long would it take to paint 40 fence panels?
- The cost of 12 printer cartridges is £90. What is the cost of five of these printer cartridges?
- The height of 500 pieces of paper is 4.9 cm. What would be the height of 800 pieces of the same type of paper?

**Applying****1**

Inverse Proportion

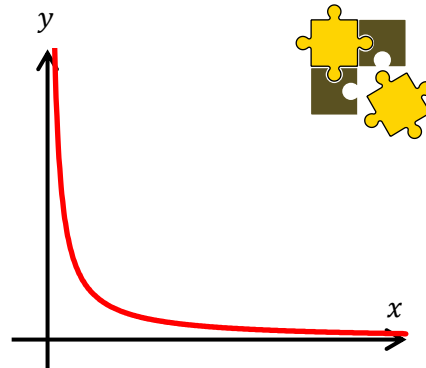
With inverse proportion, when one measure increases (e.g. average speed of a car, x), another measure decreases (e.g. the time taken to complete the journey, y). We can write this relationship as $y \propto \frac{1}{x}$. We read this as “ y is inversely proportional to x ”. The graph on the right illustrates inverse proportion.

Example

If three diggers can dig a hole in 8 hours, how long would four diggers take to dig the same hole?

Answer: To begin, we set out the information in a table, ensuring that the quantity we want to find appears in the bottom-right of the table.

Number of machines	Time
3	8 hours
4	?



By multiplying or dividing, we need to find a sum (or a series of sums) that changes 3 to be 4.

Unitary Method (always works)

Starting with 3, we divide by 3 to reach 1, then multiply by 4 to reach 4. We use the **inverse** sums with the time.

Number of machines	Time
$\div 3$ 3	8 hours $\times 3$
1	24 hours
$\times 4$ 4	6 hours $\div 4$

Inspection Method

Starting with 3, we multiply by 4 to reach 12, then divide by 3 to reach 4. We use the **inverse** sums with the time.

Number of machines	Time
$\times 4$ 3	8 hours $\div 4$
12	2 hours
$\div 3$ 4	6 hours $\times 3$

Exercise 3

1

(a) Travelling at a constant speed of 32 kilometres per hour, a journey takes 18 minutes. How long would the same journey take travelling at a constant speed of 48 kilometres per hour?

(b) It takes a team of 8 people 6 weeks to paint a bridge. How long would the painting take if 12 people were employed?

(c) Usually, a swimming pool is filled using 4 water valves, over a period of 18 hours. Today however, one of the valves cannot be used. How long will it take to fill the pool using only 3 water valves?

(d) A journey can be completed in 44 minutes if we travel at an average speed of 50 miles per hour. How long would the same journey take if we travelled at an average speed of 40 miles per hour?

(e) A supply of hay is sufficient to feed 12 horses for 15 days. How long would the same supply of hay feed 20 horses?



(f) It takes 3 combine harvesters 6 hours to harvest a crop of wheat. How long would it take to harvest the same crop using only 2 combine harvesters?

(g) It takes a team of 18 people 21 weeks to dig a canal. How long would it take to dig the canal using only 14 people?

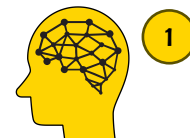
(h) A tank can be emptied using 6 pumps in 18 hours. How long would it take to empty the tank using 8 pumps?

(i) A gang of 9 bricklayers can build a wall in 20 days. How long would a gang of 15 bricklayers take to build the same wall?



Exercise 4

In this exercise, you will need to decide what type of proportion each question describes, before using an appropriate method to find the answer.



(a) The height of a stack of 150 pieces of paper is 9 mm. What would be the height of a stack of 350 pieces of similar paper?

(b) A small swimming pool can be filled in 9 minutes using 8 identical water pumps. How many pumps would be needed to fill the pool in 6 minutes?

(c) A car uses 24 litres of petrol to travel 216 km. How many litres of petrol are required to travel 162 km?

(d) A shop sells 8 apples for £1.40. What would be the price of 12 apples?

(e) A ship takes 12 days to complete a journey, travelling at a speed of 20 knots (1 knot = 1 nautical mile per hour). What speed is required to complete the journey in 10 days?

(f) For a Christmas party, a school arranges that 2 Christmas puddings are available for every 5 children. How many Christmas puddings must be purchased if there are 108 children?

(g) A car travels 180 km in 95 minutes. Find the time taken to travel 72 km at the same speed.

(h) Travelling at a speed of 84 km/hour, a train takes 2 hours to complete a journey. How long would the same journey take at a speed of 96 km/hour?

(i) If 12 pumps, all identical and working together, can empty a water tank in 60 minutes, how long would the tank take to empty if only 10 of the pumps were working?

(j) When a bike travels 145 m, each wheel rotates 58 times. How many times will each wheel rotate when completing a 1,000 m journey?

(k) It costs £1,450 to repair an 87 m long pavement. Find the cost of repairing a 72 m long pavement at the same rate.



(l) A man owning 2,400 shares in a company receives a final dividend of £128. A final dividend of £164, from the same company, was received by a woman. How many shares does she own?

(m) An electric fire uses 8 units of electricity in 3 hours. How long would the electric fire work when using 20 units of electricity?

(n) A ship takes 45 days to complete a journey travelling at a speed of 16 knots. How long would the same journey take travelling at 18 knots?

(o) It costs £10.20 to feed a cat for 14 days. Find, to the nearest penny, the cost of feeding the same cat for 30 days.

(p) A machine can fill 580 bottles in 3 minutes. How many bottles can the same machine fill in 1 hour?

(q) If 14 men can dig a ditch in 11 days, how many days would 22 men take to dig the same ditch?

(r) A bricklayer can lay 245 bricks in 3 hours. How many bricks could the bricklayer lay in 7 hours, working at the same rate?



John Napier

John Napier was born in Edinburgh, Scotland in 1550. He was a mathematician, a physicist and an astronomer. Napier was the first person to use logarithms (A Level work) and was responsible for the popularisation of the decimal point in mathematics. In 1570 he published a document that contained the following rhyme.

*Multiplication is vexation,
Division is as bad;
The Rule of Three doth puzzle me,
And practice drives me mad.*

Challenge!

Use the internet to investigate the “Rule of Three” in a mathematical context.



More than one proportion

Example

A fruit grower knows that it will take 8 hours for 20 workers to pick 420 kg of strawberries. She needs to collect 360 kg of strawberries in 5 hours. What is the minimum number of workers she should employ?

Answer: In this question, there are three things that can vary, namely time, the number of workers, and the weight of the strawberries. We can, using the methods of proportion, change two of these at any one time, whilst keeping the third measure constant.

To start, let us keep the number of workers constant (20 workers), and consider how many strawberries they can pick in 5 hours. Because time and the weight of the strawberries are in direct proportion, we can form the following table.



Time	Weight of the fruit
$\div 8$ 8 hours	420 kg
1 hour	$420 \div 8 = 52.5$ kg
$\times 5$ 5 hours	$52.5 \times 5 = 262.5$ kg



So, 20 workers can collect 262.5 kg of strawberries in 5 hours.

Next, let us keep the time constant (at 5 hours), and consider how many workers are required to collect 360 kg of strawberries. Because the weight of the fruit and the number of workers are in direct proportion, we can form the following table.

Weight of the fruit	Number of workers
$\div 262.5$ 262.5 kg	20
1 kg	$20 \div 262.5 = \frac{8}{105}$
$\times 360$ 360 kg	$\frac{8}{105} \times 360 = 27.428571$

So, we require 27.428571 workers to collect 360 kg of strawberries in 5 hours. But we must have a whole number of workers, so we must **round up** to 28 workers to ensure that 360 kg of strawberries can be collected in 5 hours.

Exercise 5

1

(a) 5 identical industrial water pumps can drain 600,000 litres of water in 8 hours. The local council wants to drain 450,000 litres of water from a flooded area. The work should not take more than 3 hours to complete. What is the minimum number of water pumps required for the task?

(b) Using their old printer, a printing company takes 12 hours to print 54,000 flyers. How long will it take to print another 72,000 flyers using a new printer that works twice as fast as the old one?

(c) A pump is used to fill empty tanks with oil. It takes 27 minutes to fill 6 identical tanks if the flow rate is 5 litres per second. Calculate how much time it would take to fill 8 of these tanks if the flow rate was 9 litres per second.

(d) A new school photocopier can photocopy 3 times as many pages as the old one in the same time. It used to take 20 minutes to copy 500 pages on the old photocopier. How much time would the new photocopier take to copy 600 pages?

(e) It takes 8 tractors 6 hours to plough 38 acres of land. What is the minimum number of tractors required to plough 76 acres of land in less than 9 hours? You may assume that each tractor works at the same rate and that all other conditions are similar.

(f) Machine A is three times as quick as Machine B at assembling identical circuit boards. Machine A is given two and a half times more circuit boards to assemble compared to Machine B. Machine B took 4 hours to complete all of its required assembly. How long did Machine A take to complete all of its required assembly? Give your answer in hours and minutes.



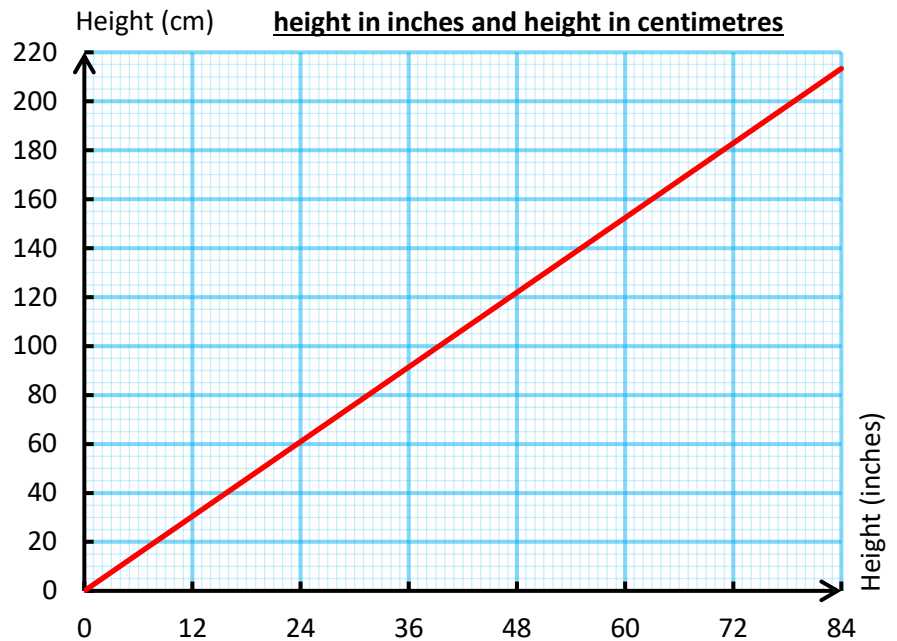
Proportion Graphs

You are required to recognise and interpret graphs that show direct proportion or inverse proportion.

Exercise 6

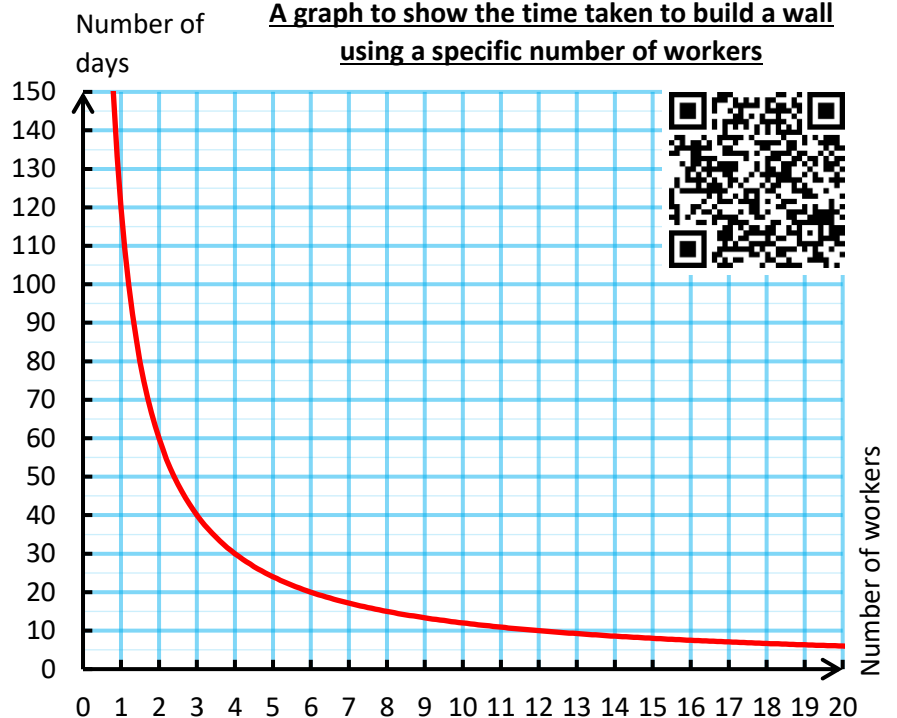
- (a) What type of proportion (direct or inverse) is shown by the graph on the right?
- (b) Siwan's height is 60 inches. What is Siwan's height in centimetres?
- (c) Ben's height is 120 cm. What is Ben's height in inches?
- (d) Huw's height is 170 cm. What is Huw's height in feet and inches?

Conversion graph for changing between height in inches and height in centimetres

**Exercise 7**

- (a) What type of proportion (direct or inverse) is shown by the graph on the right?
- (b) If 8 workers are available to build the wall, how many days will it take?
- (c) Alan wishes to build the wall in less than 10 days. What is the minimum number of workers that Alan must employ?
- (d) How long would it take for one person to build the wall?

A graph to show the time taken to build a wall using a specific number of workers



Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Proportion Equations

Direct Proportion

If two measures x and y are in **direct proportion** to each other, then it is possible to write the relationship between x and y as $y \propto x$. As an **equation**, we can write the relationship as $y = kx$, where k represents the multiplier of the proportion. Given the value of y for a specific value of x , we can solve the equation to find k , and therefore write the equation that connects x to y .



Example

y is in direct proportion to x . Given that $y = 8$ when $x = 2$, find the equation that connects x to y .

Answer: If y is in direct proportion to x , then $y \propto x$, or $y = kx$ for some multiplier k .

Substituting the values of x and y from the question, we see that $8 = k \times 2$, so that $k = \frac{8}{2}$, which gives $k = 4$.

Therefore the equation that connects x to y is $y = 4x$.

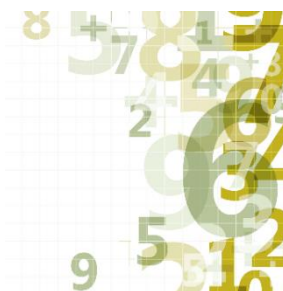
Exercise 8

(a) y is in direct proportion to x . Given that $y = 12$ when $x = 3$, find the equation that connects x to y .

(b) y is in direct proportion to x . Given that $y = 35$ when $x = 5$, find the equation that connects x to y .

(c) y is in direct proportion to x . Given that $y = 2$ when $x = 8$, find the equation that connects x to y .

(d) y is in direct proportion to x . Given that $y = \frac{1}{3}$ when $x = 7$, find the equation that connects x to y .


Skill
3


Example

y is in direct proportion to x^2 . Given that $y = 45$ when $x = 3$, find the equation that connects x to y .

Answer: If y is in direct proportion to x^2 , then $y \propto x^2$, or $y = kx^2$ for some multiplier k .

Substituting the values of x and y from the question, we see that $45 = k \times 3^2$, so that $k = \frac{45}{3^2}$, which gives $k = 5$.

Therefore the equation that connects x to y is $y = 5x^2$.

Exercise 9

(a) y is in direct proportion to x^2 . Given that $y = 80$ when $x = 4$, find the equation that connects x to y .

(b) y is in direct proportion to x^3 . Given that $y = 500$ when $x = 5$, find the equation that connects x to y .

(c) y is in direct proportion to x^2 . Given that $y = 16$ when $x = 8$, find the equation that connects x to y .

(d) y is in direct proportion to $\sqrt{x}$. Given that $y = 30$ when $x = 25$, find the equation that connects x to y .

3


Inverse Proportion

If two measures x and y are in **inverse proportion** to each other, then it is possible to write the relationship between x and y as $y \propto \frac{1}{x}$. As an **equation**, we can write the relationship as $y = \frac{k}{x}$, where k represents the multiplier of the proportion. Given the value of y for a specific value of x , we can solve the equation to find k , and therefore write the equation that connects x to y .

**Example**

y is in inverse proportion to x . Given that $y = 4$ when $x = 5$, find the equation that connects x to y .

Answer: If y is in inverse proportion to x , then $y \propto \frac{1}{x}$, or $y = \frac{k}{x}$ for some multiplier k .

Substituting the values of x and y from the question, we see that $4 = \frac{k}{5}$, so that $k = 4 \times 5$, which gives $k = 20$.

Therefore the equation that connects x to y is $y = \frac{20}{x}$.

**Exercise 10**

(a) y is in inverse proportion to x . Given that $y = 6$ when $x = 8$, find the equation that connects x to y .

(b) y is in inverse proportion to x . Given that $y = 2$ when $x = 14$, find the equation that connects x to y .

(c) y is in inverse proportion to x . Given that $y = \frac{2}{5}$ when $x = 8$, find the equation that connects x to y .



3

**Example**

y is in inverse proportion to x^2 . Given that $y = 3$ when $x = 6$, find the equation that connects x to y .

Answer: If y is in inverse proportion to x^2 , then $y \propto \frac{1}{x^2}$, or $y = \frac{k}{x^2}$ for some multiplier k .

Substituting the values of x and y from the question, we see that $3 = \frac{k}{6^2}$, so that $k = 3 \times 6^2$, which gives $k = 108$.

Therefore the equation that connects x to y is $y = \frac{108}{x^2}$.

Exercise 11

(a) y is in inverse proportion to x^2 . Given that $y = 4$ when $x = 5$, find the equation that connects x to y .

(b) y is in inverse proportion to x^2 . Given that $y = 15$ when $x = 10$, find the equation that connects x to y .

(c) y is in inverse proportion to x^3 . Given that $y = \frac{3}{4}$ when $x = 2$, find the equation that connects x to y .



3

Exercise 12

Given that $y = 5$ when $x = 4$, write an equation to show each of the following relationships.

(a) $y \propto x$ (b) $y \propto x^2$ (c) $y \propto \sqrt{x}$ (d) $y \propto \frac{1}{x}$ (e) $y \propto \frac{1}{x^3}$ (f) $y \propto \frac{1}{\sqrt{x}}$

3

Exercise 13

3

- (a) Given that y is in inverse proportion to x , and knowing that $y = 6$ when $x = 4$,
- find an expression for y in terms of x ,
 - calculate y when $x = 2$,
 - calculate x when $y = 3$.
- (b) Given that y is in proportion to x , and knowing that $y = 18$ when $x = 2$,
- find an expression for y in terms of x ,
 - calculate y when $x = 7$,
 - calculate x when $y = 27$.
- (c) Given that y is in direct proportion to x^2 , and knowing that $y = 36$ when $x = 3$,
- find an expression for y in terms of x ,
 - calculate y when $x = 4$,
 - calculate the two possible values for x when $y = 256$.
- (d) Given that y is in inverse proportion to x^3 , and knowing that $y = 5$ when $x = 4$,
- find an expression for y in terms of x ,
 - calculate y when $x = 8$,
 - calculate x when $y = 40$.

If the type of proportion is not stated, use direct proportion.

Exercise 14

Applying

3

- (a) In a science experiment, Susan takes measurements for t and m . The following table shows her results.

t	2	6	8
m	4	108	256

Susan believes that either m is in proportion to t^2 or m is in proportion to t^3 .

By considering both possibilities, find an expression for m in terms of t .

Show all of your calculations.

- (b) In an experiment, a scientist saw that the force, F , between two particles was in inverse proportion to the square of the distance, d , between the particles. The unit of force was Netwons and the unit of distance was millimetres. When the particles were 5 mm apart, the force between them was 8 Newtons. How far apart were the particles when the force between them was 12.5 Newtons?

- (c) Cerys takes people on hot air balloon trips. She knows that the pressure in the balloon, measured in atmospheres, is in inverse proportion to the square root of the height of the balloon above Earth. When the balloon is at a height of 36 metres above Earth, the pressure is 2 atmospheres.

- Write this information as an equation.
- Cerys pilots her balloon up to a height of 400 m and then down to a height of 256 m. Calculate the change in pressure during the descent.

- (d) Awel wants to paint the walls in her bedroom. The area of the walls is 75 m^2 . The paint costs £6.80 per litre and 2 litres of paint covers 30 m^2 . Write a formula that connects the area of the wall, $A \text{ m}^2$, to the number of litres of paint required, L . Use the formula to calculate the cost of painting the walls in Awel's bedroom.



Finding Proportion Equations

Example

The following table shows two measures x and y .

Find the equation that shows the proportion that is between these measurements.

x	6	8
y	18	32

Answer: As x increases, so does y , so there is direct proportion between the measurements. Considering first whether the proportion is of the form $y \propto x$, or $y = kx$, let us substitute the data from the first column:

$18 = k \times 6$, $k = \frac{18}{6}$, $k = 3$. To check whether we have a proportion of the form $y = 3x$, we must consider the data from the second column. The equation does not work for this data ($y = 3 \times 8 = 24$, not 32) so we must consider a different type of proportion. Considering whether we have a proportion of the form $y \propto x^2$, or $y = kx^2$, we must again substitute the data from the first column to find the multiplier of the proportion: $18 = k \times 6^2$, $k = \frac{18}{6^2}$, $k = \frac{1}{2}$.

To check whether we have a proportion of the form $y = \frac{1}{2}x^2$, we must again consider the data from the second column. This time, the equation does work for the data ($y = \frac{1}{2} \times 8^2 = 32$), so the equation that shows the proportion that is between the measurements x and y is $y = \frac{1}{2}x^2$.

Exercise 15

The following tables show measures x and y .

Find the equation that describes the proportion that is between the measurements.

Skill

3

(a)

x	4	6
y	12	18

(b)

x	4	6
y	3	2

(c)

x	10	6
y	15	9

(d)

x	20	15
y	3	4

(e)

x	2	3
y	20	45

(f)

x	2	3
y	18	8

(g)

x	2	3
y	32	108

(h)

x	4	9
y	14	21

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Quadratic Nth Term

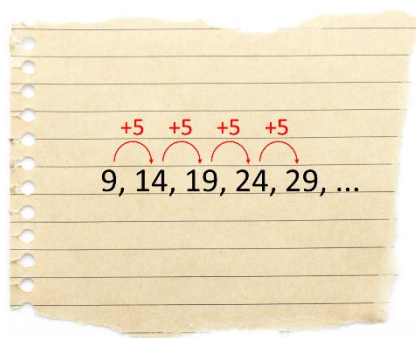
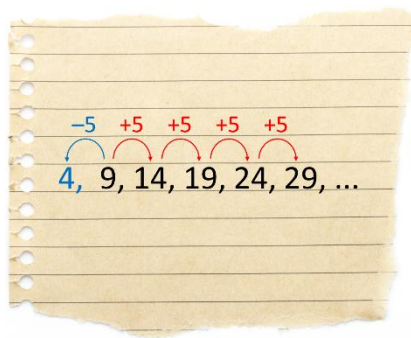
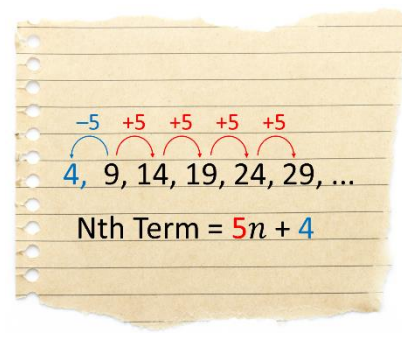
Linear Nth Term

In the Developing Algebra 1 workbook, we learnt how to find the formula for the n th term of a linear sequence such as 9, 14, 19, 24, 29, ...

1. Consider what the rule is for finding the next number. Here, we must **add five** to find the next number.

2. If another number was added at the start of the sequence, what would this number have to be? Here, the number would have to be $9 - 5 = 4$.

3. The **n th term** for this sequence is $5n + 4$. (The 5 and the 4 come from the previous steps.)



Exercise 16

Find the n th term for the following linear sequences.

(a) 4, 6, 8, 10, 12,

(b) 13, 15, 17, 19, 21,

(c) 14, 17, 20, 23, 26,

(d) 20, 18, 16, 14, 12,

(e) 34, 31, 28, 25, 22,

(f) 10, 14, 18, 22, 26,

(g) 5, 5.5, 6, 6.5, 7,

(h) 8, 9, 10, 11, 12,

(i) 3, 6, 9, 12, 15,

(j) -12, -10, -8, -6, -4,

(k) -7, -9, -11, -13, -15,

(l) -3, -1, 1, 3, 5,

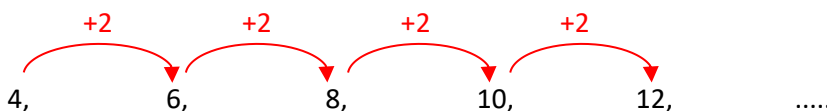
(m) -26, -30, -34, -38, -42,

(n) 2, 7, 12, 17, 22,

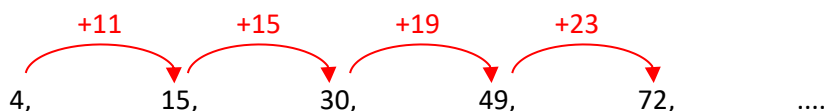
(o) 10, 9.75, 9.5, 9.25, 9,

The First Difference

The sequences in Exercise 16 were linear sequences as the difference between any two consecutive numbers was constant. For example, the common difference in question (a) was 2.



In a quadratic sequence, the difference between any two consecutive numbers is not constant. For example, in the quadratic sequence 4, 15, 30, 49, 72, the difference between two consecutive numbers increases.



We can use this **first difference** to decide whether or not a specific sequence is linear.



Revision

2

Exercise 17

Are the following sequences linear or not?

(a) 9, 11, 13, 15, 17,

(b) 1, 4, 9, 16, 25,

(c) 16, 14, 12, 10, 8,

(d) 3, 6, 11, 18, 27,

(e) 5, 7, 5, -1, -11,

(f) -20, -10, 0, 10, 20,

(g) 9, 8, 7, 6, 5,

(h) 11, 23, 43, 71, 107,

(i) 8, 7.5, 7, 6.5, 6,

Simple Quadratic Sequences

The simplest quadratic sequence is the sequence of square numbers

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \dots$$

The n th term for this sequence is n^2 . We can form another quadratic sequence by adding or subtracting the same number from each of the square numbers. For example:

Sequence	Nth Term
4, 7, 12, 19, 28,	$n^2 + 3$
-1, 2, 7, 14, 23,	$n^2 - 2$

Quadratic sequences of this form have the n th term $n^2 + a$, where a is a specific number.

Exercise 18

Find the n th term for each of the following simple quadratic sequences.

(a) 2, 5, 10, 17, 26,

(b) 11, 14, 19, 26, 35,

(c) 7, 10, 15, 22, 31,

(d) -4, -1, 4, 11, 20,

(e) -9, -6, -1, 6, 15,

(f) 0, 3, 8, 15, 24,

(g) 1.5, 4.5, 9.5, 16.5, 25.5,

(h) 1, 4, 9, 16, 25,

(i) -25, -22, -17, -10, -1,

Exercise 19

Write the first five terms of the following quadratic sequences.

(a) $n^2 + 4$

(b) $n^2 - 6$

(c) $n^2 + 13$

(d) $n^2 - \frac{1}{4}$

(e) $n^2 + 27$

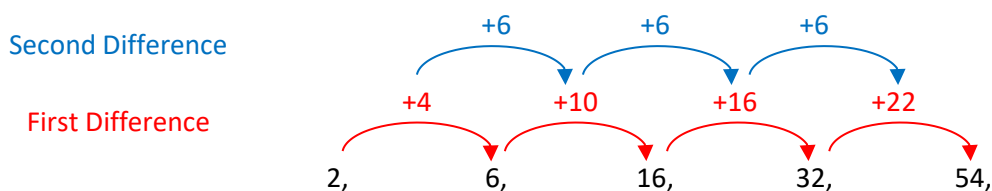
(f) $n^2 - 50$

More Complex Quadratic Sequences

Consider the quadratic sequence 2, 6, 16, 32, 54,

It is not possible to form this sequence by adding or subtracting the same number from the list of square numbers, so a different method is needed to find the n th term.

Step 1: Find the **second difference** for the sequence.



Step 2: Halve the second difference to find the coefficient¹ of n^2 in the formula for the n th term.

$$6 \div 2 = 3, \text{ so the } n\text{th term for the sequence contains the term } 3n^2.$$

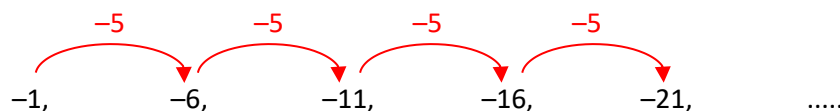
¹ The coefficient of a term is the number that appears at the start of the term.



Step 3: Form a table to find the difference between $3n^2$ and the original sequence.

Original Sequence	2,	6,	16,	32,	54,	
n^2	1,	4,	9,	16,	25,	
$3n^2$	3,	12,	27,	48,	75,	
Original Sequence – $3n^2$	-1,	-6,	-11,	-16,	-21,	

Step 4: Find the n th term of the linear sequence in the final row of the table.



The n th term of the linear sequence is $-5n + 4$, so the n th term of the quadratic sequence is $3n^2 - 5n + 4$.
(We can verify this by substituting into the formula, or by using *Table Mode* on a calculator.)

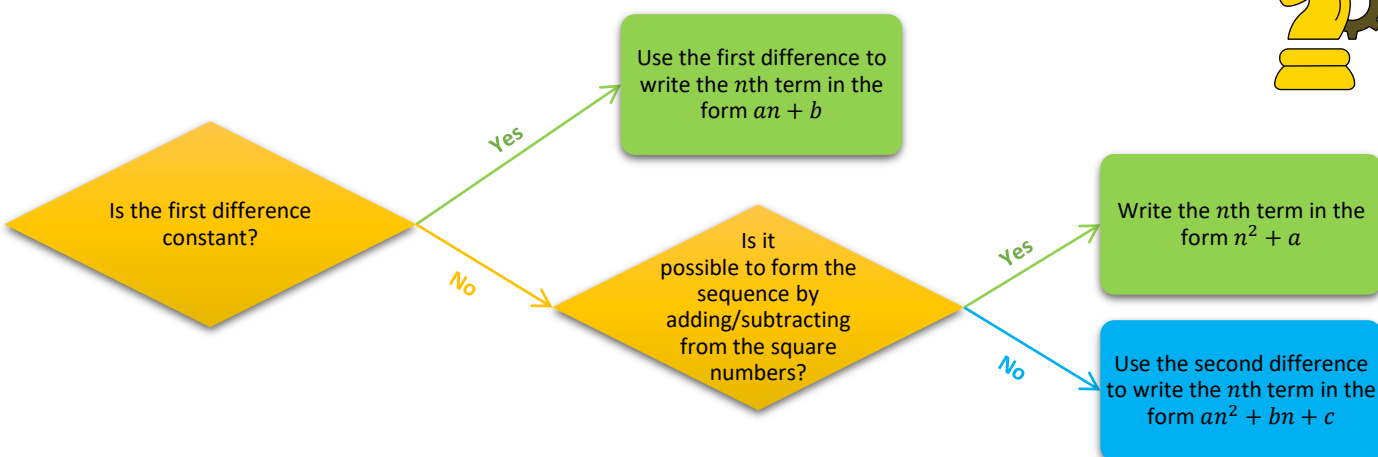
Exercise 20

2

Find the n th term of the following quadratic sequences.

- | | | |
|----------------------------------|-------------------------------------|------------------------------------|
| (a) 6, 11, 18, 27, 38, | (b) 0, 5, 12, 21, 32, | (c) 2, 3, 6, 11, 18, |
| (d) -4, -3, 0, 5, 12, | (e) 11, 22, 37, 56, 79, | (f) 1, 2, 7, 16, 29 |
| (g) 9, 18, 31, 48, 69, | (h) 6, 11, 20, 33, 50, | (i) 4, 18, 38, 64, 96, |
| (j) 6, 15, 32, 57, 90, | (k) -3, 8, 29, 60, 101, | (l) 10, 31, 64, 109, 166, |
| (m) 10, 40, 90, 160, 250, | (n) 8, 14, 24, 38, 56, | (o) 8, 22, 42, 68, 100, |
| (p) 9, 12, 13, 12, 9, | (q) 10, 9, 4, -5, -18, | (r) 3, -10, -29, -54, -85, |
| (s) 4, -8, -30, -62, -104, | (t) -15, -28, -49, -78, -115, | (u) 10.5, 17, 27.5, 42, 60.5, |

Flow Chart: Finding the n th term of a linear or quadratic sequence



Exercise 21

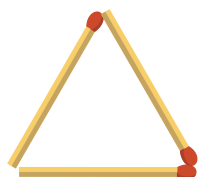
2

Write the first 5 terms of the sequences with the following n th terms.

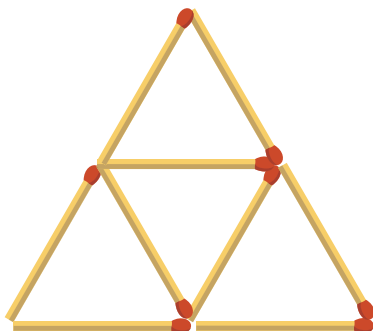
- | | | |
|---------------------|---------------------|-----------------------|
| (a) $4n + 3$ | (b) $n^2 + 9$ | (c) $4n^2$ |
| (d) $2n^2 + 6n + 5$ | (e) $5n^2 - 3n + 7$ | (f) $-3n^2 + 10n - 4$ |
| (g) $n^3 + 2$ | (h) n^4 | (i) $\frac{1}{n}$ |

Exercise 22

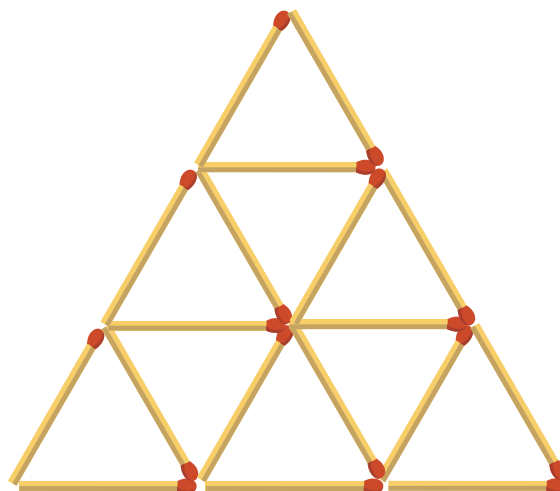
Here is a matchstick pattern.

**Applying****2**

Pattern 1



Pattern 2



Pattern 3

- (a) Draw Pattern 4 in your book.
- (b) Copy and complete the following table.

Pattern number	1	2	3	4	5	6
Number of triangles	1	4				
Number of matches	3	9				

- (c) Consider the sequence for the number of triangles. What is the n th term of this sequence?
- (d) Consider the sequence for the number of matches. What is the n th term of this sequence?
- (e) How many matches are required in order to make Pattern 20?
- (f) What is the number of the pattern that contains 100 triangles?
- (g) Steffan has 1,000 matches. What is the number of the biggest pattern that Steffan can create?
- (h) Lisa creates a pattern that contains 225 triangles. How many matches are in Lisa's pattern?

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Inequalities



If you want to buy a bag of sweets that costs 79 pence, you require **at least** 79 pence.

Perhaps you have more than this amount in your pocket. The sum in your pocket must be greater than or equal to 79 pence.

If x represents the sum of money in your pocket, then we can write the **inequality** $x \geq 79$ to show when we would be able to buy the bag of sweets.

Inequality Symbols

- The meaning of the symbol $\geq$ is 'greater than or equal to'.
- The meaning of the symbol $>$ is 'greater than'.
- The meaning of the symbol $\leq$ is 'less than or equal to'.
- The meaning of the symbol $<$ is 'less than'.



Inequalities on a Number Line

Example

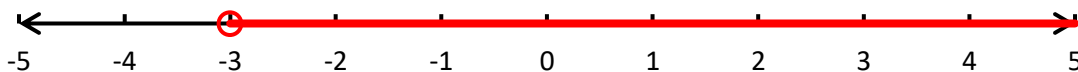
Display the following inequalities on a number line.

(a) $x > -3$

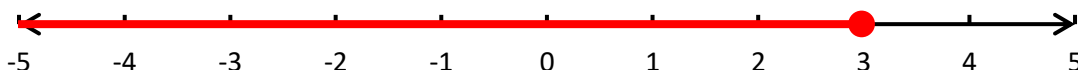
(b) $y \leq 3$

(c) $2 \leq x < 4$

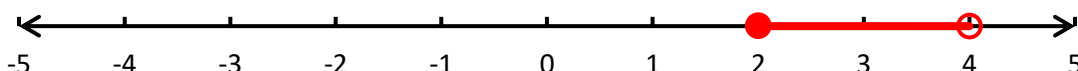
Answer: (a)



(b)



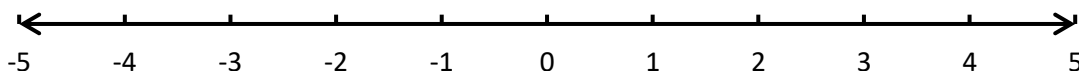
(c)



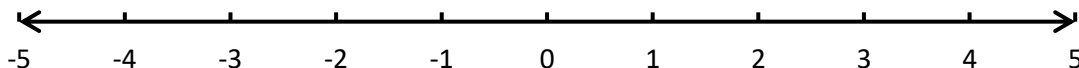
Exercise 23

Use the number lines below to display the following inequalities.

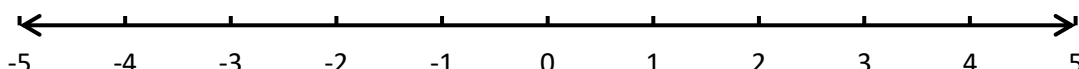
(a) $x < 4$



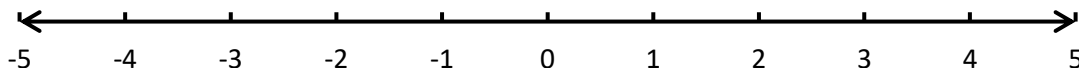
(b) $x \geq -2$



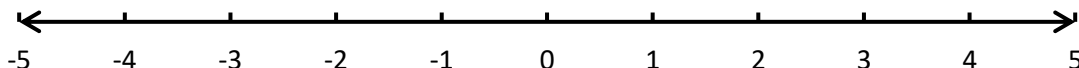
(c) $-4 < x \leq 1$



(d) $-2.5 \leq x < 3$



(e) $x > 0$



Skill

2

Exercise 24

2

Write the inequalities that are shown on the following number lines. (Use x as the variable.)

**Solving Equations**

Because solving inequalities is very similar to solving equations, it is appropriate now to revise some of the equation solving work from previous years.

Exercise 25

Solve the following equations.



Revision

A

One-step equations

(a) $x + 7 = 9$

(b) $3x = 15$

(c) $x - 4 = 14$

(d) $\frac{x}{2} = 10$

(e) $7y = 42$

(f) $\frac{12}{w} = -4$

Two-step equations

(g) $2x + 3 = 19$

(h) $3x - 1 = 17$

(i) $5y + 9 = 64$

Three-step equations

(j) $5x + 2 = 3x + 32$

(k) $4x - 5 = x + 16$

(l) $4x + 4 = 7x - 11$

Equations which require expansion first

(m) $2(x + 7) = 22$

(n) $3(y - 4) = 24$

(o) $20 = 4(x - 2)$

(p) $4(x + 2) = 2(x + 7)$

(q) $4(x - 12) + 2x = 0$

(r) $3(x - 4) = 2(x + 4) + 8$

Equations involving fractions

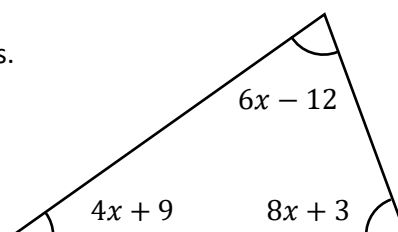
(s) $\frac{x}{2} + 5 = 9$

(t) $\frac{x+5}{2} = 4$

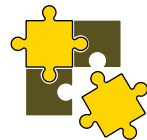
(u) $\frac{18}{x-2} = 3$

Equations in context

- (v) Each angle in the triangle on the right is measured in degrees.
Calculate the size of the smallest angle.



Solving Inequalities



Solving inequalities is exactly the same as solving equations, but there is one important additional rule:



We must **change the symbol** in the middle of an inequality if we
(a) **swap sides**; (b) **multiply or divide by a negative number**.



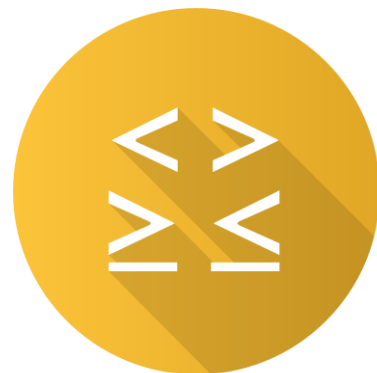
If we need to change the symbol in the middle of an inequality, then the symbol $\geq$ changes to be $\leq$; the symbol $>$ changes to be $<$; the symbol $\leq$ changes to be $\geq$; and the symbol $<$ changes to be $>$.

Why do we need to change the symbol?

(a) Consider the inequality $5 > 2$, which shows something that is true.

If we swap the inequality's sides **without** changing the symbol in the middle, then we will finish with something that is false: $2 > 5$. We must therefore change the symbol in the middle of an inequality if we swap an inequality's sides. (In the example, the correct inequality after swapping sides would be $2 < 5$.)

(b) Consider again the inequality $5 > 2$. If we multiply both sides of the inequality by -2 , we will finish with something that is false: $-10 > -4$. We must therefore change the symbol in the middle of an inequality if we multiply an inequality by a negative number. (In the example, the correct inequality after multiplying by -2 is $-10 < -4$.) The same is true if we divide an inequality by a negative number.



Example

Solve the following inequalities.

(a) $4x + 1 \geq 13$

(b) $7 - 3x < 1$

(c) $2(x + 4) \leq 5(x + 1)$

(d) $\frac{x}{2} > 6 + 2x$

Answer: (a) $4x + 1 \geq 13$

$$4x \geq 12$$

$$x \geq 3$$

[Subtract 1]

[Divide by 4]

(b) $7 - 3x < 1$

$$-3x < -6$$

$$x > 2$$

[Subtract 7]

[Divide by -3]

(c) $2(x + 4) \leq 5(x + 1)$

$$2x + 8 \leq 5x + 5$$

$$2x \leq 5x - 3$$

$$-3x \leq -3$$

$$x \geq 1$$

[Expand brackets]

[Subtract 8]

[Subtract $5x$]

[Divide by -3]

(d) $\frac{x}{2} > 6 + 2x$

$$x > 2(6 + 2x)$$

$$x > 12 + 4x$$

$$-3x > 12$$

$$x < -4$$

[Multiply by 2]

[Expand brackets]

[Subtract $4x$]

[Divide by -3]



Exercise 26

Solve the following inequalities.

(a) $x + 2 > 5$

(b) $5x \geq 20$

(c) $\frac{x}{3} < 6$

(d) $y - 4 \leq 10$

(e) $-2x < 8$

(f) $\frac{x}{-2} \leq 4$

(g) $2x + 5 > 37$

(h) $3y - 2 < 7$

(i) $4x - 4 \geq 4$

(j) $6 - 2x \geq 10$

(k) $10 - 3x < 22$

(l) $1 - x \leq 7$

(m) $4x + 6 > 2x + 18$

(n) $5x - 1 \geq 2x + 32$

(o) $3y + 4 < 2y - 10$

(p) $2x + 3 > 4x + 23$

(q) $3x - 8 \leq 5x + 20$

(r) $5y + 7 \geq y - 29$



2



Exercise 27

2

Solve the following inequalities.

(a) $3(x - 1) < 9$

(b) $2(x + 3) \leq 22$

(c) $5(3 - y) > 10$

(d) $4(x + 2) \geq 2(x + 6)$

(e) $5(x - 1) < 3(x + 5)$

(f) $5(1 - 2x) > 4(2 - 3x)$

(g) $2x + 3(x - 2) \geq 3x - 4$

(h) $z + 3(z - 4) \leq 4$

(i) $7(3 + x) < 7(3 - x)$

(j) $3x - 2(x - 1) > 4(x + 2)$

(k) $2 - 2(3 - y) \geq 6(2 - y)$

(l) $5t - 3(2 - t) < 2(3t + 10)$

Exercise 28

2

Solve the following inequalities.

(a) $\frac{x}{2} + 3 > 8$

(b) $\frac{x}{3} - 2 \leq 4$

(c) $\frac{x}{-4} + 1 > 10$

(d) $\frac{x-12}{3} > 5$

(e) $\frac{x+4}{2} \leq -4$

(f) $\frac{y-4}{3} < 2$

(g) $\frac{x}{2} < 10 - 2x$

(h) $\frac{x}{3} \geq 4 + x$

(i) $\frac{2x}{5} < x - 9$

ExampleFind the **least** whole number that satisfies the inequality $3x + 9 > x + 15$.*Answer:* To begin, let us solve the inequality:

$3x + 9 > x + 15$

$3x > x + 6$

[Subtract 9]

$2x > 6$

[Subtract x]

$x > 3$

[Divide by 2]

We can show this solution on a number line:

The **least** whole number that is greater than 3 is 4, so 4 is the answer to the question.**Exercise 29**Find the **least** whole number that satisfies the following inequalities.

(a) $x > 8$

(b) $x \geq 4$

(c) $x - 4 > 9$

(d) $2x + 6 \geq 24$

(e) $6x + 5 > 4x + 13$

(f) $6x + 4 > 4x + 13$

(g) $3x + 9 \geq x + 3$

(h) $4x - 8 \leq 5x + 5$

(i) $\frac{x+1}{2} > 5$

ExampleList the whole numbers that satisfy the inequality $5 < 2x - 1 \leq 17$.*Answer:* To begin, let us solve the inequality:

$5 < 2x - 1 \leq 17$

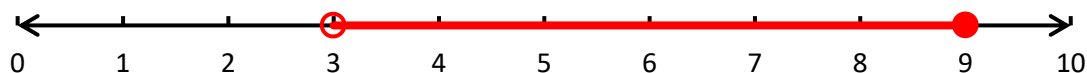
$6 < 2x \leq 18$

[Add 1]

$3 < x \leq 9$

[Divide by 2]

We can show this solution on a number line:



The whole numbers that satisfy the inequality are 4, 5, 6, 7, 8 and 9.



Exercise 30

2

List the whole numbers that satisfy the following inequalities.

(a) $5 \leq x \leq 8$

(b) $5 < x < 8$

(c) $5 < x \leq 8$

(d) $-4 \leq x \leq 2$

(e) $-4 < x < 2$

(f) $-4 \leq x < 2$

(g) $6 < 2x < 10$

(h) $6 \leq 3x < 18$

(i) $4 < 4y \leq 20$

(j) $3 \leq 2x + 1 \leq 13$

(k) $3 < 2x - 1 < 17$

(l) $5 \leq 3x - 1 < 11$

(m) $3 < 2x \leq 9$

(n) $5 \leq 2x + 4 < 15$

(o) $7 < 5x + 1 \leq 21$

Exercise 31

Applying

2

(a) Four times a number n take away 3 is less than twice the number n add 5. Write an inequality satisfied by n and solve it to find the possible values for n .

(b) Vincent and Rowena start to rent television sets at the same time. Vincent pays £14 per month for his rental television. Rowena uses a different method; she pays an upfront payment of £50 then pays rent at £8 a month. Let x represent the number of months both Vincent and Rowena have been renting their televisions.

(i) Write an inequality that is satisfied by x for the number of months the total amount paid by Vincent is **less** than the total amount paid by Rowena.

(ii) Solve the inequality. Explain what your solution tells you about Vincent and Rowena.

(c) Sali has mathematics and science homework. Let m and s represent the time that Sali intends to spend completing each of the homework tasks.

(i) Sali intends to spend less than 3 hours on her mathematics homework. Write this as an inequality.

(ii) What is the meaning of $1 < s < 2$?

(iii) What is the meaning of $m > s$?

(d) A bus can hold up to 46 people. A school intends to transport 5 adults and as many groups of 4 children as is possible to fit on the bus.

(i) Which of the following inequalities is true about the bus?

$4n + 5 > 46$

$4n + 5 \leq 46$

$4n - 5 < 46$

$4n - 5 \geq 46$

(ii) Solve the correct inequality from part (i) to find the maximum number of groups of four children that can be transported on the bus.

**Evaluation**

Key words	Corrections	I am happy with...	I need to revise...

Regions of Graphs

In this chapter, we will discuss how to shade a region of graph paper defined by a set of inequalities. In order to do this, we must revise how to plot graphs of the form $x = a$; $y = b$ and $y = mx + c$, and learn a new technique for plotting graphs of the form $ax + by + c = 0$.

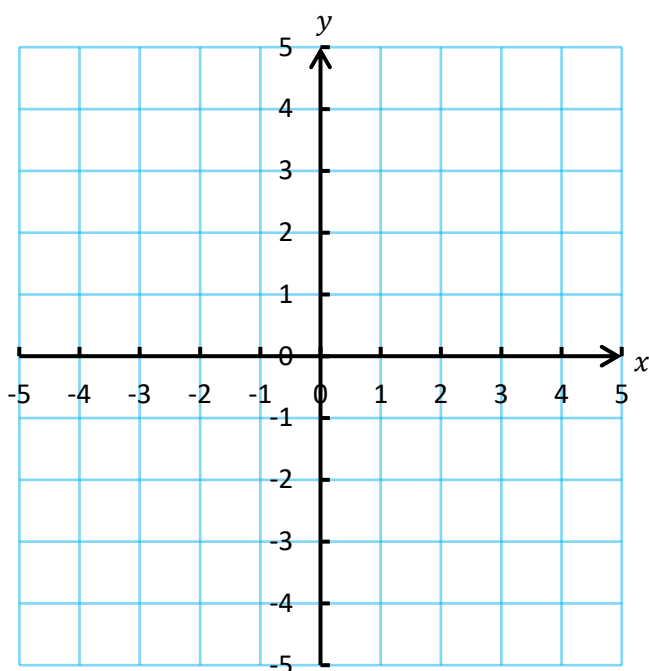
Revising plotting graphs of the form $x = a$ and $y = b$

- The graph of $x = a$ is a vertical line passing through the point $(a, 0)$.
- The graph of $y = b$ is a horizontal line passing through the point $(0, b)$.

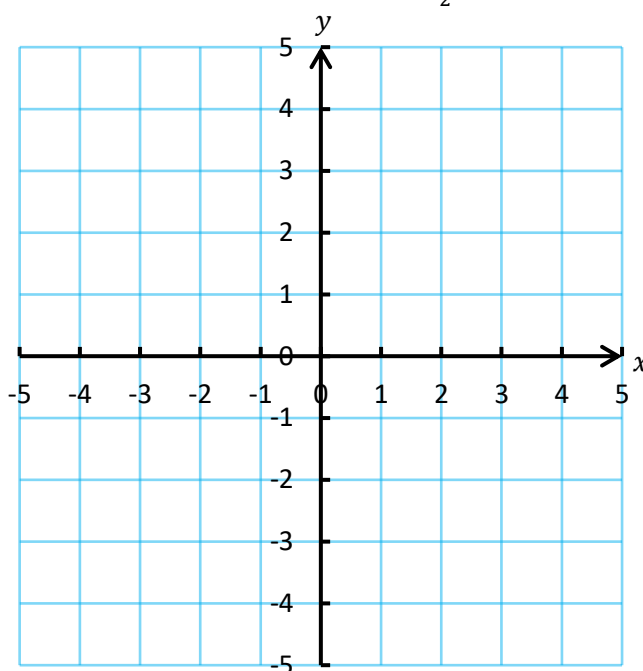
Exercise 32

Use the graph paper below to plot the following lines.

(a) $x = 3, y = 4, x = -2, y = -3$



(b) $y = 2, x = -3, y = 1.5, x = -\frac{5}{2}$



Revision

2

Revising plotting graphs of the form $y = mx + c$

Given a straight line of the form $y = mx + c$, for example $y = 3x - 2$, here are two ways of plotting the line on graph paper.

Method 1: Using a table

(a) Substitute different values of x into the equation in order to create a table of values.

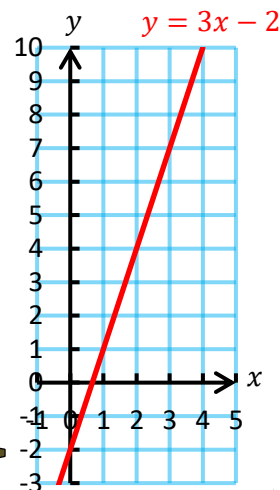
x	0	1	2	3
y	-2	1	4	7
	↑	↑	↑	↑
	$3 \times 0 - 2$	$3 \times 1 - 2$	$3 \times 2 - 2$	$3 \times 3 - 2$
	$= 0 - 2$	$= 3 - 2$	$= 6 - 2$	$= 9 - 2$
	$= -2$	$= 1$	$= 4$	$= 7$

(b) Plot the values from the table on graph paper before connecting the points with a straight line.

Method 2: Using the gradient and y-intercept

(a) For the line $y = 3x - 2$, the y-intercept is -2 , so the line passes through the point $(0, -2)$. Plot this point on the graph paper.

(b) The gradient is 3, so for each **one** unit we move to the right (starting from the point $(0, -2)$), we must move **three** units up. Plot some of these points before connecting the points with a straight line.

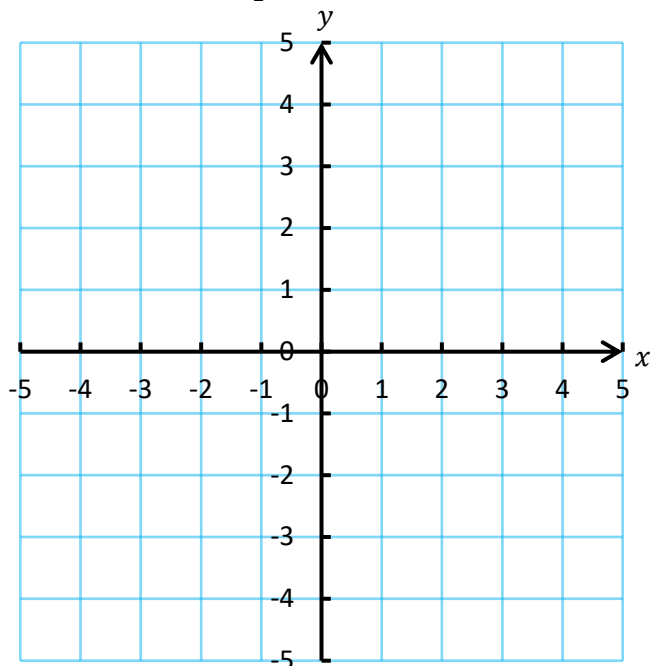


Exercise 33

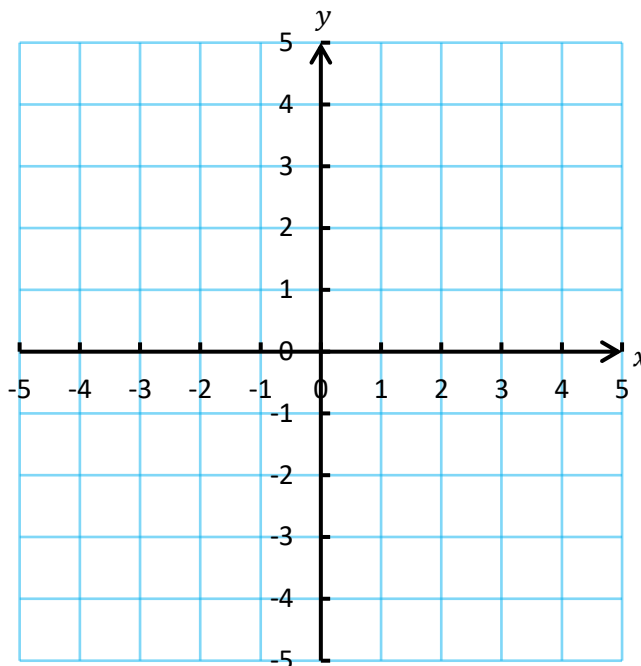
2

Use the graph paper below to plot the following lines.

