



A Level

Unit 3

Textbook

Author: Dr Gareth Evans

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Available from www.mathemateg.com:

- Microsoft Word versions of all workbooks.
- Model answers to past examination papers.
- Answers to the workbooks (available, on request, to teachers; send an e-mail from a school account to gareth@mathemateg.com).

Available from www.youtube.com/adolygumathemateg:

- Full Welsh medium revision fideos for the workbooks. Search for the playlist for 'Lefel A Uned 3'.

Available from the TikTok account 'Minute Maths':

- Live revision sessions.
- Short revision clips.



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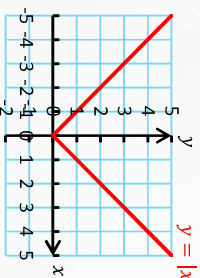
Unit 3 WJEC

Revision Sheet

Proof

Proof by contradiction, e.g. proving that $\sqrt{2}$ is irrational, proving that there are an infinite number of prime numbers.

The Modulus Function



Being able to sketch graphs of the form $y = |ax + b|$.

$$|x| = a \Rightarrow x = \pm a.$$

Solving $|ax + b| \leq c$:

$$\text{Method 1: Either } ax + b \leq c \text{ or } ax + b \geq -c.$$

Method 2: Square both sides to remove the modulus sign. (This method also works to solve equations of the form $|cx + d| = |ex + f|$.)

Functions

For $f^{-1}(x)$ to exist, $f(x)$ must be a 1-to-1 function.

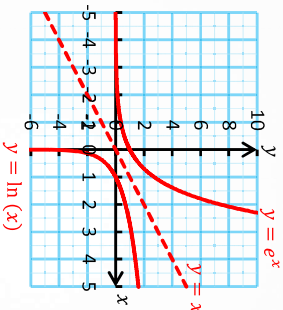
The domain of $f(x)$ is the range of $f^{-1}(x)$. The range of $f(x)$ is the domain of $f^{-1}(x)$. To draw the graph of $f^{-1}(x)$, reflect the graph of $f(x)$ in the line $y = x$.

Domain	Meaning
(a, b)	$a < x < b$
$[a, b]$	$a \leq x \leq b$

$f(g(x))$ is a composite function. If $ff(x) = x$ then $f(x) = f^{-1}(x)$.

Using functions in modelling, including consideration of limitations and refinements of the models.

The graphs of e^x and $\ln(x)$



Function	Domain	Range
$y = e^x$	$(-\infty, \infty)$	$(0, \infty)$
$y = \ln(x)$	$(0, \infty)$	$(-\infty, \infty)$

Transformations of Graphs

Combining transformations from Unit 1:

$$y = af(x), y = f(x) + a, y = f(x + a), y = f(ax).$$

Partial Fractions

$$\frac{f(x)}{(ax+b)^2(cx+d)} \equiv \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{cx+d}$$

$$\frac{f(x)}{(ax+b)^2(cx+d)} \equiv \frac{A(ax+b)(cx+d) + B(cx+d) + C(ax+b)^2}{(ax+b)^2(cx+d)}$$

$$f(x) \equiv A(ax + b)(cx + d) + B(cx + d) + C(ax + b)^2.$$

Substitute values for x or compare coefficients in order to find A, B, C .

Parametric Equations

With $x = f(t)$ and $y = g(t)$, we can use the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{g'(t)}{f'(t)}.$$

Equation of the tangent: $y - g(t) = \frac{dy}{dx}(x - f(t))$.

Equation of the normal: $y - g(t) = -\frac{dx}{dy}(x - f(t))$.

Sequences

Sequences of the form $x_{n+1} = f(x_n)$.

Increasing sequences, decreasing sequences, periodic sequences.

Understanding and using Σ notation.

Binomial Expansion

$$(a + bx)^n = a^n + na^{n-1}(bx) + \frac{n(n-1)}{2 \times 1} a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3 \times 2 \times 1} a^{n-3}(bx)^3 + \dots$$

The expansion is valid if $\left| \frac{bx}{a} \right| < 1$.

With an expression containing several parts, use the most restrictive modulus.

Arithmetic Series

First term a .

Common difference d .

n th term $t_n = a + (n - 1)d$.

Sum of the first n terms:

$$S_n = \frac{n}{2}[2a + (n - 1)d].$$

$$S_n = \frac{n}{2}[a + l].$$

Geometric Series

First term a .

Common ratio r .

n th term $t_n = ar^{n-1}$.

Sum of the first n terms:

$$S_n = \frac{a(1-r^n)}{1-r}.$$

If $|r| < 1$ then $S_\infty = \frac{a}{1-r}$.

Trigonometry

Measuring angles in radians as well as in degrees.

Length of an arc = $r\theta$.

Area of a sector = $\frac{1}{2}r^2\theta$.

Area of a triangle = $\frac{1}{2}r^2 \sin \theta$.

If $-0.115 \text{ rad} < \theta < 0.115 \text{ rad}$, then the following approximations are correct to 3 significant figures:

$$\sin \theta \approx \theta$$

$$\cos \theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan \theta \approx \theta$$

θ in radians

Angle	sin	cos	tan
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	Not defined

Graphs, Domains, Ranges

y
S
T
C

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

$$\sec^2 \theta = 1 + \tan^2 \theta$$

$$\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$$

$$R \sin(A \pm B) =$$

$$R \sin A \cos B \pm R \cos A \sin B.$$

$$R \cos(A \pm B) =$$

$$R \cos A \cos B \mp R \sin A \sin B.$$

$$R \tan(A + B) = R \left(\frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \right).$$

↑ GEOMETRIC PROOF ↑

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta.$$

$$\cos 2\theta = 2\cos^2 \theta - 1.$$

$$\cos 2\theta = 1 - 2\sin^2 \theta.$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}.$$

Differentiation

$$\frac{d^2y}{dx^2} > 0 \Rightarrow \text{minimum point.}$$

$$\frac{d^2y}{dx^2} < 0 \Rightarrow \text{maximum point.}$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow \text{further investigation.}$$

For a point of inflection, the second derivative is zero, the sign of the second derivative changes (from convex to concave, or vice-versa), and the sign of the gradient is constant.

We can solve $\frac{d^2y}{dx^2} = 0$ in order to find points of inflection.

Differentiating $\sin x$ and $\cos x$ from first principles.

PROOF

Product rule:

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\text{Quotient rule: } \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

$$\text{Chain rule: } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}.$$

Implicit differentiation:

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}.$$

Numerical Methods

Locating roots of $f(x)$ by finding a change of sign. Understanding how this can fail.

Using the ANS button on a calculator with recurrence relations of the form $x_{n+1} = f(x_n)$.

Rearranging $f(x) = 0$ to the form $x = g(x)$ so that a staircase or cobweb diagram can be used to find a root.

Newton-Raphson method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Understanding how this can fail if the gradient is too small.

The trapezium rule:

$$\int_a^b y \, dx \approx \frac{h}{2} \{ (y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1}) \}, \text{ where } h = \frac{b-a}{n}.$$

Example: $\int_1^2 \sqrt{2+x^3} \, dx$ with 5 measures: $\frac{5-1}{4} = 1$.

x	$f(x)$
1	1.732050808
1.25	1.988246715
1.5	2.318404624
1.75	2.712816802
2	3.16227766

TABLE MODE
on a calculator

$$\int_1^2 \sqrt{2+x^3} \, dx \approx \frac{0.25}{2} [1.732050808 + 3.16227766 + 2(1.988246715 + 2.318404624 + 2.712816802)]$$

$$= \frac{0.25}{2} \times 18.93326475$$

$$= 2.367 \text{ to 3 decimal places.}$$

The Examination

Length: 2 hours 30 minutes.

120 marks.

35% of the A Level qualification.

Checklist

- ☐ I have attempted all the past paper questions.
- ☐ I know which formulae appear in the **formula booklet**.
- ☐ I can check answers using a graphical calculator.



The Trapezium

Rule

It is reckoned that **Isaac Newton** knew about the Trapezium Rule in the 17th century. Newton was one of the first people to consider the theory of differentiation and integration, at Cambridge University.



Name: _____

Background

What is the work?

It is not possible to integrate every possible function using algebraic methods. For these functions, we can estimate the required area by splitting it into a number of different trapezia, and finding the sum of these trapezia's areas.

What is required before starting?

GCSE Work: Substitution, area of a trapezium.
A Level Unit 1: Integration.

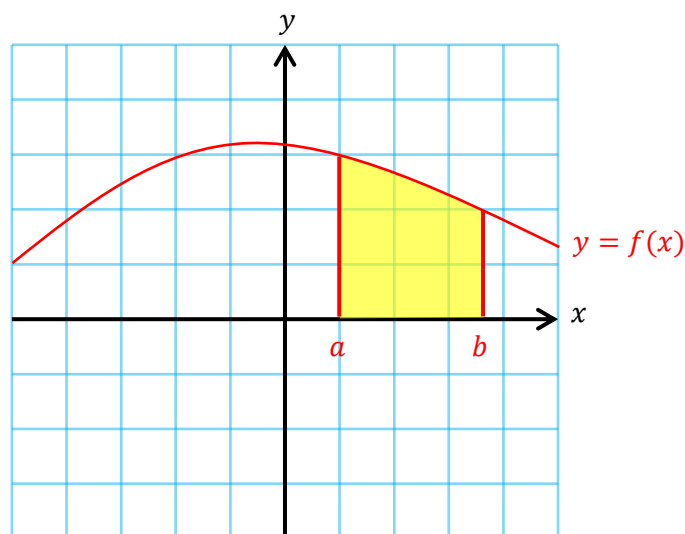
Where does this lead to?

University Work: More accurate methods of estimating the area, such as Simpson's Rule.

Applications: Finding an unusual area or volume, e.g. the amount of water displaced by a ship.

Theory

Consider any function $y = f(x)$.



Theory



Example

The area under the curve between a and b is $\int_a^b f(x)dx$. If we cannot calculate this exactly, we can use the **Trapezium Rule** to estimate it.

$$\int_a^b f(x)dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}, \text{ where } h = \frac{b-a}{n}.$$

n is the number of **trapezia**.

$n + 1$ is the number of **ordinates**.

h is the **height** of each trapezium (remember that each trapezium is on its side).

$y_0, y_1, y_2, \dots, y_{n-1}, y_n$ are the values of the function for equidistant values between a and b .

Challenge! Try to prove why the formula for the Trapezium Rule works.

In the
formula
booklet

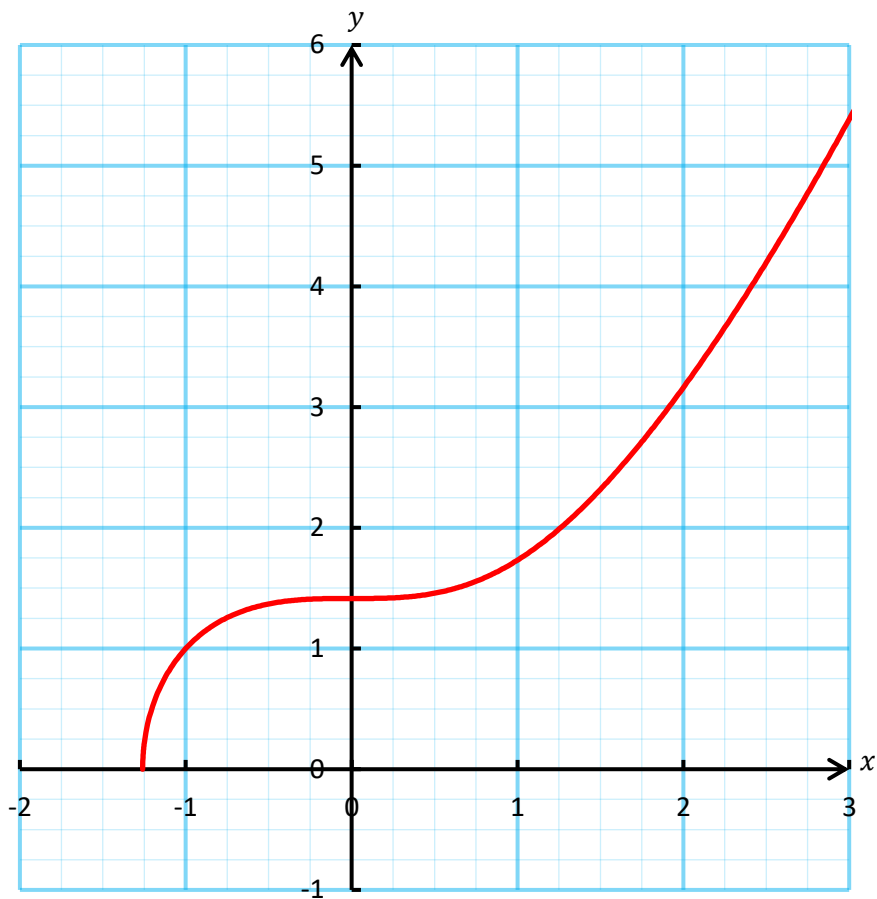
Exercise

(C2 Winter 2007)

Use the Trapezium Rule with five ordinates to find an approximate value for $\int_1^2 \sqrt{2+x^3} dx$.

Show your working and give your answer correct to three decimal places.

[4]





(C2 Summer 2005)

1. Use the Trapezium Rule with six ordinates to find an approximate value for

$$\int_0^1 \sqrt{1+x^2} \, dx.$$

Show your working and give your answer correct to three decimal places.

[4]

Handwriting practice lines for the answer.

1. Use the Trapezium Rule with six ordinates to find an approximate value for the integral

$$\int_0^1 \frac{1}{2+x^3} \, dx.$$

[4]

1. Use the Trapezium Rule with five ordinates to find an approximate value for the integral

$$\int_0^{0.4} \sqrt{1+x^4} \, dx.$$

[4]

1. Use the Trapezium Rule with five ordinates to find an approximate value for the integral

$$\int_0^1 \frac{1}{\sqrt{2+x^3}} \, dx.$$

[4]

1. Use the Trapezium Rule with **four** ordinates to find an approximate value for the integral

$$\int_0^{0.6} (1+x^2)^{\frac{3}{2}} dx.$$

[4]

1. Use the Trapezium Rule with five ordinates to find an approximate value for the integral

$$\int_{1.6}^2 \frac{1}{9-x^3} dx.$$

[4]

1. Use the Trapezium Rule with five ordinates to find an approximate value for the integral

Show your working and give your answer correct to three decimal places. [4]

1. (a) Use the Trapezium Rule with five ordinates to find an approximate value for the integral

Show your working and give your answer correct to three decimal places. [4]

$$\int_1^3 \log_{10}(3x-1)^2 dx. \quad [1]$$

1. (a) Use the Trapezium Rule with five ordinates to find an approximate value for the integral

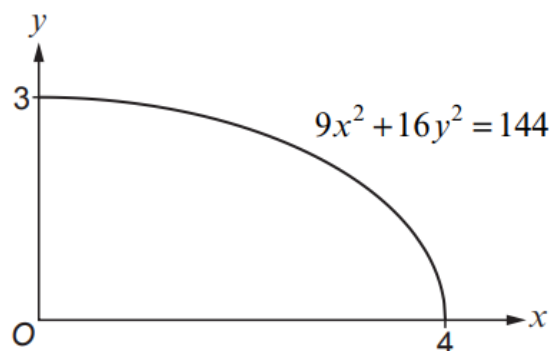
Show your working and give your answer correct to three decimal places. [4]

$$\int_1^4 \log_{10} \sqrt{(6x-1)} \, dx. \quad [1]$$

(Unit 3 Summaer 2023)

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The aerial view of a patio under construction is shown below.



The curved edge of the patio is described by the equation $9x^2 + 16y^2 = 144$, where x and y are measured in metres.

To construct the patio, the area enclosed by the curve and the coordinate axes is to be covered with a layer of concrete of depth 0.06 m.

- a) Show that the volume of concrete required for the construction of the patio is given by $0.015 \int_0^4 \sqrt{144 - 9x^2} \, dx$. [3]
- b) Use the trapezium rule with six ordinates to estimate the volume of concrete required. [4]
- c) State whether your answer in part (b) is an overestimate or an underestimate of the volume required. Give a reason for your answer. [1]

Revision Question

(C2 Summer 2007)

1. Use the Trapezium Rule with five ordinates to find an approximate value for the integral

$$\int_0^{\frac{\pi}{2}} \sqrt{1 + \sin x} \, dx \, ,$$

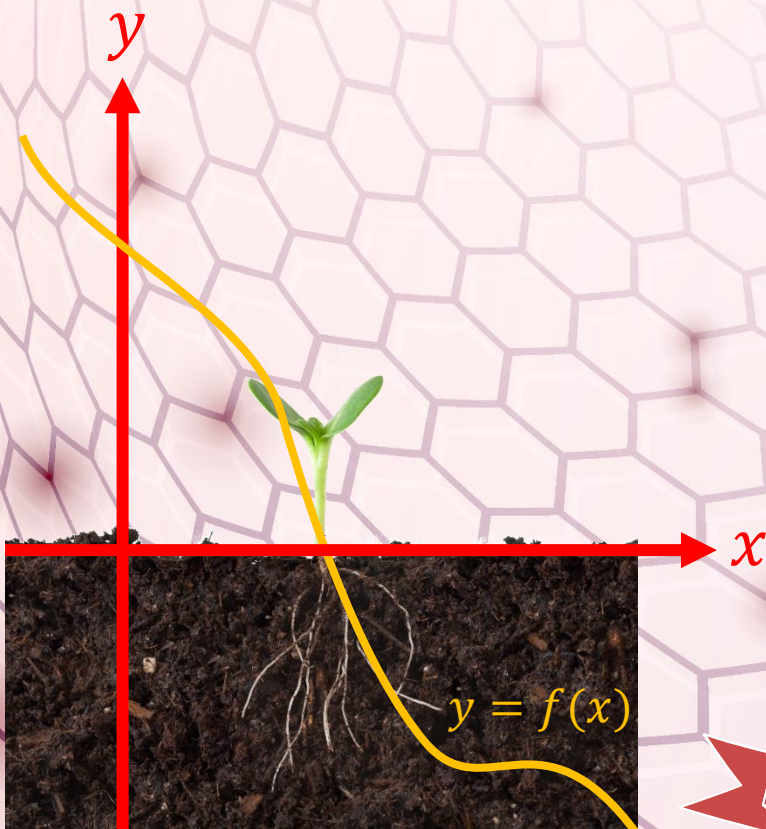
giving the value correct to three decimal places.

[4]



Roots of

Equations



Name:

Background

What is the work?

It is not possible to solve every equation using algebraic methods. For these equations, we can attempt to rearrange the equation to form a recurrence relation which, sometimes, leads to finding the roots through substitution.

What is required before starting?

GCSE Work: Substitution, rounding off, changing the subject.
A Level Unit 1: Curve sketching.

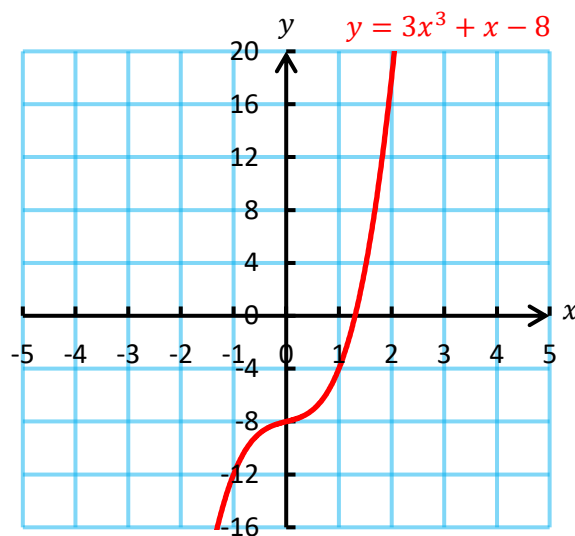
Where does this lead to?

A Level Unit 3: Cobweb and staircase diagrams; the Newton-Raphson method for solving equations.

Applications: To solve problems in context which lead to equations that cannot be solved analytically.

Theory

Consider any function $y = f(x)$, for example $y = 3x^3 + x - 8$.



Theory



Example

The **roots** of an equation are where the curve intersects the x -axis, so that $f(x) = 0$. For the above example, we see that one root exists, between 1 and 2. If we cannot solve the equation using algebraic methods, then we can attempt to form a **recurrence relation** to solve the equation. To do this, the first step is to rearrange the equation $f(x) = 0$ to make x the subject of the equation.

Example: $y = 3x^3 + x - 8$.

$$\begin{aligned} \text{Attempt 1:} \quad 3x^3 + x - 8 &= 0 \\ x &= 8 - 3x^3 \end{aligned}$$

$$\begin{aligned} \text{Attempt 2:} \quad 3x^3 + x - 8 &= 0 \\ 3x^3 &= 8 - x \\ x^3 &= \frac{8-x}{3} \\ x &= \sqrt[3]{\frac{8-x}{3}} \end{aligned}$$

The recurrence relation is formed from the equation:

$$x_{n+1} = 8 - 3x_n^3$$

$$x_{n+1} = \sqrt[3]{\frac{8-x_n}{3}}$$

We see from the graph on the previous page that the root of the equation $y = 3x^3 + x - 8$ is close to $x = 1$. So, we can choose to start at $x_0 = 1$ and use the recurrence relation to try to find the root.

$$x_1 = 8 - 3 \times x_0^3$$

$$x_1 = 5$$

$$x_2 = -367$$

$$x_3 = 148292597$$

$$x_4 = -9.783171512 \times 10^{24}$$

$$x_5 = 2.809055093 \times 10^{75}$$

$$x_6 = \text{MATH ERROR}$$

Divergent, cannot find the root.

$$x_1 = \sqrt[3]{\frac{8-x_0}{3}}$$

$$x_1 = 1.326352403$$

$$x_2 = 1.305411212$$

$$x_3 = 1.3067752$$

$$x_4 = 1.306686444$$

$$x_5 = 1.30669222$$

$$x_6 = 1.306691844 \quad \text{and so on...}$$

Converges to 1.306691867.

Use the **ANS** button to calculate these efficiently.

Exercise

(C3 Winter 2006)

Show that the equation $4x^3 + 10x - 1 = 0$ has a root α between 0 and 1.

The recurrence relation $x_{n+1} = \frac{1-4x_n^3}{10}$ with $x_0 = 0.1$ may be used to find α . Calculate and record the values of x_1, x_2, x_3 . Write down the value of x_3 correct to six decimal places and show that it is the value of α correct to six decimal places. [7]



4. (b) Show that the equation

has a root α between 1 and 2.

$$a_{n+1} = \frac{1}{2} \ln(a_n + 10)$$

Write down the value of a_4 correct to five decimal places and prove that this value is the value of α correct to five decimal places. [7]

3. (b) Show that the equation

has a root α between 0 and 1.

$$t_{n+1} = \frac{2 - t_n^3}{4}$$

with $t_0 = 0.5$ can be used to find α . Find and record the values of t_1, t_2, t_3, t_4 . Write down the value of t_4 correct to four decimal places and prove that this value is the value of α correct to four decimal places. [7]

(C3 Summer 2010)

4. Show that the equation

$$4x^3 - 2x - 5 = 0$$

has a root α between 1 and 2.

The recurrence relation

$$x_{n+1} = \left(\frac{2x_n + 5}{4} \right)^{\frac{1}{3}},$$

with $x_0 = 1.2$, may be used to find α . Find and record the values of x_1, x_2, x_3, x_4 . Write down the value of x_4 correct to five decimal places and prove that this value is the value of α correct to five decimal places. [7]

3. (b) Show that the equation

has a root α between 1 and 2.

$$t_{n+1} = \left(\frac{4t_n + 7}{2} \right)^{\frac{1}{4}}$$

with $t_0 = 1.6$ can be used to find α . Find and record the values of t_1, t_2, t_3, t_4 . Write down the value of t_4 correct to five decimal places and prove that this is the value of α correct to five decimal places. [7]

(C3 Winter 2011)

4. **You may assume** that the equation $6x^4 + 7x - 3 = 0$ has a root α between 0 and 1.
The recurrence relation

$$x_{n+1} = \frac{3 - 6x_n^4}{7}$$

with $x_0 = 0.4$ can be used to find α . Find and record the values of x_1, x_2, x_3, x_4 .
Write down the value of x_4 correct to four decimal places and show this is the value of α
correct to four decimal places.

[5]

8. You may assume that the equation

has a root α between -2 and -1 .

$$x_{n+1} = -\left(3 - e^{x_n}\right)^{\frac{1}{2}}$$

with $x_0 = -1.5$ can be used to find α . Find and record the values of x_1, x_2, x_3, x_4 . Write down the value of x_4 correct to five decimal places and prove that this is the value of α correct to five decimal places. [5]

4. (a) By sketching the graphs of $y = x^3$ and $y = 4 - x$, determine the number of real roots of the equation $x^3 + x - 4 = 0$. [3]

$$x_{n+1} = (4 - x_n)^{\frac{1}{3}}$$

with $x_0 = 1.4$ can be used to find α . Find and record the values of x_1, x_2, x_3, x_4 . Write down the value of x_4 correct to four decimal places and prove that this value is the value of α correct to four decimal places. [5]

2. (a) Sketch the graphs of $y = x^4$ and $y = 1 - 3x$. Deduce the number of real roots of the equation

(b) Show that the equation

$$x^4 + 3x - 1 = 0$$

The recurrence relation

$$x_{n+1} = \frac{1 - x_n^4}{3}$$

with $x_0 = 0.3$ can be used to find α . Find and record the values of x_1, x_2, x_3, x_4 . Write down the value of x_4 correct to five decimal places and prove that this value is the value of α correct to five decimal places. [7]

(C3 Winter 2013)

4. (a) On the same diagram, sketch the graphs of $y = \ln x$ and $y = 11 - 2x$.
Deduce the number of roots of the equation

$$\ln x + 2x - 11 = 0. \quad [3]$$

- (b) **You may assume** that the equation

$$\ln x + 2x - 11 = 0$$

has a root α between 4 and 5.

The recurrence relation

$$x_{n+1} = \frac{11 - \ln x_n}{2},$$

with $x_0 = 4.7$, can be used to find α . Find and record the values of x_1, x_2, x_3, x_4 . Write down the value of x_4 correct to five decimal places and prove that this is the value of α correct to five decimal places. [5]

(C3 Summer 2017)

4. A large tank in the form of a cuboid is used to store water. The width of the tank is denoted by x m. The length of the tank is 4 m **greater** than its width, whilst the height of the tank is 2 m **less** than its width. The volume of the tank is 150 m^3 .

(a) (i) Show that $x^3 + 2x^2 - 8x - 150 = 0$.

(ii) Show that $5 < x < 6$.

[4]

(b) The recurrence relation

$$x_{n+1} = (150 + 8x_n - 2x_n^2)^{\frac{1}{3}},$$

with $x_0 = 6$, can be used to find the value of x . Find and record the values of x_1, x_2, x_3, x_4 . Write down the value of x_4 correct to two decimal places and prove that this is the value of x correct to two decimal places.

[5]

Revision Question

(C3 Summer 2012)

4. Show that the equation

$$\cos x - 5x + 2 = 0$$

has a root α between 0 and $\frac{\pi}{4}$.

The recurrence relation

$$x_{n+1} = \frac{1}{5}(2 + \cos x_n)$$

with $x_0 = 0.6$ can be used to find α . Find and record the values of x_1, x_2, x_3, x_4 . Write down the value of x_4 correct to five decimal places and prove that this is the value of α correct to five decimal places. [7]



Unit 3, Workbook 3

13

Cobweb and

Staircase Diagrams



Name:

Background

What is the work?

In the previous workbook, we saw recurrence relations of the form $x_{n+1} = g(x_n)$. Cobweb and staircase diagrams **visualise** what happens when we use these recurrence relations to find the roots of equations.

What is required before starting?

GCSE Work: Substitution, rounding off, changing the subject.

A Level Unit 1: Curve sketching.

A Level Unit 3: Roots of equations (workbook 2).

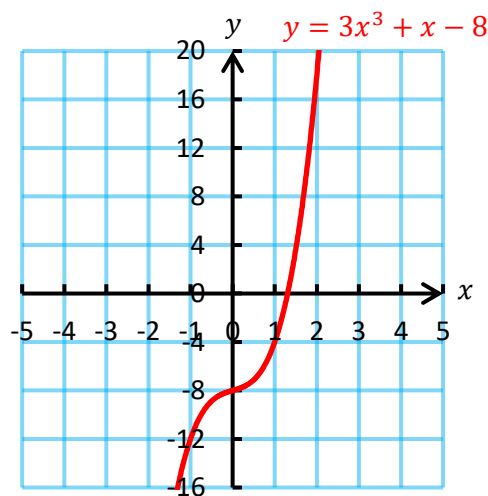
Where does this lead to?

A Level Unit 3: The Newton-Raphson method for solving equations.

Applications: To solve problems in context which lead to equations that cannot be solved analytically.

Theory

Consider any function $y = f(x)$, for example $y = 3x^3 + x - 8$.



Theory and Example

The **roots** of an equation are where the curve intersects the x -axis, so that $f(x) = 0$. For the above example, we see that one root exists, between 1 and 2. In the previous workbook, we saw that we can attempt to solve the equation $f(x) = 0$ by rearranging the equation to the form $x = g(x)$, and then forming a **recurrence relation** $x_{n+1} = g(x_n)$.

Example: $y = 3x^3 + x - 8$.

$$\begin{aligned} \text{Attempt 1:} \quad & 3x^3 + x - 8 = 0 \\ & x = 8 - 3x^3 \\ & x_{n+1} = 8 - 3x_n^3 \end{aligned}$$

$$\begin{aligned} \text{Attempt 2:} \quad & 3x^3 + x - 8 = 0 \\ & 3x^3 = 8 - x \\ & x^3 = \frac{8-x}{3} \\ & x = \sqrt[3]{\frac{8-x}{3}} \\ & x_{n+1} = \sqrt[3]{\frac{8-x_n}{3}} \end{aligned}$$

We see from the graph on the previous page that the root of the equation $y = 3x^3 + x - 8$ is close to $x = 1$. So, we can choose to start at $x_0 = 1$ and use the recurrence relation to try to find the root.

$$x_1 = 8 - 3 \times x_0^3$$

$$x_1 = 5$$

$$x_2 = -367$$

$$x_3 = 148292597$$

$$x_4 = -9.783171512 \times 10^{24}$$

$$x_5 = 2.809055093 \times 10^{75}$$

$$x_6 = \text{MATH ERROR}$$

Divergent, cannot find the root.

$$x_1 = \sqrt[3]{\frac{8-x_0}{3}}$$

$$x_1 = 1.326352403$$

$$x_2 = 1.305411212$$

$$x_3 = 1.3067752$$

$$x_4 = 1.306686444$$

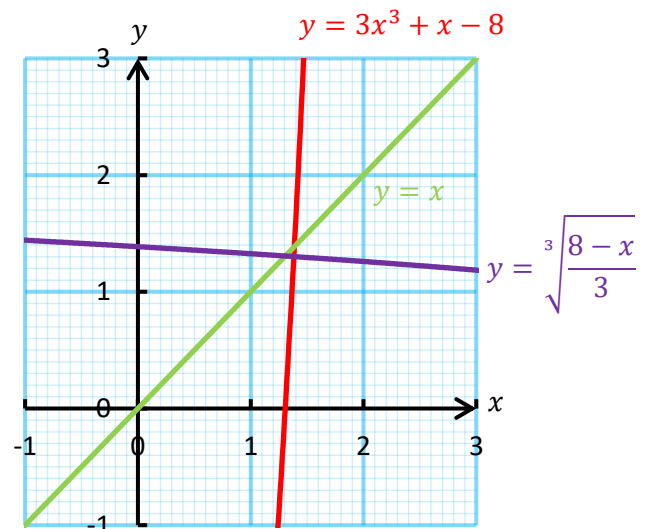
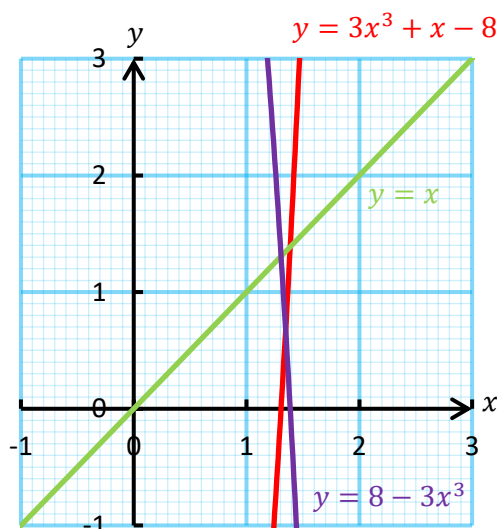
$$x_5 = 1.30669222$$

$$x_6 = 1.306691844 \quad \text{and so on...}$$

Converges to 1.306691867.

Use the **ANS** button to calculate these efficiently.

To visualise what is happening, we can plot $y = f(x)$, $y = x$ and $y = g(x)$ on the same diagram.



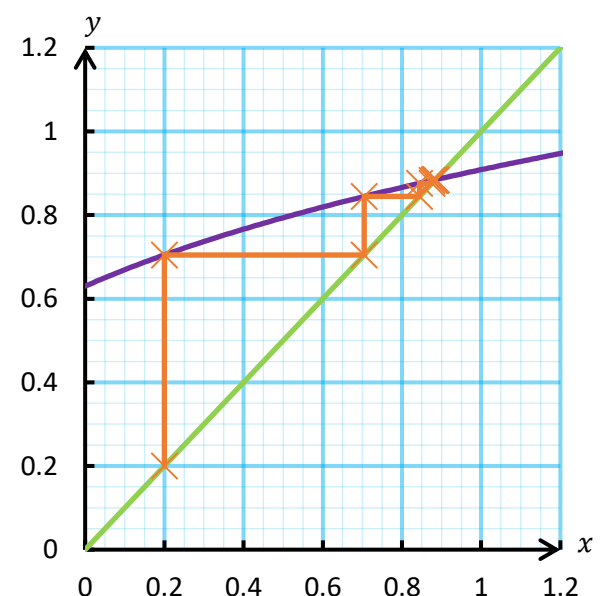
In each case, we can see that the root of $y = 3x^3 + x - 8$ is equal to the value of x where the line $y = x$ intersects the curve $y = g(x)$. This verifies that solving $f(x) = 0$ is equivalent to solving $x = g(x)$.

We can use the numbers from the recurrence relation $x_{n+1} = g(x_n)$ to form either a **cobweb diagram** or a **staircase diagram**.

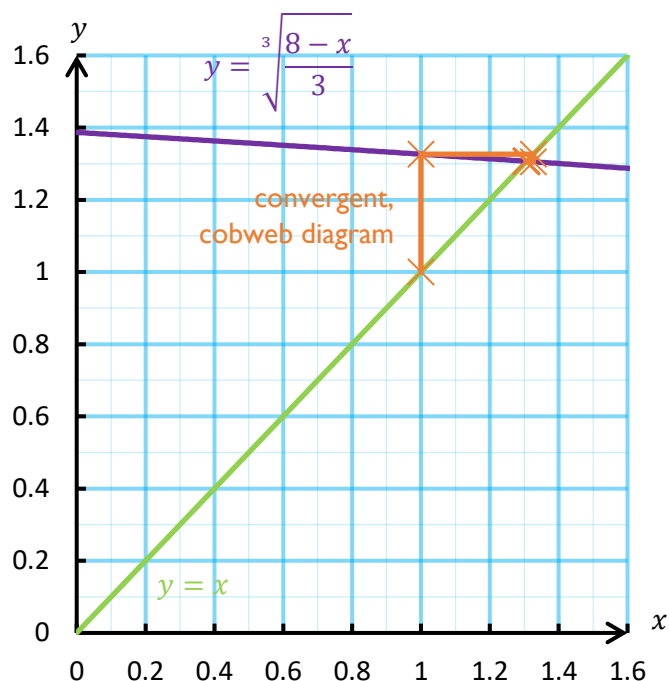
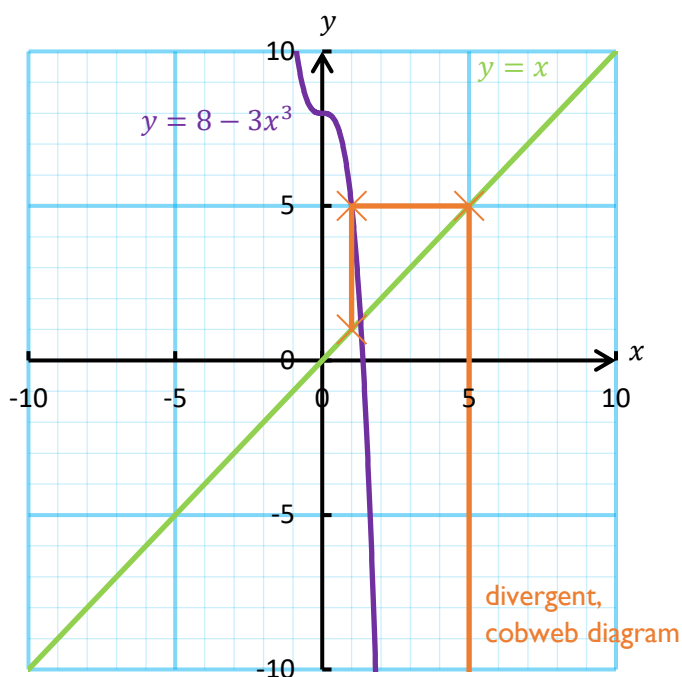
Cobweb diagram



Staircase diagram



We need to plot lines going from (x_0, x_0) to (x_0, x_1) to (x_1, x_1) to (x_1, x_2) to (x_2, x_2) to (x_2, x_3) to ...
 These lines either **converge** to a root of $y = f(x)$ or **diverge**.
 For the example on the previous page, this is what happens.



Exercises

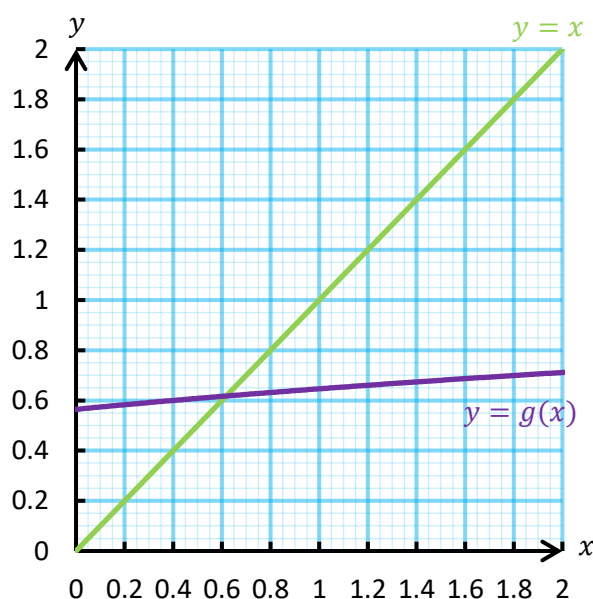
Exercise 1

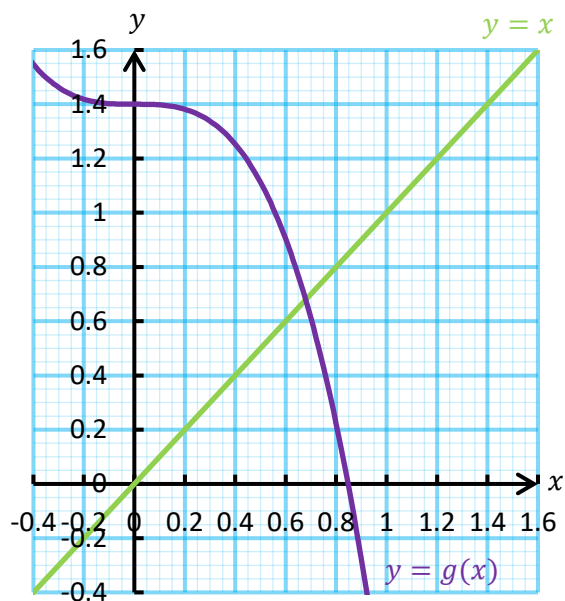
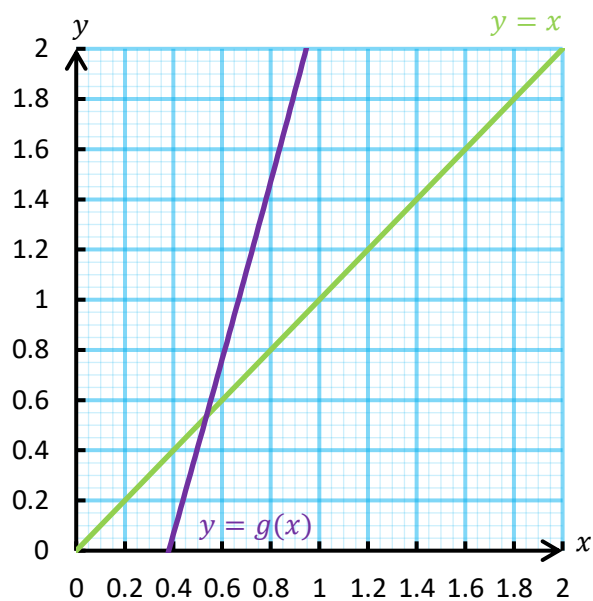
The following graphs show the line $y = x$ and a function $y = g(x)$. By starting at the stated value for x_0 , add lines to form either a cobweb diagram or a staircase diagram. Note underneath each graph whether the lines **converge** to a root of $x = g(x)$ or **diverge**.

(a) $x_0 = 0.2$

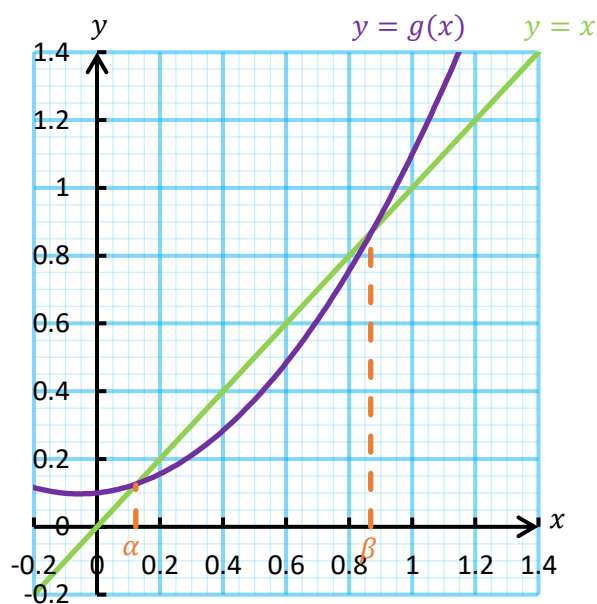


(b) $x_0 = 1.6$



(c) $x_0 = 0.6$ (d) $x_0 = 0.6$ **Exercise 2**

The graph below shows a sketch of the equation $y = g(x)$ and the line $y = x$. The equation $x = g(x)$ has roots $x = \alpha$ and $x = \beta$ as shown on the graph.



The recurrence relation $x_{n+1} = g(x_n)$ is used to try to find the roots of the equation $x = g(x)$. Describe what happens to the iteration in the case where the initial value x_0

(a) is between 0 and α

(b) is between α and β

(c) is greater than β

Exercise 3

Siwan wants to solve the cubic equation $y = 2x^3 + 2x - 3$.

(a) (i) Re-arrange the equation $2x^3 + 2x - 3 = 0$ to the form $x = g(x)$, making the first x (the red one) the subject of the equation.

(ii) Re-arrange the equation $2x^3 + 2x - 3 = 0$ to the form $x = g(x)$, making the second x (the green one) the subject of the equation.

(b) Form a recurrence relation $x_{n+1} = g(x_n)$ from the equations in part (a) of the question.

(i)

(ii)

(c) Starting at $x_0 = 0.5$, find the values of x_1, x_2, x_3, x_4, x_5 in each case.

(i) $x_1 =$

(ii) $x_1 =$

$x_2 =$

$x_2 =$

$x_3 =$

$x_3 =$

$x_4 =$

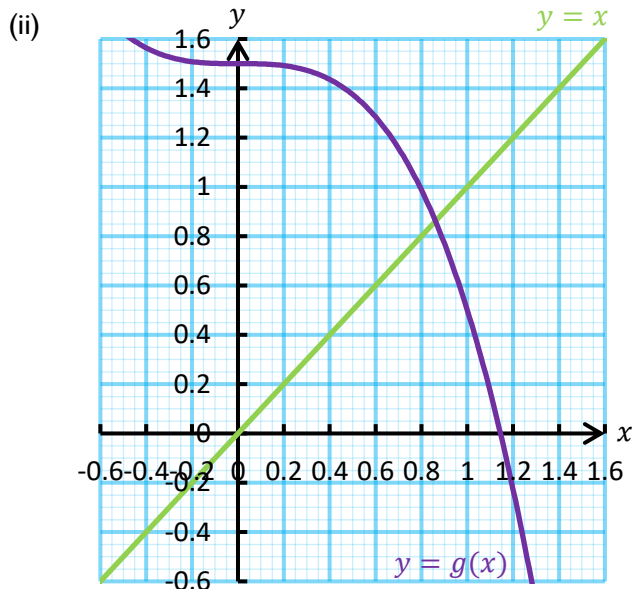
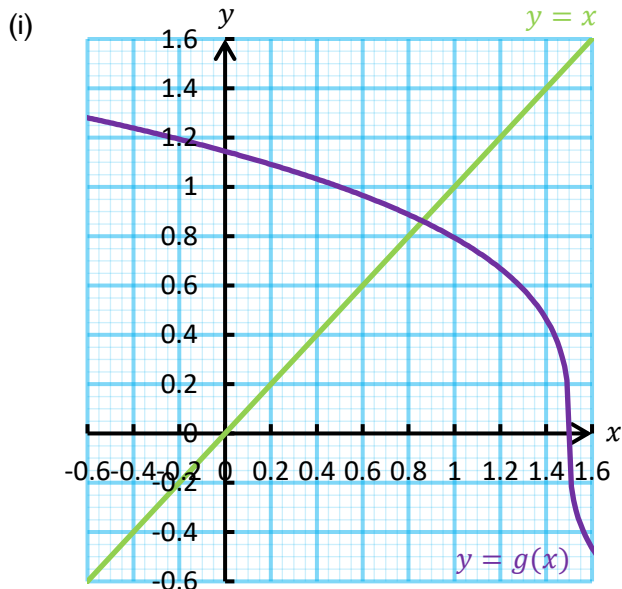
$x_4 =$

$x_5 =$

$x_5 =$

(d) Which one of the above sequences converges, and which diverges?

(e) Use the following diagrams to draw a cobweb diagram for each iteration.

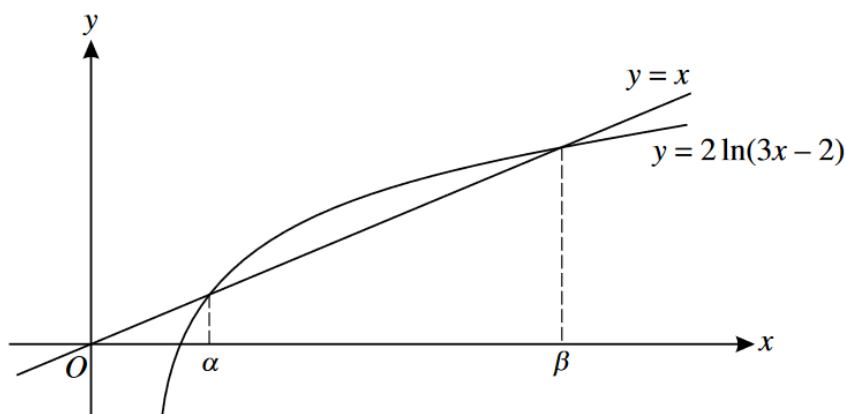


(f) For the iteration that converges, write down the value of x_5 correct to one decimal place. Prove that this is the root of Siwan's original equation correct to one decimal place.



(OCR Further Pure Mathematics 2 [4726] Summer 2010)

7



The line $y = x$ and the curve $y = 2 \ln(3x - 2)$ meet where $x = \alpha$ and $x = \beta$, as shown in the diagram.

- (i) Use the iteration $x_{n+1} = 2 \ln(3x_n - 2)$, with initial value $x_1 = 5.25$, to find the value of β correct to 2 decimal places. Show all your working. [2]
- (ii) With the help of a ‘staircase’ diagram, explain why this iteration will not converge to α , whatever value of x_1 (other than α) is used. [3]

(OCR Further Pure Mathematics 2 [4726] Summer 2011)

- 3** It is given that $F(x) = 2 + \ln x$. The iteration $x_{n+1} = F(x_n)$ is to be used to find a root, α , of the equation $x = 2 + \ln x$.

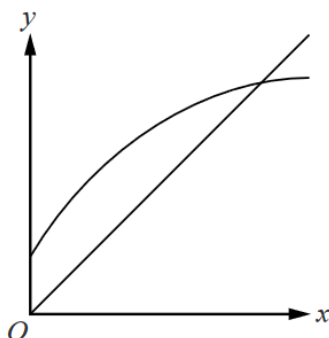
(i) Taking $x_1 = 3.1$, find x_2 and x_3 , giving your answers correct to 5 decimal places. [2]

(iii) Illustrate the iteration by drawing a sketch of $y = x$ and $y = F(x)$, showing how the values of x_n approach α . State whether the convergence is of the ‘staircase’ or ‘cobweb’ type. [3]

(OCR Further Pure Mathematics 2 [4726] Summer 2012)

- 4 It is given that the equation $x^4 - 2x - 1 = 0$ has only one positive root, α , and $1.3 < \alpha < 1.5$.

