A Level

Unit 1

Textbook

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Available from www.mathemateg.com:

- Microsoft Word versions of all workbooks.
- Interactive GeoGebra resources.
- Model answers to past examination papers.
- Answers to the workbooks (available, on request, to teachers; send an e-mail from a school account to gareth@mathemateg.com).

Available from www.youtube.com/adolygumathemateg:

- Full Welsh medium revision fideos for the workbooks. Search for the playlists for 'Lefel A Uned 1' and 'Mathemateg Ychwanegol'.

Available from the TikTok account 'Minute Maths':

- Live revision sessions.
- Short revision clips.

Contents

Unit 1 WJEC Revision Sheet

Proof

(a) Proof by **deduction**.
(b) Proof by **exhaustion**.
(c) **Disproof** by **counter-example**.

Rules of Indices

$n^a \times n^b = n^{a+b}$.
$n^a \div n^b = n^{a-b}$.
$(n^a)^b = n^{a\times b}$.
$n^0 = 1$.
$n^{-a} = \frac{1}{n^a}$.
$n^{\frac{1}{a}} = \sqrt[a]{n}$.
$\left(\sqrt[b]{n}\right)^a = n^{\frac{a}{b}} = \sqrt[b]{n^a}$.

Surds

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

$\left(\sqrt{a}\right)^2 = a$.

Rationalise the denominator:

$\frac{a+b\sqrt{c}}{d+e\sqrt{f}} = \frac{(a+b\sqrt{c})(d-e\sqrt{f})}{(d+e\sqrt{f})(d-e\sqrt{f})}$.

E.g. $\frac{2+4\sqrt{2}}{3+5\sqrt{2}} = \frac{(2+4\sqrt{2})(3-5\sqrt{2})}{(3+5\sqrt{2})(3-5\sqrt{2})}$

$= \frac{6-10\sqrt{2}+12\sqrt{2}-20\times 2}{9-15\sqrt{2}+15\sqrt{2}-25\times 2} = \frac{-34+2\sqrt{2}}{-41}$.

The Discriminant

Two distinct real roots:
$b^2 - 4ac > 0$.
Two real roots:
$b^2 - 4ac \geq 0$.
One real root (repeated):
$b^2 - 4ac = 0$.
Two complex roots /
No real roots:
$b^2 - 4ac < 0$.

Completing the Square

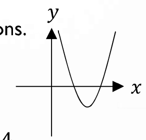
$x^2 + 2ax = (x+a)^2 - a^2$.

$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right)$.

Solving Equations and Inequalities

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Simultaneous equations.



$x^2 - 6x + 8 \geq 0$
$(x-2)(x-4) \geq 0$
Either $x \leq 2$ or $x \geq 4$.

The Remainder Theorem

When the polynomial $f(x)$ is divided by the polynomial $x - a$, where a is a constant, then the remainder at the end of the calculation is $f(a)$.

The Factor Theorem

(a) The polynomial $(x - a)$ is a **factor** of the polynomial $f(x)$ if $f(a) = 0$.
(b) If $f(a) = 0$, then the polynomial $(x - a)$ is a **factor** of the polynomial $f(x)$.

Graphs

Polynomial graphs, e.g.
$y = x^3 - 4x^2 - 17x + 60$.
Reciprocal graphs, e.g. $y = \frac{2}{x^2}$.
Asymptotes.
Transformations:
$y = f(x - a)$
Move the graph a units **right**.
$y = f(x) + a$
Move the graph a units **up**.
$y = af(x)$
Stretch along the y–axis.
$y = f(ax)$
Compress along the x–axis.

Co-ordinate Geometry

Let A = (x_1, y_1), B = (x_2, y_2).
Length of AB =
$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.
Gradient of AB: $m = \frac{y_2-y_1}{x_2-x_1}$.

Equation of AB:
$y - y_1 = m(x - x_1)$.
Mid-point of AB = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Lines L_1 and L_2 with gradients m_1 and m_2:

L_1 and L_2 are **parallel**: $m_1 = m_2$.
L_1 and L_2 are **perpendicular**:
$m_1 = -\frac{1}{m_2}$, $m_2 = -\frac{1}{m_1}$,
$m_1 m_2 = -1$.

If a **tangent** has gradient m then the **normal** has gradient $-\frac{1}{m}$.

Equation of a Circle

Equation of a circle C with centre (a, b) and radius r:
$(x-a)^2 + (y-b)^2 = r^2$ or
$x^2 + y^2 + 2gx + 2fy + c = 0$,
where $r = \sqrt{g^2 + f^2 - c}$;
$(a, b) = (-g, -f)$.

Tangent and radius meet at 90°, so the gradient of the tangent is the **negative reciprocal** of the gradient of the radius.

Intersection of two circles C_1 and C_2 with centres A, B; radii r_1, r_2:
Two intersections:
Length $AB < r_1 + r_2$.

One intersection (the circles tough **externally**):
Length $AB = r_1 + r_2$.

One intersection (the circles touch **internally**):
Length $AB = |r_1 - r_2|$.

No intersections:
Length $AB > r_1 + r_2$.
Length $AB < |r_1 - r_2|$.

Circle Theorems

(a) The angle in a **semicircle** is a right angle.
(b) The perpendicular from the centre to a chord **bisects** the chord.
(c) The radius of a circle at a given point on its circumference is **perpendicular** to the tangent to the circle at that point.

The Binomial Theorem

Pascal's Triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
etc...

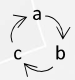
$(a + bx)^n$
$= a^n + na^{n-1}(bx)$
$+ \frac{n(n-1)}{2}a^{n-2}(bx)^2$
$+ \cdots + (bx)^n$.

Trigonometry

$S\frac{O}{H}C\frac{A}{H}T\frac{O}{A}$.

Exact values for 30°, 45°, 60°.
The graphs of sin, cos, tan.

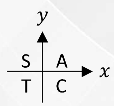
Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.



Cosine rule:
$a^2 = b^2 + c^2 - 2bc\cos A$
or $\cos A = \frac{b^2+c^2-a^2}{2bc}$.

Area of a triangle $= \frac{1}{2}ab\sin C$.

$\sin^2\theta + \cos^2\theta = 1$.
$\sin^2\theta = 1 - \cos^2\theta$.
$\cos^2\theta = 1 - \sin^2\theta$.
$\tan\theta = \frac{\sin\theta}{\cos\theta}$.



Exponentials and Logarithms

If $b^y = x$ then $y = \log_b(x)$.
Solving equations with logarithms.

The graphs of $y = a^x$, $y = e^x$ and $y = \ln(x)$.

Exponential models: $y = Ae^{kt}$ (growth) or $y = Ae^{-kt}$ (decay).
$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$.
Limitations of models.

Rules:
$\log_a(xy) = \log_a(x) + \log_a(y)$.
$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$.
$\log_a(x^n) = n\log_a(x)$.

PROOF

Curve fitting:
$y = ax^n$ (polynomial):
$\log(y) = n\log(x) + \log(a)$.
$y = kb^x$ (exponential):
$\log(y) = \log(b)x + \log(k)$.

Differentiation

From first principles:

$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Quick differentiation: If $y = ax^n$ then $\frac{dy}{dx} = nax^{n-1}$.

Finding the equation of a tangent or normal.

Identify where functions are **increasing** or **decreasing**.

Stationary Points

Solve the equation $\frac{dy}{dx} = 0$ and use the $\frac{d^2y}{dx^2}$ test to find the nature of the stationary points.

If $\frac{d^2y}{dx^2}$ is **negative**, then it is a **maximum** point; if it is **positive**, then it is a **minimum** point; if it is **zero**, more investigation is needed.

Integration

$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$.

$\int_p^q ax^n\,dx = \left[\frac{ax^{n+1}}{n+1}\right]_p^q$.

Vectors

Position vector
$\boldsymbol{a} = (x, y) = x\boldsymbol{i} + y\boldsymbol{j}$.

$|\boldsymbol{a}| = \sqrt{x^2 + y^2}$.

$\boldsymbol{AB} = -\boldsymbol{a} + \boldsymbol{b}$.

The point that divides AB in the ratio $\lambda : \mu$ is $\frac{\mu a + \lambda b}{\lambda + \mu}$.

E.g. C is the point on the line AB so that the ratio $AC : CB$ is $2 : 3$. The vector that goes from the origin to C is
$\frac{3a+2b}{5} = \frac{3}{5}\boldsymbol{a} + \frac{2}{5}\boldsymbol{b}$.

[Vector equation of the line AB:
$\boldsymbol{r} = \boldsymbol{a} + \lambda(-\boldsymbol{a} + \boldsymbol{b})$.]

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if $\boldsymbol{a} = \alpha\boldsymbol{b}$ for some number α.

The Examination

Length: 2 hours 30 minutes.
120 marks.
25% of the A Level qualification.
62.5% of the AS Level.

Checklist

- ❑ I have attempted all the past paper questions.
- ❑ I know which formulae appear in the formulae booklet.
- ❑ I can check answers using a graphical calculator.

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April 2025
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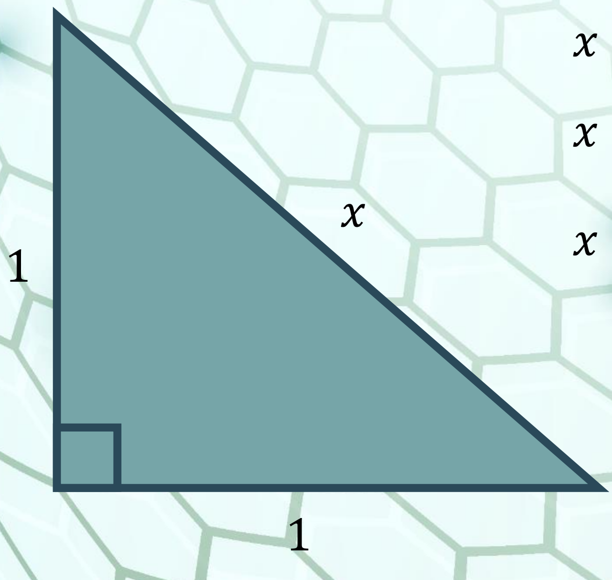
Unit 1, Workbook 1

12

Rules of Indices and Surds



$$x = \sqrt{1^2 + 1^2}$$

$$x = \sqrt{2}$$

$$x = 2^{\frac{1}{2}}$$

Name:

Background

What is the work?	Revising how to work with powers and indices.
What is required before starting?	**GCSE Work:** Pythagoras' Theorem; expansion of brackets; rules of indices; surds.
Where does this lead to?	**Unit 1:** Special values in trigonometry; logarithm proofs; curve fitting. **Unit 3:** Proof; binomial expansion. **Applications:** Electrical engineering; digital signal processing.

Theory

Welcome to the A Level Mathematics course! During the course you will build on your GCSE studies to solve increasingly more complex problems, in and out of context.

Mathematics is the art of changing the complex to the simple.

In studying mathematics at A Level, you will develop your analytical skills and your problem solving skills. These skills are valued in the workplace, and are transferable to a number of different areas.

Exercise 1

Solve the Sudoku puzzle below.

4			8	7			2	
	8					4		
		6	3			8		1
7			1				8	
6	1	2		9	8	7	3	4
				6			1	9
1	9	3	4	2	7	5		
8		7		1		3		2
	2				3			

Exercise 2

Imagine that you are standing at the seaside and looking out to sea. How far can you see? (When does the sea disappear?)

Next, imagine that you are standing on the top of Yr Wyddfa. How far can you see? Is it possible to see Blackpool Tower?

Here is a method for estimating the distance to the horizon.

In order to calculate the distance D (in km) to the horizon,

- Find (or measure) the height U between your eye level and the level of the sea, in metres.
- Multiply by 13.
- Calculate the square root.

(a) Use the above instructions to estimate the distance to the horizon if you are standing at the seaside.

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....

....

....

(b) Use the above instructions to estimate the distance to the horizon if you are standing on top of Yr Wyddfa. (The height of Yr Wyddfa is 1,085 m.)

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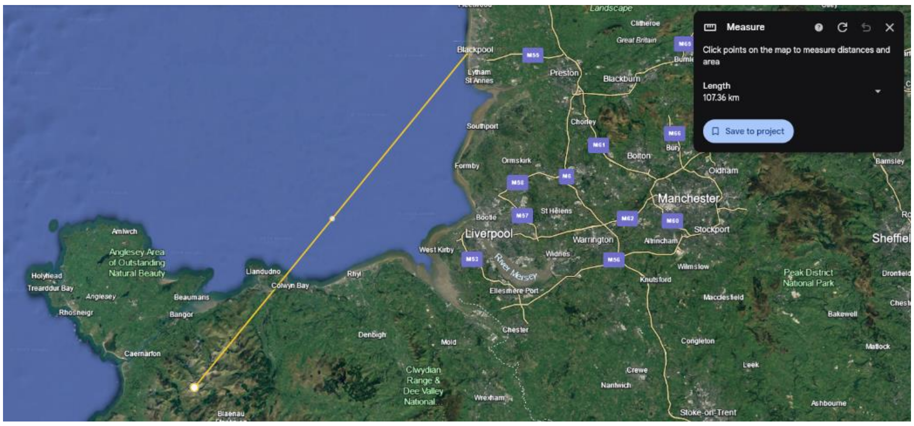
(c) Is it possible to see Blackpool Tower from the top of Yr Wyddfa?



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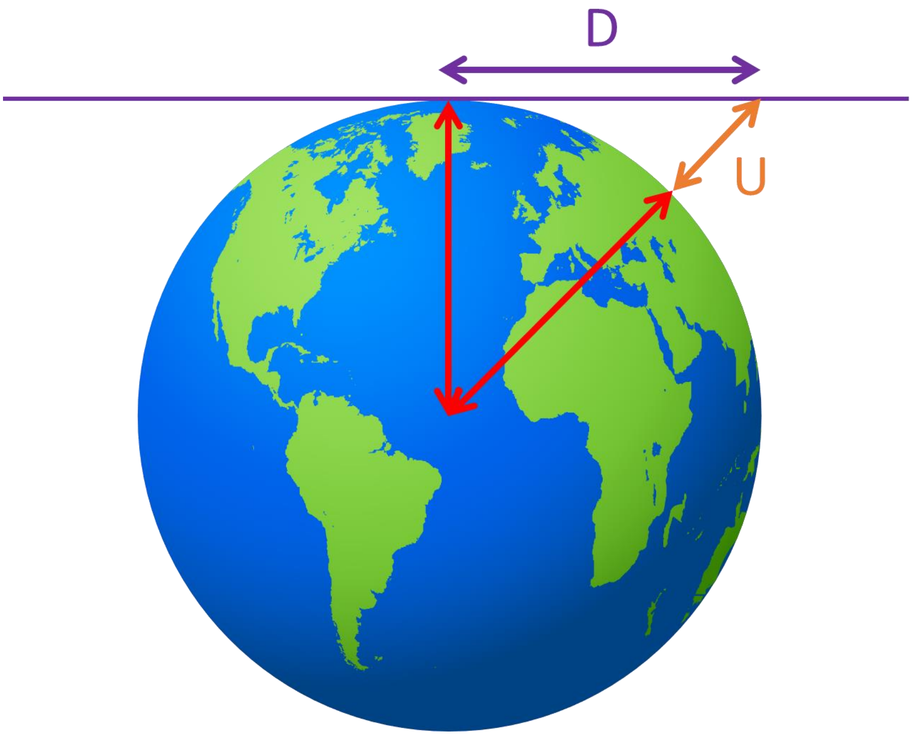
(d) Use the diagram below to explain why the method on the previous page works.
(You will need to use Pythagoras' Theorem, and the fact that the radius of the earth is approximately 6,500 km.)



Rules of Indices

GCSE Playlist

During the course, you will need to use the following rules of indices.

$n^a \times n^b = n^{a+b}$ $\quad$ $n^a \div n^b = n^{a-b} = \frac{n^a}{n^b}$

$n^0 = 1$ $\quad$ $(n^a)^b = n^{ab}$

$(nm)^a = n^a m^a$ $\quad$ $\left(\frac{n}{m}\right)^a = \frac{n^a}{m^a}$

$n^{-a} = \frac{1}{n^a}$ $\quad$ $\left(\sqrt[b]{n}\right)^a = n^{\frac{a}{b}} = \sqrt[b]{n^a}$

Exercise 3

Simplify the following.

(a) $x^4 \times x^3$

(b) $\frac{y^{12}}{y^2}$

(c) z^0

(d) $(x^5)^3$

(e) $(x^2 y)^4$

(f) $\left(\frac{2}{5}\right)^3$

(g) 3^{-2}

(h) $25^{\frac{1}{2}}$

(i) $16^{\frac{3}{4}}$

(j) $27^{-\frac{2}{3}}$

(k) $\frac{x^4 \times x^{-2}}{x^{-8}}$

(l) $\frac{2^{-1}}{\left(\frac{1}{4}\right)^{\frac{1}{2}}}$

Surds

Simplifying Surds

Expanding with Surds

The following rules build upon the GCSE work on surds.

$$\sqrt{n}\times\sqrt{n}=n$$

$$\sqrt[a]{n}\times\sqrt[a]{m}=\sqrt[a]{nm} \qquad \sqrt[a]{n}\div\sqrt[a]{m}=\sqrt[a]{\frac{n}{m}}=\frac{\sqrt[a]{n}}{\sqrt[a]{m}}$$

Rationalising the denominator: $\frac{a}{b\sqrt{c}}=\frac{a\times\sqrt{c}}{b\sqrt{c}\times\sqrt{c}}=\frac{a\sqrt{c}}{bc}$, $\quad \frac{a+b\sqrt{c}}{d+e\sqrt{f}}=\left(\frac{a+b\sqrt{c}}{d+e\sqrt{f}}\right)\times\left(\frac{d-e\sqrt{f}}{d-e\sqrt{f}}\right)$

Example

$\frac{5-2\sqrt{5}}{3+7\sqrt{10}}$

$=\left(\frac{5-2\sqrt{5}}{3+7\sqrt{10}}\right)\times\left(\frac{3-7\sqrt{10}}{3-7\sqrt{10}}\right)$

$=\frac{15-35\sqrt{10}-6\sqrt{5}+14\times\sqrt{5}\times\sqrt{10}}{9-21\sqrt{10}+21\sqrt{10}-49\times\sqrt{10}\times\sqrt{10}}$

$=\frac{15-35\sqrt{10}-6\sqrt{5}+14\times\sqrt{5}\times\sqrt{5}\times\sqrt{2}}{9-49\times 10}$

$=\frac{15-35\sqrt{11}-6\sqrt{5}+14\times 5\times\sqrt{2}}{-481}$

$=\frac{15-35\sqrt{11}-6\sqrt{5}+70\sqrt{2}}{-481}$

Exercise 4

Simplify the following.

(a) $\sqrt{6}\times\sqrt{6}$

(b) $\sqrt[3]{5}\times\sqrt[3]{25}$

(c) $\sqrt[5]{96}\div\sqrt[5]{3}$

(d) $\sqrt{45}$

(e) $\frac{5}{\sqrt{2}}$

(f) $\frac{7+\sqrt{2}}{3+\sqrt{2}}$

Top 10

(C1 Winter 2005)

2. Simplify

$$\frac{6 + \sqrt{7}}{\sqrt{7} - 2},$$

expressing your answer in surd form. [4]

(C1 Winter 2006)

2. *(a)* Simplify the following.

$$\sqrt{48} + \sqrt{27} - \frac{6}{\sqrt{3}}$$ [4]

(b) Simplify $\frac{2+\sqrt{7}}{3+\sqrt{7}}$, expressing your answer in surd form. [4]

(C1 Summer 2006)

2. Simplify each of the following, expressing your answers in surd form.

(*a*) $\dfrac{5-\sqrt{3}}{\sqrt{3}+1}$, [4]

(*b*) $\left(2+\sqrt{3}\right)\left(4-\sqrt{12}\right)$. [4]

(C1 Summer 2005)

2. Simplify each of the following, expressing your answers in surd form:

(*a*) $\sqrt{45}+\sqrt{80}-\sqrt{125}$; [3]

(*b*) $\dfrac{6+\sqrt{2}}{2+\sqrt{2}}$. [4]

(C1 Winter 2007)

2. Simplify **each** of the following expressions, expressing your answers in surd form.

(a) $2\sqrt{32} + 3\sqrt{8} - \sqrt{18}$ [3]

(b) $\dfrac{6+\sqrt{30}}{6-\sqrt{30}}$ [4]

(C1 Summer 2008)

2. Simplify

(a) $\sqrt{75} - \dfrac{9}{\sqrt{3}} + \left(\sqrt{6} \times \sqrt{2}\right)$, [4]

(b) $\dfrac{5\sqrt{5}-2}{4+\sqrt{5}}$. [4]

(C1 Summer 2010)

2. Simplify

(a) $\dfrac{5\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, [4]

(b) $\left(\sqrt{15}\times\sqrt{20}\right)-\sqrt{75}-\dfrac{\sqrt{60}}{\sqrt{5}}$. [4]

(C1 Winter 2010)

2. Simplify

(a) $\dfrac{2\sqrt{11}-3}{\sqrt{11}+2}$, [4]

(b) $\dfrac{22}{\sqrt{2}}-\sqrt{50}-\left(\sqrt{2}\right)^{5}$. [4]

(C1 Summer 2011)

2. Simplify

(*a*) $\dfrac{9}{\sqrt{3}-1}+\dfrac{7}{\sqrt{3}+1}$, [4]

(*b*) $\dfrac{90}{\sqrt{3}}-\sqrt{6}\times\sqrt{8}-\left(2\sqrt{3}\right)^{3}$. [4]

(Unit 1 Summer 2019)

0 7 Given that a, b are integers, simplify the following. Show all your working.

a) $\dfrac{2\sqrt{3}+a}{\sqrt{3}-1}$ [3]

b) $\dfrac{2\sqrt{6b^2}}{\sqrt{2}} - \sqrt{27} + \sqrt{192}$ [3]

Revision Question

(Unit 1 Summer 2018)

0 1 Showing all your working, simplify

a) $$\frac{24\sqrt{a}}{\left(\sqrt{a}+3\right)^2-\left(\sqrt{a}-3\right)^2},$$ [3]

b) $$\frac{3\sqrt{7}+5\sqrt{3}}{\sqrt{7}+\sqrt{3}}.$$ [4]

Unit 1, Workbook 2

12

Cartesian Coordinate Geometry

René Descartes was born on the 15th of March, 1596, in France. He used graphs in order to connect algebra to geometry. Graphs that use x and y axes are named **Cartes**ian graphs.

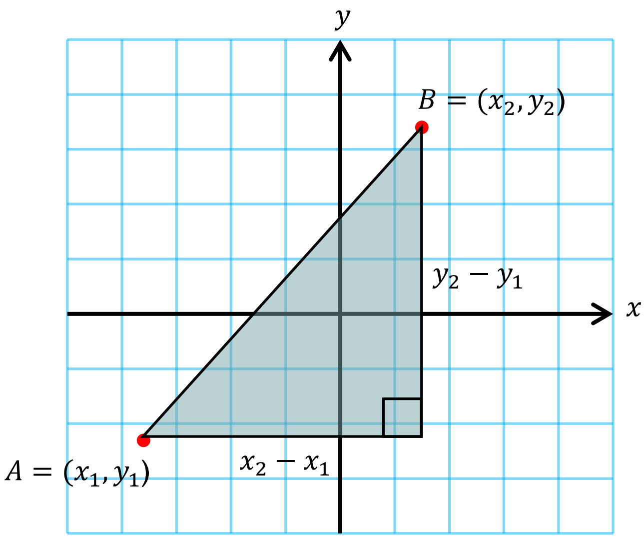
Name:

Background

What is the work?	Imagine a straight line connecting two different coordinates. This work allows us to find things such as the *equation of the straight line*; the *mid-point* of the two coordinates; the *gradient* of the straight line; and the *distance* between the two coordinates.
What is required before starting?	**GCSE Work:** Mean; coordinates; solving equations; substitution; solving equations; area; the reciprocal; Pythagoras' Theorem; gradient; equation of a straight line; trigonometry; changing the subject; surds.
Where does this lead to?	**Unit 1:** Equation of a circle; equation of the tangent and normal. **Unit 3:** Equation of the tangent and normal. **Applications:** GPS; creating and viewing PDF files.