(a) $y = 2x - 3$, $y = -\frac{1}{2}x + 1$



(b) $y = x$, $y = -x$



Plotting graphs of the form $ax + by + c = 0$

Example

Plot a straight line for the equation $2x + 3y - 12 = 0$.

Method 1: The Hiding Method

To find the value of y when $x = 0$, **hide** the term in x using your finger, and solve the equation that remains.

$$\begin{array}{lcl} 2x + 3y - 12 = 0 & \rightarrow & 3y - 12 = 0 \\ & & 3y = 12 \quad \text{[Add 12]} \\ & & y = 4 \quad \text{[Divide by 3]} \end{array}$$

So the line goes through the point $(0, 4)$.

To find the value of x when $y = 0$, **hide** the term in y using your finger, and solve the equation that remains.

$$\begin{array}{lcl} 2x + 3y - 12 = 0 & \rightarrow & 2x - 12 = 0 \\ & & 2x = 12 \quad \text{[Add 12]} \\ & & x = 6 \quad \text{[Divide by 2]} \end{array}$$

So the line goes through the point $(6, 0)$.

To plot the line for the equation $2x + 3y - 12 = 0$, plot the two points $(0, 4)$ and $(6, 0)$ on graph paper before connecting them with a straight line.

We can "hide" the terms because they disappear when substituting in 0.

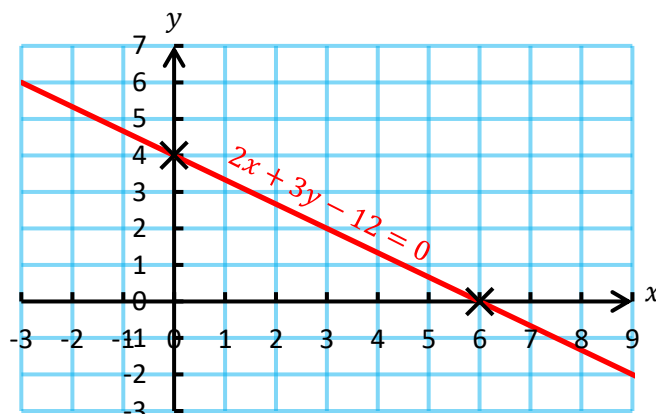
See the Developing Algebra 2 workbook to revise this topic.

Method 2: The Rearranging Method

Re-arrange the equation in order to make y the subject of the equation.

$$\begin{array}{lcl} 2x + 3y - 12 = 0 & & \\ 3y - 12 = -2x & & \text{[Subtract 2x]} \\ 3y = -2x + 12 & & \text{[Add 12]} \\ y = -\frac{2}{3}x + 4 & & \text{[Divide by 3]} \end{array}$$

We can now plot the equation, using the techniques for plotting an equation of the form $y = mx + c$.

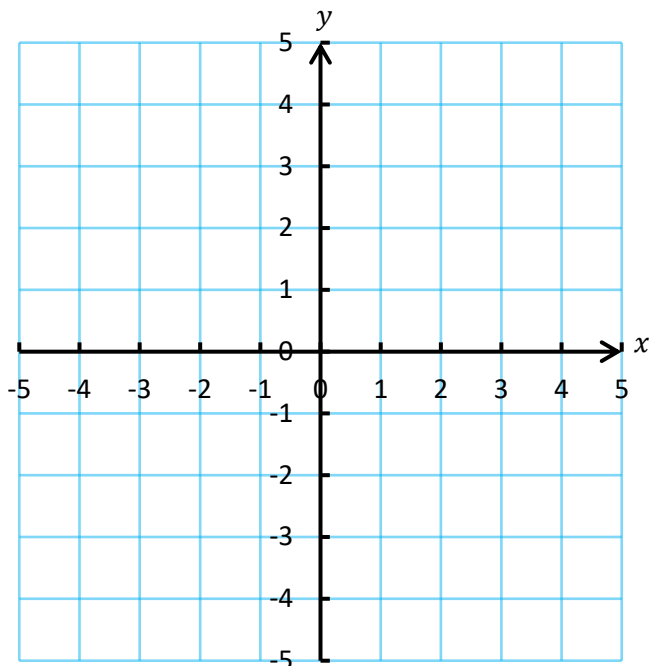
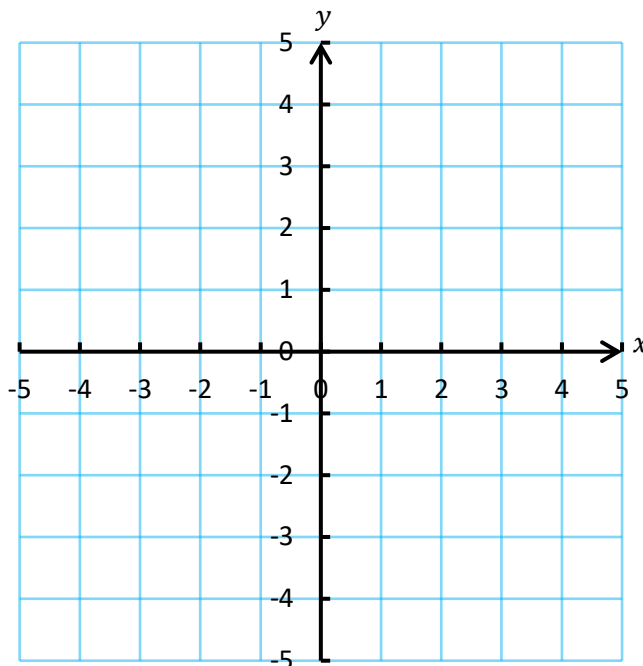


Exercise 34

Use the graph paper below to plot the following lines.

(a) $2x + 3y - 6 = 0$

(b) $4x - 2y - 8 = 0$

**Skill****2****Shading Regions**

We can now consider how to use a set of inequalities to shade a region on graph paper.

Example

Shade the region defined by the following inequalities.

$$y < 2, x \geq -1, y \geq x - 1$$

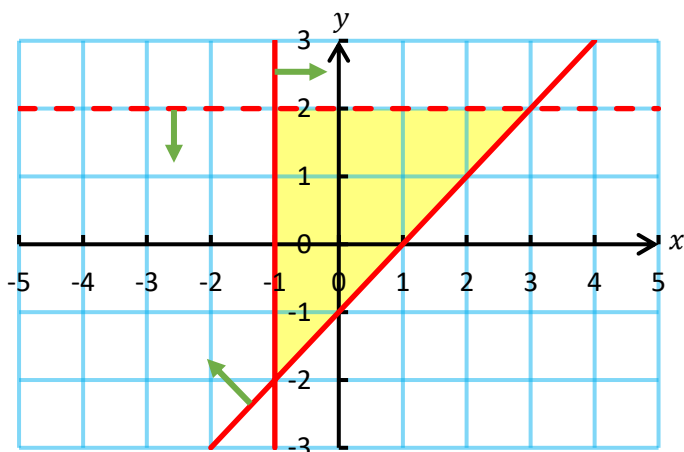
Answer: **Step 1:** Plot the graphs of $y = 2$, $x = -1$, $y = x - 1$.

**Rule:**

Use a solid line (—) for inequalities containing $\geq$ or $\leq$; and a dotted line (---) for inequalities containing $>$ or $<$.

Step 2: Show, using an arrow, a region corresponding to each of the three lines.

Step 3: Shade the region that is satisfied by all of the arrows / inequalities.



For lines that aren't vertical or horizontal, **substitute a point that doesn't lie on the line to decide which way the arrow should point.** For example, considering $y \geq x - 1$, substitute the point $(0, 0)$:

$$\begin{aligned} 0 &\geq 0 - 1 \\ 0 &\geq -1 \end{aligned}$$

This inequality is **true**, so the arrow should point **towards** the point $(0, 0)$.

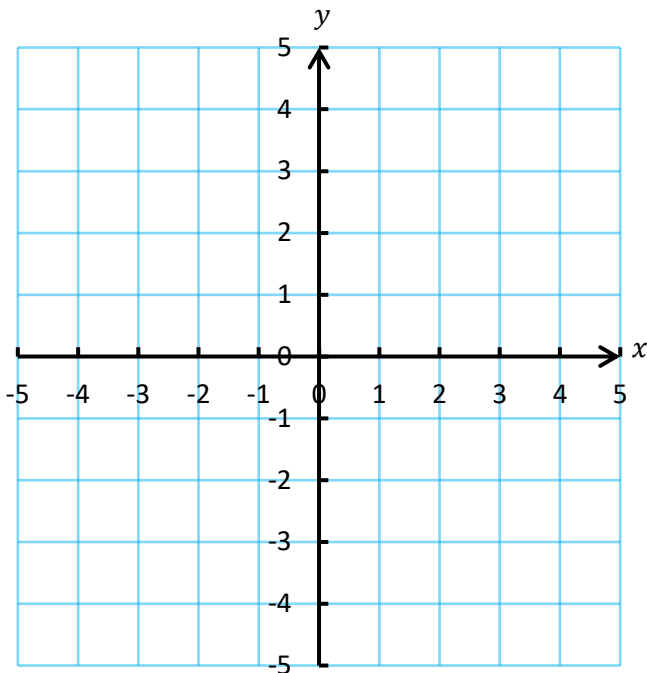


Exercise 35

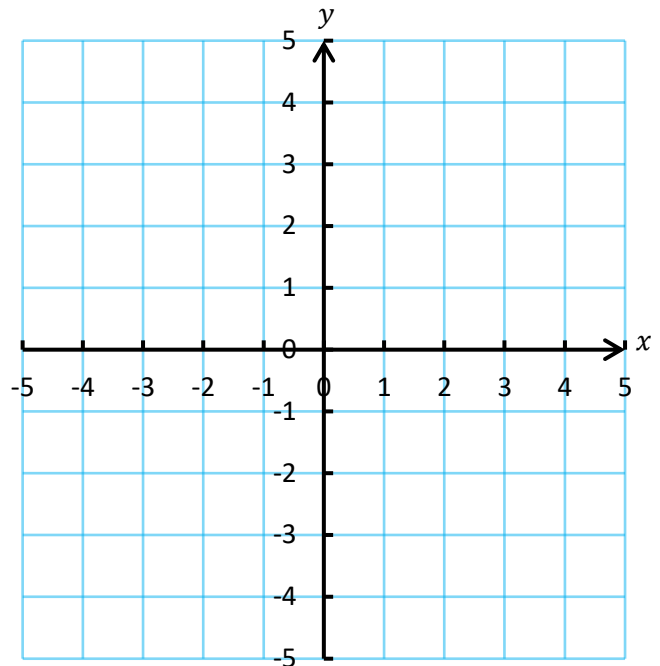
2

Shade the region that is defined by the following sets of inequalities.

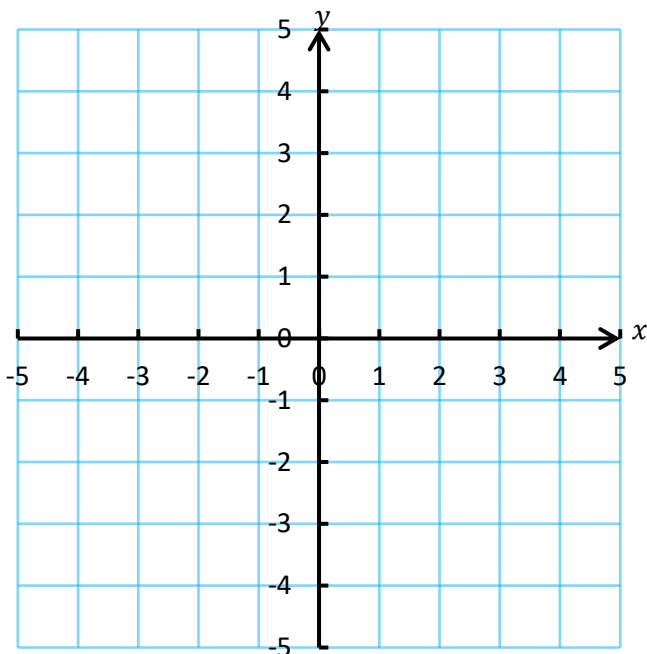
(a) $y < 4, x < 3, y \geq -2, x \geq -1$



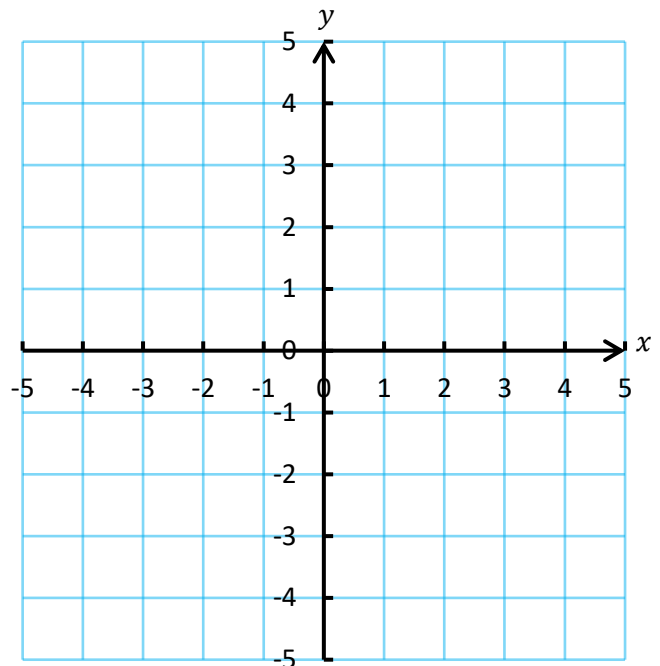
(b) $y \leq 3, x \geq 0, y \geq 2x - 3$



(c) $y > 1, x \geq 1, x + 2y - 4 \leq 0$



(d) $y > -2, y < x + 1, y < -2x + 3$

**Exercise 36**

2

Draw suitable axes in order to shade the regions that are defined by the following sets of inequalities.

(a) $x + y \leq 4, y \leq 2x + 4, y \geq 1$

(b) $y \geq 0, x < -1, y \leq x + 3$

(c) $x \geq -1, y < 4, y \geq 3x - 1$

(d) $y > -4, x < -1, y \leq 2x + 1$

(e) $y < 2, x \leq 1, y > -x + 2$

(f) $y > -3, x \geq -2, x \leq 1.5, y \leq -\frac{1}{2}x + 1$

Exercise 37

2

Draw suitable axes in order to shade the regions that are defined by the following sets of inequalities.

(a) $x \leq 2, y > -4, y \geq 2x - 2.5$

(b) $y < 2, x \geq -3, y \geq x - 1, y \geq -x - 4$

(c) $x + y < 1, x \geq -3$

(d) $y \geq x - 2, y < x + 4$

(e) $y < 2x, y \geq x + 1$

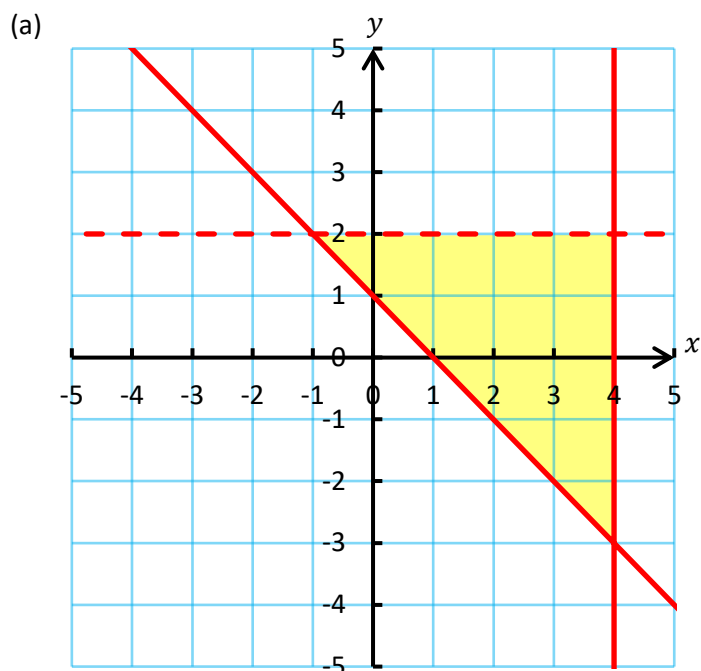
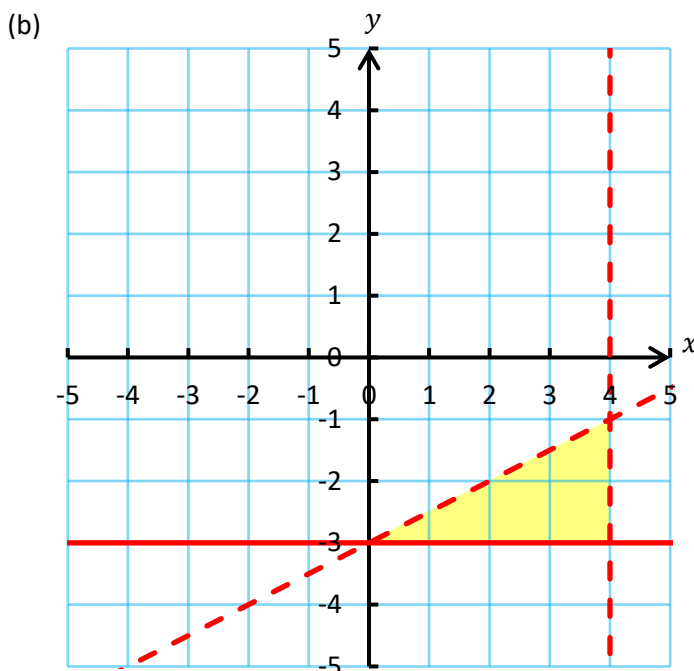
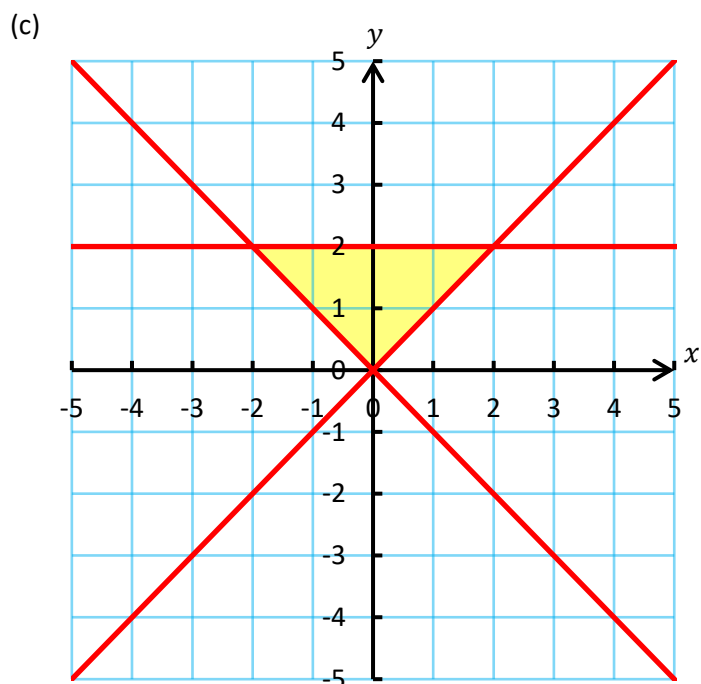
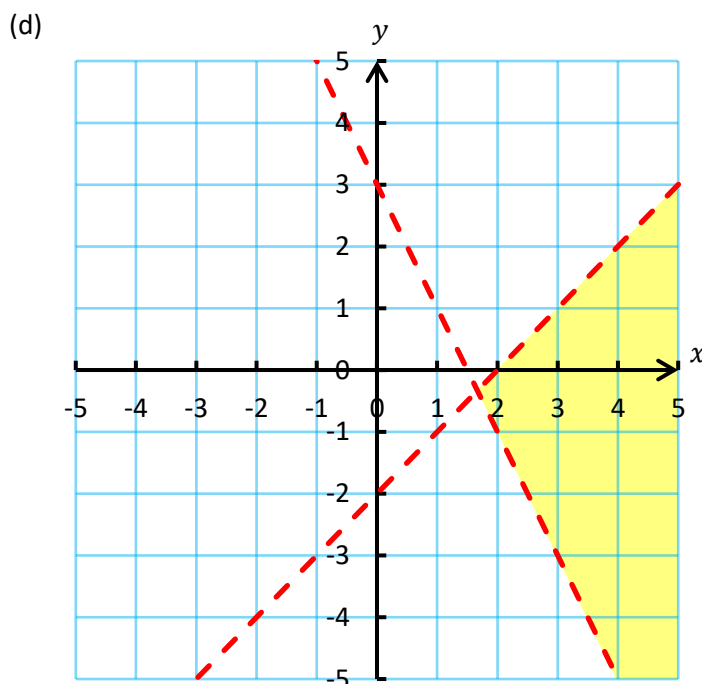
(f) $x + y > 2, y > x - 3$

(g) $x - 2y < 4, y \leq x$

(h) $2x - 3y \leq 6, 2x + 2y < 0$

**Exercise 38**

What inequalities define the following shaded regions?



Exercise 39**Applying****2**

A shop has asked a manufacturer to produce skirts and jackets.

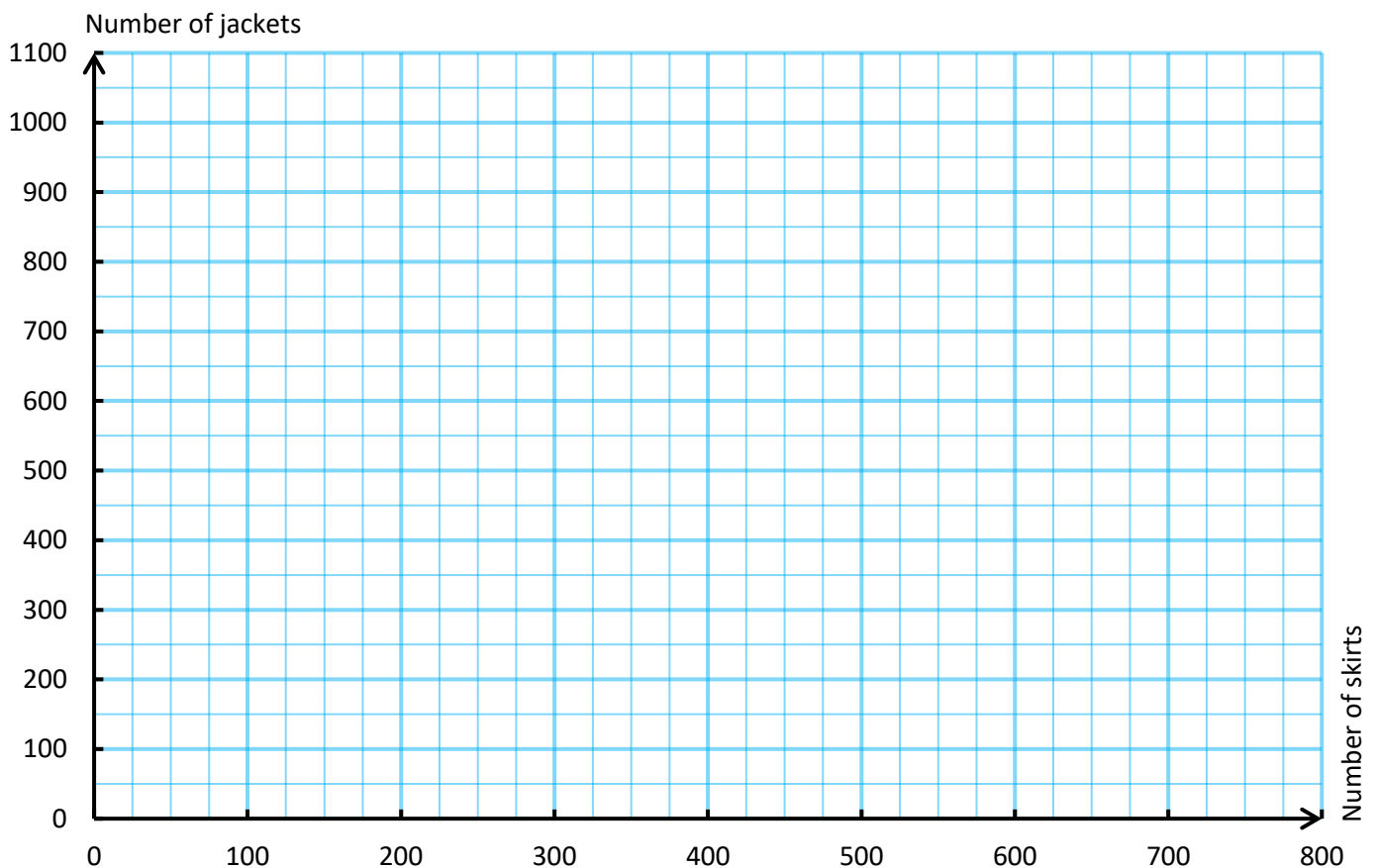
As raw materials, the manufacturer has 750 m^2 of cotton and $1,000 \text{ m}^2$ of polyester.

Each skirt requires 1 m^2 of cotton and 2 m^2 of polyester.

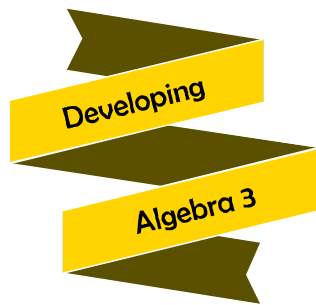
Each jacket requires 1.5 m^2 of cotton and 1 m^2 of polyester.

The price of a skirt is £50 and the price of a jacket is £40.

Assuming that everything is sold, how many skirts and jackets should the shop buy in order to maximise their profit?

**Evaluation**

Key words	Corrections	I am happy with...	I need to revise...



Name:

Percentage in the test:

	I know this. 	I need to revise this. 	Question in the test:	Correct in the test?
I know how to recognise whether the connection between two measures is a direct proportion or an inverse proportion .			1, 2, 3	
If two measures are in direct proportion, I can calculate one of the missing measures .			2	
If two measures are in inverse proportion, I can calculate one of the missing measures .			1, 3	
I can work with more than one proportion .			4	
I can recognise and use the graphs of direct proportion and inverse proportion.				
I can write and use proportion equations .			5	
Given a set of data for two measurements, I can find the equation that describes the connection between the measurements.				
I can write the nth term for simple quadratic sequences , e.g. $n^2 + 9$.			6	
I can write the nth term for more complex quadratic sequences , e.g. $4n^2 + 2n - 1$.			6	
I can illustrate an inequality on a number line .			7	
I can solve inequalities .			8	
I can find the least whole number (or the greatest) that satisfies an inequality.				
I can list all the whole numbers that satisfy an inequality.			9	
I can plot lines of the form $ax + by + c = 0$.				
Given a set of inequalities, I can shade the region that is defined by those inequalities.			10	
Given a region on graph paper, I can find the set of inequalities that define the region.				



The Mathematics Department

11

Measuring

Shapes 4

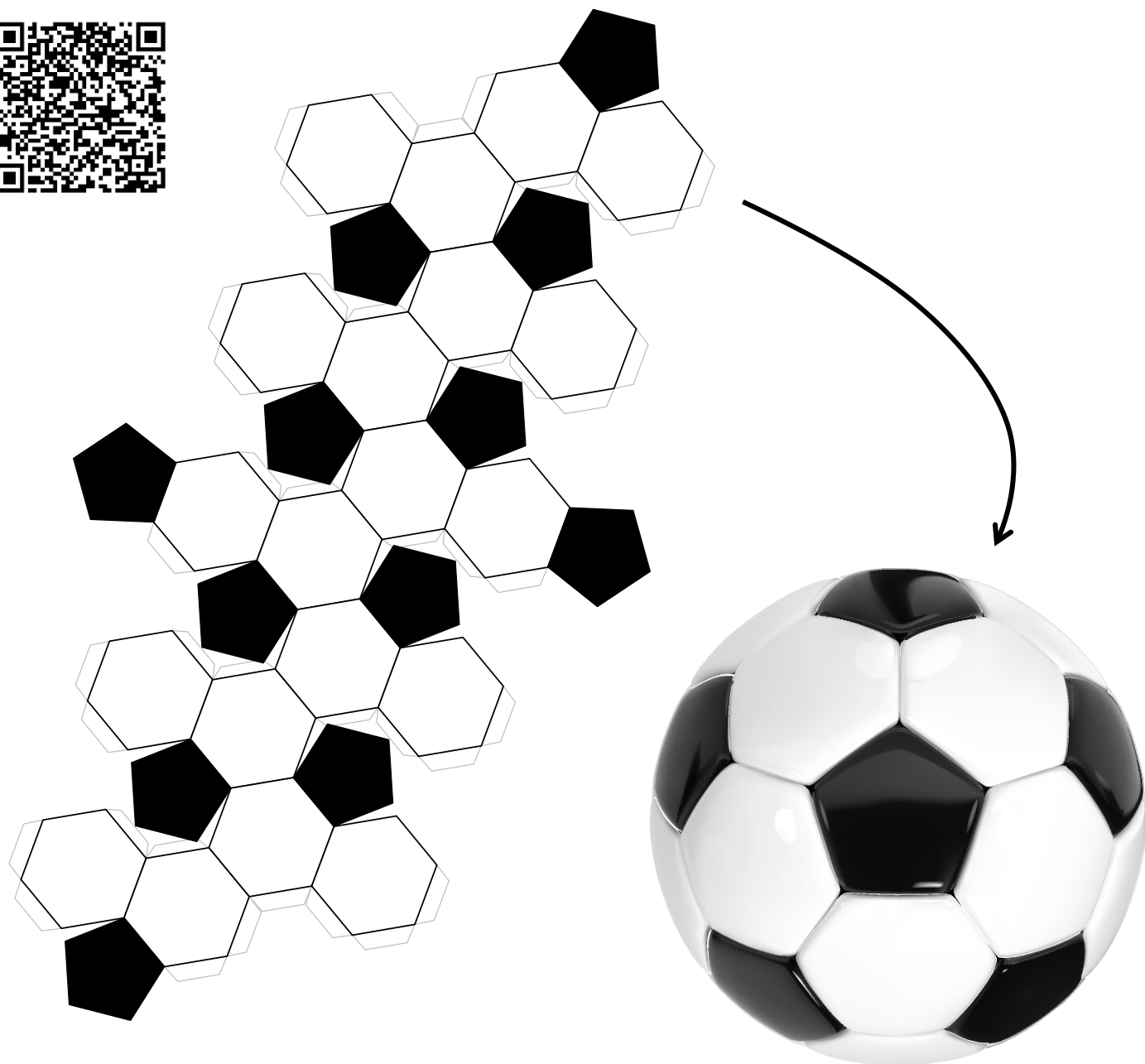
Higher Tier

Version
3.0
08/2025

Name:

Contents

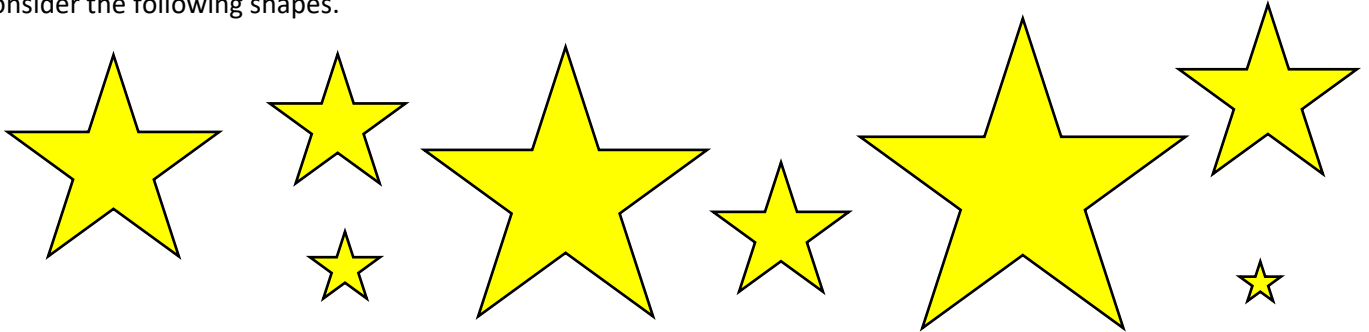
Chapter	Mathematics	Page Number
Congruent Shapes	Recognising congruent shapes. Congruent triangles proofs.	3
Angles in Polygons	The interior angles of a polygon. The exterior angles of a polygon.	8
Circle Theorems	Angles in a circle.	12
Transformations	Translation. Rotation. Reflection. Enlargement. Combinations.	18



Congruent Shapes



Consider the following shapes.



The shapes are all **similar**, so that the same shape is seen each time, but only two of the shapes are **congruent**, that is to say they have the same *size*.

Congruent shapes are the same **shape**, and the same **size**.

Exercise 1

Tick the two shapes that are congruent above.

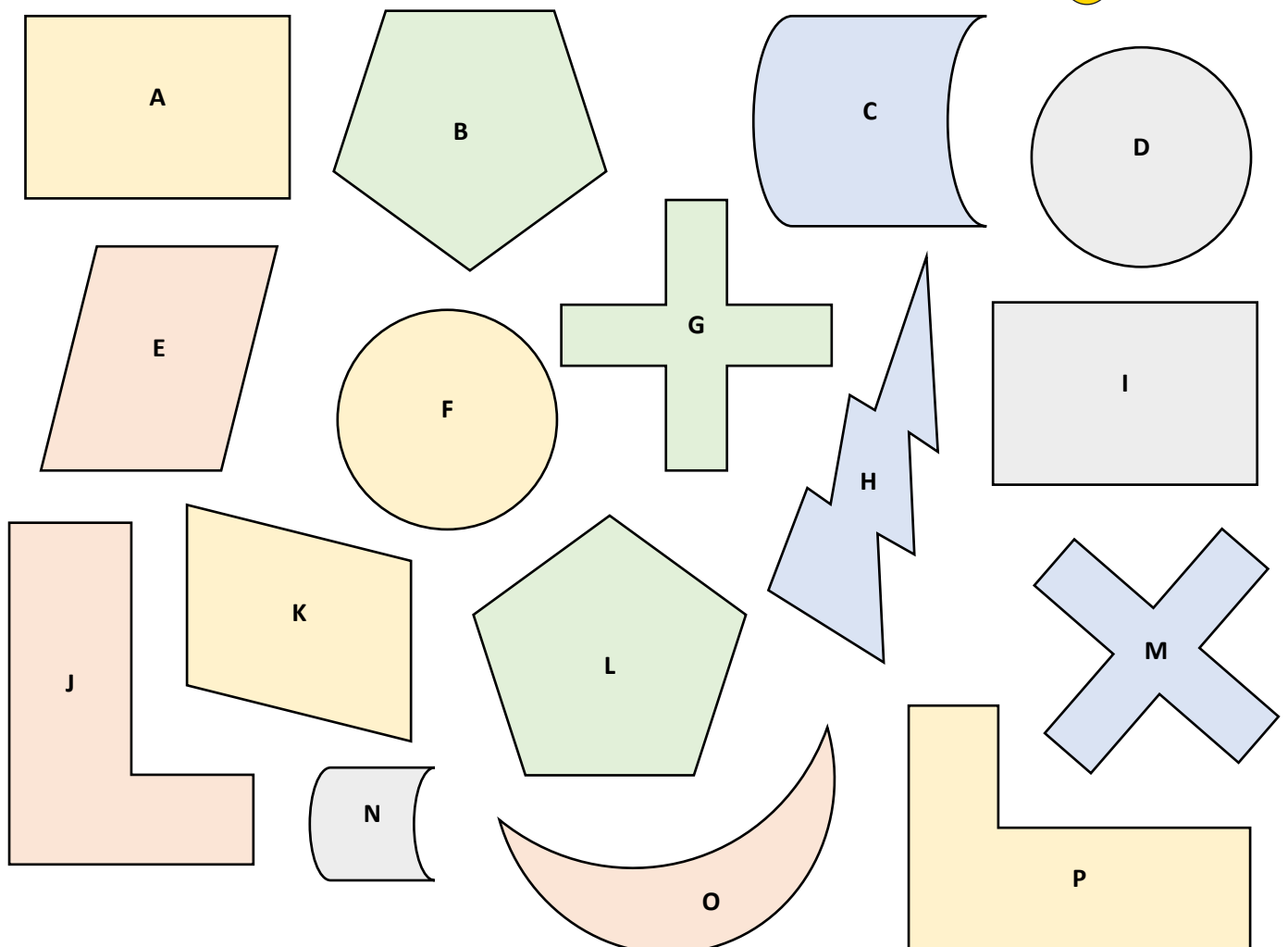
Exercise 2

Look at the following shapes. Which pairs of shapes are congruent?



Skill

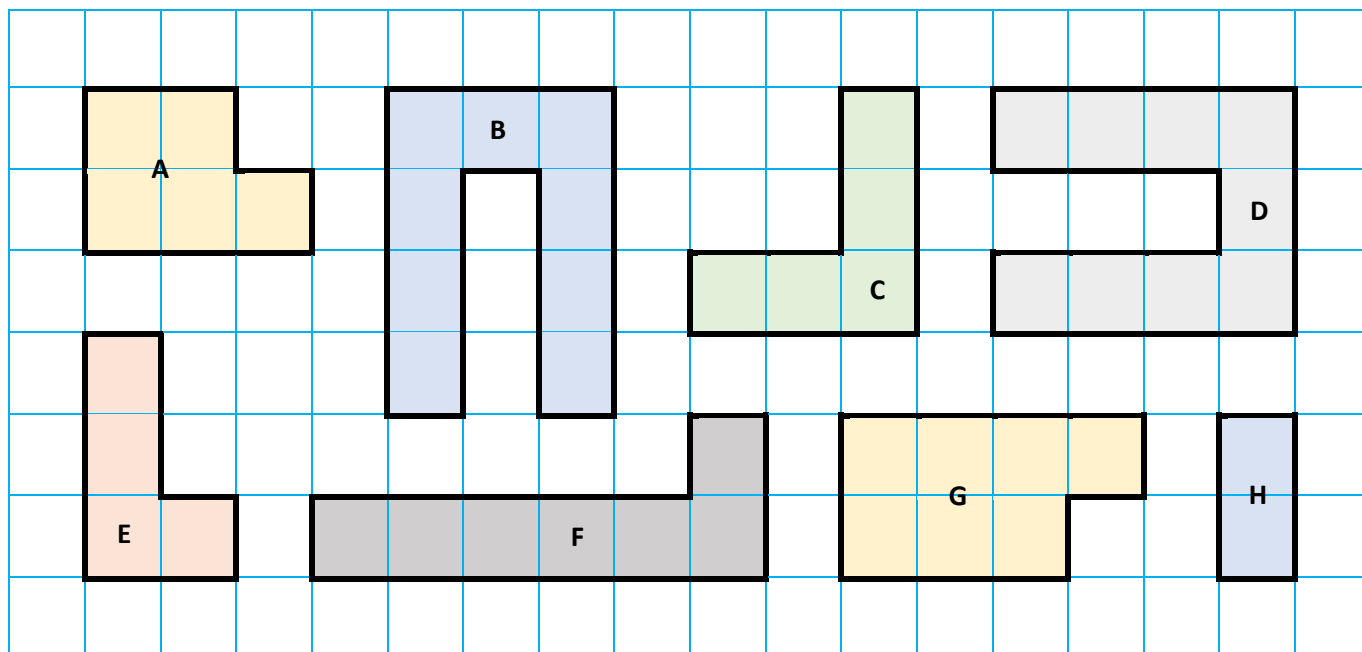
3



Exercise 3

3

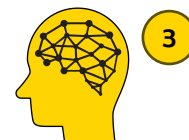
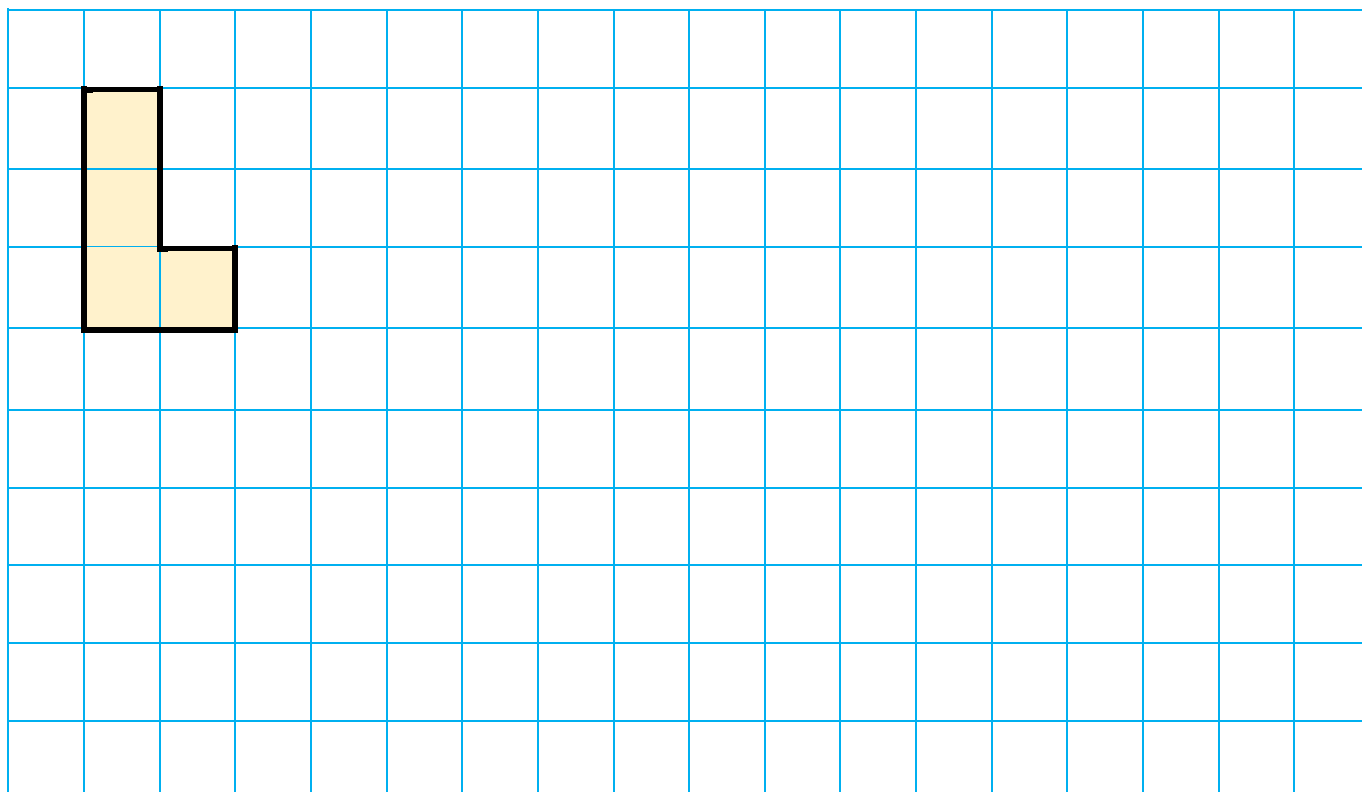
Below there is a collection of shapes drawn on a squared centimetre grid.



- Which two shapes are congruent?
- Which two shapes have an area of 5 cm^2 ?
- Which two shapes have a perimeter of 12 cm ?
- Which two shapes have an area of 7 cm^2 ?
- Which two shapes have a perimeter of 10 cm ?

Exercise 4

On the following grid, draw shapes that are congruent to the shown shape, but have different orientations. How many different orientations can be drawn?

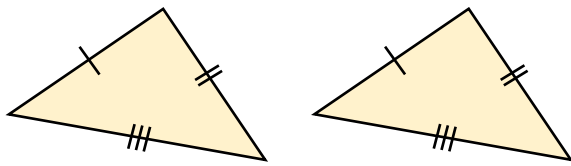




Congruent Triangles Proofs

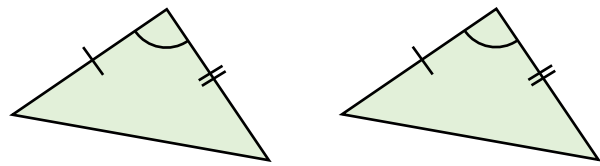
There are four ways of proving that two triangles are congruent.

(1) Side, Side, Side (SSS)



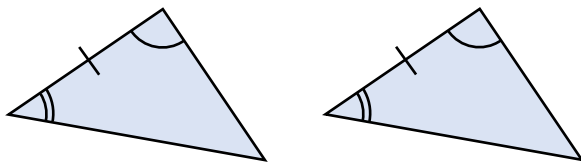
The lengths of the sides of the first triangle correspond to the lengths of the sides in the second triangle.

(2) Side, Angle, Side (SAS)



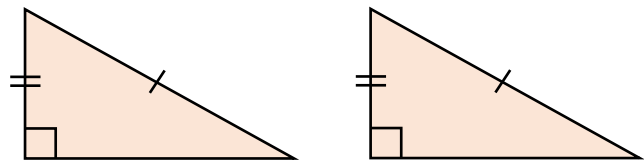
The lengths of two of the sides in the first triangle correspond to lengths of two of the sides in the second triangle, and the angles **between** the sides are equal.

(3) Angle, Side, Angle (ASA)



The size of two of the angles in the first triangle correspond to the size of two of the angles in the second triangle, and the length of the sides **between** the angles are equal.

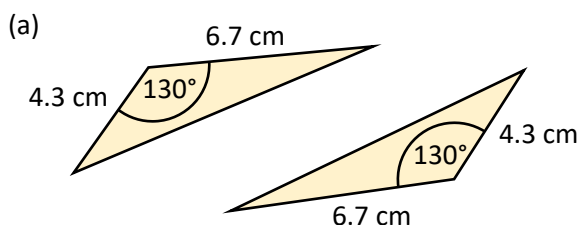
(4) Right Angle, Hypotenuse, Side (RHS)



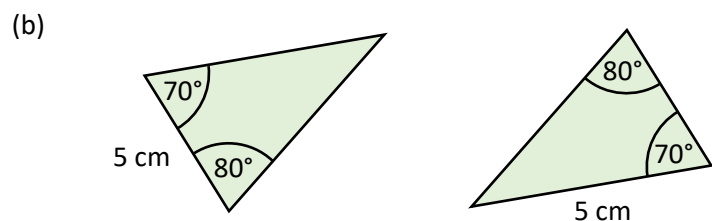
The two triangles are right-angled triangles; the lengths of the hypotenuse are equal; and the lengths of another side are equal.

Example

Explain, **noting your reasons**, if the following pairs of triangles are congruent or not.



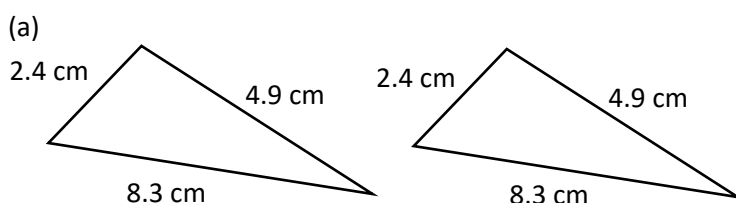
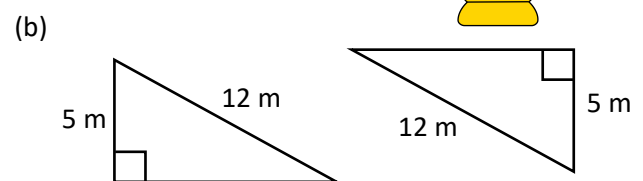
Answer: The lengths of two of the sides in the first triangle are equal to the lengths of two of the sides in the second triangle (4.3 cm, 6.7 cm). The angle between the sides (130°) is also equal, so the triangles are congruent due to the SAS rule.



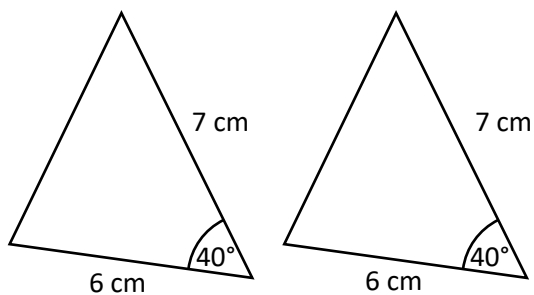
Answer: These triangles are not congruent. There is an angle of 70° , an angle of 80° and a side length of 5 cm in each triangle, but the 5 cm length is not **between** the angles in the triangle on the right. Because the longest length in a triangle is always opposite the greatest angle (80° in this case), the side between the angles must have a length less than 5 cm. So, we cannot use the ASA rule to prove that these triangles are congruent.

Exercise 5

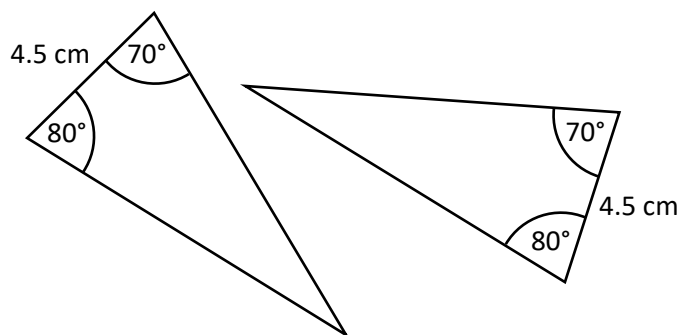
Explain, **noting your reasons**, if the following pairs of triangles are congruent or not. (The diagrams are not drawn to scale.)



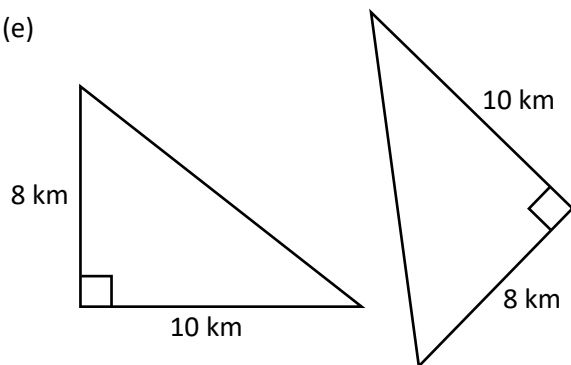
(c)



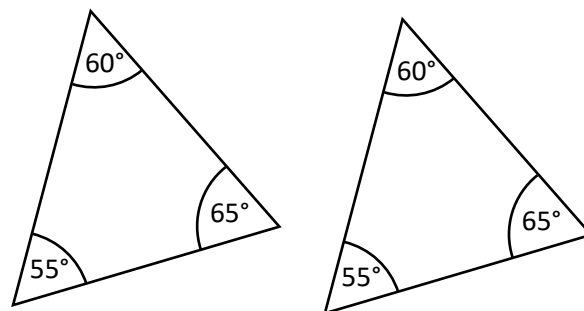
(d)



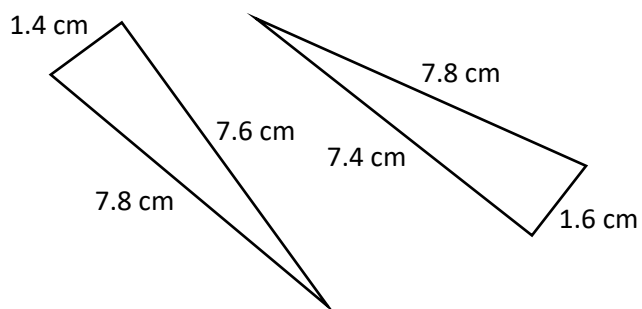
(e)



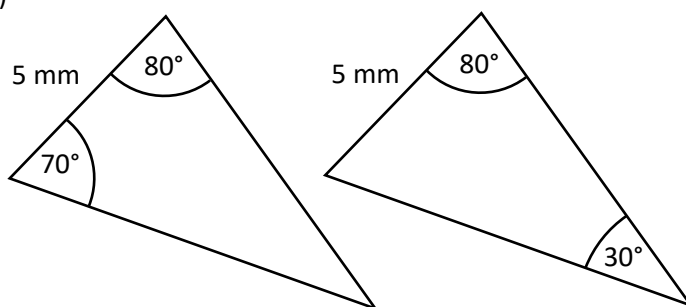
(f)



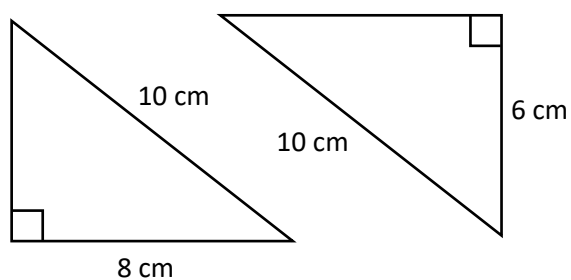
(g)



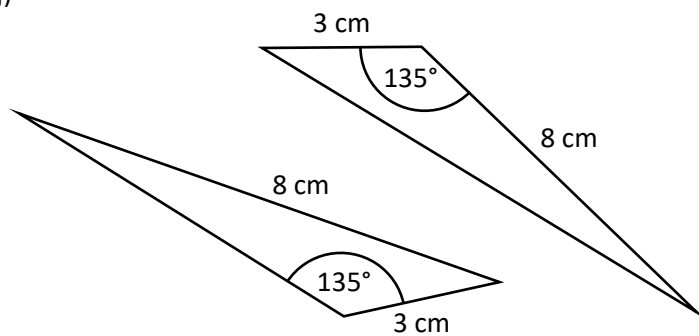
(h)



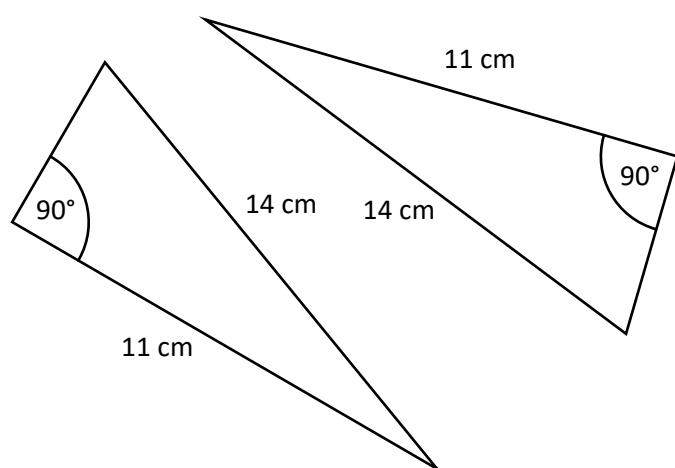
(i)



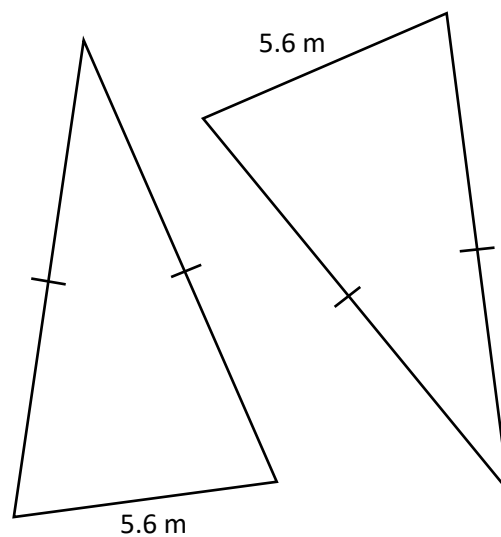
(j)



(k)



(l)



Exercise 6**Applying**

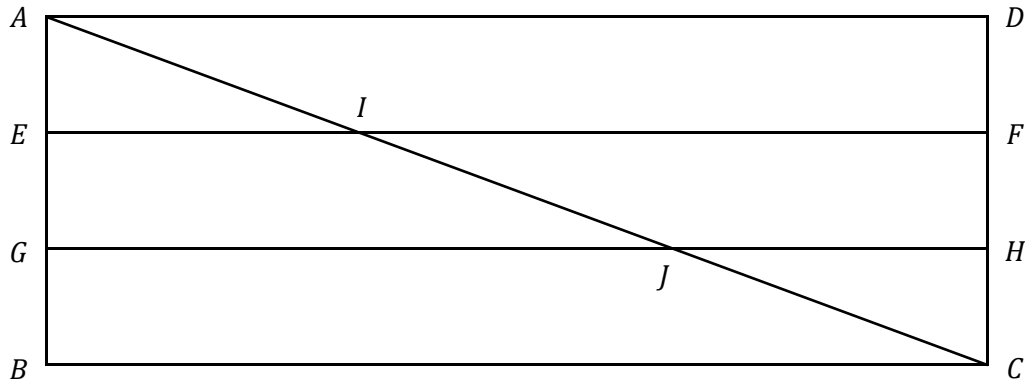
3



The following diagram shows a wooden fence, $ABCD$. The framework is strengthened by the addition of three wooden bars, EF , GH and AC .