(i)



The diagram shows a sketch of $y = x$ and $y = \sqrt[4]{2x+1}$ for $x \geq 0$. Use the iteration $x_{n+1} = \sqrt[4]{2x_n + 1}$ with $x_1 = 1.35$ to find x_2 and x_3 , correct to 4 decimal places. On the copy of the diagram show how the iteration converges to α . [3]

- (ii) For the same equation, the iteration $x_{n+1} = \frac{1}{2}(x_n^4 - 1)$ with $x_1 = 1.35$ gives $x_2 = 1.1608$ and $x_3 = 0.4077$, correct to 4 decimal places. Draw a sketch of $y = x$ and $y = \frac{1}{2}(x^4 - 1)$ for $x \geq 0$, and show how this iteration does not converge to α . [2]

4 You are given the equation $(2x-1)^2 - e^x = 0$.

- There are also two other roots, α and β , where $0 < \alpha < \beta$.

- (ii) The iterative formula $x_{r+1} = \ln(2x_r - 1)^2$ is to be used to find a root of the equation.

- (b) By drawing a ‘staircase’ diagram on your sketch, starting with a value of x that is between α and β , show that this iteration does not converge to α . [1]

- (c) Using this iterative formula with $x_1 = 3.75$, find the value of β correct to 3 decimal places. [3]

(OCR Further Pure Mathematics 2 [4726/01] Summer 2017)

8 It is required to solve the equation $f(x) = \ln(4x - 1) - x = 0$.

(i) Show that the equation has two roots, α and β , such that $0.5 < \alpha < 1$ and $1 < \beta < 2$. [1]

(ii) Use the iterative formula $x_{r+1} = \ln(4x_r - 1) + 1$ with $x_0 = 1.8$ to find x_1 , x_2 and x_3 , correct to 5 decimal places. Write down the value of β to as many decimal places as these values justify. [3]

(iii) Derive the iterative formula $x_{r+1} = \frac{(e^{x_r} + 1)}{4}$ and use it to find α correct to 4 decimal places. [4]

(iv) Show that the iterative formula in part (ii) will not find the value of α . Determine whether the iterative formula in part (iii) will find the value of β . [3]



(OCR Further Pure Mathematics 2 [4726/01] Summer 2014)

- 9 The equation $10x - 8 \ln x = 28$ has a root α in the interval $[3, 4]$. The iteration $x_{n+1} = g(x_n)$, where $g(x) = 2.8 + 0.8 \ln x$ and $x_1 = 3.8$, is to be used to find α .

(i) Find the value of α correct to 5 decimal places. You should show the result of each step of the iteration to 6 decimal places. [4]

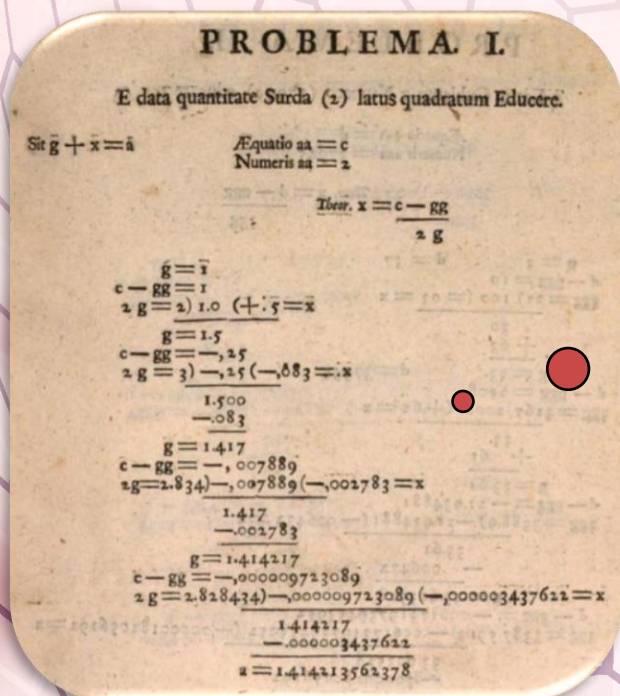
(ii) Illustrate this iteration by means of a sketch. [2]

A series of horizontal dotted lines for working space.



Newton-Raphson

Method



Joseph Raphson published the book "Analysis Aequationum Universalis" in the year 1690. Here he built on previous work by Isaac Newton on solving equations.

Name:

Background

What is the work?

Given an equation $y = f(x)$, the Newton-Raphson method uses **tangents** to try to find roots of the equation.

What is required before starting?

GCSE Work: Substitution, rounding off.
A Level Unit 1: Curve sketching, differentiation.

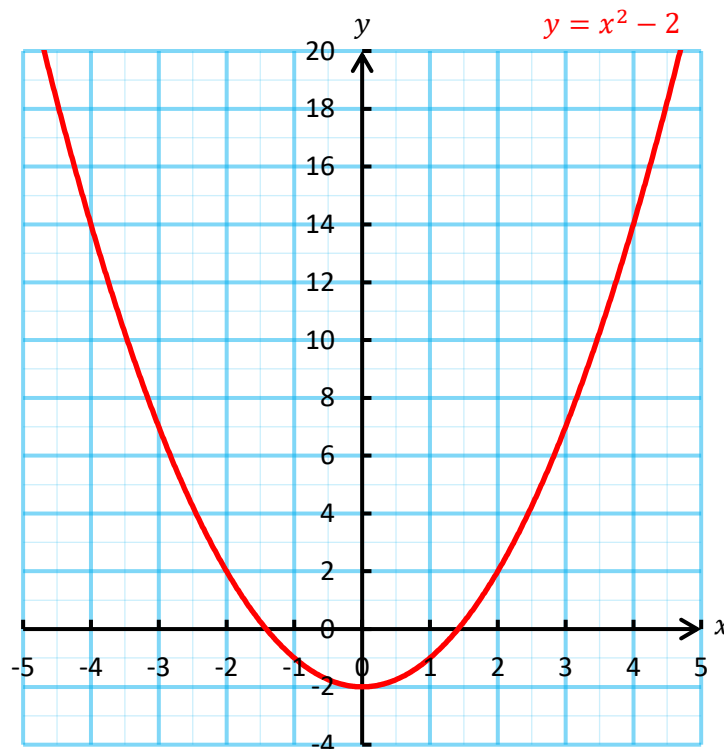
Where does this lead to?

Applications:

- To solve problems in context which lead to equations that cannot be solved analytically.
- Finding stationary points.
- Finding the reciprocal of a number without division.

Theory

Consider any function $y = f(x)$, for example $y = x^2 - 2$.



Theory and
Example



Blog

The **roots** of the equation are where the curve intersects the x -axis, so that $f(x) = 0$. For the example shown above, we see that there are two roots: a negative root between -2 and -1 , and a positive root between 1 and 2 . The Newton-Raphson method uses **tangents** to try to find a root:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This formula is
given in the formula
booklet.

Note that $f'(x_n)$ represents the function obtained by differentiating $f(x_n)$.

Example: $y = x^2 - 2$.

We see from the graph on the previous page that a root of the equation $y = x^2 - 2$ is close to $x = 2$. So, we can choose to start at $x_0 = 2$ and use the recurrence relation to try to find the root.

$$f(x_n) = x_n^2 - 2$$

$$f'(x_n) = 2x_n$$

$$x_1 = 2 - \frac{2^2 - 2}{2 \times 2}$$

$$x_1 = 1.5$$

$$x_2 = 1.5 - \frac{1.5^2 - 2}{2 \times 1.5}$$

$$x_2 = 1.41\dot{6}$$

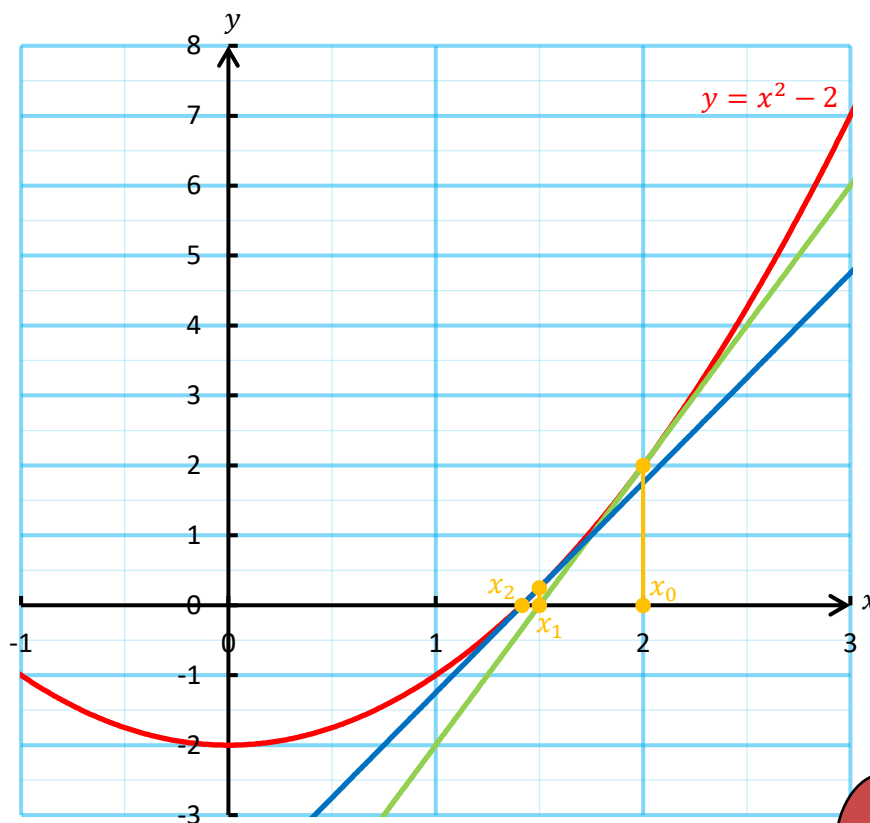
$$x_3 = 1.414215686$$

$$x_4 = 1.414213562$$

$$x_5 = 1.414213562$$

Use the **ANS** button to calculate these efficiently.

Using a normal calculator, the most precise value we can find for the root is $x = 1.414213562$. (We could use the change of sign method to prove this is the value of the root correct to nine decimal places.) To visualise the method used above, we can use the following graph.



- We start at the point $(x_0, 0)$ on the x -axis, so at the point $(2, 0)$.
- We draw the **tangent** to the curve at the point $(2, 2^2 - 2 = 2)$.
- The **tangent** crosses the x -axis at the point $(x_1, 0)$, so at $(1.5, 0)$.
- We draw the **tangent** to the curve at the point $(1.5, 1.5^2 - 2 = 0.25)$.
- The **tangent** crosses the x -axis at the point $(x_2, 0)$, so at $(1.41\dot{6}, 0)$.
- And so on...

The Newton-Raphson method finds where the next tangent crosses the x -axis.

The Newton-Raphson method usually **converges** quickly, but can be unreliable if the gradient of any tangent is **small**.

Challenge!  Prove that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is the iterative function for the Newton-Raphson method, starting with the equation of a general tangent in the form $y - y_1 = m(x - x_1)$.

Top 5

8. Mae'r ffwythiant f yn cael ei ddiffinio gan

(a) Dangoswch fod gan yr hafaliad $f(x) = 0$ wreiddyn yn y cyfwng (*interval*) $[0, 1]$. [1]

(i) ysgrifennwch werth degol x_1 yn llawn fel mae eich cyfrifiannell yn ei roi,

(ii) darganfyddwch werth y gwreiddyn hwn yn gywir i chwech lle degol. [4]

(c) Esboniwch pam dydy'r dull Newton-Raphson ddim yn gweithio os yw $x_0 = \frac{1}{3}$. [2]

4. (i) $f(x) = 3x^2 - \frac{1}{2\sqrt{x}} - 5x, \quad x > 0$

- (a) Show that the equation $f(x) = 0$ has a root α in the interval $[1, 2]$. (2)
- (b) Find $f'(x)$. (2)
- (c) Using $x_0 = 2$ as a first approximation to α , apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α , giving your answer to 2 decimal places. (2)

(Unit 3 Summer 2023)

0

4

A function f with domain $(-\infty, \infty)$ is defined by $f(x) = 6x^3 + 35x^2 - 7x - 6$.

a) Determine the number of roots of the equation $f(x) = 0$ in the interval $[-1, 1]$. [2]

b) Use the Newton-Raphson method to find a root of the equation $f(x) = 0$.
Starting with $x_0 = 1$,

i) write down the value of x_1 ,

ii) determine the value of the root correct to one decimal place. [4]

c) It is suggested that another iterative sequence

$$x_{n+1} = \sqrt{\frac{7x_n + 6 - 6x_n^3}{35}},$$

starting with $x_0 = -3$, could be used to find a root of the equation $f(x) = 0$.
Explain why this method fails. [2]

Handwriting practice area consisting of 30 horizontal dotted lines.

2.

$$f(x) = x^3 - \frac{5}{2x^{\frac{3}{2}}} + 2x - 3, \quad x > 0$$

- (a) Show that the equation $f(x) = 0$ has a root α in the interval $[1.1, 1.5]$. (2)
- (b) Find $f'(x)$. (2)
- (c) Using $x_0 = 1.1$ as a first approximation to α , apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α , giving your answer to 3 decimal places. (3)

2. $f(x) = 3x^{\frac{3}{2}} - 25x^{-\frac{1}{2}} - 125, \quad x > 0$

(2)

(b) Using $x_0 = 12.5$ as a first approximation to α , apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α , giving your answer to 3 decimal places.

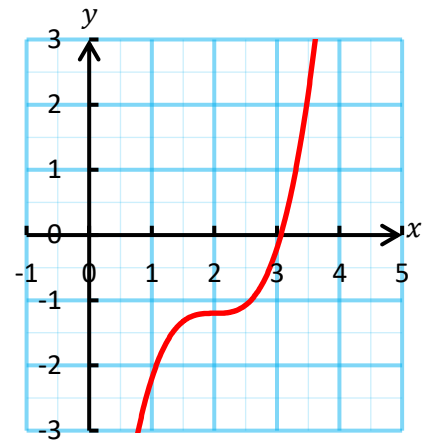
(4)

Problems

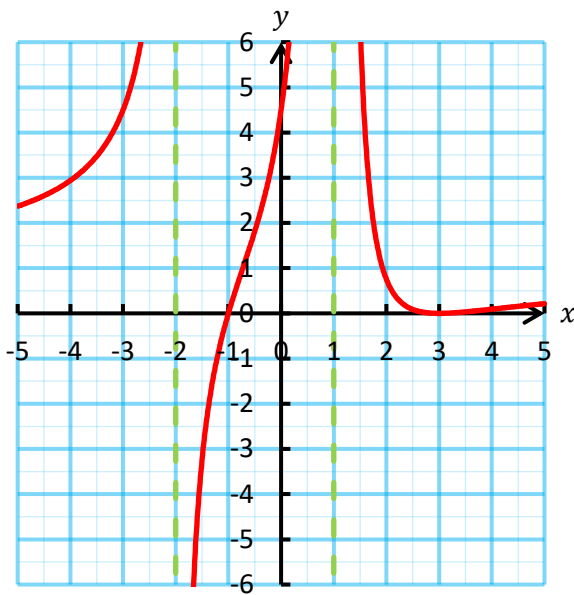
If a function crosses the x -axis, then we can show that the function has a root by looking at the sign of the function either side of the root. The **change in sign** shows that the function has a root in a specified interval.

For example, consider the function shown on the right. The change in sign between $f(2.5)$ and $f(3.5)$ shows that a root exists in the interval $[2.5, 3.5]$.

However, the change of sign method does not work every time.



Exercise 1



Consider the function shown on the left.

- (a) There is a change of sign between $f(-3)$ and $f(-1.5)$. Why is there no root over the interval $[-3, -1.5]$?

- (b) The graph shows that the function has a root when $x = 3$. Why would a change of sign method not be able to locate the root over the interval $[2, 4]$?

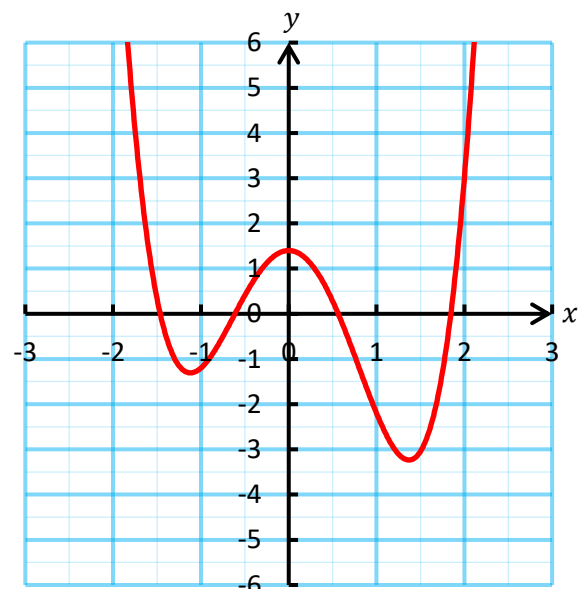
- (c) Write down an interval where the change of sign method would be able to locate a root.

Exercise 2

Consider the function shown on the right.

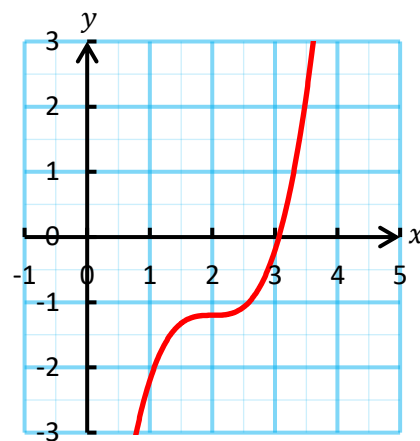
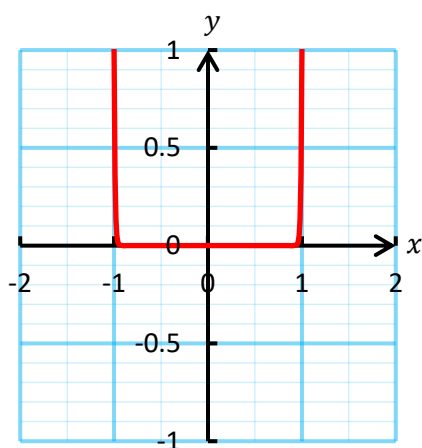
- (a) What is the problem with using a change of sign method to locate a root over the interval $[0, 2]$?

- (b) What is the problem with using a change of sign method to locate a root over the interval $[-1.75, 1.5]$?



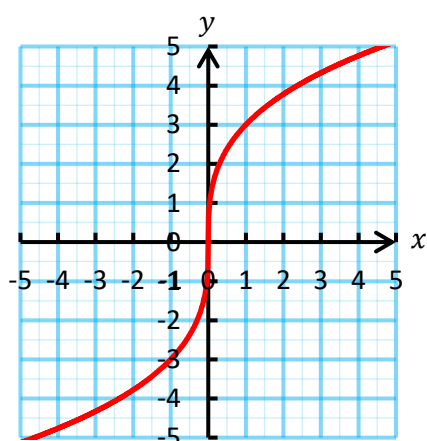
Exercise 3

For the graph on the right, what would be the problem in using the Newton-Raphson method with $x_0 = 2$?

**Exercise 4**

(a) The graph on the left shows the curve $y = x^{100}$.

A loyal supporter of the Newton-Raphson method decides to use the method to find the root of $y = x^{100}$, which is at $(0, 0)$. What happens when we use the method starting at $x_0 = 0.1$?



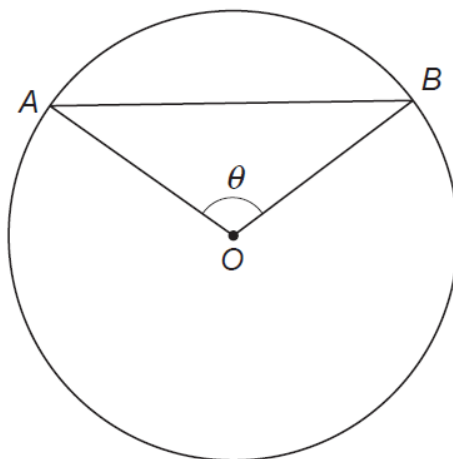
(b) The loyal supporter now decides to use the method to find the root of $y = 3x^{\frac{1}{3}}$, also located at $(0, 0)$. What happens when we use the method starting at $x_0 = 0.1$?

Revision Questions

(Unit 3 Summer 2019)

1	2
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A chord AB subtends an angle θ radians at the centre of a circle. The chord divides the circle into two segments whose areas are in the ratio 1:2.



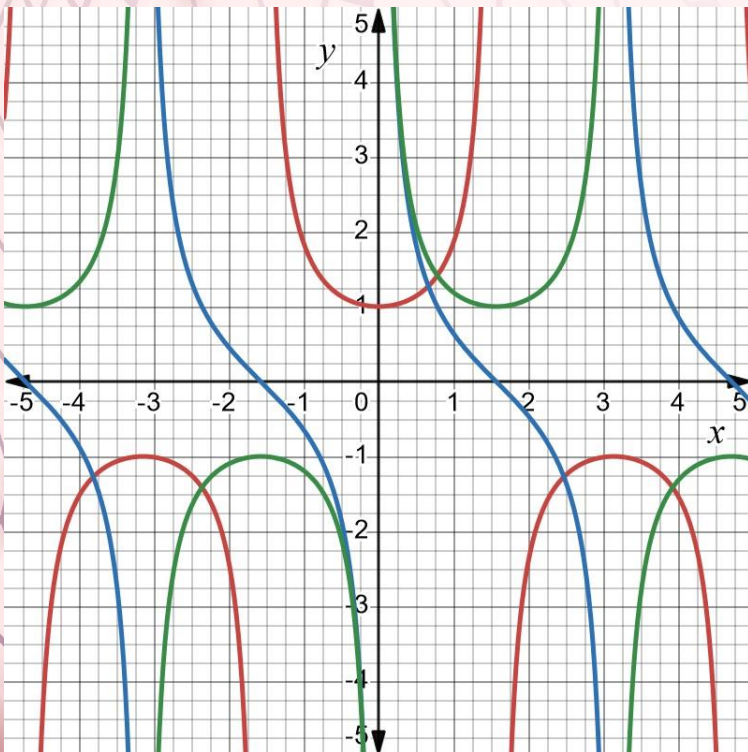
- a) Show that $\sin \theta = \theta - \frac{2\pi}{3}$. [4]
- b) i) Show that θ lies between 2.6 and 2.7.
- ii) Starting with $\theta_0 = 2.6$, use the Newton-Raphson Method to find the value of θ correct to three decimal places. [6]

1 **7** By drawing suitable graphs, show that $x - 1 = \cos x$ has only one root. Starting with $x_0 = 1$, use the Newton-Raphson method to find the value of this root correct to two decimal places. [6]



Sec, Cosec,

Cot



$$y = \sec x$$

$$y = \operatorname{cosec} x$$

$$y = \cot x$$

Name:

Background

What is the work?

Solving equations involving the trigonometric functions sec, cosec and cot.

What is required before starting?

GCSE Work: Substitution, solving equations, factorising quadratic expressions.
A Level Unit 1: Solving trigonometric equations.

Where does this lead to?

Understand the domain, range and graphs of the three functions.
 Differentiate and integrate trigonometric functions.

Theory

What is the **reciprocal** of the trigonometric functions sin, cos and tan? There is a problem with using the usual notation $x^{-1} = \frac{1}{x}$ for the reciprocal, as we have already used $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ to help us find angles. For example, if $\sin(x) = 0.5$, then

$$x = \sin^{-1}(0.5)$$

$$x = \dots, 30^\circ, \dots$$

So, $\sin^{-1}(0.5)$ does not represent $\frac{1}{\sin(0.5)}$, but rather the set of angles whose sin gives the answer 0.5. In order to represent $\frac{1}{\sin(0.5)}$, mathematicians have introduced three new trigonometric functions:

$$\sec(x) = \frac{1}{\cos(x)}, \quad \operatorname{cosec}(x) = \frac{1}{\sin(x)}, \quad \cot(x) = \frac{1}{\tan(x)}.$$

We can derive two new trigonometric identities from the identity $\sin^2(x) + \cos^2(x) = 1$.

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} = \frac{1}{\sin^2(x)} \quad (\text{Divide by } \sin^2(x))$$

$$(\text{Divide by } \cos^2(x))$$

$$1 + \cot^2(x) = \operatorname{cosec}^2(x) \quad (\text{Simplify})$$

$$(\text{Simplify})$$

$$\operatorname{cosec}^2(x) = 1 + \cot^2(x) \quad (\text{Swap sides})$$

$$(\text{Swap sides})$$

We can use the above identities when solving trigonometric equations that contain the functions sec, cosec and cot.



Theory



Example

Exercise

(C3 Summer 2005)

3. (a) Show, by counter-example, that the statement

$$\cot^2 \theta \equiv 1 + \operatorname{cosec}^2 \theta \quad [2]$$

is false.

- (b) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$10 \sec^2 \theta = 11 \tan \theta + 16. \quad [6]$$

Top 10

(C3 Summer 2008)

2. (a) Show, by counter-example, that the statement

$$\tan 2\theta \equiv \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

is false.

[2]

- (b) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$2\sec^2\theta = 8 - \tan\theta.$$

[6]

2. (a) Show, by counter-example, that the statement

is false. [2]

$$3 \sec^2 \theta = 7 - 11 \tan \theta.$$

Give your answers correct to one decimal place. [6]

2. (a) Show, by counter-example, that the statement

is false. [2]

$$3 \operatorname{cosec}^2 \theta = 11 - 2 \cot \theta. \quad [6]$$

2. (a) Show, by counter-example, that the statement

is false. [2]

$$\sec^2 \theta + 8 = 4 \tan^2 \theta + 5 \sec \theta. \quad [6]$$

2. (a) (i) Show, by counter-example, that the statement $\cos^3\theta \equiv 1 - \sin^3\theta$

(ii) **Write down** a value of θ which does satisfy the equation

$$\cos^3 \theta = 1 - \sin^3 \theta. \quad [3]$$

(b) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$4\operatorname{cosec}^2\theta = 9 - 8\cot\theta. \quad [6]$$

2. (a) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

(b) Find all values of ϕ in the range $0^\circ \leq \phi \leq 360^\circ$ satisfying

$$\sec\phi + 2\tan\phi = 0. \quad [3]$$

2. (a) Show, by counter-example, that the statement

is false.

[2]

$$15 \operatorname{cosec}^2 \theta + 2 \cot \theta = 23.$$

[6]

2. (a) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$7 \operatorname{cosec}^2 \theta - 4 \cot^2 \theta = 16 + 5 \operatorname{cosec} \theta. \quad [6]$$

- (b) Without carrying out any calculations, explain why there are no values of ϕ in the range $0^\circ \leq \phi \leq 90^\circ$ which satisfy the equation

$$4 \sec \phi + 3 \operatorname{cosec} \phi = 6. \quad [1]$$

2. (a) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

(b) Find all values of ϕ in the range $0^\circ \leq \phi \leq 360^\circ$ satisfying

$$2 \operatorname{cosec} \phi + 3 \sec \phi = 0. \quad [3]$$

2. (a) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

(b) Without carrying out any calculations, explain why there are no values of ϕ in the range $0^\circ \leq \phi \leq 90^\circ$ which satisfy the equation

$$\sec \phi + 2 \tan \phi = 0.8. \quad [1]$$

Revision Question

(Unit 3 Summer 2019)

9

- a)** Given that α and β are two angles such that $\tan \alpha = 2 \cot \beta$, show that

$$\tan(\alpha + \beta) = -(\tan\alpha + \tan\beta). \quad [2]$$

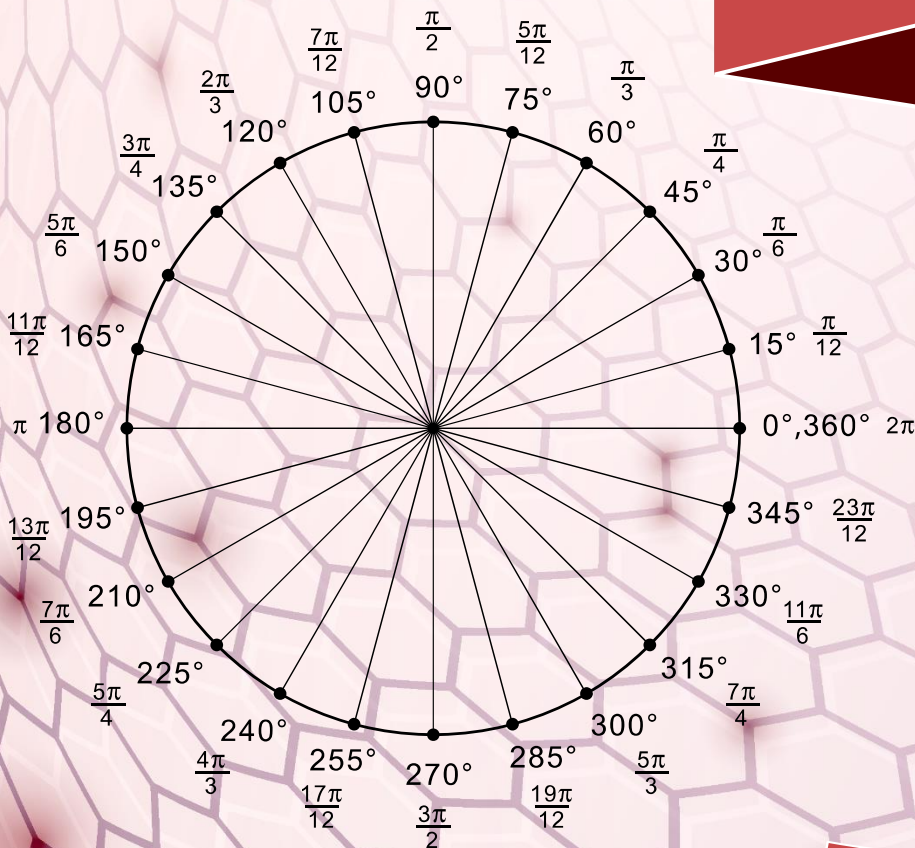
- b)** Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying the equation

$$4 \tan \theta = 3 \sec^2 \theta - 7. \quad [6]$$



Measuring Angles

in Radians



Name: _____

Background

What is the work?

In the GCSE course, we have been measuring angles in degrees, from 0° to 360° . In order to calculate the length of a circular arc or the area of a sector efficiently, it is advantageous to use a different scale to measure angles at A Level.

What is required before starting?

GCSE Work: Measuring angles, calculating the circumference and area of a circle, calculating the area of a segment in a circle.

A Level Unit 1: Trigonometry.

Where does this lead to?

A Level Unit 3:

- Small angle approximations.
- Integration of trigonometric functions.

Applications:

- Measuring angular speed in physics / further mathematics.
- Measuring the frequency of an electromagnetic wave.

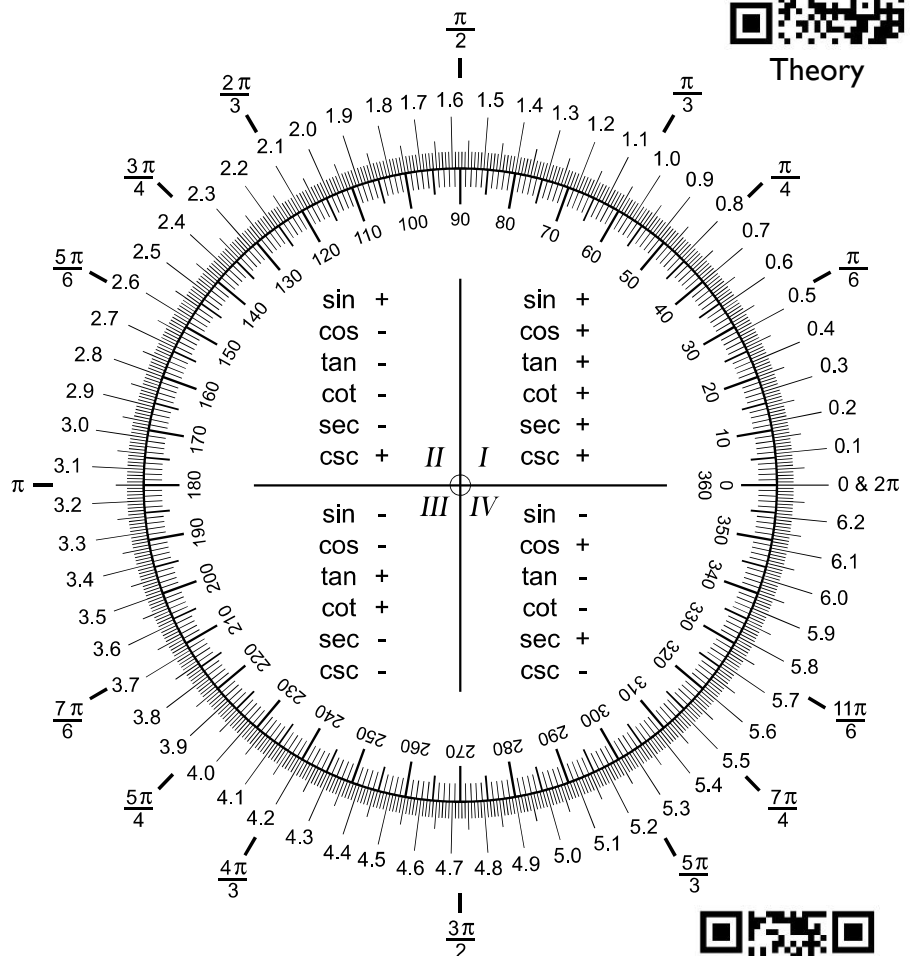
Theory



Theory

When measuring angles, we must decide on a scale to use that goes from zero to the value for a full turn. When measuring angles in **degrees**, the scale goes from 0° to 360° . 360 was chosen because of its high number of factors – for example, considering the numbers between 1 and 10, only 7 is not a factor of 360. This means it is easy to divide a full turn into 3 equal parts; 5 equal parts; and so on.

Degrees are not the only scale used to measure angles. Surveyors in Europe use **400 gradians** for a full turn (so that 100 gradians form each right angle.) This system comes from France, and appears on scientific calculators as one of the options available to measure angles. We will not, however, use gradians to measure angles in the A Level course. Instead, a system that will be of use to us is the **radians** system. Here, a full term is equal to 2π radians (which, to 2 decimal places, is equal to 6.28 radians). On first glance, using such a system will be more complicated – imagine, for example, measuring an obtuse angle using the protractor shown above. However, the major advantage of this system is that easier **formulae** appear when measuring angles in radians.



Example

Exercise 1

(a) Complete the following table. Give your answers in terms of π , if possible.

Angle in degrees	0°	90°		360°	45°		100°	
Angle in radians			π rad			$\frac{3\pi}{2}$ rad		5 rad

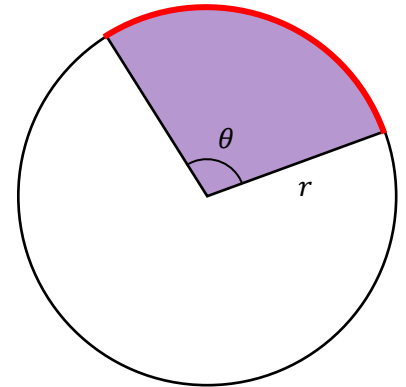
(b) Complete the following sentences.

To change an angle in degrees to be in radians, we must multiply by

To change an angle in radians to be in degrees, we must multiply by

The Length of an Arc and the Area of a Sector

Let us consider the diagram on the right, which shows a circle with radius r . To begin, let us calculate the length of the **red arc**, which is a fraction of the circle's circumference.

**In degrees**

$$\text{Length of the arc} = \frac{\theta}{360^\circ} \times \pi \times \text{diameter}$$

In radians

$$\text{Length of the arc} = \frac{\theta}{2\pi} \times \pi \times \text{diameter}$$

$$\text{Length of the arc} = \frac{\theta}{2} \times \text{diameter}$$

$$\text{Length of the arc} = \theta \times \text{radius}$$

$$\text{Length of the arc} = r\theta$$

We see that the formula for arc length is much simpler when the angle is measured in radians. This is also true when calculating the area of the **purple segment**, which is a fraction of the circle's area.

In degrees

$$\text{Area of the sector} = \frac{\theta}{360^\circ} \times \pi \times \text{radius}^2$$

In radians

$$\text{Area of the sector} = \frac{\theta}{2\pi} \times \pi \times \text{radius}^2$$

$$\text{Area of the sector} = \frac{\theta}{2} \times \text{radius}^2$$

$$\text{Area of the sector} = \frac{1}{2} r^2 \theta$$

Again the formula is simpler when measuring the angle in radians, as the two π symbols cancel. To summarise, when measuring an angle in radians,

$$\text{Length of an arc} = r\theta$$

$$\text{Area of a sector} = \frac{1}{2} r^2 \theta$$

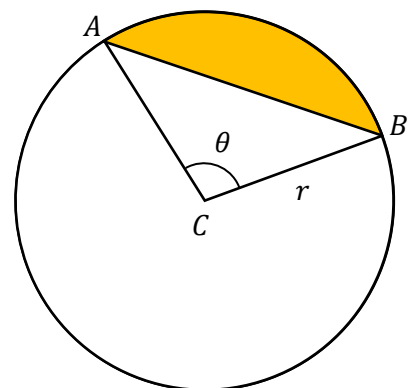
Area of a Segment of a Circle

It is also useful to be able to remember the formula for calculating the area of a **circle's segment**, as shown on the right. When measuring the angle θ in radians:

$$\text{Area of the segment} = \text{Area of the sector} - \text{Area of the triangle}$$

$$\text{Area of the segment} = \frac{1}{2} r^2 \theta - \frac{1}{2} ab \sin C$$

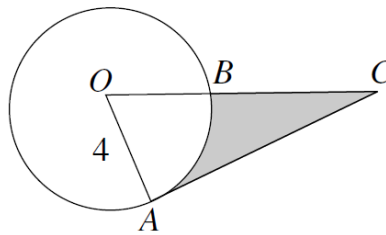
$$\text{Area of the segment} = \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin \theta$$



Exercise

(C2 Winter 2005)

9.



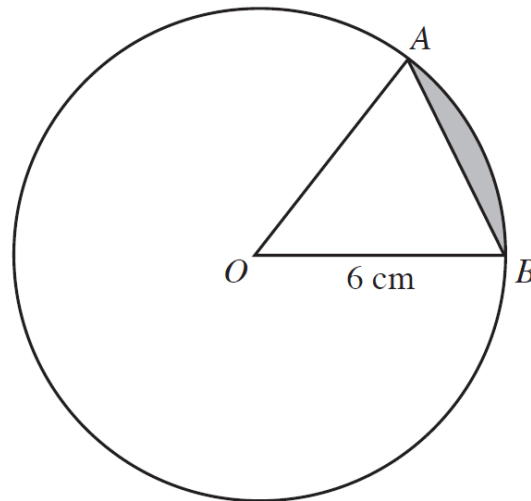
In the diagram the points A and B lie on a circle with centre O and radius 4 cm. The tangent to the circle at A intersects the line OB produced at the point C . The length of the arc AB is 3 cm.

- (a) Find $\widehat{AOB}$ in radians. [2]
- (b) Calculate the area of the shaded region. Give your answer correct to two decimal places. [4]

Top 10

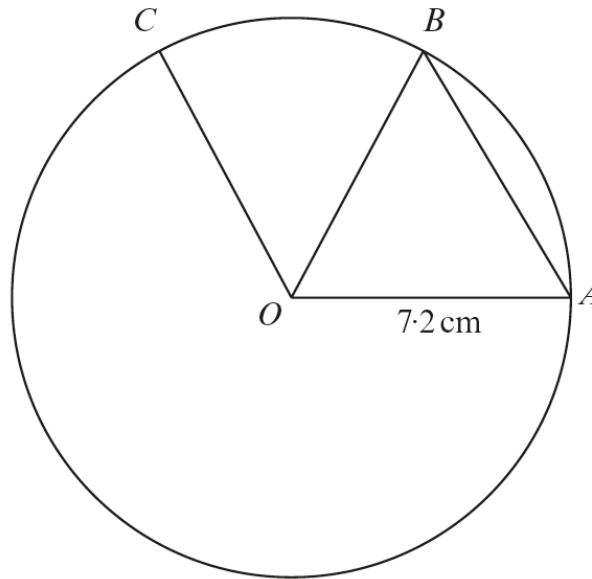
(C2 Summer 2008)

9.



The diagram shows two points A and B on a circle with centre O and radius 6 cm. The length of the **arc** AB is 5.4 cm.

- (a) Show that the area of the **sector** AOB is 16.2 cm^2 . [4]
- (b) Find the area of the shaded region, giving your answer correct to one decimal place. [3]

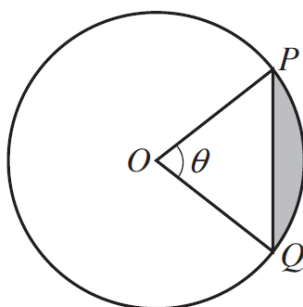


(a) Given that $\widehat{AOB} = 1.1$ radians, find the area of **triangle** AOB . Give your answer correct to one decimal place. [2]

- (b) The area of **sector** BOC is 19.44 cm^2 . Find the length of the **arc** BC . [3]

(C2 Summer 2006)

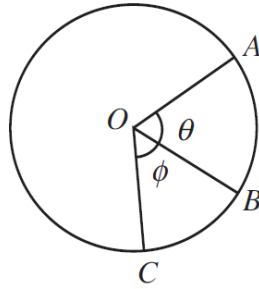
10.



The diagram shows two points P and Q on a circle, with centre O and radius 4 cm, such that $\widehat{POQ} = \theta$ radians. The perimeter and the area of the **sector** POQ are x cm and A cm², respectively. Given that $x = 3A$,

- (a) show that $\theta = 0.4$, [4]
- (b) calculate the area of the shaded region, giving your answer correct to three decimal places. [4]

9.



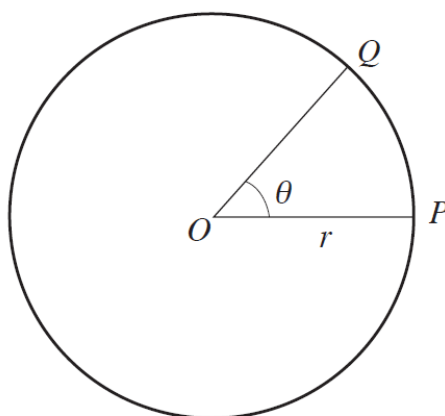
(a) Show that $\theta + \phi = 1.9$. [2]

(b) Given that the arc length AB is 3.2 cm greater than the arc length BC , find the values of θ and ϕ . [4]

- (a) show that $\theta = \pi - 2$, [4]
- (b) calculate the area of the shaded region. [3]

(C2 Winter 2008)

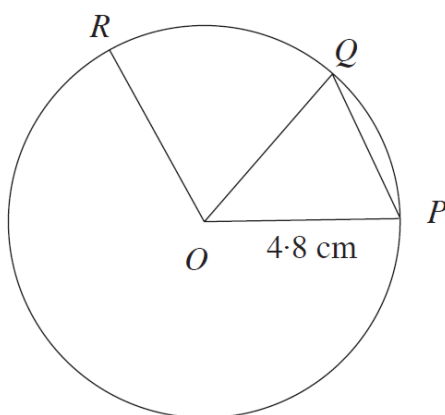
9.



The diagram shows two points P and Q on a circle with centre O . The radius of the circle is r cm and $\widehat{POQ} = \theta$ radians. The length of the arc PQ is 6 cm and the area of the sector POQ is 22.5 cm^2 . Find the values of r and θ . [5]

(C2 Winter 2009)

9.

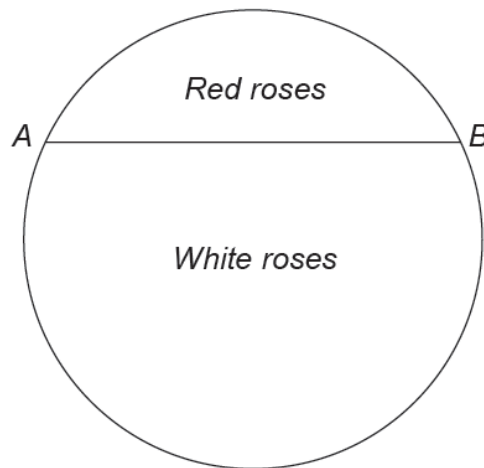


The diagram shows three points P , Q and R on a circle with centre O and radius 4.8 cm.

- (a) Given that $\widehat{POQ} = 0.7$ radians, find the area of **triangle** POQ . Give your answer correct to two decimal places. [2]
- (b) The length of the **arc** RQ is L cm and the area of **sector** ROQ is A cm². Given that $A = kL$, find the value of k . [3]

(C2 Summer 2015)

9. Gwyn wants to turn part of his garden into a circular flower bed. In order to do this, he digs out a shallow circular hole of radius r m and then divides it into two segments by means of a thin plank AB , as shown in the diagram. He plants red roses in the minor segment and white roses in the major segment.

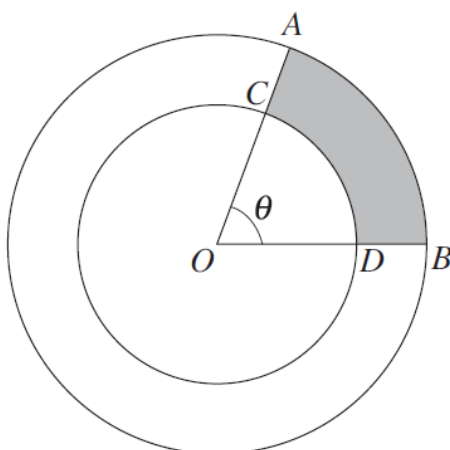


Let the centre of the flower bed be denoted by O . Show that when $\widehat{AOB}$ equals 2.6 radians, the area of the flower bed containing white roses is approximately twice the area containing red roses.

[5]

(C2 Summer 2010)

10.



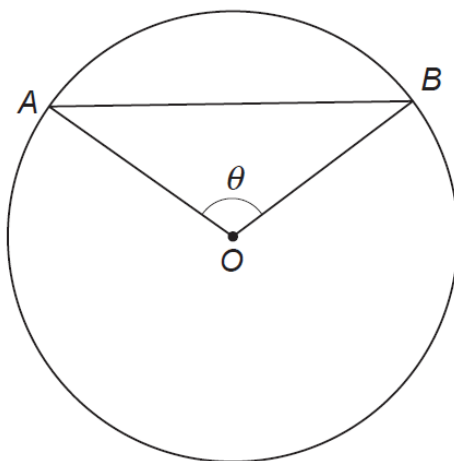
The diagram shows two concentric circles with common centre O . The radius of the larger circle is R cm and the radius of the smaller circle is r cm. The points A and B lie on the larger circle and are such that $\widehat{AOB} = \theta$ radians. The smaller circle cuts OA and OB at the points C and D respectively. The length of the arc AB is L cm **greater** than the length of the arc CD . The area of the shaded region is K cm².

- (a) (i) Write down an expression for L in terms of R , r and θ .
- (ii) Write down an expression for K in terms of R , r and θ . [2]
- (b) Use your results to part (a) to find an expression for r in terms of R , K and L . [3]

Revision Question

(Unit 3 Summer 2019)

1	2	A chord AB subtends an angle θ radians at the centre of a circle. The chord divides the circle into two segments whose areas are in the ratio 1:2.
----------	----------	---



- a) Show that $\sin \theta = \theta - \frac{2\pi}{3}$. [4]
- b) i) Show that θ lies between 2.6 and 2.7.
- ii) Starting with $\theta_0 = 2.6$, use the Newton-Raphson Method to find the value of θ correct to three decimal places. [6]

Handwriting practice area consisting of 30 horizontal dotted lines.



Trigonometric

Addition Formulae

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Bhaskara was an eminent Indian mathematician in the 12th century. He published a number of books, including *Siddhantasiromani*, a book on astronomy that contained the two formulae shown on the right.

Name:

Background

What is the work?

This workbook considers the formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$.

What is required before starting?

GCSE Work: Angles, trigonometry, similar shapes.
A Level Unit 1: Trigonometry.

Where does this lead to?

A Level Unit 3:

- Double angle formulae.
- Writing values for (e.g.) $\sin 15^\circ$ without using a calculator.

Applications:

- Vibration theory.
- Electrical circuit theory.

Theory

The trigonometric functions are not distributive. This means that (for example) $\sin(A + B) \neq \sin A + \sin B$. We can prove this by considering $A = B = 45^\circ$:

$$\begin{aligned}\text{Left hand side} &= \sin(45^\circ + 45^\circ) \\ &= \sin(90^\circ) \\ &= 1\end{aligned}$$

$$\begin{aligned}\text{Right hand side} &= \sin 45^\circ + \sin 45^\circ \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ &= \frac{2}{\sqrt{2}}\end{aligned}$$

$1 \neq \frac{2}{\sqrt{2}}$ so it is not true that $\sin(A + B) = \sin A + \sin B$. The correct formula for $\sin(A + B)$ is

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

If A and B are acute angles, then we can prove this formula by considering the triangles on the right.

OPQ and OQR are right-angled triangles containing the angles A and B .

RT and SQ are lines forming the right angles OTR and RSQ .

$O\hat{U}T = R\hat{U}Q$ therefore OTU and URQ are similar triangles.