Theory

Consider any two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in a plane.



We can use the *mean* to find the **mid-point** of AB: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

We can use *Pythagoras' Theorem* to find the **length** of AB: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The **gradient** of the line AB is $\frac{y_2-y_1}{x_2-x_1}$.

The **equation** of the line AB is $y - y_1 = m(x - x_1)$, where m represents the gradient of the line, and (x_1, y_1) represents any point on the line. We can re-arrange this equation to give the forms $y = mx + c$ or $ax + by + c = 0$, as required.

Two straight lines are **parallel** if their gradients are equal. Two straight lines are **perpendicular** if the product of their gradients is -1, or one gradient is the negative reciprocal of the other.

Length of AB

Gradient of AB

Equation of AB

Parallel and Perpendicular Gradients

Exercises

Exercise 1

For the points $A = (2, 1)$ and $B = (6, 4)$, find (a) the mid-point of the two points; (b) the distance between the two points; (c) the gradient of the straight line connecting the two points; (d) the equation of the straight line connecting the two points, in the form $ax + by + c = 0$.

Exericse 2

For the following points, find (i) the mid-point of the two points; (ii) the distance between the two points; (iii) the gradient of the straight line connecting the two points; (iv) the equation of the straight line connecting the two points, in the form $ax + by + c = 0$.

(a) $A = (3,\ 1)$, $B = (4,\ 6)$. (b) $A = (-3, 2)$, $B = (9,\ 4)$.

(c) $A = (2,\ 5)$, $B = (-5, -3)$. (d) $A = (-10, -1)$, $B = (-2, -5)$.

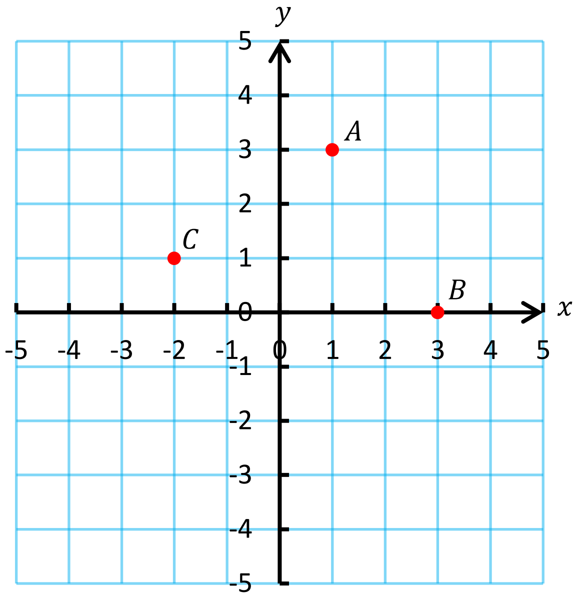
Exercise 3

Complete the following table.

A	B	Mid-point of AB
(2, 8)	(8, 14)	
(6, 8)		(10, 12)
	(20, 14)	(10, 5)
(–2, 5)		(5, 3)
	(7, –8)	(14, –16)
(–8, –5)		(–3, 4)
	(–5, 1)	(13, –2)

Exercise 4

The following diagram shows the three points $A = (1, 3)$, $B = (3, 0)$, $C = (-2, 1)$.



(a) Find the gradient of the line AB.

(b) Find the gradient of the line AC.

(c) Show that the lines AB and AC are perpendicular to each other.

(d) Find the length of the line AB.

(e) Find the length of the line AC.

(f) Find the area of the triangle ABC.

(g) Find the value of $\tan A\hat{B}C$.

(h) Find the mid-point of the line AB.

(i) Find the equation of the line AB.

(j) Find the equation of the straight line passing through the mid-point of AB and perpendicular to AB.

Top 10

(C1 Summer 2006)

1. The points A, B, C, D have coordinates $(3, 2)$, $(-4, 3)$, $(5, 6)$, $(4, -1)$, respectively.

(a) Show that the lines AC and BD are perpendicular. [5]

(b) Show that the line AC has equation

$$2x - y - 4 = 0$$

and find the equation of the line BD. [4]

(c) Find the coordinates of E, the point of intersection of AC and BD. [2]

(d) Find the length of AE. [2]

(C1 Summer 2007)

1. The points A, B, C, D have coordinates $(-1, 3)$, $(1, 7)$, $(2, -1)$, $(5, k)$, respectively. The line AB is parallel to the line CD.

(a) Find the gradient of AB. [2]

(b) Show that $k = 5$. [3]

(c) The line L is perpendicular to CD and passes through the point A. Show that the equation of L is $x + 2y - 5 = 0$. [3]

(d) The line L intersects the line CD at the point E. Find the coordinates of E. [4]

(C1 Winter 2009)

1. The points A, B, C have coordinates $(2, -1)$, $(-7, 1)$, $(5, 4)$, respectively. The line through A perpendicular to the line BC intersects BC at the point D.

(a) Show that the equation of BC is

$$x - 4y + 11 = 0,$$

and find the equation of AD. [7]

(b) Show that the coordinates of D are $(1, 3)$. [2]

(c) Find the length of CD. [2]

(d) The line AD is extended to E so that D is the mid-point of AE. Find the coordinates of E. [2]

(C1 Summer 2009)

1. The points A, B, C are such that A, B have coordinates $(-1, 5)$, $(7, 11)$, respectively and C is the mid-point of AB. The line L is the perpendicular bisector of AB.

(a) Find the gradient of AB. [2]

(b) Find the coordinates of C. [2]

(c) Show that the equation of L is

$$4x + 3y - 36 = 0.$$ [4]

(d) The line L intersects the x-axis at the point D.

(i) Find the coordinates of D.

(ii) Find the length of CD.

(iii) Find the value of $\tan C\hat{A}D$. [6]

(C1 Summer 2011)

1. The points A and B have coordinates (3, 11) and (9, –1) respectively.
The line L_1 passes through the point B and is **perpendicular** to AB.

(a) Find the gradient of AB. [2]

(b) Find the equation of L_1 and simplify your answer. [4]

The line L_2 has equation $6x + 7y + 10 = 0$.
The lines L_1 and L_2 intersect at the point C.

(c) (i) Show that C has coordinates (3, –4).

(ii) Find the length of BC.

(iii) Find the coordinates of the mid-point of BC.

(iv) Write down the equation of the line AC. [7]

(C1 Summer 2012)

1. The points A, B, C are such that A, B have coordinates $(-4, 7)$, $(2, -1)$ respectively and C is the mid-point of AB. The line L is the perpendicular bisector of AB.

(a) Find the gradient of AB. [2]

(b) Find the coordinates of C. [2]

(c) Show that the equation of L is

$$3x - 4y + 15 = 0.$$ [4]

(d) The point D lies on L and has coordinates $(7, k)$.

(i) Show that $k = 9$.

(ii) Find the length of CA and the length of DA.

(iii) Hence show that the value of $\sin A\hat{D}C$ may be expressed in the form $\frac{1}{\sqrt{a}}$, where a is an integer whose value is to be found. [7]

(C1 Winter 2014)

1. The points A and B have coordinates (6, −2) and (4, 1), respectively.
The line L_1 passes through the point B and is perpendicular to AB.

(a) (i) Find the gradient of AB.

(ii) Find the equation of L_1. [5]

(b) The line L_2 passes through A and has equation $x - 8y - 22 = 0$.
The lines L_1 and L_2 intersect at the point C.

(i) Show that C has coordinates (−2, −3).

(ii) Find the coordinates of the mid-point of AC.

(iii) Find the area of triangle ABC, simplifying your answer. [9]

(C1 Summer 2017)

1. The points A and B have coordinates $(-2, 3)$ and $(4, 5)$ respectively.
The line L_1 passes through the point B and is **perpendicular** to AB.

(a) (i) Find the gradient of AB.

(ii) Find the equation of L_1. [5]

The line L_2 has equation $x + 2y + 1 = 0$.
The lines L_1 and L_2 intersect at the point C.

(b) (i) Show that C has coordinates $(7, -4)$.

(ii) Show that the value of $\cos B\hat{C}A$ may be expressed in the form $\frac{3}{\sqrt{a}}$, where a is an integer whose value is to be found. [7]

(c) The line CB is extended to the point D so that B is the mid-point of CD.

(i) Find the coordinates of D.

(ii) **Write down** the geometrical name for the triangle ACD. [3]

(Unit 1 Summer 2018)

0 2 The points *A* and *B* have coordinates (–1, 10) and (5, 1) respectively. The straight line *L* has equation $2x - 3y + 6 = 0$.

a) The line *L* intersects the line *AB* at the point *C*. Find the coordinates of *C*. [5]

b) Determine the ratio in which the line *L* divides the line *AB*. [2]

c) The line *L* crosses the x-axis at the point *D*. Find the coordinates of *D*. [1]

d) **i)** Show that *L* is perpendicular to *AB*.

ii) Calculate the area of the triangle *ACD*. [6]

(Unit 1 Summer 2019)

0 4 The line L_1 passes through the points $A(-1, 3)$ and $B(2, 9)$. The line L_2 has equation $2y + x = 25$ and intersects L_1 at the point C. L_2 also intersects the x-axis at the point D.

a) Show that the equation of the line L_1 is $y = 2x + 5$. [3]

b) **i)** Find the coordinates of the point D.

ii) Show that L_1 and L_2 are perpendicular.

iii) Determine the coordinates of C. [5]

c) Find the length of CD. [2]

d) Calculate the angle ADB. Give your answer in degrees, correct to one decimal place. [5]

Revision Question

(Unit 1 Summer 2023)

0 3 The point A has coordinates $(-2, 5)$ and the point B has coordinates $(3, 8)$. The point C lies on the x-axis such that AC is perpendicular to AB.

a) Find the equation of AB. [3]

b) Show that C has coordinates $(1, 0)$. [3]

c) Calculate the area of triangle ABC. [4]

d) Find the equation of the circle which passes through the points A, B and C. [5]

Unit 1, Workbook 3

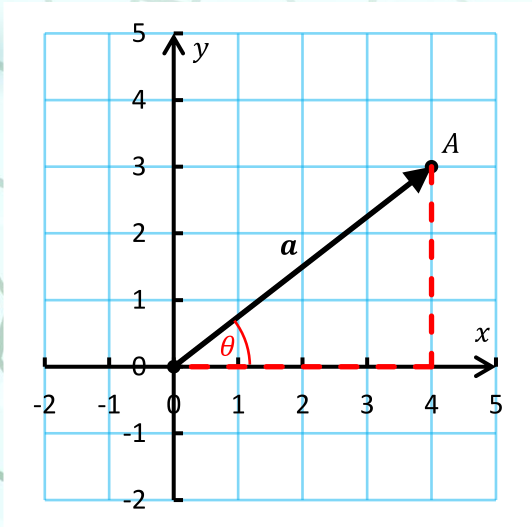
12

Introducing Vectors

$$\boldsymbol{a} = 4\boldsymbol{i} + 3\boldsymbol{j}$$

$$|\boldsymbol{a}| = \sqrt{4^2 + 3^2}$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$



Name:

Background

What is the work?	Vectors allow us to describe location and movement.
What is required before starting?	**GCSE Work:** Co-ordinates; ratio; Pythagoras' Theorem; trigonometry; inequalities.
Where does this lead to?	**Unit 2:** Working with forces. **Unit 4:** Motion in two dimensions. **Cymwysiadau:** Computer graphics; engineering; aviation.

Theory

Theory

A **scalar** measure has magnitude only, with the direction not being important. For example, *speed* is a scalar measure. If a person runs at a speed of 10 mph, then we do not know in which direction the person is running.

A **vector** measure has magnitude **and** direction. For example, *velocity* is a vector measure. If a person runs at a velocity of 10 mph, then we must also note in which direction the person is running. This could be done by using a bearing (e.g. 053°), but we could also write the vector in its **component form**.

Example

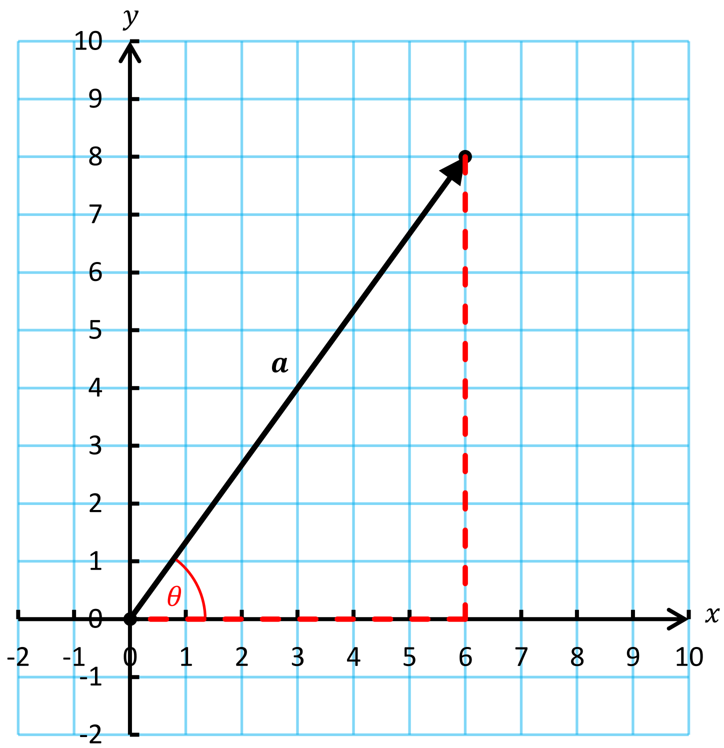
$\boldsymbol{a} = 6\boldsymbol{i} + 8\boldsymbol{j}$ is an example of a vector written in component form.

$\boldsymbol{i}$ represents moving along the x–axis by one unit.

$\boldsymbol{j}$ represents moving along the y–axis by one unit.

In the vector $\boldsymbol{a}$, 6 is the x component and 8 is the y component.

The diagram on the right shows the vector $\boldsymbol{a}$ as an arrow going from the origin to the co-ordinate (6, 8).



We can use Pythagoras' Theorem to calculate the **magnitude** $|\boldsymbol{a}|$ of the vector, which is the distance from the origin to the point (6, 8):

$$|\boldsymbol{a}| = \sqrt{6^2 + 8^2}$$
$$|\boldsymbol{a}| = \sqrt{100}$$
$$|\boldsymbol{a}| = 10$$

We can use trigonometry to calculate the angle θ the vector makes with the x–axis:

$$\tan\theta = \frac{8}{6}$$
$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$
$$\theta = 53.13° \text{ to 2 decimal places.}$$

Notation

On a computer, a vector is written using bold type:

$$\boldsymbol{a} = 6\boldsymbol{i} + 8\boldsymbol{j}$$

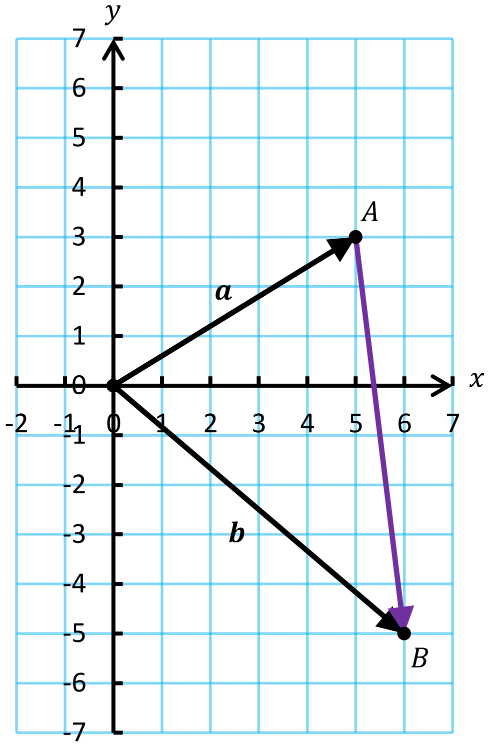
With paper and pencil, a vector is written using underlining:

$$\underline{a} = 6\underline{i} + 8\underline{j}$$

If a vector represents the journey going from the point A to the point B, then the vector is written as $\boldsymbol{AB}$ or $\overrightarrow{AB}$.

Example

In the diagram on the right, the vectors $\boldsymbol{a} = 5\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 6\boldsymbol{i} - 5\boldsymbol{j}$ are the **position vectors** of the points $A = (5, 3)$ and $B = (6, \ -5)$.



We see from the diagram that the vector going from the point A to the point B is $\boldsymbol{AB} = \boldsymbol{i} - 8\boldsymbol{j}$.

To find the vector $\boldsymbol{AB}$ from the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we can use the equation

$$\boldsymbol{AB} = -\boldsymbol{a} + \boldsymbol{b}$$

This represents the following path from the point A to the point B: start at the point A; go back to the origin (against the vector $\boldsymbol{a}$); and then follow the vector $\boldsymbol{b}$ to reach the point B.

In the example,

$$\boldsymbol{AB} = -(5\boldsymbol{i} + 3\boldsymbol{j}) + (6\boldsymbol{i} - 5\boldsymbol{j})$$
$$\boldsymbol{AB} = -5\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{i} - 5\boldsymbol{j}$$
$$\boldsymbol{AB} = \boldsymbol{i} - 8\boldsymbol{j}$$

Parallel Vectors

Two vectors are parallel if $\boldsymbol{a} = \alpha\boldsymbol{b}$ for some number $\alpha \neq 0$.
For example, the two vectors $\boldsymbol{a} = 12\boldsymbol{i} - 21\boldsymbol{j}$ and $\boldsymbol{b} = 4\boldsymbol{i} - 7\boldsymbol{j}$ are parallel because $\boldsymbol{a} = 3\boldsymbol{b}$.

Exercise 1

The diagram on the right shows the vector $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j}$.

(a) Calculate $|\boldsymbol{a}|$.

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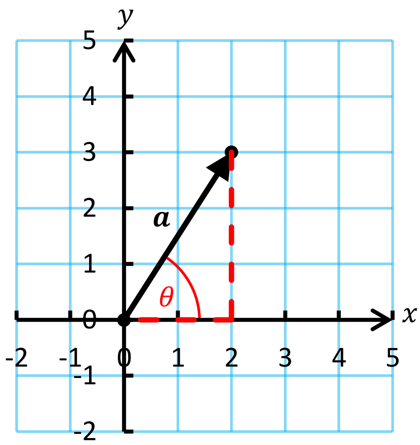

(b) Calculate the size of the angle θ the vector $\boldsymbol{a}$ makes with the x–axis.

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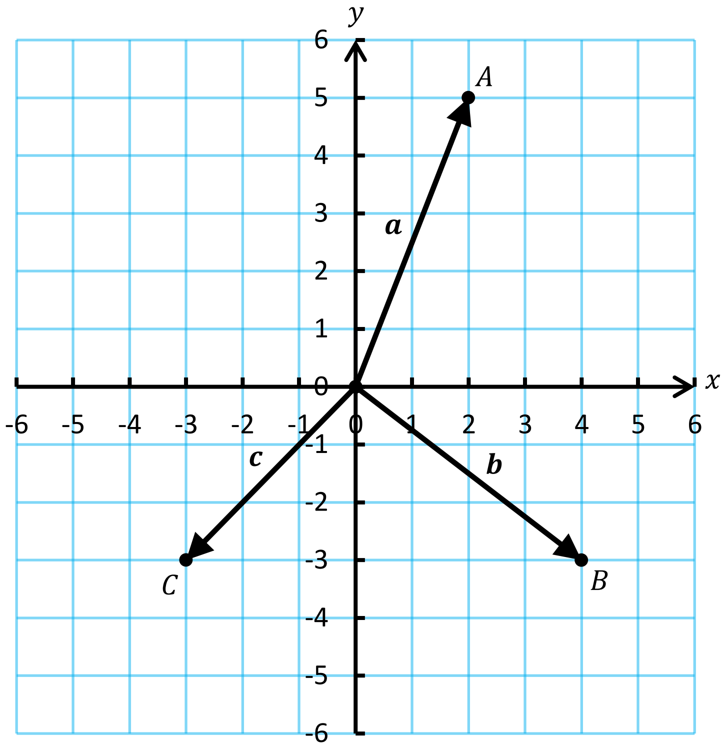
Exercise 2

The diagram on the right shows the vectors $\boldsymbol{a} = 2\boldsymbol{i} + 5\boldsymbol{j}$, $\boldsymbol{b} = 4\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{c} = -3\boldsymbol{i} - 3\boldsymbol{j}$, which are the position vectors of the points A, B and C.



(a) Calculate $|\boldsymbol{a}|$.

(b) Find the vector $\boldsymbol{AB}$.

(c) Find the vector $\boldsymbol{AC}$.

(d) Find the vector $\boldsymbol{BC}$.

(e) Find the vector $\boldsymbol{BA}$.

(f) Calculate the vector $\boldsymbol{a} + \boldsymbol{b}$.

(g) Calculate the vector $3\boldsymbol{a}$.

(h) Calculate the vector $2\boldsymbol{a} - 4\boldsymbol{b}$.

(i) Calculate the vector $3\boldsymbol{b} - 5\boldsymbol{c}$.

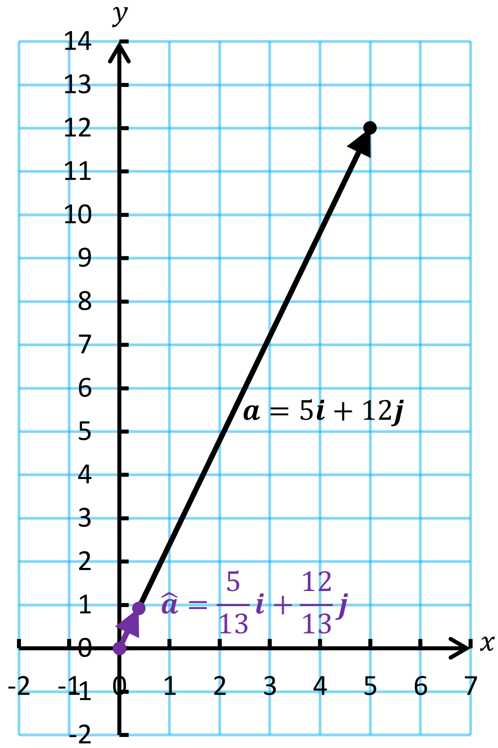
(j) Is the vector $\boldsymbol{a}$ parallel to the vector $6\boldsymbol{i} + 25\boldsymbol{j}$?

(k) Is the vector $\boldsymbol{c}$ parallel to the vector $12\boldsymbol{i} + 12\boldsymbol{j}$?

Unit Vectors

Let $\boldsymbol{a}$ represent a general vector with magnitude $|\boldsymbol{a}|$. The vector $\hat{\boldsymbol{a}}$ is the **unit vector** going in the same direction as the vector $\boldsymbol{a}$, but with length 1 unit.

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$



Example

If $\boldsymbol{a} = 5\boldsymbol{i} + 12\boldsymbol{j}$, then

$$|\boldsymbol{a}| = \sqrt{5^2 + 12^2}$$

$$|\boldsymbol{a}| = 13$$

$$\hat{\boldsymbol{a}} = \frac{5\boldsymbol{i} + 12\boldsymbol{j}}{13}$$

$$\hat{\boldsymbol{a}} = \frac{5}{13}\boldsymbol{i} + \frac{12}{13}\boldsymbol{j}$$

Exercise 3

(a) If $\boldsymbol{a} = 9\boldsymbol{i} + 12\boldsymbol{j}$, find $\hat{\boldsymbol{a}}$.