The beams AD , EF , GH and BC are parallel to each other, with equal spaces between them.

The bar AC meets EF and GH at I and J , respectively.



- (a) Name a triangle that is congruent to the triangle AGJ .
 (b) Explain clearly why these triangles are congruent.

Exercise 7

3

Circle either TRUE or FALSE for the following statements.

STATEMENT		
Every rectangle is congruent.	TRUE	FALSE
Circles with equal area are congruent.	TRUE	FALSE
Every regular pentagon is congruent.	TRUE	FALSE
Using a 100% setting, a photocopier produces congruent shapes.	TRUE	FALSE
Each triangle with a base of 5 cm and a height of 4 cm is congruent.	TRUE	FALSE
Every semicircle with a diameter of 6 cm is congruent.	TRUE	FALSE

Evaluation

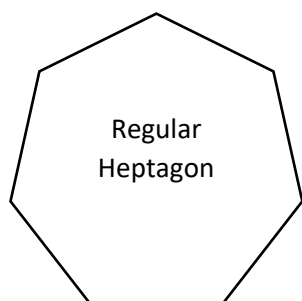
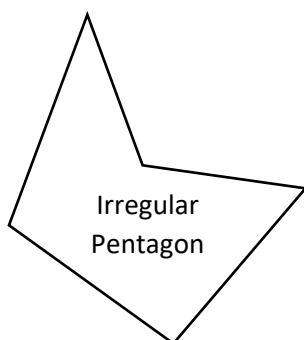
Key words	Corrections	I am happy with...	I need to revise...

Angles in Polygons

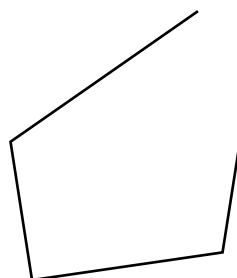
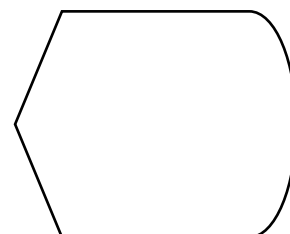


A shape which uses straight lines only is called a **polygon**. The polygon is **regular** if its sides all have the same length and its angles are also all equal. If the polygon is not regular, then it is an **irregular** polygon.

Examples of polygons



Non-examples of polygons



Exercise 8

Complete the following table.


Skill
2

Number of edges	Polygon name	Total interior angles	Interior angle for a regular polygon
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
n	Polygon with n edges		

Exercise 9

2

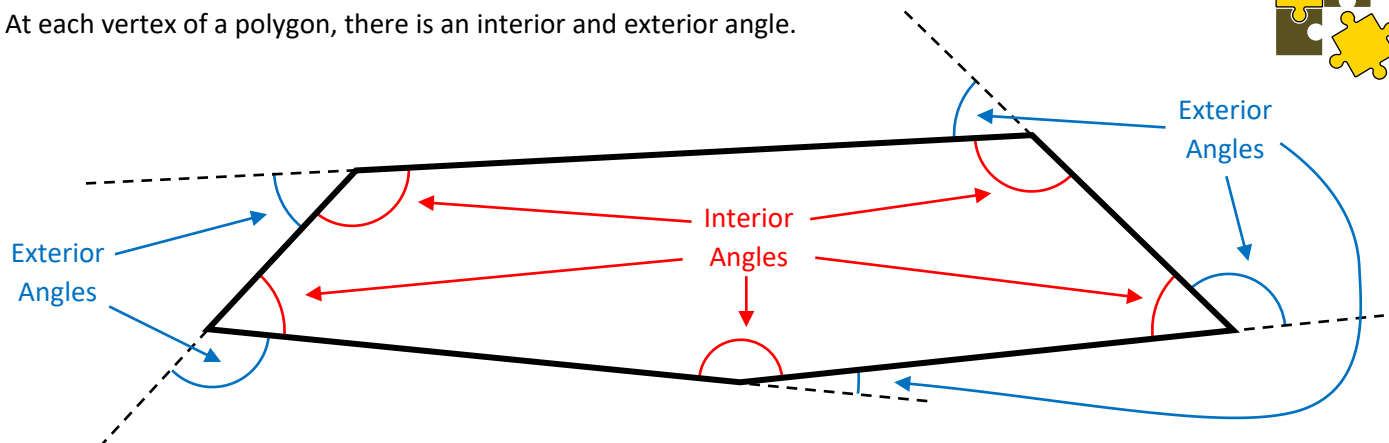
Use a ruler and a protractor to draw (a) a regular pentagon; (b) a regular hexagon; (c) a regular decagon.

Challenge!

Use an Excel spreadsheet to investigate the size of the interior angles of different regular polygons. As the number of edges increases, what happens to the interior angle? Does this pattern continue forever? What type of shape is a polygon with an ∞ number of edges?

The Exterior Angle of a Polygon

At each vertex of a polygon, there is an interior and exterior angle.



When walking along the exterior perimeter of a polygon, the exterior angle is the angle that you must turn through to continue travelling along the perimeter. For example, imagine walking around the exterior perimeter of the *Pentagon*, the headquarters of the United States Department of Defense.



The total of the interior angles is dependent upon the type of polygon. Above, the polygon is a pentagon, so the total interior angles is 540° . What is the total of the exterior angles? Again, imagine walking along the exterior perimeter. On returning to your original position, you will have rotated around a full turn, or 360° . The type of polygon is not important here, so the total exterior angles of any polygon is 360° .

Summary

For a polygon with n edges,

$$\begin{aligned}\text{Total of the exterior angles} &= 360^\circ \\ \text{Total of the interior angles} &= 180^\circ(n - 2)\end{aligned}$$

For any vertex in a polygon,

$$\text{Interior angle} + \text{exterior angle} = 180^\circ$$

If the polygon is a regular polygon,

$$\begin{aligned}\text{One exterior angle} &= \frac{360^\circ}{n} \\ \text{One interior angle} &= \frac{180^\circ(n-2)}{n} \text{ or } 180^\circ - \frac{360^\circ}{n}\end{aligned}$$



Challenge!

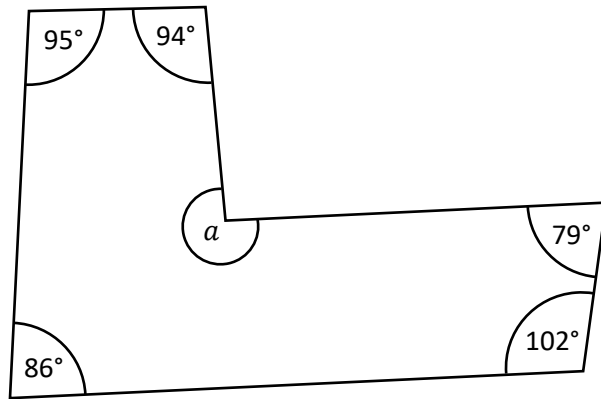
Prove that $\frac{180^\circ(n-2)}{n} \equiv 180^\circ - \frac{360^\circ}{n}$.

Exercise 10

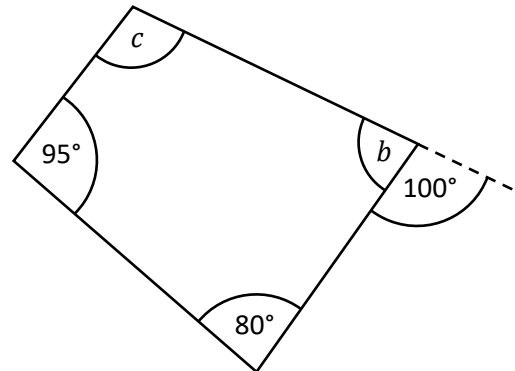
Calculate the size of the missing angles. (The diagrams are not drawn to scale.)



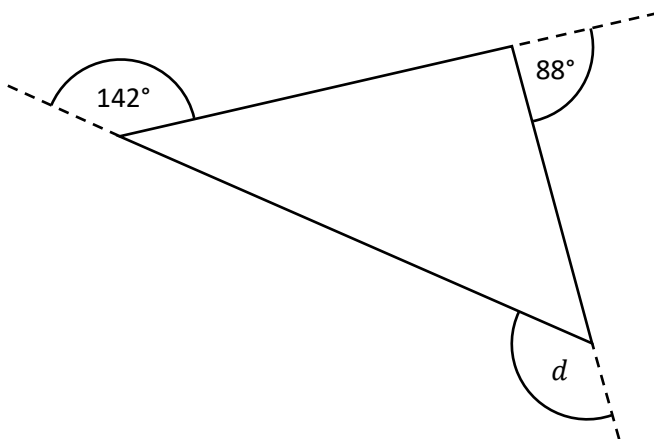
(a)



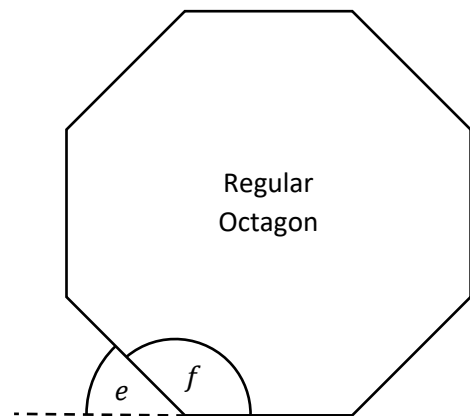
(b)



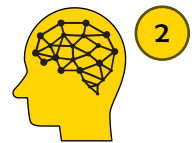
(c)



(d)

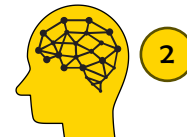
**Exercise 11**

- What is the total interior angles of any heptagon?
- What is the exterior angle of any equilateral triangle?
- What is the interior angle of any regular nonagon?
- Four of the exterior angles of a pentagon are 110° , 90° , 70° , 50° . What is the size of the fifth exterior angle?
- Four of the interior angles of a pentagon are 150° , 130° , 110° , 90° . What is the size of the fifth interior angle?

**Exercise 12**

- The size of the exterior angles of a regular polygon are 18° . How many edges does the regular polygon have?
- The size of the interior angles of a regular polygon are 156° . How many edges does the regular polygon have?
- Three of the exterior angles of a hexagon are 100° . The other three exterior angles are equal. Calculate the size of each of these exterior angles.
- Why is it not possible to draw a triangle with exterior angles 170° , 160° , 150° ?
- Four of the six interior angles of a hexagon are 130° , 140° , 150° , 160° . The other two interior angles are equal. Calculate the size of the largest **exterior** angle of the hexagon.



Exercise 13

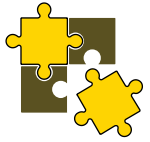
Draw polygons in the spaces below, showing clearly the size of each angle.

		Size of the least interior angle		
		Less	Equal	More
Total of the interior angles	More			
	Equal			
	Less			



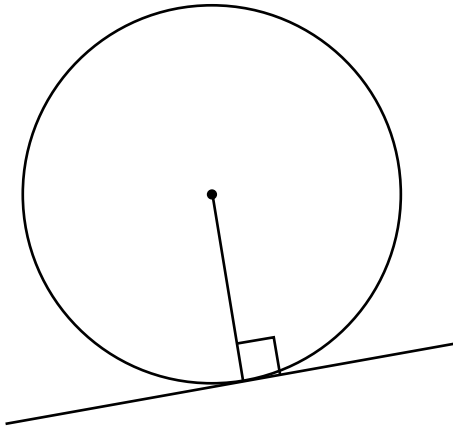
Key words	Corrections	I am happy with...	I need to revise...

Circle Theorems



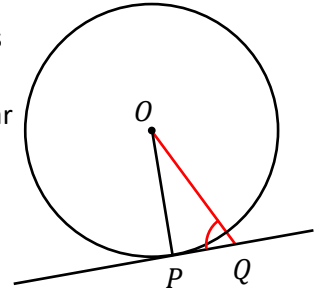
A number of facts related to angles in a circle must be learnt. (You do not have to learn the proofs.)

(1) Tangent and radius meet at a right angle.



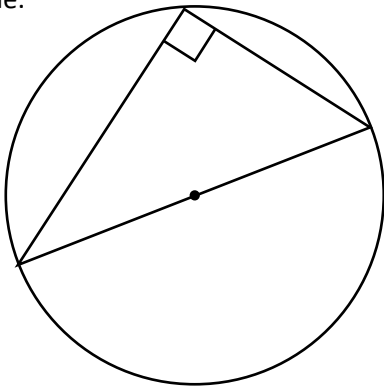
Proof

Assume that the tangent and radius do not meet at a right angle. Then we can draw the perpendicular line from the centre O to the point Q on the tangent (a point that is outside the circle), so that the angle $O\hat{Q}P = 90^\circ$.



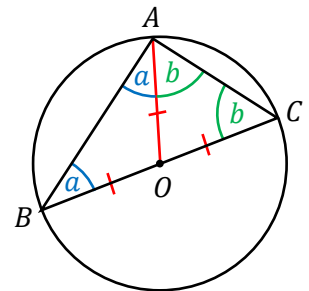
It follows that the triangle OQP is a right-angled triangle where the radius OP is the hypotenuse of the triangle. But we see that the line OQ must be longer than the line OP (as Q lies outside the circle). This goes against the mathematical fact that the hypotenuse of a right-angled triangle is the longest side, so tangent and radius must meet at a right angle.

(2) The angle in a semicircle is a right angle.



Proof

Split the triangle into two isosceles triangles, by adding a radius from the centre O to the vertex A .



The total of the angles of triangle ABC is 180° , so

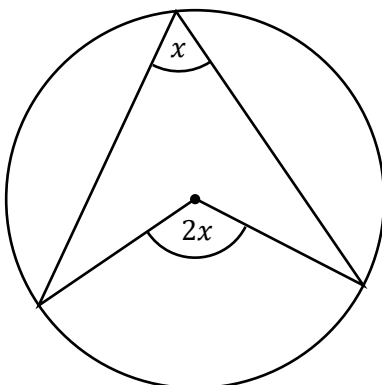
$$a + a + b + b = 180^\circ$$

$$2a + 2b = 180^\circ \quad [\text{Collect like terms}]$$

$$a + b = 90^\circ \quad [\text{Divide by 2}]$$

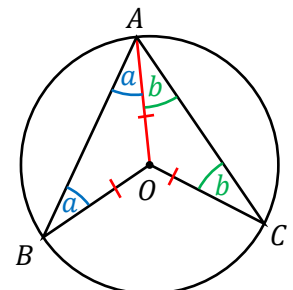
So, the angle $B\hat{A}C$ is a right angle.

(3) The angle in the centre of a circle is twice the angle on the circumference.



Proof

Split the quadrilateral into two isosceles triangles, by adding a radius from the centre O to the vertex A .



$$\text{In triangle } ABO, A\hat{O}B = 180^\circ - 2a.$$

$$\text{In triangle } ACO, A\hat{O}C = 180^\circ - 2b.$$

Using the angles around the centre O ,

$$B\hat{O}C = 360^\circ - A\hat{O}B - A\hat{O}C$$

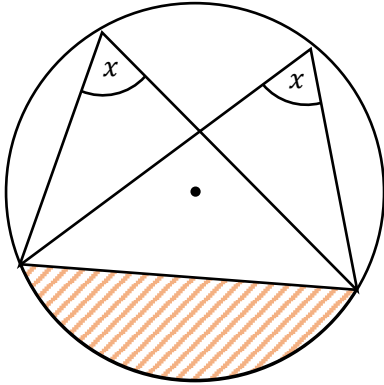
$$B\hat{O}C = 360^\circ - (180^\circ - 2a) - (180^\circ - 2b)$$

$$B\hat{O}C = 2a + 2b$$

$$B\hat{O}C = 2(a + b)$$

So, the angle in the centre of the circle ($B\hat{O}C$) is twice the angle on the circumference ($B\hat{A}C$ or $a + b$).

(4) Angles in the same segment are equal.

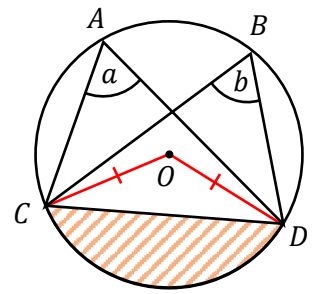


Proof

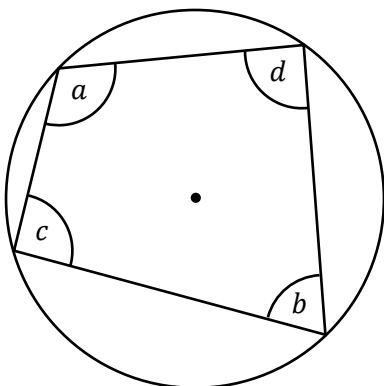
Add two radii from the centre O to the vertices C and D .

From the previous circle theorem, the size of angle $\widehat{COD}$ is twice the angle $\widehat{CAD}$. But we can also state that the angle $\widehat{COD}$ is twice the angle $\widehat{CBD}$.

It follows that the angles $\widehat{CAD}$ and $\widehat{CBD}$ are equal. So, angles in the same segment are equal.



(5) Opposite angles in a cyclic quadrilateral sum to 180° .



$$a + b = 180^\circ$$

$$c + d = 180^\circ$$

Proof

Add two radii from the centre O to the vertices C and D .

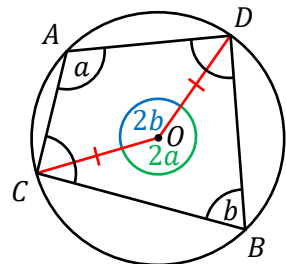
Because the angle in the centre is twice the angle on the circumference, we can state that $\widehat{COD} = 2b$, and $\widehat{COD} \text{ reflex} = 2a$.

Angles around any point sum to 360° , so

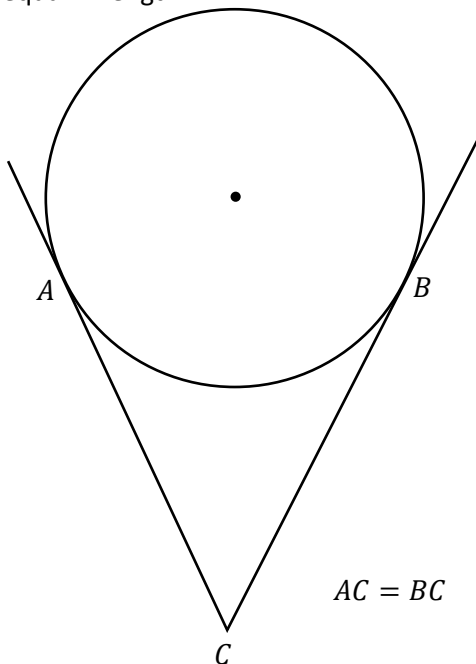
$$2a + 2b = 360^\circ$$

$$a + b = 180^\circ \quad [\text{Divide by 2}]$$

So, opposite angles in a cyclic quadrilateral sum to 180° .



(6) Tangents from an external point are equal in length.



Proof

Add two radii from the centre O to the vertices A and B . Then, add a line from the centre O to the vertex C .

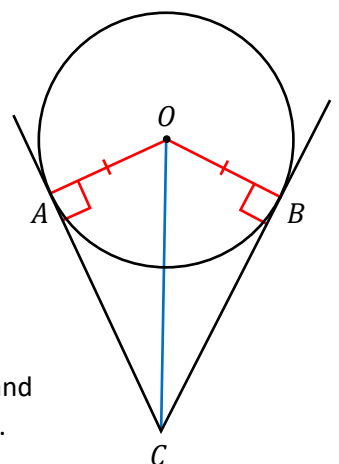
Because tangent and radius meet at a right angle, we have $\widehat{OAC} = \widehat{OBC} = 90^\circ$.

The two right-angled triangles OAC and OBC share the same hypotenuse OC .

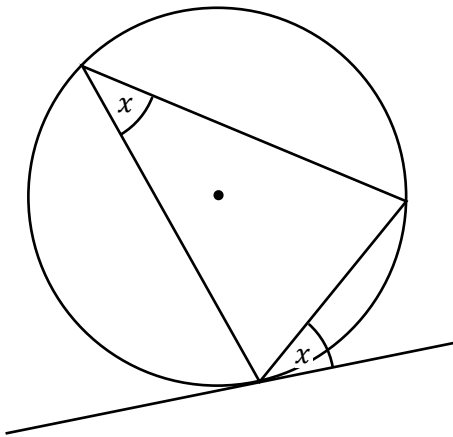
We have $OA = OB$, because they are both radii.

Using the RHS rule, we can state that the triangles OAC and OBC are congruent.

It follows that $AC = BC$, and so tangents from an external point are equal in length.



(7) The angle between chord and tangent is equal to the angle in the alternate segment.



Proof

Because angles in the same segment are equal, we can choose to prove the case where the line AC is a diameter to the circle.

Tangent and radius meet at a right angle, so $\angle CAB = 90^\circ - x$.

The angle in a semicircle is a right angle, so $\angle ABC = 90^\circ$.

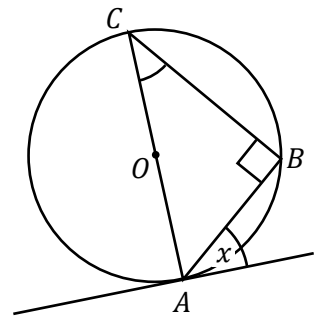
Using the triangle ABC ,

$$\angle ACB = 180^\circ - \angle ABC - \angle CAB$$

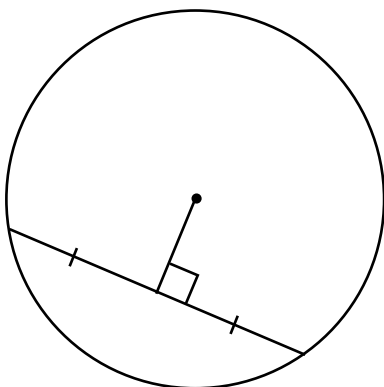
$$\angle ACB = 180^\circ - 90^\circ - (90^\circ - x)$$

$$\angle ACB = x$$

So, the angle between chord and tangent is equal to the angle in the alternate segment.



(8) The perpendicular from the centre to a chord bisects the chord.



Proof

Add two radii from the centre O to the vertices A and B .

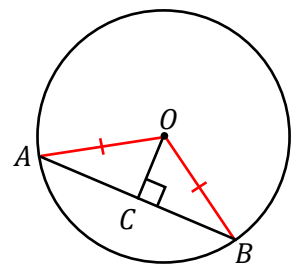
The triangles OAC and OBC are right-angled triangles.

The hypotenuse of both triangles are equal in length, since they are both radii.

The two triangles also share a side OC .

Using the RHS rule, we can state that the triangles OAC and OBC are congruent.

It follows that $AC = BC$, and so the perpendicular from the centre to a chord bisects the chord.



Exercise 14

2

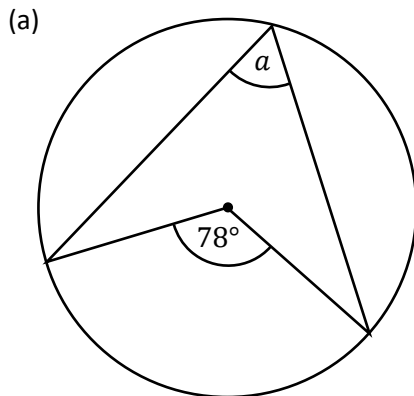
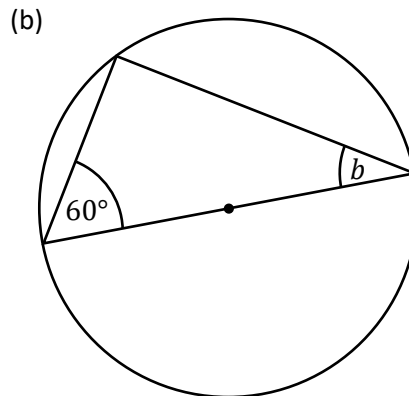
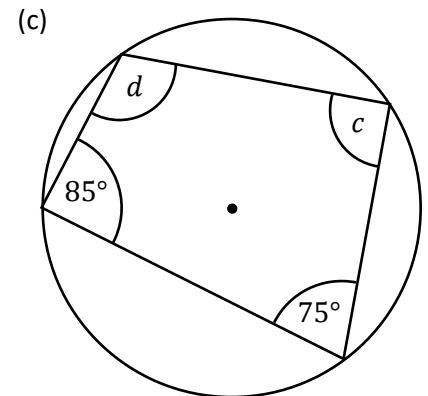
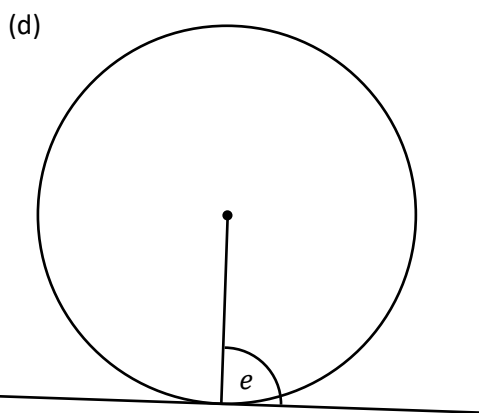
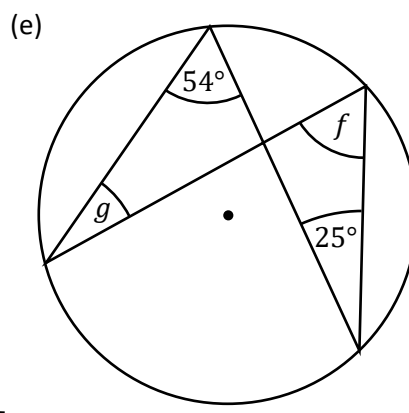
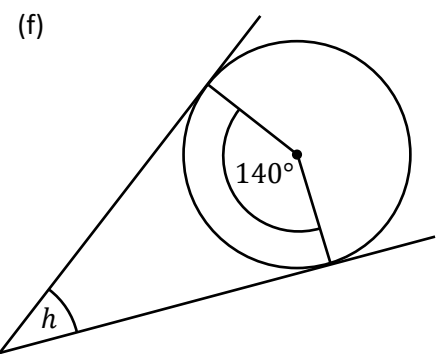
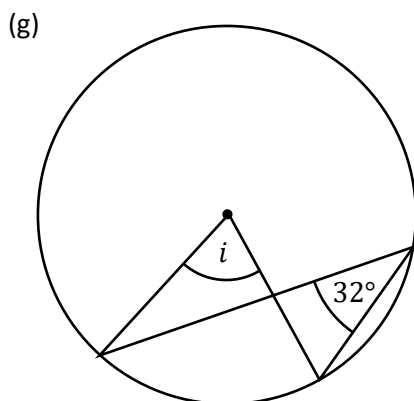
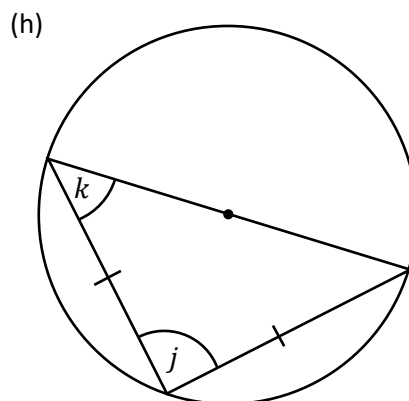
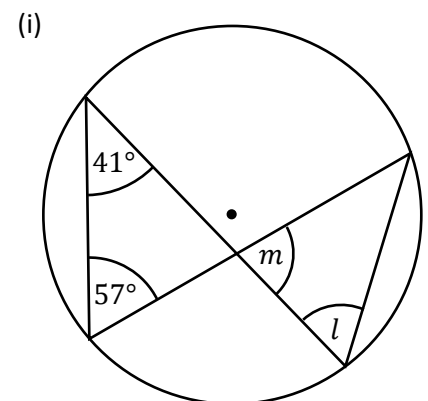
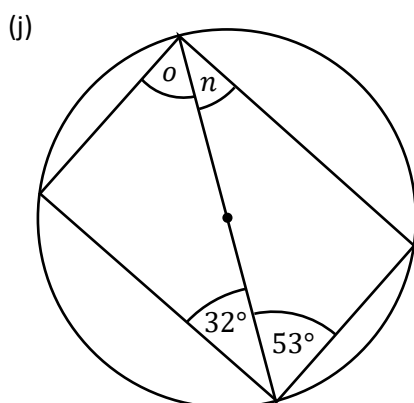
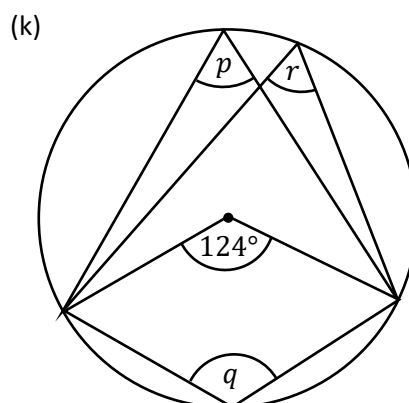
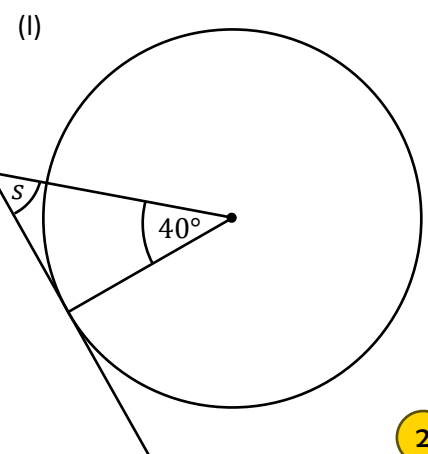
Take some time to become familiar with the circle theorems. Here are some ideas:

- Try to re-create the circle theorems using the GeoGebra software.
- Draw examples of the circle theorems in your revision book.
- Try to re-create the circle theorems using paper plates, string and colouring materials.
- Verify that the theorems are true by drawing examples using a compass, ruler and protractor.



**Exercise 15**

Use the circle theorems to find the size of the marked angles in the following diagrams. (The diagrams are not drawn to scale.)

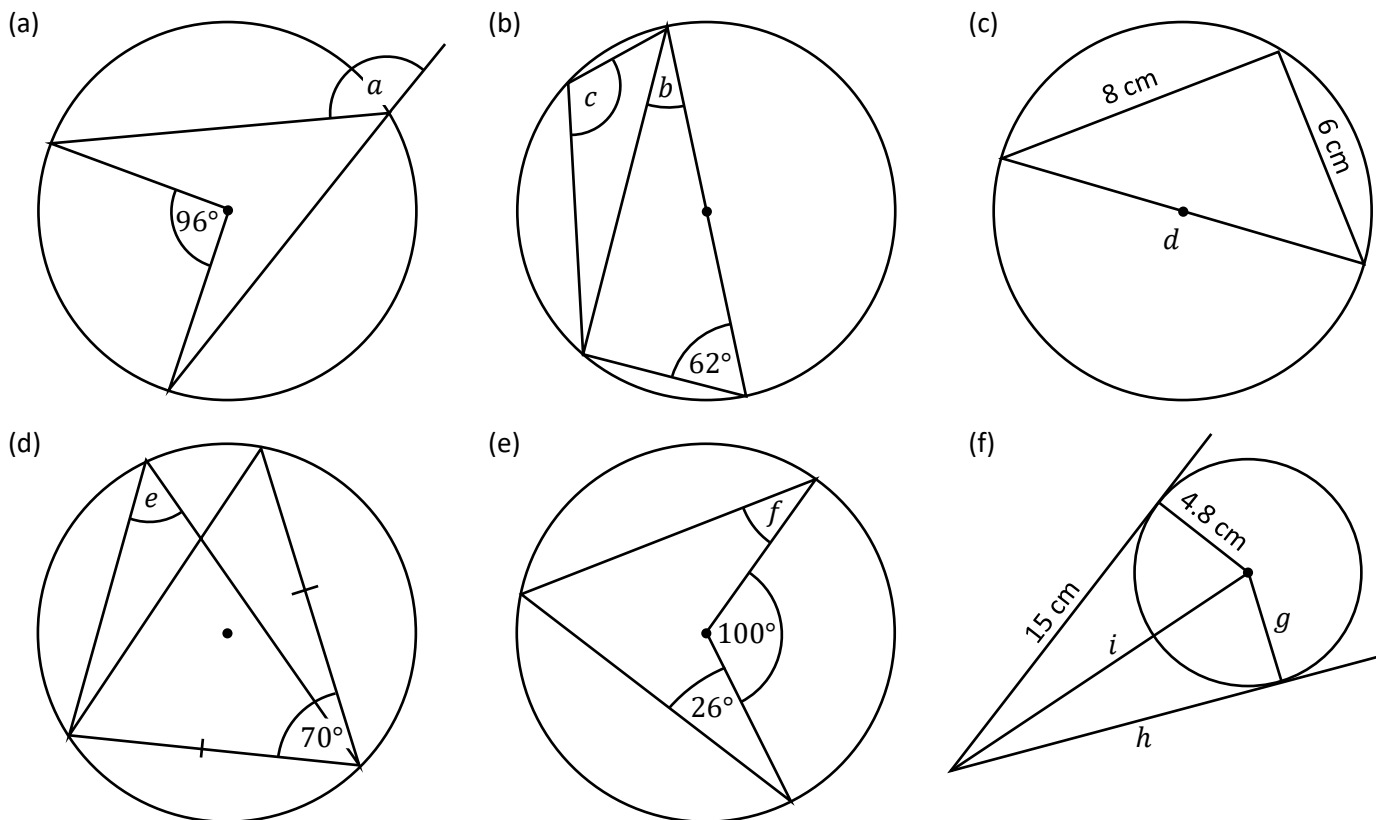
Skill**2****Exercise 16**

For each question in Exercise 15 above, note which circle theorem(s) you used in finding the missing angle(s).

Exercise 17

2

Use the circle theorems to find the size of the marked sides or angles in the following diagrams. (The diagrams are not drawn to scale.)

**Exercise 18**

2

For each question in Exercise 17 above, note which circle theorem(s) you used in finding the missing value(s).

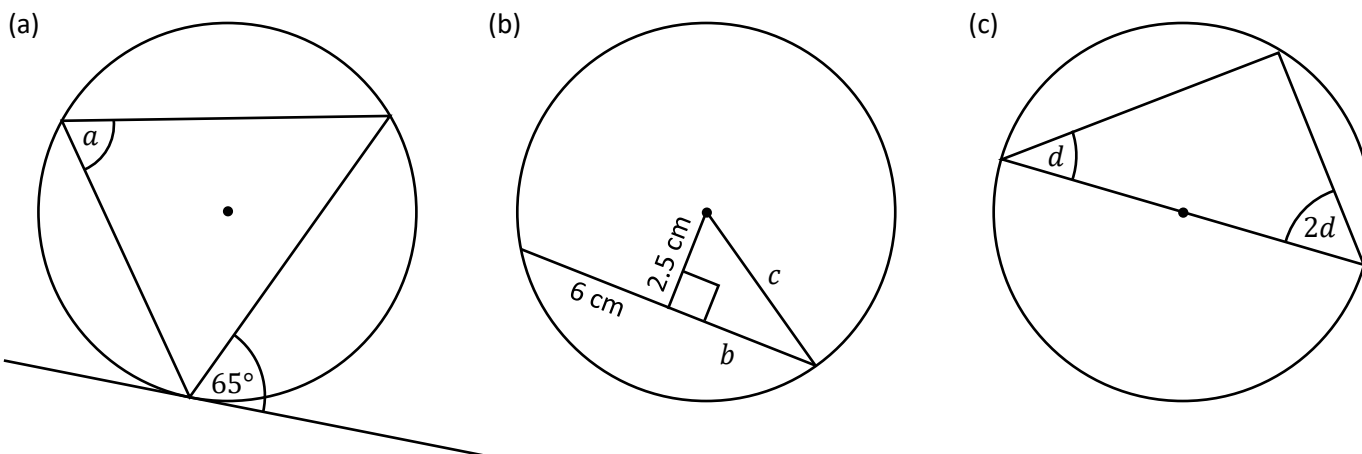
Challenge!

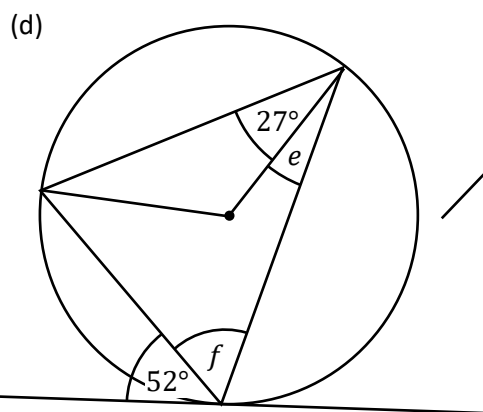
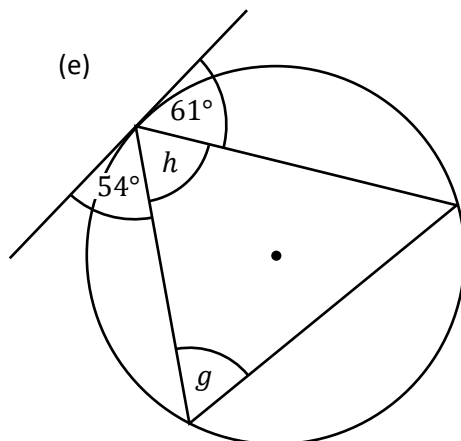
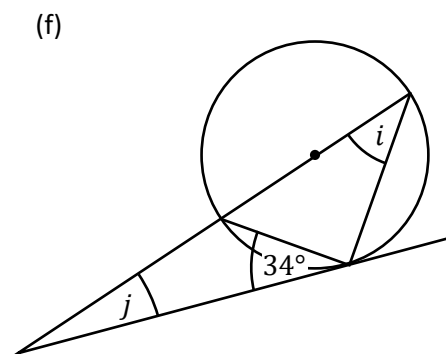
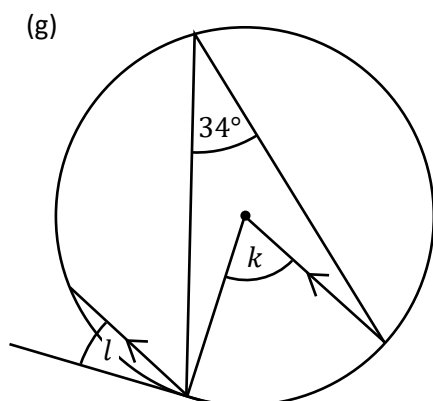
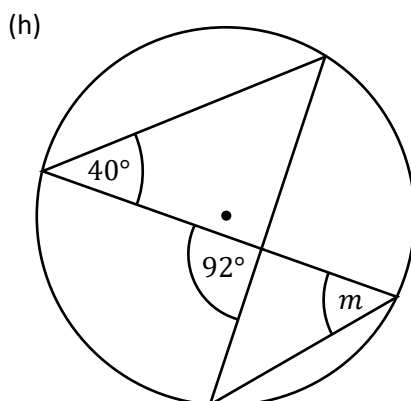
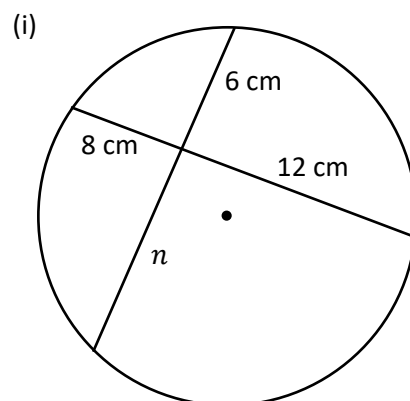
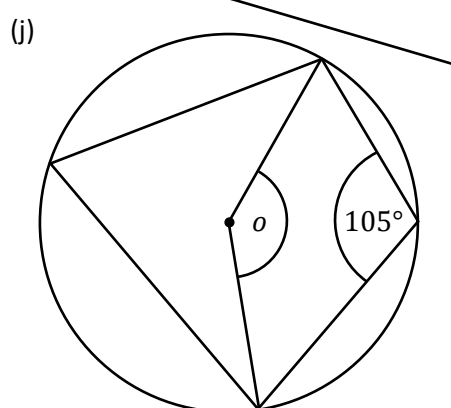
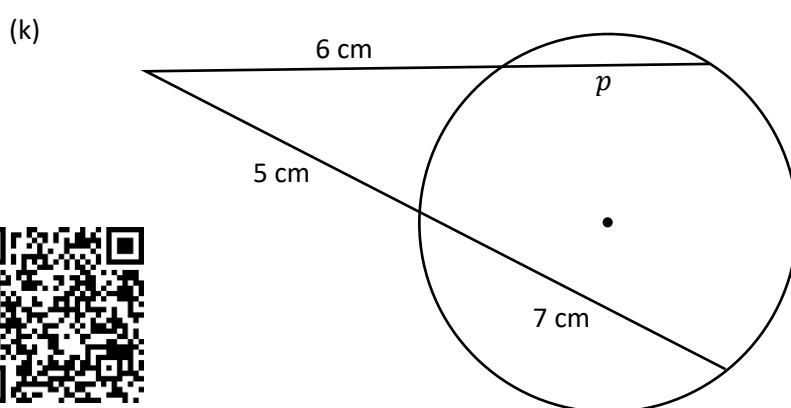
Imagine a circle with radius 1 m . 6 equilateral triangles are placed in the circle, with one vertex of each triangle in the centre of the circle, and the two other vertices on the circumference. The triangles do not overlap. What is the difference between the area of the circle and the area of the six equilateral triangles?

Exercise 19

2

Use the circle theorems to find the size of the marked sides or angles in the following diagrams. (The diagrams are not drawn to scale.)





Exercise 20

2

For each question in Exercise 19 above, note which circle theorem(s) you used in finding the missing value(s).



Key words	Corrections	I am happy with...	I need to revise...

Transformations



Over the years, we have seen four types of transformation.

Year 7	Year 8	Year 9	Year 10
Translation (moving a shape)	Rotation (turning a shape)	Reflection (reflecting a shape)	Enlargement (changing the size of a shape)

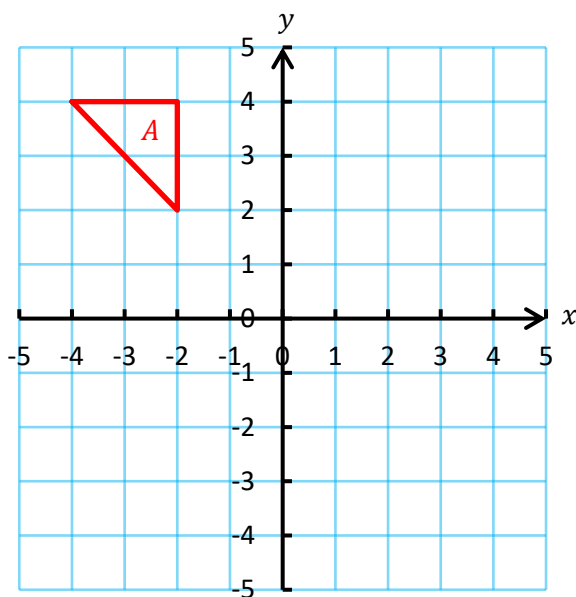
In this chapter, we will revise these transformations, before combining them in different ways.

Exercise 21

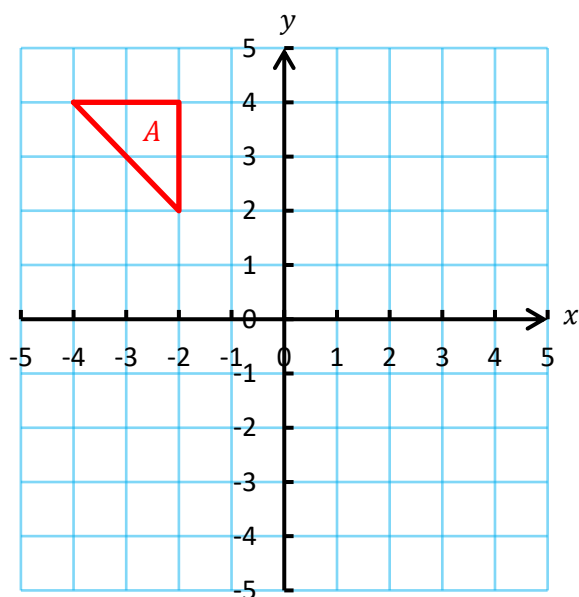
Revision

2

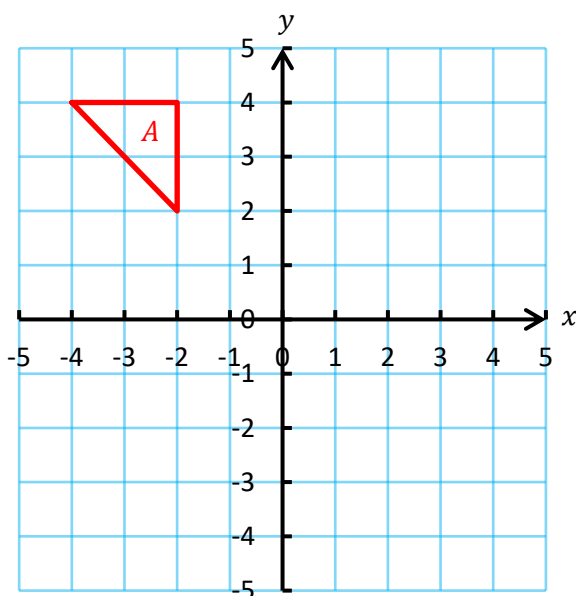
(a) Translate the triangle A 5 units to the right, 2 units down.



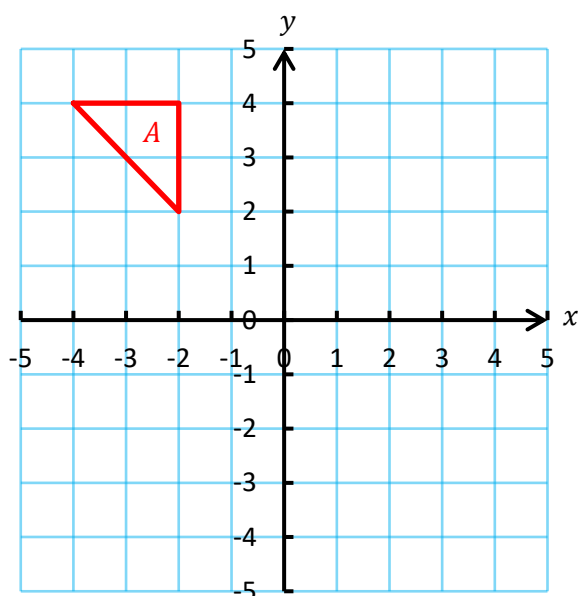
(b) Rotate the triangle A 90° clockwise around the point $(-1, 1)$.



(c) Reflect the triangle A in the line $y = 1$.



(d) Enlarge the triangle A using scale factor 2 and centre of enlargement $(-5, 3)$.

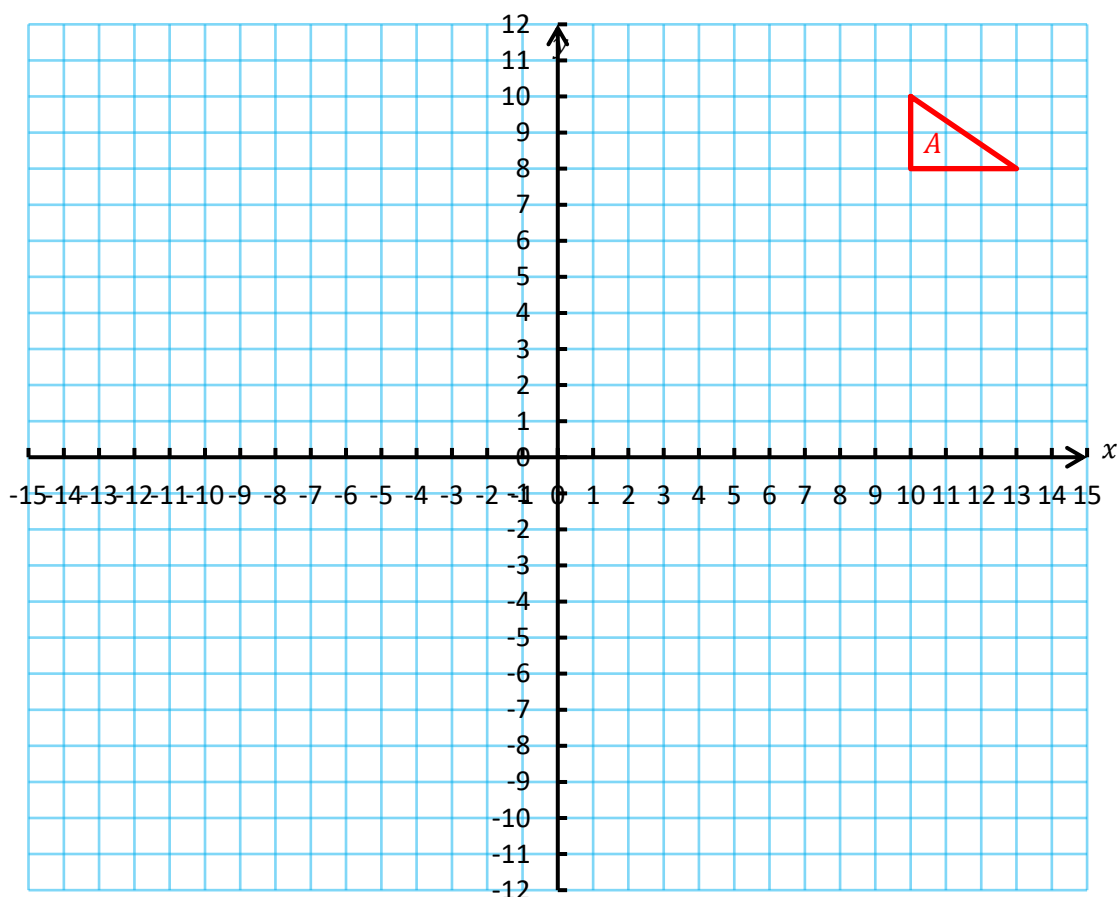


Exercise 22

Skill

2

- (a) Translate the triangle A 2 units to the left, 3 units down. Label the new triangle B .
- (b) Reflect the triangle B in the line $x = 4$. Label the new triangle C .
- (c) Rotate the triangle C 90° anticlockwise around the point $(-4, 3)$. Label the new triangle D .
- (d) Reflect the triangle D in the line $y = 1$. Label the new triangle E .
- (e) Enlarge the triangle E using scale factor 3 and centre of enlargement $(-10, -3)$. Label the new triangle F .
- (f) Reflect the triangle F in the line $x = 4$. Label the new triangle G .
- (g) Translate the triangle G 20 units to the left, 11 units up. Label the new triangle H .



Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Mesur

Siapiau 4

Reflection

Name:

Percentage in the test:

I know
this.I need to
revise this.Question
in the
test:Correct
in the
test?

I can recognise congruent shapes .			1	
I can use the SSS, SAS, ASA and RHS rules to prove when two triangles are congruent .			2, 3	
I can calculate the total interior angles of any polygon.			4, 6, 8	
I can calculate the interior angle of any regular polygon .			7, 8	
I know the total exterior angles of any polygon .			4	
I can calculate the exterior angle of any regular polygon .			5	
I know the connection between the interior and exterior angles for any vertex in a polygon.			4, 6	
I can recite the names of the circle theorems .			9, 10, 11	
I can use the circle theorems to find missing angles and sides.			9, 10, 11	
I can combine the four transformations to transform different shapes.			12	



The Mathematics Department

11

Developing

Probability

Higher Tier

Version
3.0
08/2025

Name:

Contents

Chapter	Mathematics	Page Number
Relative Frequency	Revision. Relative Frequency.	3
Combined Events	Independent events. Dependent events. Mutually exclusive events. Venn diagrams. Sample space diagrams and dependent events.	8
Tree Diagrams	Displaying combinations of two or more events. Independent events. Dependent events.	15



Relative Frequency



Exercise 1 (Revision of previous work on probability)

Revision

2

(a) Use “no chance”; “low chance”; “even chance”; “good chance” or “certain” to describe the probability of the following events.

- (i) You land on “tails” when throwing a fair coin.
- (ii) The next person you meet writes with their left hand.
- (iii) You obtain a number less than 5 when rolling a normal fair die.
- (iv) St. David’s Day will be on March 1st next year.
- (v) You will obtain 101% in your end of year mathematics examination.

(b) Draw a probability scale. Mark the points i, ii, iii, iv in order to show how probable, in your opinion, the following events are.

- (i) A man will be driving the next car you see.
- (ii) It will snow during the day tomorrow.
- (iii) A story about politics will feature on the news tonight.
- (iv) Germany will win the next football world cup.

(c) Answer with a number between 0 and 1: what is the probability that someone will walk to the top of Snowdon tomorrow?

(d) Answer the following questions using fractions.
What is the probability of obtaining...

- (i) The number 4 when rolling a normal fair die?
- (ii) “Heads” when throwing a fair coin?
- (iii) A square number when spinning a spinner showing the numbers 1 to 8?

(e) Rheinallt shuffles the 52 cards in a standard deck of playing cards and chooses one card randomly from the deck. What is the probability that the chosen card is:

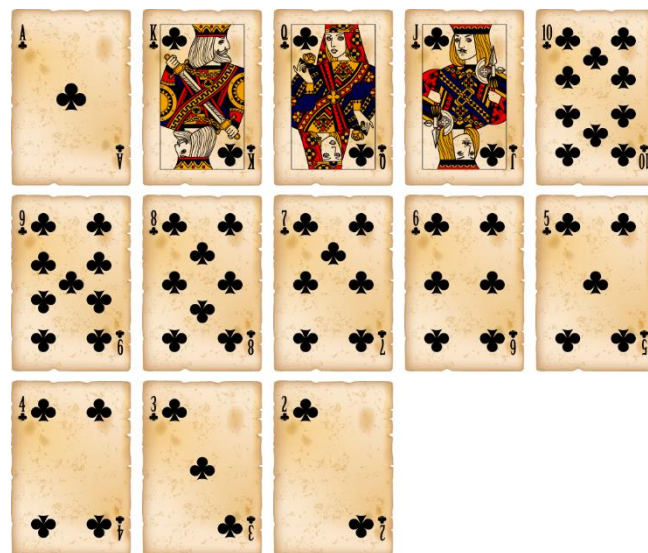
- (i) a diamond?
- (ii) 6?
- (iii) a face card?
- (iv) a spade showing an even number?
- (v) a red card less than 5?

(f) The probability that Meira goes to the shop to buy a loaf of bread tomorrow is 0.4. What is the probability that Meira does not go to the shop to buy a loaf of bread tomorrow?

(g) A red die and a blue die are labelled from 1 to 6. Gethin rolls both dice and adds the scores obtained.

- (i) Use a sample space diagram to list all the possible outcomes.
- (ii) What is the probability that the sum of both numbers is 12?
- (iii) What is the probability that the sum of both numbers is a one-digit number?

(h) The probability that Ellie goes on a training run on any day is 0.7. There are 30 days in April. On how many days can you expect Ellie to go running?





Relative Frequency

The **frequency** of an event refers to how many times the event has occurred during a number of trials.

The **relative frequency** of an event compares the frequency to the number of trials.



$$\text{Relative frequency of an event} = \frac{\text{How many times the event has occurred}}{\text{Number of trials}}$$

It is possible to use relative frequency to **estimate the probability of an event**.