Therefore, $UR\hat{Q} = A$.

$$\begin{aligned}\sin(A + B) &= \frac{TR}{OR} \\ &= \frac{TS + SR}{OR} \\ &= \frac{PQ + SR}{OR} \\ &= \frac{PQ}{OR} + \frac{SR}{OR} \\ &= \frac{PQ}{OQ} \times \frac{OQ}{OR} + \frac{SR}{QR} \times \frac{QR}{OR}\end{aligned}$$

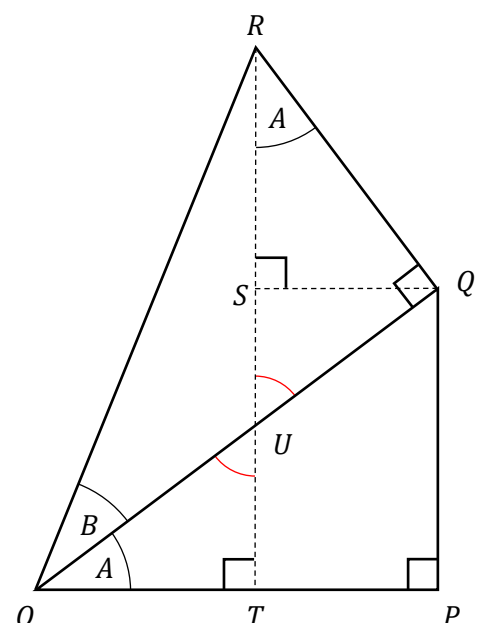
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$



Theory



Example



If we substitute $-B$ instead of B in the formula

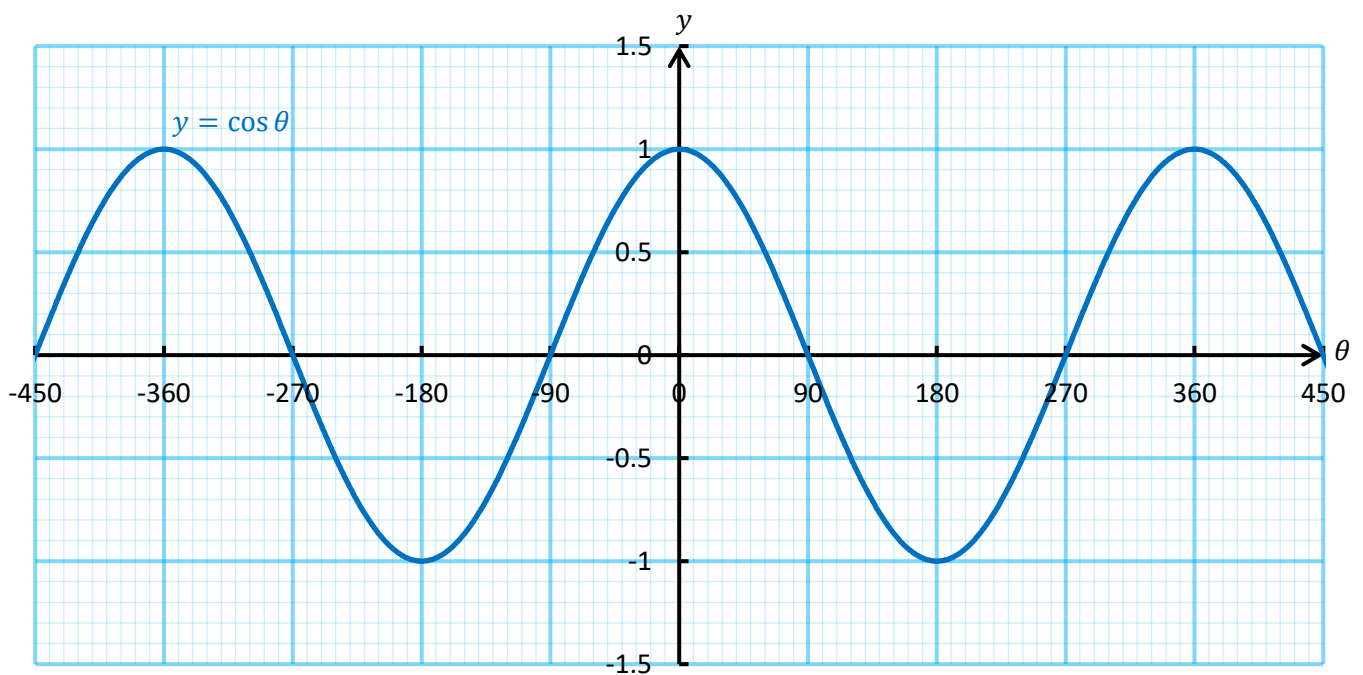
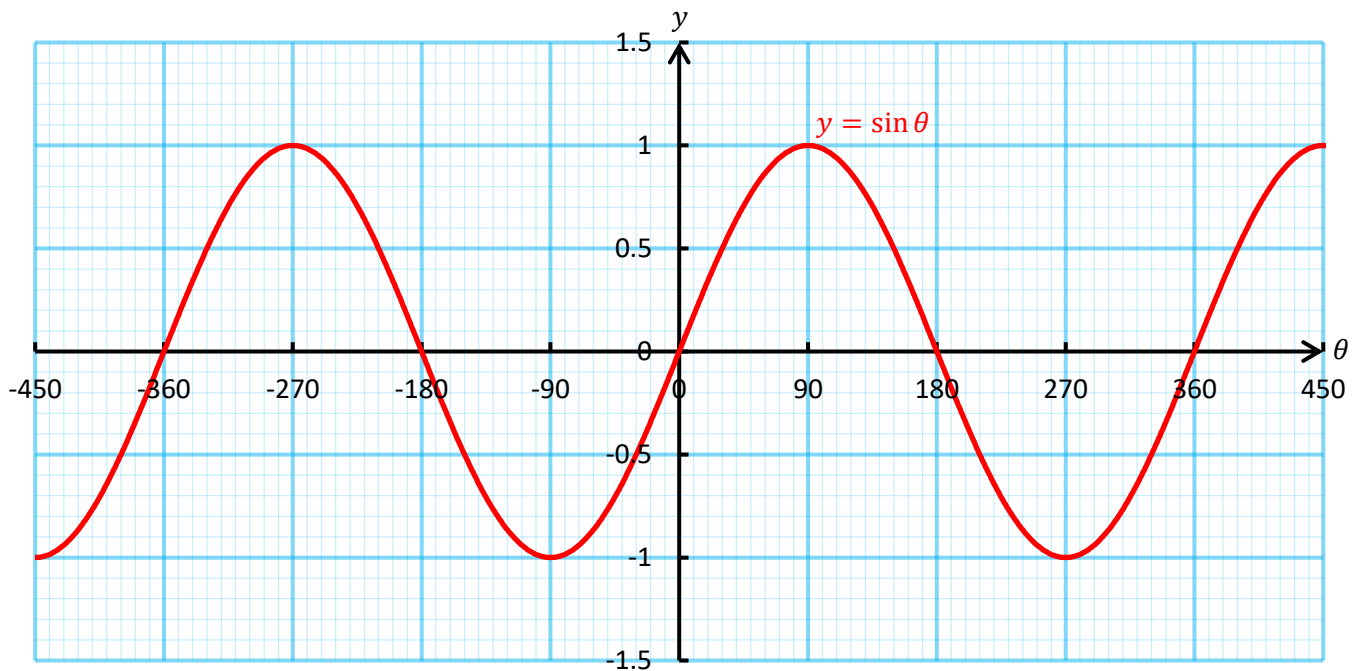
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

we obtain

$$\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B)$$

Considering the symmetry of the sin and cos graphs shown below, we see that $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$. Therefore,

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$



If we substitute $\left(\frac{\pi}{2} - A\right)$ instead of A in the formula

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

we obtain

$$\sin\left(\frac{\pi}{2} - A - B\right) = \sin\left(\frac{\pi}{2} - A\right) \cos B - \cos\left(\frac{\pi}{2} - A\right) \sin B$$

$$\sin\left(\frac{\pi}{2} - A - B\right) = \cos A \cos B - \sin A \sin B$$

(through the connection between the graphs of sin and cos)

$$\sin(\widehat{ORT}) = \cos A \cos B - \sin A \sin B$$

(considering the triangle ORT)

$$\cos\left(\frac{\pi}{2} - \widehat{ORT}\right) = \cos A \cos B - \sin A \sin B$$

(through the connection between the graphs of sin and cos)

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

(considering the triangle ORT)

Then, if we substitute $-B$ instead of B in the formula

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

we obtain

$$\cos(A - B) = \cos A \cos(-B) - \sin A \sin(-B)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

(through the symmetry of the graphs of sin and cos)

Next, remembering that $\tan \theta = \frac{\sin \theta}{\cos \theta}$, it follows that

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$\tan(A + B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

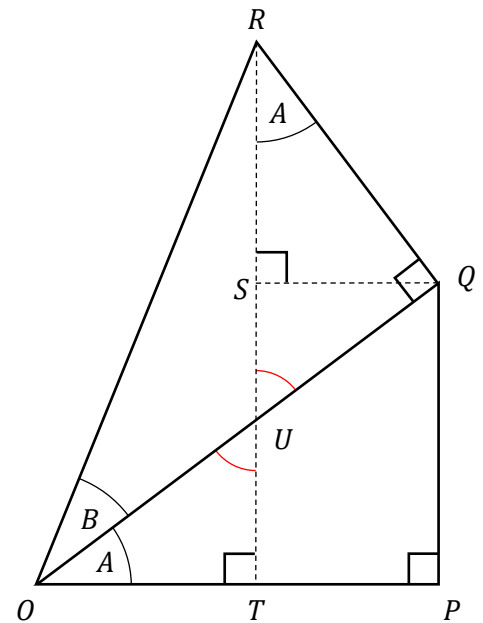
(using the previous formulae for $\sin(A + B)$ and $\cos(A + B)$)

$$\tan(A + B) = \frac{\frac{\sin A \cancel{\cos B}}{\cancel{\cos A} \cancel{\cos B}} + \frac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B}}{\frac{\cancel{\cos A} \cancel{\cos B}}{\cancel{\cos A} \cancel{\cos B}} - \frac{\sin A \sin B}{\cancel{\cos A} \cos B}}$$

(dividing each term by $\cos A \cos B$)

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(cancelling and using $\tan \theta = \frac{\sin \theta}{\cos \theta}$)



Finally, if we substitute $-B$ instead of B in the formula

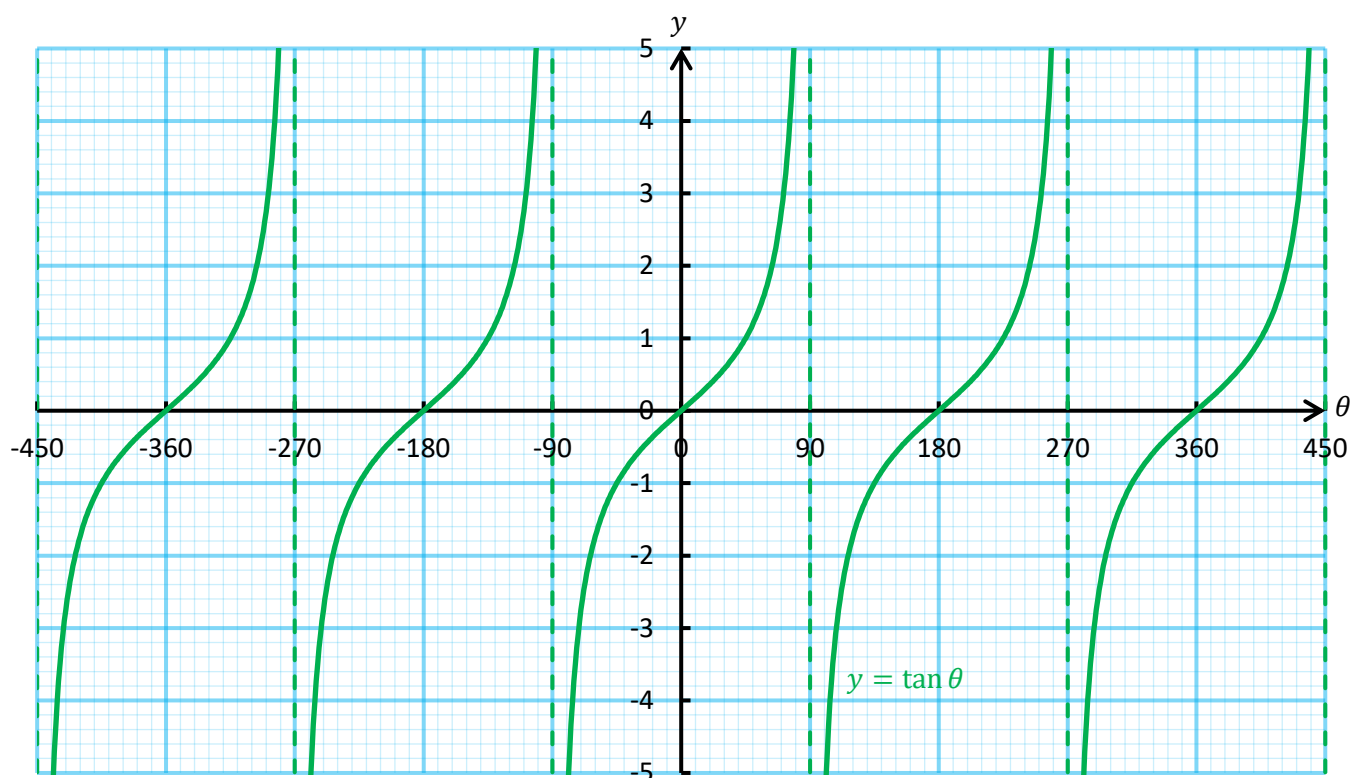
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

we obtain

$$\tan(A - B) = \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(through the symmetry of the tan graph shown below, where $\tan(-B) = -\tan B$)



To summarise:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

These formulae
are given in the
formula
booklet.

Exercises

(C4 Summer 2009)

3. (a) Express $\cos\theta + \sqrt{3}\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. [3]

- (b) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$\cos \theta + \sqrt{3} \sin \theta = 1. \quad [4]$$

(C4 Summer 2006)

4. (a) Express $4\sin x + 3\cos x$ in the form $R\sin(x + \alpha)$, where R and α are constants with $R > 0$ and $0^\circ < \alpha < 90^\circ$. [3]

- (b) Hence find the greatest value of

$$\frac{1}{4\sin x + 3\cos x + 7} \quad . \quad [2]$$



(b) Express $\sqrt{15} \cos \theta - \sin \theta$ in the form $R \cos(\theta + \alpha)$, where R and α are constants with $R > 0$ and $0^\circ < \alpha < 90^\circ$.
Hence find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$\sqrt{15} \cos \theta - \sin \theta = 3. \quad [6]$$

(C4 Summer 2015, Question 3)

- (b) (i) Express $\sqrt{13} \sin \theta - 6 \cos \theta$ in the form $R \sin(\theta - \alpha)$,
where R and α are constants with $R > 0$ and $0^\circ < \alpha < 90^\circ$.
- (ii) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$\sqrt{13} \sin \theta - 6 \cos \theta = -4.$$

[6]

3. (a) Find all values of x in the range $0^\circ \leq x \leq 180^\circ$ satisfying

$$\tan(x + 45^\circ) = 8 \tan x. \quad [5]$$

3. (a) Express $3\cos x + 2\sin x$ in the form $R\cos(x - \alpha)$, where R and α are constants with $R > 0$ and $0^\circ < \alpha < 90^\circ$. [3]

$$3\cos x + 2\sin x = 1. \quad [3]$$

(b) (i) Express $5 \sin x - 12 \cos x$ in the form $R \sin(x - \alpha)$, where R and α are constants with $R > 0$ and $0^\circ < \alpha < 90^\circ$.

$$\frac{1}{5 \sin x - 12 \cos x + 20}.$$

Write down a value for x for which this least value occurs. [6]

(b) (i) Express $21 \sin \theta - 20 \cos \theta$ in the form $R \sin (\theta - \alpha)$, where R and α are constants with $R > 0$ and $0^\circ < \alpha < 90^\circ$.

$$\frac{1}{21\sin\theta - 20\cos\theta + 31}.$$

[6]

(b) (i) Express $8 \sin x + 15 \cos x$ in the form $R \sin (x + \alpha)$, where R and α are constants with $R > 0$ and $0^\circ < \alpha < 90^\circ$.

(ii) Find all values of x in the range $0^\circ \leq x \leq 360^\circ$ satisfying $8 \sin x + 15 \cos x = 11$.

(iii) Find the greatest possible value for k so that $8 \sin x + 15 \cos x = k$ has solutions. Give a reason for your answer.

[7]

(b) Express $24 \cos \theta - 7 \sin \theta$ in the form $R \cos (\theta + \alpha)$, where R and α are constants with $R > 0$ and $0^\circ < \alpha < 90^\circ$.

$$24 \cos \theta - 7 \sin \theta = k$$

[5]

(b) Find all values of θ in the range $0 \leq \theta \leq 360^\circ$ satisfying

giving your answers in degrees correct to one decimal place. [6]



(C4 Summer 2017, Question 3)

- (b) (i) Express $\sqrt{5}\cos\phi + \sqrt{11}\sin\phi$ in the form $R\cos(\phi - \alpha)$, where R and α are constants with $R > 0$ and $0^\circ < \alpha < 90^\circ$.
- (ii) Use your result to part (i) to find the least value of

$$\frac{1}{\sqrt{5}\cos\phi + \sqrt{11}\sin\phi + 6}.$$

Write down a value for ϕ for which this least value occurs.

[6]

Handwriting practice area consisting of 30 horizontal dotted lines.

Handwriting practice lines consisting of 30 sets of three horizontal dotted lines.



Double Angle

Formulae

$$\sin(2A) = 2 \sin A \cos A$$

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

$$\cos(2A) = 1 - 2\sin^2(A)$$

$$\cos(2A) = 2\cos^2(A) - 1$$

$$\tan(2A) = \frac{2 \tan A}{1 - \tan^2 A}$$

Name:

Background

What is the work?

This workbook considers the formulae for $\sin(2A)$, $\cos(2A)$ and $\tan(2A)$.

What is required before starting?

A Level Unit 1: Trigonometry.
A Level Unit 3: Trigonometric Addition Formulae.

Where does this lead to?

A Level Unit 3:

- Integration of functions that contain (e.g.) $\sin^2(x)$.

Applications:

- Solving trigonometric equations by reducing the power of the equation.

Theory

In the previous workbook, the following formulae were introduced for trigonometric addition:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$



Theory

We will now consider what happens above when the angles A and B are equal.

$$\sin(A + A) = \sin A \cos A + \cos A \sin A$$

$$\sin(2A) = 2 \sin A \cos A$$

$$\cos(A + A) = \cos A \cos A - \sin A \sin A$$

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

$$\tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\tan(2A) = \frac{2 \tan A}{1 - \tan^2 A}$$

The above formulae are recognised as the formulae for **double angles**. In the case of the formula for $\cos(2A)$, it turns out that using the identity $\sin^2 A + \cos^2 A = 1$ from the Unit 1 course makes the formula even more useful.

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

$$\cos(2A) = (1 - \sin^2(A)) - \sin^2(A)$$

$$\cos(2A) = 1 - 2\sin^2(A)$$

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

$$\cos(2A) = \cos^2(A) - (1 - \cos^2(A))$$

$$\cos(2A) = \cos^2(A) - 1 + \cos^2(A)$$

$$\cos(2A) = 2\cos^2(A) - 1$$

$$\begin{aligned}\sin(2A) &= 2 \sin A \cos A \\ \cos(2A) &= \cos^2(A) - \sin^2(A) \\ \cos(2A) &= 1 - 2\sin^2(A) \\ \cos(2A) &= 2\cos^2(A) - 1 \\ \tan(2A) &= \frac{2 \tan A}{1 - \tan^2 A}\end{aligned}$$

Show that it is possible to write (a) $\sin(2A) = \frac{2 \tan A}{1 + \tan^2(A)}$, (b) $\cos(2A) = \frac{1 - \tan^2(A)}{1 + \tan^2(A)}$.

3. (a) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$2 \cos 2\theta = 9 \cos \theta + 7. \quad [5]$$

(C4 Summer 2011)

3. (a) Find all values of x in the range $0^\circ \leq x \leq 180^\circ$ satisfying

$$\tan 2x = 4 \tan x.$$

[5]



4. (a) Find all values of θ in the range $0 \leq \theta \leq 360^\circ$ satisfying

[4]

(C4 Summer 2009)

2. Find all the values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying $3\sin 2\theta = 2\sin \theta$. [5]

3. Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$2 + 3\cos 2\theta = \cos \theta. \quad [6]$$

3. (a) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

(C4 Summer 2013)

3. (a) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$8 \cos 2\theta + 6 = \cos^2 \theta + \cos \theta. \quad [6]$$

(C4 Summer 2014)

3. (a) Find all values of x in the range $0^\circ \leq x \leq 180^\circ$ satisfying

$$\tan 2x = 3 \cot x.$$

[4]

3. (a) Given that $\theta \neq 90^\circ$, find all values of θ in the range $0^\circ \leq \theta \leq 180^\circ$ satisfying $5 \tan 2\theta = 8 \cot \theta$.

[4]

3. (a) Find all values of θ in the range $0^\circ \leq \theta \leq 360^\circ$ satisfying

$$2 \cos 2\theta = 3 \sin^2\theta - 5 \cos^2\theta + \cos \theta + 1. \quad [6]$$

(C4 Summer 2017)

3. (a) Show that the equation

$$5\cos^2\theta + 7\sin 2\theta = 3\sin^2\theta$$

may be rewritten in the form

$$a\tan^2\theta + b\tan\theta + c = 0,$$

where a, b, c are non-zero constants whose values are to be found.

Hence, find all values of θ in the range $0^\circ \leq \theta \leq 180^\circ$ satisfying the equation

$$5\cos^2\theta + 7\sin 2\theta = 3\sin^2\theta.$$

[6]

4. (a) The angle x is such that $0^\circ \leq x \leq 180^\circ$, $x \neq 90^\circ$.

(i) show that $3 \tan^3 x - 8 \tan^2 x + 8 = 0$,

(ii) find all possible values of x , giving each answer in degrees, correct to one decimal place. [8]

Revision Question

(Sample Assessment Materials, Question 8)

(b) Use an appropriate substitution to show that

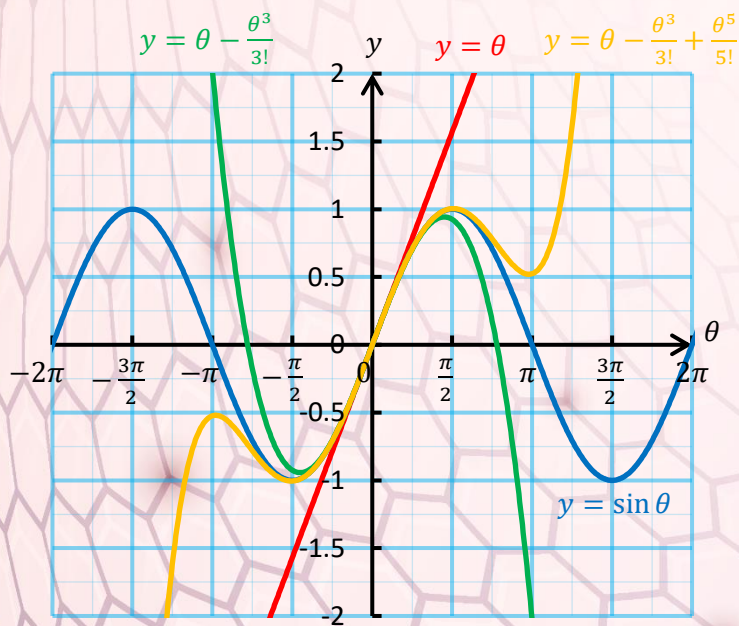
$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx = \frac{\pi}{12} - \frac{\sqrt{3}}{8}. \quad [8]$$

Handwriting practice area consisting of 30 horizontal dotted lines.



Small Angle

Approximations



$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \dots$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \dots$$

$$\tan(\theta) = \theta + \frac{2\theta^3}{3!} + \frac{16\theta^5}{5!} + \frac{272\theta^7}{7!} + \dots$$

The small angle approximations use part of these **Taylor series** for sin, cos and tan.

Name:

Background

What is the work?

If you have an angle θ that is sufficiently small, then it is possible to use alternate expressions for $\sin \theta$, $\cos \theta$ and $\tan \theta$, ones that do not require a calculator in order to calculate them.

What is required before starting?

GCSE Work: Trigonometry, area of a sector, inequalities.
A Level Unit 3: Measuring angles in radians, trigonometric addition formulae.

Where does this lead to?

A Level Unit 3:

- Differentiating $\sin \theta$ and $\cos \theta$ from first principles.

Applications:

- Measuring the distance to the stars.
- The “one in 60” rule when flying.

Theory



Theory

If we have a small angle measured in **radians**, then we can use the following approximations.

$$\sin \theta \approx \theta$$

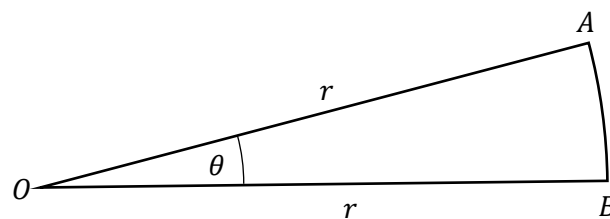
$$\cos \theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan \theta \approx \theta$$

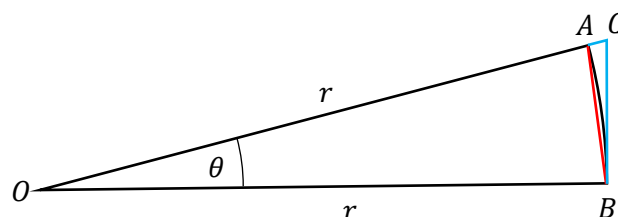
The approximations are correct to three significant figures if $-0.115 < \theta < 0.115$ (around $-6.6^\circ < \theta < 6.6^\circ$), and are correct to two significant figures if $-0.241 < \theta < 0.241$ (around $-13.8^\circ < \theta < 13.8^\circ$). The approximation for $\cos$ is the most accurate, and the approximation for $\tan$ is the least accurate.

Sin Proof

Let a small angle θ form the sector OAB of a circle. The area of the sector is $\frac{1}{2}r^2\theta$.



Let us add the **chord AB** and **extend** the radius OA to reach the point C so that OB and BC are perpendicular.



OCB is a right-angled triangle with base r and height $r \tan \theta$.

The area of the triangle OCB is $\frac{1}{2}r^2 \tan \theta$.

The area of the isosceles triangle OAB is $\frac{1}{2}r^2 \sin \theta$.

• • •

GCSE
Trigonometry

Area of triangle OAB < area of sector OAB < area of triangle OCB

$$\frac{1}{2}r^2 \sin \theta < \frac{1}{2}r^2 \theta < \frac{1}{2}r^2 \tan \theta$$

We can divide by $\frac{1}{2}r^2$ as it is always positive.

$$\sin \theta < \theta < \tan \theta$$

Because θ is a small positive angle, $\sin \theta$ is positive. We can therefore divide the inequality by $\sin \theta$.

$$\frac{\sin \theta}{\sin \theta} < \frac{\theta}{\sin \theta} < \frac{\tan \theta}{\sin \theta}$$

$$1 < \frac{\theta}{\sin \theta} < \frac{\sin \theta}{\cos \theta} \times \frac{1}{\sin \theta}$$

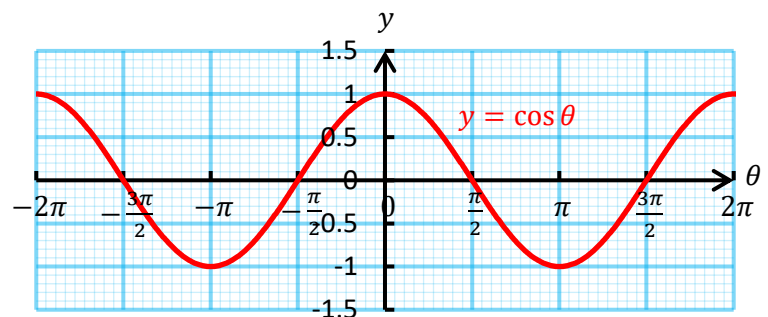
$$1 < \frac{\theta}{\sin \theta} < \sec \theta$$

As θ approaches 0, $\sec \theta$ approaches 1 (consider the graphs on the right).

Therefore, as θ approaches 0, $\frac{\theta}{\sin \theta}$ lies between 1 and a number approaching 1.

Therefore, as θ approaches 0, $\frac{\theta}{\sin \theta}$ approaches 1.

This means that $\sin \theta \approx \theta$ for small values of θ .



Tan Proof

The proof for tan is very similar to the proof for sin. We use exactly the same steps until we reach the inequality

$$\sin \theta < \theta < \tan \theta$$

Here, we divide by $\tan \theta$ (instead of by $\sin \theta$):

$$\frac{\sin \theta}{\tan \theta} < \frac{\theta}{\tan \theta} < \frac{\tan \theta}{\tan \theta}$$

$$\frac{\sin \theta}{\frac{\sin \theta}{\cos \theta}} < \frac{\theta}{\tan \theta} < 1$$

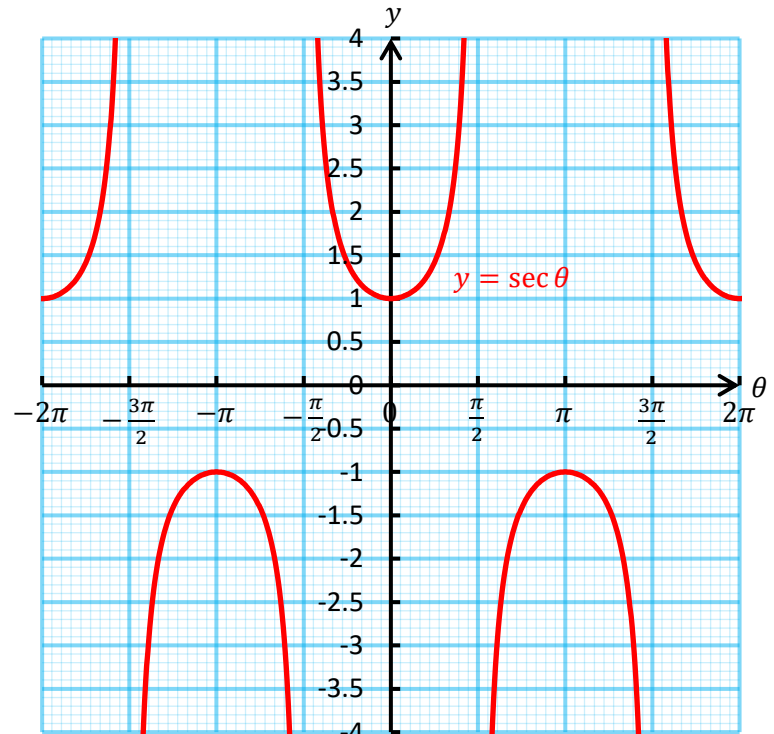
$$\cos \theta < \frac{\theta}{\tan \theta} < 1$$

As θ approaches 0, $\cos \theta$ approaches 1 (consider the graph on the right).

Therefore, as θ approaches 0, $\frac{\theta}{\tan \theta}$ lies between a number approaching 1 and 1.

Therefore, as θ approaches 0, $\frac{\theta}{\tan \theta}$ approaches 1.

This means that $\tan \theta \approx \theta$ for small values of θ .

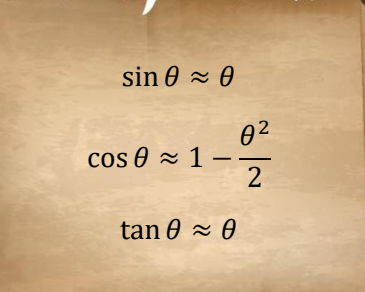


We use the double angle identity $\cos 2\theta \equiv 1 - 2\sin^2\theta$ to find the approximation for $\cos$.

$$\cos \theta \equiv 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$
$$\cos \theta \approx 1 - 2\left(\frac{\theta}{2}\right)^2$$

$$\cos \theta \approx 1 - \frac{\theta^2}{2}$$

If θ is a small angle, then



$$\sin \theta \approx \theta$$

$$\cos \theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan \theta \approx \theta$$

Exercises

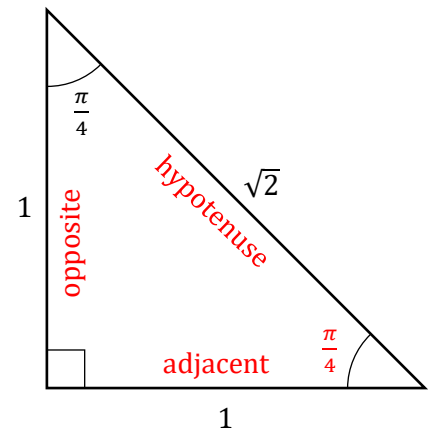
If θ is a small angle, find an approximation for the expression $\frac{\sin 3\theta}{1+\cos 2\theta}$.

Exact Values

Consider a right-angled triangle where the length of the shortest sides are 1 unit.

It is possible to use this triangle to calculate the exact values of sin, cos and tan for the angle $\frac{\pi}{4}$ (or 45°).

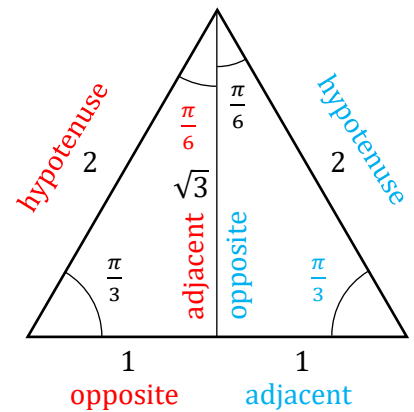
$$\begin{aligned}\sin\left(\frac{\pi}{4}\right) &= \frac{\text{opposite}}{\text{hypotenuse}} & \cos\left(\frac{\pi}{4}\right) &= \frac{\text{adjacent}}{\text{hypotenuse}} & \tan\left(\frac{\pi}{4}\right) &= \frac{\text{opposite}}{\text{adjacent}} \\ \sin\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}} & \cos\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}} & \tan\left(\frac{\pi}{4}\right) &= 1\end{aligned}$$



Next, consider an equilateral triangle where the length of the sides are 2 units.

By splitting this triangle in half, it is possible to calculate the exact values of sin, cos and tan for the angles $\frac{\pi}{6}$ (or 30°) and $\frac{\pi}{3}$ (or 60°).

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{\text{opposite}}{\text{hypotenuse}} & \cos\left(\frac{\pi}{6}\right) &= \frac{\text{adjacent}}{\text{hypotenuse}} & \tan\left(\frac{\pi}{6}\right) &= \frac{\text{opposite}}{\text{adjacent}} \\ \sin\left(\frac{\pi}{6}\right) &= \frac{1}{2} & \cos\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2} & \tan\left(\frac{\pi}{6}\right) &= \frac{1}{\sqrt{3}} \\ \sin\left(\frac{\pi}{3}\right) &= \frac{\text{opposite}}{\text{hypotenuse}} & \cos\left(\frac{\pi}{3}\right) &= \frac{\text{adjacent}}{\text{hypotenuse}} & \tan\left(\frac{\pi}{3}\right) &= \frac{\text{opposite}}{\text{adjacent}} \\ \sin\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} & \cos\left(\frac{\pi}{3}\right) &= \frac{1}{2} & \tan\left(\frac{\pi}{3}\right) &= \sqrt{3}\end{aligned}$$



Summary

	sin	cos	tan
0	0	1	0
$\frac{\pi}{6}$ (or 30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (or 45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (or 60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (or 90°)	1	0	Not defined

We can find the value of sin, cos and tan for other multiples of $\frac{\pi}{6}$ (or 30°) by using the **symmetry** in the graphs of sin, cos and tan.

If θ is a small angle, show that $\tan\left(\frac{\pi}{4} + \theta\right) \approx \frac{1+\theta}{1-\theta}$.

If θ is sufficiently small so that you can ignore θ^2 , show that $4 \sin\left(\frac{\pi}{4} - \theta\right) \approx 2\sqrt{2}(1 - \theta)$.

Given that $1^\circ \approx 0.017$ radian, find values for each of the following without using the trigonometric functions on your calculator.



(Sample Assessment Materials)

1. Find a small positive value of x which is an approximate solution of the equation.

$$\cos x - 4 \sin x = x^2. \quad [4]$$

A series of horizontal dotted lines for writing the solution.

(Unit 3 Summer 2018)

0	7
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Use small angle approximations to find the small negative root of the equation

$$\sin x + \cos x = 0.5.$$

[3]

Handwriting practice lines consisting of 25 horizontal dotted lines.

12. (a) Given that θ is small, show that $2\cos\theta + \sin\theta - 1 \approx 1 + \theta - \theta^2$. [2]

1. Given that θ is small and is measured in radians, use the small angle approximations to find an approximate value of

$$\frac{1 - \cos 4\theta}{2\theta \sin 3\theta} \quad (3)$$

4. In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Given that

- $f'(x) = 2x + \frac{1}{2} \cos x$
 - the curve has a stationary point with x coordinate α
 - α is small
- (a) use the small angle approximation for $\cos x$ to estimate the value of α to 3 decimal places.

(3)

Identities

0	9
---	---

a) Given that α and β are two angles such that $\tan \alpha = 2 \cot \beta$, show that

$$\tan(\alpha + \beta) = -(\tan\alpha + \tan\beta). \quad [2]$$

1	7
---	---

$$\cos(\alpha - \beta) + \sin(\alpha + \beta) \equiv (\cos \alpha + \sin \alpha)(\cos \beta + \sin \beta). \quad [2]$$

ii) Explain why $\frac{\cos 3\theta + \sin 5\theta}{\cos 4\theta + \sin 4\theta} \neq \cos \theta + \sin \theta$ when $\theta = \frac{3\pi}{16}$. [3]

0	6
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a) Using the trigonometric identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$, show that the **exact** value of $\cos 75^\circ$ is $\frac{\sqrt{6} - \sqrt{2}}{4}$. [3]

14. In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

$$2 \sin(x - 60^\circ) = \cos(x - 30^\circ)$$
$$\tan x = 3\sqrt{3} \quad (4)$$
$$2 \sin 2\theta = \cos (2\theta + 30^\circ)$$

giving your answers to one decimal place. (4)



Arithmetic

Series

Carl Friedrich Gauss was a famous German mathematician. In his primary school, he amazed his teacher by quickly adding the numbers between 1 and 100.



Name: _____

Background

What is the work?

Given an arithmetic series where we add the same number to obtain the next number, how can we find the n th term, or the sum of the first n terms?

What is required before starting?

GCSE Work: Solving equations, n th term.
A Level Unit 1: Solving quadratic equations.

Where does this lead to?

Applications:

- Finance: borrowing and investing money.
- Studying how populations change.
- Managing time in large projects.

Theory

A sequence of numbers is an **arithmetic sequence** if we add the same constant to obtain the next number in the sequence. For example, the sequence 5, 8, 11, 14, 17, ... is an arithmetic sequence where the first term is 5 and the common difference is 3.

Terminology

a	First term of the sequence
d	The common difference
l	Last term of the sequence, if one exists
t_n	The n th term of the sequence



Theory

For an arithmetic sequence with first term a and common difference d :

First term:	$t_1 = a$
Second term:	$t_2 = a + d$
Third term:	$t_3 = a + 2d$
Fourth term:	$t_4 = a + 3d$

Nth term: $t_n = a + (n - 1)d$

For all integers n ,

$$d = t_{n+1} - t_n$$

Exercise 1

For the arithmetic sequence 9, 14, 19, 24, 29, ..., write down

- | | |
|-----------------|-----------------|
| (a) a | (b) d |
| (c) t_7 | (d) t_n |

It is possible to consider the sum of the first n terms in an arithmetic sequence, written as S_n .

For the arithmetic sequence 5, 8, 11, 14, 17, ...

$$S_3 = 5 + 8 + 11 = 24$$

$$S_6 = 5 + 8 + 11 + 14 + 17 + 20 = 75$$

The proof for calculating S_n needs to be learnt.

1

Prove that the sum of the first n terms of an arithmetic sequence with first term a and common difference d is given by

$$S_n = \frac{1}{2}n(a + l).$$

Proof

Let us write the terms of the arithmetic series in order and then in reverse order.

$$\begin{array}{cccccccccccc} S_n & = & t_1 & + & t_2 & + & t_3 & + & \cdots & + & t_{n-1} & + & t_n \\ S_n & = & t_n & + & t_{n-1} & + & t_{n-2} & + & \cdots & + & t_2 & + & t_1 \end{array}$$

By adding the above we obtain

$$S_n + S_n = (t_1 + t_n) + (t_2 + t_{n-1}) + (t_3 + t_{n-2}) + \cdots + (t_{n-1} + t_2) + (t_n + t_1).$$

Each of the expressions in brackets is equivalent to $t_1 + t_n$. For example,

$$t_2 + t_{n-1} = (t_1 + d) + (t_n - d) = t_1 + t_n, \text{ and}$$

$$t_3 + t_{n-2} = (t_1 + 2d) + (t_n - 2d) = t_1 + t_n.$$

$$\text{Therefore } 2S_n = \underbrace{(t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + \cdots + (t_1 + t_n)}_{n \text{ times}}$$

$$2S_n = n(t_1 + t_n)$$

$$S_n = \frac{1}{2}n(t_1 + t_n).$$

But t_1 is the first term a of the arithmetic sequence and t_n is the last term l of the arithmetic sequence, so
 $S_n = \frac{1}{2}n(a + l).$

QED

2

Prove that the sum of the first n terms of an arithmetic sequence with first term a and common difference d is given by

$$S_n = \frac{1}{2}n(2a + (n - 1)d).$$

Proof

Let us write the terms of the arithmetic series in order and then in reverse order.

$$\begin{array}{cccccccccccc} S_n & = & t_1 & + & t_2 & + & t_3 & + & \cdots & + & t_{n-1} & + & t_n \\ S_n & = & t_n & + & t_{n-1} & + & t_{n-2} & + & \cdots & + & t_2 & + & t_1 \end{array}$$

By adding the above we obtain

$$S_n + S_n = (t_1 + t_n) + (t_2 + t_{n-1}) + (t_3 + t_{n-2}) + \cdots + (t_{n-1} + t_2) + (t_n + t_1).$$

Each of the expressions in brackets is equivalent to $t_1 + t_n$. For example,

$$t_2 + t_{n-1} = (t_1 + d) + (t_n - d) = t_1 + t_n, \text{ and}$$

$$t_3 + t_{n-2} = (t_1 + 2d) + (t_n - 2d) = t_1 + t_n.$$

$$\text{Therefore } 2S_n = \underbrace{(t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + \cdots + (t_1 + t_n)}_{n \text{ times}}$$

$$2S_n = n(t_1 + t_n)$$

$$S_n = \frac{1}{2}n(t_1 + t_n).$$

But the n th term of an arithmetic sequence is $t_n = a + (n - 1)d$, so

$$S_n = \frac{1}{2}n(a + (a + (n - 1)d))$$

$$S_n = \frac{1}{2}n(2a + (n - 1)d).$$

QED

Exercises

(C2 Winter 2005)

5. In an arithmetic series the sum of the first term and the fifth term is zero. The thirteenth term is 20.
- (a) Find the first term and the common difference of the series. [5]
- (b) Calculate the sum of the first twenty terms of the series. [2]

0	6
---	---

Geraint opens a savings account. He deposits £10 in the first month. In each subsequent month, the amount he deposits is 20 pence greater than the amount he deposited in the previous month.

- a) Find the amount that Geraint deposits into the savings account in the 12th month. [2]
- b) Determine the number of months it takes for the total amount in the savings account to reach £954. [3]



5. The sum of the first two terms of an arithmetic series is 3. The eighth term of the arithmetic series is 47.

(a) the first term and the common difference of the series, [4]

(b) the sum of the first twenty terms of the series. [2]

(C2 Winter 2009)

4. (a) The thirteenth term of an arithmetic series is 51. The ninth term of the series is five times the second term. Find the first term and common difference of the arithmetic series. [5]
- (b) The first term of another arithmetic series is 5 and the twentieth term is 62. Find the sum of the first twenty terms of this arithmetic series. [2]

4. The fifteenth term of an arithmetic series is seven times the fifth term. The sum of the first eleven terms of the series is 88.

- (a) Find the first term and common difference of the arithmetic series. [6]
- (b) Given that the n th term of the series is 143, find the value of n . [2]

4. (a) An arithmetic series has first term a and common difference d . Prove that the sum of the first n terms is given by

(b) The sum of the first twenty terms of an arithmetic series is 540 and the sum of the first thirty terms of the series is 1260.

- (ii) Calculate the **50th term** of the series. [6]

3. (a) An arithmetic series has first term a and common difference d . Prove that the sum of the first n terms of the series is given by

(b) Find an expression, in terms of n , for the sum of the first n terms of the arithmetic series

Simplify your answer. [2]

Handwriting practice lines consisting of 30 sets of three horizontal dotted lines.

5. (a) An arithmetic series has first term a and common difference d . Prove that the sum of the first n terms of the series is given by

(b) The first term of an arithmetic series is 4 and the common difference is 2. The sum of the first n terms of the arithmetic series is 460.

(c) The fifth term of another arithmetic series is 9. The sum of the sixth term and the tenth term of this series is 42. Find the first term and the common difference of the arithmetic series.

Handwriting practice area with horizontal dotted lines.

(C2 Summer 2011)

4. (a) The sum of the first fifteen terms of an arithmetic series is 780. The sum of the second, fourth and tenth terms of the series is 100. Find the first term and the common difference of the series. [5]
- (b) The p th term of another arithmetic series is 1023. The $(p + 4)$ th term of this series is 1059. Find the $(p + 7)$ th term of the series. [3]

4. (a) The first term of an arithmetic series is 1 and the common difference is 4.

- $$S_n = 1 + 5 + \dots + (4n - 7) + (4n - 3).$$

$$S_n = n(2n - 1). \quad [4]$$

Handwriting practice area with 30 horizontal dotted lines.

(C2 Summer 2016)

4. (a) Gwilym has decided to run in a half marathon race. In order to get himself fit, he devises a training programme whereby he runs around his local track each day, gradually increasing the distance he runs from day to day. On the first day, he runs 6 laps of the track and subsequently, on any given day, he runs 2 laps further than he did on the day before.

- (i) How many laps does he run on the 20th day of his programme?
- (ii) After how many days will the total number of laps he has run since the beginning of his training programme be equal to 750? [6]

- (b) The n th term of an **arithmetic** series is denoted by t_n . It is known that

$$t_{12} + t_{13} = 50.$$

- (i) **Write down** the value of $t_{11} + t_{14}$. [1]
- (ii) Find the sum of the first twenty-four terms of this arithmetic series. [2]

Lined area for student work.

7. Showing all your working, evaluate

[4]

Revision Questions

(Unit 3 Summer 2018)

8

[5]

(C2 Summer 2018)

4. Dafydd enters into an agreement with a finance company to provide him with a loan to buy a car. Under the terms of the agreement, he is to repay a total of £3900 over a period of two years by making 24 monthly repayments. The first repayment is £ P , and in each subsequent month, the value of the repayment for that month is £ x **less** than the repayment for the previous month. Given that Dafydd's eighth repayment is £185, find the value of P and the value of x . [5]



Geometric

Series

An infinite number of mathematicians walk into a bar. The first one orders a drink, the second one orders half a drink, the third one wants a quarter of a drink, and the next one wants just $\frac{1}{8}$ th. The bartender sees where this is going, and stops them before anyone else can order. The bartender pours two drinks, hands them over, and says "you mathematicians should really know your limits".



Name: _____

Background

What is the work?

Given a geometric series where we multiply the same number to obtain the next number, how can we find the n th term, or the sum of the first n terms?

What is required before starting?

GCSE Work: Solving equations, n th term.
A Level Unit 1: Solving quadratic equations, logarithms.

Where does this lead to?

Applications:

- The expansion rate of the universe.
- Radioactive decay.
- Compound interest.

Theory

A sequence of numbers is a **geometric sequence** if we multiply by the same constant to obtain the next number in the sequence. For example, the sequence 3, 6, 12, 24, 48, ... is a geometric sequence where the first term is 3 and the common ratio is 2.

Terminology

a	First term of the sequence
r	The common ratio
l	Last term of the sequence, if one exists
t_n	The n th term of the sequence



Theory

For a geometric series with first term a and common ratio r :

First term:	$t_1 = a$
Second term:	$t_2 = ar$
Third term:	$t_3 = ar^2$
Fourth term:	$t_4 = ar^3$

Nth term: $t_n = ar^{n-1}$

For all integers n ,

$$r = \frac{t_{n+1}}{t_n}$$

Exercise 1

For the geometric series 2, 6, 18, 54, 162, ..., write down

- | | |
|-----------------|-----------------|
| (a) a | (b) r |
| (c) t_7 | (d) t_n |

It is possible to consider the sum of the first n terms in a geometric series, written as S_n .

For the geometric series 3, 6, 12, 24, 48, ...

$$S_3 = 3 + 6 + 12 = 21$$

$$S_6 = 3 + 6 + 12 + 24 + 48 + 96 = 189$$

The proof for calculating S_n needs to be learnt.

Prove that the sum of the first n terms in a geometric series with first term a and common ratio r is given by

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Proof

1st Term: $t_1 = a$

2nd Term: $t_2 = ar$

Nth Term: $t_n = ar^{n-1}$

Sum of the first n terms:

$$\begin{aligned} S_n &= t_1 + t_2 + \dots + t_{n-1} + t_n \\ S_n &= a + ar + \dots + ar^{n-2} + ar^{n-1} \end{aligned} \quad \text{--- ①}$$

Multiply by r :

$$rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n \quad \text{--- ②}$$

Subtracting ① - ②:

$$\begin{aligned} S_n - rS_n &= a - ar^n \\ S_n(1 - r) &= a(1 - r^n) \\ S_n &= \frac{a(1 - r^n)}{1 - r} \end{aligned}$$

QED

Sum to Infinity

If $|r| < 1$, then we can consider the sum S_∞ of all the terms in a geometric series.