(b) If $\boldsymbol{a} = 2\boldsymbol{i} + 5\boldsymbol{j}$, find $\hat{\boldsymbol{a}}$.

(c) If $\boldsymbol{a} = 4\boldsymbol{i} - 6\boldsymbol{j}$, find $\hat{\boldsymbol{a}}$.

(d) If $\boldsymbol{a} = 7\boldsymbol{i}$, find $\hat{\boldsymbol{a}}$.

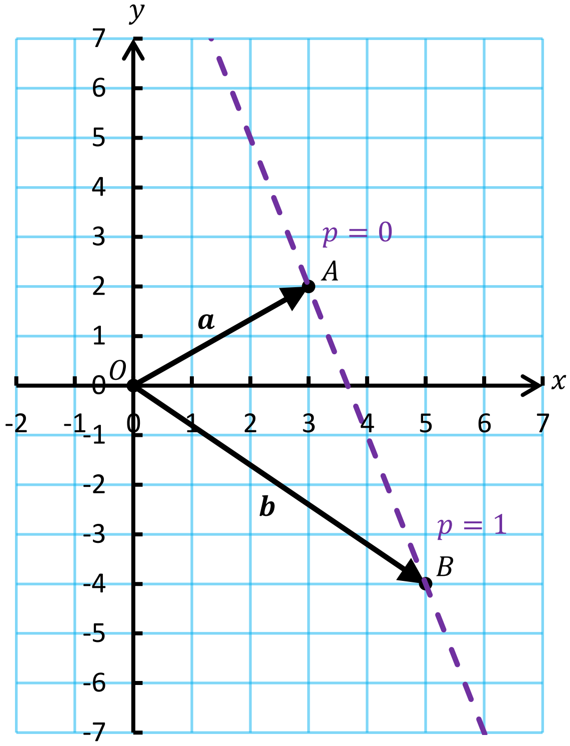
Exercise 4

If $\boldsymbol{a} = 10\boldsymbol{i} + 24\boldsymbol{j}$,

(a) Find $\hat{\boldsymbol{a}}$.

(b) Which vector in the direction of $\boldsymbol{a}$ has length 5 units?

Vector Equation of AB



Let the position vectors of the points A and B, relative to a fixed origin O, be given by $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively.

As we have seen already, the vector going from A to B is $\boldsymbol{AB} = -\boldsymbol{a} + \boldsymbol{b}$.

The **vector equation**

$$\boldsymbol{a} + p(-\boldsymbol{a} + \boldsymbol{b})$$

represents a general point on the line connecting A to B. The **parameter** p is a number controlling our position on the line. If $p = 0$, then we are at the point A. If $p = 1$, then we are at the point B. If $0 < p < 1$, then we are somewhere between A and B. If $p < 0$, then we are somewhere before A. And if $p > 1$, then we are somewhere after B.

We may think of the vector equation as follows: start at the origin O, before following the vector $\boldsymbol{a}$ to get to the point A. Then, travel a certain distance (controlled by the parameter p) along the vector $\boldsymbol{AB}$.

Example

If $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = 5\boldsymbol{i} - 4\boldsymbol{j}$, then the vector equation of AB is

$$\begin{aligned} &3\boldsymbol{i} + 2\boldsymbol{j} + p(-(3\boldsymbol{i} + 2\boldsymbol{j}) + (5\boldsymbol{i} - 4\boldsymbol{j})) \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} + p(-3\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{i} - 4\boldsymbol{j}) \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} + p(2\boldsymbol{i} - 6\boldsymbol{j}) \\ &= (3 + 2p)\boldsymbol{i} + (2 - 6p)\boldsymbol{j} \end{aligned}$$

Exercise 5

Find the vector equation of AB if

(a) $\boldsymbol{a} = 7\boldsymbol{i} + 3\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} + 4\boldsymbol{j}$

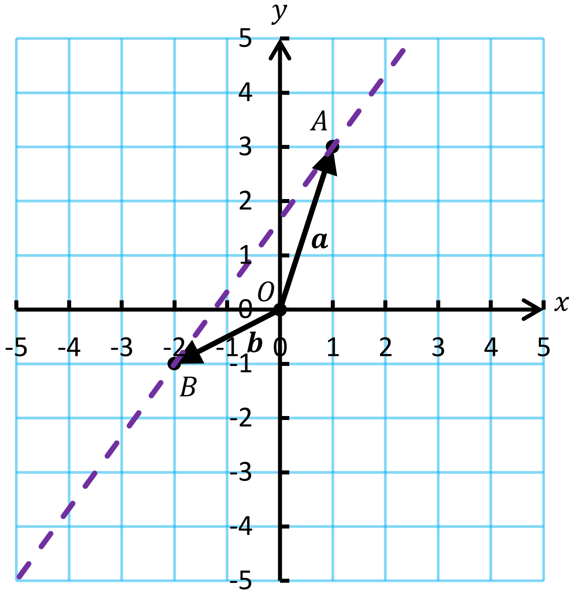
(b) $\boldsymbol{a} = 8\boldsymbol{i} + 4\boldsymbol{j}$, $\boldsymbol{b} = 4\boldsymbol{i} - 2\boldsymbol{j}$

(c) $\boldsymbol{a} = -8\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = 5\boldsymbol{i} - 8\boldsymbol{j}$

(d) $\boldsymbol{a} = 9\boldsymbol{i}$, $\boldsymbol{b} = -6\boldsymbol{i} - \boldsymbol{j}$

Exercise 6

Let the position vectors of the points A and B, relative to a fixed origin O, be given by $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = -2\boldsymbol{i} - \boldsymbol{j}$, respectively.



(a) Find the vector equation of AB.

(b) Find the mid-point of AB.

(c) If the point C divides the line AB so that $AC:CB = 2:1$, find the position vector of the point C.

A formula for dividing a line in a particular ratio

Let the position vectors of the points A and B, relative to a fixed origin O, be given by $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively.

Let the point C divide the line AB in the ratio $\lambda:\mu$.

$$\boldsymbol{OC} = \boldsymbol{OA} + \frac{\lambda}{\lambda+\mu}\boldsymbol{AB}$$

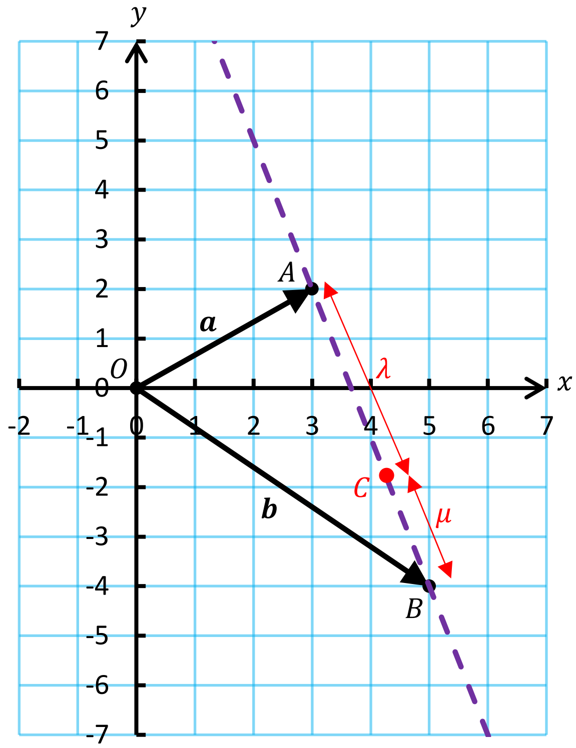
$$\boldsymbol{OC} = \boldsymbol{a} + \frac{\lambda}{\lambda+\mu}(-\boldsymbol{a}+\boldsymbol{b})$$

$$\boldsymbol{OC} = \left(\frac{\lambda+\mu}{\lambda+\mu}\right)\boldsymbol{a} + \frac{\lambda}{\lambda+\mu}(-\boldsymbol{a}+\boldsymbol{b})$$

$$\boldsymbol{OC} = \frac{(\lambda+\mu)\boldsymbol{a} + \lambda(-\boldsymbol{a}+\boldsymbol{b})}{\lambda+\mu}$$

$$\boldsymbol{OC} = \frac{\lambda\boldsymbol{a} + \mu\boldsymbol{a} - \lambda\boldsymbol{a} + \lambda\boldsymbol{b}}{\lambda+\mu}$$

$$\boldsymbol{OC} = \frac{\mu\boldsymbol{a} + \lambda\boldsymbol{b}}{\lambda+\mu}$$



The above formula is given in the formula booklet.

Example

Let the position vectors of the points A and B, relative to a fixed origin O, be given by $\boldsymbol{a} = 4\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{b} = 8\boldsymbol{i} + 3\boldsymbol{j}$, respectively. If the point C divides the line AB in the ratio $2:3$, find the position vector of the point C.

$$\boldsymbol{OC} = \frac{3(4\boldsymbol{i} + 7\boldsymbol{j}) + 2(8\boldsymbol{i} + 3\boldsymbol{j})}{2+3}$$

$$\boldsymbol{OC} = \frac{12\boldsymbol{i} + 21\boldsymbol{j} + 16\boldsymbol{i} + 6\boldsymbol{j}}{5}$$

$$\boldsymbol{OC} = \frac{28\boldsymbol{i} + 27\boldsymbol{j}}{5}$$

$$\boldsymbol{OC} = \frac{28}{5}\boldsymbol{i} + \frac{27}{5}\boldsymbol{j}$$

Comparing with $\boldsymbol{OC} = \frac{\mu\boldsymbol{a}+\lambda\boldsymbol{b}}{\lambda+\mu}$, where C divides the line AB in the ratio $\lambda:\mu$, we have $\lambda = 2$ and $\mu = 3$.

Exercise 7

(a) Let the position vectors of the points A and B, relative to a fixed origin O, be given by $\boldsymbol{a} = 3\boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{b} = 7\boldsymbol{i} + 2\boldsymbol{j}$, respectively. If the point C divides the line AB in the ratio $3:5$, find the position vector of the point C.

(b) Let the position vectors of the points A and B, relative to a fixed origin O, be given by $\boldsymbol{a} = 8\boldsymbol{i} - 2\boldsymbol{j}$ and $\boldsymbol{b} = 5\boldsymbol{i} + 4\boldsymbol{j}$, respectively. If the point C divides the line AB in the ratio $4:1$, find the position vector of the point C.

Top 10

(Edexcel Paper 1 [8MA0/01] Summer 2018)

3. Given that the point A has position vector $4\mathbf{i} - 5\mathbf{j}$ and the point B has position vector $-5\mathbf{i} - 2\mathbf{j}$,

(a) find the vector $\overrightarrow{AB}$,

(2)

(b) find $|\overrightarrow{AB}|$.

Give your answer as a simplified surd.

(2)

(Unit 1 Sample Assessment Materials)

18. (*a*) The vectors $\mathbf{u}$ and $\mathbf{v}$ are defined by $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = -4\mathbf{i} + 5\mathbf{j}$.

(i) Find the vector $4\mathbf{u} - 3\mathbf{v}$.

(ii) The vectors $\mathbf{u}$ and $\mathbf{v}$ are the position vectors of the points *U* and *V*, respectively. Find the length of the line *UV*. [4]

(*b*) Two villages *A* and *B* are 40 km apart on a long straight road passing through a desert. The position vectors of *A* and *B* are denoted by $\mathbf{a}$ and $\mathbf{b}$, respectively.

(i) Village *C* lies on the road between *A* and *B* at a distance 4 km from *B*. Find the position vector of *C* in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) Village *D* has position vector $\frac{2}{9}\mathbf{a} + \frac{5}{9}\mathbf{b}$. Explain why village *D* cannot possibly be on the straight road passing through *A* and *B*. [3]

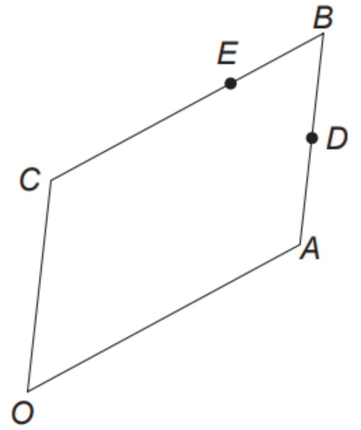
(Unit 1 Summer 2018)

1 1 **a)** The vectors $\mathbf{u}$ and $\mathbf{v}$ are defined by $\mathbf{u} = 9\mathbf{i} - 40\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$. Determine the range of values for μ such that $\mu|\mathbf{v}| > |\mathbf{u}|$. [3]

b) The point *A* has position vector $11\mathbf{i} - 4\mathbf{j}$ and the point *B* has position vector $21\mathbf{i} + \mathbf{j}$. Determine the position vector of the point *C*, which lies between *A* and *B*, such that $AC : CB$ is $2 : 3$. [3]

(Unit 1 Summer 2019)

06 *OABC* is a parallelogram with *O* as origin.



The position vector of *A* is $\mathbf{a}$ and the position vector of *C* is $\mathbf{c}$. The midpoint of *AB* is *D*. The point *E* divides the line *CB* such that $CE:EB = 2:1$.

a) Find, in terms of $\mathbf{a}$ and $\mathbf{c}$,

i) the vector **AC**,

ii) the position vector of *D*,

iii) the position vector of *E*. [3]

b) Determine whether or not *DE* is parallel to *AC*, clearly stating your reason. [2]

(Unit 1 Summer 2022)

16 The vectors **a** and **b** are defined by $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j}$.

a) Find a unit vector in the direction of **a**. [2]

b) Determine the angle **b** makes with the x-axis. [2]

c) The vector $\mu\mathbf{a} + \mathbf{b}$ is parallel to $4\mathbf{i} - 5\mathbf{j}$.

i) Find the vector $\mu\mathbf{a} + \mathbf{b}$ in terms of μ, **i** and **j**. [1]

ii) Determine the value of μ. [4]

(Edexcel Paper 1 [8MA0/01] Summer 2020)

2. [*In this question the unit vectors* **i** *and* **j** *are due east and due north respectively.*]

A coastguard station O monitors the movements of a small boat.

At 10:00 the boat is at the point $(4\mathbf{i} - 2\mathbf{j})$ km relative to O.

At 12:45 the boat is at the point $(-3\mathbf{i} - 5\mathbf{j})$ km relative to O.

The motion of the boat is modelled as that of a particle moving in a straight line at constant speed.

(a) Calculate the bearing on which the boat is moving, giving your answer in degrees to one decimal place.

(3)

(b) Calculate the speed of the boat, giving your answer in km h^{-1}

(3)

(Edexcel Paper 1 [8MA0/01] Summer 2022)

3. The triangle PQR is such that $\overrightarrow{PQ} = 3\mathbf{i} + 5\mathbf{j}$ and $\overrightarrow{PR} = 13\mathbf{i} - 15\mathbf{j}$

(a) Find $\overrightarrow{QR}$

(2)

(b) Hence find $|\overrightarrow{QR}|$ giving your answer as a simplified surd.

(2)

The point S lies on the line segment QR so that $QS : SR = 3 : 2$

(c) Find $\overrightarrow{PS}$

(2)

(Unit 1 Summer 2023)

12 The position vectors of the points A and B, relative to a fixed origin O, are given by

$$\mathbf{a} = -3\mathbf{i} + 4\mathbf{j}, \quad \mathbf{b} = 5\mathbf{i} + 8\mathbf{j},$$

respectively.

a) Find the vector **AB**. [2]

b) **i)** Find a unit vector in the direction of **a**. [2]

ii) The point C is such that the vector **OC** is in the direction of **a**. Given that the length of **OC** is 7 units, write down the position vector of C. [1]

c) Calculate the angle AOB. [3]

(Edexcel Paper 1 [8MA0/01] Summer 2023)

13. Relative to a fixed origin O

- point A has position vector $10\mathbf{i} - 3\mathbf{j}$
- point B has position vector $-8\mathbf{i} + 9\mathbf{j}$
- point C has position vector $-2\mathbf{i} + p\mathbf{j}$ where p is a constant

(a) Find $\overrightarrow{AB}$

(2)

(b) Find $\left|\overrightarrow{AB}\right|$ giving your answer as a fully simplified surd.

(2)

Given that points A, B and C lie on a straight line,

(c) (i) find the value of p,

(ii) state the ratio of the area of triangle AOC to the area of triangle AOB.

(3)

(Edexcel Paper 1 [8MA0/01] Summer 2019)

16. (i) Two non-zero vectors, **a** and **b**, are such that

$$|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$$

Explain, geometrically, the significance of this statement.

(1)

(ii) Two different vectors, **m** and **n**, are such that $|\mathbf{m}| = 3$ and $|\mathbf{m} - \mathbf{n}| = 6$
The angle between vector **m** and vector **n** is 30°

Find the angle between vector **m** and vector **m** – **n**, giving your answer, in degrees, to one decimal place.

(4)

Revision Question

(Unit 1 Summer 2024)

13. The position vectors of the points A and B, relative to a fixed origin O, are given by

$$\mathbf{a} = 4\mathbf{i} + 7\mathbf{j}, \qquad \mathbf{b} = \mathbf{i} + 3\mathbf{j},$$

respectively.

(a) Find the vector **AB**. [2]

(b) Determine the distance between the points A and B. [2]

(c) The position vector of the point *C* is given by $\mathbf{c} = -2\mathbf{i} + 5\mathbf{j}$. The point *D* is such that the distance between *C* and *D* is equal to the distance between *A* and *B*, and *CD* is parallel to *AB*. Find the possible position vectors of the point *D*. [4]

Unit 1, Workbook 4

12

Quadratic Equations

$y = 2x^2 - 6x + 4$

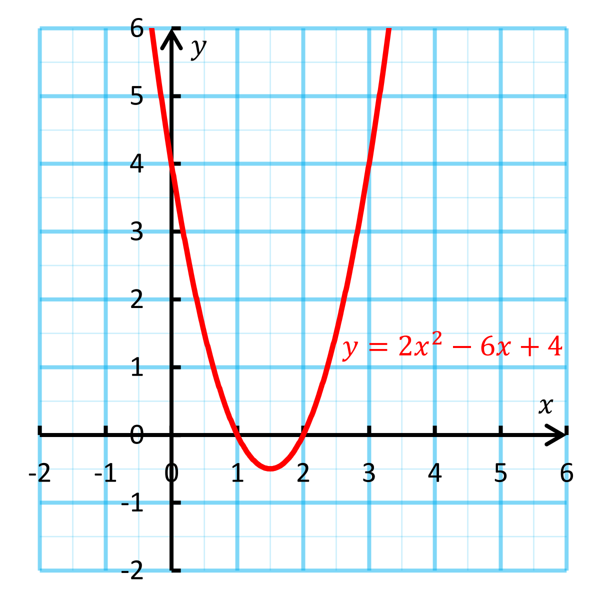
y–intercept $(0, 4)$

$y = 2(x - 1.5)^2 - 0.5$

Minimum point $(1.5, -0.5)$

$y = (2x - 4)(x - 1)$

Roots $x = 2$, $x = 1$



Name:

Background

What is the work?	For a quadratic equation of the form $ax^2 + bx + c = 0$, this workbook considers how to factorise, solve and plot the equation.
What is required before starting?	**GCSE Work:** Graph plotting; factorising quadratic expressions; solving quadratic equations with the quadratic formula.
Where does this lead to?	**Unit 1:** Solving trigonometric or exponential equations. **Unit 2:** Solving equations of motion. **Unit 3:** Composite and inverse functions. **Unit 4:** The motion of an object in two dimensions.

Theory

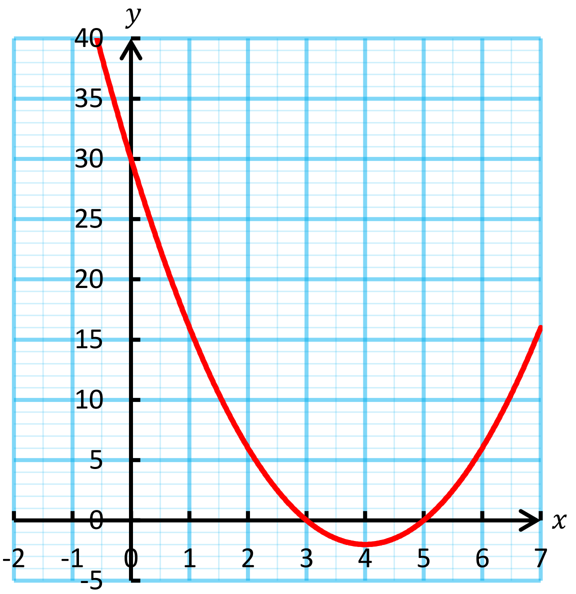
Theory

A **quadratic function** has the form $ax^2 + bx + c$, where a, b and c are constants with $a \neq 0$.
A **quadratic equation** has the form $ax^2 + bx + c = 0$.

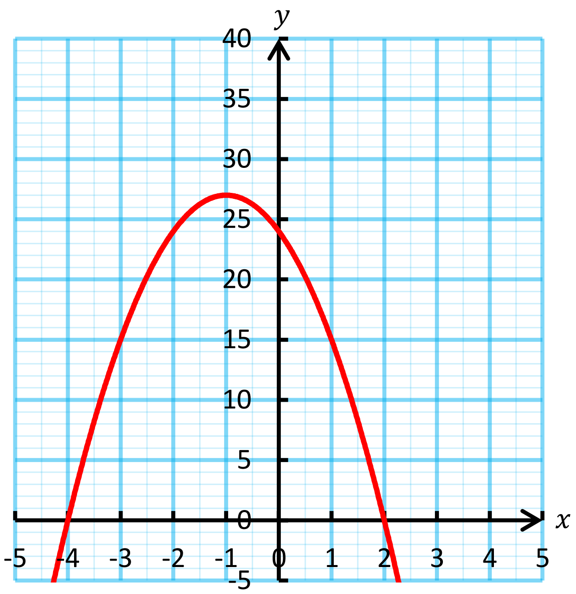
We can plot a quadratic function on graph paper, and it always takes the form of a curve with a 'u' or 'n' shape.
If $a > 0$, then the curve has a 'u' shape, and it has a minimum point.
If $a < 0$, then the curve has an 'n' shape, and it has a maximum point.

Example 1

$y = 2x^2 - 16x + 30$



$y = -3x^2 - 6x + 24$



From the functions and the graphs, we can see the following properties.

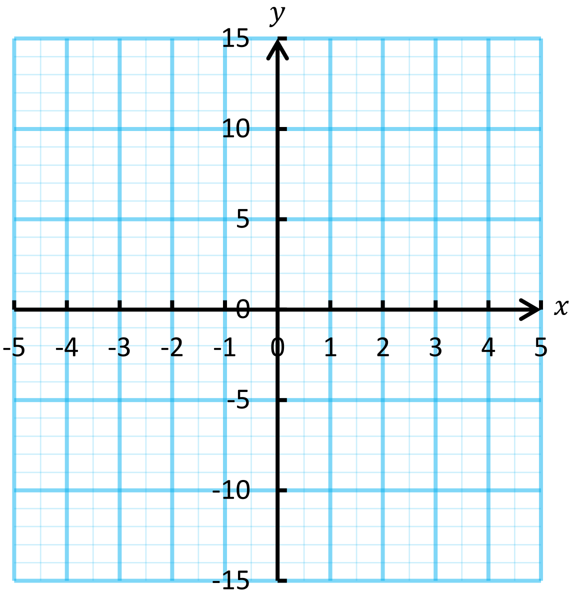
The y–intercept is $(0, 30)$.
The roots are $x = 3$ and $x = 5$.
The minimum point is $(4, -2)$.