Exercise 2

Applying

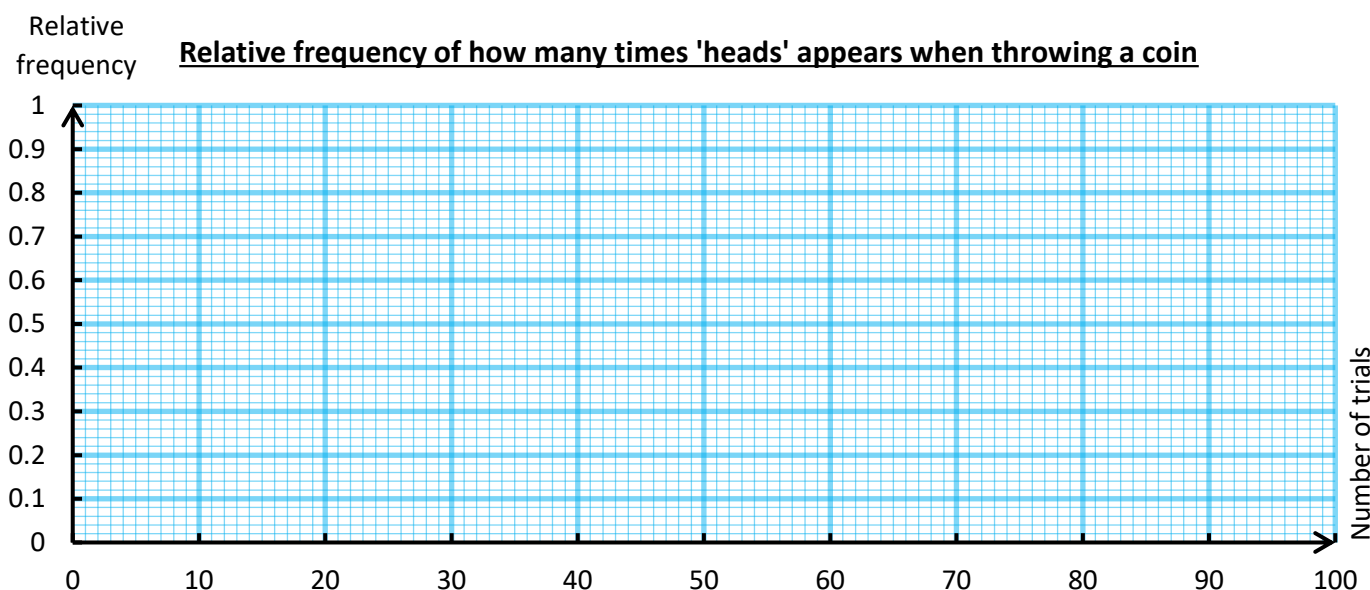
2

You will need a coin for this exercise.

(a) Throw the coin 100 times, recording in the following table, after each set of 10 throws, how many times the coin has landed showing 'heads'.

Total number of trials	Number of heads in these 10 throws	Total number of heads so far	Relative frequency of the number of heads, as a fraction	Relative frequency of the number of heads, as a decimal
10			$\frac{\quad}{10}$	
20			$\frac{\quad}{20}$	
30			$\frac{\quad}{30}$	
40				
50				
60				
70				
80				
90				
100				

(b) Plot, on the following graph paper, a line graph showing what happens to the relative frequency as the number of trials increases.



(c) If a great many more trials were held, how would you expect the graph to change?

Exercise 3 (Buffon's Needle Experiment)

2

For this exercise, you will need a plastic straw of length 3 cm, and lined paper where the lines are separated by 6 cm.

In 1777, a Frenchman called Georges-Louis Leclerc (Comte de Buffon) devised an experiment that can be used to estimate the value of π .

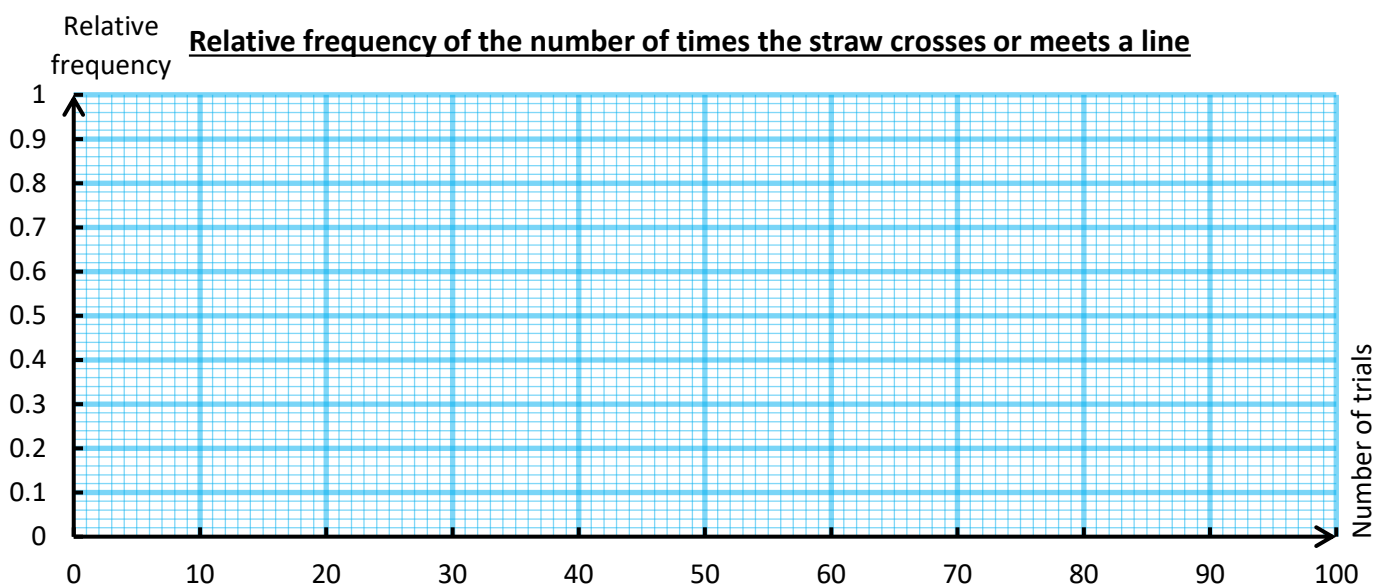
In each trial, a straw is dropped on a piece of paper. The straw should be dropped above the centre of the paper, from arm height. You should record whether or not the straw crosses (or meets) one of the lines.



(a) Perform the experiment 100 times. Record, after each set of 10 trials, how many times the straw crosses (or meets) one of the lines.

Total number of trials	Number of times the straw has crossed a line in these 10 trials	Total number of crossings so far	Relative frequency of the number of crossings, as a fraction	Relative frequency of the number of crossings, as a decimal
10			$\frac{\quad}{10}$	
20			$\frac{\quad}{20}$	
30			$\frac{\quad}{30}$	
40				
50				
60				
70				
80				
90				
100				

(b) Plot, on the graph paper below, a line graph showing what happens to the relative frequency as the number of trials increases.



(c) Calculate the **reciprocal** of your final relative frequency. How close is this value to π ?



In any experiment,

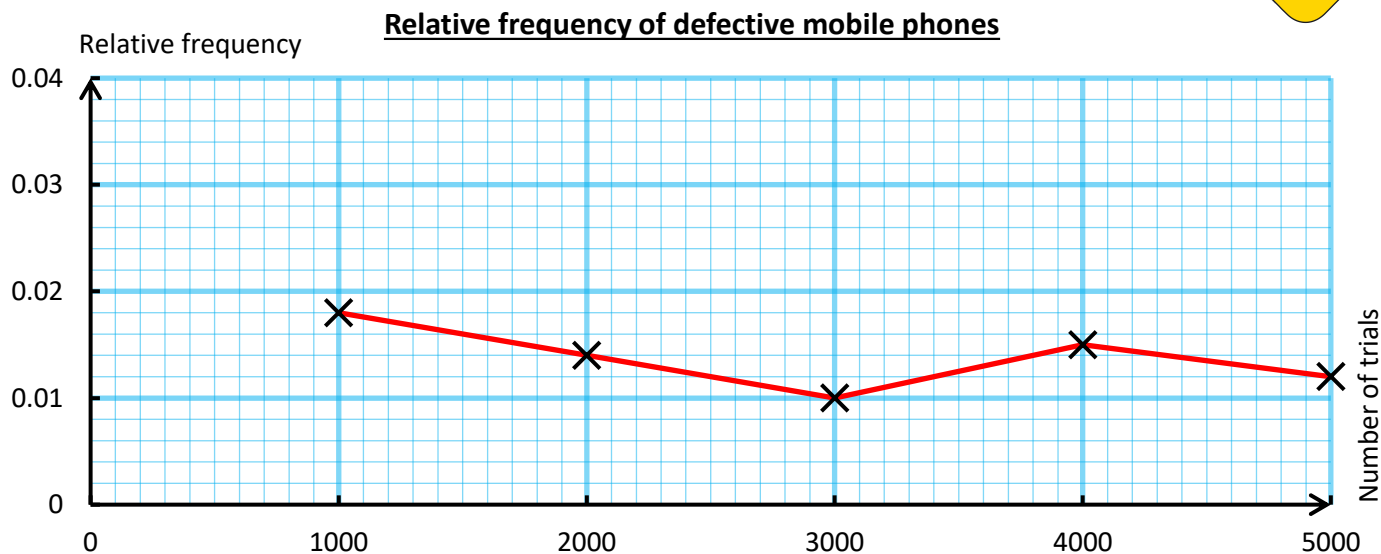
The more trials are held, the better the relative frequency is as an estimate of the probability.

Exercise 4

2

A factory produces mobile phones. The manager holds a survey to investigate the probability that the factory produces a defective mobile phone.

The relative frequency of defective mobile phones is recorded after testing a total of 1,000, 2,000, 3,000, 4,000 and 5,000 mobile phones. The results are shown in the following graph.



- (a) How many of the first 2,000 mobile phones tested were defective?
- (b) Write the best estimate for the probability that a randomly selected mobile phone is defective. You must give a reason for your answer.

Exercise 5

2

- (a) Fred throws a die 200 times and records how many times each score occurs.

Score	1	2	3	4	5	6
Frequency	29	34	35	32	34	36

- (i) Calculate the relative frequency of each score.
- (ii) In your opinion, is Fred's die fair? Give a reason for your answer.
- (b) Rhys recorded the results of his favourite football team.

Win	Draw	Lose
32	11	7



- (i) Calculate the relative frequency of each of the three possible results.
- (ii) Are your answers to part (i) a good estimate for the probability of the result of the next game? Explain your answer.

- (c) A petrol station owner noticed that 287 customers out of a total of 340 customers spent more than £30 filling up their cars. Use these figures to estimate the probability that the next customer spends

- (i) more than £30, (ii) £30 or less.

(d) In a survey, 600 people were asked about their favourite crisp flavour. The following table shows the results.

Flavour	Frequency
Ready Salted	166
Salt & Vinegar	130
Cheese & Onion	228
Other	76



- (i) Calculate the relative frequency of each flavour. Give your answers correct to 2 decimal places.
 (ii) Explain why it is reasonable to use these figures to estimate the probability of the favourite crisp flavour of the next person to be asked.

(e) A card was picked from a standard pack of cards, and its suit was noted. The card was put back and the pack was shuffled. This was repeated 250 times. The results are shown in the following table.

Suit	Frequency
Spades	52
Hearts	67
Diamonds	61
Clubs	70



Find the relative frequency of

- (i) Spades (ii) Hearts (iii) Diamonds (iv) Clubs.
 (v) What is the sum of all the relative frequencies?

(f) The following table shows the number of pictures on each page of a newspaper.

Number of pictures	Tally Marks	Frequency
0		10
1		7
2		9
3		6
4		6
5		4
6		6

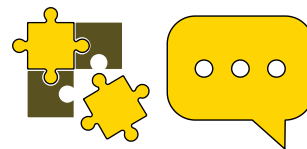


- (i) How many pages does the newspaper have?
 (ii) Find the relative frequency of one picture appearing on a page of the newspaper.

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Combined Events



Combined events involve two or more events that occur together.

For example,

- Throwing a penny and rolling a die at the same time;
- Rolling a die and then rolling it again;
- Spinning a spinner that shows the numbers 1-8, and then randomly choosing a card from a standard deck of playing cards.



Independent Events

Two events are **independent** if the result of the first event does *not* affect the probability of the second event.

For example, when throwing the same die twice, the fact that the die has landed on 6 the first time does not affect the probability of obtaining a 6 the second time.

For independent events A and B ,

$$P(A \cap B) = P(A) \times P(B)$$

The **multiplication** rule for independent events

This means the probability of A **and** B occurring is the product of A 's probability and B 's probability.

For example, the probability of obtaining heads when throwing a coin and 4 when rolling a die is $\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$.

Dependent Events

Two events are **dependent** if the result of the first event *does* affect the probability of the second event.

For example, when choosing two cards from a standard deck of playing cards, **without replacement**, the first card chosen affects the probability of the second card chosen. If the first card is a king, then the probability of the second card being a king is $\frac{3}{51}$, not $\frac{4}{52}$ as for the first card.

For dependent events A and B ,

$$P(A \cap B) = P(A) \times P(B|A)$$

The **multiplication** rule for dependent events

This means the probability of A **and** B occurring is the product of A 's probability with the probability of B occurring, given that A has already occurred. (The probability $P(B|A)$ is a **conditional** probability.)

For example, the probability of choosing two kings from a standard deck of playing cards is $\frac{4}{52} \times \frac{3}{51} = \frac{12}{2652}$, or $\frac{1}{221}$.

Mutually Exclusive Events

Two events are **mutually exclusive** if they cannot happen at the same time. For example, when rolling a normal fair die, the events "landing on an odd number" and "landing on 6" are mutually exclusive, because 6 is not an odd number. On the other hand, the events "landing on an odd number" and "landing on a prime number" are not mutually exclusive, because the numbers 3 and 5 are both odd and prime.

For mutually exclusive events A and B ,

$$P(A \cup B) = P(A) + P(B)$$

The **addition** rule for mutually exclusive events

This means the probability of A **or** B occurring is the sum of A 's probability and B 's probability.

For example, the probability of obtaining an odd number or a 6 when rolling a normal fair die is $\frac{3}{6} + \frac{1}{6} = \frac{4}{6}$, or $\frac{2}{3}$.

Exercise 6

For the following pairs of events, decide whether the events are independent or dependent.

- (a) Obtaining 'heads' when throwing a coin and obtaining '6' when rolling a normal fair die.
- (b) Choosing two queens when choosing two cards from a standard deck of playing cards, without replacement.
- (c) Choosing two queens when choosing a card from a standard deck of playing cards, returning the card to the deck, and then choosing another card.
- (d) Choosing two red balls when one ball is chosen from a bag containing 4 red balls and 5 blue balls, and the other is chosen from a bag containing 5 red balls and 4 blue balls.
- (e) Choosing two red balls from a bag containing 4 red balls and 5 blue balls, without replacement.

**Exercise 7**

For the following pairs of events, decide whether the events are mutually exclusive or not.

- (a) Obtaining 'heads' when throwing a coin and obtaining '6' when rolling a normal fair die.
- (b) When rolling a normal fair die, obtaining
 - (i) a number less than 3 and a number greater than 3.
 - (ii) an even number and a number greater than 4.
 - (iii) an odd number and a square number.
 - (iv) an even number and a cube number.
 - (v) an even number and a prime number.
- (c) Obtaining 'heads' when throwing a coin and 'heads' when throwing another coin.
- (d) On a spinner showing the numbers 1-10, landing on a multiple of 3 and landing on a multiple of 4.
- (e) On a spinner showing the numbers 1-12, landing on a multiple of 3 and landing on a multiple of 4.

**Example**

Christine has a normal fair die. She rolls the die twice.
Calculate the probability that the die lands on 3 each time.

Answer: The result of the first roll does not affect the second roll, so the events are independent.
We can therefore use the multiplication rule for independent events.

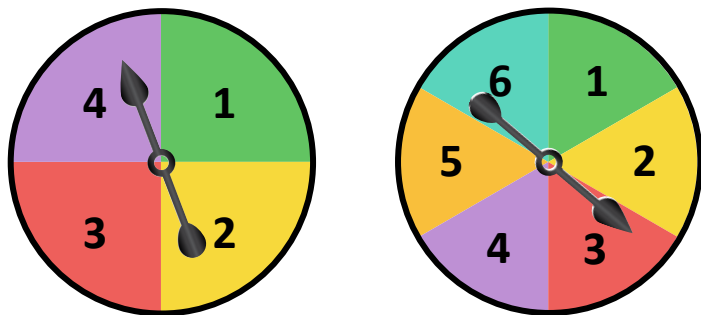
$$\begin{aligned}
 P(3 \text{ the first time, } 3 \text{ the second time}) &= P(3 \text{ the first time}) \times P(3 \text{ the second time}) \\
 &= \frac{1}{6} \times \frac{1}{6} \\
 &= \frac{1}{36}
 \end{aligned}$$

Exercise 8

- (a) David has a normal fair die. He rolls it twice.
Calculate the probability that he rolls 5 the first time and 1 the second time.
- (b) Fiona has a fair coin. She throws it 3 times.
Calculate the probability that she obtains 3 'tails'.
- (c) Rachel has a normal fair die and a fair coin. She rolls the die and throws the coin.
Calculate the probability that the die lands on 4 and the coin lands on 'tails'.



(d) A game is played where the two spinners below are spun at the same time.



- (i) What is the probability that the spinner on the left stops at 4 and the spinner on the right stops at 3?
 - (ii) What is the probability that the spinner on the left stops at an even number and the spinner on the right stops at an odd number?
 - (iii) What is the probability that both spinners stop at the same number?
- (e) A bag contains 12 counters. 3 are red, 4 are blue and the rest are green. Another bag contains 15 counters. 7 are red, 2 are blue and rest are green. What is the probability of choosing
- (i) A red counter from the first bag and a blue counter from the second bag?
 - (ii) A blue counter from the first bag and a red counter from the second bag?
 - (iii) Two green counters?

2

Exercise 9

(a) A bag contains 8 counters. 3 are red and 5 are blue. 2 counters are chosen randomly from the bag, without replacement. What is the probability of choosing

- (i) Two red counters?
- (ii) A red counter first then a blue counter?
- (iii) A blue counter first then a red counter?
- (iv) Two blue counters?



(b) Tom shuffles a standard pack of playing cards before choosing, at random, two cards from the pack without replacement. What is the probability that Tom chooses

- (i) The king of hearts first and then the queen of diamonds?
- (ii) Two hearts?
- (iii) Two cards showing 7?
- (iv) A red card first and then a black card?



(c) A class in a school has 15 girls and 12 boys. Two names are chosen at random from the register in order to represent the class in a survey. What is the probability of choosing two girls?

(d) The probability that James watches television tonight is 0.6. If James watches television tonight, the probability that he reads a book tonight is 0.2. If James does not watch television tonight, the probability that he reads a book tonight is 0.7. What is the probability that James, tonight,

- (i) Watches television and reads a book?
- (ii) Does not watch television and reads a book?
- (iii) Watches television and does not read a book?
- (iv) Does not watch television and does not read a book?

(e) An office has 20 workers. 7 of the workers wear spectacles. Two workers are chosen at random. What is the probability that the workers chosen do **not** wear spectacles?

Exercise 10

2

(a) Mariel rolls a normal fair die. What is the probability that Mariel's die lands on

- (i) 2 or 3?
- (ii) An even number or 5?
- (iii) A number less than 3 or a number greater than 4?
- (iv) A prime number or a square number?



(b) Heulwen shuffles a standard deck of playing cards before randomly choosing one card from the pack. What is the probability that Heulwen chooses

- (i) Hearts or spades?
- (ii) 3 or 5?
- (iii) A face card or a card less than 6?
- (iv) A black card or a red card?

(c) One number is chosen at random from the grid on the right. What is the probability that the number is:

- (i) 4 or 5?
- (ii) A multiple of 5 or a multiple of 7?
- (iii) A factor of 8 or a two-digit number?
- (iv) A cube number or a prime number?

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20

(d) Gareth rolls a fair die with 12 faces. What is the probability that the die lands on

- (i) 1 or 12?
- (ii) An odd number or a multiple of 4?
- (iii) A square number or 7?
- (iv) A multiple of 3 or a factor of 11?

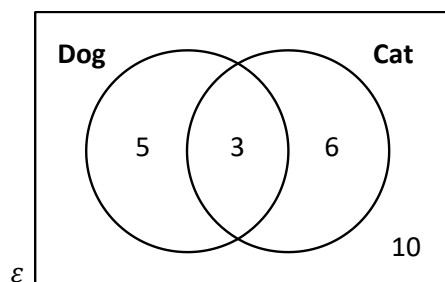


(e) A bag contains 10 counters. 3 are red, 2 are blue and the rest are purple. One counter is chosen at random from the bag. What is the probability that the chosen counter is red or purple?

Venn Diagrams**Example**

In a class of 24 learners, 8 own a dog, 9 own a cat and 3 own both a dog and a cat. One learner is chosen at random from the class. What is the probability that the chosen learner does not own either a dog or a cat?

Answer: To begin, let us draw a Venn diagram to illustrate the situation.



We see from the Venn diagram that 10 learners in the class do not own either a dog or a cat. Therefore, the probability of choosing a learner that does not own either a dog or a cat is $\frac{10}{24}$, or $\frac{5}{12}$.

Exercise 11

2

(a) In a survey someone asked 40 pupils whether they liked football or rugby.

32 pupils liked football.

25 pupils liked rugby.

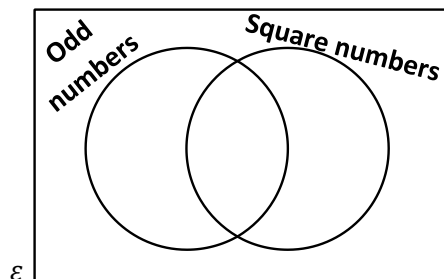
22 pupils liked both sports.

(i) Draw a Venn diagram to illustrate this information.

(ii) What is the probability that a randomly chosen pupil from this group likes rugby only?



(b) (i) Place the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 into the correct positions in the following Venn diagram.



(ii) A number is chosen at random from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

Find the probability that the chosen number

(I) is an odd number; (II) is an odd number and a square number; (III) is neither odd nor a square number.

(c) An ice cream company conducted a taste test in a supermarket. 110 people took part in the survey, where they were asked to taste strawberry, vanilla and chocolate ice creams.

65 people stated that they liked the strawberry ice cream.

80 people stated that they liked the vanilla ice cream.

60 people stated that they liked the chocolate ice cream.

55 people stated that they liked both the strawberry and vanilla flavours.

50 people stated that they liked both the vanilla and chocolate flavours.

45 people stated that they liked both the strawberry and chocolate flavours.

40 people stated that they liked all 3 flavours.

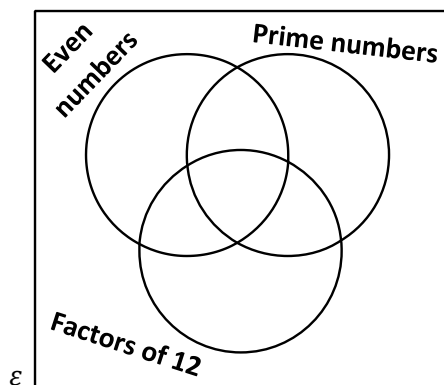
(i) Draw a Venn diagram to illustrate this information.

(ii) What is the probability that a randomly selected person from this group liked

(I) Vanilla only; (II) None of the three flavours; (III) Vanilla or Strawberry?



(d) (i) Place the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 into the correct positions in the following Venn diagram.



(ii) A number is chosen at random from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

Find the probability that the chosen number is

(I) an even number; (II) a prime number and a factor of 12; (III) a prime number but not an even number.

Sample Space Diagrams and Dependent Events



Example

Each one of the numbers 1, 2, 3, 4, 5, 6 are written on cards.



Two out of the six cards are chosen at random, **without** replacement.

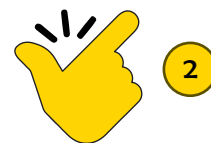
Find the probability that the **sum** of the numbers shown on the chosen cards is less than 10.

Answer: We can show all the possible combinations in a sample space diagram.

		Number on the second card					
Number on the first card	+	1	2	3	4	5	6
	1	Impossible	3	4	5	6	7
	2	3	Impossible	5	6	7	8
	3	4	5	Impossible	7	8	9
	4	5	6	7	Impossible	9	10
	5	6	7	8	9	Impossible	11
	6	7	8	9	10	11	Impossible

There are $6 \times 5 = 30$ different combinations, and of these, 26 give a sum that is less than 10. Therefore, the answer to the question is $\frac{26}{30}$, or $\frac{13}{15}$.

Exercise 12



(a) Each one of the numbers 1, 2, 3, 4, 5, 6 are written on cards.



Two out of the six cards are chosen at random, **without** replacement.

Find the probability that the **product** of the numbers shown on the chosen cards is less than 10.

(b) Each one of the numbers 1, 2, 2, 3, 3, 3 are written on cards.



Two out of the six cards are chosen at random, **without** replacement.

- Find the probability that the sum of the numbers shown on the chosen cards is less than 6.
- Find the probability that the sum of the numbers shown on the chosen cards is exactly 3.
- Find the probability that the two chosen numbers are the same.
- Find the probability that the two chosen numbers are different.
- Find the probability that the card showing 1 is chosen.

(c) A different factor of 24 is shown on each of 8 cards.

--	--	--	--	--	--	--	--

2 cards are chosen at random **without** replacement.

Find the probability that the **positive difference** between the two numbers on the chosen cards is

- (i) 4;
- (ii) an odd number;
- (iii) a one-digit number.

(d) A different factor of 18 is shown on each of 6 cards.

--	--	--	--	--	--

In a game, a player chooses two of the above cards at random, **without** replacement.

The score for the game is the **largest** of the two chosen numbers.

- (i) Draw a sample space diagram to show all the possible outcomes.
- (ii) Find the probability that the score is 18.
- (iii) A player wins the game if the score is 18.
If 120 people play the game once each, how many of them would you expect to win?
- (iv) It costs 20p to play the game once.
The prize for winning the game is 50p.
If 120 people play the game once each, how much profit would you expect the game to make?



Evaluation

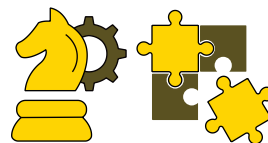
Key words	Corrections	I am happy with...	I need to revise...

Tree Diagrams

Tree diagrams are used to show **combinations of two or more events**.

Each branch is labelled on the right with the **result** and in the middle with the **probability**.

It is possible to use tree diagrams to show independent events (intermediate tier) or dependent events (higher tier).

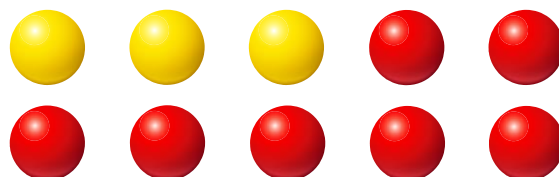


Example

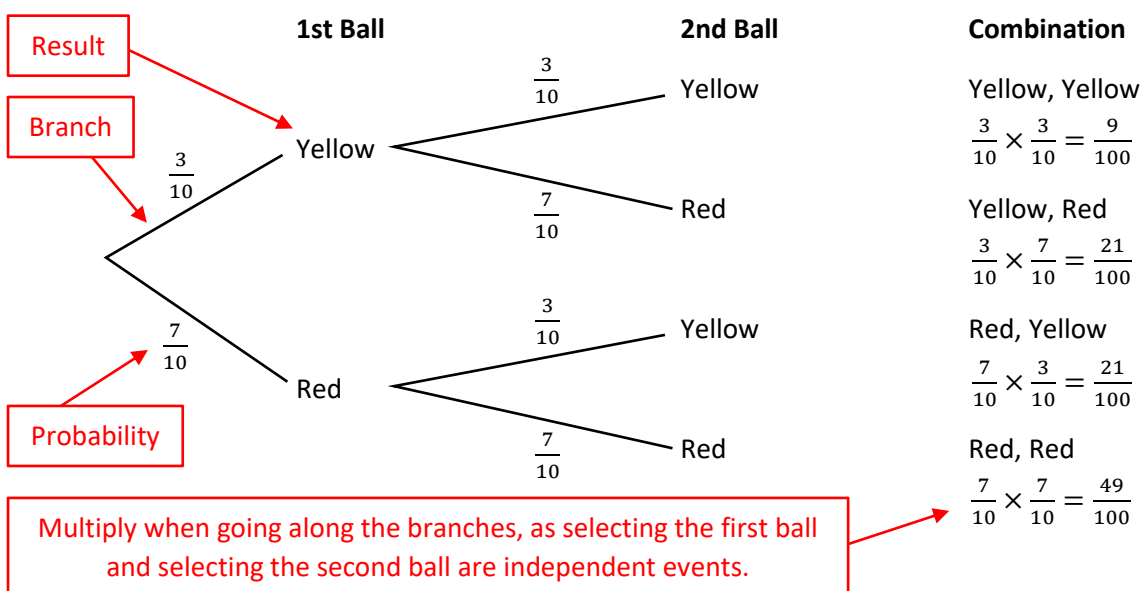
A bag contains ten balls that are indistinguishable apart from their colour. Three of the balls are yellow and the rest red. A ball is selected at random from the bag, its colour noted and returned to the bag. A second ball is chosen and its colour is also noted.

Use a tree diagram to calculate the probability that

- (a) the two selected balls are yellow;
- (b) the two selected balls have different colours;
- (c) the two selected balls are not both yellow.



Answer: Step 1: Draw a tree diagram to illustrate the situation.



Step 2: Consider which paths in the tree diagram need to be considered in order to answer the questions.

(a) The first path in the tree diagram (yellow, yellow) shows the situation in which both selected balls are yellow. The probability for this is $\frac{9}{100}$.

(b) The paths that show balls of different colours are the second path (yellow, red) or the third path (red, yellow). Because these paths are mutually exclusive, it is possible to add the probabilities to obtain the final answer:
 $\frac{21}{100} + \frac{21}{100} = \frac{42}{100}$. Note that it would be possible to simplify this fraction to obtain $\frac{21}{50}$ but, unless the question notes otherwise, fractions do not have to be simplified in questions on probability.

(c) There are two ways of answering this question:

(i) Consider all paths in the tree diagram that do **not** give two yellow balls: $\frac{21}{100} + \frac{21}{100} + \frac{49}{100} = \frac{91}{100}$;

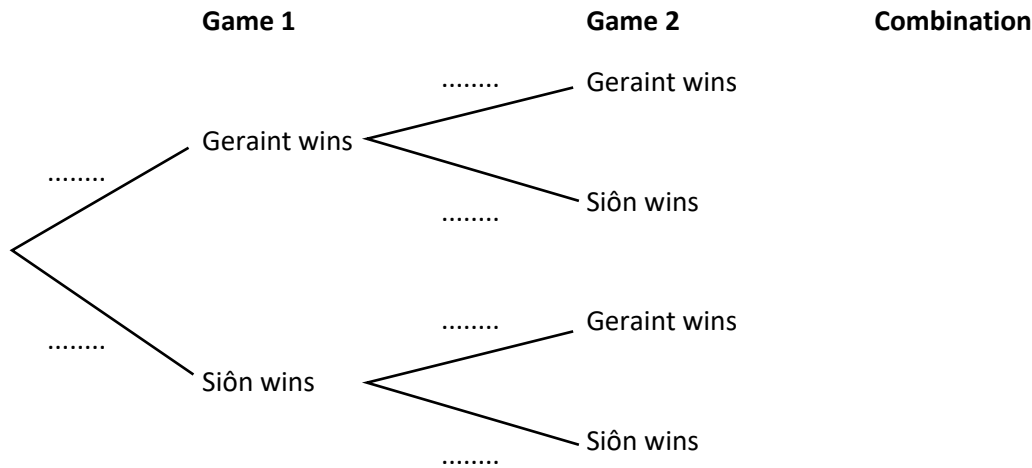
(ii) Consider the probability of selecting two yellow balls ($\frac{9}{100}$, the answer to part (a)) and subtract from 1:

$$1 - \frac{9}{100} = \frac{91}{100}.$$

Exercise 13

Whenever Geraint and Siôn play a game of 'FIFA' on a games console, the probability that Geraint wins is 0.4.

(a) Complete the following tree diagram to show the probabilities of what can happen when Geraint and Siôn play two games of 'FIFA'.



- (b) Calculate the probability that Siôn wins both games.
- (c) Calculate the probability that Siôn wins exactly one of the games.
- (d) Calculate the probability that Siôn wins neither of the games.

Exercise 14

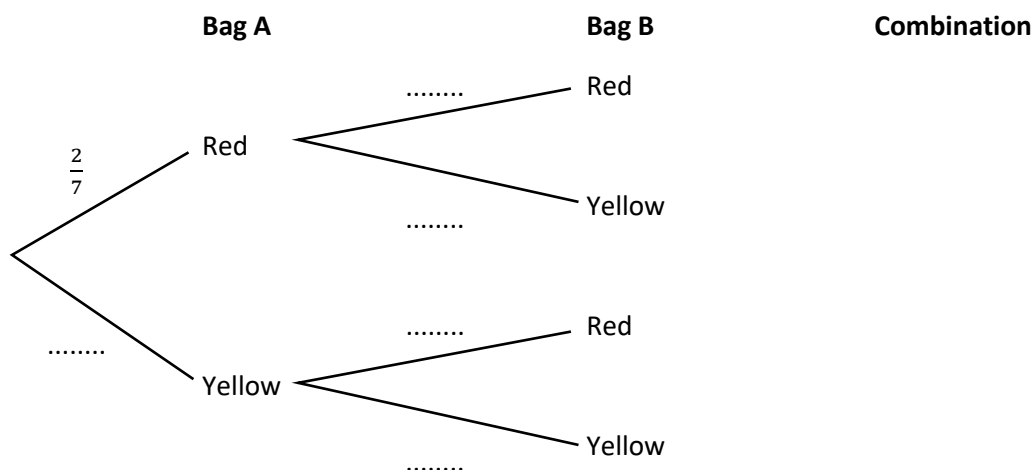
There are two bags in a game, and both bags contain coloured balls.

Bag A contains 2 red balls and 5 yellow balls.

Bag B contains 3 red balls and 2 yellow balls.

A player randomly chooses one ball from each bag.

(a) Complete the following tree diagram.



- (b) Find the probability of choosing two red balls.
- (c) Find the probability of choosing a ball of each colour.
- (d) Find the probability of not choosing two red balls.

Applying

2



2

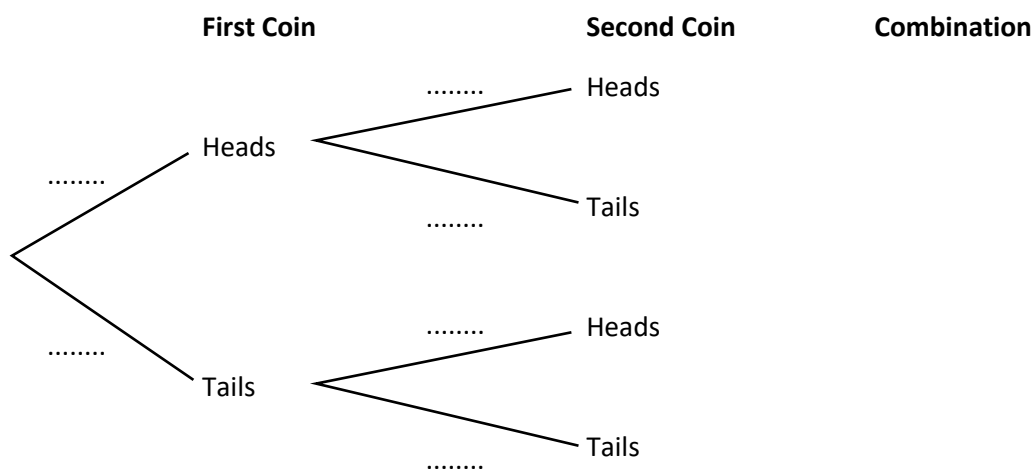


Exercise 15

2

Two biased coins are thrown. The probability of obtaining heads with the first coin is 70%.
The probability of obtaining heads with the second coin is 60%.

(a) Complete the following tree diagram.



(b) Calculate the probability that one coin shows 'Heads' and the other coin shows 'Tails'.

Example

The probability that Elin posts a picture on *Instagram* over the weekend is 0.4

The probability that Elin goes shopping over the weekend is independent of her posting a picture on Instagram over the weekend. The probability of Elin posting a picture on Instagram over the weekend, and going shopping over the weekend, is 0.12.

(a) Complete the following tree diagram.

(b) Find the probability that Elin posts a picture on Instagram over the weekend but does not go shopping over the weekend.

Answer: (a) The red text is the text that has been added.

Note in this example that the probability of one of the **combinations** has been given, so we must perform a **division** sum in order to calculate the probability of shopping over the weekend.



(b) $0.4 \times 0.7 = 0.28$ (the second path).

Exercise 16

Megan lives 5 miles from her work, and either cycles or drives to work.

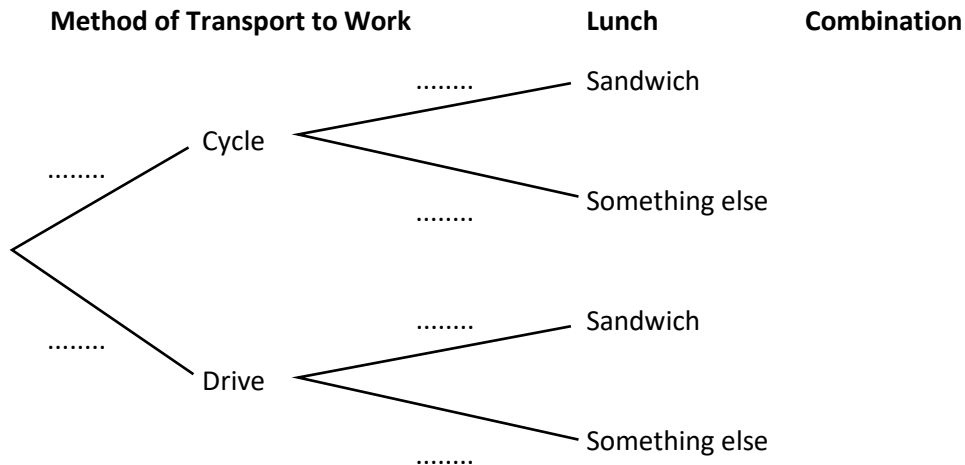
The probability that she cycles to work is 0.3.

The probability that she cycles to work and has a sandwich for lunch is 0.24.

In Megan's case, her method of transport to work is independent of what she has for lunch.

(a) Find the probability that Megan has a sandwich for lunch.

(b) Complete the following tree diagram.



2



(c) Find the probability that Megan drives to work and eats something apart from a sandwich for lunch.

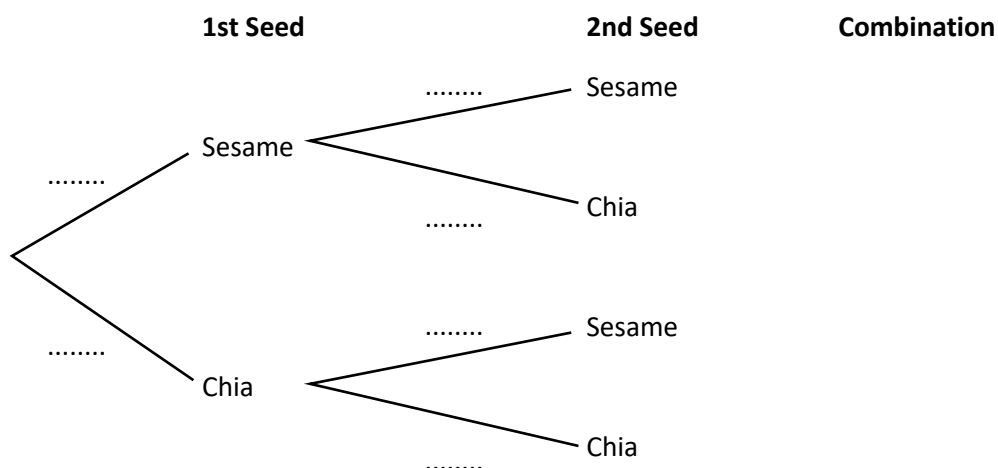
Exercise 17

A bag contains a large number of two types of seed, namely sesame and chia.

Two seeds are randomly chosen from the bag.

The probability that two sesame seeds are chosen is 0.49.

(a) Complete the following tree diagram.



2

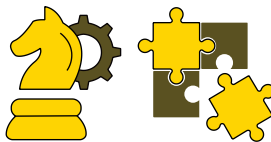


(b) Calculate the probability that one seed of each type is selected.

(c) Calculate the probability that two chia seeds are chosen.

(d) Why are the words "large number" needed at the start of the question?

(e) If three seeds were chosen from the bag, what would be the probability of choosing 3 sesame seeds?



Dependent Events

Example

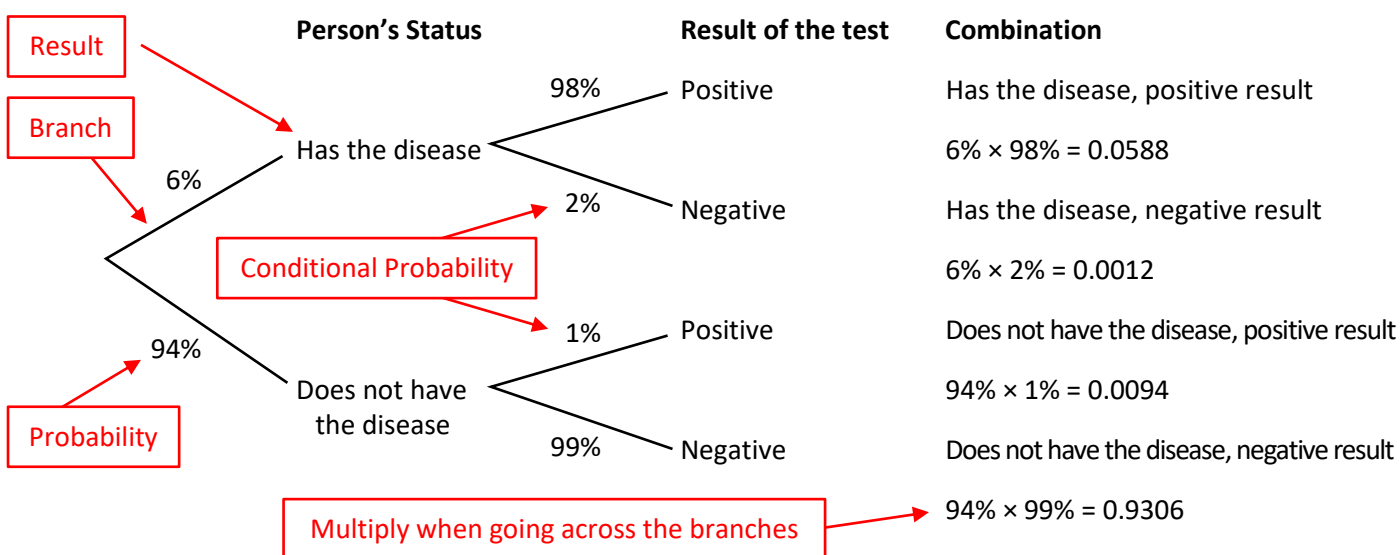
A hospital tests patients for a specific disease. If the person has the disease, the test returns a “positive” result. If the person does not have the disease, the test returns a “negative” result. The test isn’t perfect however:

- 98% of patients with the disease receive a “positive” result;
- 1% of patients without the disease receive a “positive” result;
- 6% of the population have the disease under consideration.

Use a tree diagram to calculate the probability

- that a randomly chosen person receives a “positive” result in the test;
- that the incorrect result is given to a person taking the test;
- that the correct result is given to a person taking the test.

Answer: Step 1: Draw a tree diagram to illustrate the situation.



Step 2: Consider which paths in the tree diagram need to be considered in order to answer the questions.

(a) Two paths give a positive result: either “has the disease, positive” (probability 0.0588) or “does not have the disease, positive” (probability 0.0094). We add these two probabilities to obtain the answer (they are mutually exclusive events): $0.0588 + 0.0094 = 0.0682$.

(b) Two paths give an incorrect result: either “has the disease, negative” (probability 0.0012) or “does not have the disease, positive” (probability 0.0094). We add these two probabilities to obtain the answer (they are mutually exclusive events): $0.0012 + 0.0094 = 0.0106$.

(c) There are two ways of answering this question:

- Consider all the paths in the tree diagram that give a correct result (“has the disease, positive” or “does not have the disease, negative”): $0.0588 + 0.9306 = 0.9894$;
- Consider the probability of obtaining an incorrect result (0.0106, the answer to part (b)) and subtract from 1: $1 - 0.0106 = 0.9894$.

Challenge!

Use the internet to investigate the meaning of the terms *Type I Error* and *Type II Error*. What is the connection between these statistical terms and the above example?

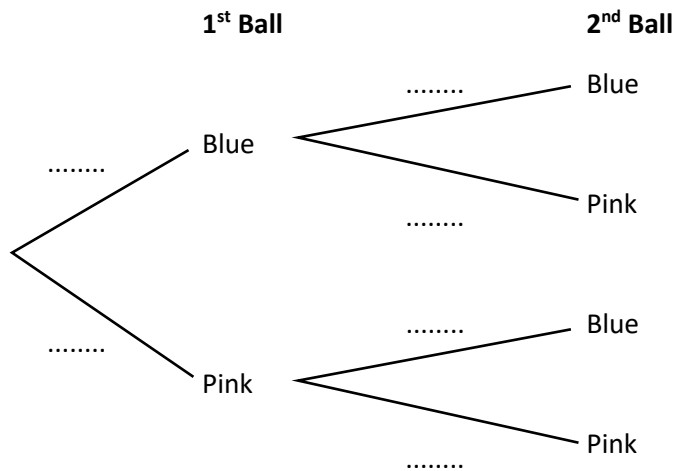


Exercise 18

A bag contains 4 blue balls and 6 pink balls.

Billy randomly chooses two balls from the bag **without replacement**.

(a) Complete the following tree diagram.



Combination



(b) Find the probability of choosing two pink balls.

(c) Find the probability of choosing one ball of each colour.

Exercise 19

A bag contains 7 yellow beads, 3 white beads and one black bead.

Two beads are chosen randomly from the bag **without replacement**.

(a) Calculate the probability that both beads are yellow.

(b) Calculate the probability that at least one white bead is chosen.

Exercise 20

A box contains 3 banana yogurts, 4 blueberry yogurts and 5 cherry yogurts.

Three yogurts are randomly chosen from the box **without replacement**.

Calculate the probability that at least one of the chosen yogurts is a cherry yogurt.

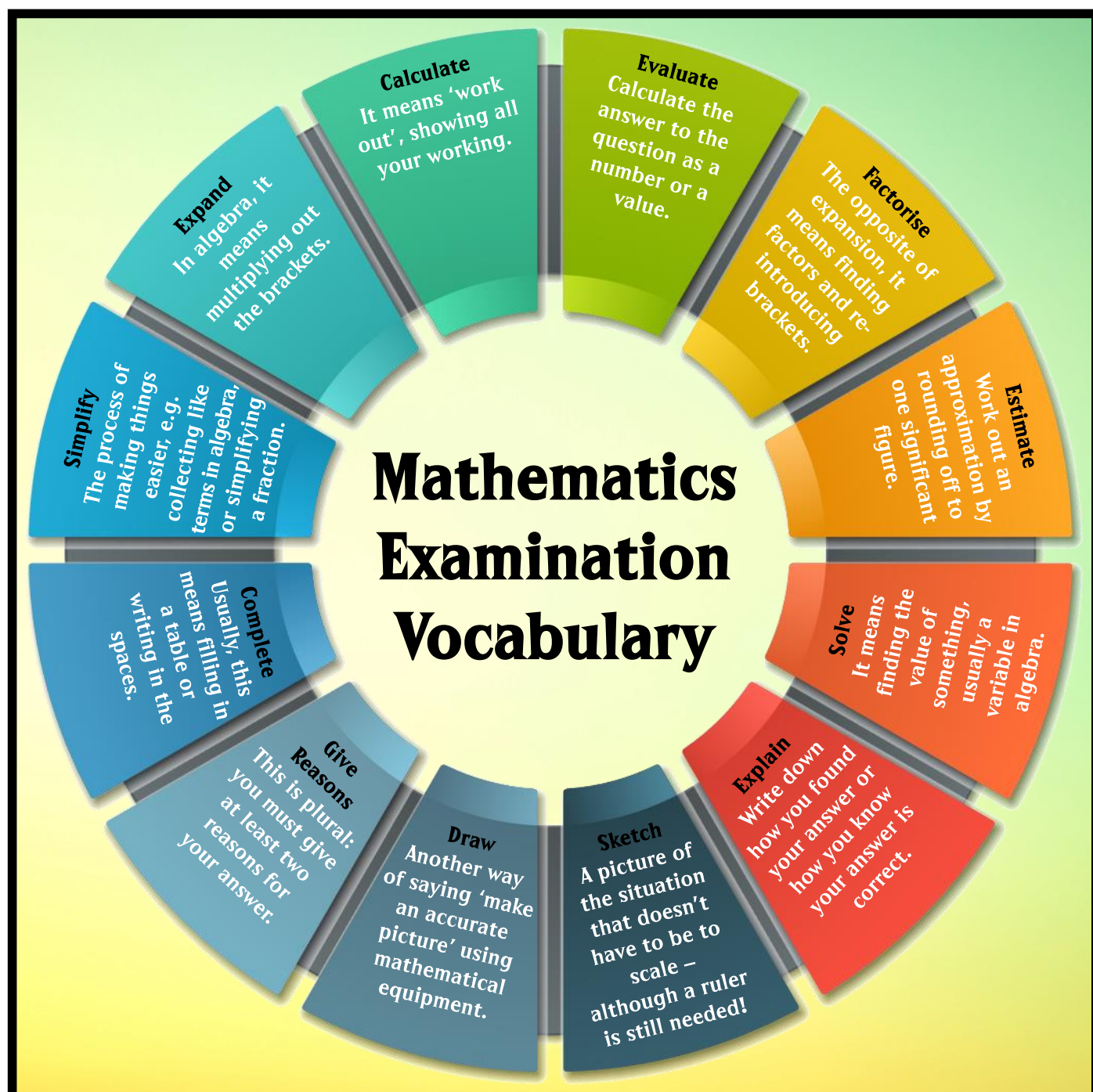
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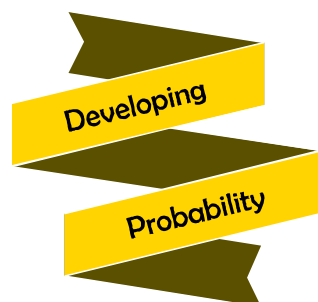


2



Key words	Corrections	I am happy with...	I need to revise...





Name:

Percentage in the test:

	I know this. 	I need to revise this. 	Question in the test:	Correct in the test?
I have revised the previous work on probability , e.g. sample space diagrams, expected frequency.			2, 6	
I know how to calculate relative frequency from experimental data .			1	
I know how to draw a graph showing relative frequency against the number of trials.			1	
I understand the more trials are held, the better the relative frequency is as an estimate of the probability.			1	
I can tell the difference between independent events and dependent events .			3	
I can tell the difference between mutually exclusive events and events that are not mutually exclusive .			3	
I can use the multiplication rule for independent events .			3	
I can use the multiplication rule for dependent events .			9	
I can use the addition rule for mutually exclusive events .			4	
I can answer questions on probability that involve Venn diagrams .			5	
I can answer questions on probability that involve sample space diagrams showing dependent events .			6	
I can draw and use tree diagrams for independent events and dependent events .			7, 9	
Given the probability of one of the combinations in a tree diagram, I can work backwards to fill in all the probabilities of the tree diagram.			8	



The Mathematics Department

11

Developing

Algebra 4

Higher Tier

Version

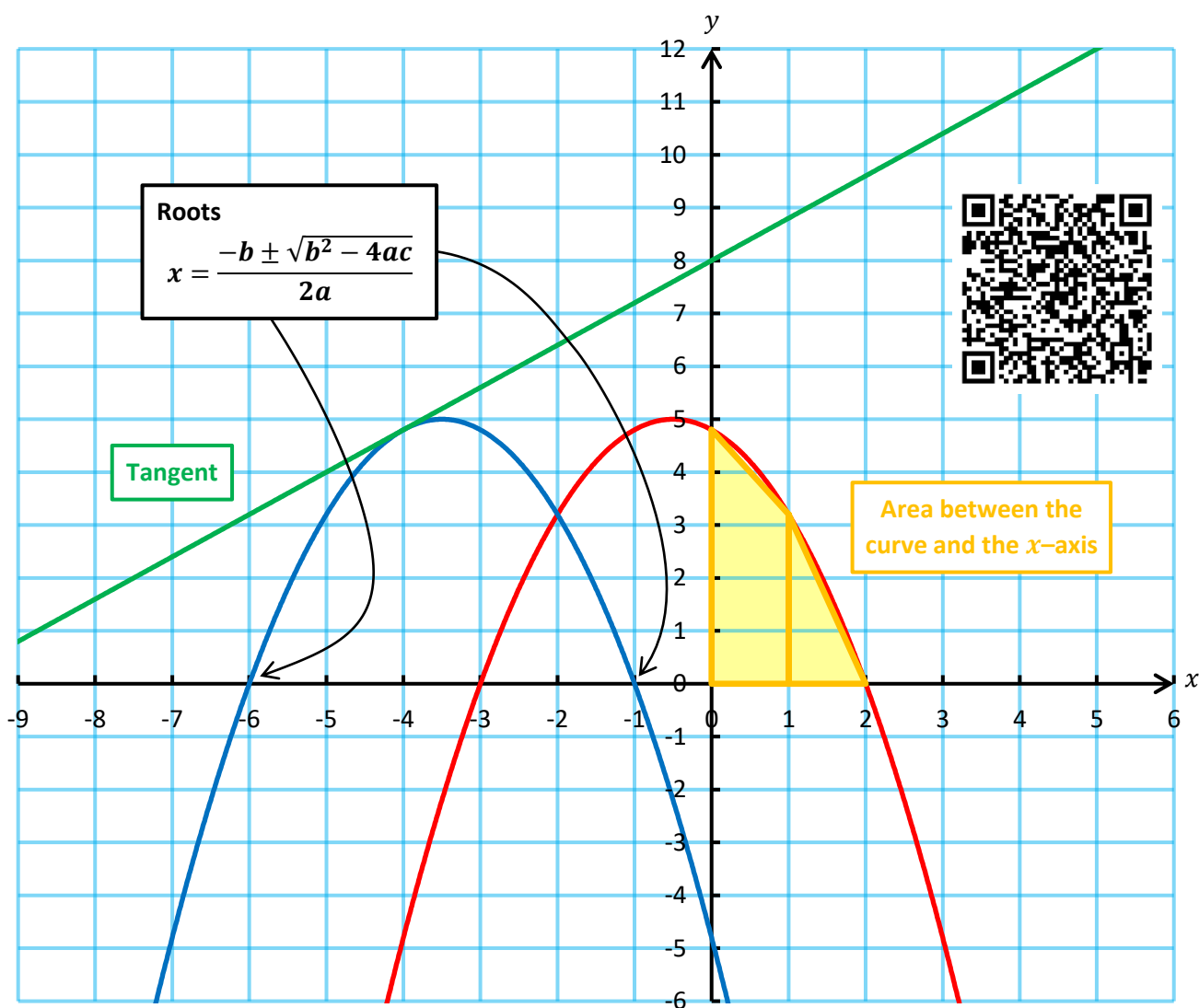
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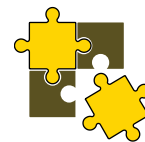
Name:

Contents

Chapter	Mathematics	Page Number
Pre-Calculus	Tangent to a function. Travel graphs. Displacement, Velocity and Acceleration. The area between a graph and the x -axis. The Trapezium Rule.	3
Further Changing the Subject	Revision of previous work. Changing the subject where the subject appears more than once.	12
The Quadratic Formula	Solving quadratic equations. Solving problems.	14
Algebraic Fractions	Numerical fractions. Algebraic fractions. Solving equations involving algebraic fractions.	17



Pre-Calculus

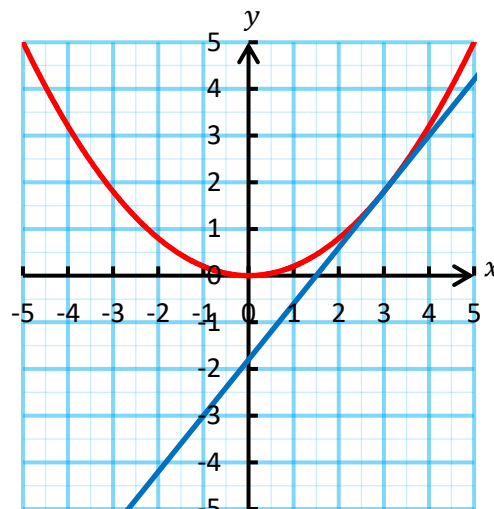


Tangent to a Function

Given a **non-linear function** (a function that is not a straight line), a **tangent** to a specific point on the function is a straight line that meets the point so that the gradient of the tangent is equal to the gradient of the function at that point.