Proof

If $|r| < 1$ (so that $-1 < r < 1$), then r^n will become closer and closer to 0 as n increases.

We say that $r^n \rightarrow 0$ as $n \rightarrow \infty$.

So, if $|r| < 1$,

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_n \rightarrow \frac{a(1 - 0)}{1 - r}$$

$$S_\infty = \frac{a}{1 - r}.$$

r^n tends to 0 as n tends to ∞ .

Exercise 2

Read the joke on the cover of the workbook.

The joke describes a geometric series where $a = 1$ and $r = \frac{1}{2}$. Write down

(a) t_6

(b) S_6

(c) S_∞



(C2 Summer 2005)

4. The sum of the first two terms of a geometric series is 6.4 , and the sum to infinity of the series is 10 .
- (a) Given that the common ratio is positive, find its value. [5]
- (b) Find, correct to three decimal places, the sum of the first eleven terms of the series. [3]

A series of horizontal dotted lines for writing the solution to the exercises.

(C2 Winter 2011)

5. (a) The second term of a geometric series is 6 and the fifth term is 384.
- (i) Find the common ratio of the series.
 - (ii) Find the sum of the first eight terms of the geometric series. [6]
- (b) The first term of another geometric series is 5 and the common ratio is $1\cdot1$.
- (i) The n th term of this series is 170, correct to the nearest integer. Find the value of n .
 - (ii) Dafydd, who has been using his calculator to investigate various properties of this geometric series, claims that the sum to infinity of the series is 940. Explain why this result cannot possibly be correct. [5]

Handwriting practice lines consisting of 30 horizontal dotted lines.



5. The second term of a geometric series is nine times the fourth term of the series.

4. A geometric series has first term a and common ratio r . The fifth term of the geometric series is 135 and the eighth term is 5.

(C2 Winter 2007)

3. (a) A geometric series has first term a and common ratio r . Write down the n th term and prove that the sum of the first n terms is given by

$$S_n = \frac{a(1-r^n)}{1-r}.$$

Given that $|r| < 1$, write down the sum to infinity of the series. [5]

- (b) The sum of the first term and the second term of a geometric series is equal to twice the sum of the second term and the third term of the series.

- (i) Given that the common ratio of the series is positive, find the value of the common ratio. [4]
(ii) The sum to infinity of the series is 12. Find, correct to two decimal places, the sum of the first eight terms of the series. [4]

5. A rich businessman makes one donation per year to a certain charity. He starts by donating £100 in the first year. In each subsequent year, the value of the donation is 1.2 times the value of the previous year's donation.

- (a) Find the value of the businessman's donation in the 12th year. Give your answer correct to the nearest pound. [2]
- (b) After receiving the n th donation, the charity's treasurer calculates that over the years, the businessman has donated a **total** of £15 474, correct to the nearest pound. Find the value of n . [5]

(C2 Winter 2013)

5. (a) The p th term of a geometric series is 16. The $(p + 1)$ th term of this series is 24. Find the $(p + 4)$ th term of the series. [3]
- (b) The sum of the first three terms of another geometric series is $22\cdot8$. The sum to infinity of the series is $18\cdot75$. Find the common ratio and the first term of this geometric series. [6]

4. (a) A geometric series has first term a and common ratio r . Write down the n th term of the series and prove that the sum of the first n terms is given by

(b) The fourth term of a geometric series is 2 and the seventh term is 54.

- (i) Find the common ratio of the series.
- (ii) Find the sum of the first ten terms of the series, giving your answer correct to one decimal place.
- (iii) Find the least value of n for the n th term to exceed 125 000. [10]

Handwriting practice area consisting of 30 horizontal dotted lines.

5. (a) A geometric series has first term a and common ratio r . Prove that the sum of the first n terms is given by

(b) The common ratio of a geometric series is positive. The sum of the first four terms of the series is $73\cdot8$. The sum to infinity of the series is 125. Find the common ratio and the first term of the geometric series. [6]

Handwriting practice lines consisting of 40 horizontal dotted lines.

6. (a) Find the sum to infinity of the geometric series

[3]

- (i) Show that r satisfies the equation

(ii) Given that $|r| < 1$, find the value of r and the corresponding value of a . [6]

Handwriting practice lines consisting of 30 sets of three horizontal dotted lines.

5

a) Find the height of the tree 10 years after it was planted. [4]

b) Determine the maximum height of the tree. [2]

7. Showing all your working, evaluate

$$(b) \sum_{r=2}^{\infty} \left(540 \times \left(\frac{1}{3} \right)^r \right).$$

[3]

Revision Questions

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- a) The 3rd, 19th and 67th terms of an arithmetic sequence form a geometric sequence. Given that the arithmetic sequence is increasing and that the first term is 3, find the common difference of the arithmetic sequence. [5]

(Unit 3 Summer 2018)

0	9
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- a) Explain why the sum to infinity of a geometric series with common ratio r only converges when $|r| < 1$. [1]
- b) A geometric progression V has first term 2 and common ratio r . Another progression W is formed by squaring each term in V . Show that W is also a geometric progression. Given that the sum to infinity of W is 3 times the sum to infinity of V , find the value of r . [6]
- c) At the beginning of each year, a man invests £5000 in a savings account earning compound interest at the rate of 3% per annum. The interest is added at the end of each year. Find the total amount of his savings at the end of the 20th year correct to the nearest pound. [3]



Types of

Series

The website <https://oeis.org/> recognises different sequences, and provides information about them.

0 1 3 6 2 7
: 13
23 : 20
10 22 11 21

THE ON-LINE ENCYCLOPEDIA
OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

The On-Line Encyclopedia of Integer Sequences (OEIS)

Enter a sequence, word, or sequence number:

1,2,3,6,11,23,47,106,235

Search Hints Welcome Video

For more information about the Encyclopedia, see the [Welcome](#) page.

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Name:



Background

What is the work?

Learning how to recognise arithmetic sequences; geometric sequences; increasing sequences; decreasing sequences; periodic sequences; and sequences of the form $x_{n+1} = f(x_n)$.

What is required before starting?

GCSE Work: Solving equations, n th term.
A Level Unit 1: Solving quadratic equations.
A Level Unit 3: Arithmetic sequences; geometric sequences.

Where does this lead to?

Applications:

- Periodic sequences can model circular situations, e.g. the motion of planets; radio waves; tides.



Theory

A sequence of numbers is an **arithmetic sequence** if we add the same constant to obtain the next number in the sequence.

$$t_n = a + (n - 1)d$$

Example 1 $t_n = 4 + 3(n - 1)$.
 The sequence is 4, 7, 10, 13, 16, ...

A sequence of numbers is a **geometric sequence** if we multiply by the same constant to obtain the next number in the sequence.

$$t_n = ar^{n-1}$$

Example 2 $t_n = 5 \times 2^{n-1}$.
 The sequence is 5, 10, 20, 40, 80, ...

A sequence of numbers is a **periodic sequence** if the sequence repeats itself after x terms. x is the **period** of the sequence.

Example 3 $t_n = (-1)^n$.
 The sequence is $-1, 1, -1, 1, -1, \dots$
 This is a periodic sequence with period 2.

A sequence of numbers is an **increasing sequence** if $t_{n+1} > t_n$ for all $n \geq 1$.
 The sequences in examples 1 and 2 are increasing sequences.

A sequence of numbers is a **decreasing sequence** if $t_{n+1} < t_n$ for all $n \geq 1$.

Note: a periodic sequence is neither increasing nor decreasing.

Are the following sequences increasing, decreasing or periodic?

	Nth Term	1st Term	2nd Term	3rd Term	4th Term	5th Term	Increasing, Decreasing or Periodic?
(a)	$t_n = 4n + 3$						
(b)	$t_n = \frac{5}{n+1}$						
(c)	$t_n = n^2 + 2n + 5$						
(d)	$t_n = \sin(30n)^\circ$						
(e)	$t_n = 3 - 2n$						
(f)	$t_n = \frac{n}{n+1}$						

For the increasing sequences in Exercise 1, prove that they are increasing, by showing that $t_{n+1} - t_n$ is always positive.

For the decreasing sequences in Exercise 1, prove that they are decreasing, by showing that $t_{n+1} - t_n$ is always negative.

For the periodic sequences in Exercise 1, find the period of the sequences.

Handwriting practice area consisting of 30 sets of horizontal dotted lines.

Top 5

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$$\mathbf{a)} \quad u_n = \sin\left(\frac{n\pi}{2}\right) \quad [2]$$

$$\textbf{b)} \quad u_6 = 33, \quad u_n = 2u_{n-1} - 1 \quad [3]$$

0	3
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a) Find the values of x_2 and x_1 . [2]

(C2 Winter 2010)

10. The n th term of a number sequence is denoted by t_n . The $(n + 1)$ th term of the sequence satisfies

$$t_{n+1} = 2t_n + 1,$$

for all positive integers n . Given that $t_4 = 63$,

(a) evaluate t_1 , [2]

(b) without carrying out any further calculations, explain why 6043582 cannot be one of the terms of this number sequence. [1]

6. The lengths of the sides of a fifteen-sided plane figure form an arithmetic sequence. The perimeter of the figure is 270 cm and the length of the largest side is eight times that of the smallest side. Find the length of the smallest side. [4]

15. Robert wants to deposit £ P into a savings account. He has a choice of two accounts.

- Account *A* offers an annual compound interest rate of 1%.
- Account *B* offers an interest rate of 5% for the first year and an annual compound interest rate of 0.6% for each subsequent year.

After n years, account A is more profitable than account B . Find the smallest value of n . [5]



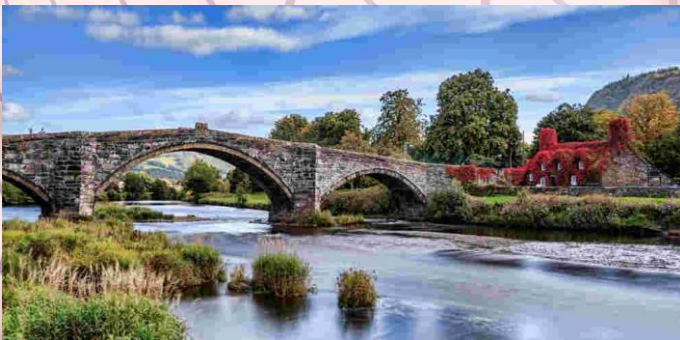
Binomial

Expansion 2

1000 × 494 pixels. JPEG 100%. 440 Kb



1000 × 494 pixels. JPEG 10%. 31 Kb



1000 × 494 pixels. JPEG 1%. 16 Kb



Name:

Background

What is the work?

Considering how to develop the Unit 1 work on expanding $(a + bx)^n$ to include cases where n is a negative number or a fraction.

What is required before starting?

GCSE Work: Expanding brackets; simplifying; substitution; rules of indices; solving inequalities.
A Level Unit 1: Binomial expansion.

Where does this lead to?

We can use this work to estimate the value of algebraic expressions or irrational numbers.

Theory

Unit 1 Revision

If n is a positive integer, then

$$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + x^n$$

$$(a + x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \dots + x^n$$

$$(a + bx)^n = a^n + \binom{n}{1}a^{n-1}(bx) + \binom{n}{2}a^{n-2}(bx)^2 + \binom{n}{3}a^{n-3}(bx)^3 + \dots + (bx)^n$$



Theory

where $\binom{n}{r}$ represents how many ways r items can be chosen from n items, and can be calculated from Pascal's triangle or the **nCr** button on a calculator.

Example 1

Expand $(1 + 3x)^4$.

Pascal's Triangle:



$$(1 + 3x)^4 = 1(1)^4(3x)^0 + 4(1)^3(3x)^1 + 6(1)^2(3x)^2 + 4(1)^1(3x)^3 + 1(1)^0(3x)^4$$

$$(1 + 3x)^4 = 1(1)^4 + 4(1)^3(3x)^1 + 6(1)^2(3x)^2 + 4(1)^1(3x)^3 + 1(3x)^4$$

$$(1 + 3x)^4 = 1 + 4(3x) + 6(9x^2) + 4(27x^3) + 1(81x^4)$$

$$(1 + 3x)^4 = 1 + 12x + 54x^2 + 108x^3 + 81x^4$$

If n is **not** a positive integer, then

- We cannot find $(a + bx)^n$ by multiplying out $(a + bx)$ n times;
- It is not possible to use Pascal's triangle or the **nCr** button on a calculator to calculate the coefficients.

We must instead use an infinite series and the general definition of $\binom{n}{r}$.

In general,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where $n! = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1$.

For example,

$$\begin{aligned}\binom{n}{4} &= \frac{n!}{4!(n-4)!} \\ \binom{n}{4} &= \frac{n \times (n-1) \times (n-2) \times (n-3) \times \cancel{(n-4)} \times \cancel{(n-5)} \times \dots \times \cancel{3} \times \cancel{2} \times \cancel{1}}{[4 \times 3 \times 2 \times 1] \times [\cancel{(n-4)} \times \cancel{(n-5)} \times \dots \times \cancel{3} \times \cancel{2} \times \cancel{1}]} \\ \binom{n}{4} &= \frac{n(n-1)(n-2)(n-3)}{4 \times 3 \times 2 \times 1}\end{aligned}$$

So, for a positive integer n ,

$$(a+bx)^n = a^n + na^{n-1}(bx) + \frac{n(n-1)}{2 \times 1}a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3 \times 2 \times 1}a^{n-3}(bx)^3 + \dots + (bx)^n$$

If n is **not** a positive integer, then we cannot say how many terms are in the series. It is however possible to write the expansion as an infinite series:

$$(a+bx)^n = a^n + na^{n-1}(bx) + \frac{n(n-1)}{2 \times 1}a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3 \times 2 \times 1}a^{n-3}(bx)^3 + \dots$$

The expansion is only sensible if the terms get progressively smaller (closer to zero). This happens when

$$\left| \frac{bx}{a} \right| < 1$$

and this is when the expansion is **valid**.

Example 2

Expand $(3+2x)^{-\frac{1}{2}}$ up to and including the term in x^2 . Note when your expansion is valid.

$$\begin{aligned}(3+2x)^{-\frac{1}{2}} &= 3^{-\frac{1}{2}} + \left(-\frac{1}{2}\right) \times 3^{-\frac{1}{2}-1} \times (2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2 \times 1} \times 3^{-\frac{1}{2}-2} \times (2x)^2 + \dots \\ &= 3^{-\frac{1}{2}} + \left(-\frac{1}{2}\right) \times 3^{-\frac{3}{2}} \times (2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2 \times 1} \times 3^{-\frac{5}{2}} \times (2x)^2 + \dots \\ &= 3^{-\frac{1}{2}} - 3^{-\frac{3}{2}}x + \frac{3}{8} \times 3^{-\frac{5}{2}} \times 4x^2 + \dots \\ &= 3^{-\frac{1}{2}} - 3^{-\frac{3}{2}}x + 3^{-\frac{5}{2}} \times \frac{3}{2}x^2 + \dots \\ &= \frac{1}{3^{\frac{1}{2}}} - \frac{1}{3^{\frac{3}{2}}}x + \frac{1}{3^{\frac{5}{2}}} \times \frac{3}{2}x^2 + \dots \\ &= \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}x + \frac{1}{9\sqrt{3}} \times \frac{3}{2}x^2 + \dots \\ &= \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}x + \frac{1}{6\sqrt{3}}x^2 + \dots \\ &= \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{9}x + \frac{\sqrt{3}}{18}x^2 + \dots \\ (3+2x)^{-\frac{1}{2}} &\approx \frac{\sqrt{3}}{3} \left(1 - \frac{x}{3} + \frac{x^2}{6}\right) \quad \text{(up to and including the term in } x^2\text{)}\end{aligned}$$

The expansion is valid if $\left| \frac{2x}{3} \right| < 1$, so that $|x| < \frac{3}{2}$.



(C4 Summer 2016)

2. (a) (i) Expand $\frac{1}{\sqrt{1+2x}}$ in ascending powers of x up to and including the term in x^2 .

(ii) State the range of values of x for which your expansion is valid. [3]

(b) Use your expansion in part (a) to find an approximate value for one root of the equation

$$\frac{6}{\sqrt{1+2x}} = 4 + 15x - x^2. \quad [2]$$

Handwriting practice lines (dotted lines) for the student's solution.

4. Expand $(1+4x)^{\frac{1}{2}} - \frac{1}{1+3x}$ as far as the term in x^2 . For what range of values of x is your expansion valid? [7]

(C4 Summer 2010)

5. Expand $\left(1 - \frac{x}{4}\right)^{\frac{1}{2}}$ in ascending powers of x up to and including the term in x^2 .

State the range of values of x for which your expansion is valid.

Hence, by writing $x = 1$ in your expansion, show that

$$\sqrt{3} \approx \frac{111}{64} . \quad [5]$$

Top 10

5. (a) (i) Expand $(1 + 6x)^{\frac{1}{3}}$ in ascending powers of x up to and including the term in x^2 .
(ii) State the range of values of x for which your expansion is valid. [3]

(b) Use your expansion in part (a) to find an approximate value for one root of the equation

$$2(1 + 6x)^{\frac{1}{3}} = 2x^2 - 15x. \quad [2]$$

(C4 Summer 2011)

6. Expand $4(1+2x)^{\frac{1}{2}} - \frac{1}{(1+3x)^2}$ in ascending powers of x up to and including the term in x^2 .

State the range of values of x for which your expansion is valid.

[7]

(C4 Summer 2014)

5. Expand

$$6\sqrt{1-2x} - \frac{1}{1+4x}$$

in ascending powers of x up to and including the term in x^2 .
State the range of values of x for which your expansion is valid.

[7]

9. Expand $\frac{1+3x}{\sqrt{1-2x}}$ in ascending powers of x up to and including the term in x^2 . State the range of x for which the expansion is valid. [5]

(C4 Summer 2005)

2. Expand $(1 - 2x)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^2 . State the range of values of x for which the expansion is valid.

Hence, by writing $x = \frac{1}{8}$ in your expansion, find an approximate value for $\sqrt{3}$ in the form $\frac{a}{b}$, where a and b are integers. [5]

(C4 Summer 2006)

10. Expand $\left(1 + \frac{x}{8}\right)^{\frac{1}{2}}$ in ascending powers of x up to and including the term in x^2 . State the range of x for which the expansion is valid. Hence by writing $x = 1$ in your expansion, show that $\sqrt{2} \approx \frac{256}{181}$. [5]

(C4 Summer 2009)

9. Expand $(1+4x)^{\frac{1}{2}}$ in ascending powers of x as far as the term in x^2 . State the range of values of x for which your expansion is valid.

Expand $(1+4k+16k^2)^{\frac{1}{2}}$ in ascending powers of k as far as the term in k^2 . [6]

(C4 Summer 2017)

5. (a) Expand $(1 + 4x)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^2 . State the range of values of x for which your expansion is valid. [3]
- (b) Use your answer to part (a) to expand $(1 + 4y + 8y^2)^{-\frac{1}{2}}$ in ascending powers of y up to and including the term in y^2 . [3]

0	2
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Expand $\frac{4-x}{\sqrt{1+2x}}$ in ascending powers of x up to and including the term in x^3 . State the range of values of x for which the expansion is valid. [6]

12. (a) Given that θ is small, show that $2\cos\theta + \sin\theta - 1 \approx 1 + \theta - \theta^2$. [2]

$$\frac{1}{2\cos\theta + \sin\theta - 1} \approx 1 + a\theta + b\theta^2,$$

where a, b are constants to be found. [4]



(C4 Summer 2012)

5. Expand $\left(1 + \frac{x}{3}\right)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^2 .

State the range of values of x for which your expansion is valid.

Hence, by writing $x = \frac{1}{5}$ in your expansion, find an approximate value for $\sqrt{15}$ in the form $\frac{a}{b}$,
where a and b are integers whose values are to be found. [5]

4. (a) Expand $\frac{1}{(1+2x)^2}$ in ascending powers of x up to and including the term in x^2 . [2]

(ii) State the range of values of x for which your expansion is valid. [4]

0	3
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 a) Express $\frac{9}{(1-x)(1+2x)^2}$ in terms of partial fractions. [4]

b) Using your answer from part (a), find the expansion of $\frac{9}{(1-x)(1+2x)^2}$ in ascending powers of x as far as the term in x^2 . State the values of x for which the expansion is valid.

[7]



Further

Differentiation

y	$\frac{dy}{dx}$
$(f(x))^n$	$n(f(x))^{n-1}f'(x)$
uv	$u \frac{dv}{dx} + v \frac{du}{dx}$
$\frac{u}{v}$	$\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$
$\sin(f(x))$	$f'(x) \cos(f(x))$
$\cos(f(x))$	$-f'(x) \sin(f(x))$
$\tan(f(x))$	$f'(x) \sec^2(f(x))$
$\sec(f(x))$	$f'(x) \sec(f(x)) \tan(f(x))$
$\operatorname{cosec}(f(x))$	$-f'(x) \operatorname{cosec}(f(x)) \cot(f(x))$
$\cot(f(x))$	$-f'(x) \operatorname{cosec}^2(f(x))$
$\sin^{-1}(f(x))$	$\frac{f'(x)}{\sqrt{1 - (f(x))^2}}$
$\cos^{-1}(f(x))$	$-\frac{f'(x)}{\sqrt{1 - (f(x))^2}}$
$\tan^{-1}(f(x))$	$\frac{f'(x)}{1 + (f(x))^2}$
$e^{f(x)}$	$f'(x)e^{f(x)}$
$a^{f(x)}$	$f'(x)a^{f(x)} \ln(a)$
$\ln(f(x))$	$\frac{f'(x)}{f(x)}$

Name:

Background

What is the work?

Consider how to develop the Unit 1 work on differentiation to include trigonometric, exponential and logarithmic functions. Understand how to differentiate combinations of functions using the product rule, the quotient rule and the chain rule.

What is required before starting?

GCSE Work: Substitution, solving equations, changing the subject.
A Level Unit 1: Differentiation.

Where does this lead to?

Finding the rate of change in physics, engineering and chemical problems (and more!).

Theory

Unit 1 Revision

If $y = ax^n$ then $\frac{dy}{dx} = nax^{n-1}$. (The notation $\frac{dy}{dx}$ or y' is used to represent the gradient.)

Example 1

y	$\frac{dy}{dx}$
$5x^2$	$10x$
$8x^5 + 3x^2 + 8x + 4$	$40x^4 + 6x + 8$
$-5x^3 + 2x^{-6}$	$-15x^2 - 12x^{-7}$

Exercise 1

Complete the following table.

y	$\frac{dy}{dx}$
$4x^7$	
$6x^3 - 12x + 5$	
$18x^3 + 12x^{-4} - \frac{6}{x}$	

Exercise 2

If $y = 4x^3 + 5$, find the value of y' if $x = 2$.

.....

.....

.....

.....

.....

.....

Trigonometric Functions

Proof

If $y = \sin x$, find $\frac{dy}{dx}$ from first principles.



From Unit 1, to differentiate from first principles we use $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

$$\begin{aligned} \text{So } \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \end{aligned}$$

using $\sin(A+B) = \sin A \cos B + \cos A \sin B$

As $h \rightarrow 0$, $\sin(h) \rightarrow h$ and $\cos(h) \rightarrow 1 - \frac{h^2}{2}$.

$$\begin{aligned} \text{So } \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\sin(x)\left(1 - \frac{h^2}{2}\right) + \cos(x)h - \sin(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\sin(x) - \sin(x)\left(\frac{h^2}{2}\right) + \cos(x)h - \sin(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{-\sin(x)\left(\frac{h^2}{2}\right) + \cos(x)h}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} -\sin(x)\left(\frac{h}{2}\right) + \cos(x) \\ \frac{dy}{dx} &= -\sin(x)\left(\frac{0}{2}\right) + \cos(x) \\ \frac{dy}{dx} &= \cos(x) \end{aligned}$$

The gradient of sin is cos

Proof

If $y = \cos x$, find $\frac{dy}{dx}$ from first principles.

From Unit 1, to differentiate from first principles we use $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

$$\begin{aligned} \text{So } \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \end{aligned}$$

using $\cos(A+B) = \cos A \cos B - \sin A \sin B$

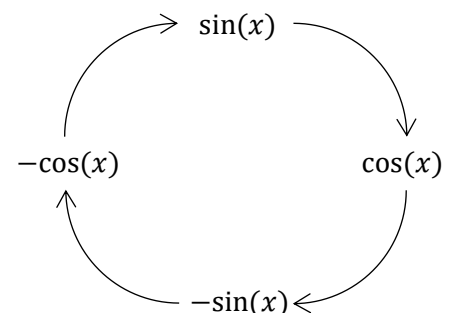
As $h \rightarrow 0$, $\sin(h) \rightarrow h$ and $\cos(h) \rightarrow 1 - \frac{h^2}{2}$.

$$\begin{aligned} \text{So } \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\cos(x)\left(1 - \frac{h^2}{2}\right) - \sin(x)h - \cos(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\cos(x) - \cos(x)\left(\frac{h^2}{2}\right) - \sin(x)h - \cos(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{-\cos(x)\left(\frac{h^2}{2}\right) - \sin(x)h}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} -\cos(x)\left(\frac{h}{2}\right) - \sin(x) \\ \frac{dy}{dx} &= -\cos(x)\left(\frac{0}{2}\right) - \sin(x) \\ \frac{dy}{dx} &= -\sin(x) \end{aligned}$$

The gradient of cos is -sin

Note

The diagram on the right may help you to remember how to differentiate the sin and cos functions.



The Chain Rule

Given two functions $y = f(t)$ and $t = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

We can use the chain rule to differentiate more complex functions than the ones seen in Unit 1.

Example 2

If $y = (2x + 4)^3$, use the chain rule to find $\frac{dy}{dx}$.

Let $t = 2x + 4$, so that $y = t^3$.

Then $\frac{dt}{dx} = 2$ and $\frac{dy}{dt} = 3t^2$.

$$\begin{aligned}\text{So } \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ \frac{dy}{dx} &= 3t^2 \times 2 \\ \frac{dy}{dx} &= 6t^2 \\ \frac{dy}{dx} &= 6(2x + 4)^2\end{aligned}$$

Example 3

If $y = \sin(3x)$, use the chain rule to find $\frac{dy}{dx}$.

Let $t = 3x$, so that $y = \sin(t)$.

Then $\frac{dt}{dx} = 3$ and $\frac{dy}{dt} = \cos(t)$.

$$\begin{aligned}\text{So } \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ \frac{dy}{dx} &= \cos(t) \times 3 \\ \frac{dy}{dx} &= 3 \cos(t)\end{aligned}$$

Exercise 3

Use the chain rule to find $\frac{dy}{dx}$ in each of the following cases.

(a) $y = (3x + 9)^5$

(b) $y = \sin(5x)$

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(c) $y = \sqrt{4x - 3}$

(d) $y = 4 \cos(9x)$

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If a car travels twice as fast as a bike and the bike is four times as fast as a person's walking speed, then the car travels $2 \times 4 = 8$ times as fast as the person.



Exercise 4

Use the chain rule to find $\frac{dy}{dx}$ in each of the following cases.

(a) $y = (f(x))^n$

(b) $y = \sin(f(x))$

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The Product Rule (Leibniz's Rule)

If $y = uv$, then

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

**Proof (no need to learn)**

Let $y = u(x)v(x)$. By defining $f(x) = u(x)v(x)$, we have

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x)v(x+h) + u(x)v(x+h) - u(x)v(x)}{h} && \text{(Note that } -u(x)v(x+h) + u(x)v(x+h) = 0 \text{)} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{v(x+h)[u(x+h) - u(x)] + u(x)[v(x+h) - v(x)]}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{u(x)[v(x+h) - v(x)] + v(x+h)[u(x+h) - u(x)]}{h} && \text{(Re-arranging the numerator)} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} u(x) \times \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} + \lim_{h \rightarrow 0} v(x+h) \times \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} \\ \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} \end{aligned}$$

QED**Example 4**

If $y = 4x^3 \sin(x)$, find $\frac{dy}{dx}$.

Let $u = 4x^3$, $v = \sin(x)$.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} \\ \frac{dy}{dx} &= 4x^3 \cos(x) + \sin(x) \times 12x^2 \\ \frac{dy}{dx} &= 4x^2(x \cos(x) + 3 \sin(x)) \end{aligned}$$

Exercise 5

(a) If $y = 5x^2 \cos(x)$, find $\frac{dy}{dx}$.

(b) If $y = 9\sqrt{x} \sin(x)$, find $\frac{dy}{dx}$.

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The Quotient RuleIf $y = \frac{u}{v}$, then

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

**Proof (no need to learn)**Let $y = \frac{u(x)}{v(x)}$. By defining $f(x) = \frac{u(x)}{v(x)}$, we have

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\frac{u(x+h)v(x) - u(x)v(x+h)}{v(x+h)v(x)}}{h}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{u(x+h)v(x) - u(x)v(x+h)}{v(x+h)v(x)h}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{u(x+h)v(x) - u(x)v(x+h)}{h} \times \lim_{h \rightarrow 0} \frac{1}{v(x+h)v(x)}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{u(x+h)v(x) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{h} \times \lim_{h \rightarrow 0} \frac{1}{v(x+h)v(x)} \quad (\text{Note that } -u(x)v(x) + u(x)v(x) = 0)$$

$$\frac{dy}{dx} = \left[\lim_{h \rightarrow 0} \frac{u(x+h)v(x) - u(x)v(x)}{h} - \lim_{h \rightarrow 0} \frac{u(x)v(x+h) - u(x)v(x)}{h} \right] \times \lim_{h \rightarrow 0} \frac{1}{v(x+h)v(x)}$$

$$\frac{dy}{dx} = \left[\lim_{h \rightarrow 0} v(x) \times \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} - \lim_{h \rightarrow 0} u(x) \times \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} \right] \times \lim_{h \rightarrow 0} \frac{1}{v(x+h)v(x)}$$

$$\frac{dy}{dx} = \left[v \times \frac{du}{dx} - u \times \frac{dv}{dx} \right] \times \frac{1}{v^2}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

QED**Example 5**If $y = \frac{\sin(x)}{x^2}$, find $\frac{dy}{dx}$.Let $u = \sin(x)$, $v = x^2$.

$$\text{Then } \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{x^2 \cos(x) - \sin(x) \times 2x}{(x^2)^2}$$

$$\frac{dy}{dx} = \frac{x^2 \cos(x) - 2x \sin(x)}{x^4}$$

$$\frac{dy}{dx} = \frac{x \cos(x) - 2 \sin(x)}{x^3}$$

Exercise 6(a) If $y = \frac{4x^3}{\sin(x)}$, find $\frac{dy}{dx}$.(b) If $y = \frac{7x^2+3}{x^5}$, find $\frac{dy}{dx}$.

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Trigonometric Functions 2

Example 6

(a) If $y = \tan(x)$, find $\frac{dy}{dx}$.

$$y = \frac{\sin(x)}{\cos(x)}$$

Let $u = \sin(x)$, $v = \cos(x)$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\cos(x) \times \cos(x) - \sin(x) \times -\sin(x)}{(\cos(x))^2}$$

$$\frac{dy}{dx} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$\frac{dy}{dx} = \frac{1}{\cos^2(x)} \quad (\text{using } \sin^2(x) + \cos^2(x) = 1)$$

$$\frac{dy}{dx} = \sec^2(x)$$

Exercise 7

(a) By writing $y = \cot(x)$ as $y = \frac{\cos(x)}{\sin(x)}$,show that $\frac{dy}{dx} = -\operatorname{cosec}^2(x)$.

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(c) If $y = \tan^{-1}(3x)$, find $\frac{dy}{dx}$ using the chain rule.

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(b) If $y = \tan^{-1}(x)$, find $\frac{dy}{dx}$.

$$y = \tan^{-1}(x)$$

$$\tan(y) = x$$

(taking tan of both sides)

$$x = \tan(y)$$

(swap sides)

$$\frac{dx}{dy} = \sec^2(y)$$

(using the result from part (a))

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

(taking the reciprocal)

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2(y)}$$

(using $\sec^2(y) = 1 + \tan^2(y)$)

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

(b) By writing $y = \sec(x)$ as $y = \frac{1}{\cos(x)}$,show that $\frac{dy}{dx} = \sec(x) \tan(x)$.

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(d) By writing $y = \operatorname{cosec}(x)$ as $\frac{1}{\sin(x)}$,show that $\frac{dy}{dx} = -\operatorname{cosec}(x) \cot(x)$.

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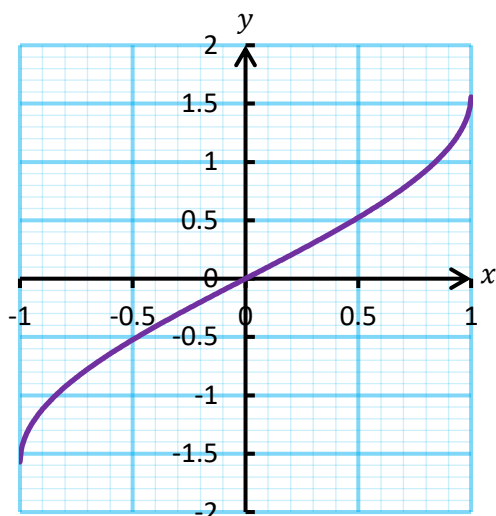
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Example 7

The graph of $y = \sin^{-1}(x)$ between -1 and 1 is shown below.



If $y = \sin^{-1}(x)$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

$$y = \sin^{-1}(x)$$

$$\sin(y) = x \quad (\text{taking sin of both sides})$$

$x = \sin(y)$ (swap sides)

$$\frac{dx}{dy} = \cos(y)$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} \quad \text{(taking the reciprocal)}$$

Now $\sin^2(y) + \cos^2(y) = 1$, so

$$\cos^2(y) = 1 - \sin^2(y)$$

$$\cos(y) = \pm \sqrt{1 - \sin^2(y)}$$

So $\frac{dy}{dx} = \pm \frac{1}{\sqrt{1-\sin^2(y)}}$

$$\frac{dy}{dx} = \pm \frac{1}{\sqrt{1-x^2}}$$

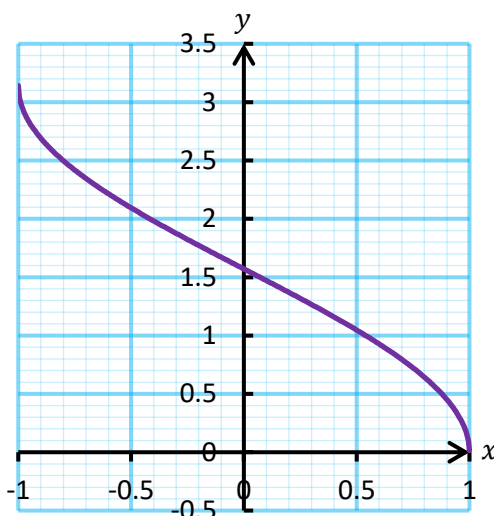
But the gradient of the graph of $y = \sin^{-1}(x)$

is positive, so $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

QED

Exercise 8

The graph of $y = \cos^{-1}(x)$ between -1 and 1 is shown below.



If $y = \cos^{-1}(x)$, show that $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$.

[illegible]

Exercise 9

(a) If $y = \sin^{-1}(8x)$, find $\frac{dy}{dx}$ using the chain rule.

[illegible]

(b) If $y = \cos^{-1}(5x^2)$, find $\frac{dy}{dx}$ using the chain rule.

[illegible]

Exponential Functions

From Unit 1, the gradient of the graph of $y = e^x$ is $\frac{dy}{dx} = e^x$.

Example 8

If $y = e^{4x^6}$, find $\frac{dy}{dx}$.

Let $t = 4x^6$, so that $y = e^t$.

Then $\frac{dt}{dx} = 24x^5$ and $\frac{dy}{dt} = e^t$.

$$\begin{aligned}\text{So } \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ \frac{dy}{dx} &= e^t \times 24x^5 \\ \frac{dy}{dx} &= 24x^5 e^{4x^6}\end{aligned}$$

Exercise 10

If $y = e^{3x^2+5x+3}$, find $\frac{dy}{dx}$.

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Exercise 11

(a) If $y = e^{f(x)}$, find $\frac{dy}{dx}$.

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(b) If $y = e^{5x} \sin(x)$, find $\frac{dy}{dx}$.

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(c) If $y = \frac{e^{x^3}}{2x+3}$, find $\frac{dy}{dx}$.

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(d) If $y = (2 + e^{\cos(x)})^3$, find $\frac{dy}{dx}$.

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If $y = a^x$, then $\frac{dy}{dx} = a^x \ln(a)$.

Proof (no need to learn)

Let $y = a^x$. From Unit 1's definition of a logarithm, we have $e^{\ln(a)} = a$ (recall that $\ln(a)$ is a logarithm with base e).

$$\begin{aligned}\text{So } y &= (e^{\ln(a)})^x \\ y &= e^{x \ln(a)} \quad (\text{rules of indices, } (n^a)^b = n^{a \times b})\end{aligned}$$

Let $t = x \ln(a)$, so that $y = e^t$.

Then $\frac{dt}{dx} = \ln(a)$ and $\frac{dy}{dt} = e^t$.

$$\begin{aligned}\text{So } \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ \frac{dy}{dx} &= e^t \times \ln(a) \\ \frac{dy}{dx} &= a^x \ln(a)\end{aligned}$$

Example 9

If $y = 5^{6x^2}$, find $\frac{dy}{dx}$.

Let $t = 6x^2$, so that $y = 5^t$.

Then $\frac{dt}{dx} = 12x$ and $\frac{dy}{dt} = 5^t \ln(5)$.

$$\begin{aligned}\text{So } \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ \frac{dy}{dx} &= 5^t \ln(5) \times 12x \\ \frac{dy}{dx} &= 12x \times 5^{6x^2} \ln(5)\end{aligned}$$

Exercise 13

(a) If $y = a^{f(x)}$, find $\frac{dy}{dx}$.

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(c) If $y = \frac{6x^2}{5x-7}$, find $\frac{dy}{dx}$.

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Exercise 12

If $y = 4^{12x}$, find $\frac{dy}{dx}$.

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(b) If $y = 2^{7x} \cos(x)$, find $\frac{dy}{dx}$.

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(d) If $y = (7 + 8^{2x})^4$, find $\frac{dy}{dx}$.

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Logarithmic Functions

If $y = \ln(x)$, then $\frac{dy}{dx} = \frac{1}{x}$.

Proof (no need to learn)

Let $y = \ln(x)$.

$$\begin{aligned}\text{So } e^y &= e^{\ln(x)} \\ e^y &= x && \text{(by the definition of a logarithm)} \\ x &= e^y && \text{(swap sides)} \\ \frac{dx}{dy} &= e^y && \text{(differentiate)} \\ \frac{dy}{dx} &= \frac{1}{e^y} && \text{(taking the reciprocal)} \\ \frac{dy}{dx} &= \frac{1}{x}\end{aligned}$$

**QED**

Example 10

If $y = \ln(3x^2 + 5)$, find $\frac{dy}{dx}$.

Let $t = 3x^2 + 5$, so that $y = \ln(t)$.

Then $\frac{dt}{dx} = 6x$ and $\frac{dy}{dt} = \frac{1}{t}$.

$$\begin{aligned} \text{So } \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ \frac{dy}{dx} &= \frac{1}{t} \times 6x \\ \frac{dy}{dx} &= \frac{6x}{3x^2+5} \end{aligned}$$

Exercise 15

(a) If $y = \ln(f(x))$, find $\frac{dy}{dx}$.

(c) If $y = \frac{\ln(7x^3+5x)}{\sin(x)}$, find $\frac{dy}{dx}$.

(e) If $y = \ln(\sin(x))$, find $\frac{dy}{dx}$.

Exercise 14

If $y = \ln(7 - 3x^4)$, find $\frac{dy}{dx}$.

(b) If $y = 4x \ln(8x)$, find $\frac{dy}{dx}$.

(d) If $y = (\ln(8x + 2) + e^x)^4$, find $\frac{dy}{dx}$.

(f) If $y = \tan^{-1}(5x) \ln(5x)$, find $\frac{dy}{dx}$.

Summary

Here is a summary of the differentiation skills and facts that have appeared in this workbook.

Name	y	$\frac{dy}{dx}$	Appearing in the formula booklet:
Function to a power	$(f(x))^n$	$n(f(x))^{n-1}f'(x)$	
The product rule (Leibniz's rule)	uv	$u \frac{dv}{dx} + v \frac{du}{dx}$	
The quotient rule	$\frac{u}{v}$	$\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$	$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
Differentiating trigonometric functions	$\sin(f(x))$ $\cos(f(x))$ $\tan(f(x))$	$f'(x) \cos(f(x))$ $-f'(x) \sin(f(x))$ $f'(x) \sec^2(f(x))$	$\frac{d}{dx}(\tan(x)) = \sec^2(x)$
Differentiating the reciprocal of trigonometric functions	$\sec(f(x))$ $\operatorname{cosec}(f(x))$ $\cot(f(x))$	$f'(x) \sec(f(x)) \tan(f(x))$ $-f'(x) \operatorname{cosec}(f(x)) \cot(f(x))$ $-f'(x) \operatorname{cosec}^2(f(x))$	$\frac{d}{dx}(\sec(x)) = \sec(x) \tan(x)$ $\frac{d}{dx}(\operatorname{cosec}(x)) = -\operatorname{cosec}(x) \cot(x)$ $\frac{d}{dx}(\cot(x)) = -\operatorname{cosec}^2(x)$
Differentiating the inverse of trigonometric functions	$\sin^{-1}(f(x))$ $\cos^{-1}(f(x))$ $\tan^{-1}(f(x))$	$\frac{f'(x)}{\sqrt{1-(f(x))^2}}$ $-\frac{f'(x)}{\sqrt{1-(f(x))^2}}$ $\frac{f'(x)}{1+(f(x))^2}$	$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$
Differentiating exponential functions	$e^{f(x)}$ $a^{f(x)}$	$f'(x)e^{f(x)}$ $f'(x)a^{f(x)} \ln(a)$	
Differentiating logarithmic functions	$\ln(f(x))$	$\frac{f'(x)}{f(x)}$	



0	2
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a) $x^3 \ln(5x)$ [3]

b) $(x + \cos 3x)^4$ [2]

5. Differentiate each of the following with respect to x , simplifying your answer where possible.

$$(b) \quad \frac{2x^2 + 1}{3x^2 + 2} \quad [3]$$

$$(d) \quad \ln(2x) \quad [2]$$

$$(e) \quad \sin^{-1}(3x). \quad [2]$$

(C3 Winter 2008)

5. Differentiate each of the following with respect to x , simplifying your answers wherever possible.

(a) $\frac{\ln x}{x^2}$

(b) $\cos^{-1} 5x$

(c) $\sqrt{1+6x^4}$

(d) $x^3 \tan 2x$

[3],[2],[2],[3]

Handwriting practice area consisting of 30 horizontal dotted lines.

(C3 Winter 2010)

5. Differentiate **each** of the following with respect to x , simplifying your answer wherever possible.

$$(a) \quad \tan^{-1} 3x$$

$$(b) \quad \ln(2x^2 - 3x + 4)$$

[2], [2]

$$(c) \quad e^{2x} \sin x$$

$$(d) \quad \frac{1 - \cos x}{1 + \cos x}$$

[3], [3]

Handwriting practice area consisting of 30 horizontal dotted lines.

(C3 Summer 2007)

6. (a) Differentiate each of the following with respect to x and simplify your answers, wherever possible.

(i) $x^2 \sin x$

(ii) $\ln(x^2 + 3)$

(iii) e^{9-2x}

(iv) $\frac{4}{(3x+7)^2}$

(v) $\sin^{-1} 3x$

[10]

- (b) Given $y = \frac{1 + \tan x}{1 - \tan x}$ ($\tan x \neq 1$), show that $\frac{dy}{dx}$ is always positive.

[4]

5. (a) Differentiate **each** of the following with respect to x and simplify your answers, wherever possible.

[8]

[4]

Handwriting practice lines consisting of 30 horizontal dotted lines.

5. (a) Differentiate each of the following with respect to x .

(b) By first writing $\cot x = \frac{\cos x}{\sin x}$, show that $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$. [3]

Handwriting practice area consisting of 25 sets of horizontal dotted lines.

5. (a) Differentiate **each** of the following with respect to x , simplifying your answer wherever possible.

(b) By first writing $\tan x = \frac{\sin x}{\cos x}$, show that

$$\frac{d}{dx}(\tan x) = \sec^2 x. \quad [3]$$

Handwriting practice lines consisting of 30 sets of three horizontal dotted lines.

6. (a) Differentiate each of the following with respect to x , simplifying your answer wherever possible.

[5]

- (b) (i) Sketch the graph of $y = \sin^{-1}x$ for values of x satisfying $-1 \leq x \leq 1$.
- (ii) By first rewriting $y = \sin^{-1}x$ as $x = \sin y$, find an expression for $\frac{dy}{dx}$ in terms of x . You should justify any choice of sign that you make. [6]

Handwriting practice lines consisting of 40 horizontal dotted lines.

6. (a) Differentiate $\cos x$ from first principles. [5]

Revision Questions

(C3 Summer 2018)

6. (a) Differentiate each of the following with respect to x , simplifying your answer wherever possible.

(i) $\ln (8 + 7x - 4x^3)$

$$\text{(ii)} \quad \sqrt[3]{(5 - 9x^2)}$$

$$\text{(iii)} \quad \frac{2+5\sin x}{4-3\cos x} \quad [7]$$

- (b) By first writing $y = \tan^{-1} 5x$ as $5x = \tan y$, find $\frac{dy}{dx}$ in terms of x . [4]

(C3 Summer 2011)

5. Differentiate each of the following with respect to x , simplifying your answer wherever possible.

$$(a) \quad (9 - 2x)^{\frac{1}{3}}$$

$$(b) \quad \ln(\cos x)$$

$$(c) \quad x^3 \tan 4x$$

$$(d) \quad \frac{e^{6x}}{(3x+2)^4}$$

[2], [3], [3], [3]

Handwriting practice lines consisting of 30 horizontal dotted lines.



Inflection

Points



Name:

Background

What is the work?

How to find a curve's **inflection points**, where $\frac{d^2y}{dx^2} = 0$ and the sign of $\frac{d^2y}{dx^2}$ changes each side of each inflection point.

What is required before starting?

GCSE Work: Substitution, solving equations, factorising quadratic expressions.

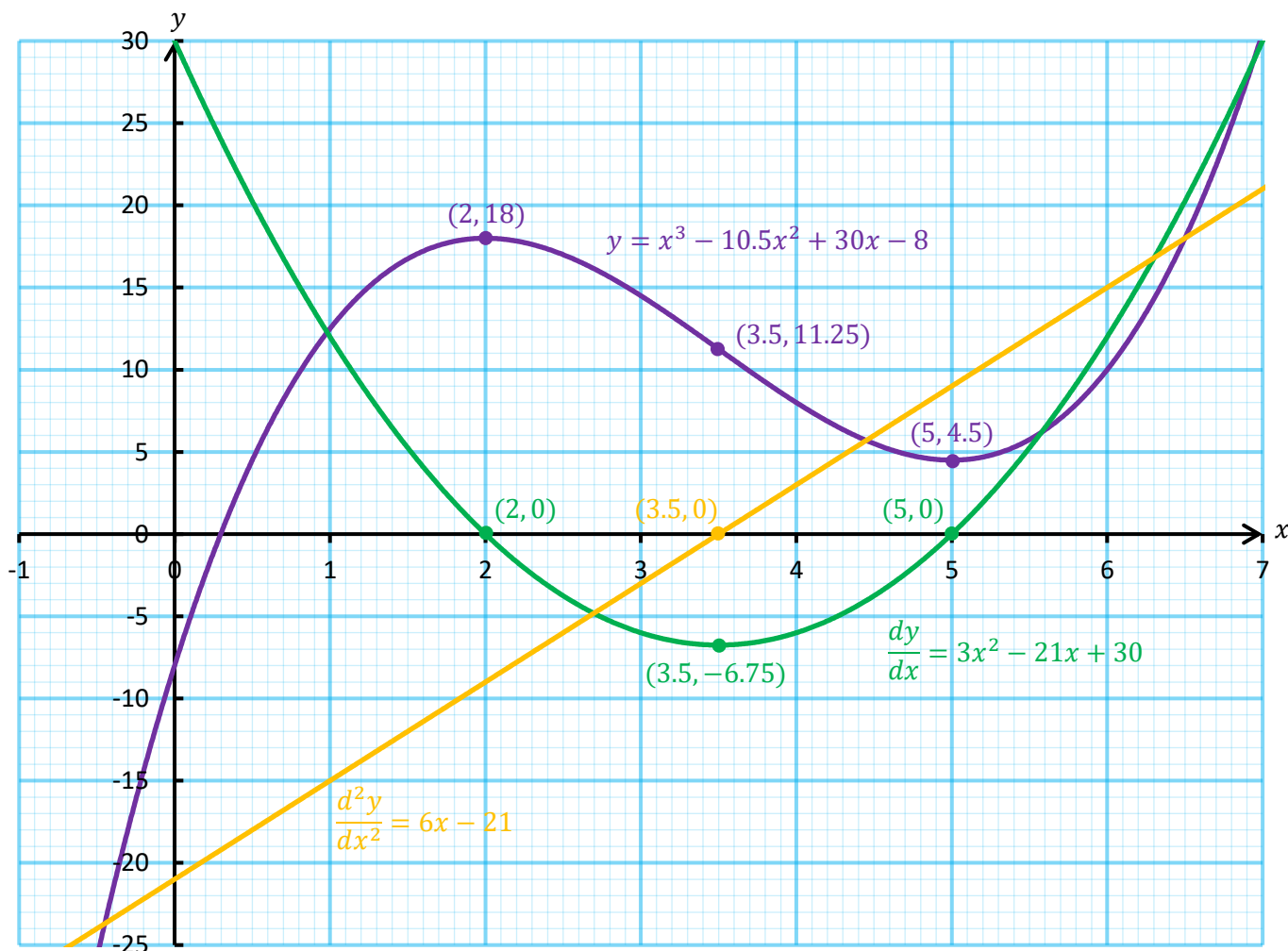
A Level Unit 1: Differentiation, finding stationary points.