The y–intercept is $(0, 24)$.
The roots are $x = -4$ and $x = 2$.
The maximum point is $(-1, 27)$.

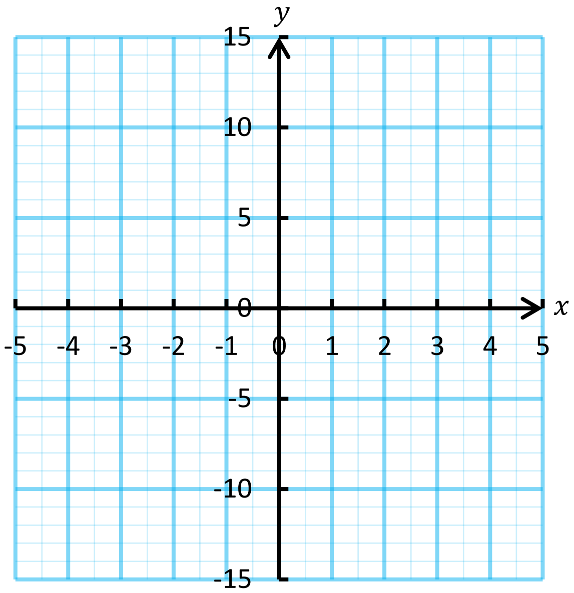
Exercise 1

Use your calculator to plot the following quadratic functions.
(On a scientific calculator, use the 'table mode' to help. On a graphical calculator, use the graph plotting function to help.)

$y = 2x^2 - 4x - 6$



$y = -3x^2 - 6x + 9$



Complete the following sentences.

The $y-$ intercept is ....

The roots are $x =$ .... and $x =$ ....

The minimum point is ....

The y–intercept is ....

The roots are $x =$ .... and $x =$ ....

The maximum point is ....

Factorising Quadratic Expressions

It is possible to factorise some quadratic expressions using the GCSE splitting method or detective method.

Example 2

Factorise the quadratic expression $3x^2 - 7x - 20$.

The Splitting Method

$3 \times -20 = -60$
The two numbers that multiply to give -60 and add to give -7 are 5 and -12.

$$\begin{aligned} 3x^2 - 7x - 20 &= 3x^2 + 5x - 12x - 20 \\ &= x(3x + 5) - 4(3x + 5) \\ &= (3x + 5)(x - 4) \end{aligned}$$

The Detective Method

$3x \times x = 3x^2$, so it is possible to factorise $3x^2 - 7x - 20$ in the form $(3x + ?)(x + ?)$.
By attempting pairs of numbers that multiply to give -20, we see that $3x^2 - 7x - 20 = (3x + 5)(x - 4)$.

Exercise 2

Factorise: (a) $2x^2 + 11x + 15$

(b) $3x^2 - 5x - 12$

(c) $4x^2 + 12x + 5$

(d) $8x^2 - 19x + 6$

Factorising Expressions with Higher Indices

It is possible to use the above techniques to factorise **some** expressions with indices that are greater than 2.

Example 3

Factorise the quartic expression $2x^4 + 9x^2 + 4$.

We can use the substitution $y = x^2$ to change the expression to be a quadratic expression:

$$\begin{aligned} 2x^4 + 9x^2 + 4 &= 2y^2 + 9y + 4 \\ &= 2y^2 + 8y + y + 4 \\ &= 2y(y+4) + 1(y+4) \\ &= (y+4)(2y+1) \\ &= (x^2+4)(2x^2+1) \end{aligned}$$

If $y = x^2$ then
$y^2 = (x^2)^2$
$= x^4$

Exercise 3

Use an appropriate substitution to factorise the following expressions.

(a) $2x^4 + 7x^2 + 6$

(b) $5x^4 + 18x^2 + 9$

(c) $3x^6 + 13x^3 + 4$

(d) $4x^8 + 21x^4 - 18$

(e) $2x^{24} - 21x^{12} + 40$

(f) $x + 3\sqrt{x} + 2$

Solving Equations

We can use the above factorising techniques to solve some equations.

Example 4

(a) Solve the equation $x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$
Either $x + 4 = 0$ or $x - 2 = 0$
$x = -4$ or $x = 2$

(b) Solve the equation $2x^2 + 3x - 20 = 0$

$2 \times -20 = -40$
$2x^2 + 8x - 5x - 20 = 0$
$2x(x + 4) - 5(x + 4) = 0$
$(x + 4)(2x - 5) = 0$
Either $x + 4 = 0$ or $2x - 5 = 0$
$x = -4$ or $x = \frac{5}{2}$

Exercise 4

Solve the following equations by factorising.

(a) $x^2 + 12x + 32 = 0$

(b) $2x^2 - 13x + 20 = 0$

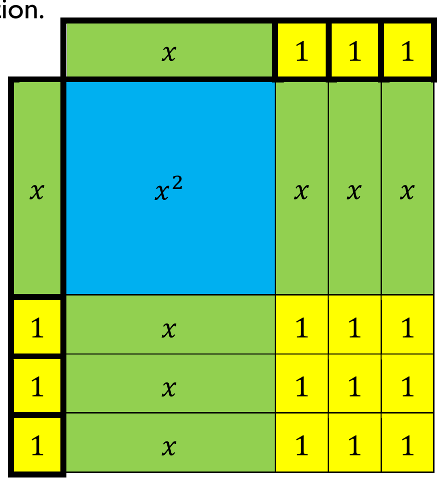
Completing the Square

It is not possible to solve every quadratic equation by using the above factorising techniques.
For these equations, we can try to **complete the square** to solve the equation.

Example 5

(a) Solve the equation $x^2 + 6x + 7 = 0$ by completing the square.

$x^2 + 6x + 7 = 0$
$(x + 3)^2 - 3^2 + 7 = 0$ [Half of 6 is 3]
$(x + 3)^2 - 9 + 7 = 0$
$(x + 3)^2 - 2 = 0$
$(x + 3)^2 = 2$
$x + 3 = \pm\sqrt{2}$
$x = -3 \pm \sqrt{2}$
Either $x = -3 + \sqrt{2}$ or $x = -3 - \sqrt{2}$



The *algebra tiles* on the right shows what is happening during the solution.

The diagram shows the sum $(x + 3)^2 = x^2 + 6x + 9$. Comparing with $x^2 + 6x + 7$, we see that we agree with the x^2 and the $6x$, but not with the constants. We must subtract 9 (or 3^2) and then add 7 to obtain what we need:

$$(x + 3)^2 = x^2 + 6x + 9$$
$$(x + 3)^2 - 9 = x^2 + 6x + 9 - 9$$
$$(x + 3)^2 - 9 = x^2 + 6x$$
$$(x + 3)^2 - 9 + 7 = x^2 + 6x + 7$$
$$(x + 3)^2 - 2 = x^2 + 6x + 7$$

Example 6

Solve the equation $3x^2 + 24x + 5 = 0$ by completing the square.

Method 1

$3x^2 + 24x + 5 = 0$

$3(x^2 + 8x) + 5 = 0$

$3((x + 4)^2 - 4^2) + 5 = 0$

$3((x + 4)^2 - 16) + 5 = 0$

$3(x + 4)^2 - 48 + 5 = 0$

$3(x + 4)^2 - 43 = 0$

$3(x + 4)^2 = 43$

$(x + 4)^2 = \frac{43}{3}$

$x + 4 = \pm\sqrt{\frac{43}{3}}$

$x = -4 \pm \sqrt{\frac{43}{3}}$

Either $x = -4 + \sqrt{\frac{43}{3}}$ or $x = -4 - \sqrt{\frac{43}{3}}$

Method 2

$3x^2 + 24x + 5 = 0$

$3\left(x^2 + 8x + \frac{5}{3}\right) = 0$

$x^2 + 8x + \frac{5}{3} = 0$

$(x + 4)^2 - 4^2 + \frac{5}{3} = 0$

$(x + 4)^2 - 16 + \frac{5}{3} = 0$

$(x + 4)^2 - \frac{48}{3} + \frac{5}{3} = 0$

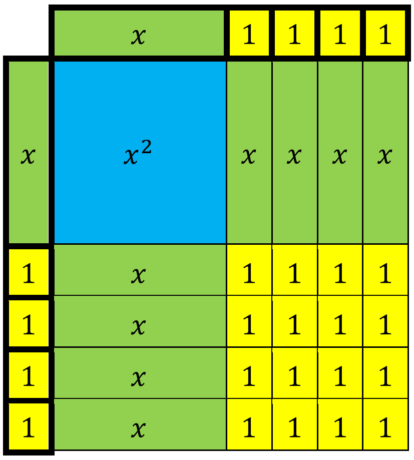
$(x + 4)^2 - \frac{43}{3} = 0$

$(x + 4)^2 = \frac{43}{3}$

$x + 4 = \pm\sqrt{\frac{43}{3}}$

$x = -4 \pm \sqrt{\frac{43}{3}}$

Either $x = -4 + \sqrt{\frac{43}{3}}$ or $x = -4 - \sqrt{\frac{43}{3}}$



Exercise 5

Solve the following equations by completing the square.

(a) $x^2 + 10x + 3 = 0$

(b) $x^2 - 8x + 9 = 0$

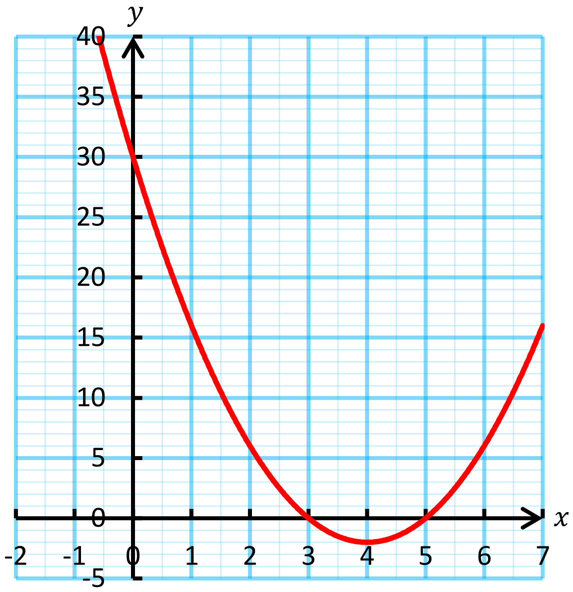
(c) $2x^2 + 12x + 1 = 0$

(d) $3x^2 + 6x - 5 = 0$

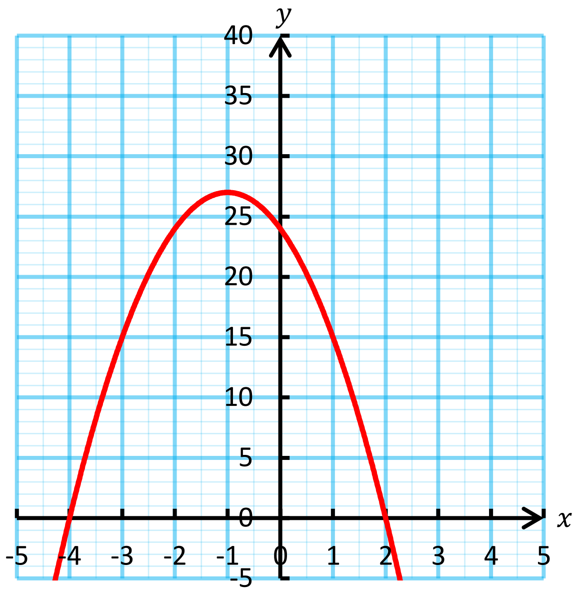
Finding a minimum point or maximum point by completing the square

Let us return to the quadratic functions we first saw in Example 1.

$y = 2x^2 - 16x + 30$



$y = -3x^2 - 6x + 24$



By completing the square, it is possible to find the roots and the minimum / maximum point.

Finding the roots:

$2x^2 - 16x + 30 = 0$
$2(x^2 - 8x) + 30 = 0$
$2((x-4)^2 - (-4)^2) + 30 = 0$
$2((x-4)^2 - 16) + 30 = 0$
$2(x-4)^2 - 32 + 30 = 0$
$2(x-4)^2 - 2 = 0$
$2(x-4)^2 = 2$
$(x-4)^2 = 1$
$x - 4 = \pm\sqrt{1}$
$x - 4 = \pm 1$
$x = 4 \pm 1$
Either $x = 4 + 1$ or $x = 4 - 1$
Either $x = 5$ or $x = 3$

$-3x^2 - 6x + 24 = 0$
$-3(x^2 + 2x) + 24 = 0$
$-3((x+1)^2 - 1^2) + 24 = 0$
$-3((x+1)^2 - 1) + 24 = 0$
$-3(x+1)^2 + 3 + 24 = 0$
$-3(x+1)^2 + 27 = 0$
$-3(x+1)^2 = -27$
$(x+1)^2 = 9$
$x + 1 = \pm\sqrt{9}$
$x + 1 = \pm 3$
$x = -1 \pm 3$
Either $x = -1 + 3$ or $x = -1 - 3$
Either $x = 2$ or $x = -4$

Minimum / maximum point:

By completing the square, we can write
$y = 2x^2 - 16x$ as $y = 2(x-4)^2 - 2$.

No matter the value of x, we have $(x-4)^2 \geq 0$.
So, the minimum value of the function is -2.
This happens when the bracket $x - 4 = 0$
$x = 4$
So, the minimum point is $(4, -2)$.

By completing the square, we can write
$y = -3x^2 - 6x + 24$ as $y = -3(x+1)^2 + 27$,
or $y = 27 - 3(x+1)^2$.
No matter the value of x, we have $(x+1)^2 \geq 0$.
So, the maximum value of the function is 27.
This happens when the bracket $x + 1 = 0$
$x = -1$
So, the maximum point is $(-1, 27)$.

Exercise 6

Use your calculations from Exercise 5 to help you find the minimum / maximum point for the following functions.

(a) $y = x^2 + 10x + 3$

(b) $y = x^2 - 8x + 9$

(c) $y = 2x^2 + 12x + 1$

(d) $y = 3x^2 + 6x - 5$

Exericse 7

Find the roots and the minimum / maximum point for the following functions.

(a) $y = 2x^2 - 4x - 6$

(b) $y = -3x^2 - 6x + 9$

Exercise 8

Verify your answers to Exercise 7 by using your previous answers to Exercise 1.

The Quadratic Formula

The general quadratic equation is $ax^2 + bx + c = 0$, with $a \neq 0$.

If $b = 0$ or $c = 0$, then solving the equation is relatively straightforward.

$b = 0$

$ax^2 + c = 0$
$ax^2 = -c$
$x^2 = -\frac{c}{a}$
$x = \pm\sqrt{-\frac{c}{a}}$
Either $x = \sqrt{-\frac{c}{a}}$ or $x = -\sqrt{-\frac{c}{a}}$

$c = 0$

$ax^2 + bx = 0$
$x(ax + b) = 0$
Either $x = 0$ or $ax + b = 0$
Either $x = 0$ or $ax = -b$
Either $x = 0$ or $x = -\frac{b}{a}$

If $b \neq 0$ and $c \neq 0$, then we can complete the square to try to solve the general equation.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Divide each term by a

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

Complete the square

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

Square the second bracket

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2} = 0$$

Multiply the top and bottom of the fraction $\frac{c}{a}$ with $4a$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Square root both sides

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Use $\sqrt{4a^2} = 2a$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is the quadratic formula

Exericse 9

Solve the following quadratic equations.

(a) $4x^2 - 9 = 0$

....

(b) $4x^2 - 9x = 0$

....

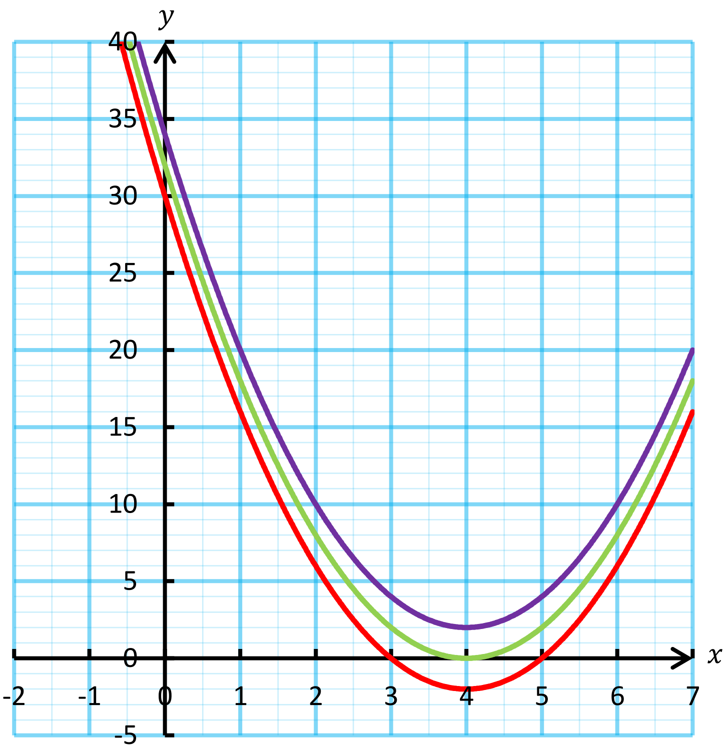
(c) $4x^2 + 9x - 6 = 0$

(d) $4x^2 - 5x - 6 = 0$

The Discriminant

Consider the graphs shown below, plotting three quadratic functions.

$y = 2x^2 - 16x + 30$ $y = 2x^2 - 16x + 32$ $y = 2x^2 - 16x + 34$



The first graph, for $y = 2x^2 - 16x + 30$, crosses the x–axis twice, at the points $(3, 0)$ and $(5, 0)$.
We say that the the function has **two real roots**.

The second graph, for $y = 2x^2 - 16x + 32$, meets the x–axis at one point, namely $(4, 0)$.
We say that the function has **one repeated real root**.

The third graph, for $y = 2x^2 - 16x + 34$, does not cross the x–axis at all.
We say that the function has **no real roots**.

We could find the above roots by using the quadratic formula.

$y = 2x^2 - 16x + 30$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$a = 2,\ b = -16,\ c = 30$
$x = \frac{-(-16)\pm\sqrt{(-16)^2-4\times2\times30}}{2\times2}$
$x = \frac{16\pm\sqrt{256-240}}{4}$
$x = \frac{16\pm\sqrt{16}}{4}$
$x = \frac{16\pm4}{4}$
Either $x = \frac{16+4}{4}$ or $x = \frac{16-4}{4}$
Either $x = 5$ or $x = 3$

$y = 2x^2 - 16x + 32$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$a = 2,\ b = -16,\ c = 32$
$x = \frac{-(-16)\pm\sqrt{(-16)^2-4\times2\times32}}{2\times2}$
$x = \frac{16\pm\sqrt{256-256}}{4}$
$x = \frac{16\pm\sqrt{0}}{4}$
$x = \frac{16\pm0}{4}$
Either $x = \frac{16+0}{4}$ or $x = \frac{16-0}{4}$
Either $x = 4$ or $x = 4$

$y = 2x^2 - 16x + 34$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$a = 2,\ b = -16,\ c = 34$
$x = \frac{-(-16)\pm\sqrt{(-16)^2-4\times2\times34}}{2\times2}$
$x = \frac{16\pm\sqrt{256-272}}{4}$
$x = \frac{16\pm\sqrt{-16}}{4}$
'Math Error' due to $\sqrt{-16}$

The part of the quadratic formula that influences the number of real roots obtained is the expression $b^2 - 4ac$, which appears within the square root. This expression is known as the **discriminant**.

If $b^2 - 4ac > 0$ then we have two real roots.

If $b^2 - 4ac = 0$ then we have one repeated real root.

If $b^2 - 4ac < 0$ then we do not have any real roots.

Exercise 10

How many real roots do the following quadratic equations have?

(a) $9x^2 + 12x + 4 = 0$

(b) $9x^2 + 12x - 4 = 0$

(c) $9x^2 + 4x + 12 = 0$

Exericse 11

Consider the quadratic equation $2x^2 + 8x + k = 0$. For which values of k does the equation have

(a) two real roots?

(b) one repeated real root?

(c) no real roots?

Exercise 12

Consider the quadratic equation $2x^2 + kx + 8 = 0$. For which values of k does the equation have

(a) two real roots? (b) one repeated real root? (c) no real roots?

Quadratic Inequalities

A quadratic inequality is an inequality that contains a quadratic function.
$2x^2 - 7x - 15 \geq 0$ and $3x^2 - 25x + 32 < 2 - 4x$ are examples of quadratic inequalities.

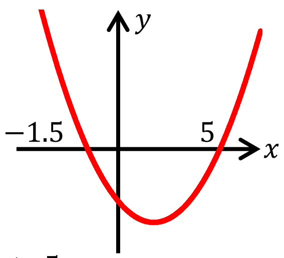
To solve a quadratic inequality:

- Re-arrange the inequality to obtain a quadratic function $f(x)$ on the left hand side and zero on the right.
- Solve the quadratic equation $f(x) = 0$ to find the **critical points**.
- Sketch the function $f(x)$ on a suitable set of axes, showing any critical points.
- Write down the final solution.

Example 7

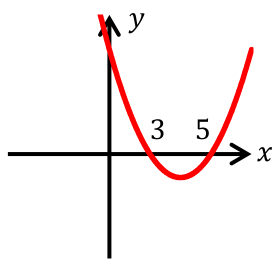
Solve the quadratic inequalities $2x^2 - 7x - 15 \geq 0$ and $3x^2 - 25x + 32 < 2 - 4x$.

$2x^2 - 7x - 15 \geq 0$
No need to re-arrange.
We have $f(x) = 2x^2 - 7x - 15$
Solving $f(x) = 0$:
$2x^2 - 7x - 15 = 0$
$2 \times -15 = -30$
$2x^2 - 10x + 3x - 15 = 0$
$2x(x - 5) + 3(x - 5) = 0$
$(x - 5)(2x + 3) = 0$
Either $x - 5 = 0$ or $2x + 3 = 0$
Either $x = 5$ or $2x = -3$
Either $x = 5$ or $x = -1.5$ (critical points)
Sketch:



Final solution: $x \leq -1.5$ or $x \geq 5$
In set notation: $\{x: x \leq -1.5\} \cup \{x: x \geq 5\}$

$3x^2 - 25x + 32 < 2 - 4x$
Re-arranging: $3x^2 - 21x + 30 < 0$
We have $f(x) = 3x^2 - 21x + 30$
Solving $f(x) = 0$:
$3x^2 - 21x + 30 = 0$
$3 \times 30 = 90$
$3x^2 - 15x - 6x + 30 = 0$
$3x(x - 5) - 6(x - 5) = 0$
$(x - 5)(3x - 6) = 0$
Either $x - 5 = 0$ or $3x - 6 = 0$
Either $x = 5$ or $2x = 6$
Either $x = 5$ or $x = 3$ (critical points)
Sketch:



Final solution: $3 < x < 5$
In set notation: $\{x: x > 3\} \cap \{x: x < 5\}$

Exercise 13

Solve the following quadratic inequalities. Give your answers using set notation.