Example

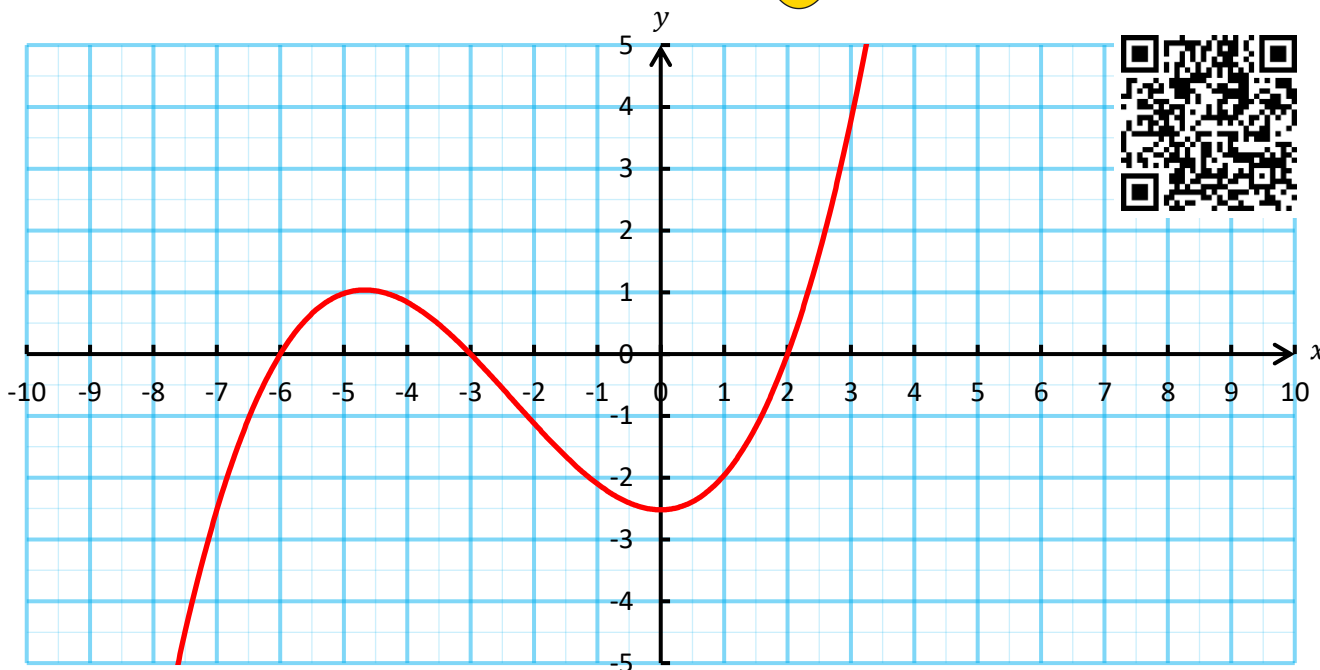
The graph on the right shows a tangent to the **red** curve at the point where $x = 3$. This tangent has a positive gradient.



Exercise 1

For the curve shown below, draw (by eye) a tangent for the points where

- (a) $x = -5$ (b) $x = 2.5$ (c) $x = 0$ (d) $x = -2$



The gradient of a tangent gives the **rate of change** for the point under consideration. That is, it represents how much the variable on the vertical axis (y) changes with respect to one unit of the unit on the horizontal axis (x). The steeper the tangent, the greater the rate of change.

For the example on the right, the gradient of the tangent is the change in the vertical distance divided by the change in the horizontal distance.

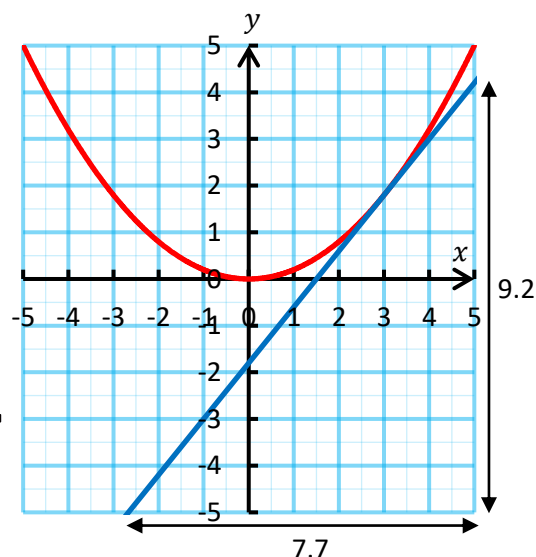
$$\text{Gradient of the tangent} = \frac{9.2}{7.7}$$

Gradient of the tangent = 1.19 to 2 decimal places.



Exercise 2

Calculate the gradient of your four tangents from Exercise 1.



Travel Graphs

We saw travel graphs for the first time in the *Movement with Sphero* workbook in year 9.

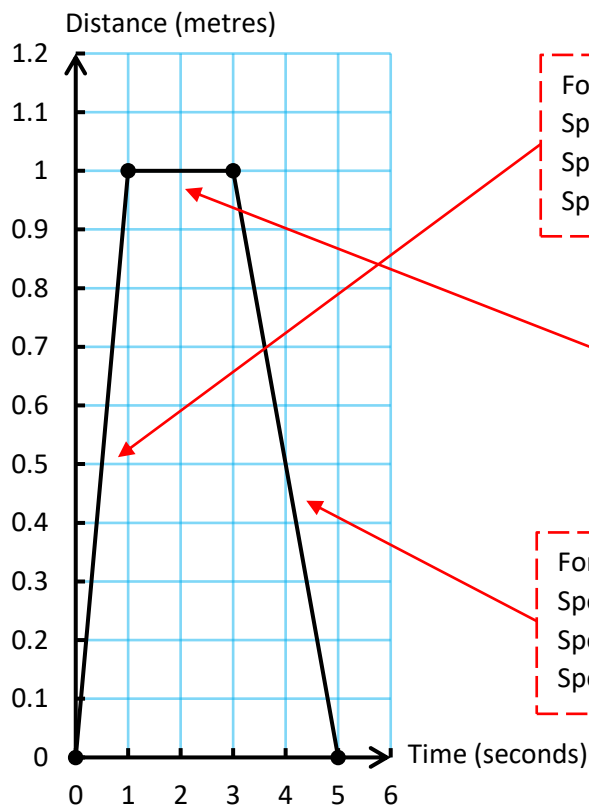
Example

The following graph shows the journey of a sphero over 5 seconds.

Watch the journey of the sphero here:



Distance-time graph for the motion of the sphero



For the **first** part of the journey,
 $\text{Speed} = \text{Change in distance} \div \text{Change in time}$
 $\text{Speed} = 1 \div 1$
 $\text{Speed} = 1 \text{ metre per second.}$

For the **second** part of the journey,
 $\text{Speed} = \text{Change in distance} \div \text{Change in time}$
 $\text{Speed} = 0 \div 2$
 $\text{Speed} = 0 \text{ metres per second.}$

For the **third** part of the journey,
 $\text{Speed} = \text{Change in distance} \div \text{Change in time}$
 $\text{Speed} = 1 \div 2$
 $\text{Speed} = 0.5 \text{ metres per second.}$



Speed is an example of a **scalar** measure, where the **direction of travel** is ignored. So, during the third part of the journey, where the sphero returns to its original position, and the gradient of the distance-time graph is negative, the speed remains positive. In order to consider the direction of travel, and therefore differentiate between positive and negative gradients in a distance-time graph, we need to consider new measures that are **vector** measures.

Displacement and Velocity

Distance and **displacement** are measures that measure how far an object is from a specific origin, but displacement takes into account the **direction of travel** as well as the distance from the origin. For example, at the end of the first part of the above journey, during which time the sphero moves away from the origin (the starting point), the distance and displacement are both 1 metre. For the final part of the journey however, where the sphero travels back to the origin, the distance travelled is 1 metre, but the displacement is -1 metre. The negative sign represents the direction of travel of the sphero, and also reflects the negative gradient of the distance-time graph for this part of the journey.

Velocity is the vector that corresponds to the scalar measure speed, and is calculated using the formula

$$\text{Velocity} = \frac{\text{Change in displacement}}{\text{Change in time}}$$

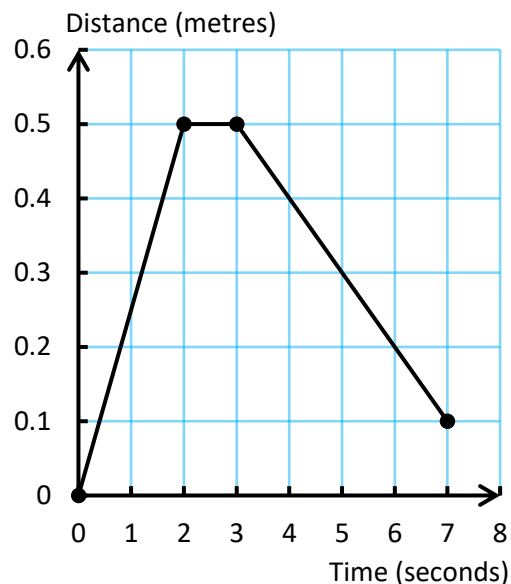
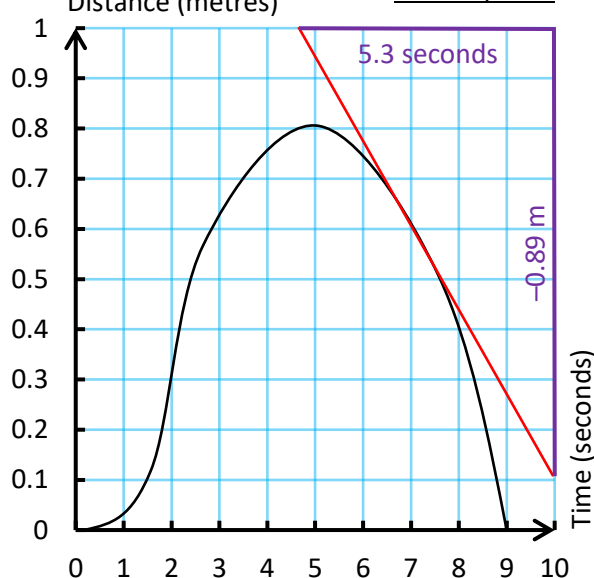
For the final part of the above journey, where the speed of the sphero is $1 \div 2 = 0.5$ metres per second, the velocity is $-1 \div 2 = -0.5$ metres per second. Again, the velocity takes into account that the distance-time graph has a negative gradient during the third part of the journey.

Exercise 3**Applying****3**

The graph on the right shows the journey of a sphero over 7 seconds. Complete the following table.

Part of the journey	Change in distance	Change in displacement	Change in time	Speed (scalar)	Velocity (vector)
1st					
2nd					
3rd					

In any **non-linear** distance-time graph, it is possible to estimate the velocity at any point by calculating the gradient of the tangent at that point.

Distance-time graph for Exercise 3Distance-time graph to show the journey of the sphero**Example**

The graph on the left shows the journey of a sphero over 9 seconds. Estimate the velocity of the sphero at time 7 seconds.

Answer: To begin, we draw (by eye) a tangent to the curve at 7 seconds. This tangent is shown in red on the graph.

Next, we complete a **right-angled triangle** around the tangent, in order to measure the change in displacement and change in time. From these we can calculate an estimate for the velocity:

$$\text{Velocity} = \frac{\text{Change in displacement}}{\text{Change in time}}$$

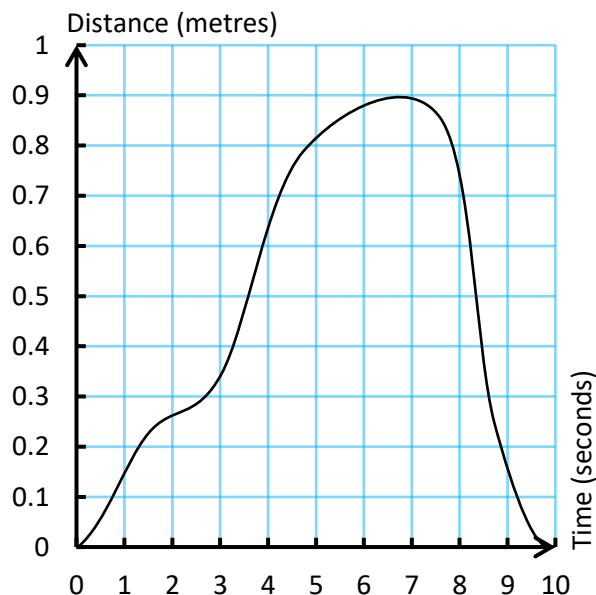
$$\text{Velocity} = \frac{-0.89}{5.3}$$

$$\text{Velocity} = -0.17 \text{ metres per second, to 2 decimal places.}$$

Exercise 4**3**

The graph on the right shows the journey of a sphero over 10 seconds.

- Estimate the velocity of the sphero at time 2 seconds.
- Estimate the velocity of the sphero at time 8 seconds.
- Estimate the speed of the sphero at time 2 seconds.
- Estimate the speed of the sphero at time 8 seconds.
- What is the total distance travelled by the sphero during its journey?
- What is the average speed for the entire journey?
- What is the average velocity for the entire journey?

Distance-time graph for Exercise 4

Acceleration

Whilst velocity is a vector that measures the rate of change of displacement with respect to time, **acceleration** is a vector that measures the rate of change of velocity with respect to time. We can use the formula

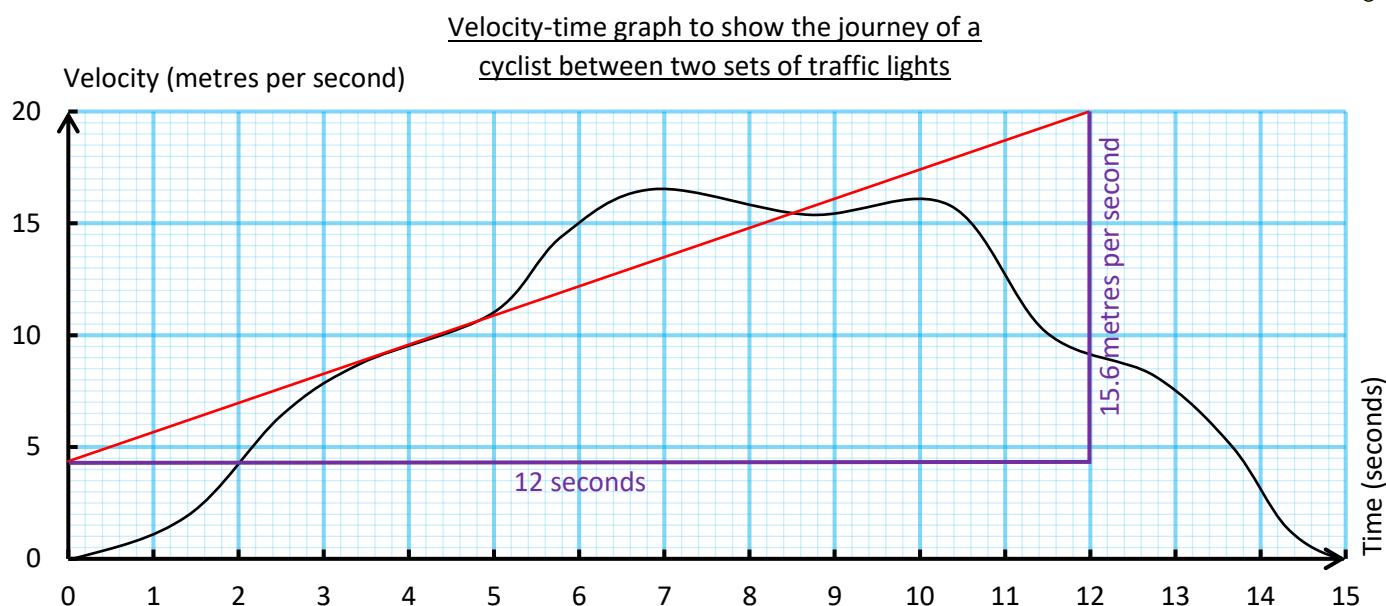


$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Change in time}}$$

to calculate acceleration, or we can estimate the gradient of a tangent to a point on a velocity-time graph.

Example

The following graph shows the velocity of a cyclist on a straight road between two sets of traffic lights. Calculate the acceleration of the cyclist at time 4 seconds. Give the units of your answer.



Answer: To begin, we draw (by eye) a tangent to the curve at time 4 seconds. This tangent is shown in red on the graph.

Next, we complete a right-angled triangle around the tangent, in order to measure the change in velocity and change in time. From these we can calculate an estimate for the acceleration:

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Change in time}}$$

$$\text{Acceleration} = \frac{15.6}{12}$$

$$\text{Acceleration} = 1.3 \text{ metres per second squared, or m/s}^2.$$

Units of Acceleration

The following table shows some common units for measuring acceleration.

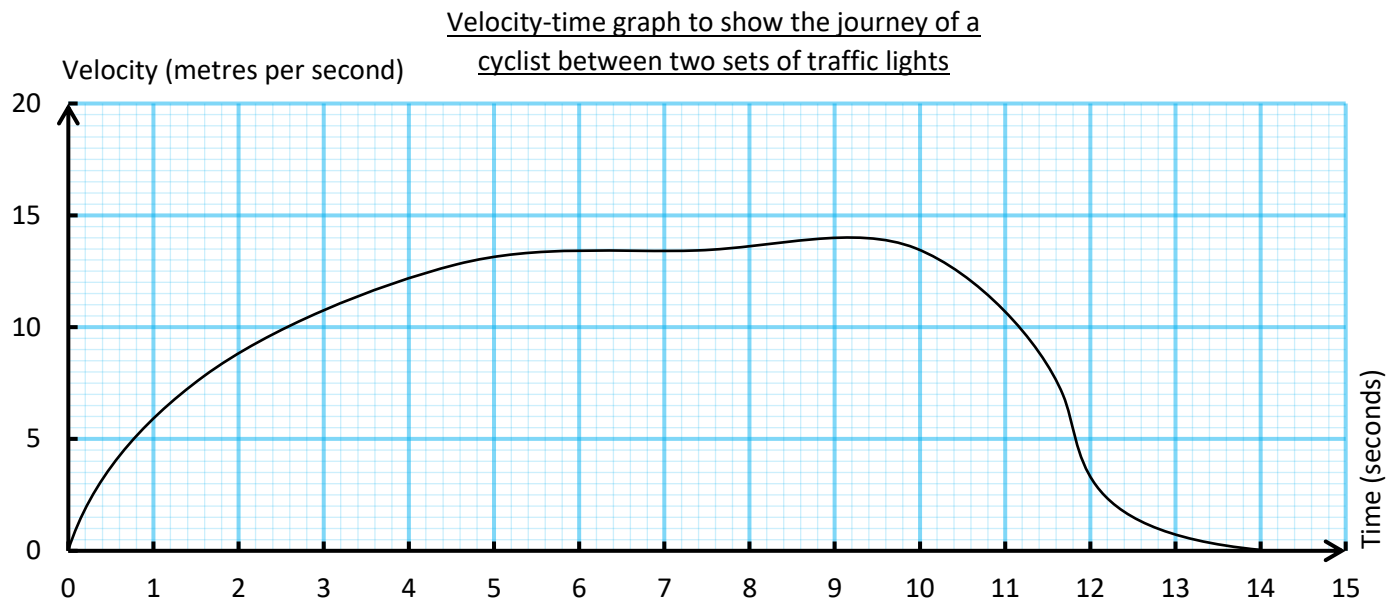
Displacement Unit	Time Unit	Velocity Unit	Acceleration Unit
Metres (m)	Seconds (s)	Metres per second (m/s)	Metres per second per second, or metres per second squared (m/s ²)
Kilometres (km)	Hours (h)	Kilometres per hour (km/h)	Kilometres per hour squared (km/h ²) or kilometres per hour per second
Miles (mi)	Hours (h)	Miles per hour (mph)	Miles per hour squared (mph ²) or miles per hour per second

The acceleration unit can depend on the horizontal axis unit of a velocity-time graph

Exercise 5

3

The following graph shows the velocity of a cyclist on a straight road between two sets of traffic lights.

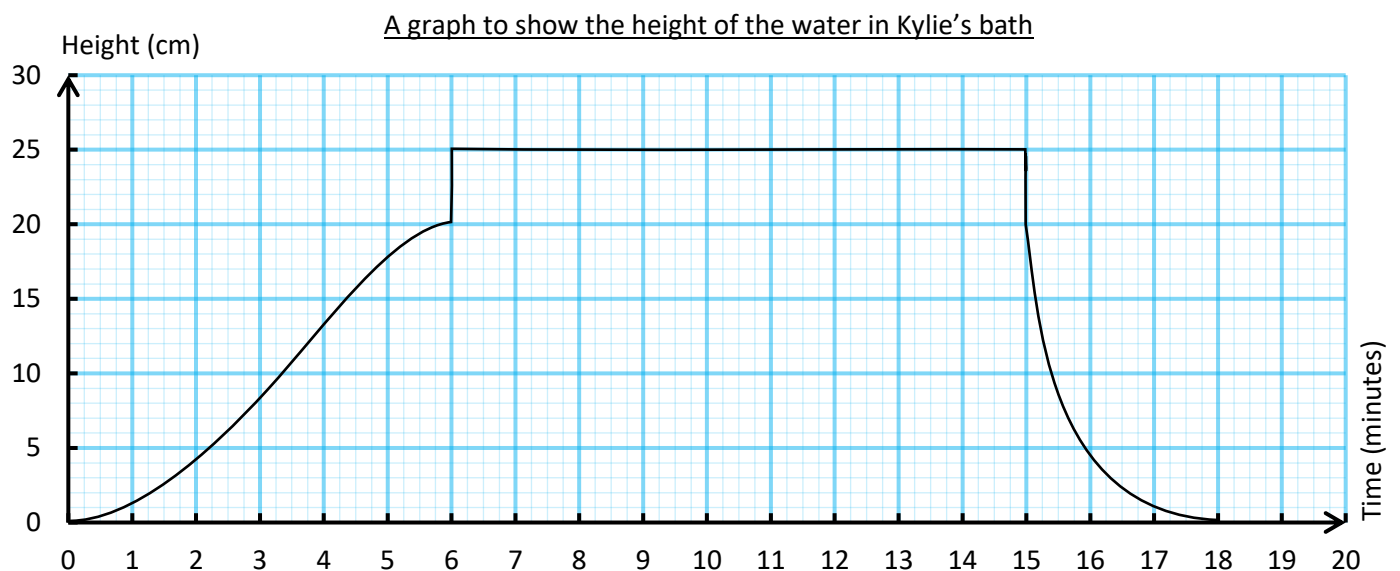


- Calculate the acceleration of the cyclist at time 2 seconds. Give the unit of your answer.
- Calculate the acceleration of the cyclist at time 11 seconds. Give the unit of your answer.
- What is the maximum velocity of the cyclist during the journey? At which time does this happen?
- Give the times at which the cyclist is travelling at a velocity of 5 metres per second.

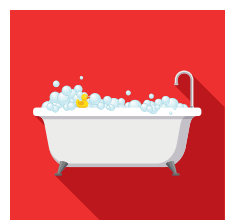
Exercise 6

3

The following graph shows the height of the water in a bath as Kylie takes a bath one Sunday night.

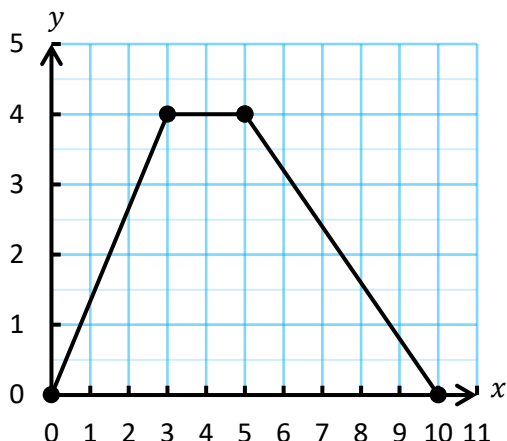


- What is the height of the water in the bath at time 2 minutes?
- What happens at time 6 minutes?
- How many minutes did Kylie spend in the bath?
- What is the rate of change of the height of the water in the bath at time 4 minutes?
- What is the rate of change of the height of the water in the bath at time 15 minutes 30 seconds?



The Area Between a Graph and the x -axis

If a graph uses straight lines only, then it is possible to find the area between the lines and the x -axis using formulae for the area of common 2-D shapes.



For example, considering the graph on the left, we can calculate the area between the lines of the graph and the x -axis by calculating the area of the trapezium:

$$\text{Formula for the area of a trapezium} = \frac{1}{2}(a + b)h$$

$$2 + 10 = 12$$

$$12 \div 2 = 6$$

$$6 \times 4 = 24$$

Calculating $a + b$

Calculating $\frac{1}{2}(a + b)$

Calculating $\frac{1}{2}(a + b)h$



So, the required area is 24 square units.

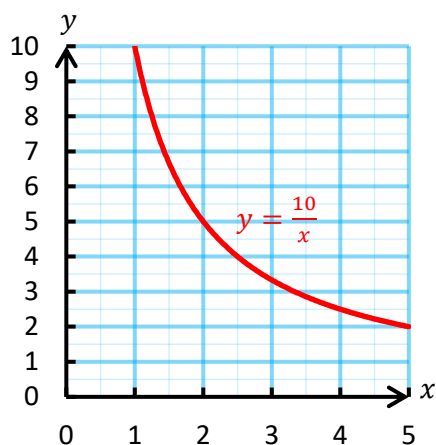
Exercise 7

For the graph shown on the right, find the area between the lines of the graph and the x -axis.

The Trapezium Rule

If we have a non-linear graph, then we cannot use a method similar to the above method to find the area between the graph and the x -axis – we must estimate the area using the **trapezium rule**.

Example



The graph on the left shows the function $y = \frac{10}{x}$ between $x = 1$ and $x = 5$. Use the ordinates $x = 1, x = 2, x = 3, x = 4$ and $x = 5$ to estimate the area enclosed by the curve $y = \frac{10}{x}$ and the x -axis, between $x = 1$ and $x = 5$.

Answer: The first step is to add vertical lines to the graph corresponding to the 5 ordinates, and then to complete the 4 trapezia formed by the ordinates. We can then read from the graph (or calculate) the values of the function for each of the five ordinates.

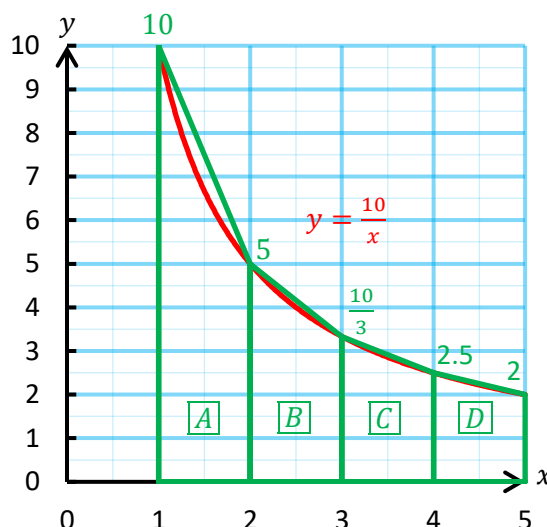
The estimate of the area between the curve and the x -axis is the total area of the four trapezia.

A $5 + 10 = 15$
 $15 \div 2 = 7.5$
 $7.5 \times 1 = 7.5$

C $\frac{10}{3} + 2.5 = \frac{35}{6}$
 $\frac{35}{6} \div 2 = \frac{35}{12}$
 $\frac{35}{12} \times 1 = \frac{35}{12}$

B $5 + \frac{10}{3} = \frac{25}{3}$
 $\frac{25}{3} \div 2 = \frac{25}{6}$
 $\frac{25}{6} \times 1 = \frac{25}{6}$

D $2.5 + 2 = 4.5$
 $4.5 \div 2 = 2.25$
 $2.25 \times 1 = 2.25$



Final answer: $7.5 + \frac{25}{6} + \frac{35}{12} + 2.25 = \frac{101}{6}$ square units (or 16.8 $\bar{3}$)

The Trapezium Rule Formula

Instead of calculating the area of all the individual trapezia, we can use the following formula to estimate the area between the curve and the x -axis:



$$\frac{1}{2}h\{y_0 + y_n + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})\}$$

Here, h is the height of each individual trapezium, and $y_0, y_1, y_2, \dots, y_n$ are the values of the function for each of the ordinates $x_0, x_1, x_2, \dots, x_n$.

For the example on the previous page, we have $h = 1$, and the following table summarises the values of x_n and y_n .

n	0	1	2	3	4
x_n	1	2	3	4	5
y_n	10	5	$\frac{10}{3}$	2.5	2

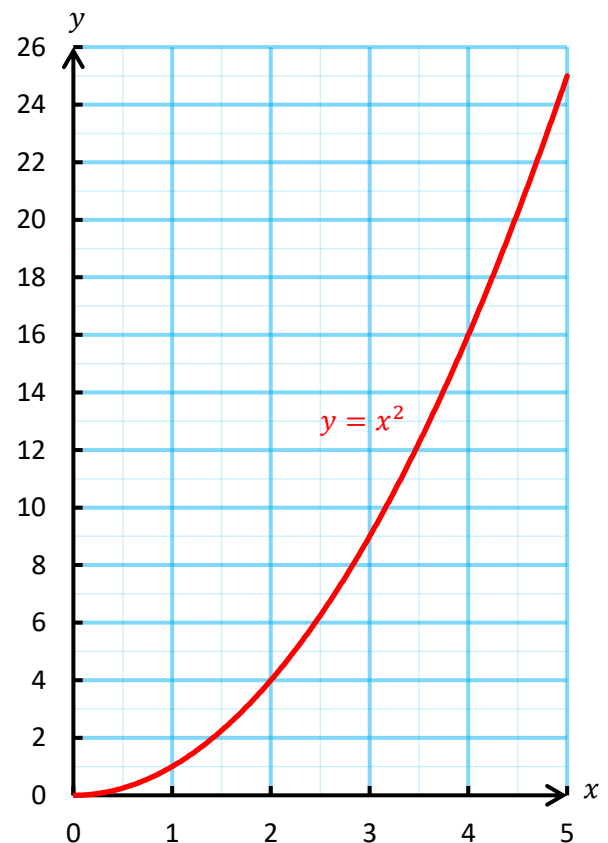
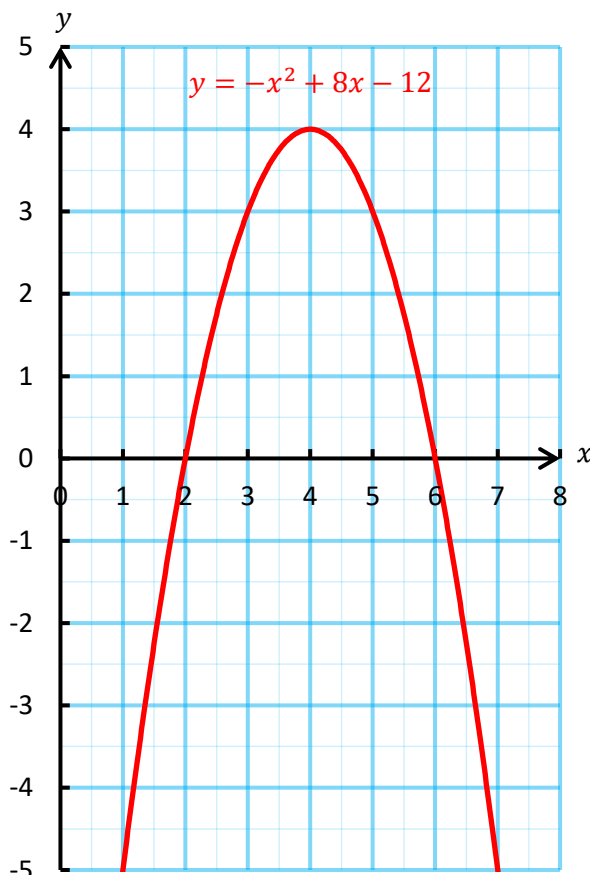
Using the formula for the trapezium rule, the estimate of the area between the curve and the x -axis is given by

$$\frac{1}{2} \times 1 \times \left\{ 10 + 2 + 2 \left(5 + \frac{10}{3} + 2.5 \right) \right\} = \frac{101}{6} \text{ square units (or } 16.8\bar{3})$$

Exercise 8

The graph on the right shows the function $y = x^2$ between $x = 0$ and $x = 5$. Use the trapezium rule and the ordinates $x = 1, x = 2, x = 3, x = 4$ and $x = 5$ to estimate the area enclosed by the curve $y = x^2$ and the x -axis, between $x = 1$ and $x = 5$.

3



Exercise 9

The graph on the left shows the function $y = -x^2 + 8x - 12$ between $x = 1$ and $x = 7$. Use the trapezium rule and the ordinates $x = 2, x = 3, x = 4, x = 5$ and $x = 6$ to estimate the area enclosed by the curve $y = -x^2 + 8x - 12$ and the x -axis, between $x = 2$ and $x = 6$.

3

Applications

Sometimes, the area between a graph and the x -axis has a special meaning.

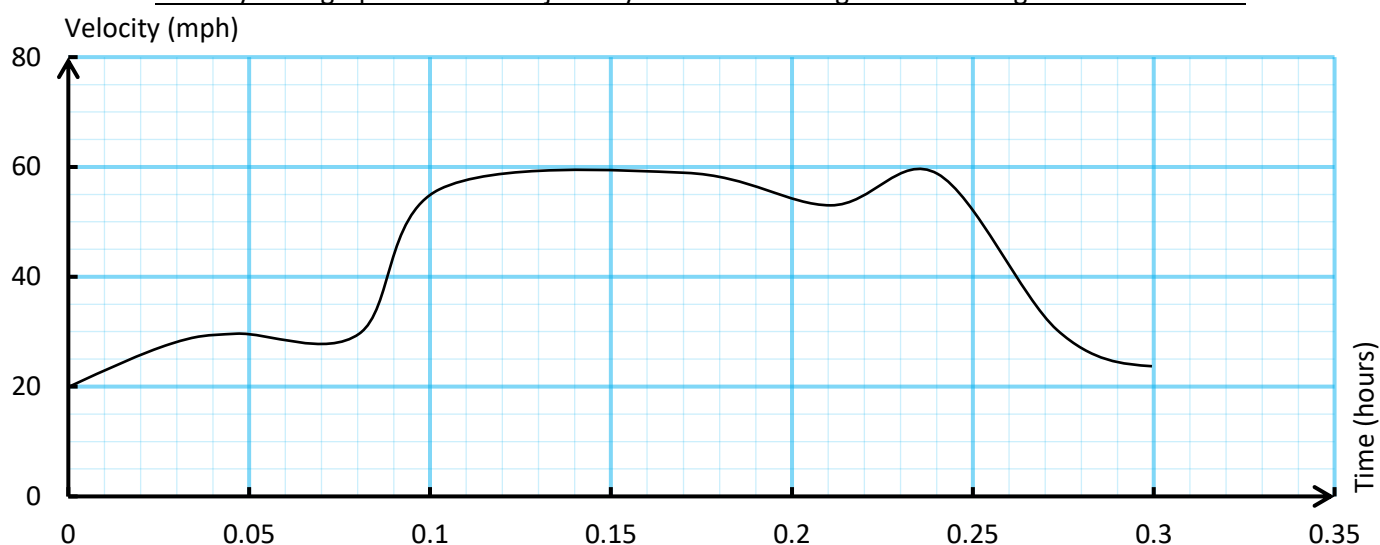
- In a **velocity-time graph**, the area between the graph and the x -axis gives the **distance travelled**.
- In a graph that plots **time** (in years) against **yearly salary**, the area between the graph and the x -axis gives the **total earnings**.
- In a graph that plots **time** (in minutes) against the **rate of filling a petrol tank** (in litres per minute), the area between the graph and the x -axis gives the **volume of petrol** that has been added to the petrol tank.

In general, the area between the graph and the x -axis represents the measure that has unit given by multiplying the unit on the vertical axis by the unit on the horizontal axis.

Exercise 10

The following graph shows the velocity of a car as it travels between Bangor and Caernarfon.

Velocity-time graph to show the journey of a car travelling between Bangor and Caernarfon



Use the trapezium rule and the ordinates $x = 0$, $x = 0.1$, $x = 0.2$ and $x = 0.3$ to estimate the distance that the car has travelled during this journey.

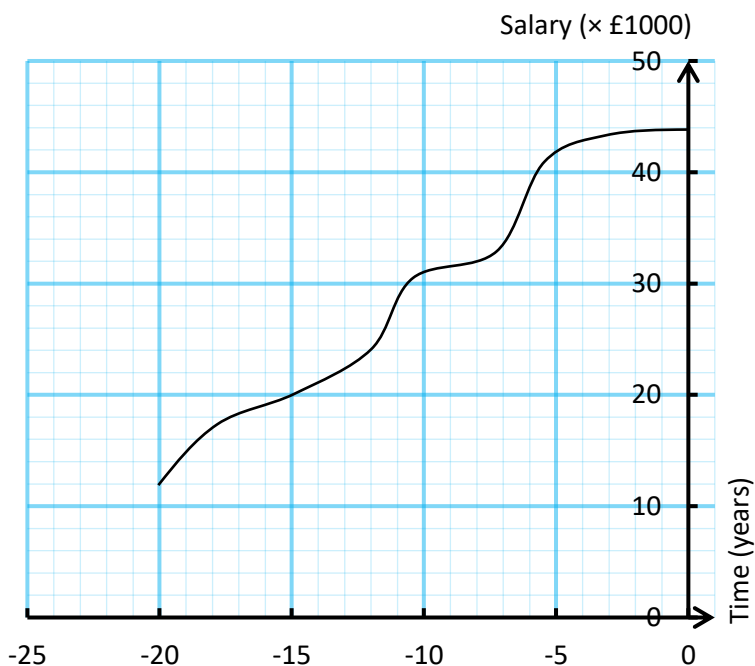
Exercise 11

3

The graph on the right shows how Trefor's yearly salary has increased during the past 20 years.

Use the trapezium rule and the ordinates $x = -20$, $x = -15$, $x = -10$, $x = -5$ and $x = 0$ to estimate Trefor's total earnings over the past 20 years.

A graph to show Trefor's salary during the past 20 years



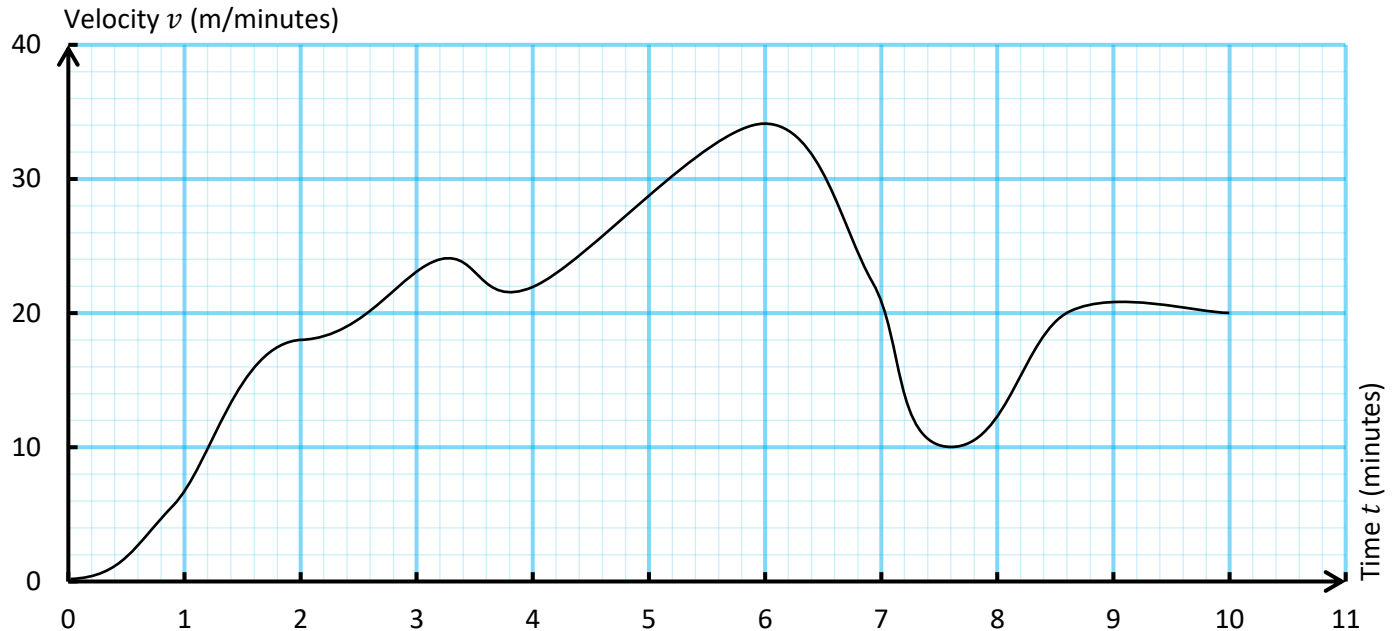
Exercise 12 (Revision)

3

Non conducted an experiment. She used equipment to measure the velocity, v , of an object during the first 10 minutes of the experiment.

The velocity-time graph for the experiment is shown below.

A graph to show the velocity of an object during an experiment



- Write down the gradient of the curve when the time is 7.6 minutes.
- Find an estimate for the acceleration of the object when the time is 4.5 minutes.
- Use the trapezium rule and the ordinates $t = 0, t = 2, t = 4, t = 6, t = 8$ and $t = 10$ to estimate the area enclosed by the curve, the positive time axis and the line $t = 10$.
- Calculate an estimate of the distance travelled by the object during the first 10 minutes of Non's experiment, giving your answer in kilometres.

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Further Changing the Subject



The purpose of **changing the subject** is to re-arrange a formula so that a particular **variable** appears on its own on the left-hand side of the formula.

Revision

2

Exercise 13

(a) Make x the subject of the formula $2y = 3z + 5x$

(b) Make s the subject of the formula $t = \frac{s}{3}$

(c) Make a the subject of the formula $F = c(a - b)$

(d) Make z the subject of the formula $\frac{x}{z} = y$

(e) Make r the subject of the formula $C = \frac{4}{3}\pi r^3$

(f) Make e the subject of the formula $g = \frac{f}{e} + c$

Changing the subject where the subject appears more than once

Example

(a) Make x the subject of the formula $3z + 4x = yx + 6y$.

Answer: $3z + 4x = yx + 6y$

$4x - yx = 6y - 3z$

[Subtract $3z$ from both sides; subtract yx from both sides]

$x(4 - y) = 6y - 3z$

[Factorise x on the left-hand side]

$x = \frac{6y - 3z}{4 - y}$

[Divide both sides by $4 - y$]

(b) Make x the subject of the formula $\frac{2x+y}{3-5x} = 2$.

Answer: $\frac{2x+y}{3-5x} = 2$

$2x + y = 2(3 - 5x)$

[Multiply both sides by $3 - 5x$]

$2x + y = 6 - 10x$

[Expand the bracket]

$2x + 10x = 6 - y$

[Subtract y from both sides; add $10x$ to both sides]

$12x = 6 - y$

[Collect like terms]

$x = \frac{6-y}{12}$

[Divide both sides by 12]



Exercise 14

(a) Make x the subject of the formula $5z + 3x = xz + 3y$

(b) Make y the subject of the formula $5y - 3x = 2y + 3z$

(c) Make z the subject of the formula $\frac{4z-5y}{6-3z} = 6$

(d) Make f the subject of the formula $11f - 1 = 4g(3f + e)$

(e) Make f the subject of the formula $7f - 5 = 3g(2f + h)$

(f) Make k the subject of the formula $5(2k - m) = ck + 5$

(h) Make u the subject of the formula $\frac{8u+3y}{3-5u} = 3z$

(i) Make r the subject of the formula $5(r + 3t) = 7(2 - 6r)$

(j) Make v the subject of the formula $\frac{2}{3}v + 5w = \frac{1}{3}(3 - 2v)$

Skill

2



Example(a) Make x the subject of the formula $z = \frac{1}{x} + \frac{1}{y}$.

Answer: $z = \frac{1}{x} + \frac{1}{y}$

$$\frac{1}{x} + \frac{1}{y} = z$$

[Swap sides]

$$\frac{1}{x} = z - \frac{1}{y}$$

[Subtract $\frac{1}{y}$ from both sides]

$$x = \frac{1}{z - \frac{1}{y}}$$

[Take reciprocal of both sides]

**Exercise 15**(a) Make x the subject of the formula $z = \frac{1}{x} - \frac{1}{y}$ (c) Make x the subject of the formula $z = \frac{y-w}{x}$ (e) Make f the subject of the formula $\frac{a}{2f+1} = \frac{b}{3f-1}$ (g) Make p the subject of the formula $q = \frac{y}{x^2-p}$ (i) Make h the subject of the formula $A = \pi r \sqrt{h^2 + r^2}$ (k) Make b the subject of the formula $m = \frac{ax+by}{a+b}$ (m) Make q the subject of the formula $y = \frac{x-np}{\sqrt{npq}}$ (o) Make x the subject of the formula $y = \frac{2}{x+3} - 5$ (b) Make x the subject of the formula $z = y + \sqrt{\frac{w}{x}}$.

Answer: $z = y + \sqrt{\frac{w}{x}}$

$$y + \sqrt{\frac{w}{x}} = z$$

[Swap sides]

$$\sqrt{\frac{w}{x}} = z - y$$

[Subtract y from both sides]

$$\frac{w}{x} = (z - y)^2$$

[Square both sides]

$$w = x(z - y)^2$$

[Multiply both sides by x]

$$x(z - y)^2 = w$$

[Swap sides]

$$x = \frac{w}{(z - y)^2}$$

[Divide by $(z - y)^2$]

2

(b) Make x the subject of the formula $z = \sqrt{\frac{w}{x}} - y$ (d) Make x the subject of the formula $z = y + \sqrt[3]{\frac{x}{w}}$ (f) Make p the subject of the formula $y = \frac{q}{p} - x$ (h) Make x the subject of the formula $w = \sqrt{\frac{x-y}{x+y}}$ (j) Make l the subject of the formula $t = 2\pi \sqrt{\frac{l}{g}}$ (l) Make y the subject of the formula $s = \sqrt{\frac{x^2+y^2}{n}}$ (n) Make x the subject of the formula $F = \frac{x^2}{1-x^2}$ (p) Make a the subject of the formula $a - b = \frac{a+2}{b}$ **Evaluation**

Key words	Corrections	I am happy with...	I need to revise...

The Quadratic Formula

It is possible to solve some quadratic equations by factorisation.

Example

Solve the following equations.

(a) $x^2 + 9x + 14 = 0$

$$x^2 + 9x + 14 = 0$$

$$(x + 2)(x + 7) = 0$$

Either $x + 2 = 0$ or $x + 7 = 0$

$$x = -2 \quad x = -7$$

(b) $3x^2 + 13x + 4 = 0$

$$3x^2 + 13x + 4 = 0$$

$$(3x + 1)(x + 4) = 0$$

Either $3x + 1 = 0$ or $x + 4 = 0$

$$3x = -1 \quad x = -4$$

$$x = -\frac{1}{3}$$

(c) $4x^2 - 9 = 0$

$$4x^2 - 9 = 0$$

$$(2x + 3)(2x - 3) = 0$$

Either $2x + 3 = 0$ or $2x - 3 = 0$

$$2x = -3 \quad 2x = 3$$

$$x = -\frac{3}{2} \quad x = \frac{3}{2}$$

Factorise using the splitting method or the detective method

Difference between two squares

Exercise 16

Solve the following equations.

(a) $x^2 + 10x + 24 = 0$

(b) $2x^2 + 21x + 40 = 0$

(c) $x^2 - 16 = 0$

(d) $x^2 + 3x - 54 = 0$

(e) $2x^2 + 7x - 15 = 0$

(f) $25x^2 - 36 = 0$

(g) $x^2 - 8x + 16 = 0$

(h) $4x^2 - 14x + 6 = 0$

(i) $8y^2 - 98 = 0$

Revision

2

If we cannot solve a quadratic equation through factorisation, then we can attempt to solve the equation using the **quadratic formula**.

THE QUADRATIC FORMULA

The solutions of $ax^2 + bx + c = 0$ where $a \neq 0$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The symbol $\pm$ means 'plus or minus'.

If $b^2 - 4ac < 0$, there are no real solutions.

Round off your answers to 2 decimal places at the end.

@MATHEMATEG

Exercise 17

The general form of a quadratic equation is $ax^2 + bx + c = 0$.
Write a , b and c for each of the following quadratic equations.

(a) $x^2 + 3x + 7 = 0$

(b) $2x^2 - 8x + 11 = 0$

(c) $9x^2 - 4 = 0$

(d) $5x - 3x^2 + 17 = 0$

(e) $23 + 5x + 3x^2 = 0$

(f) $4x^2 - 3x = 0$

Example

Using the quadratic formula, solve the equation $5x^2 + 10x - 3 = 0$.

Answer: In this case, we have $a = 5$, $b = 10$ and $c = -3$.
Substituting these values into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

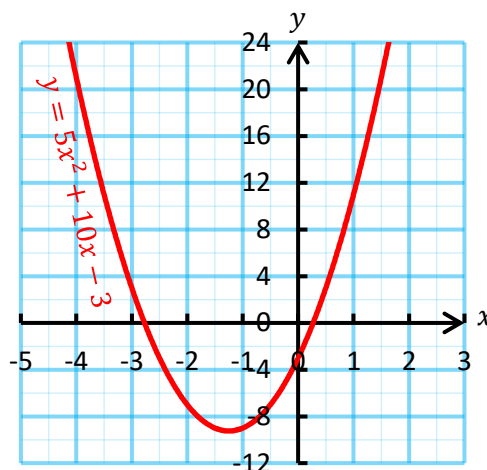
$$x = \frac{-10 \pm \sqrt{10^2 - 4 \times 5 \times -3}}{2 \times 5}$$

$$x = \frac{-10 \pm \sqrt{100 + 60}}{10}$$

$$x = \frac{-10 \pm \sqrt{160}}{10}$$

$$\text{Either } x = \frac{-10 + \sqrt{160}}{10} \text{ or } x = \frac{-10 - \sqrt{160}}{10}$$

$$\text{Either } x = 0.26 \text{ to 2 d.p., or } x = -2.26 \text{ to 2 d.p.}$$

**Exercise 18**

Using the quadratic formula, solve the following equations.
Round off your answers correct to two decimal places.
If there are no real solutions, state this.

(a) $x^2 + 8x + 5 = 0$

(b) $x^2 - 8x + 5 = 0$

(c) $x^2 + 8x - 5 = 0$

(d) $x^2 - 8x - 5 = 0$

(e) $x^2 + 8x + 25 = 0$

(f) $2x^2 + 10x + 5 = 0$

(g) $3x^2 + 8x + 1 = 0$

(h) $3x^2 - 6x - 4 = 0$

(i) $2x^2 - 9x + 8 = 0$

(j) $2x^2 + x - 20 = 0$

(k) $7x^2 + 3x - 1 = 0$

(l) $x^2 - 2x - 100 = 0$

(m) $5x^2 + 11x + 3 = 0$

(n) $4x^2 + 7x + 6 = 0$

(o) $4x^2 - 12 + 3x = 0$

Exercise 19

The area of the right-angled triangle shown on the right is 45 cm^2 .

(a) Show that x satisfies the equation $x^2 + 4x - 90 = 0$.

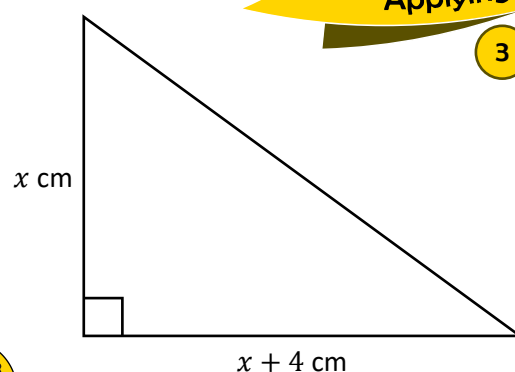
(b) Use the quadratic formula to solve the equation $x^2 + 4x - 90 = 0$, giving your answers correct to two decimal places.

(c) Using your answer to (b), find the base and height of the triangle.

(d) Calculate the length of the hypotenuse of the triangle.
Give your answer correct to one decimal place.

Challenge!

Solve the equation $5x^3 + 2x^2 - 11x = 0$



Applying

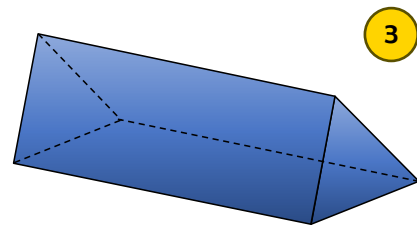
Exercise 20

The diagram shows a triangular prism.

The cross-sectional area of the triangular prism is $3x \text{ cm}^2$ and its length is $(x + 4) \text{ cm}$. The volume of the prism is 89 cm^3 .

(a) Show that x satisfies the equation $3x^2 + 12x - 89 = 0$.

(b) Use the formula method to solve the equation $3x^2 + 12x - 89 = 0$, giving your answer correct to one decimal place. Hence, write the length of the prism correct to one decimal place.

**Exercise 21**

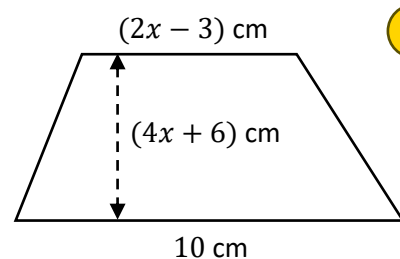
The diagram shows a trapezium.

The lengths of the parallel sides of the trapezium are 10 cm and $(2x - 3) \text{ cm}$. The height of the trapezium is $(4x + 6) \text{ cm}$ and its area is 70 cm^2 .

(a) Show that $4x^2 + 20x - 49 = 0$.

(b) Use the quadratic formula to solve the equation $4x^2 + 20x - 49 = 0$. Give your answers correct to two decimal places.

Hence, write the height of the trapezium correct to two decimal places.

**Exercise 22**

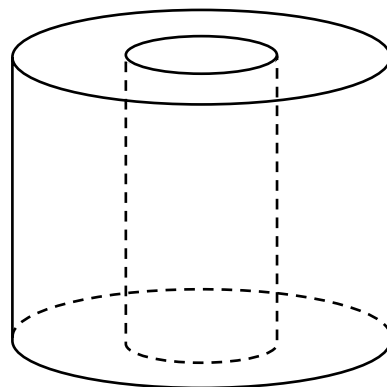
A length of plastic tube has a uniform circular cross-section.

The radius of the circular hole in the middle is $x \text{ cm}$.

The thickness of the plastic is 3 cm and the length of the plastic tube is $5x \text{ cm}$.

(a) Show that the volume of the plastic used to make the tube is $(30\pi x^2 + 45\pi x) \text{ cm}^3$.

(b) Given that the volume of plastic used to make the tube is $88\pi \text{ cm}^3$, find the length of the tube correct to one decimal place.



Evaluation

Key words	Corrections	I am happy with...	I need to revise...

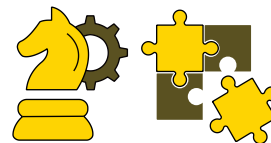
Algebraic Fractions

Numerical Fractions

In order to be able to work with fractions that include algebraic expressions, it is a good idea to first revise how to work with numerical fractions.

Example

Here are two methods of calculating $\frac{3}{4} + \frac{1}{6}$.



The Traditional Method

Step 1: Find the **lowest common denominator** of the two fractions. Here, the lowest common multiple of 4 and 6 is 12.

Step 2: Write **equivalent fractions** for $\frac{3}{4}$ and $\frac{1}{6}$, using 12 as the common denominator.

$$\frac{3}{4} \xrightarrow{\times 3} \frac{9}{12} \qquad \frac{1}{6} \xrightarrow{\times 2} \frac{2}{12}$$

Step 3: **Add** the two new fractions.

$$\frac{9}{12} + \frac{2}{12} = \frac{11}{12}$$

Step 4: Check to see if it is possible to simplify the answer. (It is not possible in this case.)

*Step 1: You can find a common denominator by multiplying the two original denominators, however this might not be the **lowest** common denominator.*

The Peanut Method

Step 1: Draw the following template. (Note that the first fraction goes on the top and the second fraction goes on the left.)

	3	4
1	×	
6		

Step 2: Fill in the gaps in the table by **multiplying** the numbers.