Where does this lead to?

In business, an inflection point can show substantial growth, or suggest that a market bubble is about to burst.

Theory

Let us consider the function $y = x^3 - 10.5x^2 + 30x - 8$. By differentiation, $\frac{dy}{dx} = 3x^2 - 21x + 30$. By differentiating again, $\frac{d^2y}{dx^2} = 6x - 21$. The graph below shows these three functions, and the connections between them.



Concentrating on the **purple curve** for $y = x^3 - 10.5x^2 + 30x - 8$, we see that we have a local maximum point at (2, 18) and a local minimum point at (5, 4.5). These correspond to the points where $\frac{dy}{dx} = 0$.

Concentrating on the **green curve** for $\frac{dy}{dx} = 3x^2 - 21x + 30$, we see that we have a minimum point at (3.5, -6.75). This corresponds to the point where $\frac{d^2y}{dx^2} = 0$. On the original **purple curve**, an **inflection point** occurs at the point (3.5, 11.25).

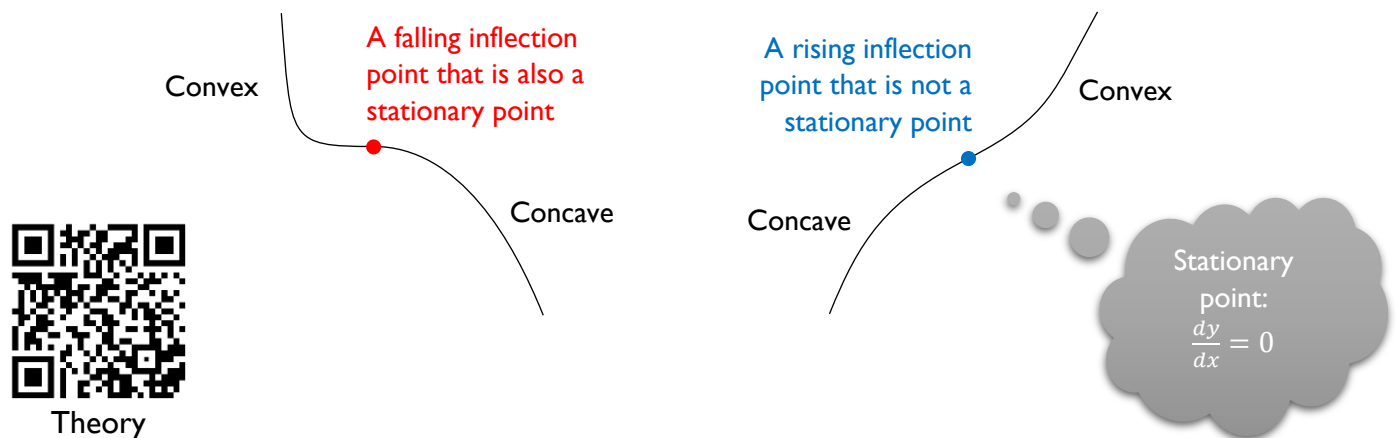
Definitions

Convex and Concave

- A curve is **convex** if $\frac{d^2y}{dx^2} > 0$, and so the tangent to the curve appears **underneath** the curve.
- A curve is **concave** if $\frac{d^2y}{dx^2} < 0$, and so the tangent to the curve appears **above** the curve.

Inflection Point

- An **inflection point** occurs when a curve changes its concavity from convex to concave, or from concave to convex. This happens when the sign of $\frac{d^2y}{dx^2}$ changes.



Looking at the **purple curve** on the previous page, the curve is concave before $x = 3.5$, because $\frac{d^2y}{dx^2} < 0$, and the tangent to the curve appears above the curve. After $x = 3.5$, the curve is convex, because $\frac{d^2y}{dx^2} > 0$, and the tangent to the curve appears underneath the curve. At $x = 3.5$, the curve therefore changes its concavity from concave to convex, and we can say that the point (3.5, 11.25) is an inflection point.

Here is a strategy to find the coordinates of a curve's inflection points:

- Solve $\frac{d^2y}{dx^2} = 0$ to find the x -coordinates of any potential inflection points.
- Verify to see if the sign of $\frac{d^2y}{dx^2}$ changes each side of each x -coordinate.
If this is true, then you have an inflection point. If this is not true, then you have another type of point (for example, a local maximum point or a local minimum point).
- For the inflection points you have found, substitute back into y to find the y -coordinates.

You need to verify that the sign of $\frac{d^2y}{dx^2}$ changes; it is not sufficient to just solve $\frac{d^2y}{dx^2} = 0$.

Example 1

- (a) Find the coordinate of the inflection point of the curve $y = x^3 - 10.5x^2 + 30x - 8$.
- (b) Show that this point is not a stationary point.

Answer: (a) Differentiate: $\frac{dy}{dx} = 3x^2 - 21x + 30$

Differentiate again: $\frac{d^2y}{dx^2} = 6x - 21$

$$\begin{aligned}\text{Solve } \frac{d^2y}{dx^2} = 0: \quad 6x - 21 &= 0 \\ 6x &= 21 \\ x &= 3.5\end{aligned}$$

$$\begin{aligned}\text{Substitute back into } y: \quad y &= 3.5^3 - 10.5 \times 3.5^2 + 30 \times 3.5 - 8 \\ y &= 11.25\end{aligned}$$

Verify that (3.5, 11.25) is an inflection point:

$$\begin{array}{ll}\text{If } x = 3.4, \text{ then } \frac{d^2y}{dx^2} = 6 \times 3.4 - 21 & \text{If } x = 3.6, \text{ then } \frac{d^2y}{dx^2} = 6 \times 3.6 - 21 \\ \frac{d^2y}{dx^2} = -0.6 & \frac{d^2y}{dx^2} = 0.6\end{array}$$

The sign of $\frac{d^2y}{dx^2}$ changes, so (3.5, 11.25) is an inflection point.

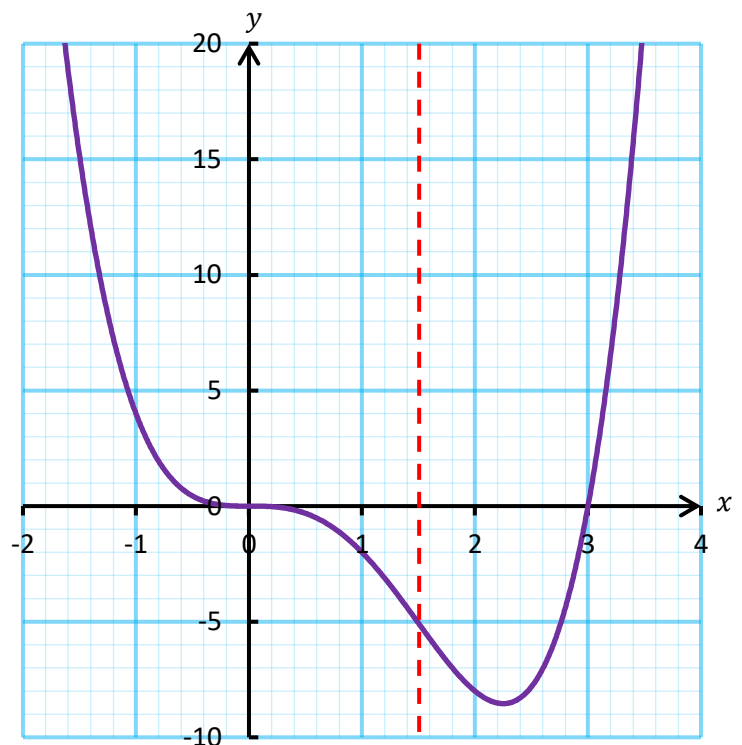
$$\begin{aligned}\text{(b) If } x = 3.5, \text{ then } \frac{dy}{dx} &= 3 \times 3.5^2 - 21 \times 3.5 + 30 \\ \frac{dy}{dx} &= -6.75\end{aligned}$$

$\frac{dy}{dx} \neq 0$ so (3.5, 11.25) is not a stationary point.

Exercise 1

The graph on the right shows the curve for $y = x^4 - 3x^3$.

- (a) Where is the curve concave?
- (b) How many inflection points does the curve have?
- (c) How many stationary points does the curve have?
- (d) Find the coordinates and the nature of each of the inflection points.
- (e) Find the coordinates and the nature of each of the stationary points.



Handwriting practice area with 30 horizontal dotted lines.

Handwriting practice area consisting of 30 horizontal dotted lines.



(Unit 3 Summer 2023)

1	1
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A curve C has equation $f(x) = 5x^3 + 2x^2 - 3x$.

- a) Find the x -coordinate of the point of inflection. State, with a reason, whether the point of inflection is stationary or non-stationary. [5]
- b) Determine the range of values of x for which C is concave. [2]

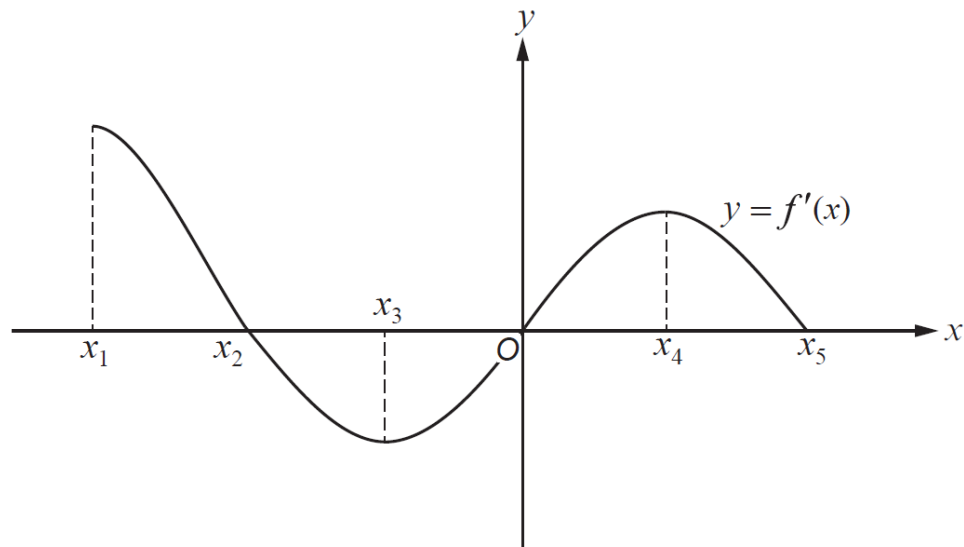
Handwriting practice lines (dotted lines) for the answer.

1	3
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 A function is defined by $f(x) = 2x^3 + 3x - 5$.

- Prove that the graph of $f(x)$ does not have a stationary point. [2]
- Show that the graph of $f(x)$ does have a point of inflection and find the coordinates of the point of inflection. [4]
- Sketch the graph of $f(x)$. [1]

13. The diagram below shows a sketch of the graph of $y = f'(x)$ for the interval $[x_1, x_5]$.



- (a) Find the interval on which $f(x)$ is both decreasing and convex. Give reasons for your answer. [2]

- (b) Write down the x -coordinate of a point of inflection of the graph of $y = f(x)$. [1]

7. The curve $y = ax^4 + bx^3 + 18x^2$ has a point of inflection at $(1, 11)$.

- Show that $2a + b + 6 = 0$. [2]
- Find the values of the constants a and b and show that the curve has another point of inflection at $(3, 27)$. [8]
- Sketch the curve, identifying all the stationary points including their nature. [6]

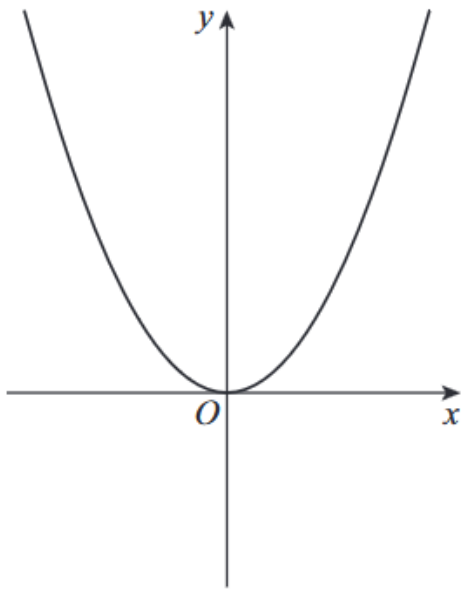
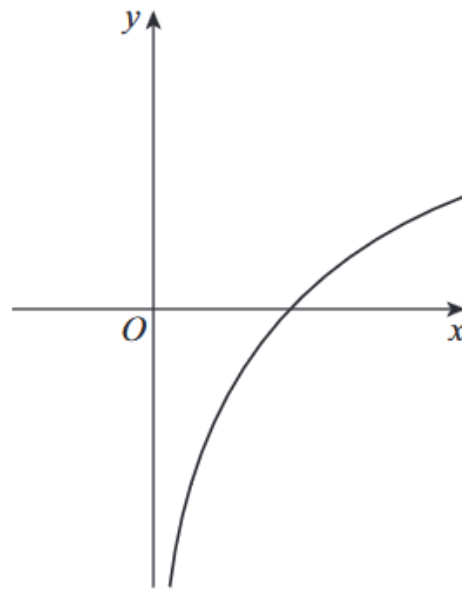
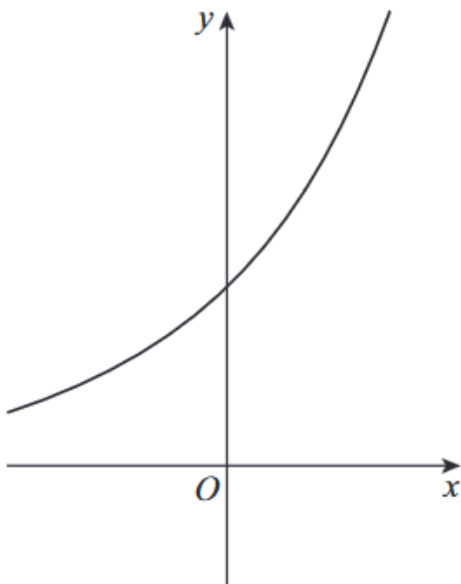
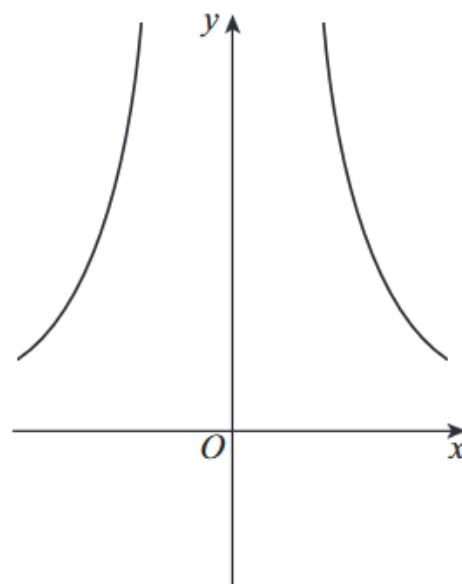
(AQA Paper 2 Summer 2022)

3 The function f is concave and is represented by one of the graphs below.

Identify the graph which represents f .

Tick (✓) **one** box.

[1 mark]

☐☐☐☐

Revision Questions

(AQA Paper 2 Summer 2021)

2 A curve has equation $y = f(x)$

The curve has a point of inflection at $x = 7$

It is given that $f'(7) = a$ and $f''(7) = b$, where a and b are real numbers.

Identify which one of the statements below must be true.

Circle your answer.

[1 mark]

$$f'(7) \neq 0$$

$$f'(7) = 0$$

$$f''(7) \neq 0$$

$$f''(7) = 0$$

(AQA Sample Assessment Materials)

8 A curve has equation $y = 2x \cos 3x + (3x^2 - 4) \sin 3x$

8 (a) Find $\frac{dy}{dx}$, giving your answer in the form $(mx^2 + n)\cos 3x$, where m and n are integers.

[4 marks]

8 (b) Show that the x -coordinates of the points of inflection of the curve satisfy the equation

$$\cot 3x = \frac{9x^2 - 10}{6x}$$

[4 marks]

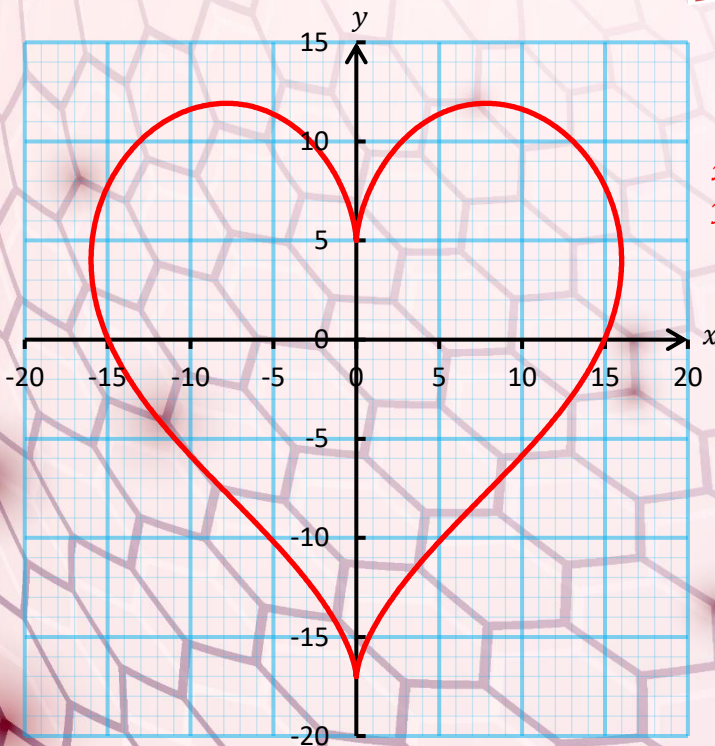
Handwriting practice lines consisting of 30 horizontal dotted lines.



Parametric

and Implicit

Equations



$$x = 16 \sin^3(t)$$

$$y = 13 \cos(t) - 5 \cos(2t) - 2 \cos(3t) - \cos(4t)$$

Name: _____

Background

What is the work?

Finding the equation of tangents and normals to curves that are defined **parametrically** or **implicitly**. Converting between Cartesian and parametric forms.

What is required before starting?

GCSE Work: Substitution, solving equations, changing the subject.
A Level Unit 3: Differentiation.

Where does this lead to?

Parametric equations can model movement that is circular in nature, or movement that returns to the same location.

Theory

Thus far, we have been differentiating functions of the form $y = f(x)$, where there is one value of y for each value of x . Not every curve is of this type however; consider the equation $x^2 + y^2 = 1$ that gives the unit circle. To try to differentiate this function, we could attempt to re-arrange it to the form $y = f(x)$:

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1 - x^2}$$

This is not an **explicit** function, as there are two values of y for each value of x . For example, if $x = 0$, we have

$$y = \pm\sqrt{1 - 0^2}$$

$$y = \pm 1$$

This can be seen on the graph as the points $(0, 1)$ and $(0, -1)$ lie on the circle. Because it is not explicit what the value of y is for each value of x , we say that $x^2 + y^2 = 1$ is an **implicit** equation.

Differentiating Implicit Equations

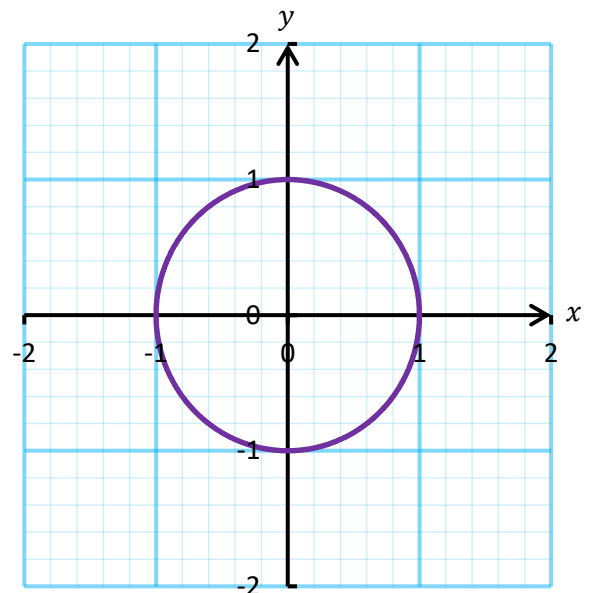
How do we differentiate $x^2 + y^2 = 1$ with respect to x ? There is no problem in differentiating x^2 with respect to x : this would give us $2x$. We could also differentiate 1 with respect to x to give 0. But how do we differentiate y^2 with respect to x ? To see how, let us define z as $z = y^2$. We can differentiate z with respect to y :

$$\frac{dz}{dy} = 2y$$

Then, we can use the chain rule to differentiate z with respect to x :

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

$$\frac{dz}{dx} = 2y \times \frac{dy}{dx}$$



Theory

So, to differentiate z with respect to x , we differentiate it with respect to y , and then multiply by $\frac{dy}{dx}$.

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx}$$

In general, to differentiate any function of y with respect to x , we differentiate it with respect to y , and then multiply by $\frac{dy}{dx}$.

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}$$

Returning to the equation of a unit circle, $x^2 + y^2 = 1$, when we differentiate it with respect to x , we obtain

$$2x + 2y \times \frac{dy}{dx} = 0$$

We can re-arrange this to find $\frac{dy}{dx}$:

$$\begin{aligned} 2y \times \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

Given a point that lies on the circle, we can substitute the point into $\frac{dy}{dx}$ to find the gradient of the circle at that point. For example, on the previous page we saw that the point $(0, 1)$ lies on the circle. Here we have

$$\begin{aligned} \frac{dy}{dx} &= -\frac{0}{1} \\ \frac{dy}{dx} &= 0 \end{aligned}$$

We can now (for example) find the equation of the tangent to the circle at the point $(0, 1)$:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 0(x - 0) \\ y - 1 &= 0 \\ y &= 1 \end{aligned}$$

This is a horizontal line that goes through the point $(0, 1)$.

Example 1

Find $\frac{dy}{dx}$ for the curve $3x^2 + 5y^2 = 7$.

Answer: By differentiating implicitly: $6x + 10y \times \frac{dy}{dx} = 0$

$$\begin{aligned} 10y \times \frac{dy}{dx} &= -6x \\ \frac{dy}{dx} &= \frac{-6x}{10y} \\ \frac{dy}{dx} &= -\frac{3x}{5y} \end{aligned}$$

Exercise 1

(a) Find $\frac{dy}{dx}$ for the curve $8x^2 + 2y^2 = 10$.

(b) The point $(-1, 1)$ lies on the curve $8x^2 + 2y^2 = 10$.

Find the equation of the tangent to the curve $8x^2 + 2y^2 = 10$ at the point $(-1, 1)$.

Exercise 2

Find the equation of the normal to the curve $4x^2 + 3y^2 - 7x + 2y = 13$ at the point $(1, 2)$.

Write your answer in the form $y = mx + c$.

Parametric Equations

The **parametric equations** for a curve define the coordinates of the curve using a third variable, usually t , which represents the **parameter**.

$$x = f(t)$$

$$y = g(t)$$

For example, we could represent the unit circle using the following parametric equations.

$$x = \cos(t)$$

$$y = \sin(t)$$

The following table shows the value of x and y for various values of t , starting at $t = 0$, and going up in increments of 0.1. (The values are shown correct to four decimal places, with the angle t in radians). These are plotted as the **purple coordinates** on the graph on the right. (To plot the entire circle, the value of t would need to reach 2π .)

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
x	1	0.9950	0.9801	0.9553	0.9211	0.8776	0.8253	0.7648	0.6967	0.6216	0.5403
y	0	0.0998	0.1987	0.2955	0.3894	0.4794	0.5646	0.6442	0.7174	0.7833	0.8415

To find the gradient of the curve at any point, we can differentiate the functions $f(t)$ and $g(t)$ with respect to t :

$$\frac{dx}{dt} = f'(t)$$

$$\frac{dy}{dt} = g'(t)$$

We can then use the chain rule to find the gradient:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

where

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

For the above example, we have

$$\frac{dx}{dt} = -\sin(t)$$

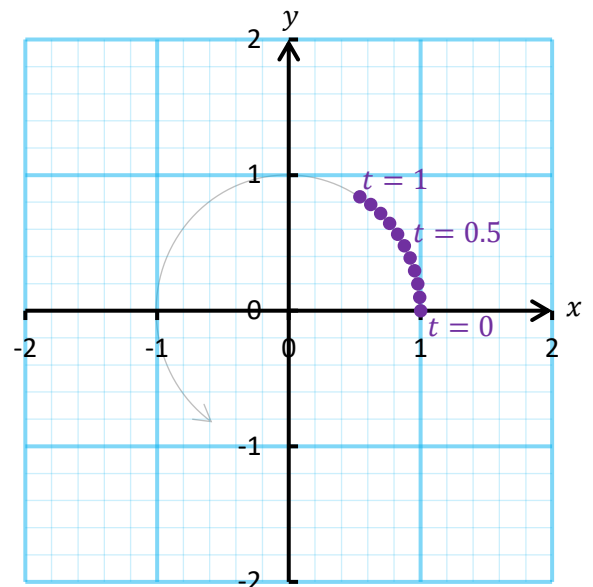
$$\frac{dy}{dt} = \cos(t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \cos(t) \times \frac{1}{-\sin(t)}$$

$$\frac{dy}{dx} = -\cot(t)$$

If (for example) $t = 1$, then $\frac{dy}{dx} = -\cot(1)$
 $\frac{dy}{dx} = -0.6421$ to 4 decimal places.



The table uses values of t from zero upwards, but it is important to note that t could also take negative values.



Theory

To find the equation of the normal to the curve at the point where $t = 1$, we can use the following information from the previous page:

$$\text{If } t = 1, \text{ then (to 4 decimal places) } x = 0.5403, y = 0.8415, \frac{dy}{dx} = -0.6421$$

The gradient of the normal is $-\frac{1}{-0.6421} = 1.5574$ (which is the negative reciprocal of $\frac{dy}{dx}$).

The equation of the normal at the point where $t = 1$ is $y - y_1 = m(x - x_1)$
 $y - 0.8415 = 1.5574(x - 0.5403)$
 $y - 0.8415 = 1.5574x - 0.8415$
 $y = 1.5574x$

Exercise 3

A curve is defined by the following parametric equations.

$$x = t^2$$

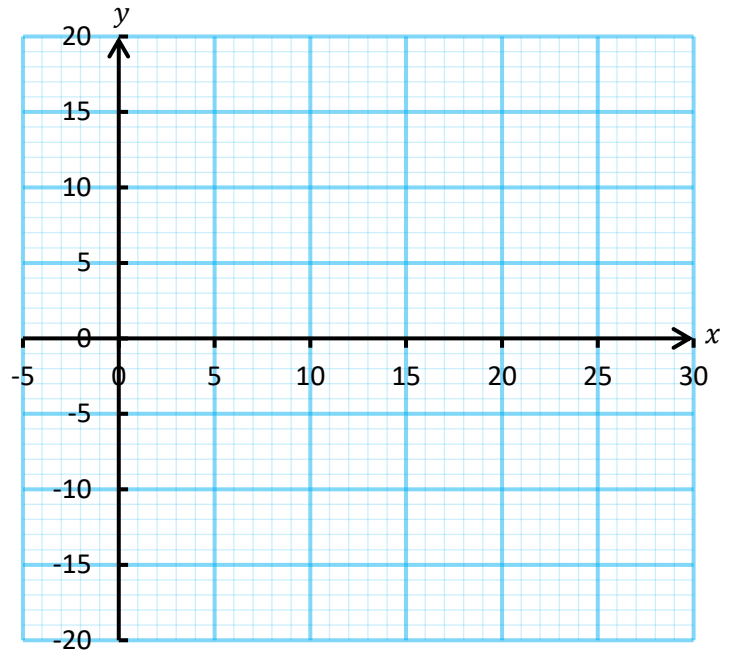
$$y = 2t + 3$$

(a) Complete the following table.

t	0	1	2	3	4	5
x						
y						

(b) Use the above table to plot a part of the curve on the graph paper on the right.

(c) Find the gradient $\frac{dy}{dx}$ of the curve, writing your answer in terms of t .



(d) Find the equation of the tangent to the curve at the point where $t = 2$.

(e) Find the equation of the normal to the curve at the point where $t = 4$.

Writing Parametric Equations as a Cartesian Equation

Sometimes, it is possible to transform a set of parametric equations to be one **Cartesian equation**, which avoids using the original parameter, and uses only x and y .

Example 2

On page 5, the following parametric equations were used to define the unit circle.

$$\begin{aligned}x &= \cos(t) \\ y &= \sin(t)\end{aligned}$$

To delete the parameter t from these equations, we can use the identity $\sin^2(t) + \cos^2(t) = 1$ from Unit 1:

$$\begin{aligned}y^2 + x^2 &= 1 \\ x^2 + y^2 &= 1\end{aligned}$$

This is the equation for the unit circle from page 2.

Example 3

In Exercise 3, the following parametric equations were used to define a curve.

$$\begin{aligned}x &= t^2 \\ y &= 2t + 3\end{aligned}$$

To delete the parameter t from these equations, we can re-arrange the second equation to make t its subject:

$$\begin{aligned}y - 3 &= 2t \\ \frac{y - 3}{2} &= t\end{aligned}$$

We can then substitute for t into the first equation:

$$\begin{aligned}x &= \left(\frac{y - 3}{2}\right)^2 \\ x &= \frac{(y - 3)^2}{2^2} \\ x &= \frac{(y - 3)(y - 3)}{4} \\ 4x &= y^2 - 3y - 3y + 9 \\ 4x &= y^2 - 6y + 9 \\ 4x - y^2 + 6y - 9 &= 0\end{aligned}$$

Exercise 4

A curve is defined by the parametric equations

$$\begin{aligned}x &= t^2 - 3 \\ y &= t + 2\end{aligned}$$

Find the Cartesian equation that defines this curve.



3. (a) The curve C is defined by

Find the value of $\frac{dy}{dx}$ at the point (2, 1).

[4]

3. (a) A function is defined parametrically by

(i) Find $\frac{dy}{dx}$ in terms of t .

[5]

4. Given that $x = 2e^t - 5$, $y = 8e^{-t} + 3e^t - 4$, find the value of t when $\frac{dy}{dx} = -1$.

[7]

3. (a) Given that $x = \cos t$, $y = \sin 2t$, find $\frac{dy}{dx}$ in terms of t . [4]

$$x^4 + 2x^2y + y^2 = 21,$$

find $\frac{dy}{dx}$ in terms of x and y . [4]

3. (a) Given that $x = t^4 + 1$, $y = e^{2t} + 5$, find $\frac{dy}{dx}$ in terms of t . [4]

(b) Given that $x^4 + \sin y + x^2y^3 = 9$, find $\frac{dy}{dx}$ in terms of x and y . [3]

3. Given that

find the value of $\frac{dy}{dx}$ at the point $(1, \pi)$.

[4]

11. A curve is defined parametrically by

$$x = 2\theta + \sin 2\theta, \quad y = 1 + \cos 2\theta.$$

(a) Show that the gradient of the curve at the point with parameter θ is $-\tan \theta$. [6]

(b) Find the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$. [4]

(Unit 3 Summer 2023)

1	3
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The curve C_1 has parametric equations $x = 3p + 1$, $y = 9p^2$.The curve C_2 has parametric equations $x = 4q$, $y = 2q$.Find the Cartesian coordinates of the points of intersection of C_1 and C_2 .

[7]

1	6
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 The parametric equations of the curve C are

a) Find the coordinates of the points where C meets the y -axis. [3]

b) Show that the x -axis is a tangent to the curve C . [5]

1	0
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 The equation of a curve C is given by the parametric equations

$$x = \cos 2\theta, \quad y = \cos \theta.$$

- a) Find the Cartesian equation of C . [2]
- b) Show that the line $x - y + 1 = 0$ meets C at the point P , where $\theta = \frac{\pi}{3}$, and at the point Q , where $\theta = \frac{\pi}{2}$. Write down the coordinates of P and Q . [5]
- c) Determine the equations of the tangents to C at P and Q . Write down the coordinates of the point of intersection of the two tangents. [7]

Handwriting practice lines consisting of 30 horizontal dotted lines.

Handwriting practice area with 30 horizontal dotted lines.

Revision Questions

(Unit 3 Summer 2019)

6

A curve C has parametric equations $x = \sin \theta$, $y = \cos 2\theta$.

- b)** Find the coordinates of the points of intersection of the curve C and the straight line $x + y = 1$. [5]

Handwriting practice lines consisting of 30 horizontal dotted lines.

(C3 Winter 2009)

3. (a) Given that $x^2 + 3xy + 2y^2 - 2x = 13$, find the value of $\frac{dy}{dx}$ at the point (1, 2). [4]

(b) Given that $x = 2e^t + 6$, $y = 4e^{2t} + 3e^t + 1$, find the value of t when $\frac{dy}{dx} = 6$, giving your answer correct to three decimal places. [7]



Further

Integration 1

$$\int (ax + b)^n dx = \frac{1}{(n+1)a} (ax + b)^{n+1} + c$$

$$\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + c$$

$$\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + c$$

$$\int \sec^2(ax + b) dx = \frac{1}{a} \tan(ax + b) + c$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$$

$$\int a^{ax+b} \ln(a) dx = \frac{1}{a} a^{ax+b} + c$$

$$\int \frac{1}{ax + b} dx = \frac{1}{a} \ln|ax + b| + c$$

Name: _____

Background

What is the work?

Using the previous work on differentiating trigonometric; exponential and logarithmic functions to **integrate** similar functions. Using definite integrals to find the area between two curves.

What is required before starting?

A Level Unit 1: Differentiation and integration.
A Level Unit 3: Further differentiation.

Where does this lead to?

The questions in this workbook can appear when modelling population growth; radioactive decay and financial growth.

Theory

Here are some of the differentiation techniques that appeared in a previous workbook on further differentiation.

Name	y	$\frac{dy}{dx}$	Appearing in the formula booklet:
Function to a power	$(f(x))^n$	$n(f(x))^{n-1}f'(x)$	
Differentiating trigonometric functions	$\sin(f(x))$ $\cos(f(x))$ $\tan(f(x))$	$f'(x) \cos(f(x))$ $-f'(x) \sin(f(x))$ $f'(x) \sec^2(f(x))$	$\frac{d}{dx}(\tan(x)) = \sec^2(x)$
Differentiating exponential functions	$e^{f(x)}$ $a^{f(x)}$	$f'(x)e^{f(x)}$ $f'(x)a^{f(x)} \ln(a)$	
Differentiating logarithmic functions	$\ln(f(x))$	$\frac{f'(x)}{f(x)}$	

The above techniques all use the **chain rule**: Given two functions $y = f(t)$ and $t = g(x)$, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

The following set of differentiation questions all use the table on the previous page in cases where the function $f(x)$ is of the form $ax + b$ for some constants a and b .

Example 1

Differentiate the following functions.

(a) $y = (3x - 5)^4$

$$\frac{dy}{dx} = 4(3x - 5)^3 \times 3$$

$$\frac{dy}{dx} = 12(3x - 5)^3$$

(b) $y = \sin(6x - 4)$

$$\frac{dy}{dx} = 6 \times \cos(6x - 4)$$

$$\frac{dy}{dx} = 6 \cos(6x - 4)$$

(c) $y = e^{8x+9}$

$$\frac{dy}{dx} = 8 \times e^{8x+9}$$

$$\frac{dy}{dx} = 8e^{8x+9}$$

(d) $y = 6^{5x+3}$

$$\frac{dy}{dx} = 5 \times 6^{5x+3} \times \ln(6)$$

$$\frac{dy}{dx} = 5 \ln(6) \times 6^{5x+3}$$

(e) $y = \ln(7x + 3)$

$$\frac{dy}{dx} = \frac{7}{7x + 3}$$

Exercise 1

Differentiate the following functions.

(a) $y = (6x + 11)^7$

(b) $y = \cos(9x + 7)$

(c) $y = e^{x-3}$

(d) $y = 9^{2x-9}$

(e) $y = \ln(9x + 25)$

Remembering that integration reverses differentiation, we can use the above work to write some integration questions:

$$\int 12(3x - 5)^3 dx = (3x - 5)^4 + c$$

$$\int 6 \cos(6x - 4) dx = \sin(6x - 4) + c$$

$$\int 8e^{8x+9} dx = e^{8x+9} + c$$

and so on...

These integration sums all **undo** the chain rule. In general, if

$$\int f(x) dx = F(x) + c,$$

then

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + c$$

Proof (no need to remember)

Let $y = \frac{1}{a}F(ax + b) + c$. By defining $t = ax + b$, we have $y = \frac{1}{a}F(t) + c$.

By differentiating, $\frac{dt}{dx} = a, \frac{dy}{dt} = \frac{1}{a}f(t)$.

Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$
 $\frac{dy}{dx} = \frac{1}{a}f(t) \times a$
 $\frac{dy}{dx} = f(t)$
 $\frac{dy}{dx} = f(ax + b)$

So, $\int f(ax + b) dx = \frac{1}{a}F(ax + b) + c$.

QED

Practically, when we use the above technique to integrate, we think about what **differentiates** to give $f(ax + b)$, multiplying by a constant as required.

Example 2

Integrate the following.

(a) $\int 18(3x + 2)^5 dx$

Differentiating $(3x + 2)^6$ gives $6 \times (3x + 2)^5 \times 3$
 $= 18(3x + 2)^5$, so
 $\int 18(3x + 2)^5 dx = (3x + 2)^6 + c$.

(b) $\int 32(2x - 5)^3 dx$

Differentiating $(2x - 5)^4$ gives $4 \times (2x - 5)^3 \times 2$
 $= 8(2x - 5)^3$. But we need $32(2x - 5)^3$, which is 4
times greater, so we need to differentiate $4(2x - 5)^4$
to give $32(2x - 5)^3$. So,
 $\int 32(2x - 5)^3 dx = 4(2x - 5)^4 + c$.

(c) $\int \sin(7x - 1) dx$

Differentiating $-\cos(7x - 1)$ gives
 $7 \times \sin(7x - 1)$. But we need $\sin(7x - 1)$, which is 7
times less, so we need to differentiate $-\frac{1}{7}\cos(7x - 1)$
to give $\sin(7x - 1)$. So,
 $\int \sin(7x - 1) dx = -\frac{1}{7}\cos(7x - 1) + c$.

(d) $\int e^{5x+7} dx$

Differentiating e^{5x+7} gives $5 \times e^{5x+7}$. But we need
 e^{5x+7} , which is 5 times less, so we need to differentiate
 $\frac{1}{5}e^{5x+7}$ to give $\frac{1}{5} \times 5 \times e^{5x+7} = e^{5x+7}$. So,
 $\int e^{5x+7} dx = \frac{1}{5}e^{5x+7} + c$.

(e) $\int 15 \ln(7) \times 7^{5x-2} dx$

Differentiating 7^{5x-2} gives $5 \times 7^{5x-2} \times \ln(7)$. But we
need $15 \ln(7) \times 7^{5x-2}$, which is 3 times greater, so we
need to differentiate $3 \times 7^{5x-2}$ to give
 $3 \times 5 \times 7^{5x-2} \times \ln(7) = 15 \ln(7) \times 7^{5x-2}$. So,
 $\int 15 \ln(7) \times 7^{5x-2} dx = 3 \times 7^{5x-2} + c$.

Exercise 2

Integrate the following.

(a) $\int 10(2x - 5)^4 dx$

(b) $\int 70(5x + 1)^6 dx$

(c) $\int \cos(3x + 2) dx$

(d) $\int e^{-2x+3} dx$

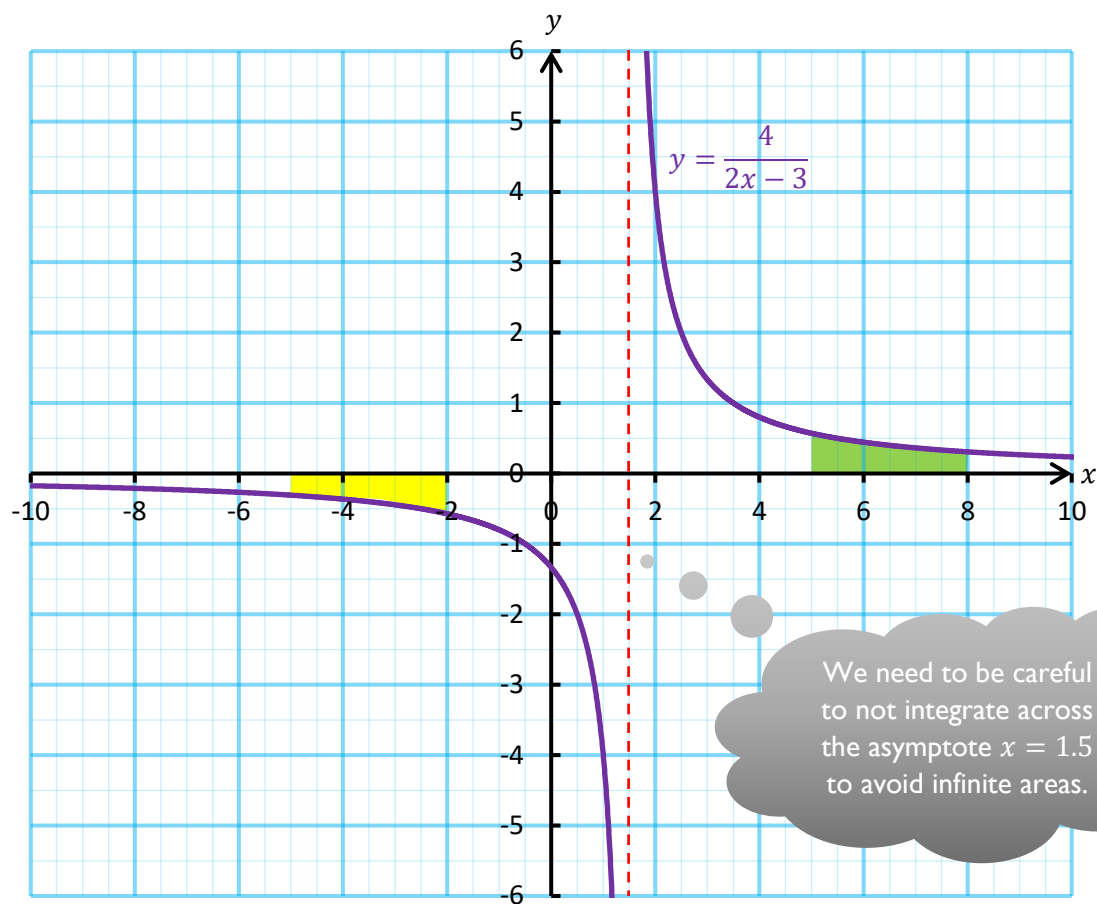
(e) $\int 18 \ln(3) \times 3^{2x+4} dx$



Theory

Example 3

Evaluate $\int_{-5}^{-2} \frac{4}{2x-3} dx$.



Answer: If we write the integrand $\frac{4}{2x-3}$ in the form $2 \times \frac{2}{2x-3}$, we see that the fraction is of the form $\frac{f'(x)}{f(x)}$.

The fraction therefore integrates to give a natural logarithm, as $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$.

The rule that we use to integrate a fraction of the form $\frac{f'(x)}{f(x)}$ is

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

In the example, we have

$$\begin{aligned} & \int_{-5}^{-2} \frac{4}{2x-3} dx \\ &= \int_{-5}^{-2} 2 \times \frac{2}{2x-3} dx \\ &= [2 \ln|2x-3|]_{-5}^{-2} \\ &= (2 \ln|2 \times -2 - 3|) - (2 \ln|2 \times -5 - 3|) \\ &= (2 \ln|-7|) - (2 \ln|-13|) \\ &= 2 \ln(7) - 2 \ln(13) \\ &= -1.238 \text{ to 3 decimal places} \end{aligned}$$

This corresponds to the **yellow area** in the above graph.

Consider that we attempt to use the rule $\int \frac{f'(x)}{f(x)} dx = \ln(f(x)) + c$ to integrate the fraction $\frac{f'(x)}{f(x)}$, not $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$. The calculations for evaluating $\int_{-5}^{-2} \frac{4}{2x-3} dx$ would begin as follows:

$$\begin{aligned} & \int_{-5}^{-2} \frac{4}{2x-3} dx \\ &= \int_{-5}^{-2} 2 \times \frac{2}{2x-3} dx \\ &= [2 \ln(2x-3)]_{-5}^{-2} \\ &= (2 \ln(2 \times -2 - 3)) - (2 \ln(2 \times -5 - 3)) \\ &= (2 \ln(-7)) - (2 \ln(-13)) \end{aligned}$$

It would be impossible to proceed from this point, as it is not possible to evaluate $\ln(-7)$ or $\ln(-13)$. (Remember that the natural logarithm $\ln(x)$ finds the power p of the number e that gives x , so that $e^p = x$. Because the value of e^p is always positive no matter what the value of p is, there is no possible value for $\ln(x)$ if x is negative.)

So, we need to use the modulus symbols when integrating $\frac{f'(x)}{f(x)}$ to give $\ln|f(x)| + c$, but why does this work?

Consider again the graph for $y = \frac{4}{2x-3}$ on the previous page. When evaluating $\int_{-5}^{-2} \frac{4}{2x-3} dx$, we are finding the yellow area below the x -axis between $x = -5$ and $x = -2$. By the symmetry of the graph, the magnitude of the yellow area is the same as the magnitude of the green area on the right, between $x = 5$ and $x = 8$. Compare the following calculations for integrating the function between -5 and -2 to begin with, and then between 5 and 8 .

$$\begin{aligned} & \int_{-5}^{-2} \frac{4}{2x-3} dx \\ &= \int_{-5}^{-2} 2 \times \frac{2}{2x-3} dx \\ &= [2 \ln|2x-3|]_{-5}^{-2} \\ &= (2 \ln|2 \times -2 - 3|) - (2 \ln|2 \times -5 - 3|) \\ &= (2 \ln|-7|) - (2 \ln|-13|) \\ &= 2 \ln(7) - 2 \ln(13) \\ &= -1.238 \text{ to 3 decimal places} \end{aligned}$$

$$\begin{aligned} & \int_5^8 \frac{4}{2x-3} dx \\ &= \int_5^8 2 \times \frac{2}{2x-3} dx \\ &= [2 \ln|2x-3|]_5^8 \\ &= (2 \ln|2 \times 8 - 3|) - (2 \ln|2 \times 5 - 3|) \\ &= (2 \ln|13|) - (2 \ln|7|) \\ &= 2 \ln(13) - 2 \ln(7) \\ &= 1.238 \text{ to 3 decimal places} \end{aligned}$$

Note that the calculations in the penultimate line both use the numbers 7 and 13, but in a different order. On the left, the calculations give a negative answer, corresponding to the **yellow area underneath** the x -axis. On the right, the calculations give a positive answer, corresponding to the **green area above** the x -axis.

Evaluate $\int_{-6}^{-2} \frac{2}{4x+5} dx$.

Summary

Name	Integrand	Answer
Function to a power	$\int (ax + b)^n dx$	$\frac{1}{(n+1)a}(ax + b)^{n+1} + c$
Integrating trigonometric functions	$\int \cos(ax + b) dx$ $\int \sin(ax + b) dx$ $\int \sec^2(ax + b) dx$	$\frac{1}{a}\sin(ax + b) + c$ $-\frac{1}{a}\cos(ax + b) + c$ $\frac{1}{a}\tan(ax + b) + c$
Integrating exponential functions	$\int e^{ax+b} dx$ $\int a^{ax+b} \ln(a) dx$	$\frac{1}{a}e^{ax+b} + c$ $\frac{1}{a}a^{ax+b} + c$
Integrating to give logarithmic functions	$\int \frac{1}{ax + b} dx$	$\frac{1}{a}\ln ax + b + c$

Area between two curves**Example 4**

The diagram on the right shows a sketch of the curves $y = x^2 + 4$ and $y = 12 - x^2$. Find the area of the region bounded by the two curves.

Answer: Step 1: Find where the curves intersect.

$$x^2 + 4 = 12 - x^2$$

$$2x^2 = 8$$

$$x^2 = 4$$

$$x = \pm\sqrt{4}$$

$$x = \pm 2$$

Either $x = 2$ or $x = -2$.