(a) $2x^2 + 4x - 30 > 0$

(b) $3x^2 + 9x \leq 2(2 - x)$

(c) $-2x^2 + 6x - 4 > 0$

(d) $x^2 < x - 1$

Top 10

(C1 Summer 2016)

5. *(a)* Express $x^2 + 4x - 8$ in the form $(x + a)^2 + b$, where a and b are constants whose values are to be found. [2]

(b) Use an algebraic method to solve the simultaneous equations $y = x^2 + 4x - 8$ and $y = 2x + 7$. [4]

(c) Draw a sketch illustrating geometrically the results of both part *(a)* and part *(b)*. [4]

(C1 Summer 2018)

6. Find the range of values of x satisfying the inequality

$$9x^2 + 16x - 4 > 0.$$ [3]

(Unit 1 Summer 2022)

0 4 Solve the inequality $x^2 + 3x - 6 > 4x - 4$. [4]

(C1 Winter 2007)

7. Given that the equation

$$kx^2 - 4x + (k - 3) = 0$$

has real roots, show that

$$k^2 - 3k - 4 \leqslant 0.$$

Find the range of values of k satisfying this inequality. [7]

(C1 Winter 2009)

5. Given that the quadratic equation

$$(3k-2)x^2 + 8x + k = 0$$

has no real roots, show that

$$3k^2 - 2k - 16 > 0.$$

Find the range of values of k satisfying this inequality. [7]

(C1 Summer 2016)

6. *(a)* Find the range of values of k for which the quadratic equation

$$9x^2 + 8x - 2k = 0$$

has **two distinct real roots**. [4]

(b) Solve the inequality $x(5x - 7) \geqslant 6$. [4]

(C1 Summer 2011)

5. The curve C has equation

$$y = x^2 + (4k+3)x + 7,$$

and the line L has equation

$$y = x + k,$$

where k is a constant.
Given that L and C intersect at two distinct points,

(a) show that $4k^2 + 5k - 6 > 0$, [6]

(b) find the range of values of k satisfying this inequality. [3]

(C1 Summer 2015)

5. *(a)* Find the range of values of k for which the quadratic equation

$$kx^2 + (2k - 5)x + (k - 6) = 0$$

has **no real roots**. [4]

(b) Without carrying out any further calculation, write down the value of k for which the quadratic equation

$$kx^2 + (2k - 5)x + (k - 6) = 0$$

has **two equal roots**. [1]

(C1 Winter 2014)

6. Given that the quadratic equation

$$(2k-3)x^2+8x+(2k+3)=0$$

has no real roots, show that k satisfies an inequality of the form

$$m-nk^2<0,$$

where m, n are integers whose values are to be found.

Hence find the range of values of k such that the quadratic equation

$$(2k-3)x^2+8x+(2k+3)=0$$

has no real roots. [6]

(C1 Summer 2013)

6. *(a)* (i) Assuming that the quadratic equation

$$(k + 1)x^2 + (4k + 1)x + (k - 5) = 0$$

has **two equal** roots, show that

$$4k^2 + 8k + 7 = 0.$$

(ii) Hence show that there are **no real** values of k such that the quadratic equation

$$(k + 1)x^2 + (4k + 1)x + (k - 5) = 0$$

has two equal roots. [6]

(b) Find the range of values of x satisfying the inequality

$$4x^2 - 9x - 9 \geqslant 0.$$ [3]

Revision Question

(Unit 1 Summer 2023)

0 7 The curve *C* has equation $y = 2x^2 + 5x - 12$ and the line *L* has equation $y = mx - 14$, where m is a real constant.

a) Given that *L* is a tangent to *C*,

i) show that m satisfies the equation

$$m^2 - 10m + 9 = 0,$$ [5]

ii) find the coordinates of the two possible points of contact of *C* and *L*. [6]

b) Given instead that *L* intersects *C* at two distinct points, find the range of values of m. [2]

Unit 1, Workbook 5

12

Differentiation from First Princples

Isaac Newton

Gottfried Wilhelm Leibniz

In the 18th century a huge row developed between Leibniz and Newton regarding who was the first person to develop the idea of differentiation.

Name:

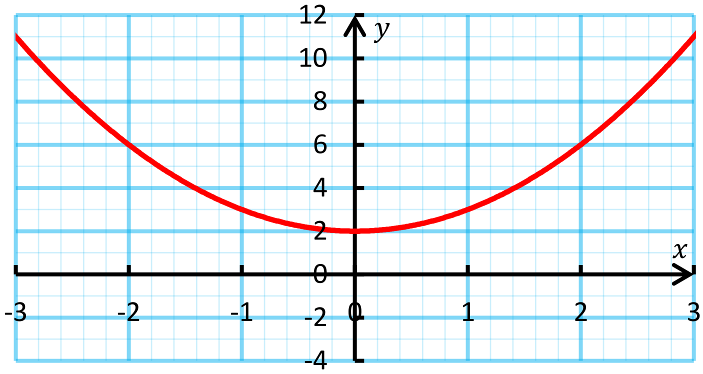
Background

What is the work?	Finding the gradient of a curve using algebraic techniques.
What is required before starting?	**GCSE Work:** Expanding brackets; simplifying; finding the gradient of a curve.
Where does this lead to?	**Units 1–4:** Differentiating and integrating various functions.

Theory

Consider the function $y = x^2 + 2$. It is possible to draw a graph of this function by forming a table of values and plotting them on graph paper.

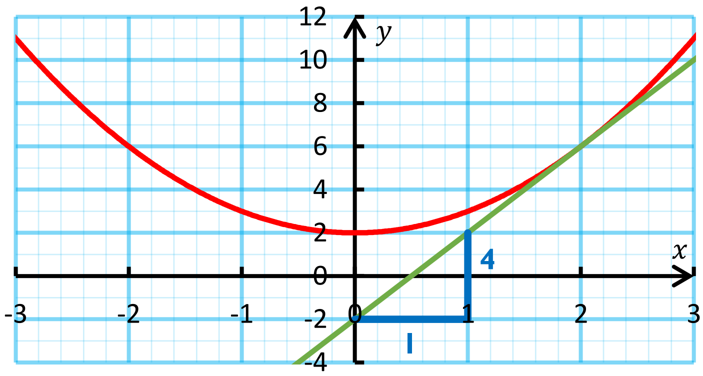
x	-3	-2	-1	0	1	2	3
y	11	6	3	2	3	6	11



The gradient of a graph can be very useful. For example, the gradient of a displacement-time graph gives the velocity.

We see that the **gradient** of the curve (how steep the curve is) changes as we move along the curve. If $x = 0$, then the gradient is zero. If x is negative, then the gradient is negative, and if x is positive, then the gradient is positive.

What is the gradient of the graph if $x = 2$? One way of finding this would be to draw a tangent to the graph at the point (2, 6) and find the gradient of the tangent by drawing a triangle.

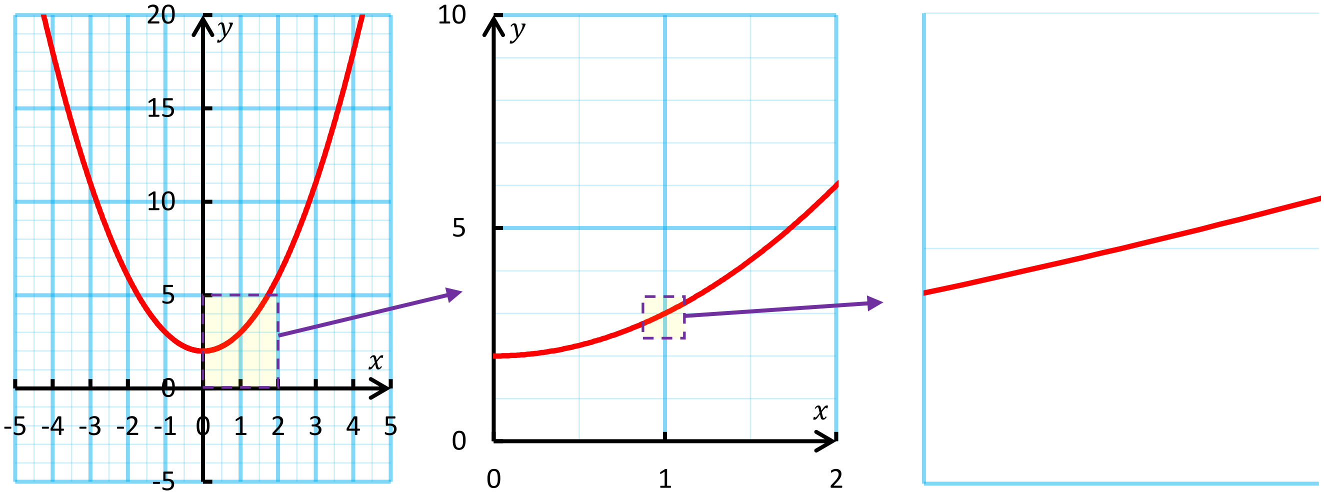


We see from the diagram on the left that the gradient of the graph when $x = 2$ is 4.

Differentiating is a method for finding the gradient of a curve at a particular point **without** having to draw a tangent and forming a triangle. We use special notation to represent this gradient: if y represents any function, then $\frac{dy}{dx}$ or y' represents the gradient of the function.

Differentiating from first principles

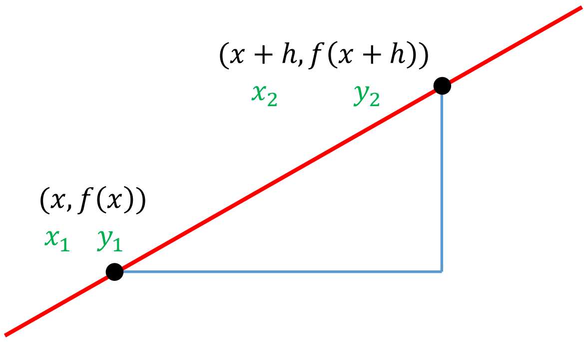
Consider again the function $y = x^2 + 2$ from the previous page. If we zoom in enough on the graph, then it will look like a straight line. (Try this yourself on the website www.desmos.com/calculator.)



This is true for any part of the graph, so to find the gradient of the graph at any point,

- Zoom in to the point – enough so that you see a straight line;
- Find the gradient of the straight line by adding a right-angled triangle to the line.

Let $(x, f(x))$ represent the co-ordinate of the point we are zooming in to. (In the above example, we are zooming in on the point $(1, 3)$, where $f(1) = 1^2 + 2 = 3$.) Consider that we have zoomed in enough so that we see the following straight line.



Theory

Let h represent a very small distance along the x–axis. We can move right from the point $(x, f(x))$ to reach a distance $x + h$ along the x–axis. The point on the function corresponding to $x + h$ is $(x + h,\ f(x + h))$. Considering that h is small enough so that the function behaves like a straight line between x and $x + h$, we can use the fraction $\frac{y_2 - y_1}{x_2 - x_1}$ to calculate the gradient of the straight line.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x + h) - f(x)}{x + h - x}$$
$$= \frac{f(x + h) - f(x)}{h}$$

The problem now is to ensure that h is small enough to give a straight line each time. To make sure of this, we use the idea of a **limit** so that h can be as close to zero as we want. So, the gradient $\frac{dy}{dx}$ of the function at the point $\big(x, f(x)\big)$ is

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Note: Some textbooks and worksheets use δx instead of h to represent the small distance along the x–axis.

Example 1

From first principles, find $\frac{dy}{dx}$ if $y = x^2 + 2$.

We need to construct $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ step-by-step.

We have $f(x) = x^2 + 2$

> We can use FOIL to expand the brackets.

So
$$f(x+h) = (x+h)^2 + 2$$
$$f(x+h) = (x+h)(x+h) + 2$$
$$f(x+h) = x^2 + xh + xh + h^2 + 2$$
$$f(x+h) = x^2 + 2xh + h^2 + 2$$

> The x^2 cancels the $-x^2$ and $+2$ cancels -2.

Now
$$f(x+h) - f(x) = x^2 + 2xh + h^2 + 2 - (x^2 + 2)$$
$$f(x+h) - f(x) = \cancel{x^2} + 2xh + h^2 \cancel{+2} - \cancel{x^2} \cancel{-2}$$
$$f(x+h) - f(x) = 2xh + h^2$$

Next
$$\frac{f(x+h)-f(x)}{h} = \frac{2xh+h^2}{h}$$
$$\frac{f(x+h)-f(x)}{h} = 2x + h$$

> After simplifying as much as we can, we now use the limit by substituting $h = 0$.

Finally
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$
$$\frac{dy}{dx} = \lim_{h \to 0} 2x + h$$
$$\frac{dy}{dx} = 2x$$

Example 2

From first principles, find $\frac{dy}{dx}$ if $y = x^2 + 4x$.

We need to construct $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ step-by-step.

We have $f(x) = x^2 + 4x$

So
$$f(x+h) = (x+h)^2 + 4(x+h)$$
$$f(x+h) = (x+h)(x+h) + 4(x+h)$$
$$f(x+h) = x^2 + xh + xh + h^2 + 4x + 4h$$
$$f(x+h) = x^2 + 2xh + h^2 + 4x + 4h$$

Now
$$f(x+h) - f(x) = x^2 + 2xh + h^2 + 4x + 4h - (x^2 + 4x)$$
$$f(x+h) - f(x) = \cancel{x^2} + 2xh + h^2 + \cancel{4x} + 4h - \cancel{x^2} - \cancel{4x}$$
$$f(x+h) - f(x) = 2xh + h^2 + 4h$$

Next
$$\frac{f(x+h)-f(x)}{h} = \frac{2xh+h^2+4h}{h}$$
$$\frac{f(x+h)-f(x)}{h} = 2x + h + 4$$

Finally
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$
$$\frac{dy}{dx} = \lim_{h \to 0} 2x + h + 4$$
$$\frac{dy}{dx} = 2x + 4$$

Example 3

From first principles, find $\frac{dy}{dx}$ if $y = 2x^2 - 6x + 9$.

We need to construct $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ step-by-step.

We have $f(x) = 2x^2 - 6x + 9$

So

$$f(x+h) = 2(x+h)^2 - 6(x+h) + 9$$
$$f(x+h) = 2(x+h)(x+h) - 6(x+h) + 9$$
$$f(x+h) = 2(x^2 + xh + xh + h^2) - 6x - 6h + 9$$
$$f(x+h) = 2x^2 + 4xh + 2h^2 - 6x - 6h + 9$$

Now

$$f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 - 6x - 6h + 9 - (2x^2 - 6x + 9)$$
$$f(x+h) - f(x) = \cancel{2x^2} + 4xh + 2h^2 \cancel{- 6x} - 6h + \cancel{9} - \cancel{2x^2} + \cancel{6x} - \cancel{9}$$
$$f(x+h) - f(x) = 4xh + 2h^2 - 6h$$

Next

$$\frac{f(x+h)-f(x)}{h} = \frac{4xh + 2h^2 - 6h}{h}$$
$$\frac{f(x+h)-f(x)}{h} = 4x + 2h - 6$$

Finally

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$
$$\frac{dy}{dx} = \lim_{h \to 0} 4x + 2h - 6$$
$$\frac{dy}{dx} = 4x - 6$$

Exercise 1

From first principles, find $\frac{dy}{dx}$ if $y = x^2 + 7x$.

Top 10

(Unit I Summer 2023)

0 9 **a)** Given that $y = x^2 - 3x$, find $\frac{dy}{dx}$ from first principles. [5]

(Unit 1 Summer 2019)

0 8 **a)** Given that $y = 2x^2 - 5x$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ from first principles. [5]

(C1 Winter 2005)

7. Differentiate $x^2 + 4x + 3$ from first principles. [5]

(C1 Summer 2005)

5. Given $y = x^2 - 7x + 2$, find $\frac{dy}{dx}$ from first principles. [5]

(C1 Winter 2009)

8. *(a)* Given that $y = 7x^2 + 5x - 2$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ from first principles. [5]

(C1 Summer 2012)

7. *(a)* Given that $y = 3x^2 - 7x + 5$, find $\frac{dy}{dx}$ from first principles. [5]

(C1 Summer 2015)

7. *(a)* Given that $y = 9x^2 - 8x - 3$, find $\frac{dy}{dx}$ from first principles. [5]

(C1 Winter 2013)

6. *(a)* Given that $y = -x^2 + 4x - 6$, find $\frac{dy}{dx}$ from first principles. [5]

(C1 Summer 2017)

9. *(a)* Given that $y = -5x^2 - 7x + 13$, find $\frac{dy}{dx}$ from first principles. [5]

(Sample Assessment Materials)

3. Given that $y = x^3$, find $\frac{dy}{dx}$ from first principles. [6]

Revision Question

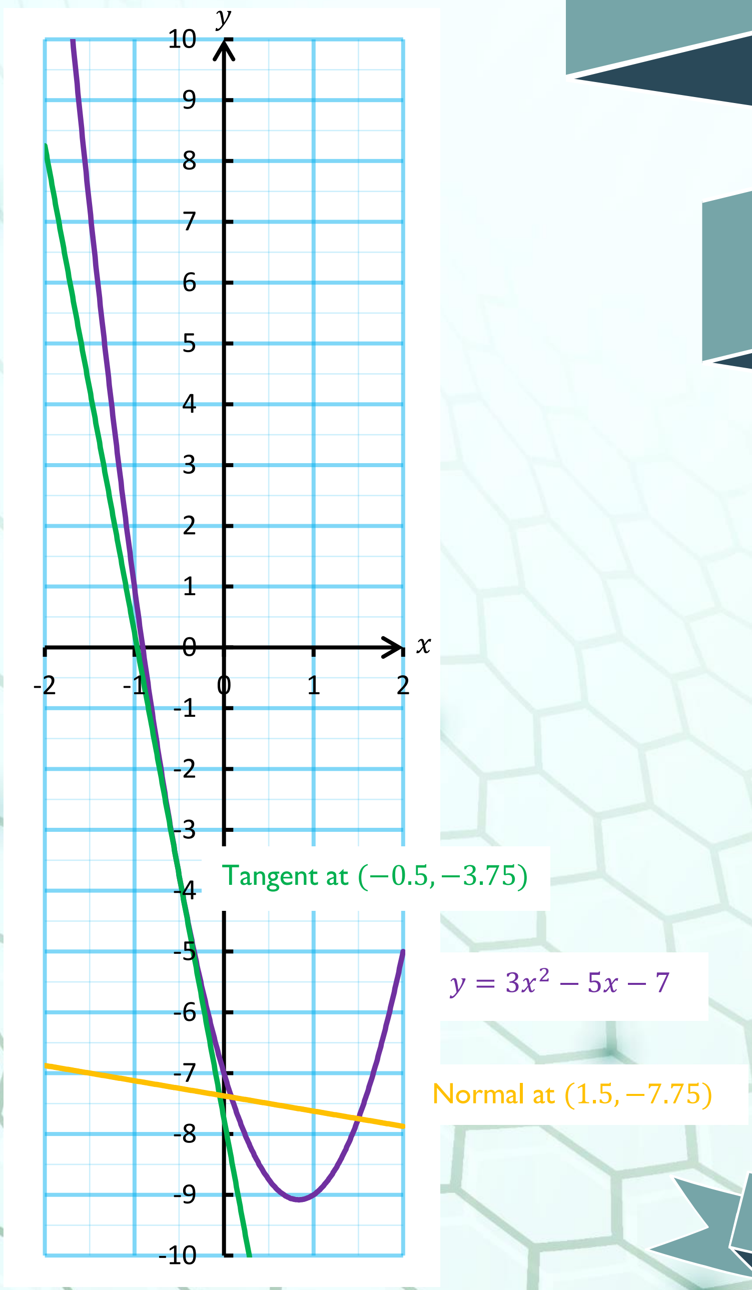
(C1 Summer 2018)

7. *(a)* Given that $y = 9x^2 - 7x - 8$, find $\frac{dy}{dx}$ from first principles. [5]

Unit 1, Workbook 6

12

Quick Differentiation



Name:

Background

What is the work?	Differentiating a function without having to go from first principles, and using this technique to find the equation of tangents and normals.
What is required before starting?	**GCSE Work:** Solving linear equations. **A Level Unit 1:** Solving quadratic equations.
Where does this lead to?	**Units 1–4:** Differentiating and integrating various functions.

Theory

In the previous workbook, the technique of differentiating from first principles was introduced. Did you notice, in this work, the connection between each question and its answer? For example, we can differentiate from first principles to obtain the following answers.

Theory

y	$\frac{dy}{dx}$
$3x^2 + 8x$	$6x + 8$
$5x^2 - 9x$	$10x - 9$
$4x^2 + 11x + 3$	$8x + 11$
$7x^3$	$21x^2$
$26x + 18$	26

We can use the following rule to quickly differentiate a function without having to go from first principles:

If $y = ax^n$, then $\frac{dy}{dx} = nax^{n-1}$.

Example 1

y	a	n	$\frac{dy}{dx} = nax^{n-1}$
$6x^2$	6	2	$2 \times 6 \times x^1 = 12x$
$4x^3$	4	3	$3 \times 4 \times x^2 = 12x^2$
$-14x$	-14	1	$1 \times -14 \times x^0 = -14$
23	23	0	$0 \times 23 \times x^{-1} = 0$
$4x^{-6}$	4	-6	$-6 \times 4 \times x^{-7} = -24x^{-7}$
$-643x^{215}$	-643	215	$215 \times -643 \times x^{214} = -138245x^{214}$

Given a function that is a sum or difference of a series of terms, we can differentiate the terms individually to differentiate the function.

Example 2

y	$\frac{dy}{dx}$
$7x^4 + 8x^2 - 4$	$28x^3 + 16x$
$10x^5 + 8x^{\frac{1}{2}} - 7x^{-3}$	$50x^4 + 4x^{-\frac{1}{2}} + 21x^{-4}$

Exercise 1

Complete the following table.

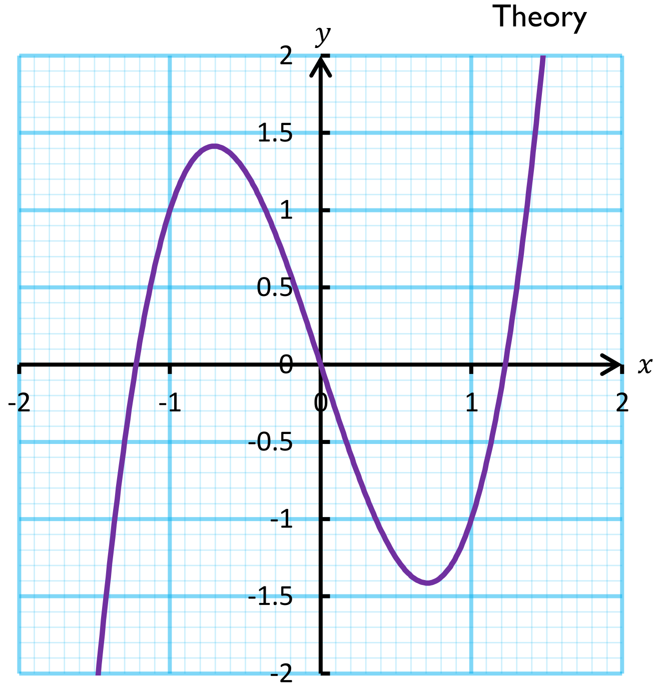
y	$\frac{dy}{dx}$
$11x^3$	
$5x^2 + 3$	
$18x^3 - 24x^2 + 18x - 2$	
$28x^{\frac{1}{2}}$	
$6 + 2x^{-1}$	
$19x^3 - 3 + 4x^2 - 19x$	
$24x^{-\frac{3}{4}}$	
8	
$\sqrt{x} + \frac{4}{x^2}$	

We can use the process of differentiation to find the equation of the **tangent** or **normal** to a curve at a particular point.

Theory

Example 3

The curve for $y = 2x^3 - 3x$ is shown on the right.



(a) Find the equation of the tangent to the curve at the point where $x = 1$.

(b) Find the equation of the normal to the curve at the point where $x = -1$.