	3	4
1	×	4
6	18	24

Step 3: **Add** the two numbers that form the peanut shape.

	3	4	18 + 4 = 22
1	×	4	
6	18	24	The answer is $\frac{22}{24}$.

Step 4: Check to see if it is possible to simplify the answer. Here, $\frac{22}{24}$ simplifies to give the final answer $\frac{11}{12}$.

Revision

A

Exercise 23

Calculate the following. Give your answers in their simplest form.

(a) $\frac{2}{5} + \frac{1}{3}$

(b) $\frac{1}{4} + \frac{3}{8}$

(c) $\frac{9}{10} - \frac{3}{5}$

(d) $\frac{3}{7} + \frac{1}{6}$

(e) $\frac{7}{8} - \frac{2}{5}$

(f) $\frac{2}{9} + \frac{2}{3}$

(g) $\frac{11}{12} - \frac{1}{2}$

(h) $2\frac{1}{3} + 4\frac{3}{4}$

Algebraic Fractions

Algebraic fractions contain at least one numerator or denominator that is an algebraic expression.

Examples of numerical fractions	$\frac{3}{4}$	$\frac{4}{9}$	$\frac{12}{5}$	$-\frac{4}{5}$
Examples of algebraic fractions	$\frac{x+2}{3}$	$\frac{5}{x-2}$	$\frac{x+7}{x^2+x-1}$	$-\frac{1}{y}$

Example

Here are two methods of writing $\frac{2}{3x+4} + \frac{3}{x+2}$ as a single fraction in its simplest form.

**The traditional method**

Step 1: Find the **lowest common denominator** of $3x + 4$ and $x + 2$. Here, the smallest expression that is a multiple of both $3x + 4$ and $x + 2$ is the product $(3x + 4)(x + 2)$.

Step 2: Write **equivalent fractions** for $\frac{2}{3x+4}$ and $\frac{3}{x+2}$, with $(3x + 4)(x + 2)$ appearing in the denominator.

$$\begin{array}{lcl} \frac{2}{3x+4} & \xrightarrow{\times (x+2)} & \frac{2(x+2)}{(3x+4)(x+2)} \\ \frac{3}{x+2} & \xrightarrow{\times (3x+4)} & \frac{3(3x+4)}{(3x+4)(x+2)} \end{array}$$

Step 3: Add the two new fractions.

$$\begin{aligned} & \frac{2(x+2)}{(3x+4)(x+2)} + \frac{3(3x+4)}{(3x+4)(x+2)} \\ &= \frac{2(x+2)+3(3x+4)}{(3x+4)(x+2)} \\ &= \frac{2x+4+9x+12}{(3x+4)(x+2)} \\ &= \frac{11x+16}{(3x+4)(x+2)} \end{aligned}$$

Step 4: Check to see if it is possible to **simplify** the answer, e.g. through factorising. (This is not possible in this case.)

The peanut method

Step 1: Draw the following template. (Note that the first fraction goes on the top and the second fraction goes on the left.)

	2	3x + 4
3	×	
x + 2		

Step 2: Fill in the gaps in the table by **multiplying** the expressions.

	2	3x + 4
3	×	3(3x + 4)
x + 2	2(x + 2)	(x + 2)(3x + 4)

Cam 3: Add the two expressions that form the peanut shape.

$$\begin{aligned} & 2(x + 2) + 3(3x + 4) \\ &= 2x + 4 + 9x + 12 \\ &= 11x + 16 \end{aligned}$$

The answer is $\frac{11x+16}{(3x+4)(x+2)}$

Step 4: Check to see if it is possible to **simplify** the answer, e.g. through factorising. (This is not possible in this case.)

Exercise 24

Write each of the following as a single fraction. Give your answers in their simplest form.

(a) $\frac{5}{3x+2} + \frac{4}{x+1}$

(b) $\frac{7}{2x+3} + \frac{6}{x+4}$

(c) $\frac{1}{x+5} + \frac{2}{2x+7}$

(d) $\frac{5}{3x+2} + \frac{4}{x-1}$

(e) $\frac{7}{2x-3} + \frac{6}{x+4}$

(f) $\frac{1}{x+5} - \frac{2}{2x+7}$

(g) $\frac{5}{4x+2} - \frac{3}{x-8}$

(h) $\frac{3}{x-5} - \frac{4}{x+2}$

(i) $\frac{9}{x-3} + \frac{6}{2x-5}$

(j) $\frac{x+1}{2} + \frac{x+4}{3}$

(k) $\frac{x+3}{5} - \frac{x+1}{2}$

(l) $\frac{x-2}{4} + \frac{x-5}{3}$

(m) $\frac{3x+1}{4} + \frac{2x-5}{2}$

(n) $\frac{x+2}{3} + \frac{2}{x-1}$

(o) $\frac{1}{x-2} + \frac{x+3}{4}$

(p) $\frac{2x+5}{3} - \frac{2}{x-5}$

(q) $\frac{3x+2}{4x-1} + \frac{2}{7}$

(r) $\frac{5+3x}{2x-3} + \frac{1}{4}$

(s) $\frac{2}{x+3} + \frac{x+1}{x}$

(t) $\frac{x+2}{3x} + \frac{x}{x+3}$

(u) $\frac{x+3}{x-4} + \frac{x-3}{x+4}$

(v) $\frac{2x+3}{x-3} - \frac{x-2}{x-5}$

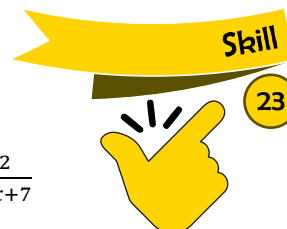
(w) $\frac{x+2}{3x-4} + \frac{x-3}{x+2}$

(x) $\frac{2}{2x+1} + \frac{3x+5}{x+2}$

(y) $\frac{4x+17}{x+3} - \frac{2x-15}{x-3}$

(z) $\frac{2x+3}{x-1} + \frac{4-x}{3x-5}$

(α) $\frac{3x-4}{x+1} - \frac{x+2}{5x+3}$



Solving Equations involving Algebraic Fractions**One fraction equal to another fraction****Example**

Solve $\frac{4x}{3x+1} = \frac{3}{x+5}$.

Answer: $\frac{4x}{3x+1} = \frac{3}{x+5}$
 $\frac{4x(x+5)}{3x+1} = 3$

$$4x(x+5) = 3(3x+1)$$

$$4x^2 + 20x = 9x + 3$$

$$4x^2 + 11x - 3 = 0$$

$$(4x-1)(x+3) = 0$$

$$\text{Either } 4x-1 = 0 \text{ or } x+3 = 0$$

$$4x = 1 \quad x = -3$$

$$x = \frac{1}{4}$$

Technique: Multiply by each denominator in turn.

[Multiply both sides by $x+5$]

[Multiply both sides by $3x+1$]

[Expand the brackets]

[Subtract $9x$ from both sides; subtract 3 from both sides]

[Factorise]

[Solve]

**Exercise 25**

Solve the following equations.

(a) $\frac{x+4}{x-1} = \frac{x}{x-3}$

(b) $\frac{6}{x-4} = \frac{5}{x-3}$

(c) $\frac{1}{2x+3} = \frac{1}{3x-2}$

(d) $\frac{3}{2(2x-1)} = \frac{4}{3x+2}$

(e) $\frac{x}{4x+3} = \frac{1}{2x+9}$

(f) $\frac{3x}{4-x} = \frac{2}{x-4}$

(g) $\frac{3x}{x+1} - \frac{x+4}{3} = 0$

(h) $\frac{1}{3x-2} - \frac{2x+3}{x-4} = 0$

(i) $\frac{2}{3x+1} - \frac{5}{x+3} = 0$

Fractions where all the denominators are numbers**Example**

Solve $\frac{2x+3}{6} + \frac{x-1}{3} = \frac{7}{12}$.

Answer: $\frac{2x+3}{6} + \frac{x-1}{3} = \frac{7}{12}$
 $12\left(\frac{2x+3}{6}\right) + 12\left(\frac{x-1}{3}\right) = 12\left(\frac{7}{12}\right)$

$$2(2x+3) + 4(x-1) = 7$$

$$4x + 6 + 4x - 4 = 7$$

$$8x + 2 = 7$$

$$8x = 5$$

$$x = \frac{5}{8}$$

Technique: Multiply by the lowest common denominator of the fractions.

[Multiply each term by 12]

[Simplify]

[Expand the brackets]

[Collect like terms]

[Subtract 2 from both sides]

[Divide both sides by 8]

**Exercise 26**

Solve the following equations.

(a) $\frac{x+3}{6} + \frac{2x-5}{3} = \frac{2}{9}$

(b) $\frac{x-2}{5} - \frac{2x+5}{4} = \frac{1}{4}$

(c) $\frac{3-x}{4} + \frac{2x+5}{3} = 1$

(d) $\frac{2x-1}{3} + \frac{x-2}{6} = \frac{3x}{4}$

(e) $\frac{x+2}{3} - \frac{x-3}{2} = 2$

(f) $\frac{2x-3}{4} - \frac{x+1}{3} = \frac{3}{4}$

(g) $\frac{2x+5}{4} + \frac{x-4}{6} = \frac{x}{3}$

(h) $\frac{3-x}{8} + \frac{3}{16} = \frac{3x+2}{4}$

(i) $\frac{3x+5}{10} - \frac{2-x}{25} = \frac{3}{5}$

Example

Solve the equation $\frac{2x}{x+3} + \frac{3x+1}{2x-1} = 3$.

Answer: $\frac{2x}{x+3} + \frac{3x+1}{2x-1} = 3$

$$(x+3)(2x-1) \times \frac{2x}{x+3} + (x+3)(2x-1) \times \frac{3x+1}{2x-1} = 3(x+3)(2x-1)$$

$$(2x-1)2x + (x+3)(3x+1) = 3(x+3)(2x-1)$$

$$4x^2 - 2x + (3x^2 + x + 9x + 3) = 3(2x^2 - x + 6x - 3)$$

$$4x^2 - 2x + (3x^2 + 10x + 3) = 3(2x^2 + 5x - 3)$$

$$7x^2 + 8x + 3 = 6x^2 + 15x - 9$$

$$x^2 - 7x + 12 = 0$$

$$(x-3)(x-4) = 0$$

$$\text{Either } x-3 = 0 \text{ or } x-4 = 0$$

$$x = 3 \quad x = 4$$

Technique: Multiply by the lowest common denominator of the fractions.

[Multiply by $(x+3)(2x-1)$]

[Simplify]

[Expand the brackets]

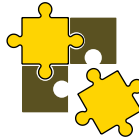
[Collect like terms]

[Collect like terms; expand]

[Subtract $6x^2 + 15x - 9$]

[Factorise]

[Solve]

**Exercise 27**

Solve the following equations.

(a) $\frac{2}{2x+3} + \frac{1}{x+2} = 3$

(b) $\frac{3x}{x+4} + \frac{2x}{5x-2} = \frac{3}{2}$

(c) $\frac{2x}{x-3} - \frac{x}{x-2} = 6$

(d) $\frac{6x}{3x-1} + \frac{15}{2x+3} = 5$

(e) $\frac{4x}{5x-2} + \frac{3}{3x+1} = 3$

(f) $\frac{x+3}{x+1} + \frac{3}{x-3} = 2$

(g) $\frac{8x}{4x-3} + \frac{20}{3x+2} = 10$

(h) $\frac{2x}{x-5} + \frac{x-1}{3x} = 2$

(i) $\frac{2}{x} + \frac{1}{x+1} = 5$

Challenge!

Ava runs a distance of 26 miles at an average speed of x mph.

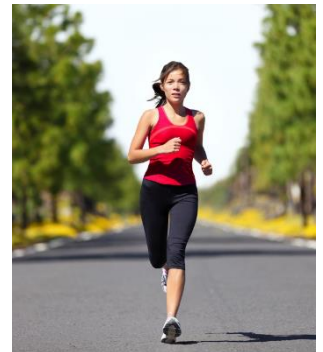
Delyth runs the same distance at an average speed that is 2 mph slower than Ava.

The difference between their times is exactly 1 hour.

(a) Show that x satisfies the equation $x^2 - 2x - 52 = 0$.

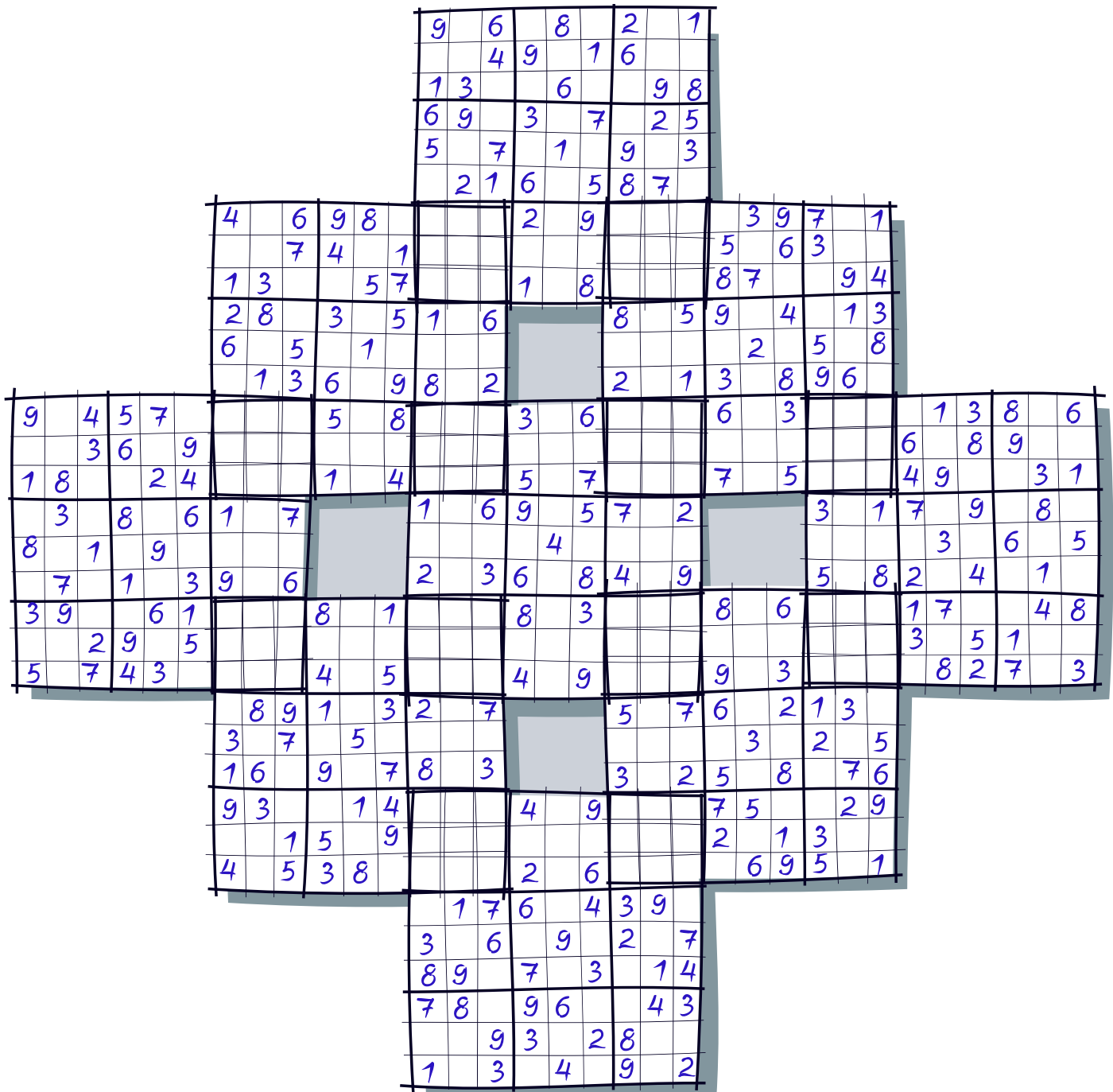
(b) Use the quadratic formula to find Ava's speed.

Give your answer correct to 2 decimal places.

**Evaluation**

Key words	Corrections	I am happy with...	I need to revise...

Giant Sudoku Puzzle



Developing

Algebra 4

Reflection

Name:

Percentage in the test:

I know
this.I need to
revise this.Question
in the
test:Correct
in the
test?

I know how to draw a tangent to a function and measure its gradient .			1	
I know and can use the definitions for velocity and acceleration .			1	
I know how to find the area between a graph and the x-axis using the trapezium rule .			1	
I know when the area between a graph and the x -axis has a special meaning .			1	
I can change the subject of a formula when the subject appears more than once in the formula.			2	
I can change the subject of a formula in questions involving fractions and roots .			3, 4	
I can use the quadratic formula to solve quadratic equations.			5, 7	
I can combine two algebraic fractions to give a single fraction in its simplest form.			7	
I can solve equations that include algebraic fractions .			7, 8, 9	



The Mathematics Department

11

Measuring

Shapes 5

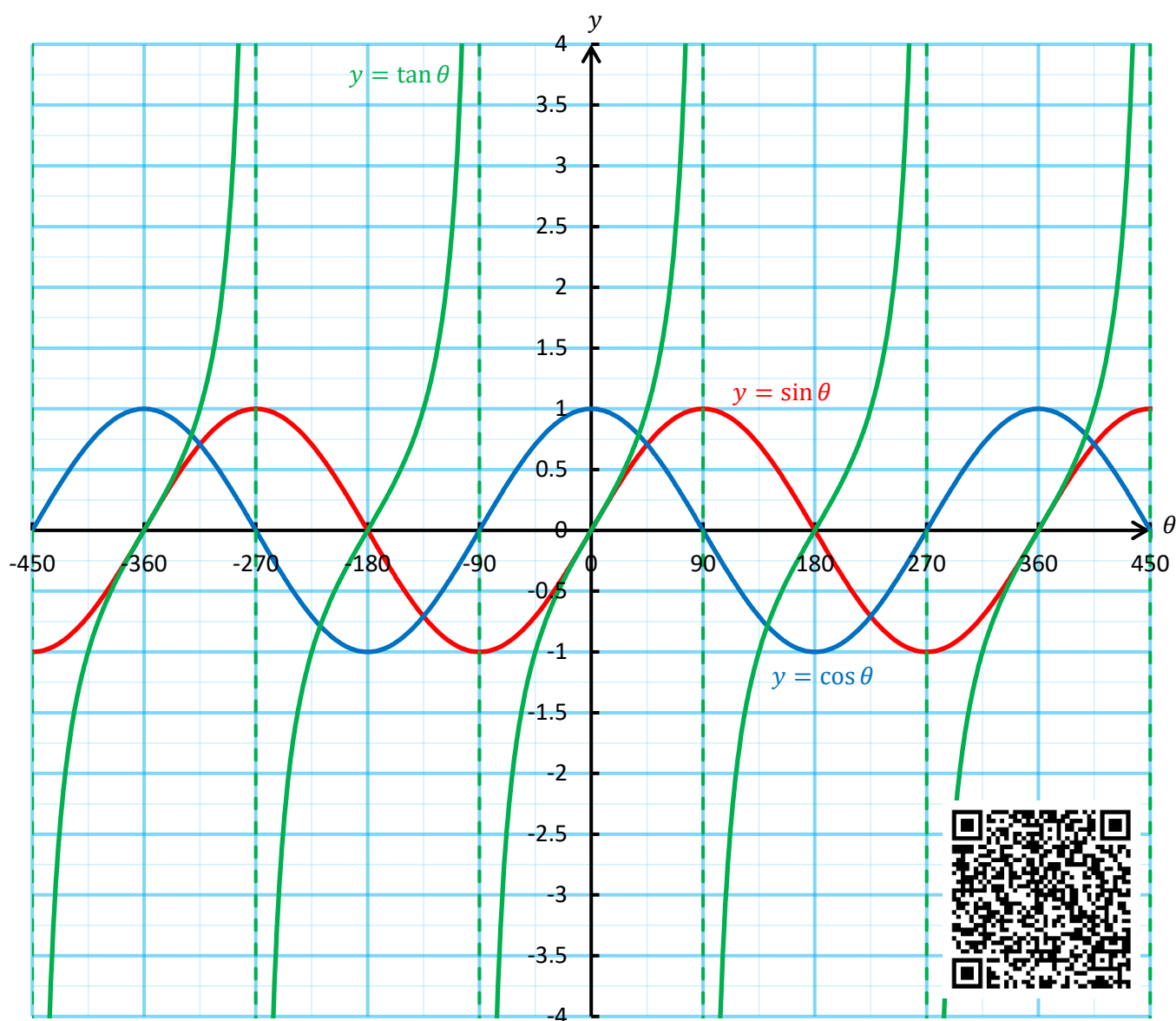
Higher Tier

Version
3.0
08/2025

Name:

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Trigonometric Graphs	The unit circle. Trigonometric graphs. Solving trigonometric equations. Function transformations.	17



3-D Trigonometry

2-D Trigonometry (Revision)



In mathematics, we worship the King **SOHCAHTOA**.

S **O** **C** **A** **T** **O**
H **H** **A**

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

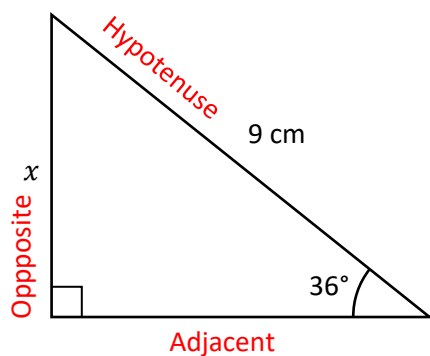
$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

Example

Calculate the missing side x or the missing angle θ . (The diagrams are not drawn to scale.)

(a)



Answer: Label the sides in red.

We know the hypotenuse, and need to find the opposite: choose

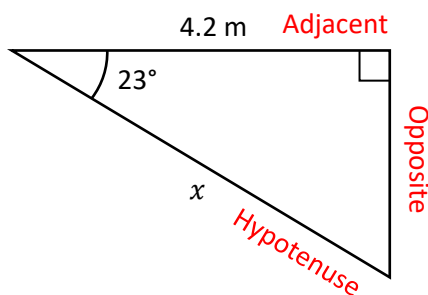
$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 36^\circ = \frac{x}{9}$$

$$x = 9 \times \sin 36^\circ$$

$$x = 5.29 \text{ cm to 2 decimal places.}$$

(b)



Answer: Label the sides in red.

We know the adjacent, and need to find the hypotenuse: choose

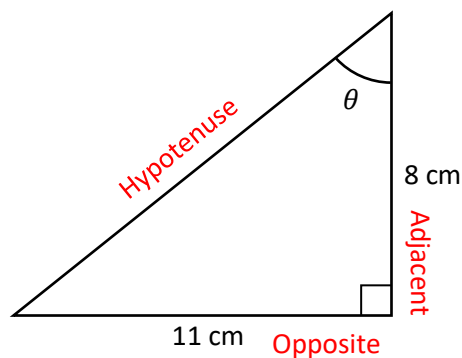
$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 23^\circ = \frac{4.2}{x}$$

$$x = 4.2 \div \cos 23^\circ$$

$$x = 4.56 \text{ m to 2 decimal places.}$$

(c)



Answer: Label the sides in red.

We know the opposite and adjacent: choose

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

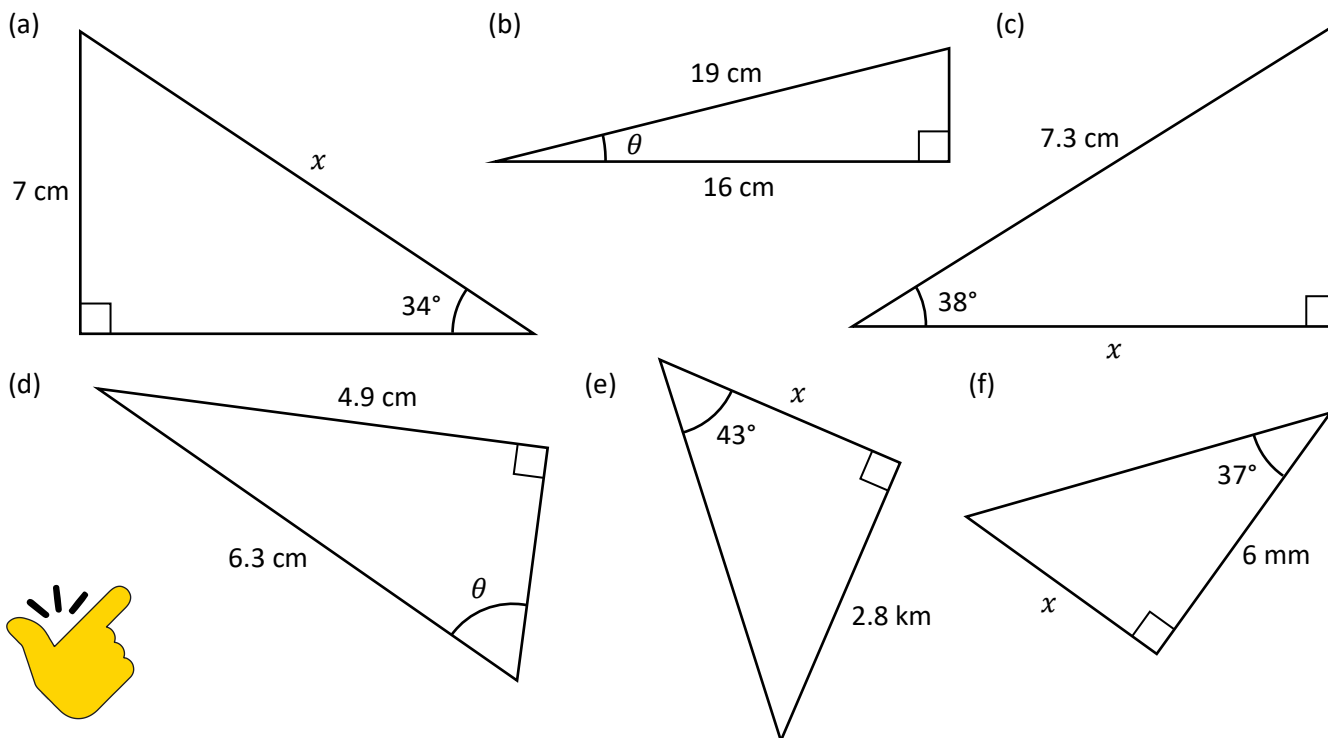
$$\tan \theta = \frac{11}{8}$$

$$\theta = \tan^{-1} \left(\frac{11}{8} \right)$$

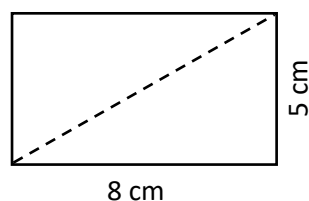
$$\theta = 53.97^\circ \text{ to 2 decimal places.}$$

x on **top** of the fraction leads to a **multiplication** sum.

x at the **bottom** of the fraction leads to a **division** sum.

Exercise 1 (Revision)**Revision****3**Calculate the missing side x or the missing angle θ . (The diagrams are not drawn to scale.)**3-D Trigonometry**

It is possible to use trigonometry to find missing sides and angles in three dimensional shapes.

ExampleFor the cuboid shown on the right, calculate the size of the angle $\widehat{ABC}$.*Answer:* To start, let us use Pythagoras' Theorem to calculate the length of the diagonal on the base of the cuboid, which is the diagonal of this rectangle:

$$\begin{array}{l} a^2 \\ b^2 \\ c^2 \end{array} \quad \begin{array}{r} 5^2 = 25 \\ 8^2 = + 64 \\ \hline 89 \end{array}$$

$$\sqrt{89} = 9.43 \text{ cm to 2 decimal places.}$$

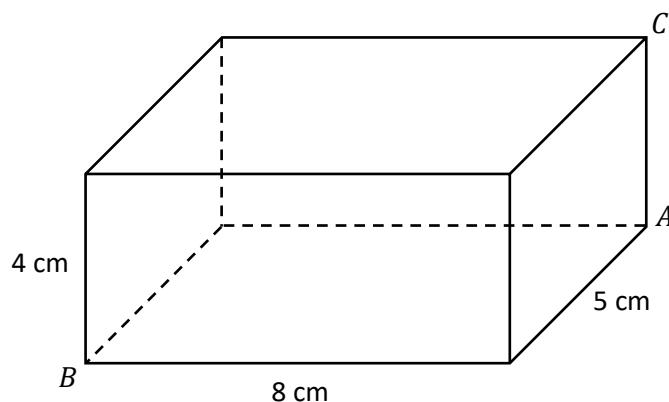
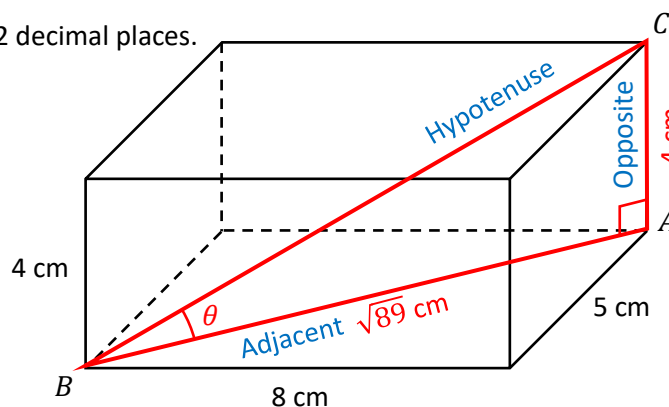
Next, we need to consider the **red** right-angled triangle shown on the right.

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan \theta = \frac{4}{\sqrt{89}}$$

$$\theta = \tan^{-1}\left(\frac{4}{\sqrt{89}}\right)$$

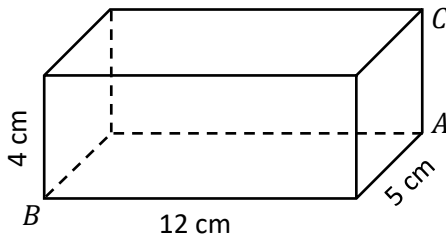
$$\theta = 22.98^\circ \text{ to 2 decimal places.}$$



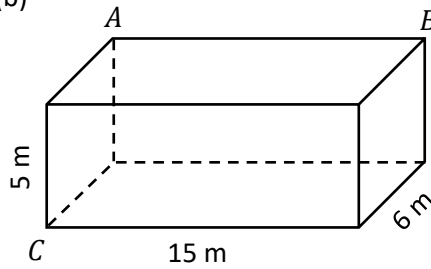
Exercise 2

For the following cuboids, calculate the size of the angle $\hat{ABC}$.

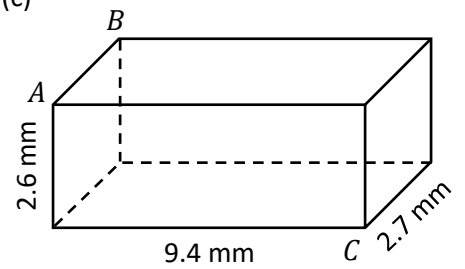
(a)



(b)

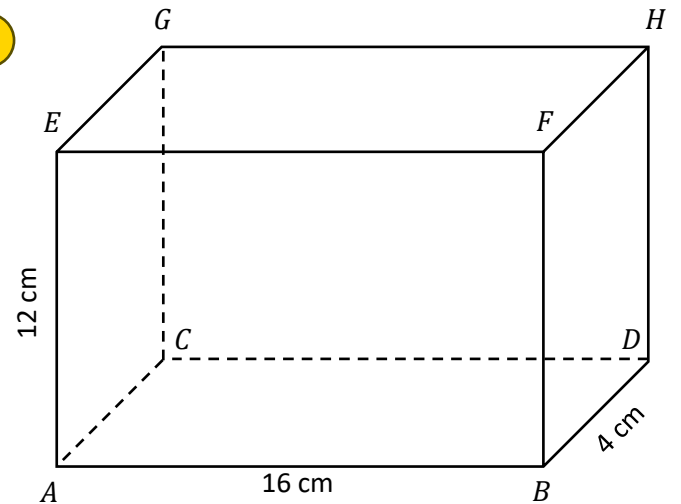


(c)

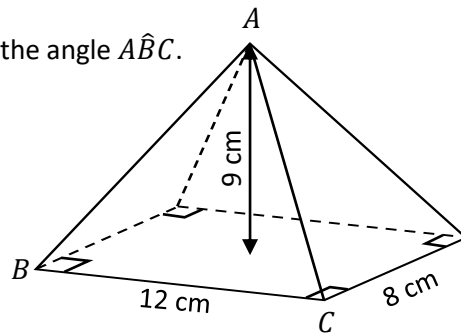
**Skill****3****Exercise 3**

The diagram on the right shows a cuboid. Calculate the size of the following angles.

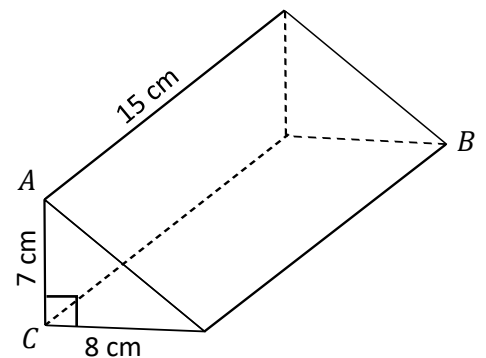
- (a) $\hat{ABC}$
- (b) $\hat{AEC}$
- (c) $\hat{AFE}$
- (d) $\hat{GAB}$
- (e) $\hat{DAH}$
- (f) $\hat{GDE}$

3**Exercise 4**

(a) Calculate the size of the angle $\hat{ABC}$.



(b) Calculate the size of the angle $\hat{ABC}$.

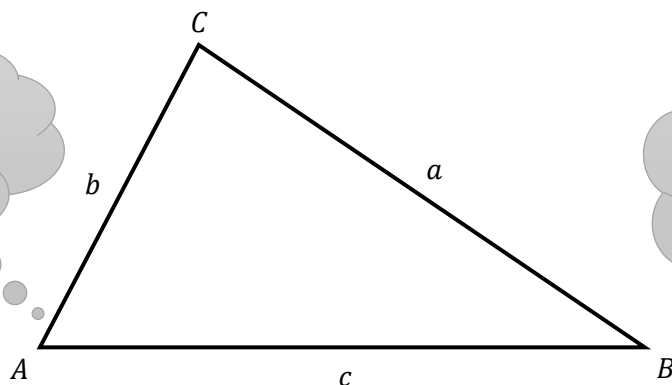
**Evaluation**

Key words	Corrections	I am happy with...	I need to revise...

The Sine Rule, The Cosine Rule

The **Sine Rule** and the **Cosine Rule** are used to calculate the size of angles and sides in triangles that are not right-angled triangles.

A general triangle with sides a, b, c and angles A, B, C .



The formulae for sides are given on page 2 of the Unit 3 paper.

Sine Rule for finding sides:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

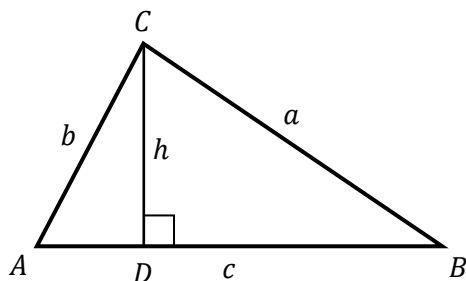
Sine Rule for finding angles:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

[Take the reciprocal]

Proof (for triangles without an obtuse angle¹)

Draw the perpendicular from C to the base AB .



Using the triangle CDB , $\sin B = \frac{h}{a}$
so that $h = a \sin B$.

Using the triangle CAD , $\sin A = \frac{h}{b}$
so that $h = b \sin A$.

Using the two expressions for the height of the triangle h , we have $a \sin B = b \sin A$.

Therefore, $\frac{a}{\sin A} = \frac{b}{\sin B}$ [Divide by $\sin A$ and $\sin B$]

It would be possible to repeat the above ("Draw the perpendicular from A to the base BC ...") to obtain $\frac{b}{\sin B} = \frac{c}{\sin C}$. We can combine the two formulae that use fractions to obtain the Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine Rule for finding sides:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

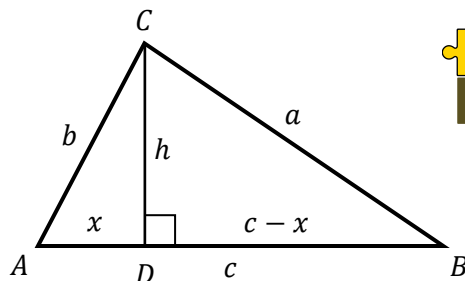
Cosine Rule for finding angles:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

[Re-arrange the formula]

Proof (for triangles without an obtuse angle¹)

Draw the perpendicular from C to the base AB .



Pythagoras' Theorem for triangle BCD :

$$a^2 = (c - x)^2 + h^2 \text{ so that } h^2 = a^2 - (c - x)^2.$$

Pythagoras' Theorem for triangle ACD :

$$b^2 = x^2 + h^2 \text{ so that } h^2 = b^2 - x^2.$$

Using the two expressions for h^2 :

$$\begin{aligned} a^2 - (c - x)^2 &= b^2 - x^2 \\ a^2 - (c - x)(c - x) &= b^2 - x^2 \\ a^2 - (c^2 - cx - cx + x^2) &= b^2 - x^2 \\ a^2 - c^2 + 2cx - x^2 &= b^2 - x^2 \\ a^2 &= b^2 + c^2 - 2cx \end{aligned}$$

Using the triangle ACD , $\cos A = \frac{x}{b}$
so that $x = b \cos A$.

So, $a^2 = b^2 + c^2 - 2c(b \cos A)$ which gives the Cosine Rule

$$a^2 = b^2 + c^2 - 2bc \cos A$$

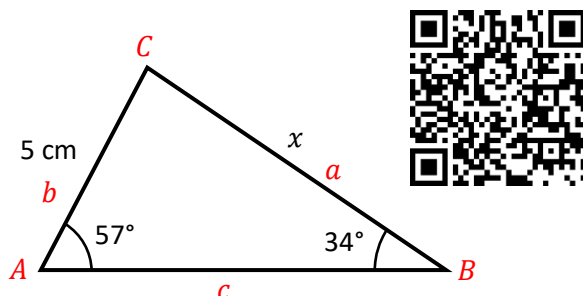
¹ Search on the internet for a proof in the case of a triangle that includes an obtuse angle.

The Sine Rule

Example

Calculate the missing side x or the missing angle θ . (The diagrams are not drawn to scale.)

(a)



Answer: To start, **label the angles** and then the **corresponding sides**. Because we want to find the length of the side x , we write the Sine Rule for finding sides:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

We do not know the length of the side c nor the size of the angle C , so we cross out this fraction from the formula:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \cancel{\frac{c}{\sin C}}$$

By substituting into the two fractions that are left, we obtain

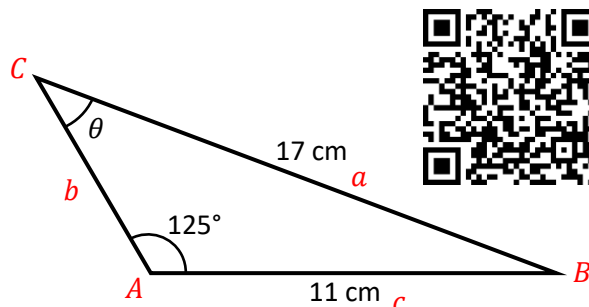
$$\frac{x}{\sin 57^\circ} = \frac{5}{\sin 34^\circ}$$

We can solve this equation by multiplying each side by $\sin 57^\circ$:

$$x = \left(\frac{5}{\sin 34^\circ} \right) \times \sin 57^\circ$$

$$x = 7.50 \text{ cm to 2 decimal places.}$$

(b)



Answer: To start, **label the angles** and then the **corresponding sides**. Because we want to find the size of the angle θ , we write the Sine Rule for finding angles:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

We do not know the length of the side b nor the size of the angle B , so we cross out this fraction from the formula:

$$\frac{\sin A}{a} = \cancel{\frac{\sin B}{b}} = \frac{\sin C}{c}$$

By substituting into the two fractions that are left, we obtain

$$\frac{\sin 125^\circ}{17} = \frac{\sin \theta}{11}$$

We can solve this equation by multiplying each side by 11:

$$\sin \theta = \left(\frac{\sin 125^\circ}{17} \right) \times 11$$

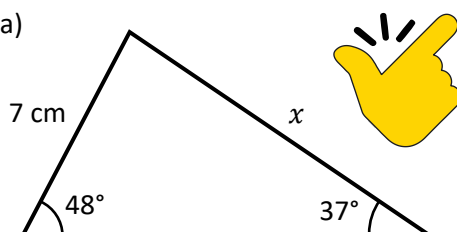
$$\theta = \sin^{-1} \left(\left(\frac{\sin 125^\circ}{17} \right) \times 11 \right)$$

$$\theta = 32.01^\circ \text{ to 2 decimal places.}$$

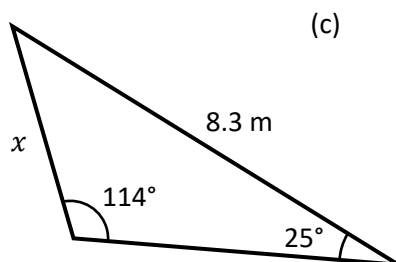
Exercise 5

Calculate the length of the missing side x . (The diagrams are not drawn to scale.)

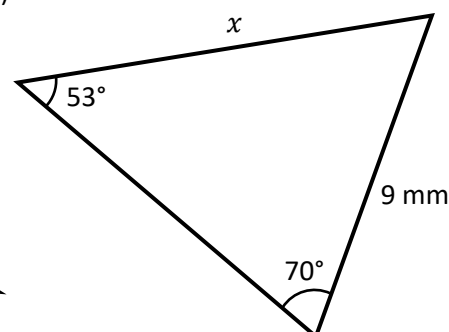
(a)



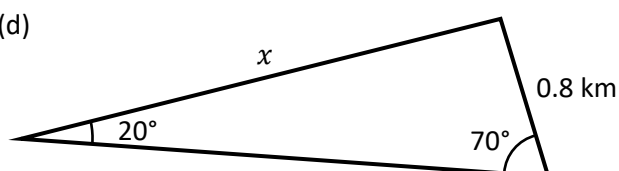
(b)



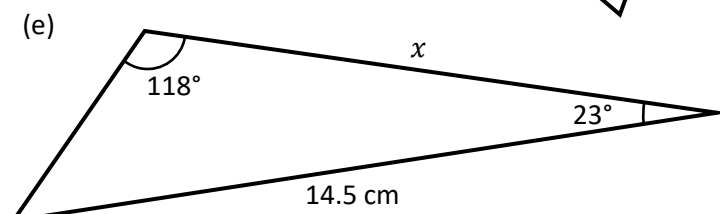
(c)



(d)



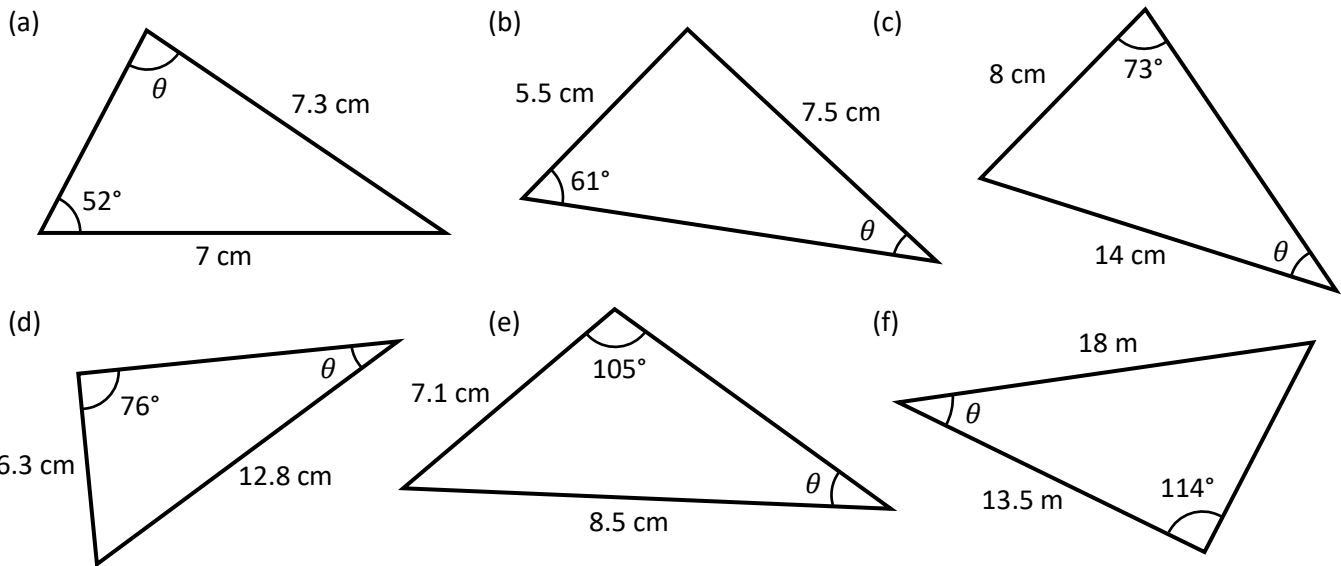
(e)



Exercise 6

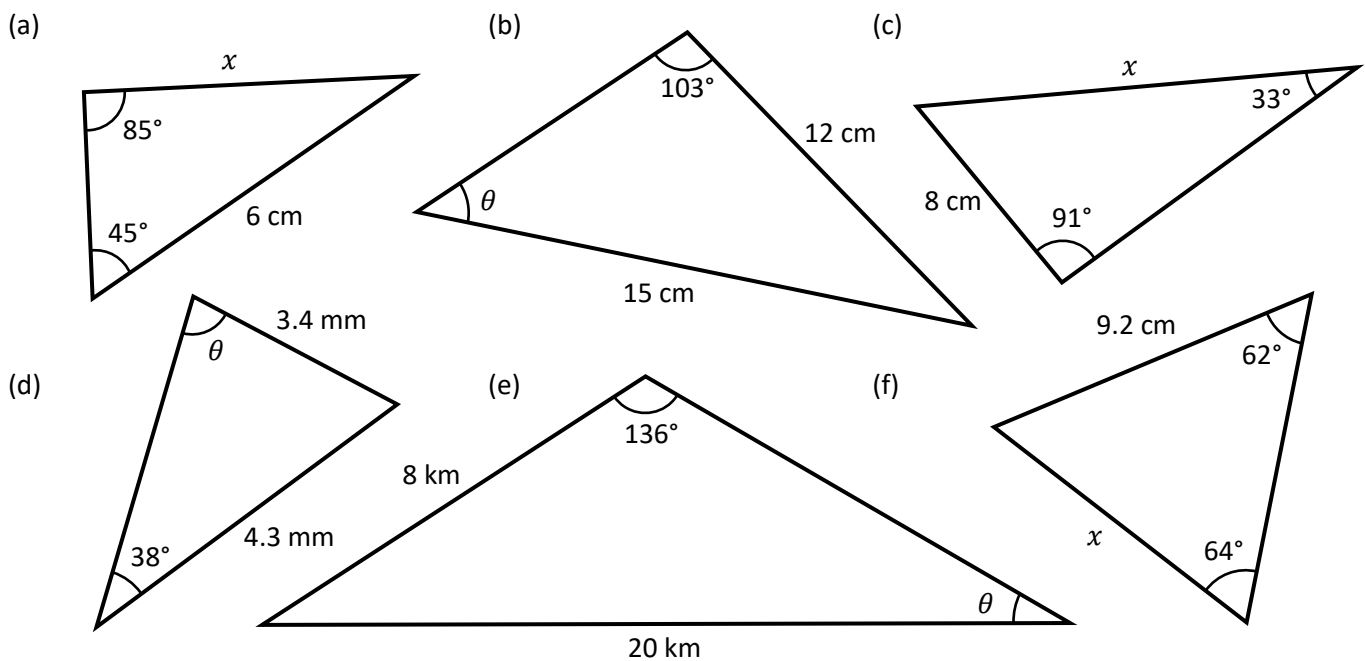
3

Calculate the size of the missing angle θ . (The diagrams are not drawn to scale.)

**Exercise 7**

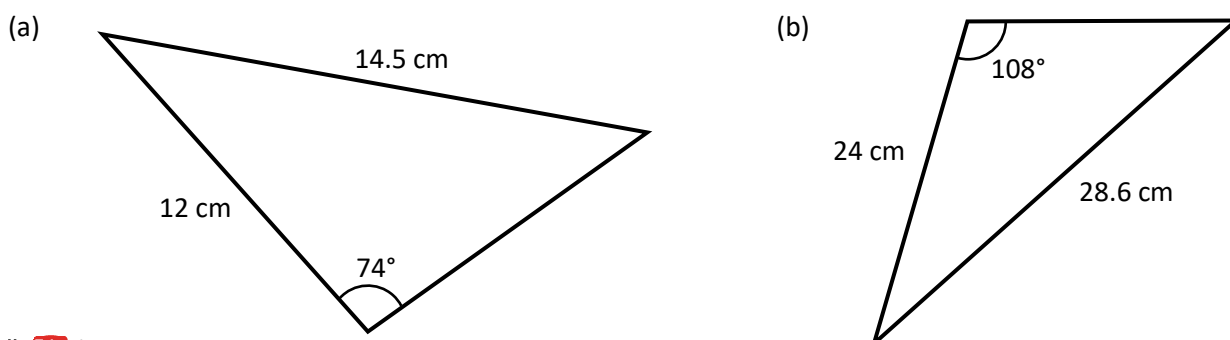
3

Calculate the missing side x or the missing angle θ . (The diagrams are not drawn to scale.)

**Exercise 8**

3

Calculate the size of each of the missing sides and angles in the following diagrams. (The diagrams are not drawn to scale.)

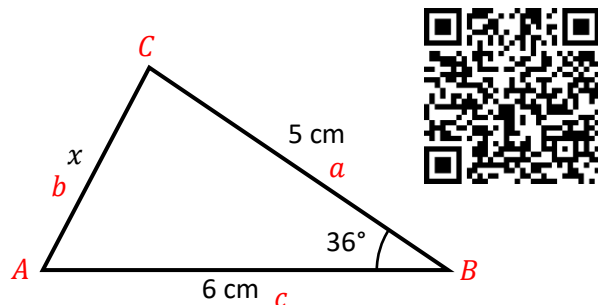


The Cosine Rule

Example

Calculate the missing side x or the missing angle θ . (The diagrams are not drawn to scale.)

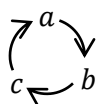
(a)



Answer: To start, **label the angles** and then the **corresponding sides**. Because we want to find the length of the side x , we write the Cosine Rule for finding sides:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

The formula does not fit the labels we have chosen, so we change the variables in the Cosine Rule by cycling around the following circle once.

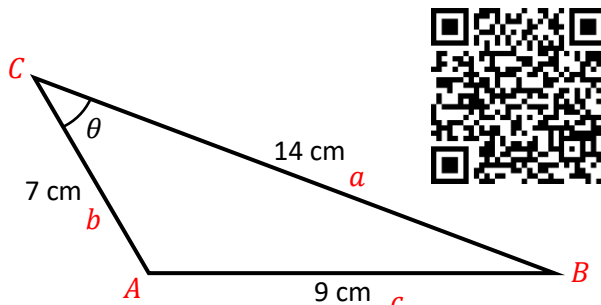


$$b^2 = c^2 + a^2 - 2ca \cos B$$

We can now substitute in the values from the triangle:

$$\begin{aligned} x^2 &= 6^2 + 5^2 - 2 \times 6 \times 5 \times \cos 36^\circ \\ x &= \sqrt{12.45898034} \\ x &= 3.53 \text{ cm to 2 decimal places.} \end{aligned}$$

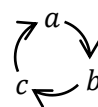
(b)



Answer: To start, **label the angles** and then the **corresponding sides**. Because we want to find the size of the angle θ , we write the Cosine Rule for finding angles:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

The formula does not fit the labels we have chosen, so we change the variables in the Cosine Rule by cycling around the following circle twice.



$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

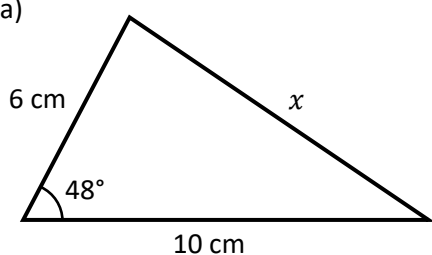
We can now substitute in the values from the triangle:

$$\begin{aligned} \cos \theta &= \frac{14^2 + 7^2 - 9^2}{2 \times 14 \times 7} \\ \theta &= \cos^{-1} \left(\frac{41}{49} \right) \\ \theta &= 33.20^\circ \text{ to 2 decimal places.} \end{aligned}$$

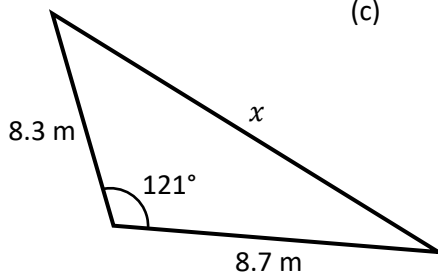
Exercise 9

Calculate the length of the missing side x . (The diagrams are not drawn to scale.)