Step 2: The area between the curves

$$= (\text{Area underneath } 12 - x^2 \text{ between } -2 \text{ and } 2) - (\text{Area underneath } x^2 + 4 \text{ between } -2 \text{ and } 2)$$

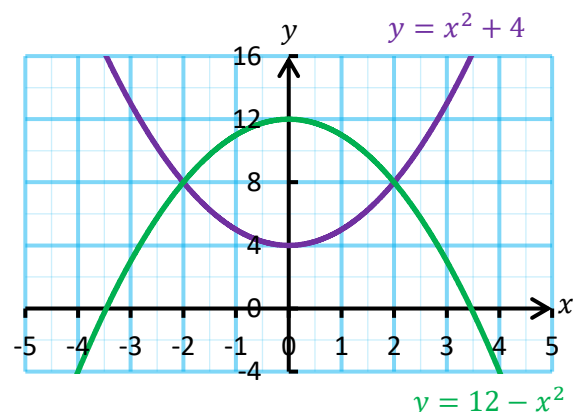
Area underneath $12 - x^2$ between -2 and 2 :

$$\begin{aligned} & \int_{-2}^2 12 - x^2 dx \\ &= \left[12x - \frac{x^3}{3} \right]_{-2}^2 \\ &= \left(12 \times 2 - \frac{2^3}{3} \right) - \left(12 \times -2 - \frac{(-2)^3}{3} \right) \\ &= \left(24 - \frac{8}{3} \right) - \left(-24 + \frac{8}{3} \right) \\ &= \frac{64}{3} - -\frac{64}{3} \\ &= \frac{128}{3} \end{aligned}$$

Area underneath $x^2 + 4$ between -2 and 2 :

$$\begin{aligned} & \int_{-2}^2 x^2 + 4 dx \\ &= \left[\frac{x^3}{3} + 4x \right]_{-2}^2 \\ &= \left(\frac{2^3}{3} + 4 \times 2 \right) - \left(\frac{(-2)^3}{3} + 4 \times -2 \right) \\ &= \left(\frac{8}{3} + 8 \right) - \left(-\frac{8}{3} - 8 \right) \\ &= \frac{32}{3} - -\frac{32}{3} \\ &= \frac{64}{3} \end{aligned}$$

So, the area between the two curves is $\frac{128}{3} - \frac{64}{3} = \frac{64}{3}$ square units.





(C3 Summer 2007)

7. (a) Find (i) $\int \frac{1}{(5-2x)} dx$, (ii) $\int (3x+2)^{20} dx$,
(iii) $\int e^{7x} dx$. [7]
- (b) Evaluate $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{3}\right) dx$. [4]

7. (a) Find

(ii) $\int \cos 2x \, dx.$ [2]

(C3 Summer 2008)

7. (a) Find (i) $\int \sin 3x \, dx$, (ii) $\int \frac{2}{3x+5} \, dx$, (iii) $\int e^{3x+4} dx$. [6]

(b) Evaluate $\int_0^1 \frac{1}{(2x+1)^4} \, dx$. [4]

(C3 Winter 2007)

7. (a) Find

(i) $\int \frac{1}{(2x+3)^5} dx$

(ii) $\int e^{2-3x} dx.$ [4]

(b) Evaluate $\int_0^2 \frac{6}{3x+2} dx$, expressing your answer as a single logarithm. [4]

(c) Evaluate $\int_0^{\frac{\pi}{4}} \cos \left(3x + \frac{\pi}{4}\right) dx.$ [4]

(C3 Winter 2010)

6. (a) Find

$$(i) \int \frac{1}{4x-7} dx, \quad (ii) \int e^{3x-1} dx, \quad (iii) \int \frac{5}{(2x+3)^4} dx. \quad [6]$$

$$(b) \text{ Evaluate } \int_0^{\frac{\pi}{4}} \sin\left(2x + \frac{\pi}{4}\right) dx, \text{ expressing your answer in surd form.} \quad [4]$$

(C3 Summer 2012)

6. Find

$$(a) \quad (i) \int 3e^{2-\frac{x}{4}} dx \quad (ii) \int \frac{9}{(2x-3)^6} dx \quad (iii) \int \frac{7}{3x+1} dx \quad [6]$$

(b) Given that $0 < a < \frac{\pi}{2}$ and that

$$\int_0^a \sin 2x \, dx = \frac{1}{4},$$

find the value of the constant a .

[5]

(C3 Summer 2019)

6. (a) Find each of the following integrals, simplifying your answer wherever possible.

(i) $\int \frac{7}{8x+1} dx$ (ii) $\int \frac{10}{(3x-2)^5} dx$ (iii) $\int \sec^2 3x dx$ [6]

- (b) Given that $\frac{\pi}{3} < a < 3$ and that

$$\int_{\frac{\pi}{3}}^a \cos \frac{1}{2}x dx = 0.92,$$

find the value of the constant a . Give your answer correct to three decimal places. [5]

(C3 Summer 2015)

7. (a) Find each of the following integrals, simplifying your answer wherever possible.

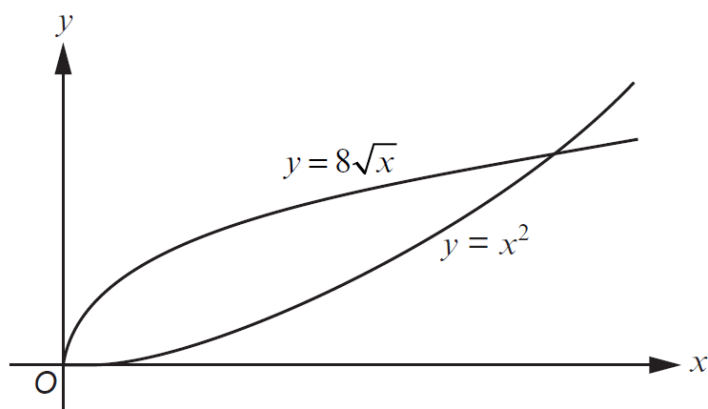
(i) $\int \frac{(7x^2 - 2)}{x} dx$

(ii) $\int \sin\left(\frac{2x}{3} - \pi\right) dx$ [5]

(b) Evaluate $\int_3^6 \frac{1}{\sqrt[4]{(5x - 14)}} dx$. [4]

(Unit 3 Summer 2022)

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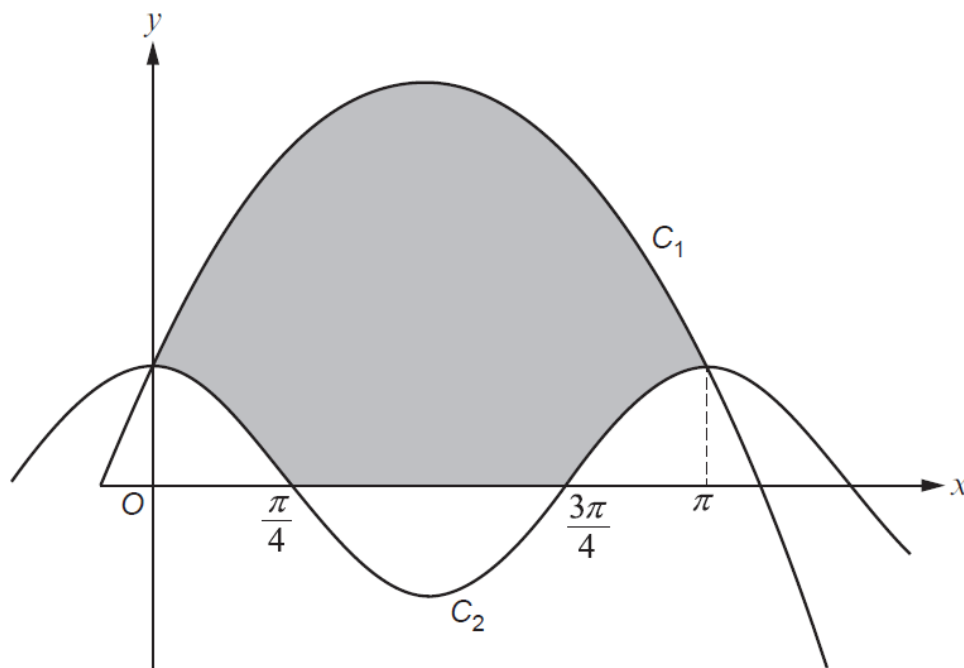
The diagram below shows a sketch of the curves $y = x^2$ and $y = 8\sqrt{x}$.

Find the area of the region bounded by the two curves.

[7]

(Unit 3 Summer 2024)

9. The diagram below shows a sketch of the curve C_1 with equation $y = -x^2 + \pi x + 1$ and a sketch of the curve C_2 with equation $y = \cos 2x$. The curves intersect at the points where $x = 0$ and $x = \pi$.



Calculate the area of the shaded region enclosed by C_1 , C_2 and the x -axis. Give your answer in terms of π . [9]

Revision Question

7. (a) Find each of the following, simplifying your answer wherever possible.

$$(i) \int e^{\frac{5x}{6}} dx, \quad (ii) \int \sqrt[3]{8x+1} dx, \quad (iii) \int \sin\left(1 - \frac{x}{3}\right) dx. \quad [6]$$

(b) Given that $a > 2$, and that

$$\int_2^a \frac{1}{4x-1} dx = 0.284,$$

find the value of the constant a . Give your answer correct to one decimal place. [5]

Handwriting practice area consisting of 20 sets of horizontal dotted lines.



Further

Integration 2

$$\int f'(g(x))g'(x) dx = f(g(x)) + c$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Name:

Background

What is the work?

Reversing the chain rule and the product rule to **integrate by substitution** and **integrate by parts**.

What is required before starting?

A Level Unit 1: Differentiation and integration.
A Level Unit 3: Further differentiation and integration.

Where does this lead to?

Integration by parts can be extended to produce reduction formulae than can integrate expressions such as $\cos^n x$.

Theory

In cases where we are integrating something that includes **a function and its derivative**, we can try to find the answer through **integration by substitution**. We show how this works using the following example.

Example 1

$$\int (24x^2 + 9)e^{8x^3+9x} dx$$

We recognise that the integral contains the function $f(x) = 8x^3 + 9x$ and its derivative $f'(x) = 24x^2 + 9$. We choose the substitution based on this function: $u = 8x^3 + 9x$.

By differentiating u , we obtain

$$\frac{du}{dx} = 24x^2 + 9$$

We can re-arrange the above to make dx the subject:

$$\begin{aligned} du &= (24x^2 + 9) dx \\ \frac{du}{24x^2 + 9} &= dx \end{aligned}$$

We now substitute u instead of $8x^3 + 9x$ and $\frac{du}{24x^2+9}$ instead of dx in the original integral:

$$\begin{aligned} &\int (24x^2 + 9)e^{8x^3+9x} dx \\ &= \int \cancel{(24x^2 + 9)} e^u \frac{du}{\cancel{24x^2 + 9}} \\ &= \int e^u du \end{aligned}$$

e^u integrates (with respect to u) to give e^u . Therefore

$$\int e^u du = e^u + c$$



Theory

To finish, we substitute back for u :

$$\int e^u du = e^{8x^3+9x} + c$$

Therefore

$$\int (24x^2 + 9)e^{8x^3+9x} dx = e^{8x^3+9x} + c$$

Exercise 1

Use integration by substitution to show that

$$\int 12x \cos(6x^2 - 4) dx = \sin(6x^2 - 4) + c$$

In general, we can **integrate by substitution** when we integrate an expression of the form $f'(g(x))g'(x)$.

$$\int f'(g(x))g'(x) dx = f(g(x)) + c$$

To show why the above is true, we can use the chain rule to differentiate $y = f(g(x)) + c$.

Let $u = g(x)$ so that $y = f(u) + c$.

By differentiating, $\frac{du}{dx} = g'(x)$ and $\frac{dy}{du} = f'(u)$.

Chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ \frac{dy}{dx} &= f'(u) \times g'(x) \\ \frac{dy}{dx} &= f'(g(x))g'(x)\end{aligned}$$

So, $\int f'(g(x))g'(x) dx = f(g(x)) + c$, and we can think of integration by substitution as the **reverse** of the chain rule.

Example 2

Evaluate $\int_0^{0.5} x \cos(3x^2 + 5) dx$.

We recognise that the integral contains the function $f(x) = 3x^2 + 5$ and a **multiple** of its derivative $f'(x) = 6x$.

We choose the substitution based on this function:

$$u = 3x^2 + 5.$$

By differentiating u , we obtain

$$\frac{du}{dx} = 6x$$

We can re-arrange the above to make dx the subject:

$$\begin{aligned} du &= 6x dx \\ \frac{du}{6x} &= dx \end{aligned}$$

We now substitute u instead of $3x^2 + 5$ and $\frac{du}{6x}$ instead of dx in the original integral:

$$\begin{aligned} &\int_0^{0.5} x \cos(3x^2 + 5) dx \\ &= \int_0^{0.5} \cancel{x} \cos(u) \frac{du}{\cancel{6x}} \\ &= \int_0^{0.5} \frac{1}{6} \cos(u) du \end{aligned}$$

Before integrating with respect to u , we must ensure that the **limits** are also in terms of u , not x . Currently, we are integrating between $x = 0$ and $x = 0.5$. Remembering that $u = 3x^2 + 5$,

$$\begin{aligned} \text{If } x = 0 \text{ then } u &= 3 \times 0^2 + 5 \\ u &= 5 \end{aligned}$$

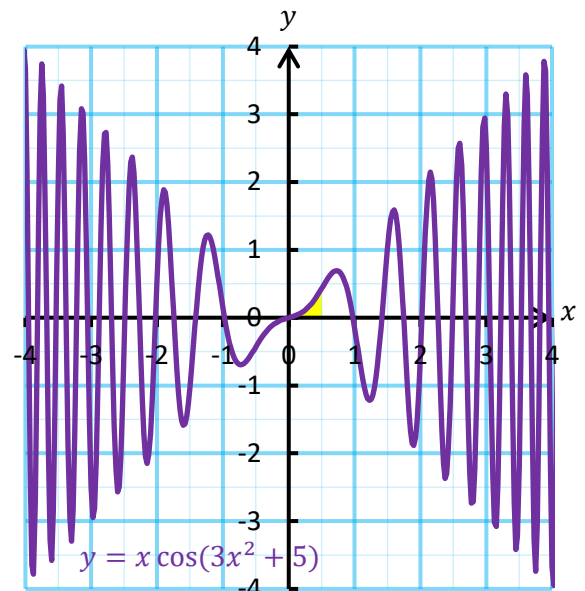
$$\begin{aligned} \text{If } x = 0.5 \text{ then } u &= 3 \times 0.5^2 + 5 \\ u &= 5.75 \end{aligned}$$

So, the integral in terms of u is

$$\int_5^{5.75} \frac{1}{6} \cos(u) du$$

$\cos(u)$ integrates (with respect to u) to give $\sin(u)$. Therefore

$$\begin{aligned} &\int_5^{5.75} \frac{1}{6} \cos(u) du \\ &= \left[\frac{1}{6} \sin(u) \right]_5^{5.75} \\ &= \left(\frac{1}{6} \times \sin(5.75) \right) - \left(\frac{1}{6} \times \sin(5) \right) \\ &= -0.08471317958 - -0.1598207124 \\ &= 0.07510753282 \\ &= 0.075 \text{ square units, to 3 decimal places} \end{aligned}$$



Exercise 2

Evaluate $\int_1^2 12x e^{3x^2-4} dx$.

Integrating $\sin^2(\theta)$ and $\cos^2(\theta)$

We remember from previous work that

$$\cos(2\theta) = 2 \cos^2(\theta) - 1$$

or

$$\cos(2\theta) = 1 - 2 \sin^2(\theta)$$

By re-arranging,

$$\frac{\cos(2\theta)+1}{2} = \cos^2(\theta)$$

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

Therefore

$$\begin{aligned} \int \cos^2(\theta) d\theta &= \int \frac{\cos(2\theta)+1}{2} d\theta \\ &= \int \frac{1}{2} \cos(2\theta) + \frac{1}{2} d\theta \\ &= \frac{1}{4} \sin(2\theta) + \frac{\theta}{2} + c \end{aligned}$$

$$\begin{aligned} \int \sin^2(\theta) d\theta &= \int \frac{1-\cos(2\theta)}{2} d\theta \\ &= \int \frac{1}{2} - \frac{1}{2} \cos(2\theta) d\theta \\ &= \frac{\theta}{2} - \frac{1}{4} \sin(2\theta) + c \end{aligned}$$

Example 3

(a) Use the substitution $x = 4 \sin(\theta)$ to show that

$$\int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{(16-x^2)}} dx = \int_0^a b \sin^2 \theta d\theta$$

where a and b are constants whose values are to be determined.

(b) **Hence**, evaluate

$$\int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{(16-x^2)}} dx$$

Give your answer in the form $c\pi + d$, where c and d are integers whose values are to be determined.

(a) If $x = 4 \sin(\theta)$ then $\frac{dx}{d\theta} = 4 \cos(\theta)$ and $x^2 = 16 \sin^2(\theta)$
 $dx = 4 \cos(\theta) d\theta$

Limits: If $x = 0$ then $0 = 4 \sin(\theta)$ If $x = 2\sqrt{2}$ then $2\sqrt{2} = 4 \sin(\theta)$
 $0 = \sin(\theta)$ $\frac{\sqrt{2}}{2} = \sin(\theta)$
 $\theta = \sin^{-1}(0)$ $\theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
 $\theta = 0$ $\theta = \frac{\pi}{4}$

Therefore

$$\begin{aligned} & \int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{(16-x^2)}} dx \\ &= \int_0^{\frac{\pi}{4}} \frac{16 \sin^2(\theta)}{\sqrt{(16-16 \sin^2(\theta))}} \times 4 \cos(\theta) d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{16 \sin^2(\theta)}{\sqrt{16(1-\sin^2(\theta))}} \times 4 \cos(\theta) d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{16 \sin^2(\theta)}{\sqrt{16 \cos^2(\theta)}} \times 4 \cos(\theta) d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{16 \sin^2(\theta)}{\cancel{4 \cos(\theta)}} \times \cancel{4 \cos(\theta)} d\theta \\ &= \int_0^{\frac{\pi}{4}} 16 \sin^2(\theta) d\theta \end{aligned}$$

So, $a = \frac{\pi}{4}$ and $b = 16$.

(b)

$$\begin{aligned} & \int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{(16-x^2)}} dx \\ &= \int_0^{\frac{\pi}{4}} 16 \sin^2(\theta) d\theta \\ &= \int_0^{\frac{\pi}{4}} 16 \left(\frac{1 - \cos(2\theta)}{2} \right) d\theta \\ &= \int_0^{\frac{\pi}{4}} 8(1 - \cos(2\theta)) d\theta \\ &= \int_0^{\frac{\pi}{4}} 8 - 8 \cos(2\theta) d\theta \\ &= [8\theta - 4 \sin(2\theta)]_0^{\frac{\pi}{4}} \\ &= \left(8 \times \frac{\pi}{4} - 4 \times \sin\left(2 \times \frac{\pi}{4}\right) \right) - (8 \times 0 - 4 \times \sin(2 \times 0)) \\ &= (2\pi - 4 \times 1) - (0 - 4 \times 0) \\ &= (2\pi - 4) - (0 - 0) \\ &= 2\pi - 4 \end{aligned}$$

$$\begin{aligned} \cos(2\theta) &= 1 - 2 \sin^2(\theta) \\ \sin^2(\theta) &= \frac{1 - \cos(2\theta)}{2} \end{aligned}$$

So, $c = 2$ and $d = -4$.

Use an appropriate substitution to show that

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx = \frac{\pi}{12} - \frac{\sqrt{3}}{8}.$$

Example 4

Use the substitution $u = 3x - 2$ to evaluate

$$\int_1^2 \frac{x}{(3x-2)^4} dx.$$

This time the x on top of the fraction will not cancel.

$$\begin{aligned} \text{If } u = 3x - 2 \quad \text{then} \quad \frac{du}{dx} &= 3 \\ du &= 3 \, dx \\ \frac{du}{3} &= dx \end{aligned}$$

If $u = 3x - 2$ then $u + 2 = 3x$
 $\frac{u+2}{3} = x$

Limits: If $x = 1$ then $u = 3 \times 1 - 2$
 $u = 1$

If $x = 2$ then $u = 3 \times 2 - 2$
 $u = 4$

Therefore

$$\begin{aligned} & \int_1^2 \frac{x}{(3x-2)^4} dx \\ &= \int_1^4 \frac{\left(\frac{u+2}{3}\right)}{u^4} \frac{du}{3} \\ &= \frac{1}{9} \int_1^4 \frac{u+2}{u^4} du \\ &= \frac{1}{9} \int_1^4 u^{-3} + 2u^{-4} du \\ &= \frac{1}{9} \left[\frac{u^{-2}}{-2} + \frac{2u^{-3}}{-3} \right]_1^4 \\ &= \frac{1}{9} \left[\left(\frac{4^{-2}}{-2} + \frac{2 \times 4^{-3}}{-3} \right) - \left(\frac{1^{-2}}{-2} + \frac{2 \times 1^{-3}}{-3} \right) \right] \\ &= \frac{1}{9} \left[\left(\frac{1}{-32} + \frac{2}{-192} \right) - \left(\frac{1}{-2} + \frac{2}{-3} \right) \right] \\ &= \frac{1}{9} \left[-\frac{1}{24} - \frac{7}{6} \right] \\ &= \frac{1}{9} \times \frac{9}{8} \\ &= \frac{1}{8} \end{aligned}$$

Exercise 4

Use the substitution $u = 2x - 3$ to evaluate

$$\int_2^3 \frac{x}{(2x-3)^3} dx.$$

Choosing an appropriate substitution

The substitution is not always given in an examination question. Here are **some** appropriate substitutions.

Integral	Substitution
$\int \sqrt{a^2 - x^2} dx$	$x = a \sin \theta$
$\int \frac{1}{\sqrt{a^2 - x^2}} dx$	$x = a \sin \theta$
$\int \frac{1}{a^2 + x^2} dx$	$x = a \tan \theta$

Otherwise, choose a substitution for u so that

- u and its *derivative* appears in the integral
- u appears in a *bracket*
- u appears in the *denominator*
- u is a function raised to the *highest exponent*

Integration by Parts

In a previous workbook on further differentiation, we saw the **product rule**, which was the technique for differentiating an expression of the form $y = uv$.

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

We can integrate the above with respect to x to give the following.

$$\int \frac{dy}{dx} dx = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$$

$$y = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$$

$$uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$$

$$uv - \int v \frac{du}{dx} dx = \int u \frac{dv}{dx} dx$$

This is given in the formula booklet.

By swapping sides, we arrive at the formula for **integrating by parts**:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

We use this technique to integrate a product made up of two parts, u and $\frac{dv}{dx}$.

As a rule, choose u as the function that appears **highest** in the following 'LIATE' list.

L	L ogarithmic functions, e.g. $\ln(3x)$
I	I nverse trigonometric functions, e.g. $\tan^{-1}(2x)$
A	A lgebraic functions (simple polynomials), e.g. $3x^2 - 5x + 8$
T	T rigonometric functions, e.g. $\sin(5x)$
E	E xponential functions, e.g. e^{7x}

Example 5

Find

$$\int 5x \cos x \, dx.$$

Answer: The integral is a product of two expressions, so we can attempt to use integration by parts to find a solution.

Let $u = 5x$ $\frac{dv}{dx} = \cos x$
 $\frac{du}{dx} = 5$ $v = \sin x$

Then

$$\begin{aligned} \int u \frac{dv}{dx} dx &= uv - \int v \frac{du}{dx} dx \\ \int 5x \cos x \, dx &= 5x \times \sin x - \int \sin x \times 5 \, dx \\ \int 5x \cos x \, dx &= 5x \sin x - \int 5 \sin x \, dx \\ \int 5x \cos x \, dx &= 5x \sin x - -5 \cos x + c \\ \int 5x \cos x \, dx &= 5x \sin x + 5 \cos x + c \\ \int 5x \cos x \, dx &= 5(x \sin x + \cos x) + c \end{aligned}$$



Theory

Exercise 5

Find

$$\int 7xe^{3x} dx.$$

Example 6

Find

$$\int x^2 e^x dx.$$

Answer: Let

$$\begin{array}{ll} u = x^2 & \frac{dv}{dx} = e^x \\ \frac{du}{dx} = 2x & v = e^x \end{array}$$

Then

$$\begin{aligned} \int u \frac{dv}{dx} dx &= uv - \int v \frac{du}{dx} dx \\ \int x^2 e^x dx &= x^2 \times e^x - \int e^x \times 2x dx \\ \int x^2 e^x dx &= x^2 e^x - \int 2xe^x dx \end{aligned}$$

Now for $\int 2xe^x dx$ we must integrate by parts again.

Let

$$\begin{array}{ll} u = 2x & \frac{dv}{dx} = e^x \\ \frac{du}{dx} = 2 & v = e^x \end{array}$$

Then

$$\begin{aligned} \int u \frac{dv}{dx} dx &= uv - \int v \frac{du}{dx} dx \\ \int 2x e^x dx &= 2x \times e^x - \int e^x \times 2 dx \end{aligned}$$

We can continue to add the constant of integration (Why?)

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - (2x e^x - 2e^x + c) \\ \int x^2 e^x dx &= x^2 e^x - 2x e^x + 2e^x + c \\ \int x^2 e^x dx &= e^x(x^2 - 2x + 2) + c\end{aligned}$$

Find

$$\int e^x \sin x \, dx.$$

Top 10

(C4 Summer 2010)

7. (a) Find $\int x^3 \ln x \, dx$.

[4]

(b) Use the substitution $u = 2x - 3$ to evaluate $\int_1^2 x(2x - 3)^4 \, dx$.

[5]

7. (a) Use the substitution $u = 2x - 1$ to evaluate

(b) (i) Find $\int x \cos 2x \, dx$. [4]

$$\int x \cos^2 x \, dx \quad . \quad [3]$$

Handwriting practice area with 30 horizontal dotted lines.

7. (a) Find $\int x \sin 2x \, dx$. [4]

$$\int_0^2 \frac{x}{(5-x^2)^3} dx. \quad [4]$$

7. (a) Find $\int x e^{-2x} dx$. [4]

$$\int_1^e \frac{1}{x(1+3\ln x)} dx.$$

Give your answer correct to four decimal places. [4]

6. (a) Find $\int (2x + 1)e^{-3x} dx$. [4]

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{4+5\tan x}}{\cos^2 x} dx. \quad [4]$$

7. (a) Find $\int x^4 \ln 2x \, dx$. [4]

$$\int_0^{\frac{\pi}{3}} \sqrt{(10 \cos x - 1)} \sin x \, dx. \quad [4]$$

(C4 Summer 2008)

6. (a) Find $\int (3x + 1) e^{2x} dx$. [4]

(b) Use the substitution $x = 3\sin\theta$ to show that

$$\int_{1.5}^3 \sqrt{9 - x^2} dx = \int_a^b k \cos^2 \theta d\theta \quad ,$$

where the values of the constants a , b and k are to be found.

Hence evaluate $\int_{1.5}^3 \sqrt{9 - x^2} dx$. [8]

Handwriting practice lines consisting of 30 horizontal dotted lines.

(C4 Summer 2007)

7. (a) Find $\int x^2 \ln x \, dx$. [4]

(b) Use the substitution $x = 2\sin\theta$ to show that

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} \, dx = \int_0^a k \sin^2 \theta \, d\theta ,$$

where the values of a and k are to be determined.

Hence, or otherwise, evaluate $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} \, dx$. [8]

1	8
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 a) Use a suitable substitution to find

$$\int \frac{x^2}{(x+3)^4} dx. \quad [5]$$

b) Hence evaluate $\int_0^1 \frac{x^2}{(x+3)^4} dx$. [2]

(Unit 3 Summer 2018)

1	4
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 Evaluate

a) $\int_1^2 x^3 \ln x \, dx$. [6]

b) $\int_0^1 \frac{2+x}{\sqrt{4-x^2}} \, dx$. [6]

Revision Question

(Unit 3 Summer 2019)

1	4
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 a) Find $\int (e^{2x} + 6\sin 3x) dx$. [2]

b) Find $\int 7(x^2 + \sin x)^6(2x + \cos x) dx$. [1]

c) Find $\int \frac{1}{x^2} \ln x \, dx$. [4]

d) Use the substitution $u = 2\cos x + 1$ to evaluate

$$\int_0^{\frac{\pi}{3}} \frac{\sin x}{(2 \cos x + 1)^2} dx. \quad [4]$$



Differential

Equations



Name:

Background

What is the work?

Solving **first order differential equations**, which are equations containing a function and its derivative.

What is required before starting?

A Level Unit 1: Differentiation and integration; logarithms.
A Level Unit 3: Further integration.

Where does this lead to?

Further Mathematics: solving second order differential equations.
Applications: Modelling electrical circuits; forces on beams.

Theory

A **differential equation** connects a function y to its derivative $\frac{dy}{dx}$. By separating the variables and integrating, we can find an equation that connects x and y .

Example (C4 Summer 2005)

The size P of a population of bacteria at time t days is to be modelled as a continuous variable such that the rate of increase of P is directly proportional to P .

- (a) Write down a differential equation that is satisfied by P .
- (b) Given that the initial size of the population is P_0 , show that $P = P_0 e^{kt}$, where k is a positive constant.
- (c) Two days after the start, the population is $1.2P_0$. Find when the population will be $2P_0$.

Answer: (a) If the rate of increase of P at time t days is directly proportional to P , then

$$\begin{aligned}\frac{dP}{dt} &\propto P \\ \frac{dP}{dt} &= kP\end{aligned}$$

- (b) Separating the variables in the differential equation above,

$$\begin{aligned}dP &= kP dt \\ \frac{1}{P} dP &= k dt\end{aligned}$$

Integrating,

$$\begin{aligned}\int \frac{1}{P} dP &= \int k dt \\ \ln|P| &= kt + c\end{aligned}$$

The initial size of the population is P_0 , so $P = P_0$ when $t = 0$.

$$\begin{aligned}\ln|P_0| &= k(0) + c \\ \ln|P_0| &= c\end{aligned}$$



Theory

Therefore

$$\begin{aligned}\ln|P| &= kt + \ln|P_0| \\ \ln|P| - \ln|P_0| &= kt \\ \ln\left|\frac{P}{P_0}\right| &= kt \\ \frac{P}{P_0} &= e^{kt} \\ P &= P_0 e^{kt}\end{aligned}$$

(c) If $t = 2$, $P = 1.2P_0$. Therefore

$$\begin{aligned}1.2P_0 &= P_0 e^{k(2)} \\ 1.2 &= e^{2k} \\ \ln(1.2) &= 2k \\ \frac{1}{2}\ln(1.2) &= k\end{aligned}$$

Therefore

$$P = P_0 e^{\frac{1}{2}\ln(1.2)t}$$

If $P = 2P_0$,

$$\begin{aligned}2P_0 &= P_0 e^{\frac{1}{2}\ln(1.2)t} \\ 2 &= e^{\frac{1}{2}\ln(1.2)t} \\ \ln(2) &= \frac{1}{2}\ln(1.2)t \\ \frac{2\ln(2)}{\ln(1.2)} &= t \\ t &= 7.60 \text{ days, to 2 decimal places.}\end{aligned}$$

Establishing a Differential Equation

Exercise I

Complete the following table.

Description	Proportional Relationship	Differential Equation
The rate of increase of P at time t is directly proportional to P^2 .	$\frac{dP}{dt} \propto P^2$	$\frac{dP}{dt} = kP^2$
The rate of increase of P at time t is inversely proportional to P^2 .	$\frac{dP}{dt} \propto \frac{1}{P^2}$	$\frac{dP}{dt} = \frac{k}{P^2}$
The rate of decrease of P at time t is directly proportional to P^2 .	$\frac{dP}{dt} \propto -P^2$	$\frac{dP}{dt} = -kP^2$
The rate of decrease of P at time t is inversely proportional to P^2 .	$\frac{dP}{dt} \propto -\frac{1}{P^2}$	$\frac{dP}{dt} = -\frac{k}{P^2}$
The rate of increase of P at time t is directly proportional to P^3 .		
The rate of decrease of P at time t is inversely proportional to P .		
The rate of increase of P at time t is inversely proportional to P^4 .		
The rate of decrease of P at time t is directly proportional to $\sqrt{P}$.		

Top 10

8. Water leaks from a hole at the bottom of a large water tank. The depth of the water at time t minutes is x metres. The rate of decrease of x is directly proportional to $\sqrt{x}$.

- $$kt = 6 - 2\sqrt{x} \quad ,$$

(c) Given that the depth of water in the tank is 4 metres when $t = 20$, find the time taken for the tank to empty. [3]

Handwriting practice area consisting of 30 horizontal dotted lines.

8. The price $\pounds P$ of an item at time t years is to be modelled as a continuous variable such that the rate of increase of P is directly proportional to P .

7. The value of an electronic component may be modelled as a continuous variable. The value of the component at time t years is $\text{£}P$. The rate of decrease of P is directly proportional to P^3 .

- (b) The value of the component when $t = 0$ is £20. Show that

where A is a positive constant. [5]

- (c) Given that the value of the component when $t = 1$ is £10, find the time when the value is £5. [4]

8. The value, £ V , of a car may be modelled as a continuous variable. At time t years, the rate of decrease of V is directly proportional to V^2 .

- (b) Given that $V = 12000$ when $t = 0$, show that

where a is a constant. [4]

- (c) The value of the car at the end of two years is £9000. Find the value of the car at the end of four years. [4]

Handwriting practice area consisting of 30 horizontal dotted lines.

8. Part of the surface of a small lake is covered by green algae. The area of the lake covered by the algae at time t years is $A \text{ m}^2$. The rate of increase of A is directly proportional to $\sqrt{A}$.

- (a) Write down a differential equation satisfied by A . [1]
- (b) The area of the lake covered by the algae at time $t = 3$ is 64m^2 and the area covered at time $t = 5.5$ is 196m^2 . Find an expression for A in terms of t . [6]

Handwriting practice lines consisting of 40 horizontal dotted lines.

9. A bookseller values a rare book at $\pounds A$ on August 1st 2010. The value, $\pounds P$, of the book t years after this date may be modelled as a continuous variable. The rate of increase of P may be assumed to be directly proportional to P^2 .

- (b) Show that

$$\frac{1}{k} \left(\frac{P - A}{PA} \right) = t,$$

(c) The value of the book is £800 on August 1st 2013 and £900 on August 1st 2014. Find the value of A . [3]

Handwriting practice lines consisting of 30 sets of three horizontal dotted lines.

7. The value, $\text{£}V$, of a particular car may be modelled as a continuous variable. At time t years, the rate of decrease of V is directly proportional to V^3 .

- (b) Given that the initial value of the car is £ A , show that

where b is a constant. [4]

- (c) When $t = 2$, the value of the car has fallen to a half of its initial value. Find the value of t when the value of the car will have fallen to a quarter of its initial value. [4]

(C4 Summer 2018)

8. The value of a painting on January 1st 2000 was £900. The value, £ V , of the painting t years after this date may be modelled as a continuous variable. The rate of increase of V may be assumed to be directly proportional to $V^{\frac{3}{2}}$.

(a) Write down a differential equation satisfied by V . [1]

(b) The value of the painting on January 1st 2003 was £1600. Find its value on January 1st 2008. [8]

1 **5** The variable y satisfies the differential equation

Given that $y = 1$ when $x = 0$, find an expression for y in terms of x . [5]

1	3
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a) Write down a differential equation that is satisfied by A . [1]

Revision Question

(Unit 3 Summer 2023)

1	2
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The rate of change of a variable y with respect to x is directly proportional to y .

- a) Write down a differential equation satisfied by y . [1]
- b) When $x = 1$ and $y = 0.5$, the rate of change of y with respect to x is 2. Find y when $x = 3$. [6]

Handwriting practice area consisting of 30 horizontal dotted lines.



Partial

Fractions

Rational Function	Partial Fractions
$\frac{px + q}{(ax + b)(cx + d)}$	$\frac{A}{ax + b} + \frac{B}{cx + d}$
$\frac{px + q}{(ax + b)^2}$	$\frac{A}{ax + b} + \frac{B}{(ax + b)^2}$
$\frac{px^2 + qx + r}{(ax + b)(cx + d)(ex + f)}$	$\frac{A}{ax + b} + \frac{B}{cx + d} + \frac{C}{ex + f}$
$\frac{px^2 + qx + r}{(ax + b)^2(cx + d)}$	$\frac{A}{ax + b} + \frac{B}{(ax + b)^2} + \frac{C}{cx + d}$

Name:

Background

What is the work?

Separating an algebraic fraction into a fraction sum with linear denominators. Using this work to integrate algebraic fractions.

What is required before starting?

A Level Unit 1: Differentiation and integration.
A Level Unit 3: Further integration.

Where does this lead to?

Applications: Reducing the noise in an audio clip; compressing computer files; studying how a drug decays in the body.

Theory

Simplifying Rational Expressions

Before using the techniques for partial fractions that follow, it is important to check whether an algebraic expression can be simplified using previous techniques.

Example 1

Simplify $\frac{15x+20}{(3x+4)(2x-3)}$.

$$\text{Answer: } \frac{15x+20}{(3x+4)(2x-3)} = \frac{5(3x+4)}{(3x+4)(2x-3)} = \frac{5}{(2x-3)}$$

Example 3

Simplify $\frac{6x^3-5x^2+2x+3}{2x+1}$.

Answer: By using a division frame,

$$\begin{array}{r} 3x^2 - 4x + 3 \\ 2x + 1 \overline{) 6x^3 - 5x^2 + 2x + 3} \\ \underline{6x^3 + 3x^2} \\ -8x^2 + 2x + 3 \\ \underline{-8x^2 - 4x} \\ 6x + 3 \\ \underline{6x + 3} \\ 0 \end{array}$$

$$\text{So, } \frac{6x^3-5x^2+2x+3}{2x+1} = 3x^2 - 4x + 3.$$

Example 2

Simplify $\frac{x^2+8x+15}{x^2+2x-15}$.

$$\text{Answer: } \frac{x^2+8x+15}{x^2+2x-15} = \frac{(x+5)(x+3)}{(x+5)(x-3)} = \frac{x+3}{x-3}$$

Example 4

Simplify $\frac{6x^3-5x^2+2x+5}{2x+1}$.

Answer: By using a division frame,

$$\begin{array}{r} 3x^2 - 4x + 3 \\ 2x + 1 \overline{) 6x^3 - 5x^2 + 2x + 5} \\ \underline{6x^3 + 3x^2} \\ -8x^2 + 2x + 5 \\ \underline{-8x^2 - 4x} \\ 6x + 5 \\ \underline{6x + 3} \\ 2 \end{array}$$

$$\text{So, } \frac{6x^3-5x^2+2x+5}{2x+1} = 3x^2 - 4x + 3 + \frac{2}{2x+1}.$$



Exercise 2

Simplify $\frac{x^2+7x+12}{x^2-2x-15}$.

Exercise 4

Simplify $\frac{6x^3+19x^2+37x+15}{3x+2}$.

Exercise 6

Solve the equation $\frac{6x^6+11x^5-4x^4-4x^3}{2x+1} = 0$.

Example 5

Express the fraction $\frac{7x-10}{(x+2)(x-6)}$ in terms of partial fractions.

Answer: It is possible to write the fraction in the form $\frac{A}{x+2} + \frac{B}{x-6}$, where A and B are numbers to be found.

$$\frac{7x-10}{(x+2)(x-6)} \equiv \frac{A}{x+2} + \frac{B}{x-6}$$

$$7x-10 \equiv \frac{A(x+2)(x-6)}{x+2} + \frac{B(x+2)(x-6)}{x-6}$$

$$7x-10 \equiv A(x-6) + B(x+2)$$

Multiply by $(x+2)(x-6)$

- Substitute values into the identity, usually values that eliminate some of the terms;
- Compare coefficients, e.g. the coefficients of x .

$$\begin{aligned} 7(6) - 10 &\equiv A(6 - 6) + B(6 + 2) \\ 42 - 10 &\equiv A(0) + B(8) \\ 32 &\equiv 8B \\ B &\equiv 4 \end{aligned}$$
$$\begin{aligned} 7(-2) - 10 &\equiv A(-2 - 6) + B(-2 + 2) \\ -14 - 10 &\equiv A(-8) + B(0) \\ -24 &\equiv -8A \\ A &\equiv 3 \end{aligned}$$

$$\text{So, } \frac{2x}{(x+2)(x-6)} = \frac{3}{x+2} + \frac{4}{x-6}.$$

Exercise 7

Express the fraction $\frac{13x-1}{(x+3)(x-7)}$ in terms of partial fractions.

If the denominator contains a squared factor, we must consider the factor both individually and squared.

Example 6 (C4 Summer 2005)

(a) Express the fraction $\frac{8x^2+x-5}{(2x-1)^2(x+2)}$ in terms of partial fractions.

(b) Find $\int \frac{8x^2+x-5}{(2x-1)^2(x+2)} dx$.

Answer: (a) It is possible to write the fraction in the form $\frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x+2}$, where A , B and C are numbers to be found.

$$\begin{aligned}\frac{8x^2+x-5}{(2x-1)^2(x+2)} &\equiv \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x+2} \\ 8x^2+x-5 &\equiv \frac{A(2x-1)^2(x+2)}{2x-1} + \frac{B(2x-1)^2(x+2)}{(2x-1)^2} + \frac{C(2x-1)^2(x+2)}{x+2} \\ 8x^2+x-5 &\equiv A(2x-1)(x+2) + B(x+2) + C(2x-1)^2\end{aligned}$$

Multiply by
 $(2x-1)^2(x+2)$

By substituting $x = -2$ into the identity, the terms containing A and B will disappear, and we can calculate the value of C .

$$\begin{aligned}8(-2)^2 + (-2) - 5 &\equiv A(2(-2) - 1)(-2 + 2) + B(-2 + 2) + C(2(-2) - 1)^2 \\ 32 - 2 - 5 &\equiv A(0) + B(0) + C(25) \\ 25 &\equiv 25C \\ C &\equiv 1\end{aligned}$$

By substituting $x = 0.5$ into the identity, the terms containing A and C will disappear, and we can calculate the value of B .

$$\begin{aligned}8(0.5)^2 + 0.5 - 5 &\equiv A(2(0.5) - 1)(0.5 + 2) + B(0.5 + 2) + C(2(0.5) - 1)^2 \\ 2 + 0.5 - 5 &\equiv A(0) + B(2.5) + C(0) \\ -2.5 &\equiv 2.5B \\ B &\equiv -1\end{aligned}$$

$$\begin{aligned}(2x-1)(x+2) &= \\ 2x^2 + 3x - 2, \\ (2x-1)^2 &= \\ 4x^2 - 4x + 1\end{aligned}$$

By comparing the coefficients of x^2 , we can calculate the value of A .

$$\begin{aligned}8 &\equiv 2A + 0B + 4C \\ 8 &\equiv 2A + 4(1) \\ 4 &\equiv 2A \\ A &\equiv 2\end{aligned}$$

$$\text{So, } \frac{8x^2+x-5}{(2x-1)^2(x+2)} = \frac{2}{2x-1} - \frac{1}{(2x-1)^2} + \frac{1}{x+2}.$$

$$\begin{aligned}\text{(b) } \int \frac{8x^2+x-5}{(2x-1)^2(x+2)} dx &= \int \frac{2}{2x-1} - \frac{1}{(2x-1)^2} + \frac{1}{x+2} dx \\ &= \int \frac{2}{2x-1} dx - \int (2x-1)^{-2} dx + \int \frac{1}{x+2} dx \\ &= \ln|2x-1| + \frac{1}{2}(2x-1)^{-1} + \ln|x+2| + c \\ &= \ln|(2x-1)(x+2)| + \frac{1}{2(2x-1)} + c\end{aligned}$$

Which other
methods could be
used to find the
value of A ?

Top 10

1. Given that $f(x) = \frac{x^2 + x + 13}{(x+2)^2(x-3)}$,

- $$\int_6^7 f(x)dx,$$

Handwriting practice area with 30 horizontal dotted lines.

1. Given that $f(x) = \frac{3x^2 - 3x - 8}{x(x-2)^2}$,

- (b) evaluate

$$\int_6^9 f(x) \, dx,$$

giving your answer correct to two decimal places. [3]

(C4 Summer 2007)

1. (a) Express $\frac{x+3}{x^2(x-1)}$ in terms of partial fractions. [4]

(b) Find $\int \frac{x+3}{x^2(x-1)} dx$. [2]

Handwriting practice area with 30 horizontal dotted lines.

1. The function f is defined by

$$f(x) = \frac{17 + 4x - x^2}{(2x - 1)(x - 3)^2}.$$

- (a) Express $f(x)$ in terms of partial fractions. [4]
- (b) Use your result to part (a) to find an expression for $f'(x)$. [2]

1. The function f is defined by

$$f(x) = \frac{8 - x - x^2}{x(x - 2)^2}.$$

- (b) Use your result to part (a) to find the value of $f'(1)$. [3]

Handwriting practice area consisting of 30 horizontal dotted lines.

1. The function f is defined by

$$f(x) = \frac{6 + x - 9x^2}{x^2(x + 2)}.$$

- (a) Express $f(x)$ in terms of partial fractions. [4]
- (b) Using your result to part (a),
 - (i) find an expression for $f'(x)$,
 - (ii) verify that $f(x)$ has a stationary value when $x = 2$. [3]

(C4 Summer 2014)

2. (a) Express $\frac{5x^2 + 7x + 17}{(x+1)^2(x-4)}$ in terms of partial fractions. [4]

(b) Use your answer to part (a) to express $\frac{5x^2 + 9x + 9}{(x+1)^2(x-4)}$ in terms of partial fractions. [2]

Handwriting practice area consisting of multiple horizontal dotted lines.

1. (a) Express $\frac{8x^2 + 7x - 25}{(x-1)^2(x+4)}$ in terms of partial fractions. [4]

(b) Use your result to part (a) to express $\frac{9x^2 + 5x - 24}{(x-1)^2(x+4)}$ in terms of partial fractions. [3]

(Unit 3 Summer 2022)

1	0
---	---

Solve the equation

$$\frac{6x^5 - 17x^4 - 5x^3 + 6x^2}{(3x + 2)} = 0.$$

[5]

0	8
---	---

 The function f is defined by $f(x) = \frac{4x^2 + 12x + 9}{2x^2 + x - 3}$, where $x > 1$.

- b)** Hence find the exact value of $\int_3^7 f(x)dx$. [4]

Handwriting practice area consisting of 30 sets of horizontal dotted lines.

Revision Question

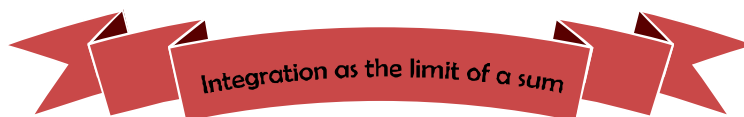
1. The function f is given by

$$f(x) = \frac{25x + 32}{(2x - 5)(x + 1)(x + 2)}.$$

[4]

- (b) Show that $\int_1^2 f(x) dx = -\ln P$, where P is an integer whose value is to be found. [5]

- (c) Show that the sign of $f(x)$ changes in the interval $x = 2$ to $x = 3$. Explain why the change of sign method fails to locate a root of the equation $f(x) = 0$ in this case. [2]



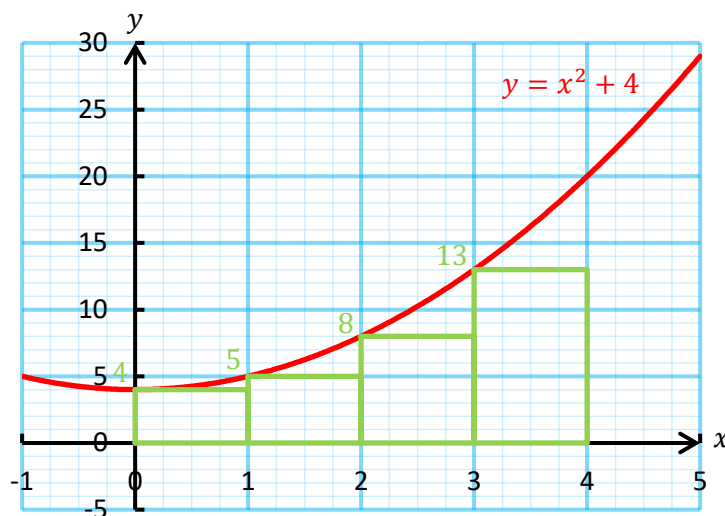
We can use integration to find the area bounded by the curve $y = f(x)$, the x -axis, the line $x = a$ and the line $x = b$.

$$\int_a^b y \, dx$$

It would be possible to estimate this area by splitting it into rectangles of constant width δx .

Example 7

The following diagram shows the curve $y = x^2 + 4$ between $x = -1$ and $x = 5$.



We can estimate the value of $\int_0^4 x^2 + 4 \, dx$ by using the green rectangles above, all of width 1 unit. The height of each rectangle is the value of the function to the left of the rectangle.

$$\begin{aligned} \int_0^4 x^2 + 4 \, dx &\approx (1 \times 4) + (1 \times 5) + (1 \times 8) + (1 \times 13) \\ \int_0^4 x^2 + 4 \, dx &\approx 30 \end{aligned}$$

We see from the diagram that the estimate is an underestimate, as the red curve lies above the rectangle. In reality, we have

$$\begin{aligned} \int_0^4 x^2 + 4 \, dx &= \left[\frac{x^3}{3} + 4x \right]_0^4 \\ \int_0^4 x^2 + 4 \, dx &= \left[\frac{4^3}{3} + 4 \times 4 \right] - \left[\frac{0^3}{3} + 4 \times 0 \right] \\ \int_0^4 x^2 + 4 \, dx &= \frac{112}{3} \end{aligned}$$

So, our estimate is $\frac{112}{3} - 30 = \frac{22}{3}$ square units too small. It would be possible to improve the estimate by using more rectangles, of smaller width.