Answer: (a) If $y = 2x^3 - 3x$, then $\frac{dy}{dx} = 6x^2 - 3$.
When $x = 1$, we have $y = 2(1^3) - 3(1) = -1$, and $\frac{dy}{dx} = 6(1^2) - 3 = 3$.
We can use $y - y_1 = m(x - x_1)$ to find the equation of the tangent to the curve:

$y - (-1) = 3(x - 1)$
$y + 1 = 3x - 3$
$y = 3x - 4$

(b) When $x = -1$, we have $y = 2(-1)^3 - 3(-1) = 1$, and $\frac{dy}{dx} = 6(-1)^2 - 3 = 3$. Given that the **normal** is always **perpendicular** to the tangent, we can say that the gradient of the normal is $-\frac{1}{3}$ (which is the negative of the reciprocal of 3). We can use $y - y_1 = m(x - x_1)$ to find the equation of the normal to the curve:

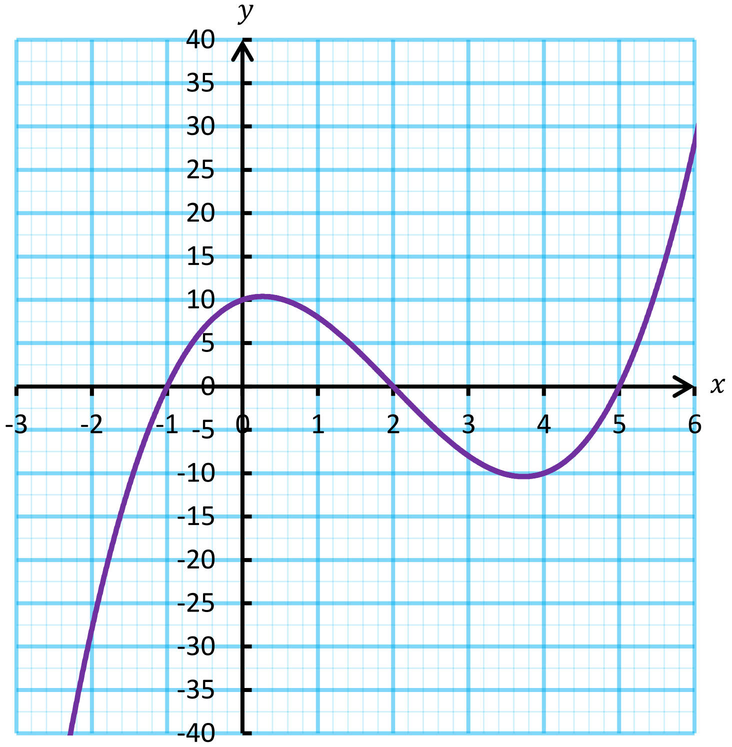
$y - 1 = -\frac{1}{3}(x - (-1))$
$y - 1 = -\frac{1}{3}(x + 1)$
$y - 1 = -\frac{1}{3}x - \frac{1}{3}$
$y = -\frac{1}{3}x + \frac{2}{3}$

Exercise 2

The curve for $y = x^3 - 6x^2 + 3x + 10$ is shown on the right.

(a) Find the equation of the tangent to the curve at the point where $x = -1$.

(b) Find the equation of the normal to the curve at the point where $x = 3$.



Top 10

(C1 Summer 2006, Question 7)

(b) Differentiate $\frac{2}{x^2} + 7\sqrt{x}$ with respect to x. [2]

(C1 Winter 2010, Question 6)

(b) Given that $y = ax^{\frac{5}{2}}$ and $\frac{dy}{dx} = -2$ when $x = 4$, find the value of the constant a. [3]

(C1 Summer 2016, Question 8)

(b) Given that $y = 4\sqrt{x} + \frac{45}{x}$, find the value of $\frac{dy}{dx}$ when $x = 9$. [4]

(C1 Winter 2008)

3. The curve C has equation $y = 2x^2 - 10x + 16$. The point P has coordinates (3, 4) and lies on C. Find the equation of the tangent to C at P. [4]

(C1 Winter 2005)

8. The curve C has equation $y = 3x^{\frac{3}{2}} - \frac{32}{x}$.

(a) Find the equation of the tangent to C at the point where $x = 4$. [7]

(b) Find the equation of the normal to C at the point where $x = 4$. [2]

(C1 Winter 2009)

3. The curve C has equation $y = x^2 - 9x + 13$.

(a) The point P has coordinates $(6, -5)$ and lies on C. Find the equation of the **tangent** to C at P. [4]

(b) The point Q lies on C and is such that the gradient of the **normal** to C at Q is $\frac{1}{7}$.

Find the x-coordinate of Q. [3]

(C1 Summer 2013)

3. The curve C has equation $y = 2x^2 - 10x + 7$.

(a) The point P has coordinates $(3, -5)$ and lies on C. Find the equation of the **normal** to C at P. [5]

(b) The point Q lies on C and is such that the **tangent** to C at Q is parallel to the x-axis. Find the x-coordinate of Q. [2]

(C1 Summer 2010)

3. The curve C has equation $y = x^2 - 8x + 10$.

(a) The point P has coordinates $(3, -5)$ and lies on C. Find the equation of the **normal** to C at P. [5]

(b) The point Q lies on C and is such that the **tangent** to C at Q has equation

$$y = 4x + c,$$

where c is a constant. Find the coordinates of Q and the value of c. [4]

(C1 Winter 2011)

8. The curve C has equation $y = x^2 - 6x + 7$.

(a) The point P, whose x-coordinate is 5, lies on the curve C. Find the equation of the tangent to C at P. [5]

The line L has equation $y = \frac{1}{2}x - 2$.

(b) (i) Find the coordinates of the two points of intersection of C and L.

(ii) Verify that L is in fact the normal to C at one of these points of intersection. [8]

(Unit 1 Summer 2023, Question 9)

b) The function f is defined by $f(x) = 4x^{\frac{3}{2}} + \frac{6}{\sqrt{x}}$ for $x > 0$.

i) Find $f'(x)$. [2]

ii) When $x > k$, $f(x)$ is an increasing function. Determine the least possible value of k. Give your answer correct to two decimal places. [4]

Revision Question

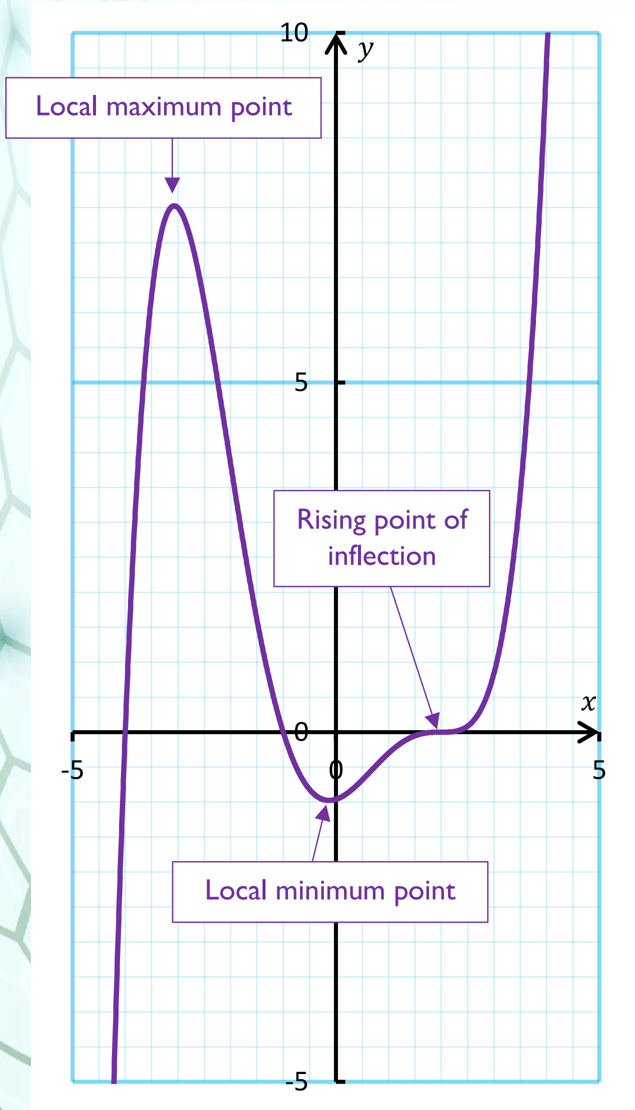
(Unit 1 Summer 2018)

16 The curve C has equation $y = 7 + 13x - 2x^2$. The point P lies on C and is such that the tangent to C at P has equation $y = x + c$, where c is a constant. Find the coordinates of P and the value of c. [5]

Unit 1, Workbook 7

12

Stationary Points



Name:

Background

What is the work?	How to find minimum points; maximum points and points of inflection for a particular function.
What is required before starting?	**GCSE Work:** Solving linear equations. **A Level Unit 1:** Differentiation; solving quadratic equations.
Where does this lead to?	**Unit 3:** Finding points of inflection that are not stationary points. **Applications:** Finding the maximum profit or minimum cost in the business world.

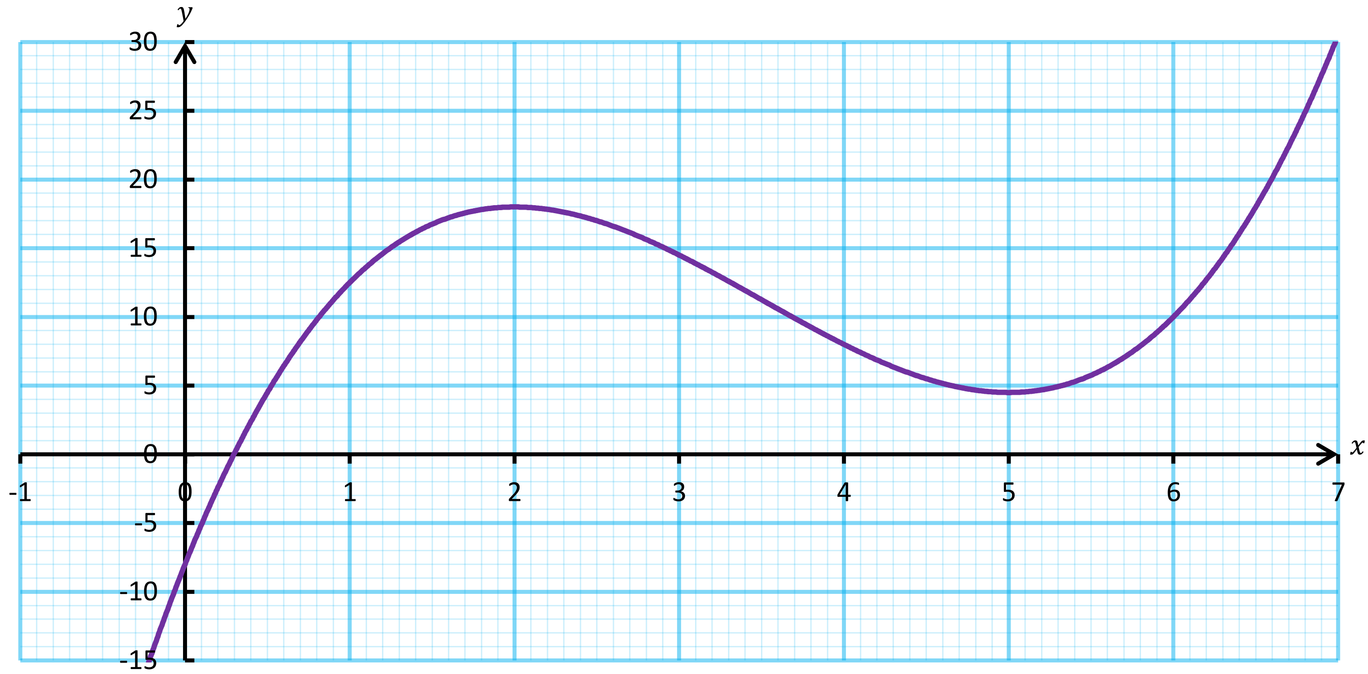
Theory

The Second Derivative

Given a function of the form $y = f(x)$, the gradient $\frac{dy}{dx} = f'(x)$ tells us what the **rate of change** of y is for a particular value of x. This represents how much y would change when moving one unit across to the right.

Theory

For example, consider the graph below which shows the function $y = x^3 - 10.5x^2 + 30x - 8$.



We can differentiate the function to give $\frac{dy}{dx} = 3x^2 - 21x + 30$. The table below shows some of the values of $\frac{dy}{dx}$ for particular values of x.

x	–1	–0.5	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7
$\frac{dy}{dx}$	54	41.25	30	20.25	12	5.25	0	–3.75	–6	–6.75	–6	–3.75	0	5.25	12	20.25	30

We see that $\frac{dy}{dx}$ is sometimes positive, which means that the rate of change of y is positive, and so the curve is **increasing**. In set notation, the curve increases for the following values of x:

$$\{x: x < 2\} \cup \{x: x > 5\}.$$

At other values of x, $\frac{dy}{dx}$ is negative, which means that the rate of change of y is negative, and so the curve is decreasing. In set notation, the curve decreases for the following values of x:

$$\{x: x > 2\} \cap \{x: x < 5\}.$$

The **second derivative** $\frac{d^2y}{dx^2}$ represents the **rate of change of** $\boldsymbol{\frac{dy}{dx}}$, which is how much the gradient is changing at a particular point. We can find it by differentiating $\frac{dy}{dx}$, and we use the notation $\frac{d^2y}{dx^2} = f''(x)$ to represent it.
In the above case where $\frac{dy}{dx} = 3x^2 - 21x + 30$, the second derivative is $\frac{d^2y}{dx^2} = 6x - 21$.
The table below shows some of the values of $\frac{d^2y}{dx^2}$ for particular values of x.

x	–1	–0.5	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7
$\frac{d^2y}{dx^2}$	–27	–24	–21	–18	–15	–12	–9	–6	–3	0	3	6	9	12	15	18	21

Up to 3.5, the value of $\frac{dy}{dx}$ is decreasing, and then after 3.5, the value of $\frac{dy}{dx}$ is increasing.

Exercise 1

Complete the following table.

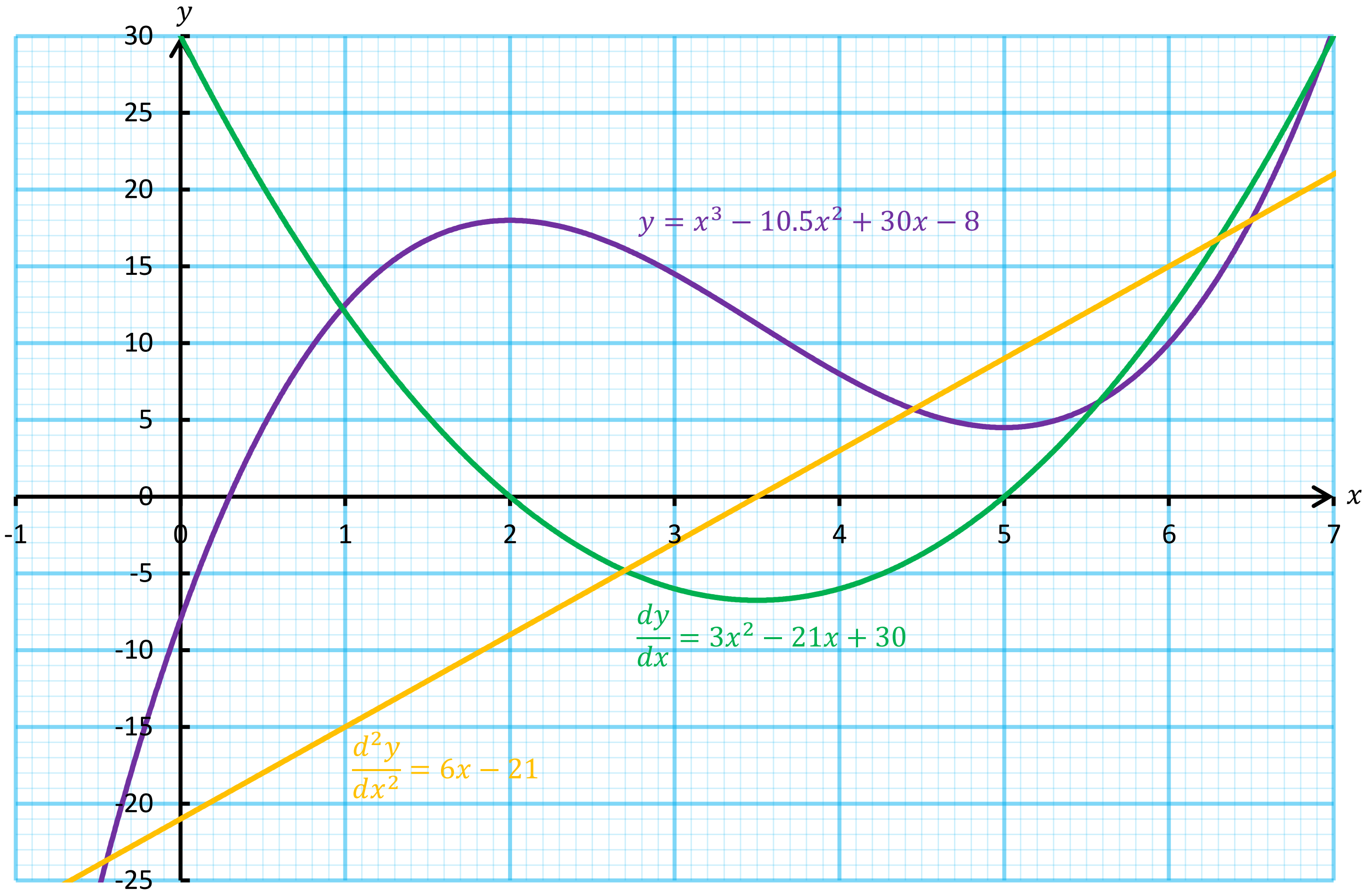
y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
$8x^3$		
$20x^5$		
$x^3 + 4x^2 - 3x + 5$		
$\sqrt{x}$		
$\frac{4}{x^2}$		

Exercise 2

For the function $y = x^3 + 4x^2 - 3x + 5$, find where the function is increasing, and when it is decreasing.
Write your answers in set notation.

Stationary Points

Let us consider again the example from the previous pages, where $y = x^3 - 10.5x^2 + 30x - 8$, $\frac{dy}{dx} = 3x^2 - 21x + 30$, and $\frac{d^2y}{dx^2} = 6x - 21$. The graph below shows these three functions, and the connections between them.



Any point where $\frac{dy}{dx} = 0$ is called a **stationary** point. At these points, the tangent to the curve is horizontal, and the curve does not increase or decrease. For the curve $y = x^3 - 10.5x^2 + 30x - 8$, we see from the above graph that there are two stationary points, where $x = 2$ and $x = 5$. For the first stationary point, where $x = 2$,

- the purple curve reaches a local maximum;
- the value of the green curve is zero;
- the value of the orange line is negative.

For the second stationary point, where $x = 5$,

- the purple curve reaches a local minimum;
- the value of the green curve is zero;
- the value of the orange line is positive.

We can use the above information to form an initial strategy for finding the coordinates and the nature of a curve's stationary points:

- Solve $\frac{dy}{dx} = 0$ to find the x–coordinate of each stationary point.
- Substitute back into y to find the y–cooordinates.
- Substitute into $\frac{d^2y}{dx^2}$ to find the nature of each stationary point. If $\frac{d^2y}{dx^2} > 0$, then the stationary point is a local minimum point. If $\frac{d^2y}{dx^2} < 0$, then the stationary point is a local maximum point.

Example 1

Find the coordinates and the nature of the stationary points for the curve $y = x^3 - 10.5x^2 + 30x - 8$.

Differentiate: $\frac{dy}{dx} = 3x^2 - 21x + 30$.

$-15 \times -6 = 90$
$-15 + -6 = -21$

Solve $\frac{dy}{dx} = 0$:

$3x^2 - 21x + 30 = 0$
3 × 30 = 90
$3x^2 - 15x - 6x + 30 = 0$
$3x(x - 5) - 6(x - 5) = 0$
$(x - 5)(3x - 6) = 0$
Either $x - 5 = 0$ or $3x - 6 = 0$
$x = 5$ $\quad$ $3x = 6$
$x = 2$

Substitute back into y: If $x = 5$ then $y = 5^3 - 10.5 \times 5^2 + 30 \times 5 - 8$
$y = 4.5$
If $x = 2$ then $y = 2^3 - 10.5 \times 2^2 + 30 \times 2 - 8$
$y = 18$

Differentiate again: $\frac{d^2y}{dx^2} = 6x - 21$.

Substitute into $\frac{d^2y}{dx^2}$: If $x = 5$ then $\frac{d^2y}{dx^2} = 6 \times 5 - 21$
$\frac{d^2y}{dx^2} = 9$
$\frac{d^2y}{dx^2}$ is positive so $(5, 4.5)$ is a local minimum point.

If $x = 2$ then $\frac{d^2y}{dx^2} = 6 \times 2 - 21$
$\frac{d^2y}{dx^2} = -9$
$\frac{d^2y}{dx^2}$ is negative so $(2, 18)$ is a local maximum point.

Exercise 3

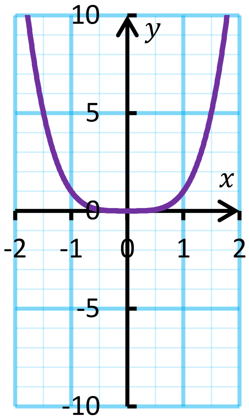
Find the coordinates and the nature of the stationary points for the curve $y = x^3 - 3x^2 - 9x + 2$.

Zero Second Derivative

In a case where $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, it is not possible to come to an immediate conclusion concerning the type of stationary point we have. Instead, we must look at the sign of $\frac{dy}{dx}$ each side of the stationary point.

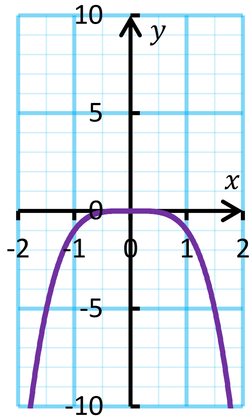
Example 2

(a) $y = x^4$



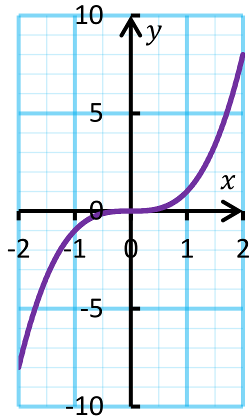
$\frac{dy}{dx} = 4x^3$

(b) $y = -x^4$



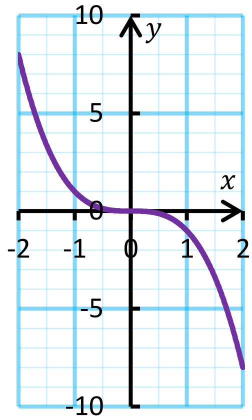
$\frac{dy}{dx} = -4x^3$

(c) $y = x^3$



$\frac{dy}{dx} = 3x^2$

(d) $y = -x^3$



$\frac{dy}{dx} = -3x^2$

Stationary points occur when $\frac{dy}{dx} = 0$.

(a)
$4x^3 = 0$
$x^3 = 0$
$x = \sqrt[3]{0}$
$x = 0$

(b)
$-4x^3 = 0$
$x^3 = 0$
$x = \sqrt[3]{0}$
$x = 0$

(c)
$3x^2 = 0$
$x^2 = 0$
$x = \pm\sqrt{0}$
$x = 0$

(d)
$-3x^2 = 0$
$x^2 = 0$
$x = \pm\sqrt{0}$
$x = 0$

In each case, there is a stationary point on the curve when $x = 0$.