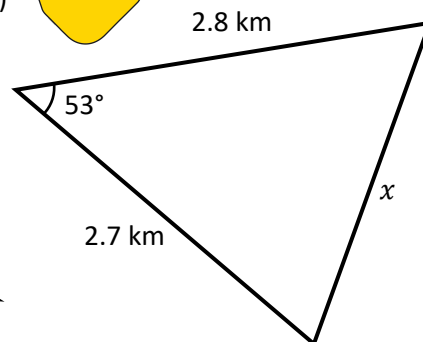
(a)



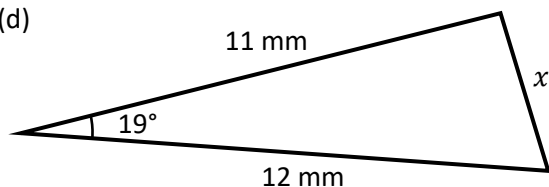
(b)



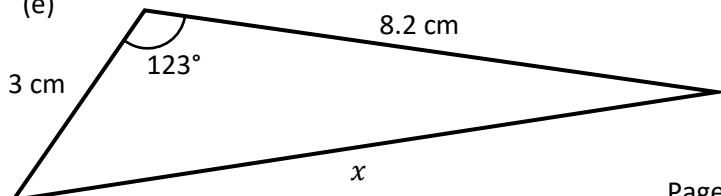
(c)



(d)

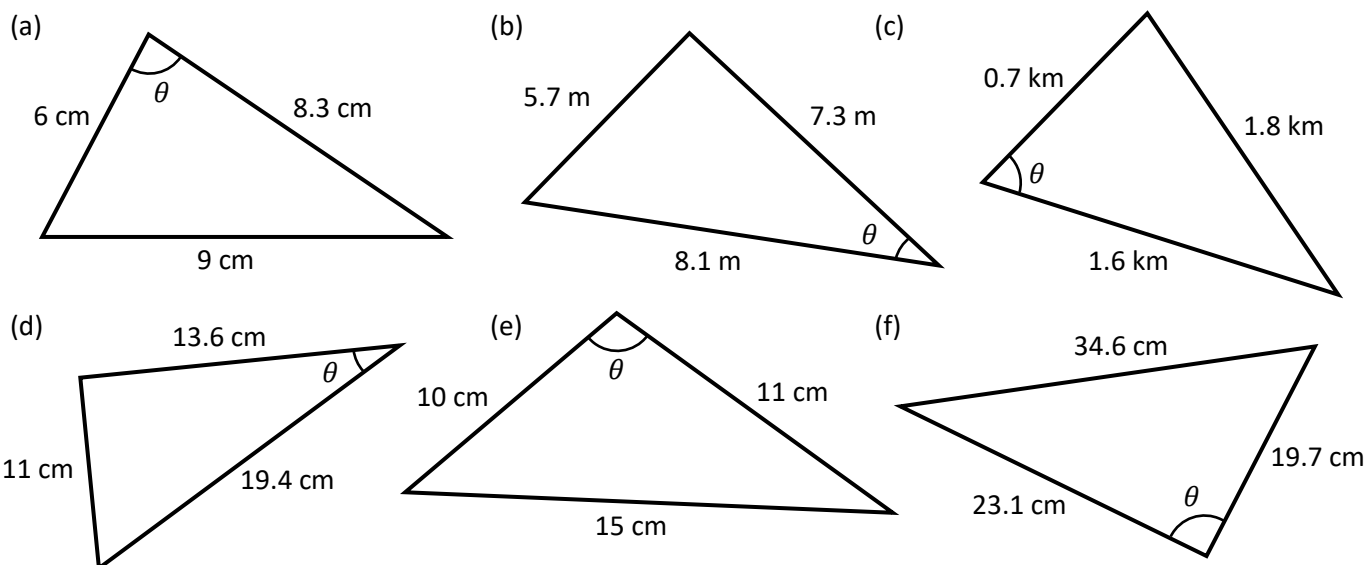


(e)

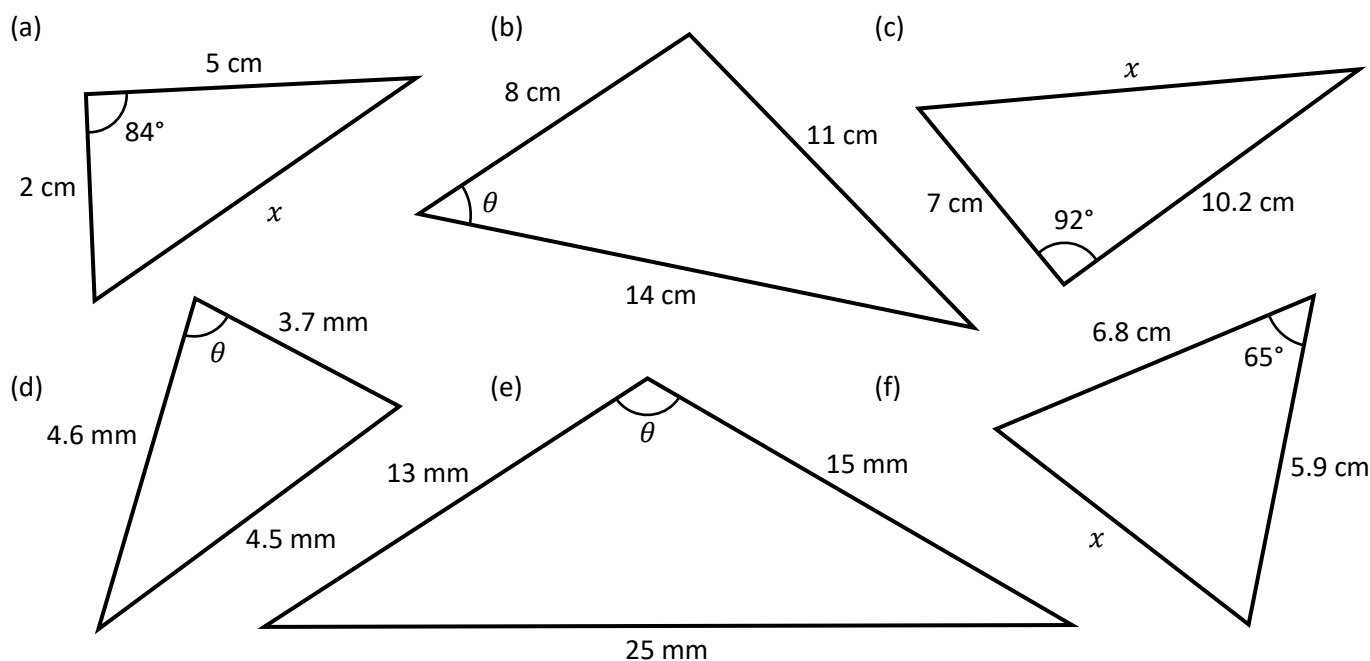


Exercise 10

3

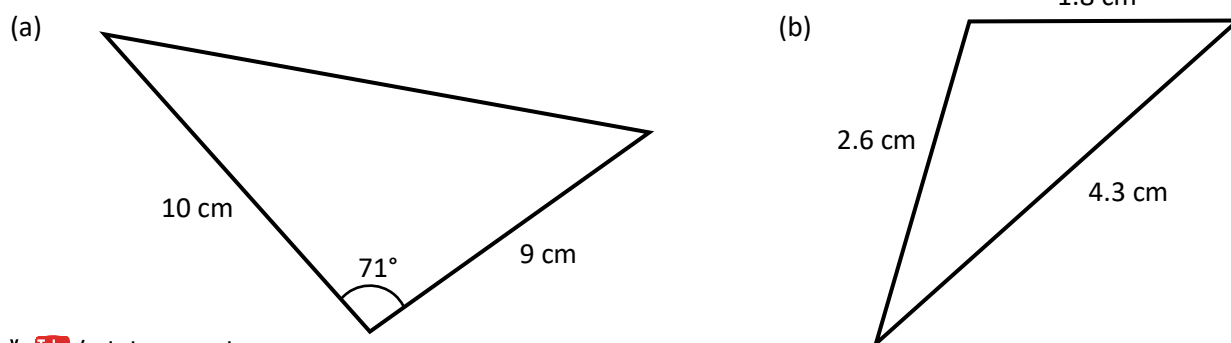
Calculate the size of the missing angle θ . (The diagrams are not drawn to scale.)**Exercise 11**

3

Calculate the missing side x or the missing angle θ . (The diagrams are not drawn to scale.)**Exercise 12**

3

Calculate the size of each of the missing sides and angles in the following diagrams. (The diagrams are not drawn to scale.)



Sine Rule or Cosine Rule?

The Sine Rule works for any triangle where we know

- The lengths of two sides and the size of an angle that is not between the two sides
- The length of one side and the size of any two angles (because that means in reality we know all three angles).

The Cosine Rule works for any triangle where we know

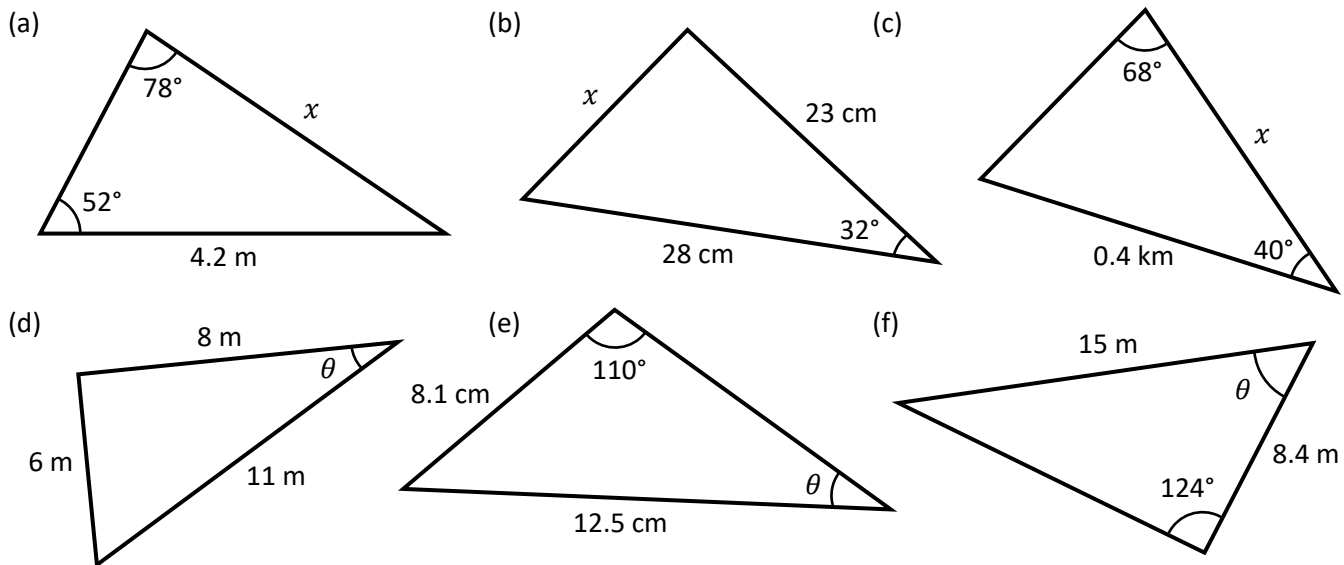
- The lengths of the three sides
- The lengths of two of the sides and the size of the angle between the sides.



3

Exercise 13

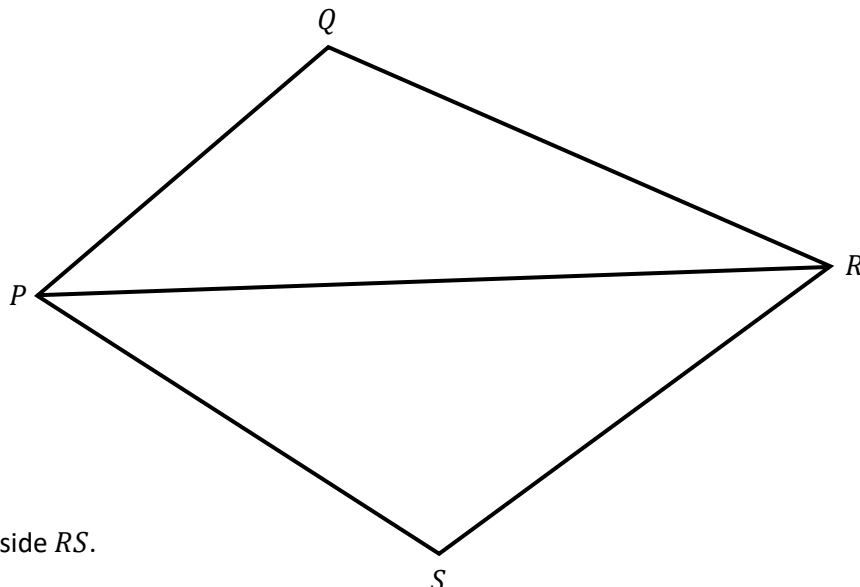
For the following triangles, decide whether we need to use the Sine Rule or the Cosine Rule to calculate the length of the missing side x or the size of the missing angle θ .

**Exercise 14**

For the triangles in Exercise 13, calculate the missing side x or the missing angle θ .
(The diagrams are not drawn to scale.)

Exercise 15

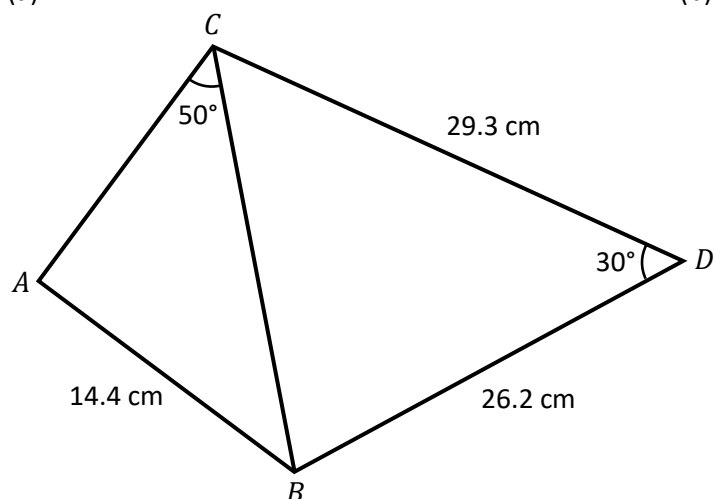
The following diagram shows two triangles PQR and PRS with $PQ = 24$ cm, $QR = 18$ cm, $\hat{PQR} = 124^\circ$, $\hat{SPR} = 36^\circ$ and $\hat{PSR} = 112^\circ$.



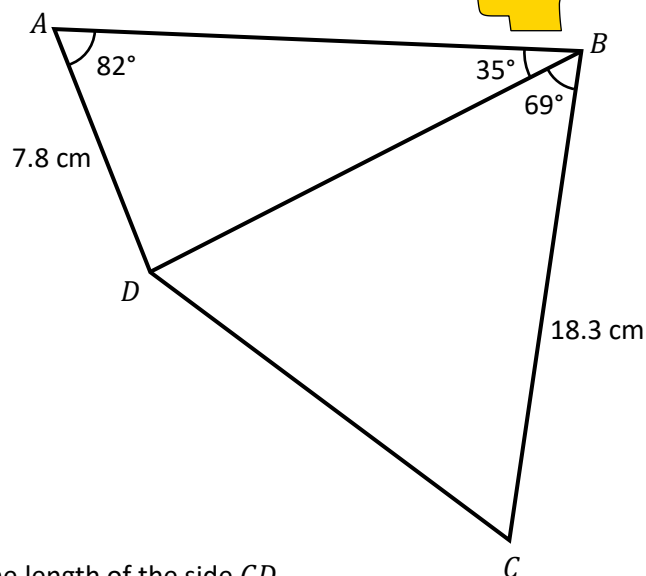
Find the length of the side RS .

Exercise 16 (The diagrams are not drawn to scale)

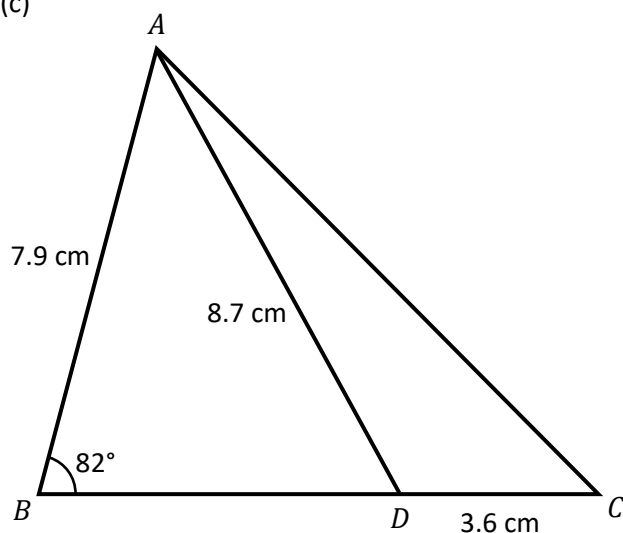
(a)

Find the size of the angle $\hat{CAB}$.

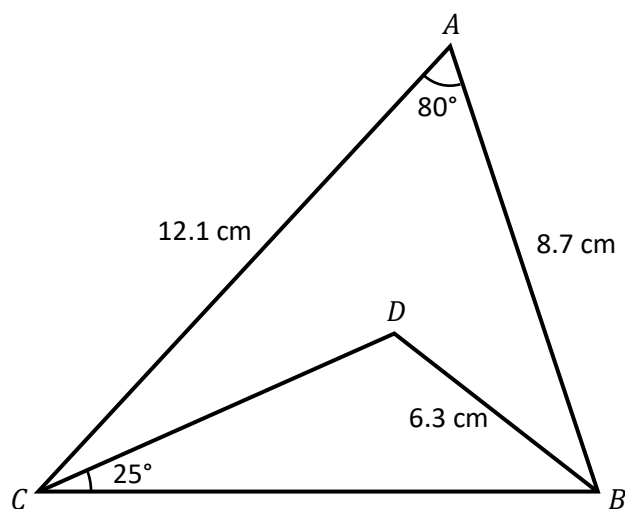
(b)

Find the length of the side CD .

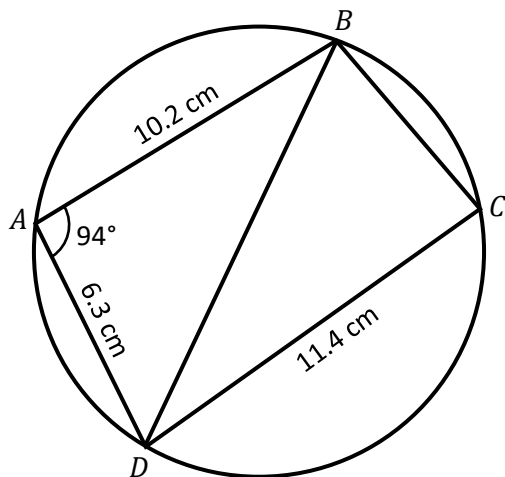
(c)

Given that BC is a straight edge, find the length AC .

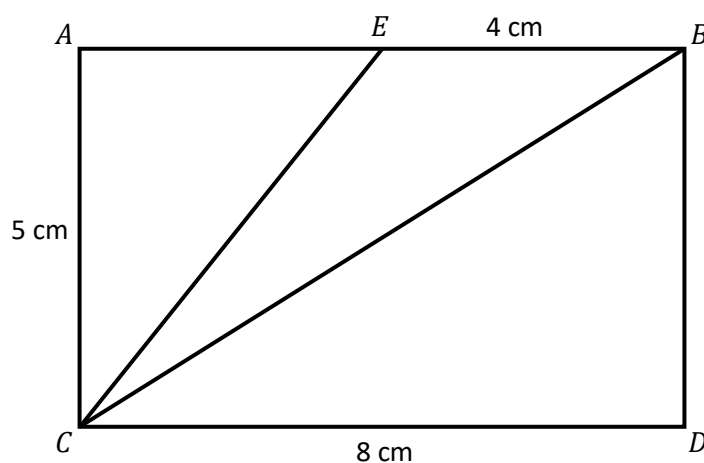
(d)

Find the size of the angle $\hat{CDB}$.

(e)

Find the size of the angle $\hat{DBC}$.

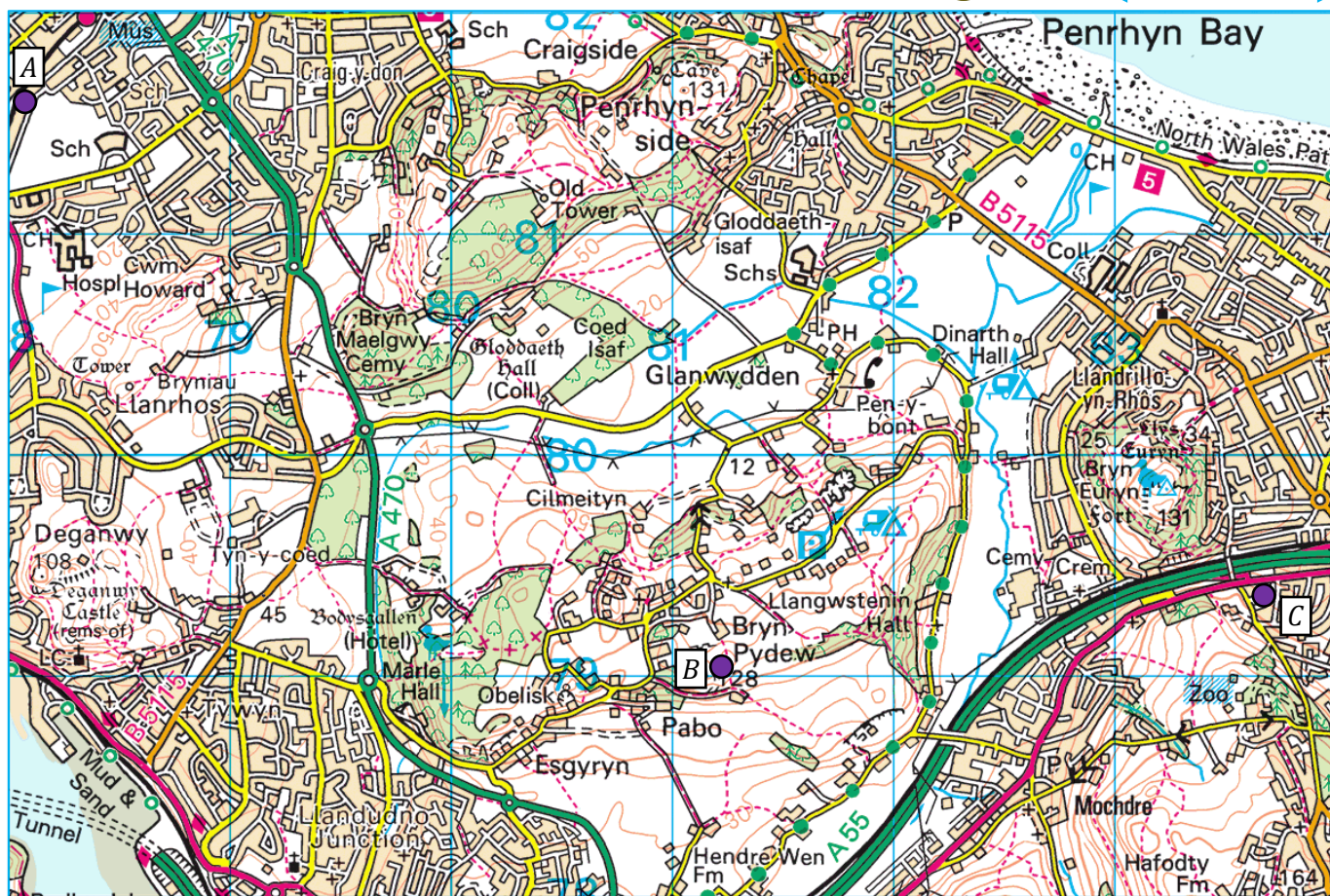
(f)

Given that $ABCD$ is a rectangle, calculate the size of $\hat{CEB}$.

Exercise 17 (Triangulation of a Mobile Phone Location)

3

1 km



The above map shows the location of three mobile phone masts A , B , C belonging to Vodafone.

- Steve's mobile phone reports that it is exactly 3 km from mast A . Use a compass to plot Steve's possible locations on the map.
- Steve's mobile phone also reports that it is exactly 2 km from mast B . Use a compass to find Steve's two possible locations.
- Given that Steve is around 3 km from mast C , mark Steve's location on the map with the letter S .
- Given that the distance between masts A and B is 4,064 m, find the size of the angle $\hat{A}SB$.

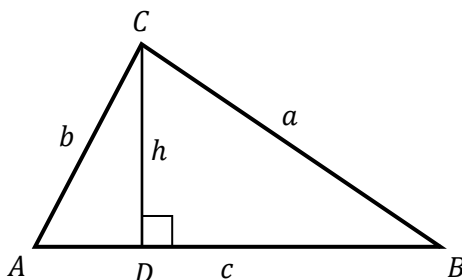
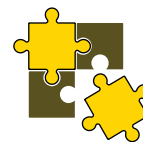


Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Area of a Triangle

The following diagram shows a general triangle with sides a , b , c and angles A , B , C .



Let us draw the perpendicular from the vertex C to the base AB . Using the triangle CAD that is formed, we see that $\sin A = \frac{h}{b}$, and so $h = b \sin A$. Using the formula Area of a Triangle = $\frac{\text{base} \times \text{height}}{2}$,

$$\text{Area of a Triangle} = \frac{c \times b \sin A}{2}$$

$$\text{Area of a Triangle} = \frac{1}{2} bc \sin A$$

Or, changing the variables using $\begin{matrix} a & \curvearrowright & b \\ c & \curvearrowleft & \end{matrix}$,

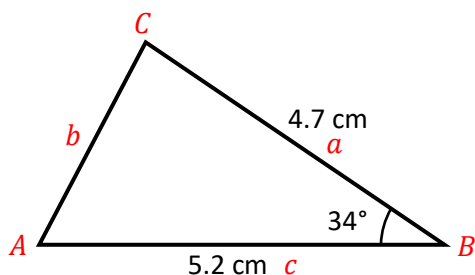
$$\text{Area of a Triangle} = \frac{1}{2} ac \sin B$$

$$\text{Area of a Triangle} = \frac{1}{2} ab \sin C$$

This is the version given on page 2 of the Unit 3 paper.

Example

(a) Calculate the area of the triangle below.



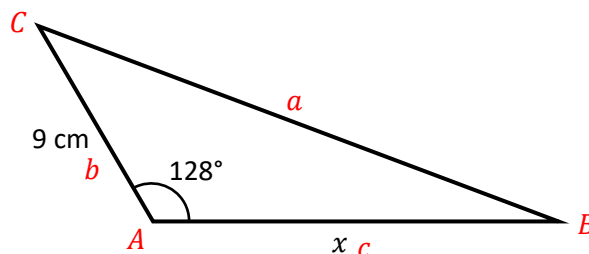
Answer: To start, we **label the angles** and then the **corresponding sides**. Using the formula

$$\text{Area of a Triangle} = \frac{1}{2} ac \sin B$$

$$\text{Area of a Triangle} = \frac{1}{2} \times 4.7 \times 5.2 \times \sin 34^\circ$$

$$\text{Area of a Triangle} = 6.83 \text{ cm}^2 \text{ to 2 decimal places.}$$

(b) Given that the area of the triangle below is 27 cm^2 , calculate the length x of the base of the triangle.



Answer: To start, we **label the angles** and then the **corresponding sides**. Using the formula

$$\text{Area of a Triangle} = \frac{1}{2} bc \sin A$$

$$27 = \frac{1}{2} \times 9 \times x \times \sin 128^\circ$$

$$27 \times 2 = 9 \times x \times \sin 128^\circ$$

$$\frac{54}{9 \times \sin 128^\circ} = x$$

$$x = 7.61 \text{ cm to 2 decimal places.}$$

We can use the formula

$$\text{Area of a Triangle} = \frac{1}{2} ab \sin C$$

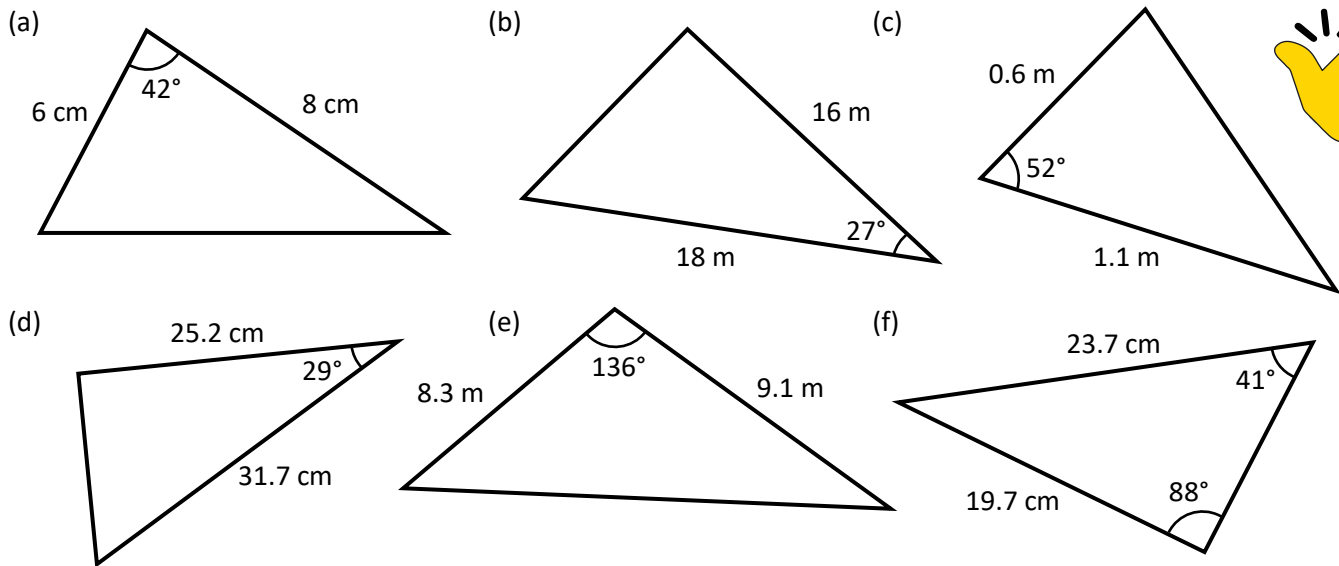
if we know the length of two of the sides of a triangle, and the size of the angle between the sides.

Exercise 18

Calculate the area of the following triangles. (The diagrams are not drawn to scale.)

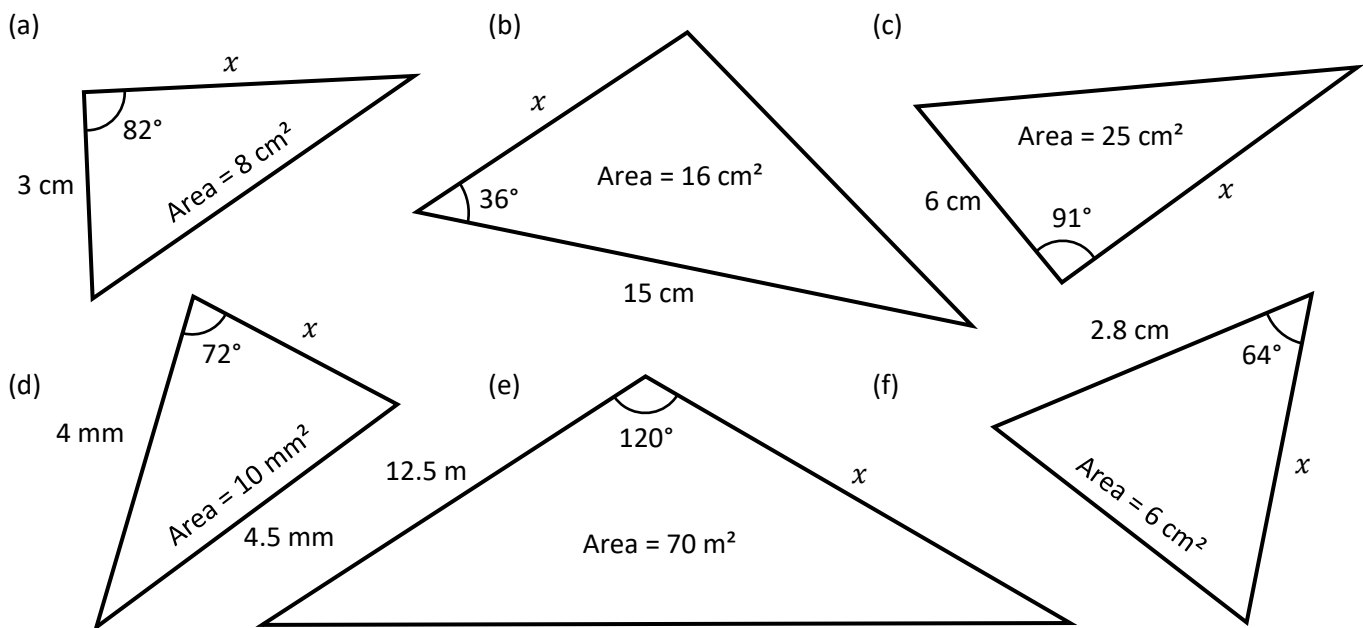
Skill

3

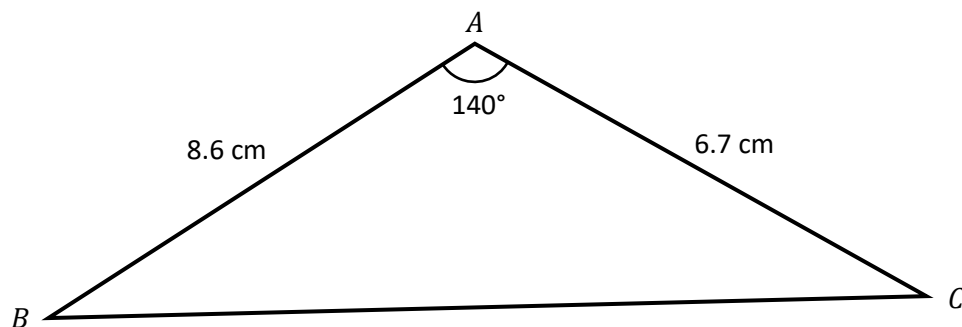
**Exercise 19**

3

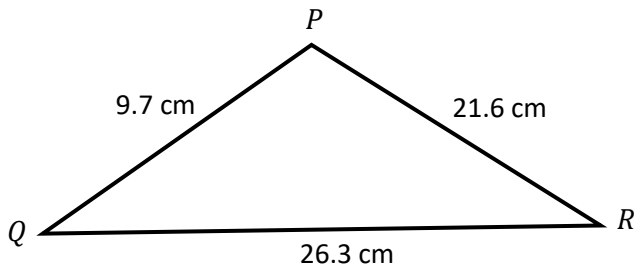
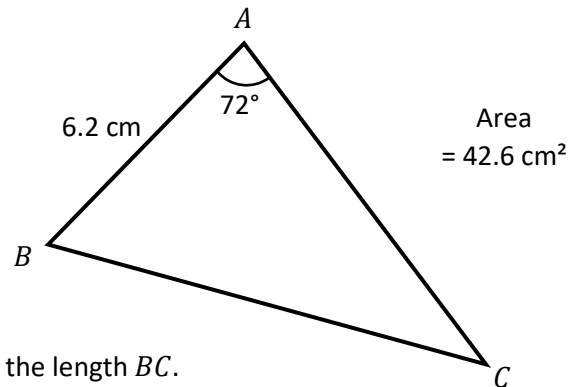
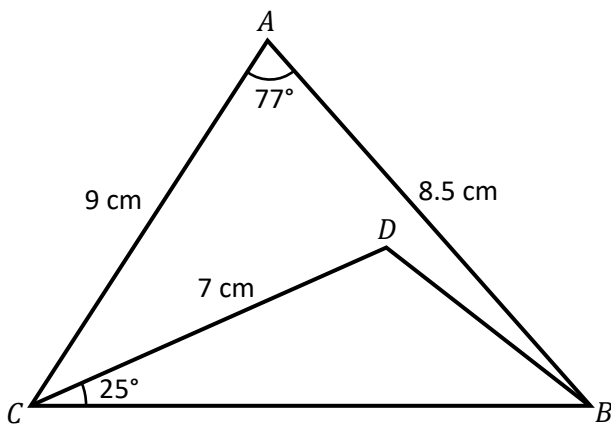
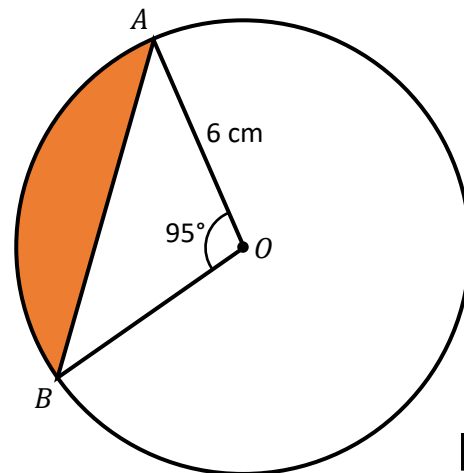
Calculate the length of the missing side x . (The diagrams are not drawn to scale.)

**Exercise 20**

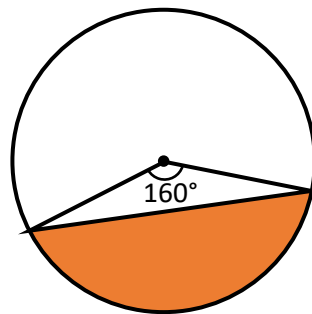
3



- Find the length of the side BC .
- Calculate the area of the triangle ABC .
- Hence, find the perpendicular distance between A and BC .

**Exercise 21 (The diagrams are not drawn to scale.)**(a) Calculate the area of the triangle PQR .(b) The following diagram shows the triangle ABC .Calculate the length BC .(c) Calculate the area of the quadrilateral $ABCD$.(d) Calculate the area of the **minor segment** AB .**Challenge!**

The area of the segment on the right is 50 cm^2 .
Calculate the length of the radius of the circle.

**Evaluation**

Key words	Corrections	I am happy with...	I need to revise...

Trigonometric Graphs

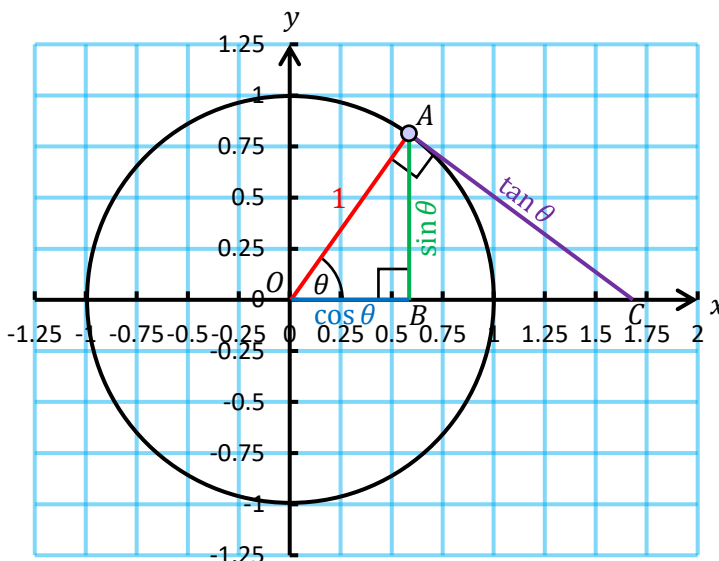


The Unit Circle

Let us consider a unit circle (a circle where the radius is 1 unit) where the centre of the circle O is located at the origin of a set of x and y axes.

If A represents a general point on the circle's circumference, let θ represent the angle between the radius OA and the positive x -axis. Then

- $\sin \theta$ is the vertical displacement from the x -axis to the point A ;
- $\cos \theta$ is the horizontal displacement from the origin to the x -coordinate of A ;
- $\tan \theta$ is the length of the tangent to the point A , measuring from A to the x -axis.



Why is this true?

Let us consider the right-angled triangle OAB to begin with.

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin \theta = \frac{AB}{1}$$

$$1 \times \sin \theta = AB$$

$$AB = \sin \theta$$

So, the height of the triangle is $\sin \theta$.

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos \theta = \frac{OB}{1}$$

$$1 \times \cos \theta = OB$$

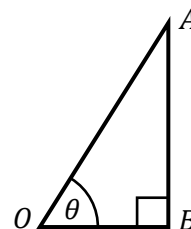
$$OB = \cos \theta$$

So, the base of the triangle is $\cos \theta$.

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan \theta = \frac{AB}{OB}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$



Next, let us consider the right-angled triangle ABC .

We have $\hat{BAC} = \theta$ because the triangles OAB , OAC and ABC are all similar triangles (they share the same angles).

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

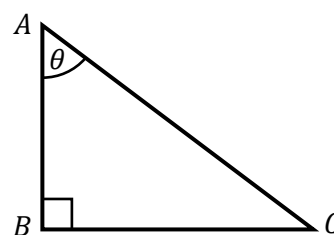
$$\cos \theta = \frac{AB}{AC}$$

$$AC \times \cos \theta = AB$$

$$AC = \frac{AB}{\cos \theta}$$

$$AC = \frac{\sin \theta}{\cos \theta} \quad [\text{from the triangle } OAB]$$

But the triangle OAB also tells us that $\tan \theta = \frac{\sin \theta}{\cos \theta}$, so we must have $AC = \tan \theta$.



Exercise 22

Experiment with the unit circle using GeoGebra: <https://www.geogebra.org/m/fGsz9sfN>

Move the point A around the circle.

What happens to the values of $\sin \theta$, $\cos \theta$ and $\tan \theta$ as you move A ?

When are $\sin \theta$, $\cos \theta$ and $\tan \theta$ positive, and when are they negative? What are their lowest and highest values?

Write a paragraph summarising your findings.

Skill

3

Exercise 23

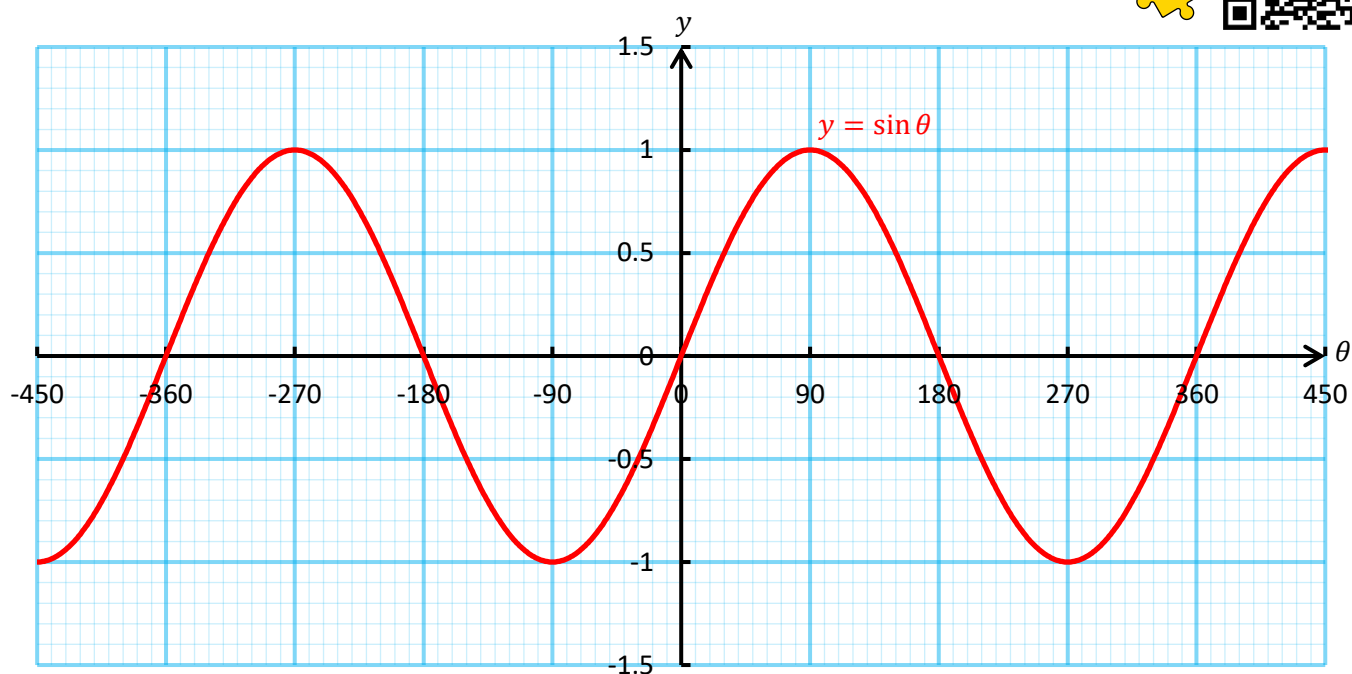
3

Use your calculator to complete the following table. Give your answers correct to 4 decimal places.

Angle θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0		
30°		0.8660	
60°			1.7321
90°			
120°	0.8660		
150°		-0.8660	
180°			0
210°			
240°	-0.8660		
270°		0	
300°			-1.7321
330°			
360°	0		

Trigonometric Graphs

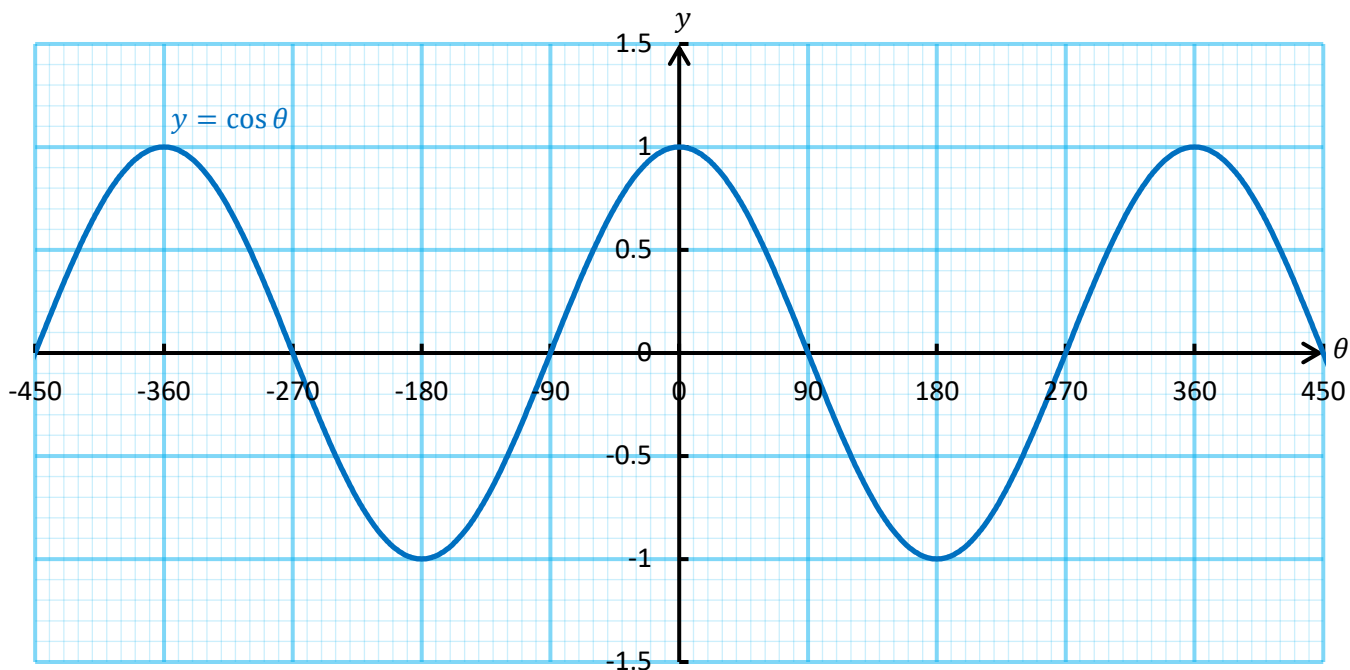
$$y = \sin \theta$$



Properties

- The greatest **amplitude** (height) of the graph is 1 unit. The graph varies between -1 and 1 .
- The **period** of the graph is 360° . (The graph repeats every 360° .)

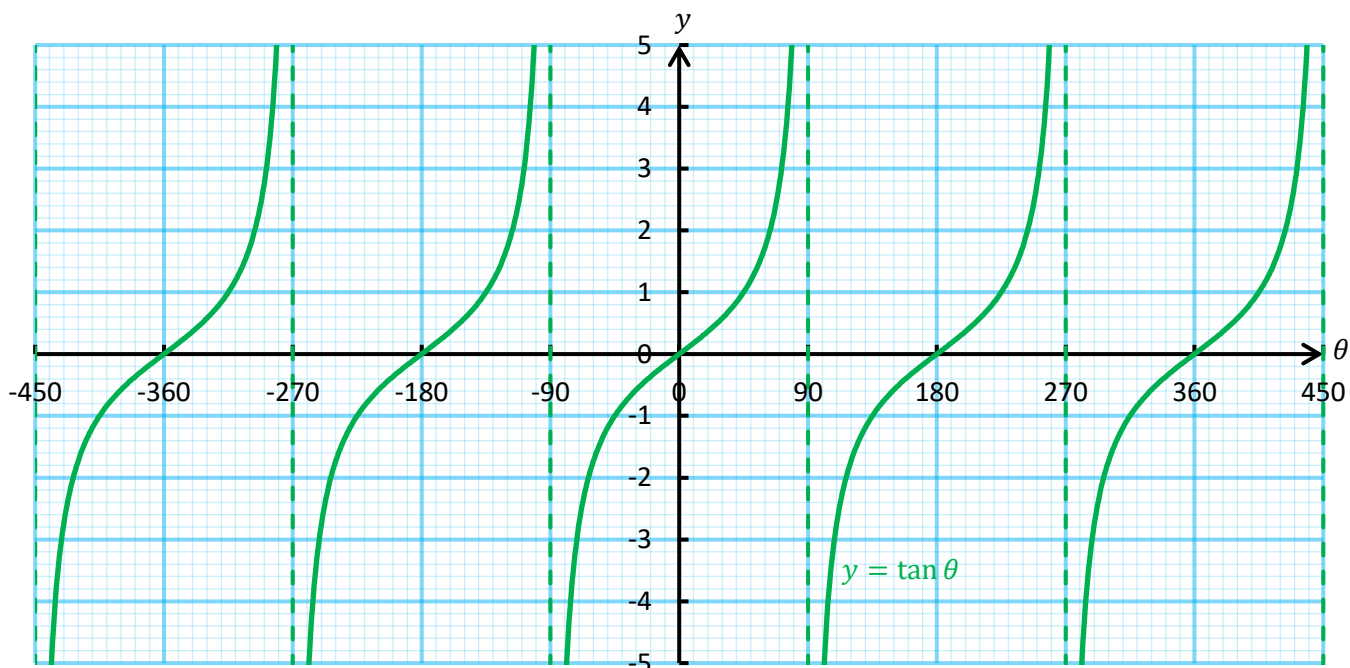
$$y = \cos \theta$$



Properties

- The greatest **amplitude** (height) of the graph is 1 unit. The graph varies between -1 and 1 .
- The **period** of the graph is 360° . (The graph repeats every 360° .)

$$y = \tan \theta$$



Properties

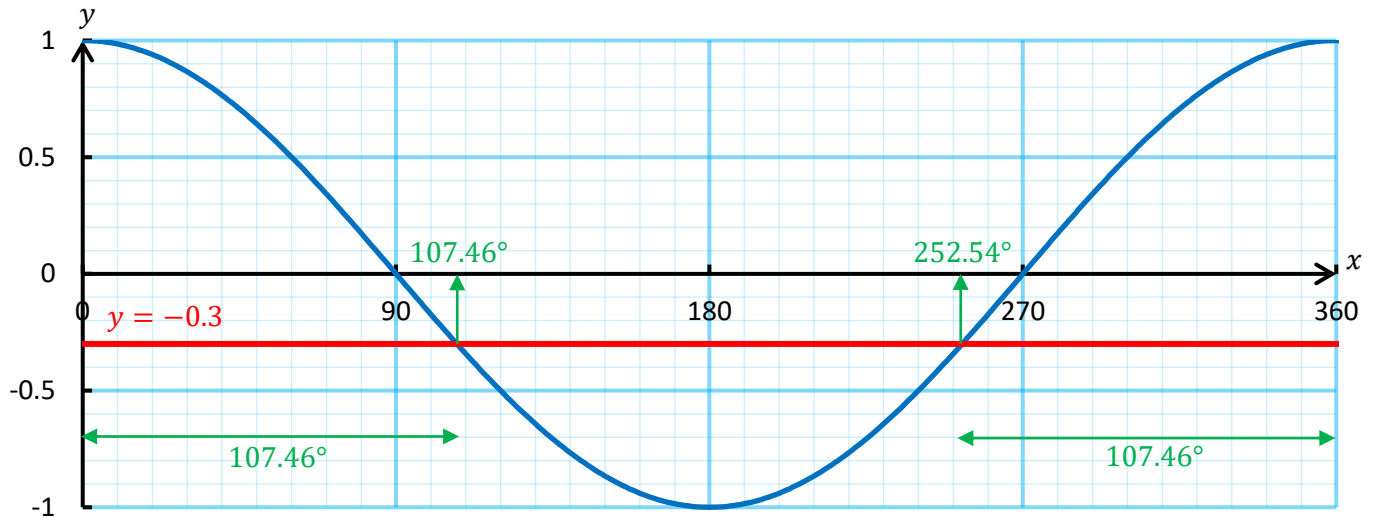
- The **amplitude** (height) of the graph is not defined. The graph varies between $-\infty$ and ∞ .
- The **period** of the graph is 180° . (The graph repeats every 180° .)
- The graph has **asymptotes** every 180° , e.g. at -90° or at 90° . $y = \tan \theta$ is not defined at these angles.

Solving Trigonometric Equations

Example



The following diagram shows a sketch of $y = \cos x$ for values of x between 0° and 360° .



Find all the solutions of the following equation between 0° and 360° .

$$\cos x = -0.3$$

Answer: To begin, draw the **horizontal red line $y = -0.3$** on the graph. This line intersects the **blue curve** at two different points, so there are two solutions to the equation between 0° and 360° . We can find one of the solutions by using a calculator:

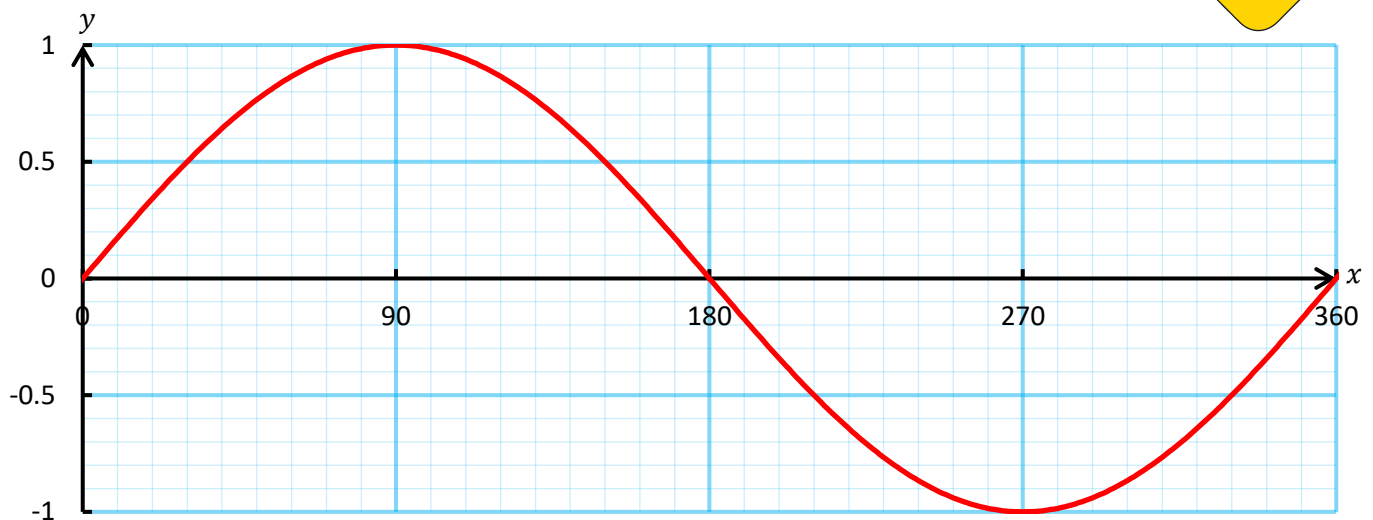
$$\begin{aligned} x &= \cos^{-1}(-0.3) \\ x &= 107.46^\circ \text{ to 2 decimal places} \end{aligned}$$

We can find the second solution by using the symmetry of the graph of $y = \cos x$. If 107.46° is a solution, then $360^\circ - 107.46^\circ = 252.54^\circ$ must also be a solution. So, the answers (to 2 decimal places) are 107.46° and 252.54° .

Exercise 24



The following diagram shows a sketch of $y = \sin x$ for values of x between 0° and 360° .



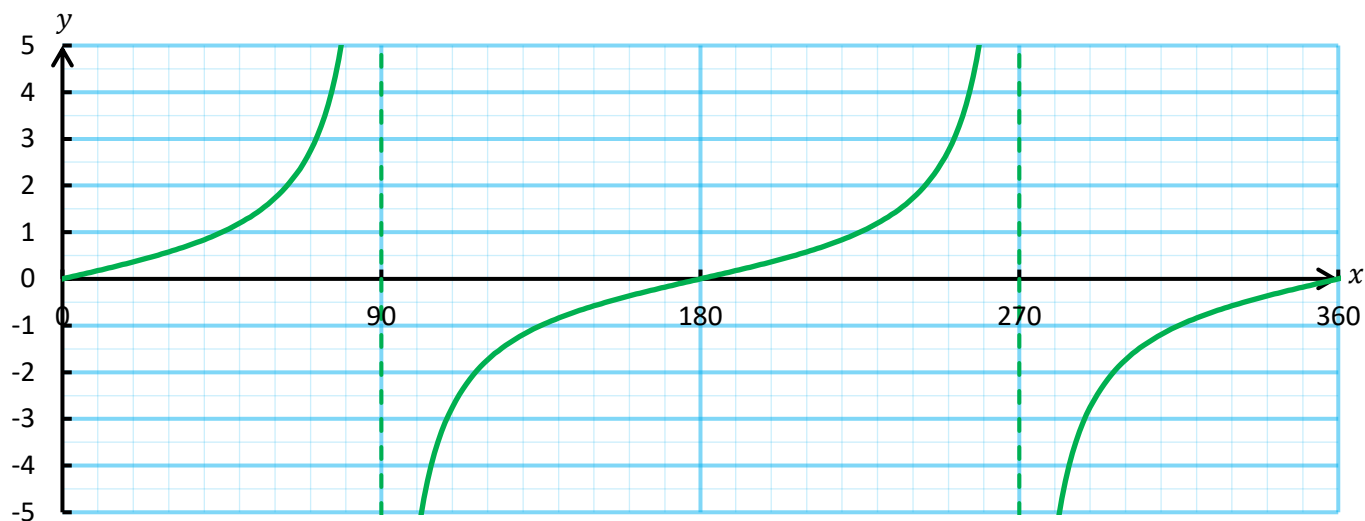
Find all the solutions of the following equation between 0° and 360° .

$$\sin x = 0.75$$

Exercise 25

3

The following diagram shows a sketch of $y = \tan x$ for values of x between 0° and 360° .