In general, if we have n rectangles, each of width δx , then the estimate of the area A is

$$A \approx y_1 \delta x + y_2 \delta x + \cdots + y_n \delta x$$

We can write the above sum using sigma notation:

$$A \approx \sum_{i=1}^n y_i \delta x$$

If we increase the number n of rectangles, the width δx of each rectangle decreases, and the sum gives a better estimate of the area. Taking the limit,

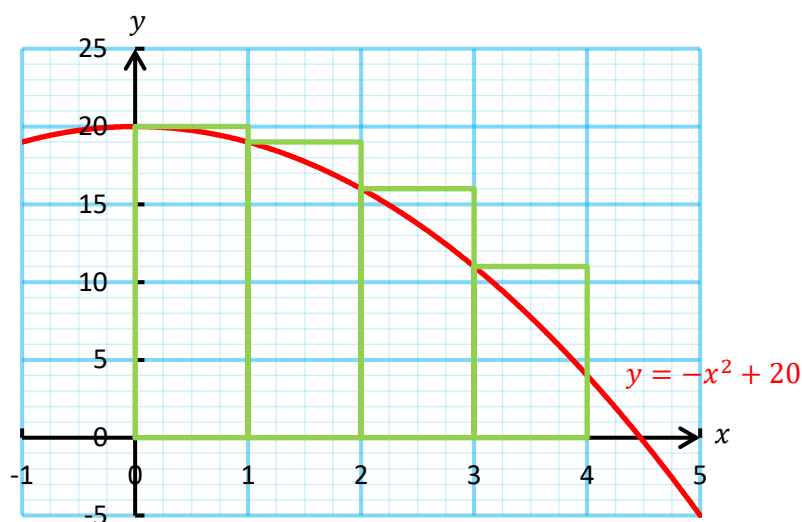
$$\lim_{n \rightarrow \infty} \sum_{i=1}^n y_i \delta x = \int_a^b y \, dx$$

Because the values of y_i come from the function $y = f(x)$, we can write

$$\lim_{\delta x \rightarrow 0} \sum_{x=a}^b f(x) \delta x = \int_a^b y \, dx$$

Exercise 8

The following diagram shows the curve $y = -x^2 + 20$ between $x = -1$ and $x = 5$.



Find an estimate for $\int_0^4 -x^2 + 20 \, dx$ using the four green rectangles in the diagram. Is your estimate an underestimate or an overestimate?



Proof by

Contradiction

“Let us prove that
this is true...”



Name:

Background

What is the work?

Proving that something is true by showing that the opposite cannot be true.

What is required before starting?

GCSE: Solving equations and inequalities.
A Level Unit 1: Proof.

Where does this lead to?

Applications: Showing that there is an infinite number of prime numbers; showing that $\sqrt{2}$ cannot be written in the form $\frac{a}{b}$.

Theory



Theory

Proof by contradiction

We can show that something is true by showing that the opposite can never be true.

Example 1 (C4 Summer 2005)

Complete the following proof by contradiction to show that $x + \frac{25}{x} \geq 10$ when x is real and positive.

Assume that $x + \frac{25}{x} < 10$, where x is real and positive.

Since x is positive, multiplication of both sides of the inequality by x gives $x^2 + 25 < 10x$.

Answer: By subtracting $10x$ from both sides, we obtain $x^2 - 10x + 25 < 0$.

By factorising, $(x - 5)(x - 5) < 0$.
 $(x - 5)^2 < 0$.

Squaring any real number gives a number that is greater than or equal to 0, so there is no real and positive value of x such that $(x - 5)^2$ is less than zero. This contradicts the assumption that x is real, so we must have $x + \frac{25}{x} \geq 10$ if x is real and positive.

Example 2 (C4 Summer 2006)

Complete the following proof by contradiction to show that $\sqrt{2}$ is irrational.

Assume that $\sqrt{2}$ is rational. Then $\sqrt{2}$ may be written in the form $\frac{a}{b}$, where a and b are positive integers having no common factor.

$\therefore a^2 = 2b^2$.
 $\therefore a^2$ has a factor of 2.
 $\therefore a$ has a factor of 2 so that $a = 2k$, where k is an integer.

$\therefore$ is 'therefore'

Answer: Substituting $a = 2k$ into $a^2 = 2b^2$, we obtain

$(2k)^2 = 2b^2$.
 $4k^2 = 2b^2$.
 $2k^2 = b^2$.

$\therefore b^2$ has a factor of 2.

$\therefore b$ has a factor of 2.

If 2 is a factor of both a and b , then this contradicts the earlier assumption that a and b were positive integers having **no** common factors. (We have found a common factor, namely 2.)

So, $\sqrt{2}$ is not rational.

Example 3

Use proof by contradiction to show that there is an infinite number of prime numbers.

Answer: Assume, to the contrary, that a finite number n of prime numbers exist.

We can list the prime numbers like this: $2, 3, 5, 7, \dots, p_n$.

Let x be the product of all of these prime numbers, and then add one.

$$x = (2 \times 3 \times 5 \times 7 \times \dots \times p_n) + 1.$$

2 is not a factor of x as it has a remainder of 1 when we divide x by 2.

We can use the same argument to show that all the other prime numbers, up to p_n , are also not factors of x .

So, x must be a prime number that is greater than p_n .

This contradicts the earlier assumption that the greatest prime number is p_n .

So, there must be an infinite number of prime numbers.

Example 4 (C4 Summer 2014)

Complete the following proof by contradiction to show that

$$\sin \theta + \cos \theta \leq \sqrt{2}$$

for all values of θ .

Assume that there is a value of θ for which $\sin \theta + \cos \theta > \sqrt{2}$.

Then squaring both sides, we have:

Answer: $(\sin \theta + \cos \theta)^2 > 2$

$$(\sin \theta + \cos \theta)(\sin \theta + \cos \theta) > 2$$

$$\sin^2 \theta + \sin \theta \cos \theta + \cos \theta \sin \theta + \cos^2 \theta > 2$$

$$\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta > 2$$

$$\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta > 2$$

$$1 + 2 \sin \theta \cos \theta > 2$$

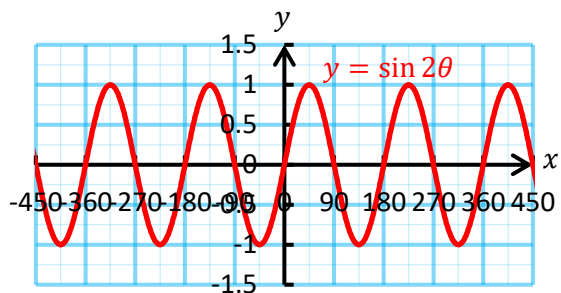
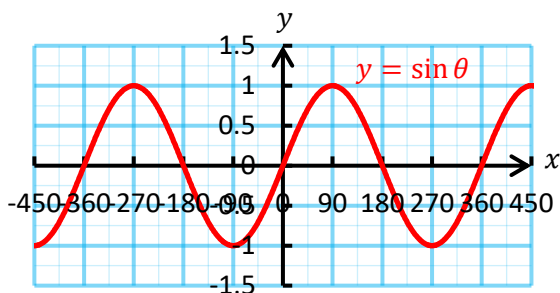
$$(using \sin^2 \theta + \cos^2 \theta = 1)$$

$$2 \sin \theta \cos \theta > 1$$

$$\sin 2\theta > 1$$

$$(using 2 \sin \theta \cos \theta = \sin 2\theta)$$

The graph of $\sin \theta$ varies between -1 and 1 , so the graph of $\sin 2\theta$ will also vary between -1 and 1 .



It is therefore impossible to find a value of θ so that $\sin 2\theta > 1$.

This contradicts the earlier assumption that $\sin \theta + \cos \theta > \sqrt{2}$.

So, we must have $\sin \theta + \cos \theta \leq \sqrt{2}$ for all values of θ .



10. Prove by contradiction the following proposition.

$$x + \frac{49}{x} \geq 14 .$$

Assume that there is a positive and real value of x such that

$$x + \frac{49}{x} < 14 \quad . \quad [4]$$

(C4 Summer 2010)

10. Prove by contradiction the following proposition.

If a, b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

The first line of the proof is given below.

Assume that positive real numbers a, b exist such that $a + b < 2\sqrt{ab}$.

[3]

10. Prove by contradiction the following proposition.

$$(5x - 3)^2 + 1 \geq (3x - 1)^2.$$

Assume that there is a real value of x such that

$$(5x-3)^2 + I < (3x-1)^2. \quad [3]$$

5. Prove by contradiction the following proposition:

(C4 Summer 2009)

10. Complete the following proof by contradiction to show that $\sqrt{3}$ is irrational.

Assume that $\sqrt{3}$ is rational. Then $\sqrt{3}$ may be written in the form $\frac{a}{b}$ where a and b are integers having no common factors.

$$\therefore a^2 = 3b^2.$$

$$\therefore a^2 \text{ has a factor } 3.$$

$$\therefore a \text{ has a factor } 3 \text{ so that } a = 3k, \text{ where } k \text{ is an integer.} \quad [4]$$

1	5
---	---

[5]

1	1
---	---

 Prove by contradiction that, for every real number x such that $0 \leq x \leq \frac{\pi}{2}$,

$$\sin x + \cos x \geq 1. \quad [4]$$

10. Complete the following proof by contradiction to show that, if n is a positive integer and $3n + 2n^3$ is odd, then n is odd. [2]

Assume that n is even so that $n = 2k$.

(C4 Summer 2015)

10. Prove by contradiction the following proposition.

If a and b are odd integers such that 4 is a factor of $a - b$,
then 4 is **not** a factor of $a + b$.

The first lines of the proof are given below.

Assume that 4 is a factor of $a + b$.

Then there exists an integer c such that $a + b = 4c$.

[3]

1	7
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 a) Prove that

b) i) Hence show that $\frac{\cos 3\theta + \sin 5\theta}{\cos 4\theta + \sin 4\theta}$ can be expressed as $\cos \theta + \sin \theta$.

ii) Explain why $\frac{\cos 3\theta + \sin 5\theta}{\cos 4\theta + \sin 4\theta} \neq \cos \theta + \sin \theta$ when $\theta = \frac{3\pi}{16}$. [3]



(C4 Summer 2017)

10. Complete the following proof by contradiction to show that $\sqrt{7}$ is irrational.

Assume that $\sqrt{7}$ is rational. Then $\sqrt{7}$ may be written in the form $\frac{a}{b}$,

where a, b are integers having no factors in common.

$\therefore a^2 = 7b^2$.

$\therefore a^2$ has a factor 7.

$\therefore a$ has a factor 7 so that $a = 7k$, where k is an integer.

[3]

10. Prove by contradiction the following proposition.

$$\left|4x + \frac{1}{x}\right| \geq 4.$$

Assume that there is a real value of x such that

$$\left| 4x + \frac{1}{x} \right| < 4.$$

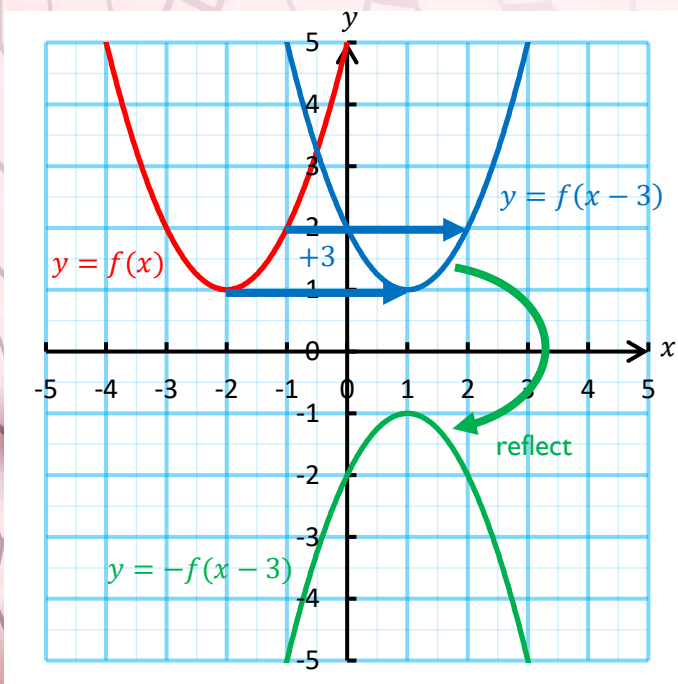
[3]



Combining

Transformations of

Graphs



Name: _____

Background

What is the work?

Combining the transformations of graphs from Unit 1.

What is required before starting?

GCSE: Plotting graphs; transformations of graphs.
A Level Unit 1: Transformations of graphs.

Where does this lead to?

Applications: Unit conversion (e.g. temperature in Celsius, Fahrenheit and Kelvin); simple harmonic motion.

Theory

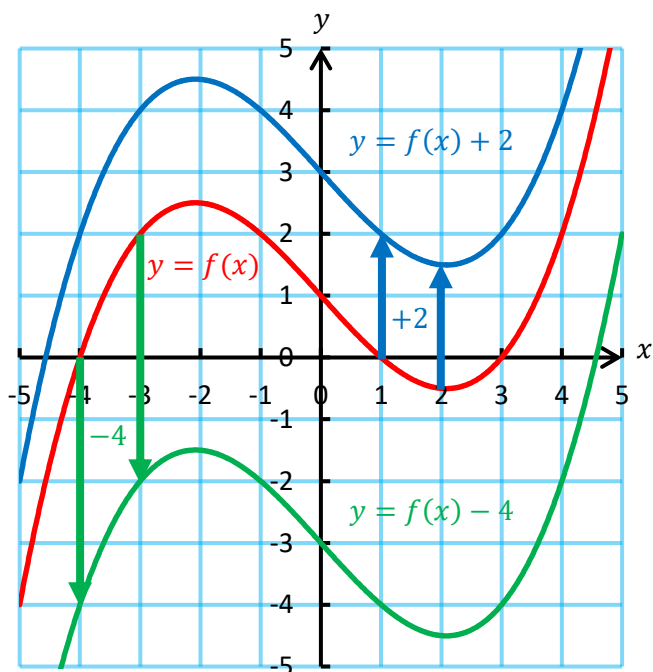
The following theory comes from the GCSE notes, or the Unit 1 notes.
 The questions in this workbook will **combine** the parts that follow.



Transformations of Graphs

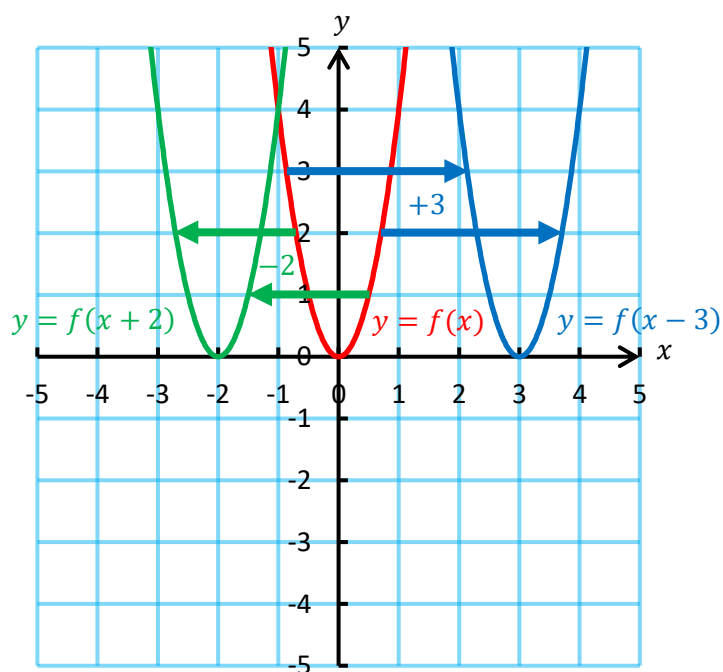
We can transform the function $y = f(x)$ using the following techniques.

$$y = f(x) + a$$



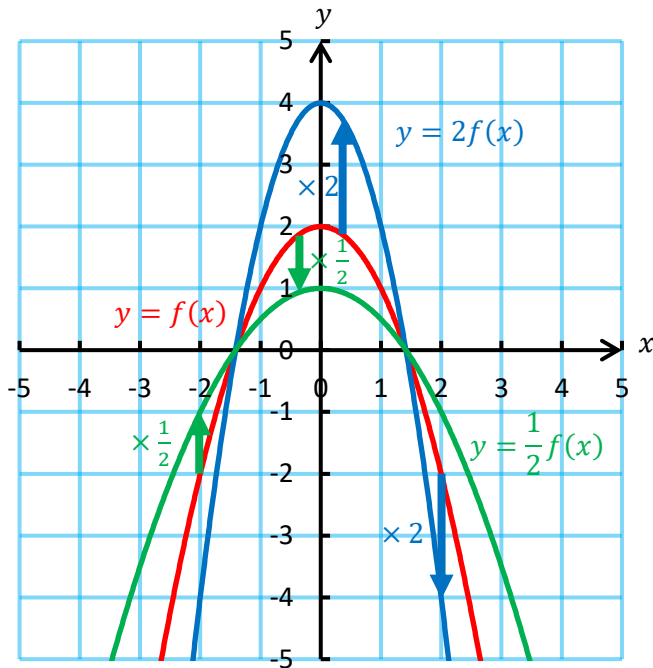
The transformation $y = f(x) + a$ moves the graph a units up (if a is positive) or a units down (if a is negative).

$$y = f(x + a)$$



The transformation $y = f(x + a)$ moves the graph a units to the left (if a is positive) or a units to the right (if a is negative).

$$y = af(x)$$

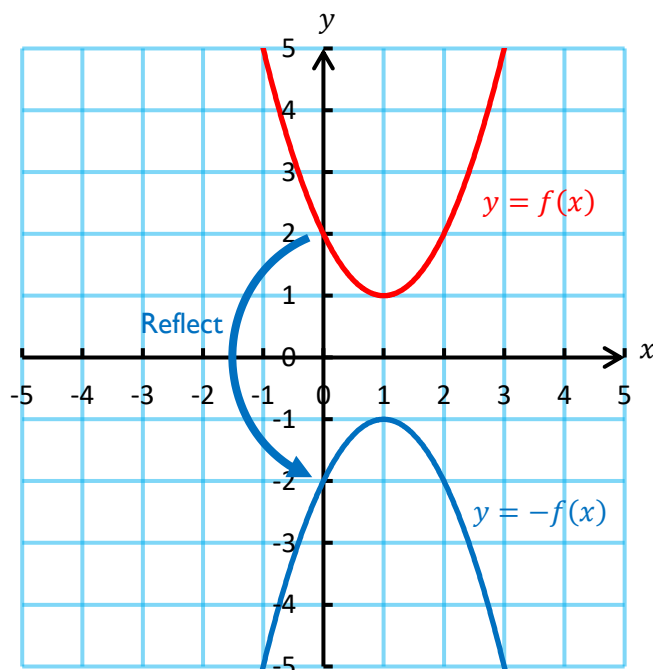


The transformation $y = af(x)$ stretches the graph in the direction of the y -axis (if $a > 1$) or compresses the graph in the direction of the y -axis (if $0 < a < 1$).

As general advice, we need to **follow** anything that takes place **outside** parentheses, and **undo** anything that takes place **inside** parentheses. So, for example, we follow $y = f(x) + 3$ and translate the graph 3 units up, but undo $y = f(x + 3)$ by translating the graph 3 units to the left.

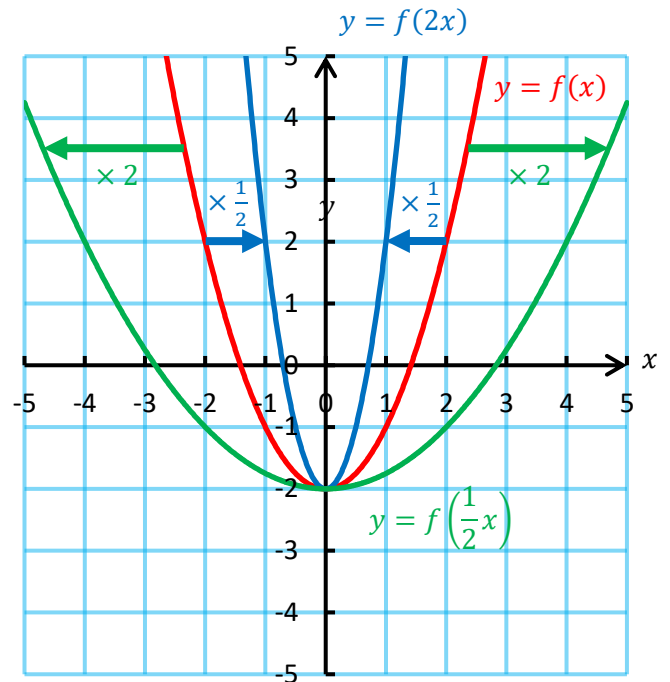
Special Cases

$$y = -f(x)$$



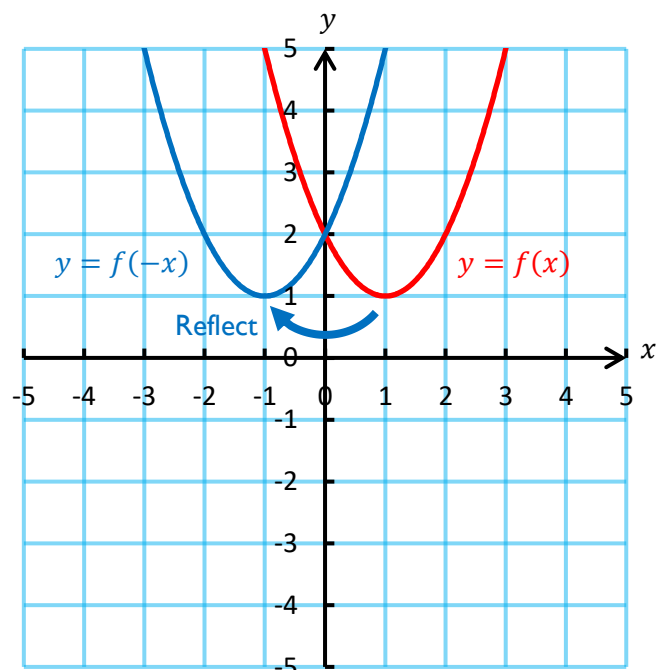
The transformation $y = -f(x)$ reflects the graph $y = f(x)$ in the x -axis.

$$y = f(ax)$$



The transformation $y = f(ax)$ compresses the graph in the direction of the x -axis (if $a > 1$) or stretches the graph in the direction of the x -axis (if $0 < a < 1$).

$$y = f(-x)$$



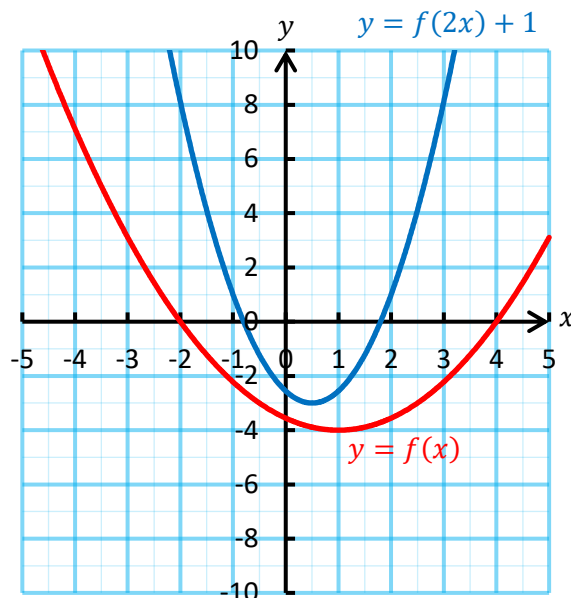
The transformation $y = f(-x)$ reflects the graph $y = f(x)$ in the y -axis.

Combining the Transformations

Consider the graph of $y = f(x)$ that is shown on the right.

To plot $y = f(2x) + 1$, we must combine two transformations.

- To begin, we must form $y = f(2x)$, which compresses the graph in the direction of the x -axis. Under this transformation, the points $(-2, 0)$, $(1, -4)$ and $(4, 0)$ move to the points $(-1, 0)$, $(\frac{1}{2}, -4)$ and $(2, 0)$.
- Then, we must form $y = f(2x) + 1$, which moves the graph one unit up. The points $(-1, 0)$, $(\frac{1}{2}, -4)$ and $(2, 0)$ move to the points $(-1, 1)$, $(\frac{1}{2}, -3)$ and $(2, 1)$.



Exercise 1

Complete the following table, which considers how to combine two transformations of graphs. In each case, show what happens to the point $(8, 12)$ under the transformations.

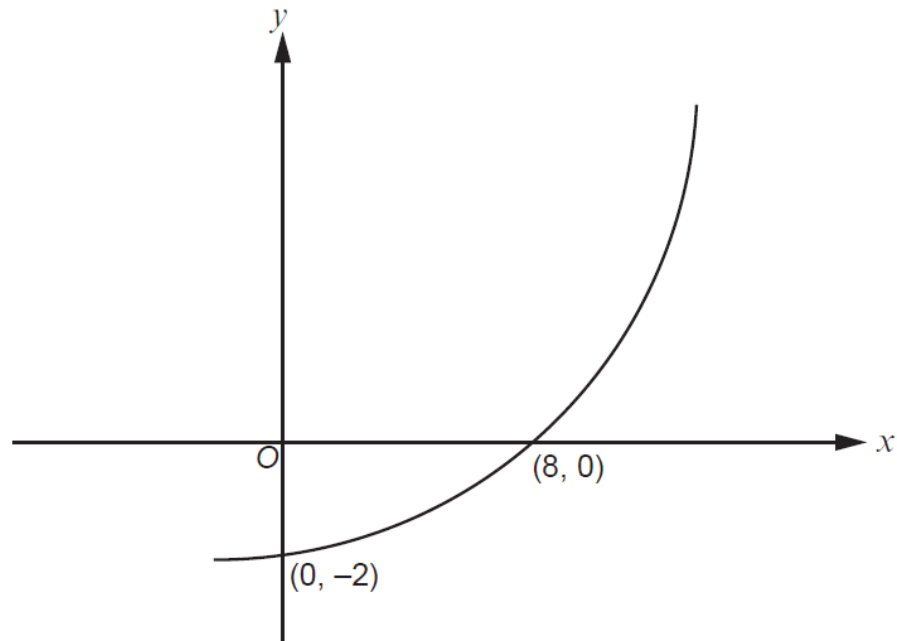
Transformation	First step	Effect on $(8, 12)$	Second step	Effect on $(8, 12)$
$y = 3f(x - 2)$	$y = f(x - 2)$ Move 2 units to the right	$(10, 12)$	$y = 3f(x - 2)$ Stretch along the y -axis	$(10, 36)$
$y = f(x + 1) - 4$				
$y = f(2x) + 3$				
$y = 5f(4x)$				
$y = \frac{1}{2}f\left(\frac{1}{2}x\right)$				
$y = -f(x + 4)$				
$y = 2f(-x)$				
$y = 3 - f(x)$				
$y = f\left(\frac{1}{3}x\right) + 7$				
$y = -2f(-3x)$				

Top 10

(Unit 3 Summer 2019)

0	7
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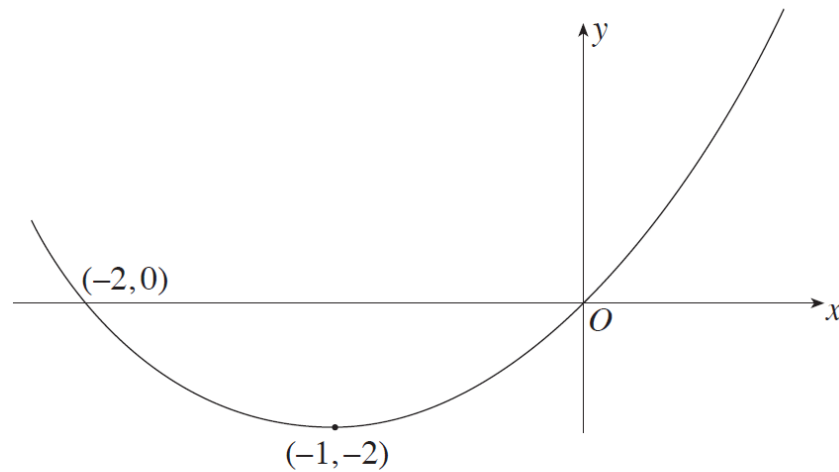
The diagram below shows a sketch of the graph of $y = f(x)$. The graph crosses the y -axis at the point $(0, -2)$, and the x -axis at the point $(8, 0)$.



- a) Sketch the graph of $y = -4f(x + 3)$. Indicate the coordinates of the point where the graph crosses the x -axis and the y -coordinate of the point where $x = -3$. [3]
- b) Sketch the graph of $y = 3 + f(2x)$. Indicate the y -coordinate of the point where $x = 4$. [2]

(C3 Winter 2007)

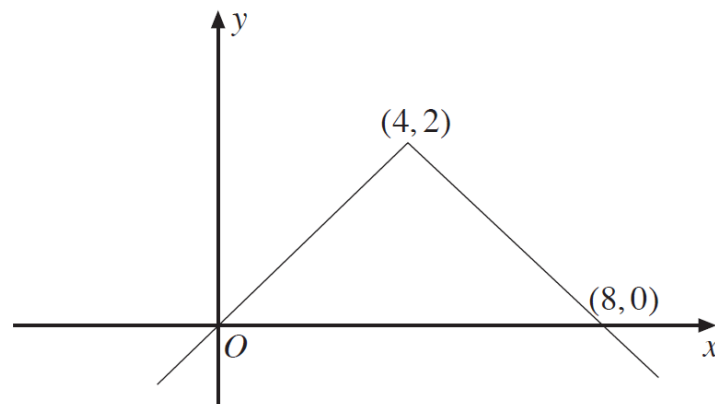
8. The diagram below shows a sketch of the graph of $y = f(x)$. The graph passes through the origin and the point $(-2, 0)$, and has a minimum point at $(-1, -2)$.



- (a) Sketch the graph of $y = 2f(x - 3)$. Indicate the coordinates of the stationary point and of the points where the graph crosses the x -axis. [3]
- (b) On a separate diagram, sketch the graph of $y = -f(x) + 1$. Indicate the coordinates of the stationary point and the coordinates of the point where the graph crosses the y -axis. [3]

(C3 Winter 2008)

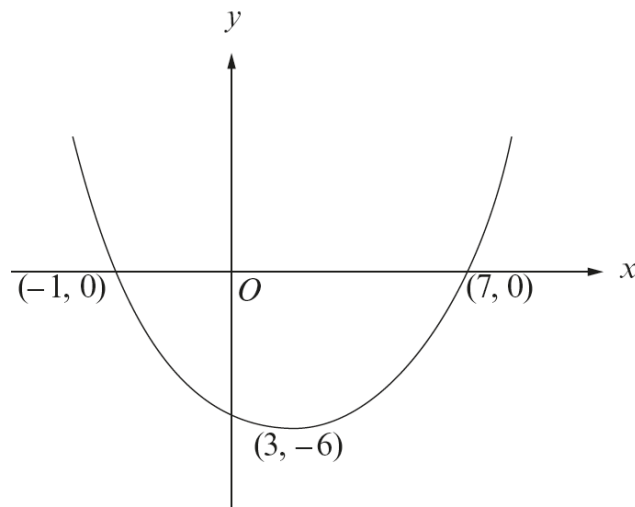
8. The diagram shows the graph of $y = f(x)$. The graph has its highest point at $(4, 2)$ and it intersects the x -axis at the points $(0, 0)$ and $(8, 0)$.



- (a) Sketch the graph of $y = 2f(x + 3)$, indicating the coordinates of the highest point and of the points where the graph intersects the x -axis. [3]
- (b) On a separate diagram, sketch the graph of $y = f(2x) + 1$, indicating the coordinates of the highest point and of the point where the graph intersects the y -axis. [3]

(C3 Summer 2013)

9. The diagram shows a sketch of the graph of $y = f(x)$. The graph passes through the points $(-1, 0)$ and $(7, 0)$ and has a minimum point at $(3, -6)$.



Sketch the graph of $y = -\frac{2}{3}f(x+4)$, indicating the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x -axis. [3]

(C3 Winter 2010)

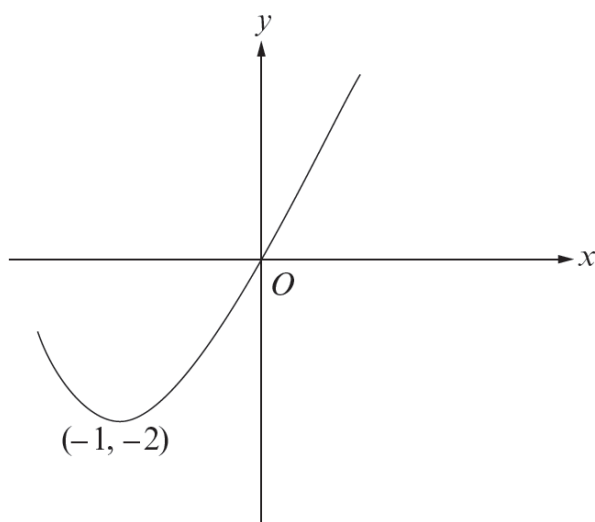
8. Given that $f(x) = e^x$, sketch, on the same diagram, the graphs of $y = f(x)$ and $y = 2f(x) - 3$. Label the coordinates of the point of intersection of each of the graphs with the y -axis. Indicate the behaviour of each of the graphs for large positive and negative values of x . [5]

(C3 Winter 2009)

8. Given that $f(x) = \ln x$, sketch the graphs of $y = f(x)$ and $y = -f(x + 1)$ on the same diagram. Label the coordinates of the points of intersection with the x -axis and indicate the behaviour of the graphs for large positive and negative values of y . [5]

(C3 Summer 2012)

8. The diagram shows a sketch of the graph of $y = f(x)$. The graph passes through the origin and has a minimum point at $(-1, -2)$.

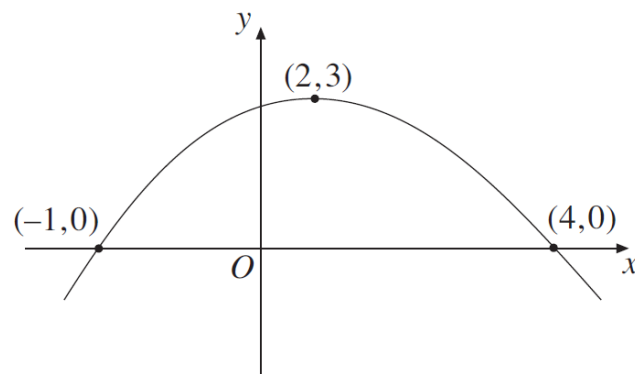


Sketch the graph of $y = -4f(2x)$, indicating the coordinates of the stationary point.

[3]

(C3 Summer 2009)

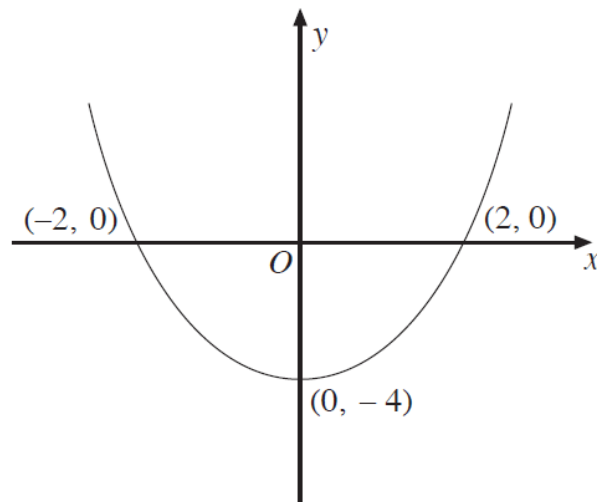
8.



The diagram shows a sketch of the graph of $y = f(x)$. The graph has its highest point at $(2, 3)$ and intersects the x -axis at the points $(-1, 0)$ and $(4, 0)$. Sketch the graph of $y = 3f(x - 2)$, indicating the coordinates of three points on the graph. [3]

(C3 Summer 2008)

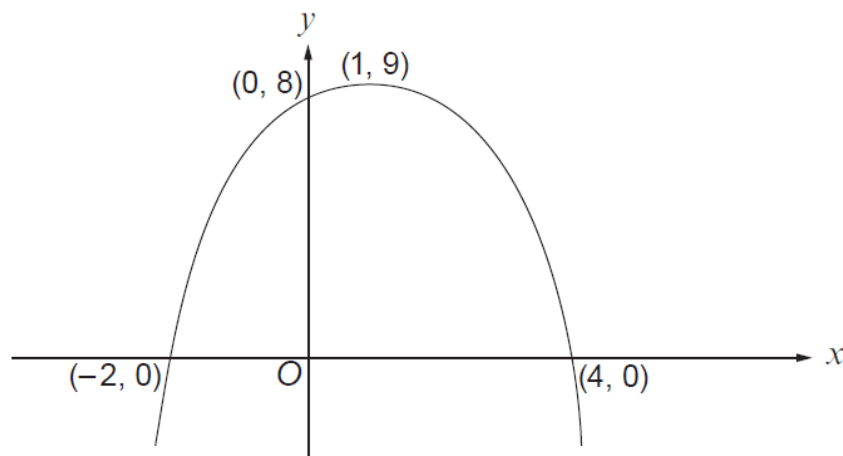
6. (a) The diagram shows the graph of $y = f(x)$. The graph has a stationary point at $(0, -4)$ and it intersects the x -axis at the points $(-2, 0)$ and $(2, 0)$.



Sketch the graph of $y = 3f(x - 1)$, indicating the coordinates of the stationary point and of the points where the graph crosses the x -axis. [3]

(Unit 3 Summer 2018)

- | | |
|---|---|
| 0 | 3 |
|---|---|
- The diagram below shows a sketch of the graph of
- $y = f(x)$
- . The graph passes through the points
- $(-2, 0)$
- ,
- $(0, 8)$
- ,
- $(4, 0)$
- and has a maximum point at
- $(1, 9)$
- .

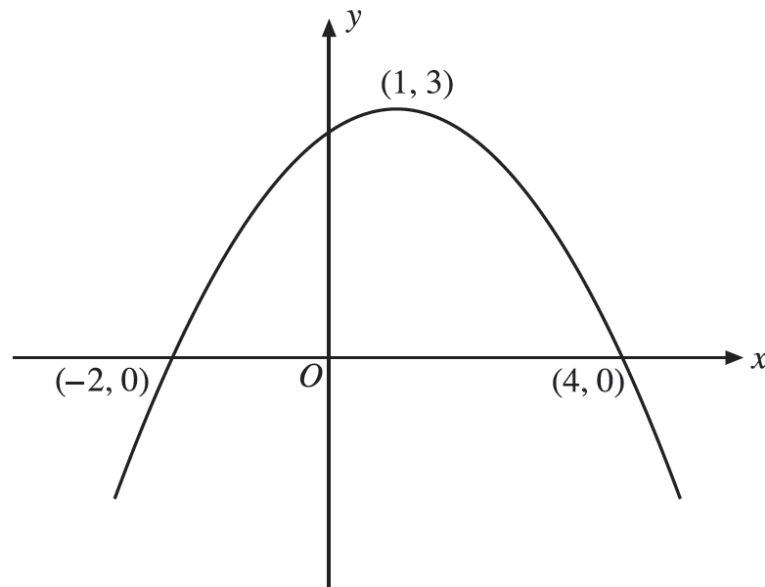


- a) Sketch the graph of $y = 2f(x + 3)$. Indicate the coordinates of the stationary point and the points where the graph crosses the x -axis. [3]
- b) Sketch the graph of $y = 5 - f(x)$. Indicate the coordinates of the stationary point and the point where the graph crosses the y -axis. [3]

Revision Questions

(C3 Winter 2011)

8. The diagram shows a sketch of the graph of $y = f(x)$. The graph passes through the points $(-2, 0)$ and $(4, 0)$ and has a maximum point at $(1, 3)$.



Sketch the graph of $y = -3f(x + 2)$, indicating the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x -axis. [3]

7. (a) Solve the inequality $|3x + 1| \leq 5$. [3]

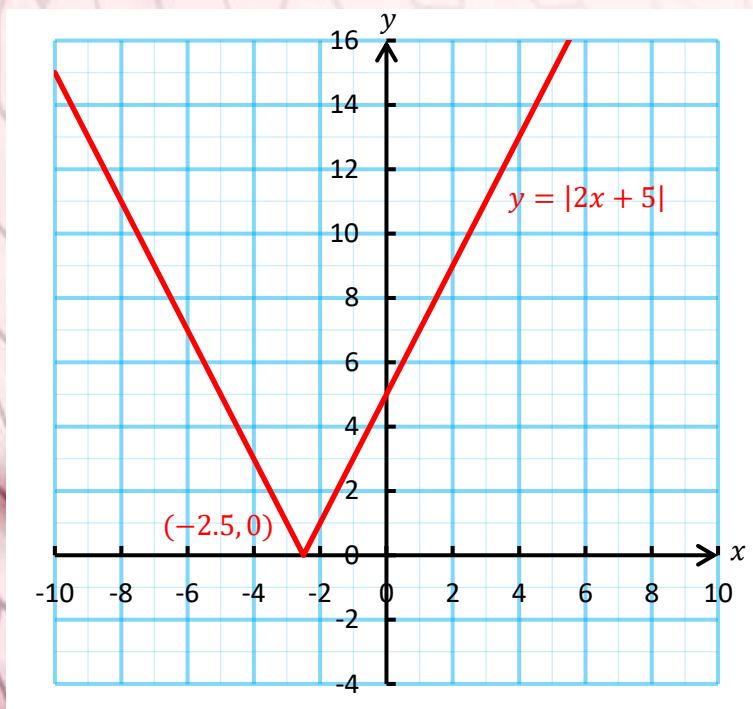
(i) Sketch the graph of $y = f(x)$.

(ii) On a separate set of axes, sketch the graph of $y = f(x - 3) + 2$. On your sketch, indicate the coordinates of the point on the graph where the value of the y -coordinate is least and the coordinates of the point where the graph crosses the y -axis. [4]



The Modulus

Function



Name: _____

Background

What is the work?

Introducing the modulus function, which ignores the sign of a number.

What is required before starting?

GCSE Work: Graph plotting; transformations of graphs; solving inequalities.

Where does this lead to?

Applications: Finding the angle between two vectors; interpreting the results of a statistical test.

Theory

The modulus function $y = |x|$ gives the magnitude of a number, independent of its sign (positive or negative).

For example, if $x = 3$, we have $|3| = 3$; and if $x = -3$, we have $|-3| = 3$.

The graph of $y = |x|$ is 'v'-shaped, as shown on the right. The gradient is not defined if $x = 0$.

Solving Equations

We can solve equations that contain the modulus function.

Example 1

Solve the inequality $|3x + 4| > 5$.

Answer: To visualise the answer, we can plot the graph for $y = |3x + 4|$, as shown on the right.

There are two parts to the answer:

$$3x + 4 > 5$$

and

$$3x + 4 < -5$$

$$3x > 1$$

$$3x < -9$$

$$x > \frac{1}{3}$$

$$x < -\frac{9}{3}$$

$$x < -3$$

This is the part of the graph above the blue line $y = 5$ after $x = \frac{1}{3}$.

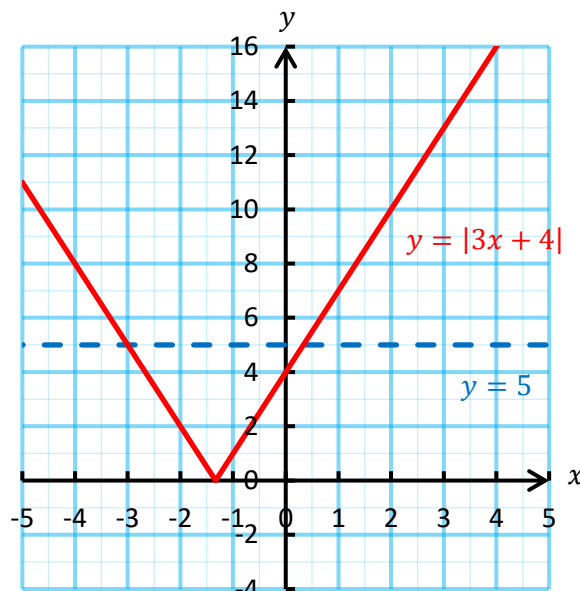
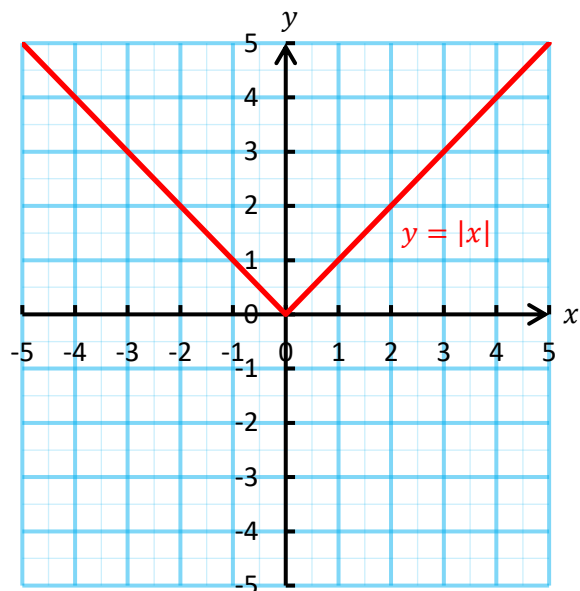
This is the part of the graph above the blue line $y = 5$ before $x = -3$.

In order to find this part we consider where the graph $y = 3x + 4$ is less than -5 .

Final solution: $x < -3$ or $x > \frac{1}{3}$.



Theory



Exercise 1Solve the inequality $|4x + 1| > 13$.

Example 2Solve the equation $3|x| + 4 = 6 - 2|x|$.Answer: $3|x| + 4 = 6 - 2|x|$

$$5|x| + 4 = 6$$

add $2|x|$ to each side

$$5|x| = 2$$

subtract 4 from both sides

$$|x| = 0.4$$

divide by 5

Either $x = 0.4$ or $x = -0.4$ **Exercise 2**Solve the equation $5|x + 1| + 2 = 17 + 2|x + 1|$.

Example 3Solve the equation $|2x + 3| = |3x - 2|$.

Answer: We can see the two solutions in the graph on the right.
To find them, we must square each side of the equation to eliminate the modulus symbols.

$$(2x + 3)^2 = (3x - 2)^2$$

$$(2x + 3)(2x + 3) = (3x - 2)(3x - 2)$$

$$4x^2 + 6x + 6x + 9 = 9x^2 - 6x - 6x + 4$$

$$4x^2 + 12x + 9 = 9x^2 - 12x + 4$$

$$0 = 5x^2 - 24x - 5$$

$$5 \times -5 = -25$$

$$0 = 5x^2 - 25x + x - 5$$

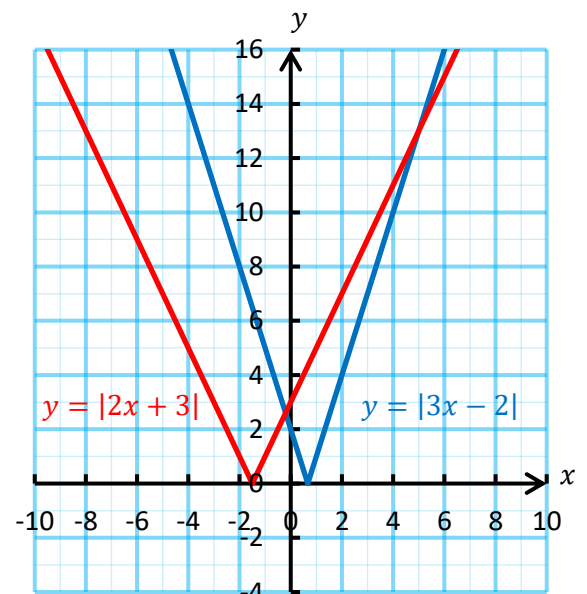
$$0 = 5x(x - 5) + 1(x - 5)$$

$$0 = (x - 5)(5x + 1)$$

Either $x - 5 = 0$ or $5x + 1 = 0$

$$x = 5$$

$$x = -\frac{1}{5}$$





7. Solve the following.

$$(b) \quad |5x - 8| \geq 3 \quad [3]$$

(Unit 3 Summer 2018)

0	1
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 Solve the equation

$$|2x + 1| = 3|x - 2|.$$

[4]

6. Solve the following.

$$(b) \quad |5x + 7| \leq 4 \quad [3]$$

(C3 Winter 2012)

7. Solve the following.

$$(a) \quad |4x - 5| \geq 3, \quad [3]$$

$$(b) \quad (3|x| + 1)^{\frac{1}{3}} = 4. \quad [2]$$

(C3 Winter 2006)

6. (a) Solve the inequality $|3x - 8| \leq 5$. [3]

(b) Given that $f(x) = |x|$, sketch the graph of $y = f(x)$. On the same diagram, sketch the graph of $y = f(x + 2) + 1$, indicating its position. [4]

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(Unit 3 Summer 2019)

0	5
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a) Find the range of values of x for which $|1 - 3x| > 7$.