(a)
$\frac{d^2y}{dx^2} = 12x^2$
If $x = 0$, then
$\frac{d^2y}{dx^2} = 12 \times 0^2$
$\frac{d^2y}{dx^2} = 0$

(b)
$\frac{d^2y}{dx^2} = -12x^2$
If $x = 0$, then
$\frac{d^2y}{dx^2} = -12 \times 0^2$
$\frac{d^2y}{dx^2} = 0$

(c)
$\frac{d^2y}{dx^2} = 6x$
If $x = 0$, then
$\frac{d^2y}{dx^2} = 6 \times 0$
$\frac{d^2y}{dx^2} = 0$

(d)
$\frac{d^2y}{dx^2} = -6x$
If $x = 0$, then
$\frac{d^2y}{dx^2} = -6 \times 0$
$\frac{d^2y}{dx^2} = 0$

In each case, $\frac{d^2y}{dx^2} = 0$. This is not positive (a minimum point) or negative (a maximum point), so we need to look at the sign of $\frac{dy}{dx}$ each side of the stationary point, in order for us to recognise its type.

(a)
If $x = -1$, then
$\frac{dy}{dx} = 4 \times (-1)^3$
$\frac{dy}{dx} = -4$ (negative)
If $x = 1$, then
$\frac{dy}{dx} = 4 \times 1^3$
$\frac{dy}{dx} = 4$ (positive)

The gradient is negative before and positive afterwards so $(0, 0)$ is a **minimum** point.

(b)
If $x = -1$, then
$\frac{dy}{dx} = -4 \times (-1)^3$
$\frac{dy}{dx} = 4$ (positive)
If $x = 1$, then
$\frac{dy}{dx} = -4 \times 1^3$
$\frac{dy}{dx} = -4$ (negative)

The gradient is positive before and negative afterwards so $(0, 0)$ is a **maximum** point

(c)
If $x = -1$, then
$\frac{dy}{dx} = 3 \times (-1)^2$
$\frac{dy}{dx} = 3$ (positive)
If $x = 1$, then
$\frac{dy}{dx} = 3 \times 1^2$
$\frac{dy}{dx} = 3$ (positive)

The gradient is positive before and positive afterwards so $(0, 0)$ is a **rising point of inflection**.

(d)
If $x = -1$, then
$\frac{dy}{dx} = -3 \times (-1)^2$
$\frac{dy}{dx} = -3$ (negative)
If $x = 1$, then
$\frac{dy}{dx} = -3 \times 1^2$
$\frac{dy}{dx} = -3$ (negative)

The gradient is negative before and negative afterwards so $(0, 0)$ is a **falling point of inflection**.

We can now form a final strategy for finding the coordinates and the nature of a curve's stationary points:

- Solve $\frac{dy}{dx} = 0$ to find the x–coordinate of each stationary point.
- Substitute back into y to find the y–coordinates.
- Substitute into $\frac{d^2y}{dx^2}$ to find the nature of each stationary point.
    - If $\frac{d^2y}{dx^2} > 0$, then the stationary point is a **local minimum point**.
    - If $\frac{d^2y}{dx^2} < 0$, then the stationary point is a **local maximum point**.
    - If $\frac{d^2y}{dx^2} = 0$, then we need to consider the sign of $\frac{dy}{dx}$ each side of the stationary point.
        - If $\frac{dy}{dx} < 0$ before and $\frac{dy}{dx} > 0$ afterwards, then the stationary point is a **local minimum point**.
        - If $\frac{dy}{dx} > 0$ before and $\frac{dy}{dx} < 0$ afterwards, then the stationary point is a **local maximum point**.
        - If $\frac{dy}{dx} > 0$ before and $\frac{dy}{dx} > 0$ afterwards, then the stationary point is a **rising point of inflection**.
        - If $\frac{dy}{dx} < 0$ before and $\frac{dy}{dx} < 0$ afterwards, then the stationary point is a **falling point of inflection**.

Theory

Exercise 4

Find the coordinates and the nature of the stationary point of the curve $y = x^3 - 6x^2 + 12x - 8$.

Top 10

(C1 Summer 2007)

10. The curve C has equation

$$y = x^3 - x^2 - x + 2.$$

Find the coordinates of the stationary points of C and determine the nature of each of these stationary points. [7]

(C1 Summer 2008)

9. The curve C has equation

$$y = -2x^3 + 3x^2 + 12x - 5.$$

Find the coordinates and nature of each of the stationary points of C. [7]

(C1 Summer 2010)

10. The curve C has equation

$$y = \frac{1}{2}x^3 - 6x + 3.$$

Find the coordinates and the nature of each of the stationary points of C. [6]

(C1 Summer 2009)

10. The curve C has equation

$$y = x^3 - 3x^2 + 3x + 5.$$

(a) Show that C has only one stationary point. Find the coordinates of this point. [4]

(b) Verify that this stationary point is a point of inflection. [2]

(C1 Winter 2011)

10. The curve C has equation

$$y = x^3 + kx^2 - 9x - 10,$$

where k is a constant. The two stationary points on the graph of C are denoted by Q and R. The x-coordinate of Q is -1.

(a) Find $\frac{dy}{dx}$ and hence show that $k = -3$. [3]

(b) Find the x-coordinate of R. [2]

(c) Determine the nature of each of the stationary points Q and R. [2]

(C1 Winter 2005)

3. A curve C has equation

$$y = 2x^3 - 6x^2 + 12.$$

(a) Find the coordinates of the stationary points of C and determine the nature of each of those points. [7]

(b) Sketch C. [3]

(c) State, giving a reason, the number of real roots of the equation

$$2x^3 - 6x^2 + 12 = 0.$$ [2]

(C1 Winter 2010)

10. The curve C has equation

$$y = x^3 - 6x^2 + 20.$$

(a) Find the coordinates and the nature of each of the stationary points of C. [6]

(b) Sketch C, indicating the coordinates of each of the stationary points. [2]

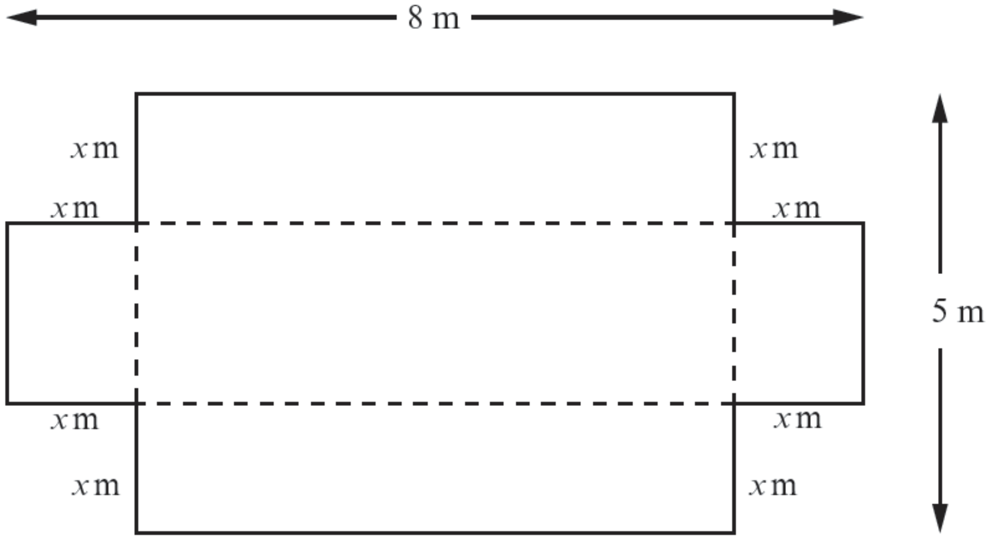
(c) Given that the equation

$$x^3 - 6x^2 + 20 = k$$

has three **distinct** real roots, find the range of possible values for k. [2]

(C1 Summer 2011)

10. A rectangular sheet of metal has length 8 m and width 5 m. Four squares, each of side x m, where $x < 2{\cdot}5$, have been cut away from the corners of the rectangular sheet, as shown in the diagram below. The rest of the metal sheet is now bent along the dotted lines to form an open tank in the form of a cuboid.



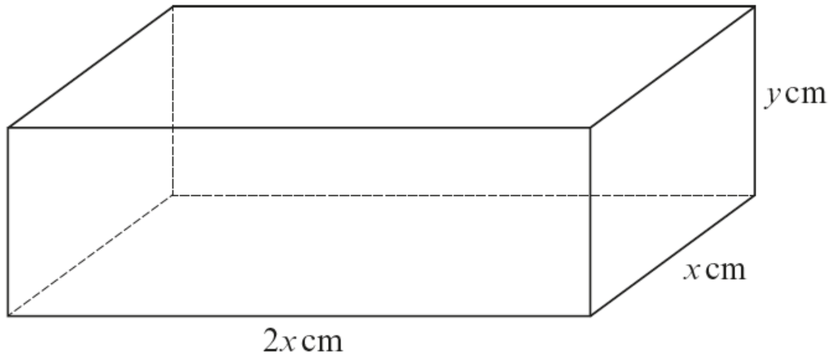
(a) Show that the volume V m^3 of this tank is given by

$$V = 4x^3 - 26x^2 + 40x\,.$$ [2]

(b) Find the maximum value of V, showing that the value you have found is a maximum value. [5]

(C1 Summer 2013)

10. The diagram shows a **closed** box in the form of a cuboid. The length of the box is $2x$ cm, its width is x cm and its height is y cm.



The total surface area of the box is $108\,\text{cm}^2$.

(a) (i) Write down an equation involving x and y and hence show that

$$xy = 18 - \frac{2}{3}x^2.$$

(ii) Hence show that the volume $V\,\text{cm}^3$ of the box is given by

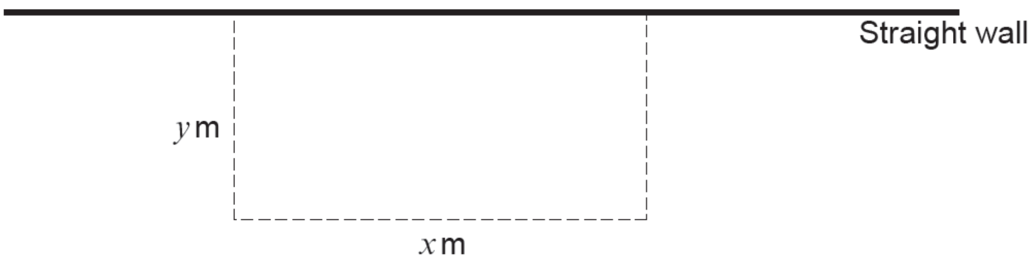
$$V = 36x - \frac{4}{3}x^3.$$

[3]

(b) Find the maximum value of V, showing that the value you have found is a maximum value. [5]

(C1 Summer 2015)

10. A sheep farmer wishes to construct a rectangular enclosure for his animals. He decides to use a straight wall as one side of the enclosure and fencing for the other three sides. The area of the enclosure is to be 800 m^2. The lengths of the sides of the rectangular enclosure are x m and y m, as shown in the diagram, and the total length of the **fencing** is L m.



(a) Show that $L = x + \dfrac{1600}{x}$. [2]

(b) Find the minimum value of L, showing that the value you have found is a minimum value. [5]

Revision Questions

(Unit 1 Summer 2024)

12. A curve C has equation $y = -x^3 + 12x - 20$.

(a) Find the coordinates of the stationary points of C and determine their nature. [7]

(b) Determine the range of values of x for which the curve is decreasing.
Give your answer in set notation. [3]

(Unit 1 Summer 2024)

10. Water is being emptied out of a sink. The depth of water, y cm, at time t seconds, may be modelled by

$$y = t^2 - 14t + 49 \qquad 0 \leqslant t \leqslant 7.$$

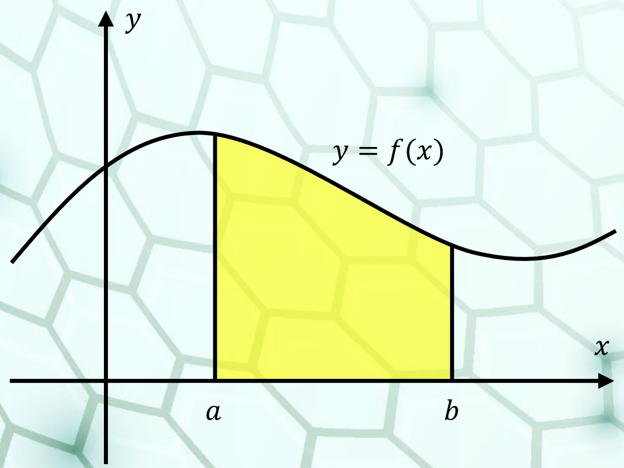
(a) Find the value of t when the depth of water is 25 cm. [3]

(b) Find the rate of decrease of the depth of water when $t = 3$. [3]

Unit 1, Workbook 8

12

Introducing Integration



$$A = \int_a^b f(x)\,dx$$
$$A = [F(x)]_a^b$$
$$A = F(b) - F(a)$$

Name:

Background

What is the work?	Introducing the process of integration as the inverse of differentiation, and using this to find the area beneath curves.
What is required before starting?	**GCSE Work:** Rules of indices; simultaneous equations. **A Level Unit 1:** Differentiation.
Where does this lead to?	**Unit 3:** Further integration. **Applications:** Finding areas; volumes; mass; location; cumulative distribution functions in probability.

Theory

Indefinite Integration

Integration is the process of reversing differentiation.

Exercise 1

Complete the following table.

Theory

y	$\frac{dy}{dx}$
$5x^2+9x+3$	
$5x^2+9x+4$	
$5x^2+9x-3$	
$5x^2+9x+\frac{3}{4}$	
$5x^2+9x+\pi$	

In completing the above exercise, you will see the problem facing us when attempting to integrate: whilst differentiation always gives a unique answer, this will not be the case for integration. Above, every function y in the first column differentiates to give the same answer. This leads to the question: if we are integrating $10x+9$, what should the answer be? To deal with this problem, we introduce a **constant of integration**. Different textbooks use different letters for the constant of integration: sometimes c, sometimes k, sometimes something else. This workbook will use c as the constant of integration.

When integrating $10x+9$, we say that the answer is $5x^2+9x+c$, where c represents any number, because any expression of the form $5x^2+9x+c$ would differentiate to give $10x+9$. The formal notation for this process is

$$\int 10x+9\,dx = 5x^2+9x+c$$

where dx denotes we are integrating with respect to the variable x, and $\int$ is the symbol for integration.

The rule for integrating a term of the form ax^n is the following:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

In words, we add one to the power n, and then divide by the new power $n+1$.

Example 1

Question	Answer
$\int 28x^3\,dx$	$\frac{28x^4}{4} + c = 7x^4 + c$
$\int 35x^4 - 6x + 2\,dx$	$\frac{35x^5}{5} - \frac{6x^2}{2} + 2x + c = 7x^5 - 3x^2 + 2x + c$

Exercise 2

Complete the following table.

Question	Answer
$\int 15x^2\,dx$	
$\int 12x - 9\,dx$	
$\int 24x^5 - 20x^3 + 10x\,dx$	

Often, we need to use rules of indices to re-write a question before we can integrate.

Example 2

Question	Re-writing the question using rules of indices	Answer
$\int \sqrt[3]{x}\,dx$	$\int x^{\frac{1}{3}} dx$	$\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{3}{4}x^{\frac{4}{3}} + c$
$\int \frac{6}{x^3} dx$	$\int 6x^{-3} dx$	$\frac{6x^{-2}}{-2} + c = -3x^{-2} + c$
$\int \sqrt{x} + \frac{2}{x^2} dx$	$\int x^{\frac{1}{2}} + 2x^{-2} dx$	$\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^{-1}}{-1} + c = \frac{2}{3}x^{\frac{3}{2}} - 2x^{-1} + c$

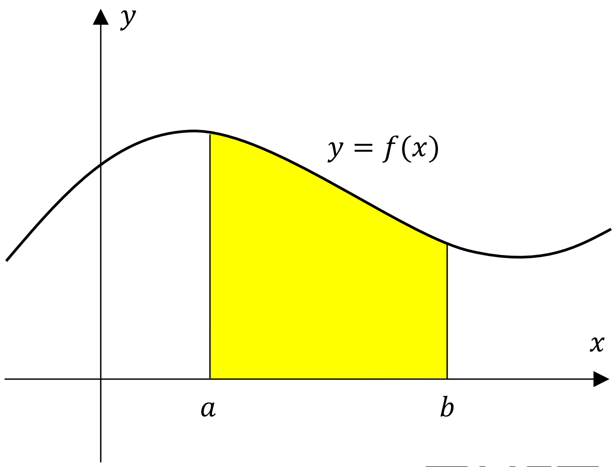
Exercise 3

Complete the following table.

Question	Re-writing the question using rules of indices	Answer
$\int \frac{8}{x^2} dx$		
$\int \sqrt[4]{x}\,dx$		
$\int 9\sqrt{x} + \frac{14}{x^{\frac{3}{4}}} dx$		

Definite Integration

It is possible to use integration to find the area between a curve and the x–axis.



Let us consider a general curve of the form $y = f(x)$.
The yellow area on the right is bounded by the curve, the x–axis and the lines $x = a$, $x = b$.
To find the size of the area, we integrate between a and b using the notation

$$\int_a^b f(x)\,dx$$

We say that a is the **lower limit** and b is the **upper limit**.

Theory

The **Fundamental Theorem of Calculus** tells us how to calculate the above integral:

If $\int f(x)\,dx = F(x) + c$, then

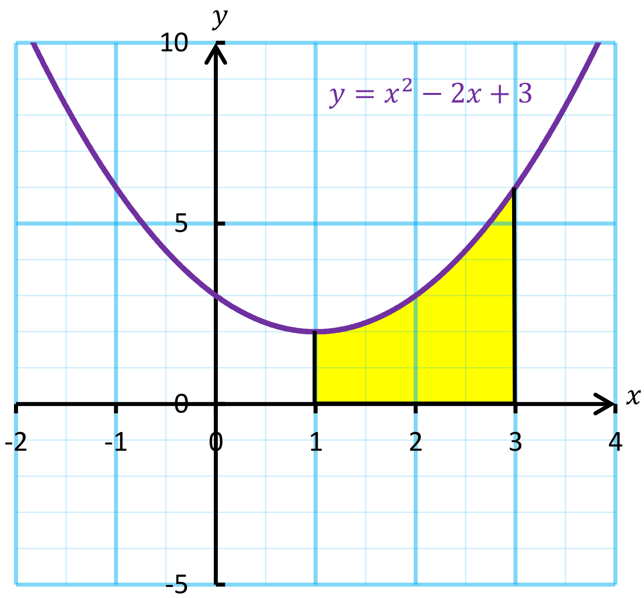
$$\int_a^b f(x)\,dx$$
$$= [F(x) + c]_a^b$$
$$= [F(b) + c] - [F(a) + c]$$
$$= F(b) - F(a)$$

Notice that the constants of integration cancel out, so we can write

$$\int_a^b f(x)\,dx$$
$$= [F(x)]_a^b$$
$$= F(b) - F(a)$$

Example 3

Find the area underneath the curve $y = x^2 - 2x + 3$ between $x = 1$ and $x = 3$.

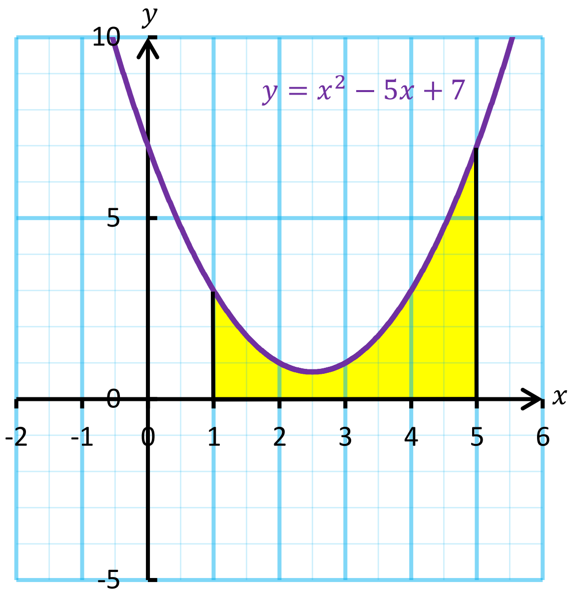


Answer: We need to find the value of

$$\int_1^3 x^2 - 2x + 3\,dx$$
$$= \left[\frac{x^3}{3} - \frac{2x^2}{2} + 3x\right]_1^3$$
$$= \left[\frac{x^3}{3} - x^2 + 3x\right]_1^3$$
$$= \left[\frac{3^3}{3} - 3^2 + 3 \times 3\right] - \left[\frac{1^3}{3} - 1^2 + 3 \times 1\right]$$
$$= [9 - 9 + 9] - \left[\frac{1}{3} - 1 + 3\right]$$
$$= 9 - \frac{7}{3}$$
$$= \frac{20}{3} \text{ square units}$$

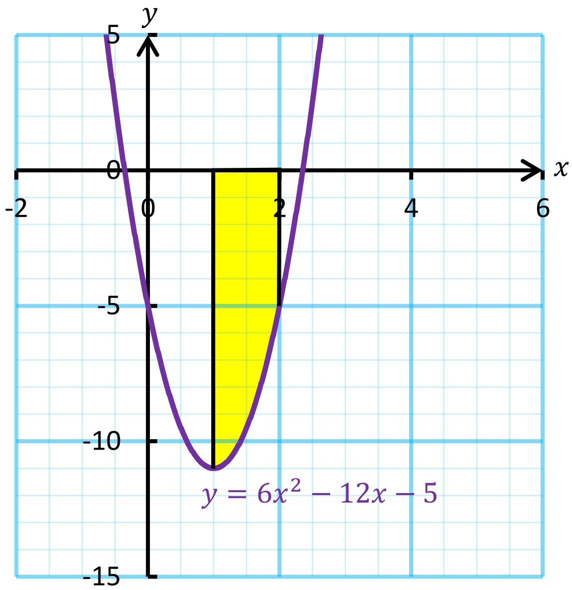
Exercise 4

Find the area underneath the curve
$y = x^2 - 5x + 7$ between $x = 1$ and $x = 5$.



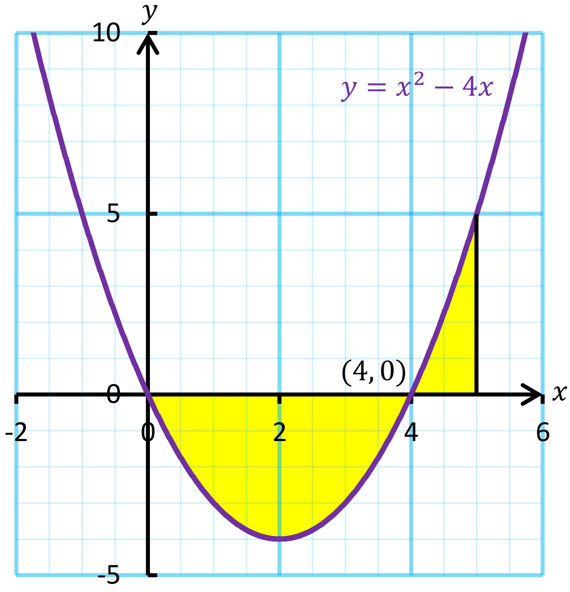
Exercise 5

Find the area above the curve
$y = 6x^2 - 12x - 5$ between $x = 1$ and $x = 2$.



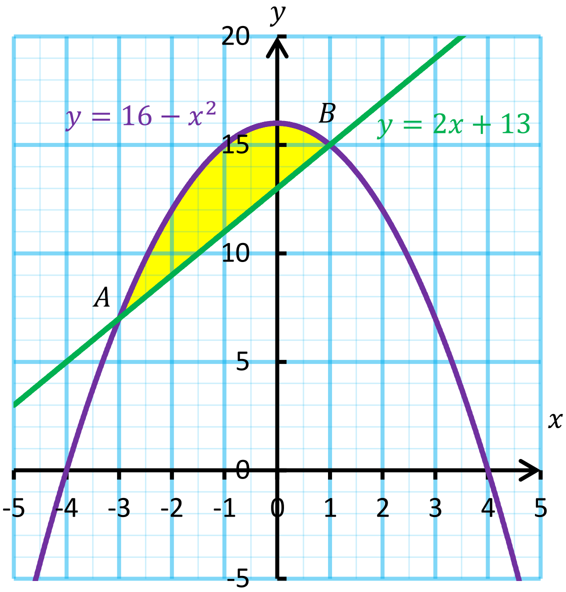
Exercise 6

Find the total area between the curve $y = x^2 - 4x$ and the x–axis between $x = 0$ and $x = 5$.