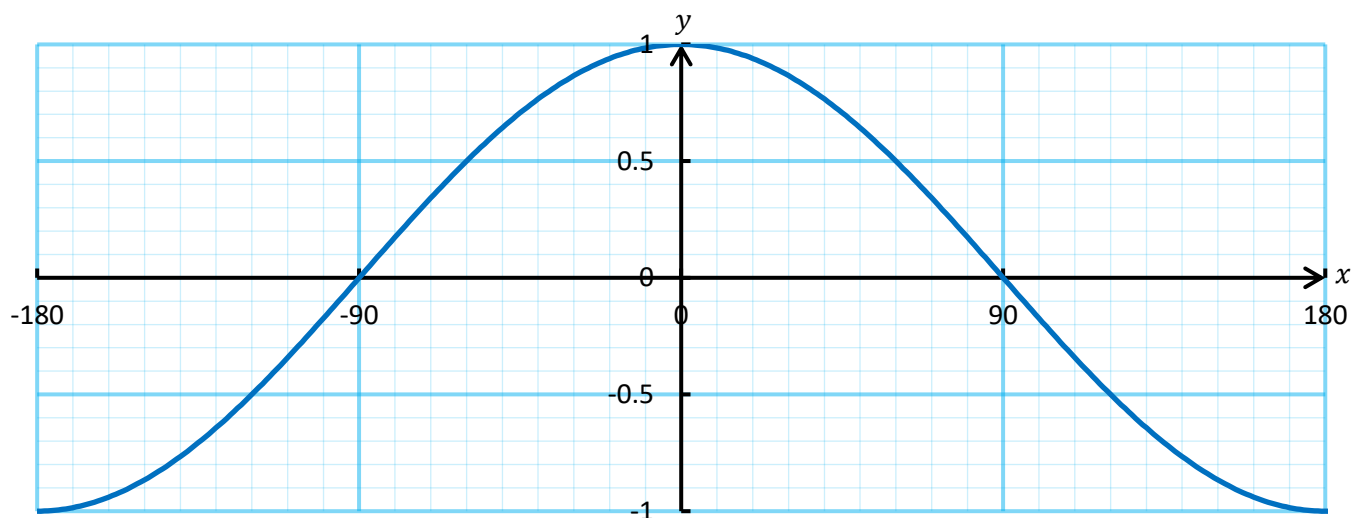
Find all the solutions of the following equation between 0° and 360° .

$$\tan x = 3$$

Exercise 26

3

The following diagram shows a sketch of $y = \cos x$ for values of x between -180° and 180° .



Find all the solutions of the following equation between -180° and 180° .

$$\cos x = -0.8$$

Exercise 27

3

Use suitable graphs to solve the following trigonometric equations.

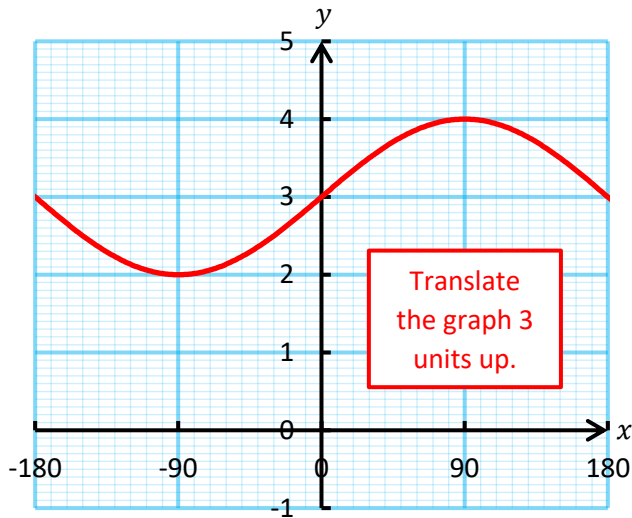
- (a) $\sin x = 0.1$ between 0° and 360°
- (b) $\cos x = 0.83$ between 0° and 360°
- (c) $\tan x = 3.14$ between 0° and 360°
- (d) $\sin x = 0.36$ between -180° and 180°
- (e) $\cos x = -0.4$ between -180° and 180°
- (f) $\tan x = -1.3$ between -180° and 180°

Function Transformations

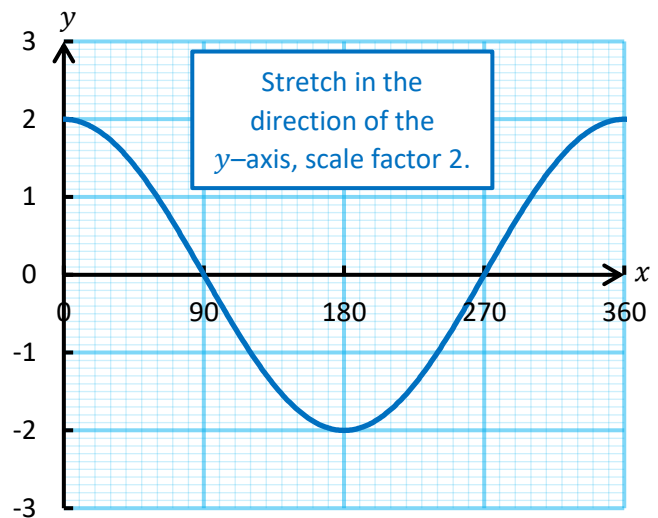
Here are some examples of transformations that use trigonometric functions.



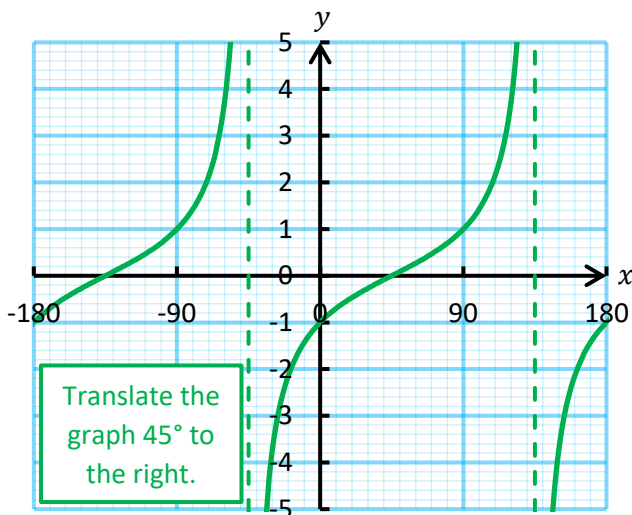
(a) $y = \sin(x) + 3$ between -180° and 180° .



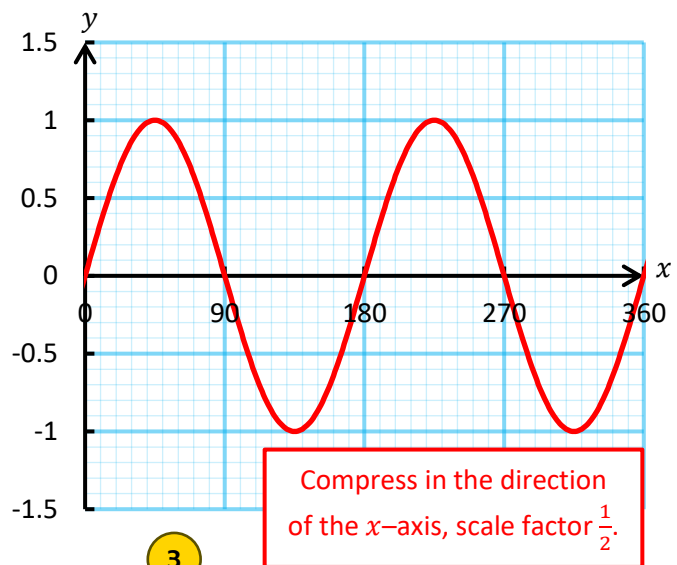
(b) $y = 2 \cos(x)$ between 0° and 360° .



(c) $y = \tan(x - 45^\circ)$ between -180° and 180° .



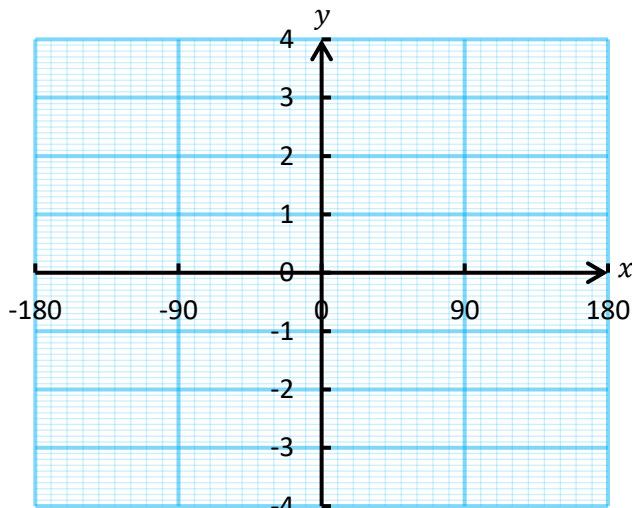
(d) $y = \sin(2x)$ between 0° and 360° .



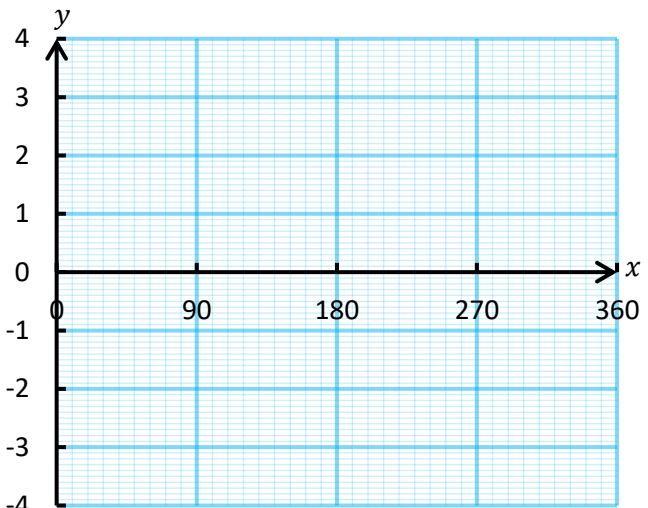
Exercise 28

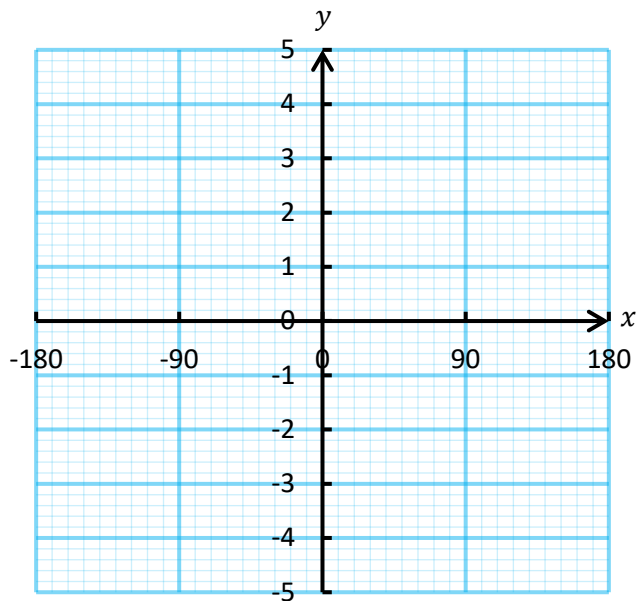
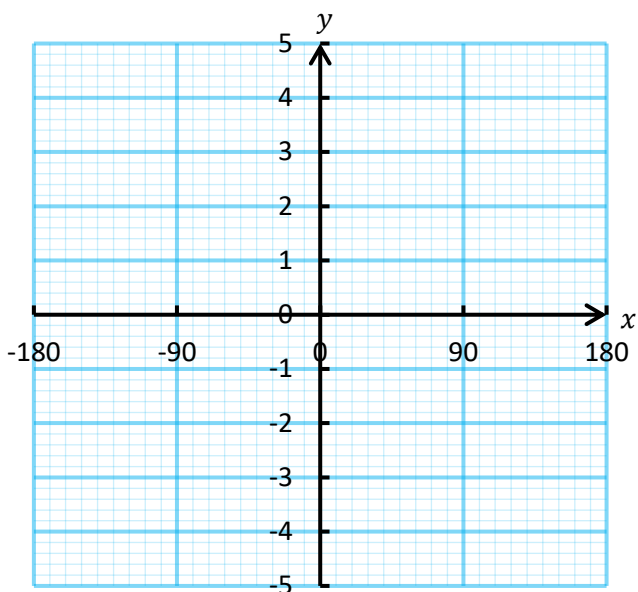
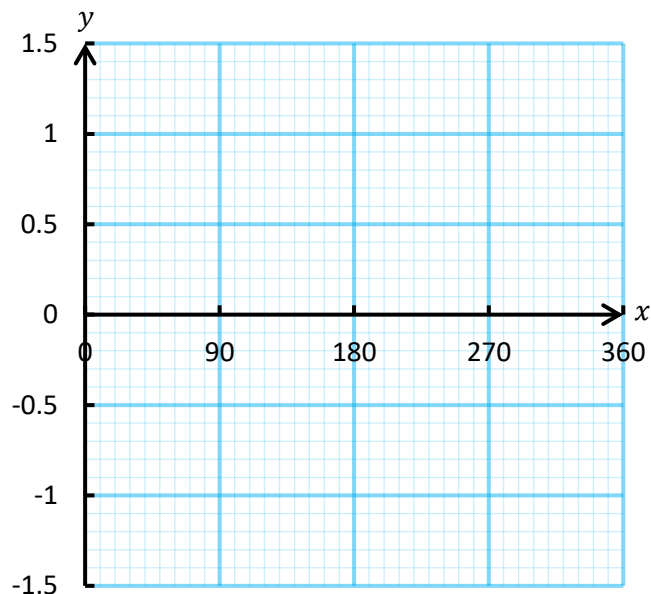
Draw, on the graph paper provided, graphs for the following functions.

(a) $y = \cos(x) - 2$ between -180° and 180° .

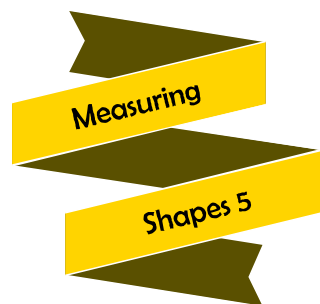


(b) $y = -3 \sin(x)$ between 0° and 360° .



(c) $y = \tan(2x)$ between -180° and 180° .(d) $y = \cos(x + 45^\circ)$ between 0° and 360° .(e) $y = -\tan(x)$ between -180° and 180° .(f) $y = \sin(-x)$ between 0° and 360° .

Key words	Corrections	I am happy with...	I need to revise...



Name:

Percentage in the test:

	I know this. 	I need to revise this. 	Question in the test:	Correct in the test?
I know how to use trigonometry to find angles and lengths in three dimensional shapes .			1	
I know how to use the Sine Rule to find lengths in triangles.			3	
I know how to use the Sine Rule to find angles in triangles.			4, 7	
I know how to use the Cosine Rule to find lengths in triangles.			4	
I know how to use the Cosine Rule to find angles in triangles.			2, 7	
I know when to use the Sine Rule or Cosine Rule to solve problems.			2, 3, 4, 7	
I know how to use the formula $\frac{1}{2}ab \sin C$ to find the area of a triangle .			2, 3, 7	
I know how to find the area of a segment of a circle .			3	
I can recognise and sketch the trigonometric graphs $y = \sin \theta$, $y = \cos \theta$ and $y = \tan \theta$.			5	
I can solve trigonometric equations with the help of a trigonometric graph.			5	
I can draw transformations of trigonometric functions.			6	



The Mathematics Department

11

The End of

Year 11

Higher Tier

Version
3.0
08/2025

Name:

Contents

Chapter	Mathematics	Page Number
Trial and Improvement	Solving equations using trial and improvement (to the nearest unit; to one decimal place; to two decimal places). Using the <i>table mode</i> on a calculator.	3
AER, APR	Annual Equivalent Rate (AER). Alternative method of calculating AER. Annual Percentage Rate (APR).	9
Mopping	Topics from the syllabus that have not appeared in previous workbooks.	15



Trial and Improvement

If we cannot solve an equation by algebraic methods, then we can attempt to solve the equation using **trial and improvement**.

In this method, we start with a sensible guess for the solution (a **trial**), and then **improve** this guess, repeating until we find the solution to a **specified degree of accuracy**.

Example

Let us attempt to solve the equation

$$x^3 + 4x - 29 = 0$$

As we have not seen an equation of this type before (and therefore we do not have an algebraic method of solving the equation), let us use the trial and improvement method to solve the equation. We will find the solution correct to the nearest unit and allow ourselves to use a calculator to perform the calculations.

To begin, let us **guess** that $x = 5$ is the solution to the equation. To see whether this is correct, we need to substitute $x = 5$ into the left-hand side of the equation, and check to see whether we obtain zero.

Buttons to press (Casio fx-83GT CW)	Calculator screen

The sum gives the answer 116, which is too **too high** (116 is greater than zero). Therefore, we must trial a number that is less than 5. Let us try $x = 3$.

Buttons to press	Calculator screen

The answer is still too high (10 is greater than 0), so we must trial an even smaller number, say $x = 2$.

Buttons to press	Calculator screen

This time, we see that the answer is **too low** (–13 is less than 0), so we can say that the solution to the equation is between $x = 2$ and $x = 3$. Because we have decided to solve the equation to the nearest unit, we must now decide whether the true solution is closer to $x = 2$ or to $x = 3$. It *appears* that the solution is closer to $x = 3$ (because 10 is closer to zero than –13), however this is not sufficient evidence to **prove** that the solution (to the nearest unit) is $x = 3$. Instead, we must consider the number that is **half way** between 2 and 3 (namely 2.5) and substitute this number into the left-hand side of the equation.

Buttons to press	Calculator screen

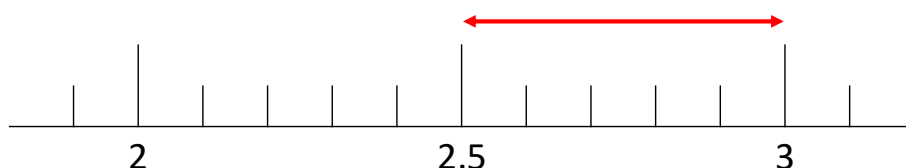
Because –3.375 is less than zero, we can say that the true solution to the equation lies between $x = 2.5$ and $x = 3$. Because every number in this range would round off (to the nearest unit) to be 3, we can now state, without doubt, that the solution to the equation, to the nearest unit, is $x = 3$.

Here's how to set out the solution in the form of a table.

Trial	The value of $x^3 + 4x - 29$	Too high / Too low?
5	116	Too high
3	10	Too high
2	-13	Too low
2.5	-3.375	Too low

 The solution to the equation is between $x = 2.5$ and $x = 3$ so, to the nearest unit, the solution to the equation is $x = 3$.

We can also show the location of the true solution of the equation on a number line.



Exercise 1

The following equations have a solution between $x = 0$ and $x = 10$.

Use the trial and improvement method to find the solution correct to the nearest unit.

(a) $x^3 + 8x - 15 = 0$

(b) $x^3 - 8x - 15 = 0$

(c) $x^3 - 2x - 15 = 0$

(d) $x^2 + 7x - 100 = 0$

(e) $x^2 + 2x - 50 = 0$

(f) $2x^2 - 11x - 80 = 0$

(g) $2x^3 - 2x - 15 = 0$

(h) $0.2x^3 - 10x + 4 = 0$

(i) $x^4 - 60x^2 - 14 = 0$

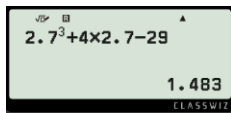
Skill

3

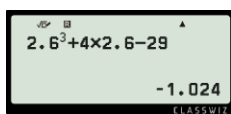
Trial and improvement to one decimal place

Let us return to the previous example of solving the equation $x^3 + 4x - 29 = 0$. We know, from previous work, that the solution to the nearest unit is $x = 3$. Let us now find a more detailed solution, to one decimal place.

From the table on the top of this page, we know that substituting $x = 2.5$ into the equation gives an answer that is too low (-3.375 is less than zero). Let us therefore try to substitute a value that is greater than 2.5, let's say $x = 2.7$.

Buttons to press	Calculator screen
	

As 1.483 is greater than zero, we must trial a number that is less than 2.7, let's say $x = 2.6$.

Buttons to press	Calculator screen
	

Now -1.024 is less than zero, so we can say that the solution lies between $x = 2.6$ and $x = 2.7$. As before, we must now decide whether the solution lies closer to $x = 2.6$ or to $x = 2.7$. To do this, we must consider the number that is **half way** between 2.6 and 2.7 (namely 2.65) and substitute this number into the left-hand side of the equation.

Buttons to press	Calculator screen

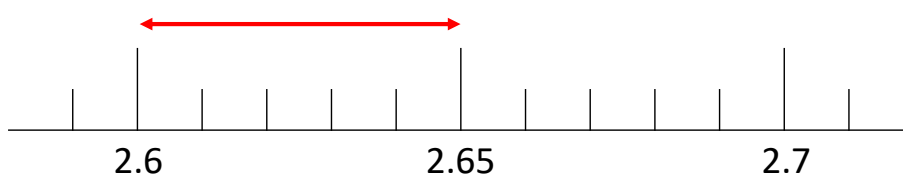
Because 0.209625 is greater than zero, we can say that the true solution to the equation lies between $x = 2.6$ and $x = 2.65$. Because every number in this range would round off (to one decimal place) to be 2.6, we can now state, without doubt, that the solution to the equation, correct to one decimal place, is $x = 2.6$.

Here's how to set out the solution to this question in the form of a table (continuing the previous work).

Trial	The value of $x^3 + 4x - 29$	Too high / Too low?
5	116	Too high
3	10	Too high
2	-13	Too low
2.5	-3.375	Too low
2.7	1.483	Too high
2.6	-1.024	Too low
2.65	0.209625	Too high

The solution to the equation is between $x = 2.6$ and $x = 2.65$ so, to one decimal place, the solution to the equation is $x = 2.6$.

We can also show the location of the true solution of the equation on a number line.



Exercise 2

The following equations have a solution between $x = 0$ and $x = 10$.

Use the trial and improvement method to find the solution correct to one decimal place.

(a) $x^3 + 8x - 15 = 0$

(b) $x^3 - 7x - 120 = 0$

(c) $x^3 - 3x - 93 = 0$

(d) $x^3 + 3x = 15$

(e) $x^2 + x - 79 = 0$

(f) $x^2 + 3x = 39$

(g) $2x^3 - 2x - 11 = 0$

(h) $2x(x + 5) = 110$

(i) $x^4 - 43x^2 - 25 = 0$



Trial and improvement to two decimal places

Let us again return to the example of solving the equation $x^3 + 4x - 29 = 0$. We know, from previous work, that the solution to one decimal place is $x = 2.6$. Let us now find a more detailed solution, to two decimal places.



From the table on the top of this page, we see that substituting $x = 2.65$ into the equation gives an answer that is too high (0.209625 is greater than zero). Let us therefore try to substitute a value that is smaller than 2.65, let's say $x = 2.64$.

Buttons to press	Calculator screen

Now -0.040256 is less than zero, so we can say that the solution lies between $x = 2.64$ and $x = 2.65$. As before, we must now decide whether the solution is closer to $x = 2.64$ or to $x = 2.65$. To do this, we must consider the number that is **half way** between 2.64 and 2.65 (namely 2.645) and substitute this number into the left-hand side of the equation.

Buttons to press	Calculator screen

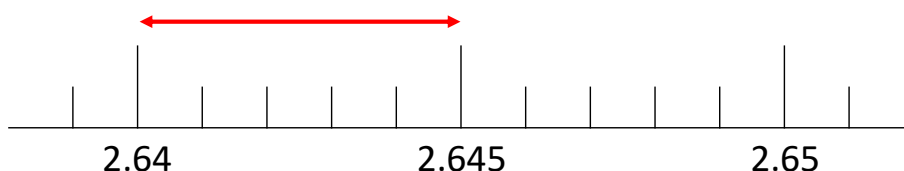
Because 0.084486125 is greater than zero, we can say that the true solution to the equation lies between $x = 2.64$ and $x = 2.645$. Because every number in this range would round off (to two decimal places) to be 2.64, we can now state, without doubt, that the solution to the equation, correct to two decimal places, is $x = 2.64$.

Here's how to set out the solution to this question in the form of a table (continuing the previous work).

Trial	The value of $x^3 + 4x - 29$	Too high / Too low?
5	116	Too high
3	10	Too high
2	-13	Too low
2.5	-3.375	Too low
2.7	1.483	Too high
2.6	-1.024	Too low
2.65	0.209625	Too high
2.64	-0.040256	Too low
2.645	0.084486125	Too high

The solution to the equation is between $x = 2.64$ and $x = 2.645$ so, to two decimal places, the solution to the equation is $x = 2.64$.

We can also show the location of the true solution of the equation on a number line.



Exercise 3

The following equations have a solution between $x = 0$ and $x = 10$.

Use the trial and improvement method to find the solution correct to two decimal places.

(a) $x^3 + 8x - 15 = 0$

(b) $x^3 - 5x - 120 = 0$

(c) $x^3 - 2x - 65 = 0$

(d) $x^3 + 7x = 24$

(e) $x^2 + x - 43 = 0$

(f) $x^2 + 6x = 24$

(g) $2x^3 - 3x - 25 = 0$

(h) $x(24 + x) = 18$

(i) $x^4 - x^2 - 48 = 0$

Using the 'Table Mode'

Some scientific calculators have a 'table mode' that can be used in the solution of trial and improvement questions. The advantage of using the *table mode* is to eliminate the need for any guessing as it can show us the answer to many trials at once.



Let us now use the *table mode* to solve the example from earlier in the chapter, namely to solve the equation $x^3 + 4x - 29 = 0$ correct to two decimal places.

Buttons to press (Casio fx-83GT CW)	Calculator screen	Notes
		Enter the Table Mode and start to define the function.
		Finish defining the function.
		Substitute different numbers into the function.
		Continue to enter numbers to solve correct to two decimal places. (You can use the arrows to see the answers in full.)

Using the calculator's *table mode*, this is what we would have to write as the solution to this question.

Trial	The value of $x^3 + 4x - 29$	Too high / Too low?
2	-13	Too low
3	10	Too high
2.6	-1.024	Too low
2.7	1.483	Too high
2.64	-0.040256	Too low
2.65	0.209625	Too high
2.645	0.084486125	Too high

 The solution to the equation lies between $x = 2.64$ and $x = 2.645$ so, to two decimal places, the solution to the equation is $x = 2.64$.

Exercise 4

3

The following equations have a solution between $x = 0$ and $x = 10$.

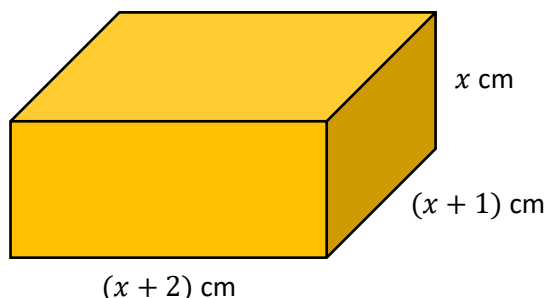
Using the trial and improvement method, and the *table mode* on your calculator, find the solution to the required degree of accuracy.

- (a) $x^2 - 4x - 3 = 0$ to the nearest unit
- (b) $x^3 + 6x - 31 = 0$ to one decimal place
- (c) $x^3 + 8x - 15 = 0$ to two decimal places
- (d) $x^2 - \sqrt{9x} - 1 = 0$ to the nearest whole number
- (e) $x^5 - 63x - 17 = 0$ to one decimal place
- (f) $x^3 - 6x - 19 = 0$ to two decimal places



Exercise 5

- (a) The length of a cuboid is $(x + 2)$ cm, its width is $(x + 1)$ cm and its height is x cm. The volume of the cuboid is 140 cm^3 .

**Applying**

3



Calculate the length of the cuboid correct to one decimal place.

- (b) Find the whole number that satisfies the equation $x^3 + 4x = 240$.
- (c) Huw enters a number into his calculator, before adding the cube of the same number. The calculator screen shows 190.893. Find the number that Huw initially entered into his calculator. Show your method.
- (d) Here is a graph of the equation $y = x^3 - 7x^2 + 4x + 3$.



Use the trial and improvement method to find, correct to one decimal place, each of the solutions to the equation $x^3 - 7x^2 + 4x + 3 = 0$.

Evaluation

Key words	Corrections	I am happy with...	I need to revise...



AER, APR

When **borrowing** or **investing** money, it is important to consider the **interest rate** that is used to calculate the **interest** that is added to the loan or investment. Another factor is important however – **how frequently** interest is added. This makes it difficult to directly compare interest rates if the periods for adding interest are different. For example, consider that you want to invest a sum of money. Which option is best for you: an interest rate of 4% paid every six months, or an interest rate of 2% paid every quarter? In order to compare interest rates of this type fairly, we use the special percentages **AER** and **APR**.



AER = Annual Equivalent Rate

AER is used to note the percentage of interest earned in a period of **one year**. It allows you to compare different accounts that pay interest at different times, e.g. every month, every quarter, every six months. The following method for calculating AER is given on page 2 of the examination paper.

AER, as a decimal, is calculated using the formula $\left(1 + \frac{i}{n}\right)^n - 1$, where i is the nominal interest rate per annum as a decimal and n is the number of compounding periods per annum.

Note that sometimes the 'nominal interest rate' is referred to as the 'gross interest rate'.

Exercise 6

Complete the following table. (The first row has been completed for you.)



Skill

1

Interest rate	Number of compounding periods per annum	Nominal interest rate per annum
3%	4	12%
4%	2	
2.5%		7.5%
	12	6%

Example

Calculate the AER for the following two savings accounts: one with an interest rate of 4% paid every six months, and another with an interest rate of 2% paid every quarter.

Interest rate 4% paid every six months

There are 2 compounding periods during the year.
The nominal interest rate per annum is $4\% \times 2 = 8\%$.
As a decimal, this is 0.08.

$$\text{AER} = \left(1 + \frac{0.08}{2}\right)^2 - 1$$

$$\text{AER} = 0.0816$$

As a percentage, the AER is 8.16%.

Interest rate 2% paid every quarter

There are 4 compounding periods during the year.
The nominal interest rate per annum is $2\% \times 4 = 8\%$.
As a decimal, this is 0.08.

$$\text{AER} = \left(1 + \frac{0.08}{4}\right)^4 - 1$$

$$\text{AER} = 0.08243216$$

As a percentage, the AER is 8.24%, correct to 2 decimal places.



By comparing the two AER values, it is possible to see that the second account (2% interest paid every quarter) is the better option, as the AER is higher.

Exercise 7

Calculate the AER for the following savings accounts.

- (a) An interest rate of 5% paid every six months. (b) An interest rate of 3% paid every quarter.
 (c) An interest rate of 7% paid every 4 months. (d) An interest rate of 2% paid every month.
 (e) An interest rate of 8.4% paid every quarter. (f) An interest rate of 0.25% paid every quarter.

**Example**

Susan intends to invest £2,500 into a savings account for one year.

HSBC bank offer a nominal interest rate of 3% a year, with interest paid every quarter.

- (a) Calculate the AER for HSBC's account.
 (b) If Susan decides to invest her money with HSBC for one year, how much money will be in her account at the end of the year?

Answer: (a) With interest paid every quarter, there are 4 compounding periods during the year.

$$\text{AER} = \left(1 + \frac{0.03}{4}\right)^4 - 1$$

$$\text{AER} = 0.03033919066 \dots$$

AER = 3.03%, to 2 decimal places.

(b) Method 1: Use the AER.

$$£2,500 \times 103.033919066\% = £2,575.85, \text{ to the nearest penny.}$$

Method 2: Use the nominal interest rate.

3% a year so $3 \div 4 = 0.75\%$ a quarter.

$$£2,500 \times 100.75\%^4 = £2,575.85, \text{ to the nearest penny.}$$

Use the percentage
before rounding off for
an accurate answer.

The £2,500 increases by
0.75% four times.

Exercise 8

(a) Dave intends to invest £4,000 into a savings account for one year. Barclays bank offer a nominal interest rate of 2% a year, with interest to be paid every quarter.

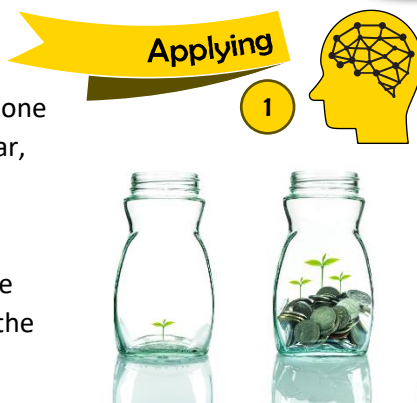
- (i) Calculate the AER for Barclays' account.
 (ii) If Dave decides to invest the money with Barclays for one year, how much money will be in his account at the end of the year?

(b) Victoria intends to invest £2,500 into a savings account for one year. HSBC bank offer a nominal interest rate of 5% a year, with interest to be paid every month.

- (i) Calculate the AER for HSBC's account.
 (ii) If Victoria decides to invest the money with HSBC for one year, how much money will be in her account at the end of the year?

(c) Which is better: investing money into an account that offers AER at a rate of 4%, or investing money into an account that offers an interest rate of 1% paid every three months?

(d) Always / sometimes / never: AER is always greater than the nominal interest rate.



Example

(a) Morgan invests £400 with Barclays bank at an AER of 2.4%. How much money will Morgan have in the bank after 3 years?

(b) Four years ago, Mari invested a sum of money into HSBC bank at an AER of 4.5%. The money is now worth £4,000. What is the minimum amount of money that Mari had to invest in order to accomplish this?

Answer: (a) $£400 \times 102.4\%^3 = £429.50$, to the nearest penny.

(b) $? \times 104.5\%^4 = £4,000$

$? = £4,000 \div 104.5\%^4$

$? = £3,354.25$, to the nearest penny.

**Exercise 9**

1

(a) Ffion invests £800 into Lloyds bank at an AER of 3.1%. How much money will Ffion have in the bank after 5 years?

(b) Three years ago, Jac invested a sum of money into Santander bank at an AER of 2.3%. The money is now worth £1,400. What is the minimum amount of money that Jac had to invest in order to accomplish this?

(c) Meical invests £6,500 into Halifax bank at an AER of 1.7%. How much money will Meical have in the bank after 2 years?

(d) Nine years ago, Catrin invested a sum of money into Barclays bank at an AER of 6.25%. The money is now worth £20,000. What is the minimum amount of money that Catrin had to invest in order to accomplish this?

(e) Megan has £5,000 to invest in HSBC bank at an AER of 6.4%. In how many years will Megan's money be worth more than £7,000?

**Alternative method of calculating AER**

As well as the method shown on page 2 of a GCSE examination paper, it is possible to use the following method for calculating AER.



$$\text{AER} = \frac{\text{Interest accrued over one year}}{\text{Initial value}} \times 100\%$$

Example

Calculate the AER for a savings account that offers an interest rate of 4% paid every quarter.

Answer: Imagine that we decide to invest £1,000 into this savings account. After one year, the money will be worth $£1,000 \times 104\%^4 = £1,169.86$ (to the nearest penny), so the interest accrued over one year is £169.86 (to the nearest penny). So, the AER is $\frac{169.86}{1000} \times 100\% = 16.99\%$, to 2 decimal places.

(The previous method gives the same answer, as $(1 + \frac{0.16}{4})^4 - 1 = 0.16985856 = 16.99\%$, to 2 decimal places.)

Exercise 10

Skill

Use the alternative method of calculating AER to calculate the AER for the following savings accounts.

(a) An interest rate of 5% paid every six months.

(b) An interest rate of 3% paid every quarter.

(c) An interest rate of 7% paid every 4 months.

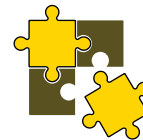
(d) An interest rate of 2% paid every month.

(e) An interest rate of 8.4% paid every quarter.

(f) An interest rate of 0.25% paid every quarter.



$$\text{APR} = \text{Annual Percentage Rate}$$



APR is used to compare accounts where there is a charge for the account, or these are additional costs associated with the account.

For a savings account,

$$\text{APR} = \frac{\text{Interest accrued over one year} - \text{costs}}{\text{Initial value}} \times 100\%$$

For a borrowing account,

$$\text{APR} = \frac{\text{Interest accrued over one year} + \text{costs}}{\text{Initial value}} \times 100\%$$



In most cases, there are **no costs** associated with a **savings** account, so the AER and APR rates are equal to each other. This explains why we see AER rates advertised alongside savings accounts.

In most cases, there **are** costs associated with a **borrowing** account, so we must use the APR rate. This explains why we see APR rates advertised alongside borrowing accounts such as mortgages, credit cards and loans from the bank.

Example

Huw intends to borrow £4,800 from the company *Loans4U*. The company offers an interest rate of 4% a month, and charges an annual fee of £150 to use the account.

(a) How much interest will this loan accrue over a period of one year?

(b) Calculate the APR for this loan.

Answer: (a) There are 12 compounding periods during the year. $£4,800 \times 104\%^{12} = £7,684.95$, to the nearest penny.
So, $£7,684.95 - £4,800 = £2,884.95$ of interest is accrued during the year.

$$(b) \text{APR} = \frac{\text{Interest accrued over one year} + \text{costs}}{\text{Initial value}} \times 100\%$$

$$\text{APR} = \frac{2884.95 + 150}{4800} \times 100\%$$

$$\text{APR} = 63.2\%, \text{ to one decimal place.}$$



Applying

1

Exercise 11

(a) Lisa intends to borrow £7,000 from the company *BestLoans*. The company offers an interest rate of 2% a month, and charges an annual fee of £200 to use the account.

(i) How much interest will this loan accrue over a period of one year?

(ii) Calculate the APR for this loan.

(b) Deiniol intends to borrow £24,000 from the company *LoanKing*. The company offers an interest rate of 3% every six months, and charges a monthly fee of £15 for using the account.

(i) How much interest will this loan accrue over a period of one year?

(ii) Calculate the APR for this loan.

(c) Sophie intends to borrow £154,000 from the company *MorgaisGorau*.

The company offers an interest rate of 0.4% a month, and charges an annual fee of £300 for using the account.

(i) How much interest will this loan accrue over a period of one year?

(ii) Calculate the APR for this loan.



Example

Calculate the AER or APR for each of the following situations.

Situation 1: Savings account with no costs.

Method 1: Use the formula $AER = \left(1 + \frac{i}{n}\right)^n - 1$

Nominal interest rate per annum $3\% \times 4 = 12\%$.

$$AER = \left(1 + \frac{0.12}{4}\right)^4 - 1$$

$$AER = 0.12550881 \dots$$

AER = 12.55% to 2 decimal places.

Method 2: Use the formula

$$AER = \frac{\text{Interest accrued over one year}}{\text{Initial value}} \times 100\%$$

Value at the end of the year

$$= £2,500 \times 103\%^4$$

$$= £2,813.77 \text{ to the nearest penny.}$$

Interest accrued over one year

$$= £2,813.77 - £2,500$$

$$= £313.77$$

$$AER = \frac{£313.77}{£2,500} \times 100\%$$

AER = 12.55% to 2 decimal places.

Situation 2: Savings account with costs of £40 a year.

We must use the formula

$$APR = \frac{\text{Interest accrued over one year} - \text{costs}}{\text{Initial value}} \times 100\%$$

Value at the end of the year

$$= £2,500 \times 103\%^4$$

$$= £2,813.77 \text{ to the nearest penny.}$$

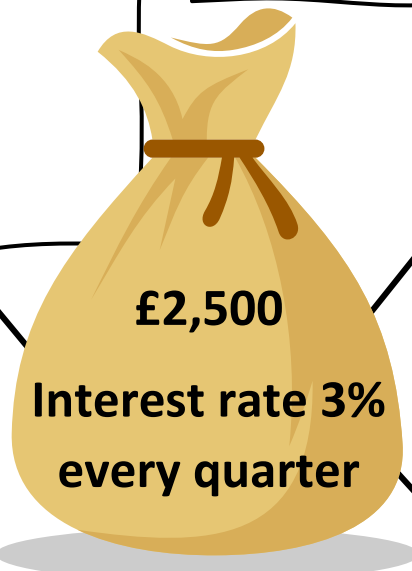
Interest accrued over one year

$$= £2,813.77 - £2,500$$

$$= £313.77$$

$$APR = \frac{£313.77 - £40}{£2,500} \times 100\%$$

APR = 10.95% to 2 decimal places.

**Situation 3: Borrowing account with no charges.**

The calculations are exactly the same as for situation 1. So, the AER is 12.55% to 2 decimal places.

Because there are no costs, the APR is also 12.55% to 2 decimal places.

Situation 4: Borrowing account with costs of £40 a year.

We must use the formula

$$APR = \frac{\text{Interest accrued over one year} + \text{costs}}{\text{Initial value}} \times 100\%$$

Loan at the end of the year

$$= £2,500 \times 103\%^4$$

$$= £2,813.77 \text{ to the nearest penny.}$$

Interest accrued over one year

$$= £2,813.77 - £2,500$$

$$= £313.77$$

$$APR = \frac{£313.77 + £40}{£2,500} \times 100\%$$

APR = 14.15% to 2 decimal places.

Exercise 12

Calculate the AER or APR for each of the following situations.

- (a) An investment of £1,500 into a savings account that offers an interest rate of 2% per quarter.
- (b) An investment of £2,400 into a savings account that offers an interest rate of 5% per year and annual costs of £50.
- (c) A loan of £3,500 from an account that offers an interest rate of 3.2% per quarter.
- (d) A loan of £15,000 from an account that offers an interest rate of 1.2% per month and annual costs of £150.
- (e) A loan of £140,000 from an account that offers an interest rate of 1.8% per quarter and quarterly costs of £50.
- (f) An investment of £250,000 into a savings account that offers an interest rate of 0.4% per month and monthly costs of £5.

**Challenge!**

HSBC's website shows the following information for a personal loan of £10,000 taken over 12 months.

How much would you like to borrow?
Enter a value between 1,000 and 25,000

GBP 10,000

GBP 1,000 GBP 25,000

Over how many months?
Enter a value between 12 and 60

12

12 months 60 months

Representative example*

Total amount payable
GBP 10,340.09

Monthly repayment
GBP 861.67

Representative
6.4% APR

Interest rate p.a. fixed
6.4%

<https://www.hsbc.co.uk/loans/products/personal/> , 11/05/2025

6.4% of £10,000 is £640. Why is the total amount payable not £10,640? Investigate...



Key words	Corrections	I am happy with...	I need to revise...

Mopping

Here is a collection of questions on topics from the GCSE syllabus that don't quite fit into any of the previous workbooks.

Exercise 13

A

(a) Which fraction of 25 is 20? Simplify your answer.

(b) Which percentage of 25 is 20?

Exercise 14

23

Simplify the following algebraic fractions.

(a) $\frac{4x}{8}$

(b) $\frac{4x+6}{2}$

(c) $\frac{2x+3}{7} + \frac{3x+7}{7}$

(d) $\frac{2x+3}{5} + \frac{3x+7}{5}$

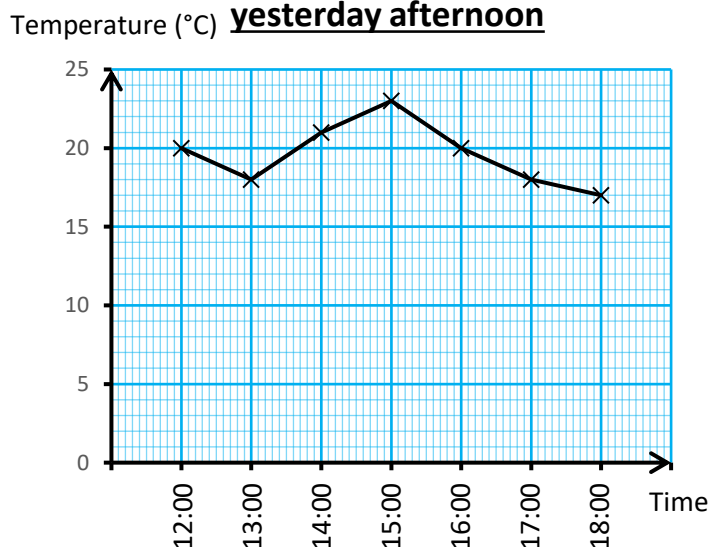
(e) $\frac{2x+3}{2} + \frac{3x+7}{3}$

Exercise 15

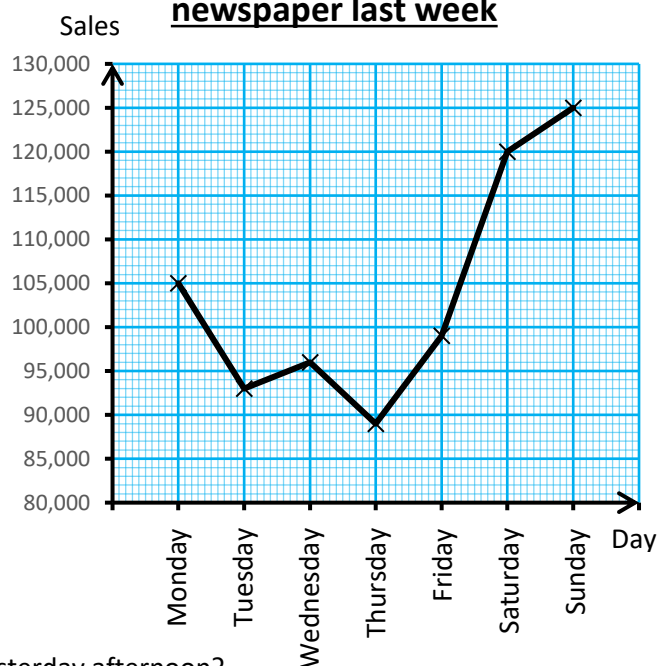
13

Look at the following line graphs.

**Line Graph to show the
temperature in Llandudno
yesterday afternoon**



**Line Graph to show the sales of a
newspaper last week**



(a) What was the temperature in Llandudno at 4 o'clock yesterday afternoon?

(b) How many newspapers were sold on Sunday?

(c) The **intermediate values** in a line graph are the values on the line **between** any two plotted points.

(i) Do the intermediate values in Llandudno's temperature graph have any meaning?

(ii) Do the intermediate values in the newspaper sales graph have any meaning?

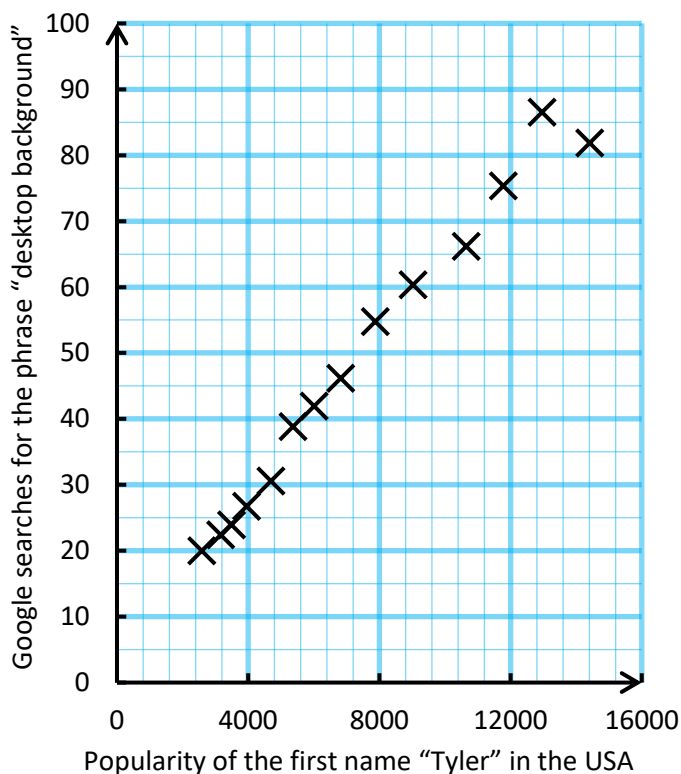
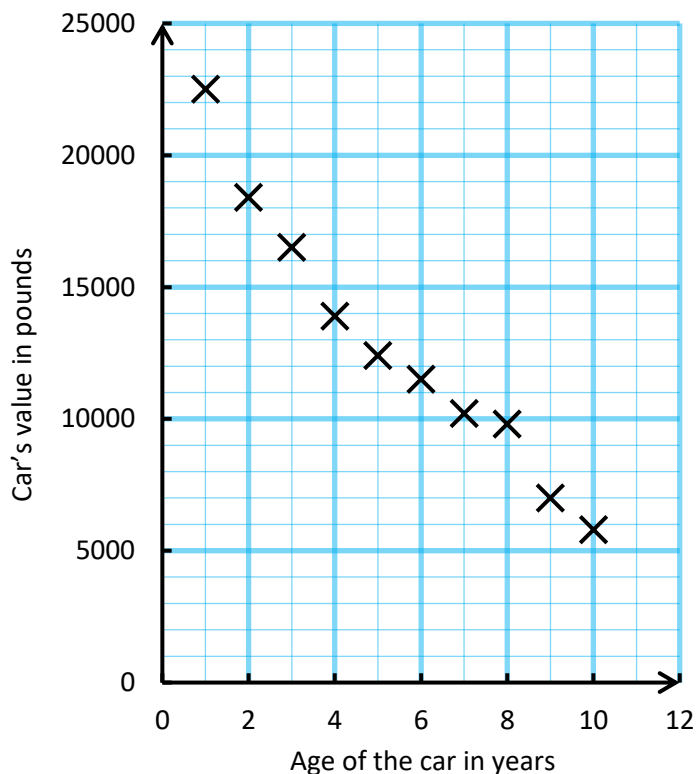
(d) The owner of the newspaper says "Look! The graph shows there has been an enormous increase in sales over the weekend." Explain how the graph could mislead the owner to say this.

(e) List different ways that graphs can mislead people.

Exercise 16

3

Look at the following scatter diagrams. The first one shows, between 2007 and 2022, how many people named “Tyler” were born in the USA, against the relative volume of Google searches for “desktop background”. The second one shows the monetary value of a car against its age, over a 10 year period.

Scatter Diagram for Exercise 16**Scatter Diagram for Exercise 16**

- Which type of correlation is shown in the first scatter diagram (the one on the left)?
- Which type of correlation is shown in the second scatter diagram (the one on the right)?
- Does the correlation in the first scatter diagram suggest **causation**? In other words, do you believe that there is an actual connection between the popularity of the first name “Tyler” and how many people search for “desktop background” on Google?
- Does the correlation in the second scatter diagram suggest causation?

Exercise 17

23

Expand the following algebraic expressions.

- $(x + 2)(x^2 + 3)$
- $(2x + 4)(x^2 - 3x + 5)$
- $(6x - 1)(2x^2 + 7x - 3)$

Exercise 18

13

Convert the following units.

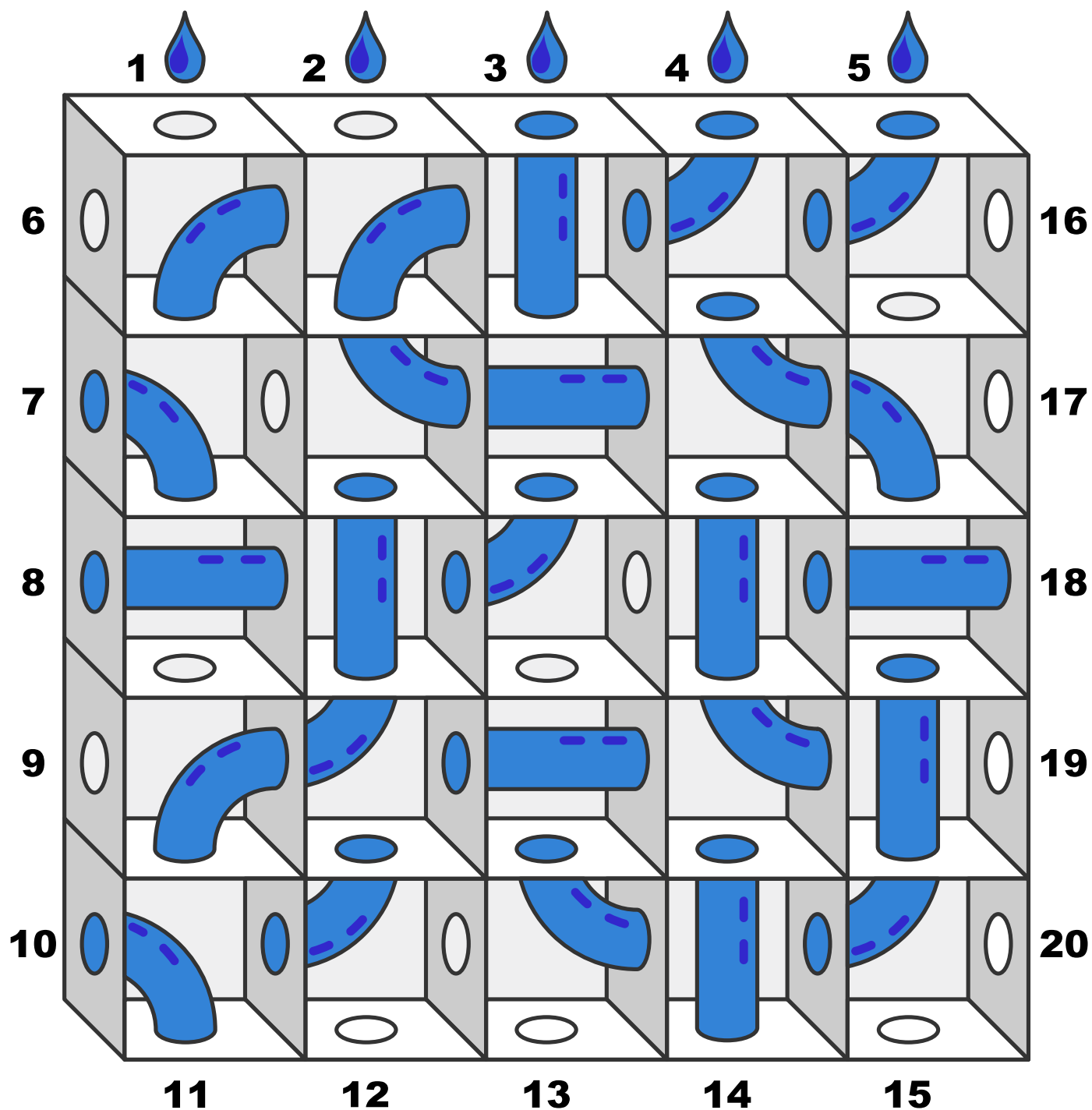
- 1 m^2 into cm^2
- 2.4 cm^2 into mm^2
- 8 km^2 into m^2
- 1 m^3 into cm^3
- 2.4 cm^3 into mm^3
- 8 km^3 into m^3
- 5 km^2 into mm^2
- 9.2 m^3 into mm^3
- 24 mm^2 into cm^2

Puzzle

The front of the tank below is solid and transparent.

Where will the liquid pour out if it is poured into hole 1?

What about hole 2? Hole 3? Hole 4? Hole 5?

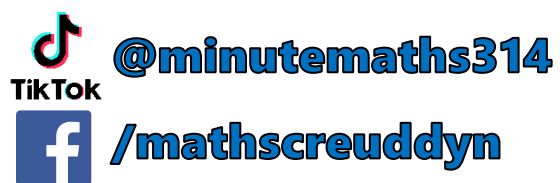




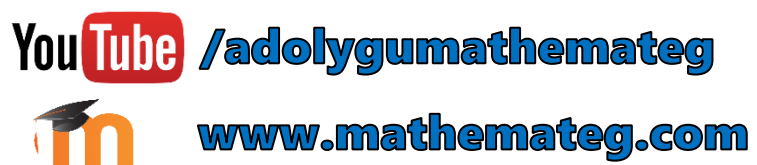
Name:

Percentage in the test:

	I know this. 	I need to revise this. 	Question in the test:	Correct in the test?
I can use the trial and improvement method to solve equations correct to the nearest unit ; to one decimal place ; or to two decimal places .			1	
I can use the trial and improvement method to solve equations , e.g. find the whole number that satisfies the equation $x^3 - 2x = 115$.				
I can calculate the AER for a savings account using the formula $\left(1 + \frac{i}{n}\right)^n - 1$.			2	
I can use the AER to calculate how much money is in a savings account, at the end or start of an investment.			2, 3, 4	
I can calculate the APR for savings or borrowing accounts where there is an additional charge associated with the account.			5	



 @minutemaths314



 /adolygumathemateg