[3]

b) Sketch the graph of $y = |1 - 3x| - 7$. Clearly label the minimum point and the points where the graph crosses the x -axis. [4]

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8. (a) Find all values of x satisfying the inequality $|3x - 5| \leq 1$. [3]

$$\left| \frac{3}{y} - 5 \right| \leq 1. \quad [2]$$

0	7
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a) The graphs of $y = 5x - 3$ and $y = 2x + 3$ intersect at the point A. Show that the coordinates of A are (2, 7). [2]

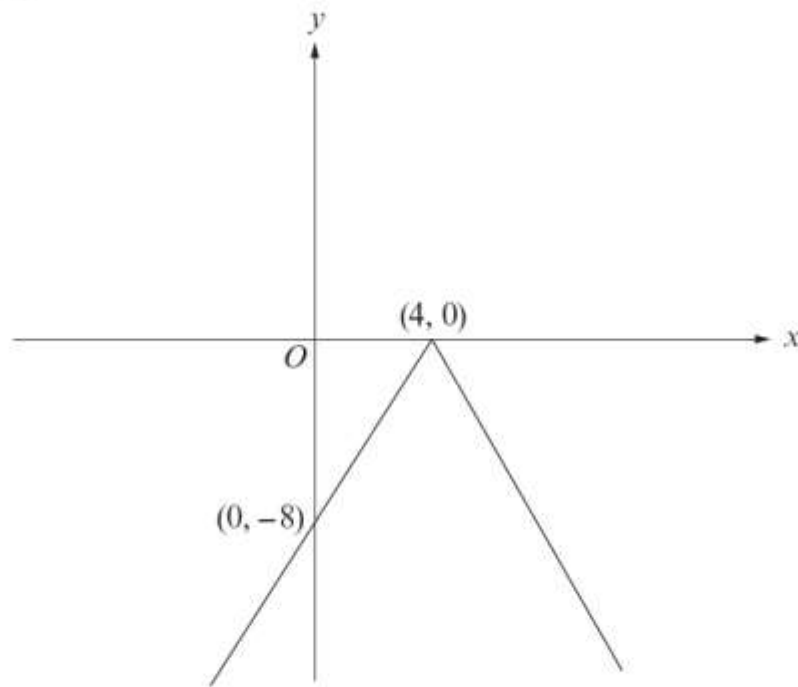
b) On the same set of axes, sketch the graphs of $y = |5x - 3|$ and $y = |2x + 3|$, clearly indicating the coordinates of the points of intersection of the two graphs and the points where the graphs touch the x -axis. [4]

c) Calculate the area of the region satisfying the inequalities

$$y \geq |5x - 3| \quad \text{and} \quad y \leq |2x + 3|. \quad [4]$$

(C3 Winter 2013)

7. (a) Solve the inequality $|3x - 4| > 5$. [3]
- (b) (i) Sketch the graph of $y = |x|$.
- (ii) The diagram below shows a sketch of the graph of $y = a|x + b|$, where a and b are constants. The graph meets the x -axis at the point $(4, 0)$ and the y -axis at the point $(0, -8)$.



Find the value of a and the value of b .

[3]

[illegible]

7. (a) Show, by counter-example, that the statement

is false. [2]

$$|x^2 - 10| \leq 6. \quad [4]$$

Revision Questions

(C3 Summer 2008)

6. (b) Solve $3|x| + 1 = 2 - |x|$. [2]

(c) Solve $|2x - 9| > 3$. [4]

10. The functions f and g are defined for all values of x by

$$g(x) = |2x + 1| + 2.$$

[5]



Composite

Functions



Ffion wants to buy a new mobile phone on the internet. The cost of the mobile phone is £500.

Ffion has a '10% off' code for the website, and also a £50 voucher.

Should Ffion enter the '10% off' code first, and then use the £50 voucher, or enter the £50 voucher first, and then use the '10% off' code?

Enter a gift card, voucher or promotional code



Enter Code

Apply

Name:

Background

What is the work?

How to combine two or more functions into a composite function.

What is required before starting?

GCSE Work: Substitution; solving equations; solving inequalities; changing the subject.

Where does this lead to?

Applications: Any situation where a series of calculations is required, and the result of one calculation directly affects the input of the next calculation.

Theory

A **function** f maps elements from the set X (the **domain**) to elements of the set Y (the **range**). The input $x \in X$ maps to one (and only one) output $y \in Y$.

For $f(x) = x^2 - 7$,
the domain is $(-\infty, \infty)$ and the range is $[-7, \infty)$.

For $f(x) = \frac{1}{\sqrt{x-3}}$,
the domain is $(3, \infty)$ and the range is $(0, \infty)$.

Notation

$\in$ 'is an element of'
 $[2, 5)$ All numbers between 2 and 5,
 including 2 but not including 5.



Theory

In a **one-to-one function**, every element of the range corresponds to **exactly one element** of the domain.

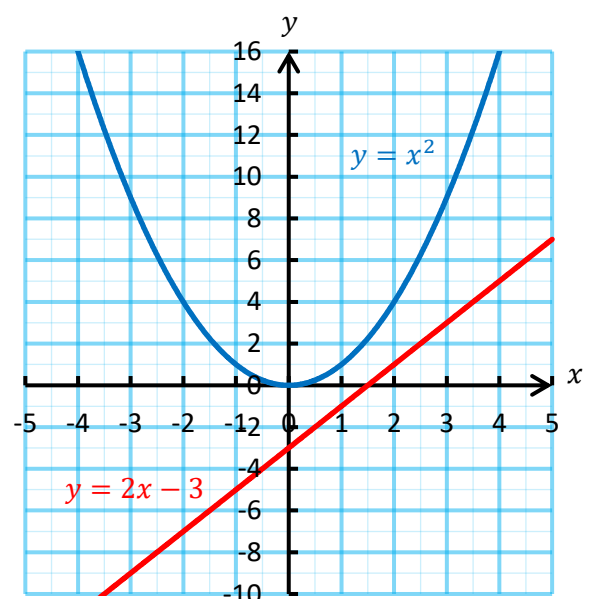
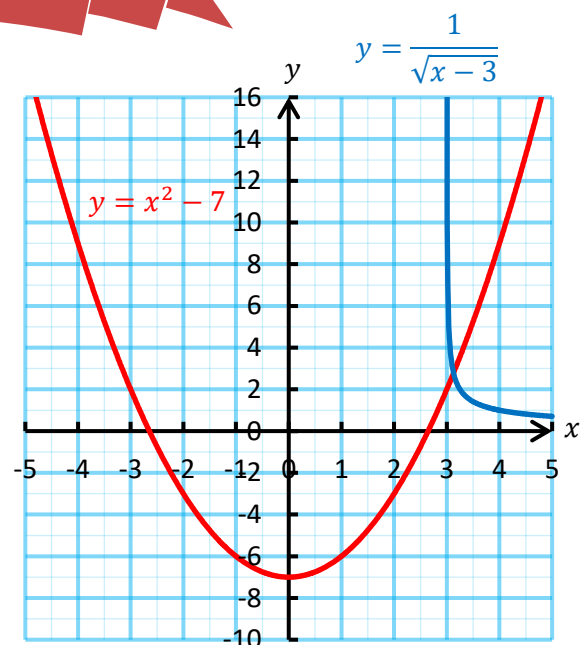
Example 1

$$f(x) = 2x - 3.$$

In a **many-to-one function**, each element of the range can correspond to **more than one element** of the domain.

Example 2

$$f(x) = x^2.$$



Composite Functions

A **composite function** combines two or more functions.

Given functions $f(x)$ and $g(x)$, the composite function $fg(x)$ means 'apply f to the results of $g(x)$ '.

The composite function $gf(x)$ means 'apply g to the results of $f(x)$ '.

The order is important. $fg(x)$ is not necessarily the same as $gf(x)$.

Example 3

If $f(x) = x^2$, $g(x) = 5x + 3$, calculate

- (a) $f(4)$ (b) $g(4)$ (c) $fg(4)$ (d) $gf(4)$

Answer:

(a) $f(4) = 4^2$	(b) $g(4) = 5 \times 4 + 3$	(c) $fg(4) = f(g(4))$	(d) $gf(4) = g(f(4))$
$= 16$	$= 20 + 3$	$= f(23)$	$= g(16)$
	$= 23$	$= 23^2$	$= 5 \times 16 + 3$
		$= 529$	$= 80 + 3$
			$= 83$

Exercise 1

If $f(x) = 3x + 1$, $g(x) = \sqrt{x}$, calculate

- (a) $f(16)$ (b) $g(16)$ (c) $fg(16)$ (d) $gf(16)$

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Example 4 (C3 Summer 2005)

The functions f and g have domains $(0, \infty)$ and $(5, \infty)$ respectively, and are defined by

$$f(x) = x^2 + 1$$

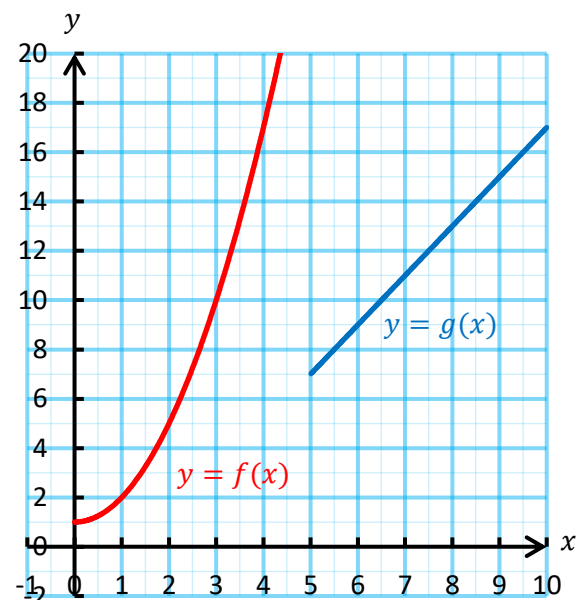
$$g(x) = 2x - 3$$

- (a) Write down the ranges of f and g .
 (b) Give the reason why $gf(1)$ cannot be formed.
 (c) Solve the equation $fg(x) = 3x^2 - 6x + 17$.

Answer: (a) Considering that $f(x)$ and $g(x)$ are increasing functions throughout their domain,
 Range of $f(x) = (1, \infty)$
 Range of $g(x) = (7, \infty)$

(b) $gf(1) = g(f(1))$
 $= g(1^2 + 1)$
 $= g(1 + 1)$
 $= g(2)$

$gf(1)$ cannot be formed as 2 is not in the domain of $g(x)$.



$$\begin{aligned}
 \text{(c) } fg(x) &= 3x^2 - 6x + 17 \\
 f(g(x)) &= 3x^2 - 6x + 17 \\
 f(2x - 3) &= 3x^2 - 6x + 17 \\
 (2x - 3)^2 + 1 &= 3x^2 - 6x + 17 \\
 (2x - 3)(2x - 3) + 1 &= 3x^2 - 6x + 17 \\
 4x^2 - 6x - 6x + 9 + 1 &= 3x^2 - 6x + 17 \\
 4x^2 - 12x + 10 &= 3x^2 - 6x + 17 \\
 x^2 - 6x - 7 &= 0 \\
 (x - 7)(x + 1) &= 0 \\
 \text{Either } x - 7 = 0 \text{ or } x + 1 &= 0 \\
 x = 7 \quad x = -1
 \end{aligned}$$

But -1 is not in the domain of $g(x)$, so the only solution is $x = 7$.

This is difficult to explain in words! It is easier to understand by considering the following two examples...

The domain of $fg(x)$

Which numbers can be input to $fg(x)$?

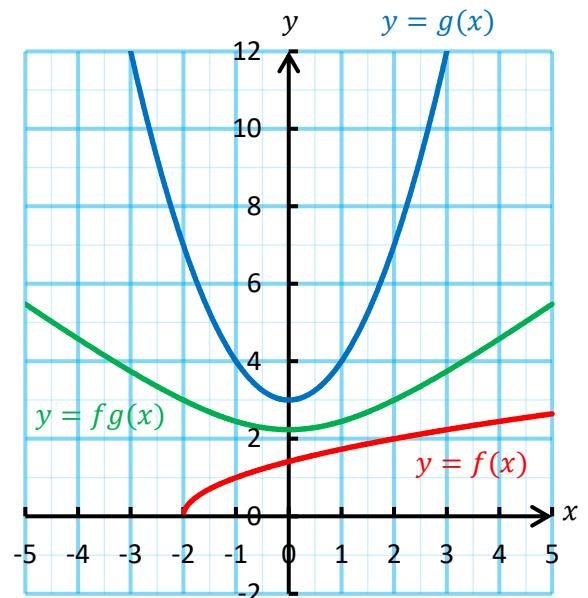
- (1) Find the domain of g and the domain of f .
- (2) Find all values of x from the domain of g such that $g(x)$ is in the domain of f .
- (3) The domain of $fg(x)$ is all the values from step (2) such that the function $fg(x)$ is defined.

Example 3

$$f(x) = \sqrt{x+2}, \quad g(x) = x^2 + 3$$

Function	Domain	Range
$f(x)$	$[-2, \infty)$	$[0, \infty)$
$g(x)$	$(-\infty, \infty)$	$[3, \infty)$
$fg(x)$	$(-\infty, \infty)$	$[\sqrt{5}, \infty)$

$$\begin{aligned}
 fg(x) &= \sqrt{g(x) + 2} \\
 fg(x) &= \sqrt{(x^2 + 3) + 2} \\
 fg(x) &= \sqrt{x^2 + 5}
 \end{aligned}$$



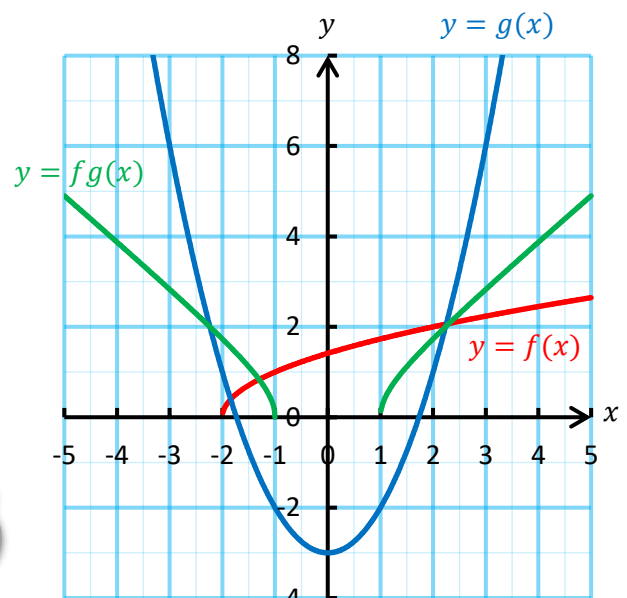
Example 4

$$f(x) = \sqrt{x+2}, \quad g(x) = x^2 - 3$$

Function	Domain	Range
$f(x)$	$[-2, \infty)$	$[0, \infty)$
$g(x)$	$(-\infty, \infty)$	$[-3, \infty)$
$fg(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \infty)$

$$\begin{aligned}
 fg(x) &= \sqrt{g(x) + 2} \\
 fg(x) &= \sqrt{(x^2 - 3) + 2} \\
 fg(x) &= \sqrt{x^2 - 1}
 \end{aligned}$$

$fg(x)$ is not defined for $(-1, 1)$





10. The functions f and g have domains $(0, \infty)$ and $(2, \infty)$ respectively and are defined by

$$\begin{aligned} f(x) &= x^2 - 1, \\ g(x) &= 2x - 1. \end{aligned}$$

- (ii) Write down the domain and range of fg . [4]

(C3 Summer 2011)

10. The functions f and g have domains $(-\infty, 0)$ and $(6, \infty)$ respectively and are defined by

$$f(x) = x^2 - 19,$$

$$g(x) = 1 - \frac{1}{2}x.$$

(a) Write down the range of f and the range of g . [2]

(b) Write down the domain and range of fg . [2]

(c) (i) Write down an expression for $fg(x)$.

(ii) Hence, solve the equation

$$fg(x) = 2x - 26. \quad [4]$$

10. The function g has domain $(-\infty, \infty)$ and is defined by

$$g(x) = \sqrt{3x^2 + 7}.$$

$$gg(x) = 8.$$

[5]

10. The functions f and g have domains $[0, \infty)$ and $(-\infty, \infty)$ respectively and are defined by

$$\begin{aligned} f(x) &= e^x, \\ g(x) &= 4x^3 + 7. \end{aligned}$$

- (a) Find and simplify an expression for $gf(x)$. [2]
- (b) Find the domain and range of gf . [2]
- (c) (i) Solve the equation $gf(x) = 18$. Give your answer correct to three decimal places.
- (ii) Giving a reason, write down a value for k so that $gf(x) = k$ has no solution. [3]

(C3 Summer 2017)

9. The function f has domain $[2, \infty)$ and is defined by

$$f(x) = 4x + k,$$

where k is a constant.

- (a) Write down, in terms of k , the range of f . [1]

The function g has domain $[-3, \infty)$ and is defined by

$$g(x) = x^2 - 9.$$

- (b) Find the least value of k so that the function gf can be formed. [2]

- (c) (i) Write down an expression, in terms of k , for $gf(x)$.

- (ii) Given that $gf(2) = 7$, find the value of k .

[5]

9. The function f has domain $(-\infty, \infty)$ and is defined by

The function g has domain $(0, \infty)$ and is defined by

(a) Write down the domain and range of fg . [2]

$$fg(x) = 12. \quad [5]$$

(C3 Winter 2007)

10. The functions f and g are defined for all values of x by

$$f(x) = x + 5,$$

$$g(x) = |2x + 1| + 2.$$

Solve the inequality $fg(x) > 10$.

[5]

8. The functions f and g have domains $[0, \infty)$ and $(-\infty, \infty)$ respectively, and are defined by

$$g(x) = x^2 + 1.$$

(C3 Winter 2014)

10. The functions f and g have domains $(0, \infty)$ and $(-\infty, -2)$ respectively and are defined by

$$f(x) = \sqrt{x^2 + 5},$$

$$g(x) = \frac{-4}{x+1}.$$

(a) By considering $g'(x)$, show that g is an increasing function. [2]

(b) Write down the range of g . [2]

(c) Write down the domain and range of fg . [2]

(d) (i) Write down an expression for $fg(x)$.

(ii) Hence, solve the equation

$$fg(x) = 3. \quad [5]$$

Handwriting practice lines consisting of 30 horizontal dotted lines.

(C3 Winter 2013)

9. (a) The functions f and g have domains $(-\infty, \infty)$ and $(0, \infty)$ respectively and are defined by

$$\begin{aligned}f(x) &= x^2 - 25, \\g(x) &= 2x - 3.\end{aligned}$$

- (i) Write down the domain of fg .
- (ii) Write down the range of fg .
- (iii) Write down an expression for $fg(x)$.
- (iv) Solve the equation $fg(x) = 0$.

[7]

- (b) The function h is defined by

$$h(x) = \frac{2x + 7}{5x - 2}.$$

- (i) Show that $hh(x) = x$.
- (ii) **Hence** write down an expression for $h^{-1}(x)$.

[3]

Handwriting practice lines consisting of 30 horizontal dotted lines.

Revision Question

(C3 Summer 2010)

10. The functions f and g have domains $[-3, \infty)$ and $(-\infty, \infty)$ respectively and are defined by

$$f(x) = \sqrt{x+4} \text{ ,}$$

$$g(x) = 2x^2 - 3.$$

- Write down the range of f and the range of g . [2]
- Find an expression for $gf(x)$. Simplify your answer. [2]
- Solve the equation $fg(x) = 17$. [4]



Inverse

Functions

1
2
3
4
5

$f(x)$



-1
5
15
29
47

$f^{-1}(x)$



1
2
3
4
5

Name:

Background

What is the work?

How to find the inverse function $f^{-1}(x)$ in order to reverse the effect of a function $f(x)$.

What is required before starting?

GCSE Work: Graph plotting; changing the subject.
A Level Unit 3: Composite functions.

Where does this lead to?

Applications: When typing a sum into a computer, the computer will change the sum to be in binary ($f(x)$) in order to be able to perform the sum, before changing back to a decimal number ($f^{-1}(x)$) in order to show you the answer.

Theory

Given a function $f(x)$, the **inverse function** $f^{-1}(x)$ reverses the effect of the function $f(x)$.

It is possible to find the inverse function by changing the subject of $f(x)$ to be x .

Example 1

$$\begin{aligned} f(x) &= 4x + 3 \\ 4x + 3 &= f(x) && \text{[swap sides]} \\ 4x &= f(x) - 3 && \text{[subtract 3]} \\ x &= \frac{f(x)-3}{4} && \text{[divide by 4]} \end{aligned}$$

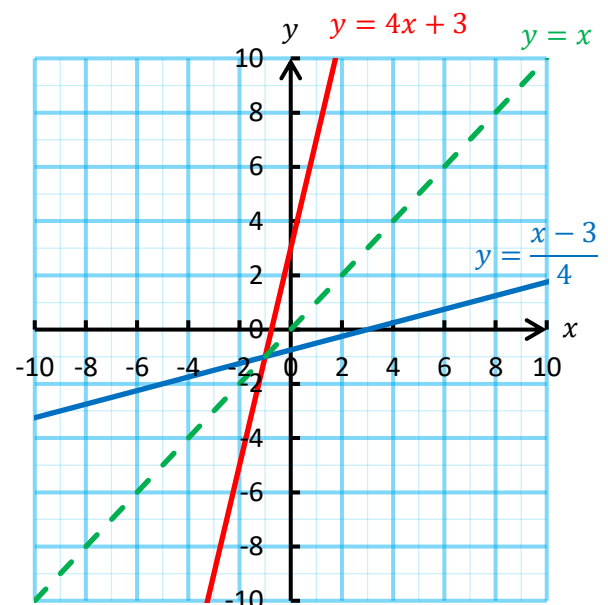
$$\text{So, } f^{-1}(x) = \frac{x-3}{4}.$$

(Hint: You can change $f(x)$ to be y in order to make the algebraic manipulation easier.

$$\begin{aligned} y &= 4x + 3 \\ 4x + 3 &= y && \text{[swap sides]} \\ 4x &= y - 3 && \text{[subtract 3]} \\ x &= \frac{y-3}{4} && \text{[divide by 4]} \end{aligned}$$



Theory



The graph for $f^{-1}(x)$ is a reflection of the graph of $f(x)$ in the line $y = x$.

The domain of $f^{-1}(x)$ is the range of $f(x)$, and the range of $f^{-1}(x)$ is the domain of $f(x)$.

On taking the inverse, many-to-one functions become one-to-many relations.

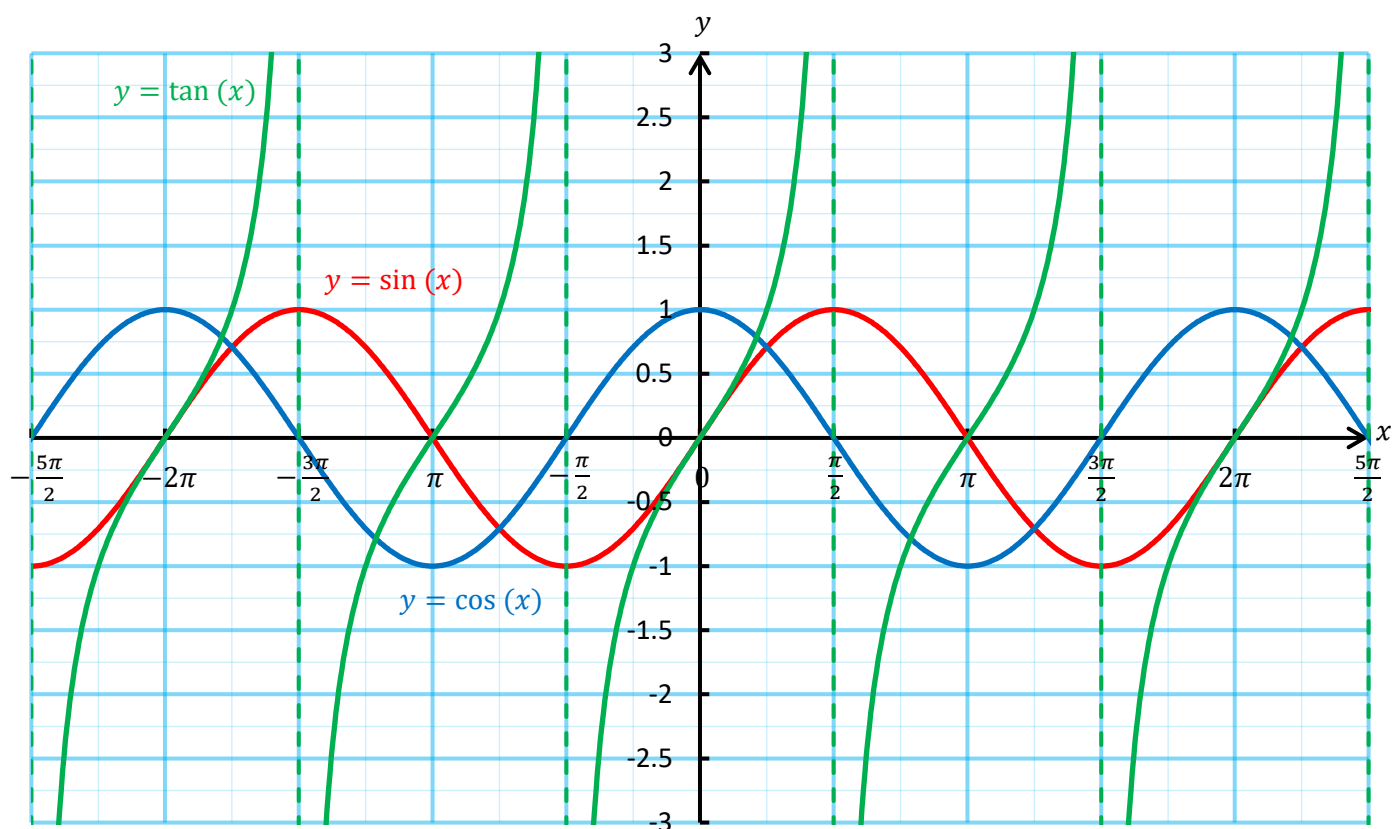
So, without reducing the domain, an inverse function does not exist for a many-to-one function.

🔗 The function $f(x)$ has an inverse $f^{-1}(x)$ if $f(x)$ is a one-to-one function. 🔗

If a function has an inverse function, then $ff^{-1}(x) = f^{-1}f(x) = x$.

Trigonometric Functions

You will be familiar with the shape of the graphs of **sin**, **cos** and **tan** from previous work.



The graphs of sec, cosec and cot

$$\sec(x) = \frac{1}{\cos(x)}$$

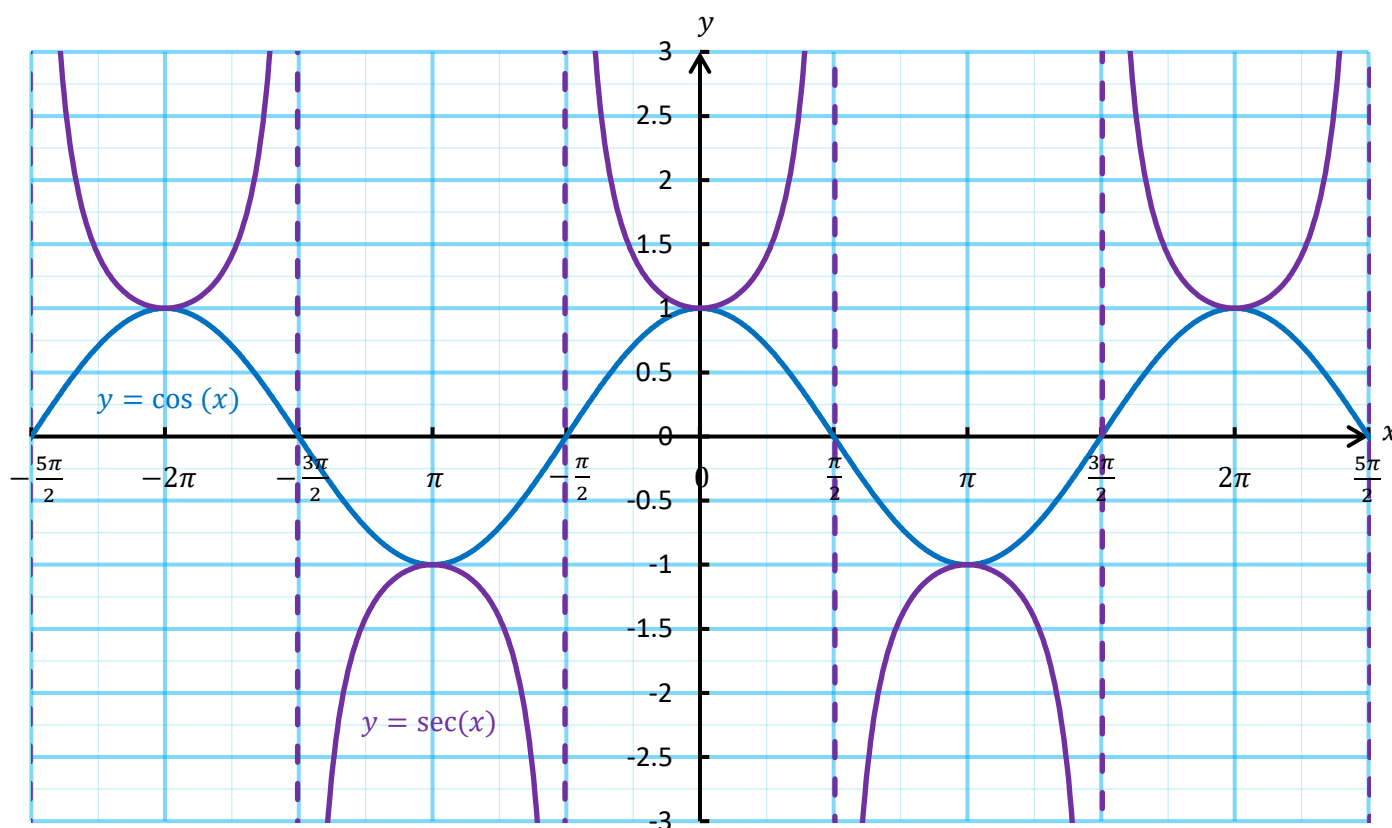
$$\operatorname{cosec}(x) = \frac{1}{\sin(x)}$$

$$\cot(x) = \frac{1}{\tan(x)}$$

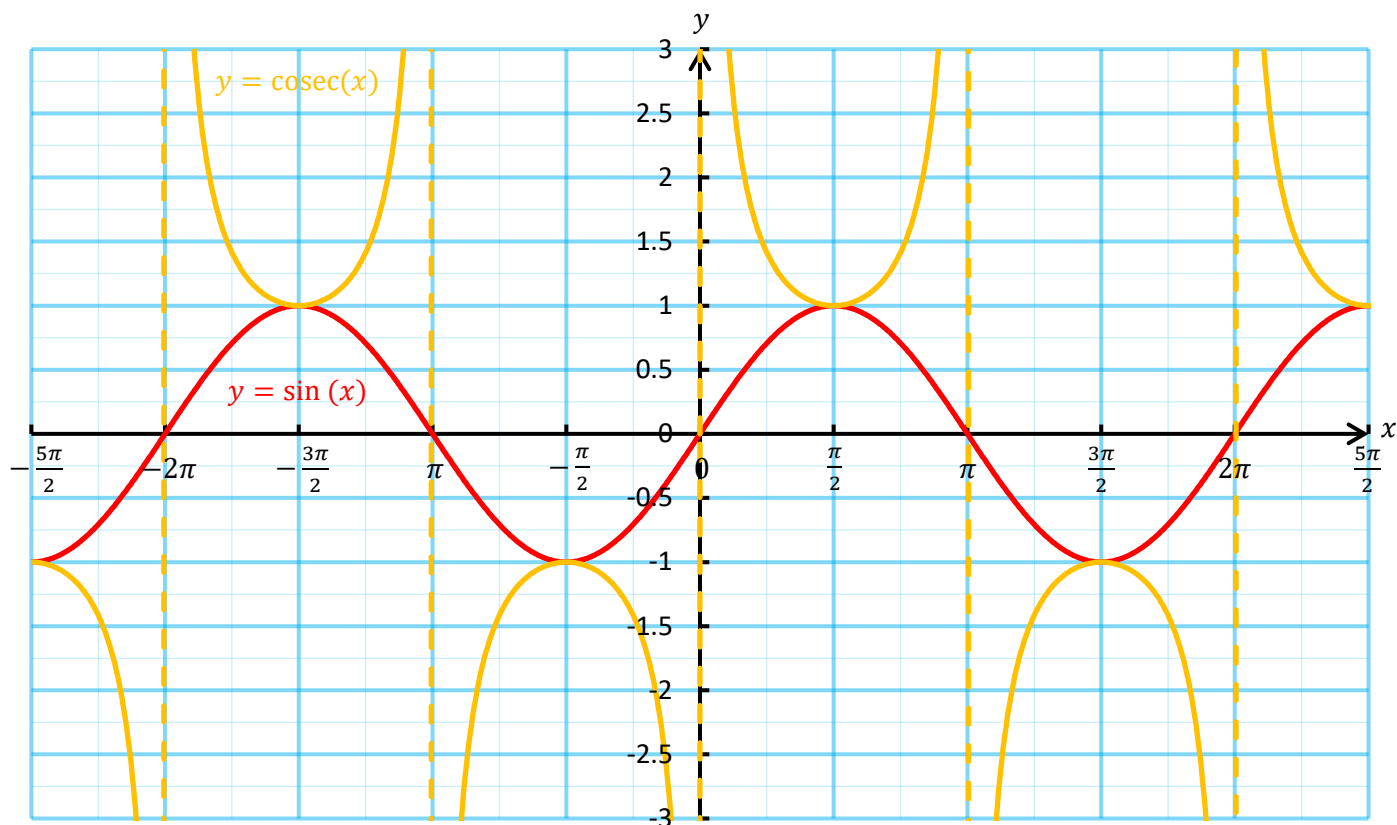
$y = \sec(x)$ Domain $(-\infty, \infty)$, $x \neq \frac{\pi}{2} + k\pi$

Range $(-\infty, -1] \cup [1, \infty)$

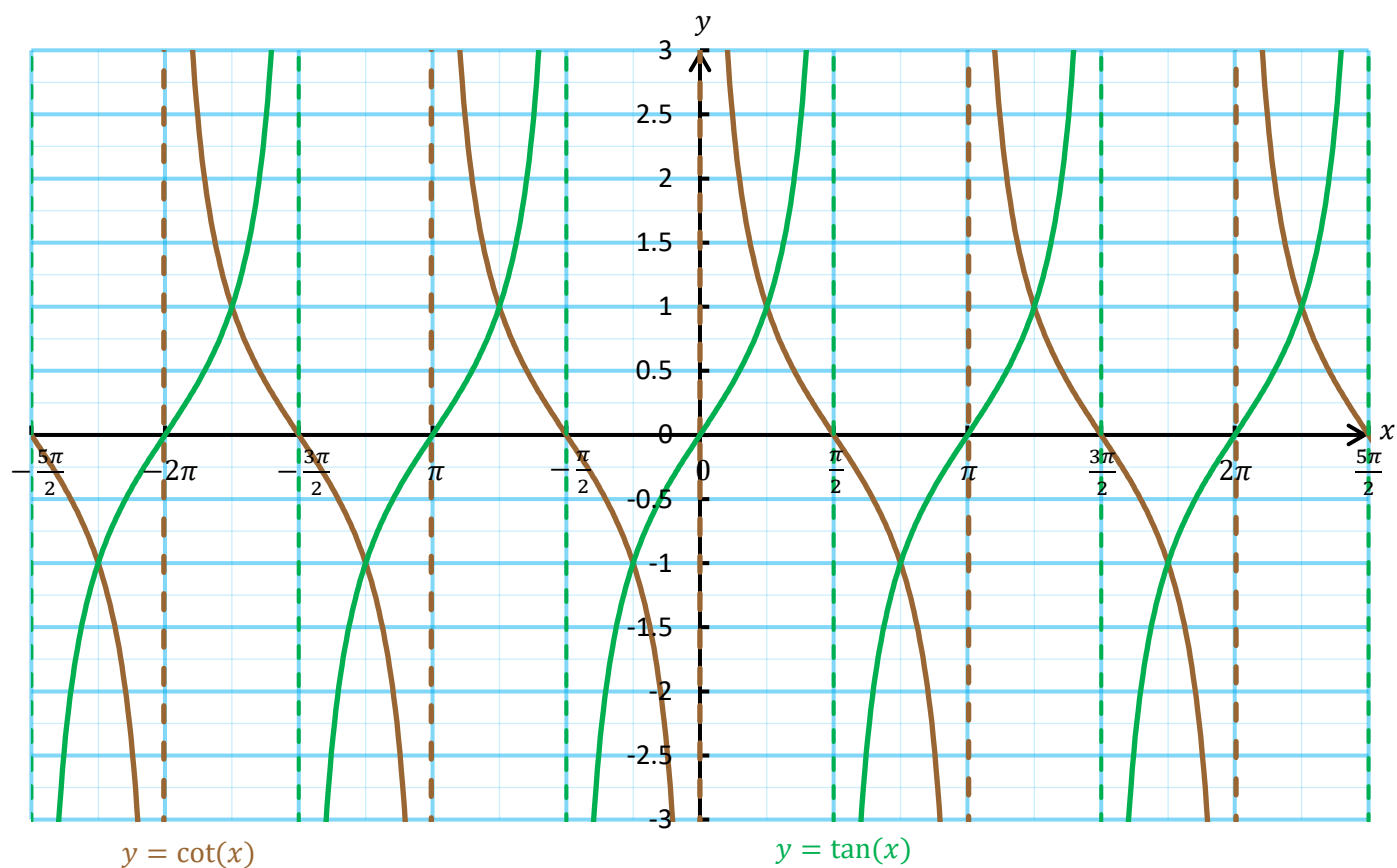
Vertical asymptotes every 180°



$y = \operatorname{cosec}(x)$

Domain $(-\infty, \infty)$, $x \neq k\pi$ Range $(-\infty, -1] \cup [1, \infty)$ Vertical asymptotes every 180° 

$y = \cot(x)$

Domain $(-\infty, \infty)$, $x \neq k\pi$ Range $(-\infty, \infty)$ Vertical asymptotes every 180° 

The graphs of $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$

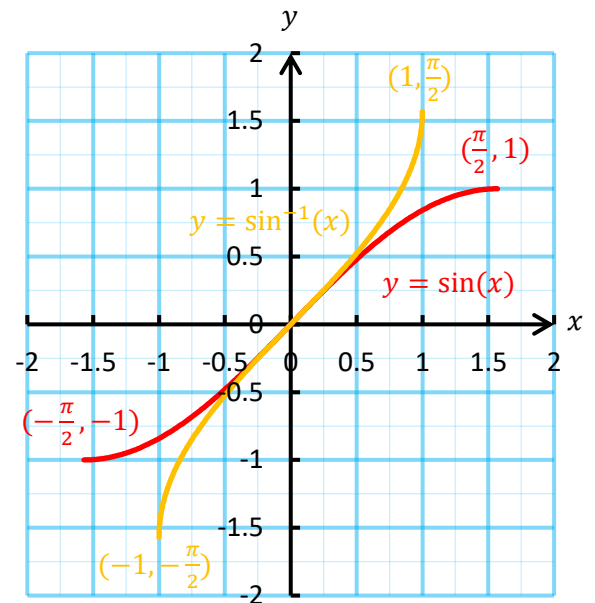
The function $f(x) = \sin(x)$ is a many-to-one function, because a number of angles give the same value.

For example, $\sin(0) = 0$ and also $\sin(\pi) = 0$.

So, the inverse of $f(x) = \sin(x)$ does not exist unless we reduce its domain.

By reducing the domain to be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (which is between -90° and 90°), we can form the function $f^{-1}(x) = \sin^{-1}(x)$.

Function	Domain	Range
$f(x)$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$
$f^{-1}(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$



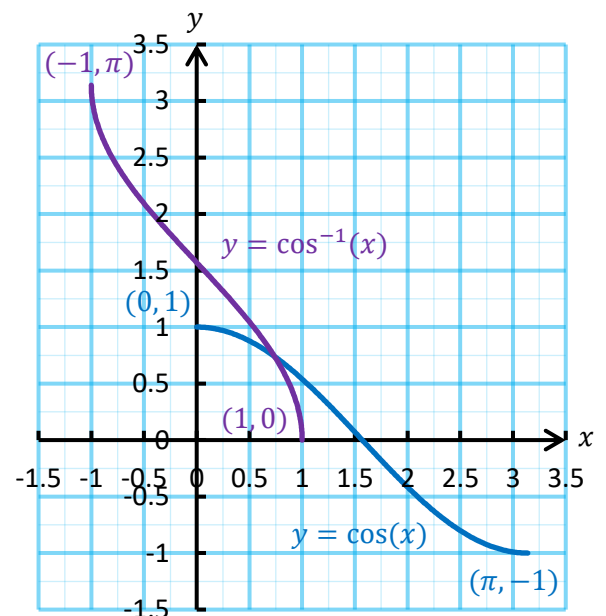
The function $f(x) = \cos(x)$ is a many-to-one function, because a number of angles give the same value.

For example, $\cos(0) = 1$ and also $\cos(2\pi) = 1$.

So, the inverse of $f(x) = \cos(x)$ does not exist unless we reduce its domain.

By reducing the domain to be between 0 and π (which is between 0° and 180°), we can form the function $f^{-1}(x) = \cos^{-1}(x)$.

Function	Domain	Range
$f(x)$	$[0, \pi]$	$[-1, 1]$
$f^{-1}(x)$	$[-1, 1]$	$[0, \pi]$



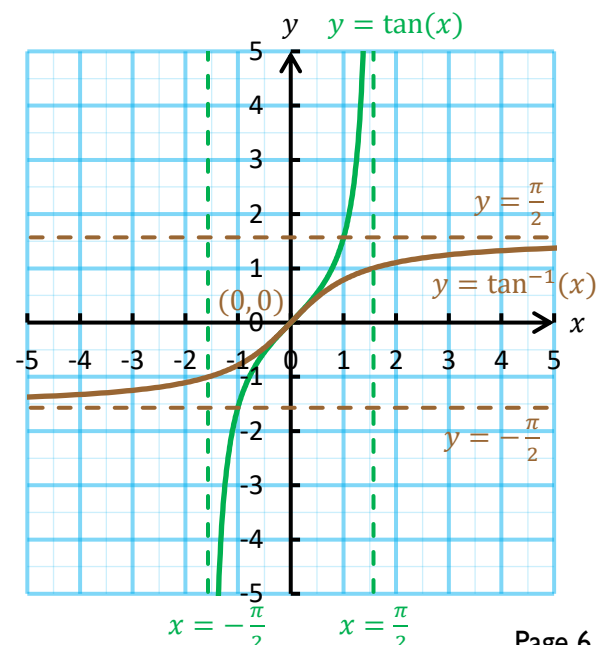
The function $f(x) = \tan(x)$ is a many-to-one function, because a number of angles give the same value.

For example, $\tan(0) = 0$ and also $\tan(\pi) = 0$.

So, the inverse of $f(x) = \tan(x)$ does not exist unless we reduce its domain.

By reducing the domain to be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (which is between -90° and 90°), we can form the function $f^{-1}(x) = \tan^{-1}(x)$.

Function	Domain	Range
$f(x)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$
$f^{-1}(x)$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$



Top 10

9. The function f has domain $(-\infty, 0]$ and is defined by

(a) Find an expression for $f^{-1}(x)$. [4]

(b) Write down the domain of f^{-1} . [2]

8. The function f is defined for $x \geq 0$ by $f(x) = 3x^2 + 4$.

- (a) Find an expression for f^{-1} , stating the range and domain of f^{-1} . [6]
- (b) Sketch the graphs of f and f^{-1} using the same axes. [3]

9. The function f has domain $x \leq -1$ and is defined by

(a) Find the range of f . [1]

(b) Find an expression for $f^{-1}(x)$. State the domain and range of f^{-1} . [6]

(Unit 3 Summer 2018)

1	2
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a) Given that f is a function,**i)** state the condition for f^{-1} to exist,**ii)** find $ff^{-1}(x)$.

[2]

b) The functions g and h , are given by

$$g(x) = x^2 - 1,$$

$$h(x) = e^x + 1.$$

i) Suggest a domain for g such that g^{-1} exists.**ii)** Given the domain of h is $(-\infty, \infty)$, find an expression for $h^{-1}(x)$ and sketch, using the same axes, the graphs of $h(x)$ and $h^{-1}(x)$. Indicate clearly the asymptotes and the points where the graphs cross the coordinate axes.**iii)** Determine an expression for $gh(x)$ in its simplest form.

[8]

Handwriting practice lines consisting of 20 horizontal dotted lines.

1	1
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$$f(x) = \frac{\sqrt{x^2 - 1}}{x}$$

a) Find an expression for $f^{-1}(x)$. State the domain for f^{-1} and sketch both $f(x)$ and $f^{-1}(x)$ on the same diagram. [6]

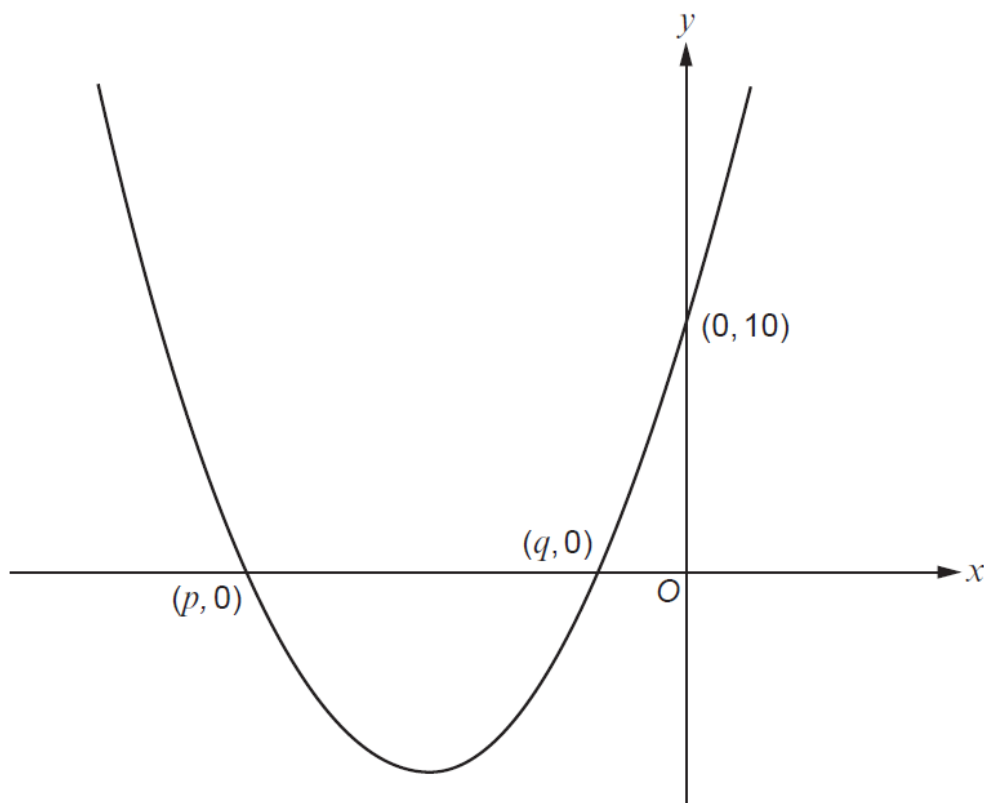
b) Explain why the function $ff(x)$ cannot be formed. [1]

(Unit 3 Summer 2022)

1	2
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The diagram below shows a sketch of the graph of $y = f(x)$, where

$$f(x) = 2x^2 + 12x + 10.$$

The graph intersects the x -axis at the points $(p, 0)$, $(q, 0)$ and the y -axis at the point $(0, 10)$.

- a) Write down the value of $ff(p)$. [1]
- b) Determine the values of p and q . [2]
- c) Express $f(x)$ in the form $a(x+b)^2 + c$, where a, b, c are constants whose values are to be found. Write down the coordinates of the minimum point. [3]
- d) Explain why $f^{-1}(x)$ does not exist. [1]
- e) The function $g(x)$ is defined as
- $$g(x) = f(x) \quad \text{for} \quad -3 \leq x < \infty.$$
- i) Find an expression for $g^{-1}(x)$. [4]
- ii) Sketch the graph of $y = g^{-1}(x)$, indicating the coordinates of the points where the graph intersects the x -axis and the y -axis. [2]

Handwriting practice lines consisting of 30 horizontal dotted lines.

1	0
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 Two real functions are defined as

$$g(x) = (x-2)^2 \quad \text{for} \quad -\infty < x < \infty.$$

- a)**
- i)** Find an expression for $fg(x)$. [2]
 - ii)** Determine the values of x for which $fg(x)$ does not exist. [3]
- b)** Find an expression for $f^{-1}(x)$. [3]

(Unit 3 Summer 2024)

10. The function f has domain $[4, \infty)$ and is defined by

$$f(x) = \frac{2(3x+1)}{x^2-2x-3} + \frac{x}{x+1}.$$

(a) Show that $f(x) = \frac{x+2}{x-3}$.

[4]

(b) Determine the range of $f(x)$.

[2]

(c) Find an expression for $f^{-1}(x)$ and write down the domain and range of f^{-1} . [4]

(d) Find the value of x when $f(x) = f^{-1}(x)$. [4]

9. The function f has domain $[4, \infty)$ and is defined by

$$f(x) = \frac{1}{2} \sqrt{x-3} \quad .$$

- (a) Find an expression for $f^{-1}(x)$. Write down the range and domain of f^{-1} . [5]
- (b) Sketch the graph of $y = f^{-1}(x)$. On the same diagram, sketch the graph of $y = f(x)$. [3]

9. The function f has domain $(-\infty, -\frac{1}{2})$ and is defined by

(a) Find an expression for $f^{-1}(x)$. [4]

(b) Write down the domain of f^{-1} . [2]

Revision Questions

(C3 Summer 2019)

8. The function f has domain $(-\infty, 1]$ and is defined by

$$f(x) = \frac{4x+3}{7-5x} \text{ .}$$

- (a) Find an expression for $f^{-1}(x)$. [4]
- (b) Write down the domain of f^{-1} . [2]
- (c) (i) Find the exact value of $f^{-1}(0.5)$.
- (ii) By carrying out an appropriate calculation involving f , verify that your answer to (c) (i) is correct. [3]

9. The function f has domain $(-\infty, 4)$ and is defined by

(a) Express $f(x)$ in the form

where a, b are constants whose values are to be found. [1]

(b) Hence or otherwise, find an expression for $f^{-1}(x)$. [4]