Example 4

The diagram shows a sketch of the curve $y = 16 - x^2$ and the line $y = 2x + 13$. The line and curve intersect at the points A and B.



(a) Find the coordinates of A and B.

(b) Find the area of the shaded region.

Answer: (a) To find the coordinates of A and B, we need to find where $y = 16 - x^2$ and $y = 2x + 13$ intersect (where they are equal).

$$\begin{aligned} 16 - x^2 &= 2x + 13 \\ 0 &= x^2 + 2x + 13 - 16 \\ 0 &= x^2 + 2x - 3 \\ 0 &= (x+3)(x-1) \end{aligned}$$

Either $x + 3 = 0$ or $x - 1 = 0$
$x = -3$ $\quad$ $x = 1$

Substitute back into the line: $y = 2 \times -3 + 13$ or $y = 2 \times 1 + 13$
$y = 7$ $\quad$ $y = 15$

So $A = (-3, 7)$ and $B = (1, 15)$.

(b) We need to consider how the yellow region is formed. We start with the area beneath the purple curve between $x = -3$ and $x = 1$, and then subtract the area beneath the green line between $x = -3$ and $x = 1$.

Area beneath the purple curve:

$$\begin{aligned} &\int_{-3}^{1} 16 - x^2 \, dx \\ &= \left[16x - \frac{x^3}{3}\right]_{-3}^{1} \\ &= \left[16 \times 1 - \frac{1^3}{3}\right] - \left[16 \times -3 - \frac{(-3)^3}{3}\right] \\ &= \left[16 - \frac{1}{3}\right] - [-48 + 9] \\ &= \frac{47}{3} - -39 \\ &= \frac{164}{3} \end{aligned}$$

Area beneath the green line:

$$\begin{aligned} &\int_{-3}^{1} 2x + 13 \, dx \\ &= \left[\frac{2x^2}{2} + 13x\right]_{-3}^{1} \\ &= [x^2 + 13x]_{-3}^{1} \\ &= [1^2 + 13 \times 1] - [(-3)^2 + 13 \times -3] \\ &= [1 + 13] - [9 - 39] \\ &= 14 - -30 \\ &= 44 \end{aligned}$$

So, the area of the shaded region is $\frac{164}{3} - 44 = \frac{32}{3}$ square units.

We could also find this area by calculating the area of the trapezium:
$7 + 15 = 22$
$22 \div 2 = 11$
$11 \times 4 = 44$

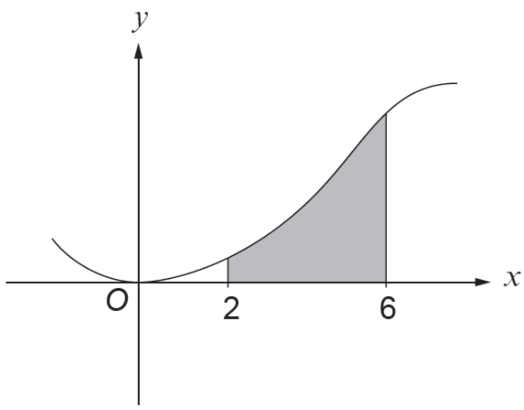
Top 10

(C2 Winter 2014)

6. *(a)* Find $\int \left(\frac{5}{x^3} - 2x^{\frac{1}{3}} - 4 \right) \mathrm{d}x.$ [3]

(b) The diagram below shows a sketch of the curve with equation $y = 3x^2 - \frac{1}{4}x^3$.

The shaded region is bounded by the curve, the x-axis and the lines $x = 2$, $x = 6$. Find the area of this shaded region. [4]



(C2 Summer 2016)

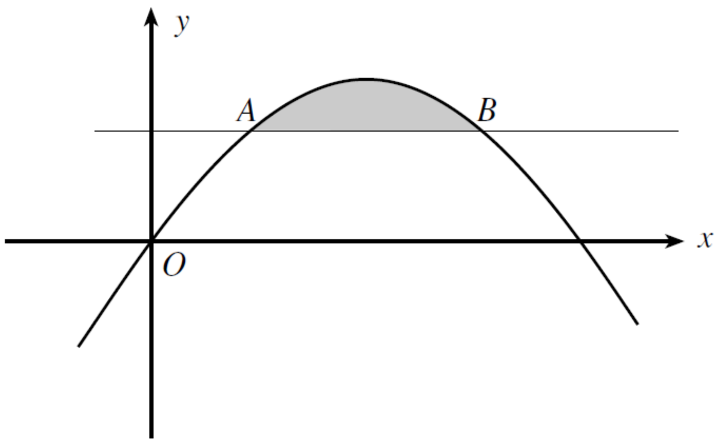
6. *(a)* Find $\int\left(\frac{3}{\sqrt[4]{x}} - 9x^{\frac{5}{2}}\right)\mathrm{d}x$. [2]

(b) The region *R* is bounded by the curve $y = 2x^2 + \frac{6}{x^2}$, the x-axis and the lines x = 1, x = 4. Find the area of *R*. [5]

(C2 Summer 2005)

7. *(a)* Find $\int \left(2x^{\frac{3}{4}} + \frac{7}{x^{\frac{1}{2}}}\right) \mathrm{d}x$. [2]

(b)



The diagram shows a sketch of the curve $y = 6x - x^2$ and the line $y = 5$. The line and the curve intersect at the points A and B.

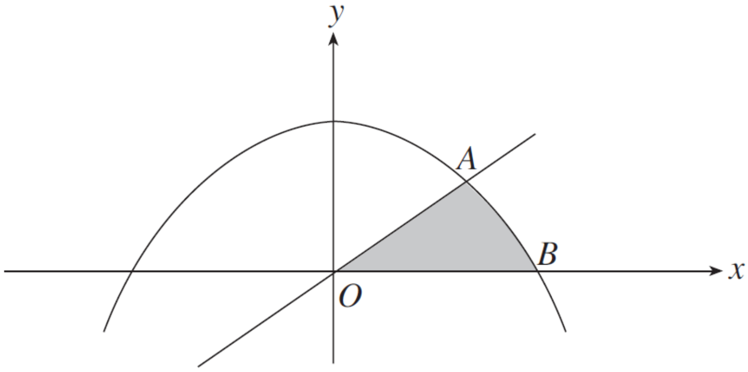
(i) Showing your working, find the coordinates of A and B.

(ii) Find the area of the shaded region. [10]

(C2 Winter 2006)

6. Integrate $5x^{\frac{1}{3}} + 3x^{-3}$ with respect to x. [2]

7.



The diagram shows the curve $y = 4 - x^2$ and the line $y = 3x$ intersecting at the point A. The curve $y = 4 - x^2$ intersects the x-axis at B.

(a) Find the coordinates of A and B, showing your working. [5]

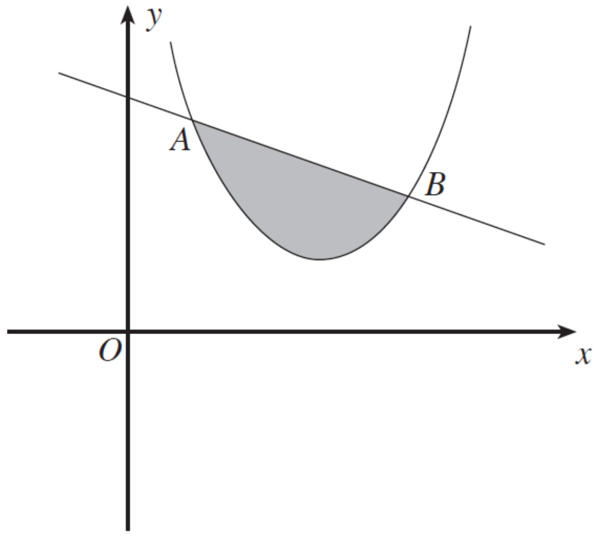
(b) Evaluate the area of the shaded region. [7]

(C2 Winter 2008)

7. *(a)* Find $\int\left(4x^{\frac{2}{3}} - \frac{7}{\sqrt{x}}\right)\mathrm{d}x.$

[2]

(b)



The diagram shows a sketch of the curve $y = x^2 - 6x + 11$ and the line $y = -x + 7$. The curve and the line intersect at the points A and B.

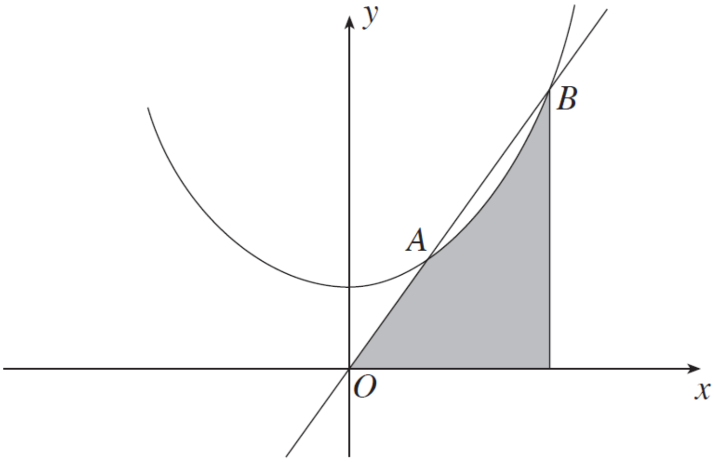
(i) Showing your working, find the coordinates of A and B.

(ii) Find the area of the shaded region. [11]

(C2 Summer 2007)

6. *(a)* Find $\int \left(2x^{\frac{3}{2}} + \frac{9}{x^4}\right) \mathrm{d}x$. [2]

(b)



The diagram shows a sketch of the curve $y = x^2 + 2$ and the line $y = 3x$.
The line and the curve intersect at the points A and B.

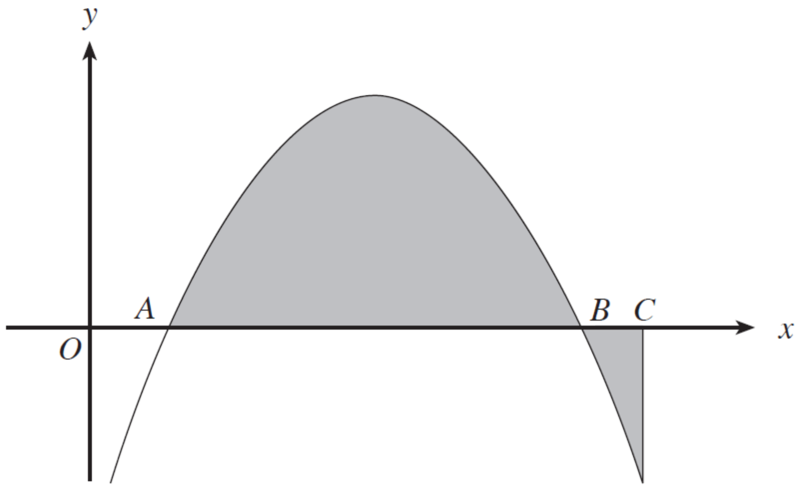
(i) Find the coordinates of the points A and B. [4]

(ii) Evaluate the area of the shaded region. [7]

(C2 Winter 2009)

6. *(a)* Find $\int \left(\frac{3}{x^2} - 2\sqrt{x} \right) dx.$ [2]

(b)



The diagram shows a sketch of the curve $y = 5x - 4 - x^2$.
The curve intersects the x-axis at the points A and B. The point C has coordinates (5, 0).

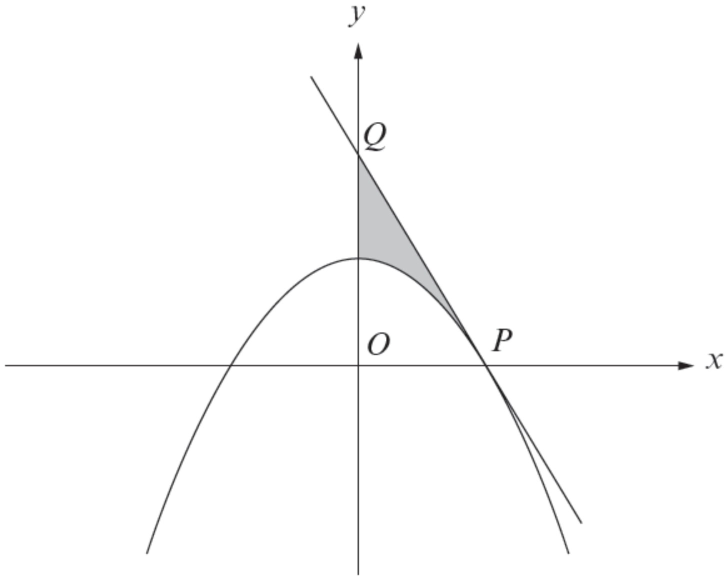
(i) Find the x-coordinates of the points A and B. [3]

(ii) Find the **total** area of the shaded regions. [7]

(C2 Winter 2013)

6. *(a)* Find $\int\left(\frac{5}{x^4} - 7x^{\frac{2}{3}}\right)\mathrm{d}x.$ [2]

(b)



The diagram shows a sketch of the curve $y = 9 - x^2$ which intersects the positive x-axis at the point $P(a, 0)$.

(i) Find the value of a.

The tangent to the curve at P intersects the y-axis at the point $Q(0, b)$.

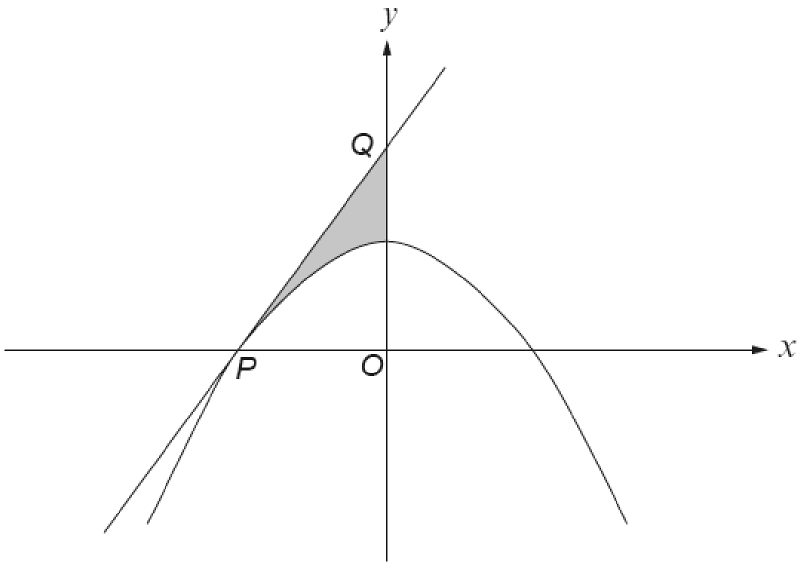
(ii) Show that $b = 18$.

(iii) Find the area of the shaded region. [10]

(C2 Summer 2017)

6. *(a)* Find $\int\left(\frac{2}{x^5} - 6x^{\frac{3}{4}}\right)dx$. [2]

(b)



The diagram shows a sketch of the curve $y = 16 - x^2$ which intersects the negative x-axis at the point $P(a, 0)$.

(i) Write down the value of a.

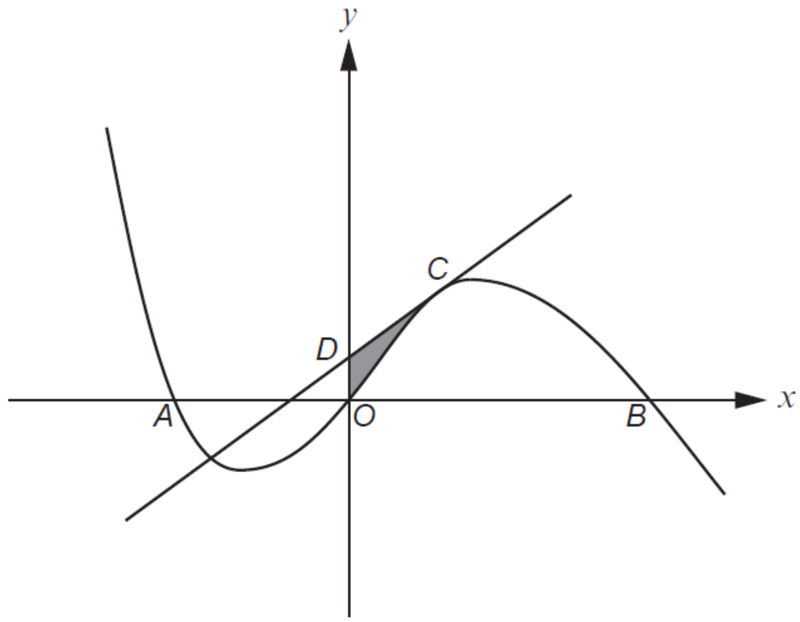
The tangent to the curve at P intersects the y-axis at the point $Q(0, b)$.

(ii) Show that $b = 32$.

(iii) Find the area of the shaded region. [10]

(Unit 1 Summer 2022)

1 1 The diagram below shows a sketch of the curve $y = f(x)$, where $f(x) = 10x + 3x^2 - x^3$. The curve intersects the x-axis at the origin O and at the points $A(-2, 0)$, $B(5, 0)$. The tangent to the curve at the point $C(2, 24)$ intersects the y-axis at the point D.



a) Find the coordinates of D. [5]

b) Find the area of the shaded region. [6]

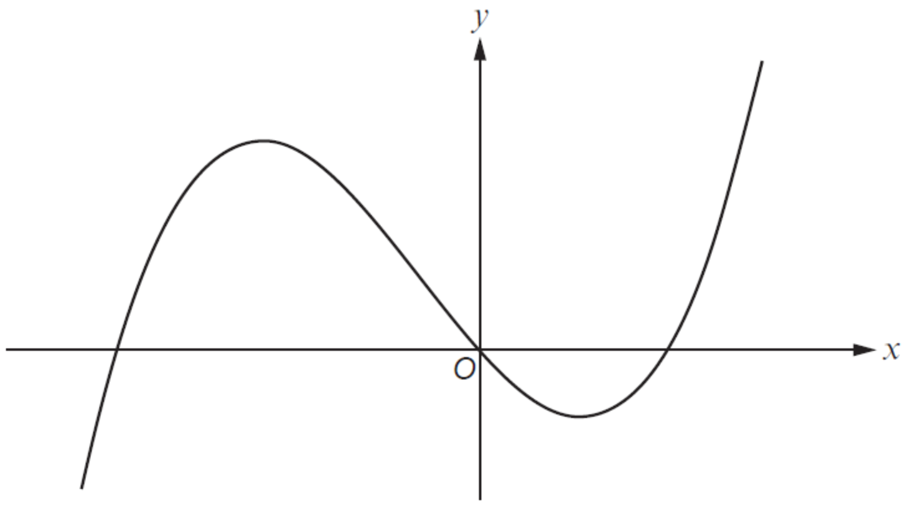
c) Determine the range of values of x for which $f(x)$ is an increasing function. [4]

Revision Questions

(Unit 1 Summer 2023)

1 3 **a)** Find $\int\left(4x^{-\frac{2}{3}}+5x^{3}+7\right)\mathrm{d}x$. [3]

b) The diagram below shows the graph of $y=x(x+6)(x-3)$.

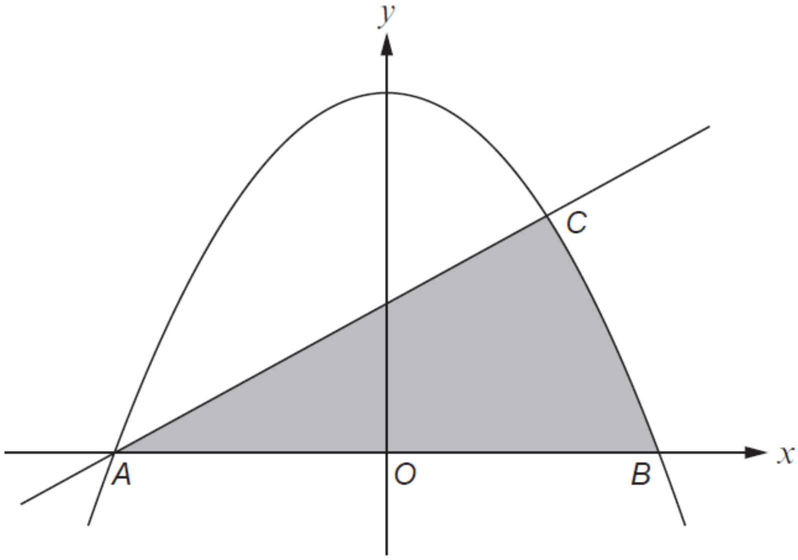


Calculate the total area of the regions enclosed by the graph and the x-axis. [9]

(C2 Summer 2018)

6. *(a)* Find $\int \left(\sqrt[3]{x} - \frac{4}{x^{\frac{7}{2}}} \right) \mathrm{d}x$. [2]

(b)



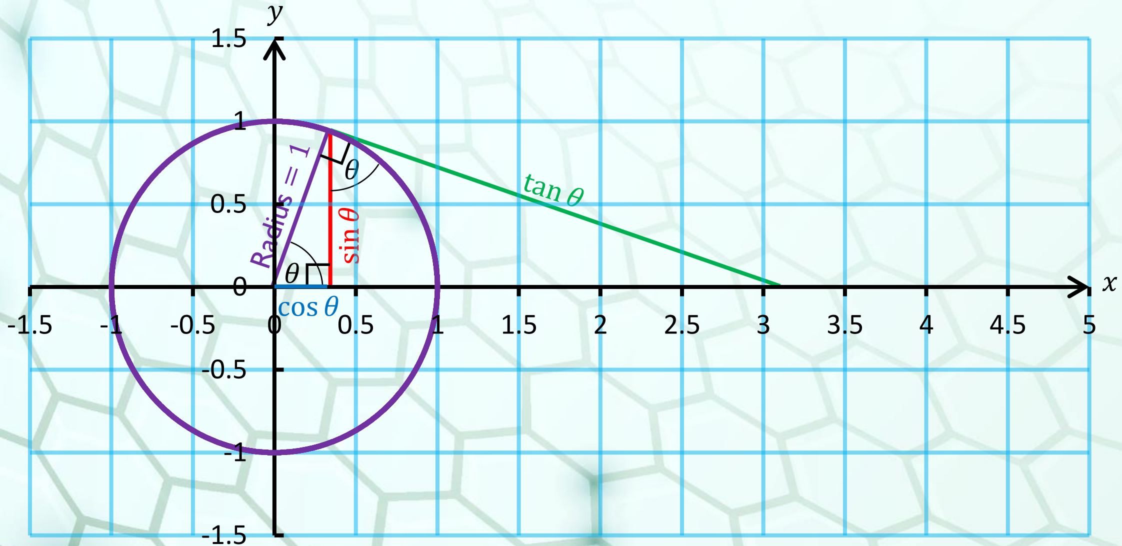
The diagram shows a sketch of the curve $y = 25 - x^2$ and the line $y = 2x + 10$. The curve and the line intersect at the points *A* and *C*. The curve intersects the x-axis at the points *A* and *B*. The coordinates of *A*, *B* and *C* are (– 5, 0), (5, 0) and (3, 16) respectively.
Find the area of the shaded region. [6]

Unit 1, Workbook 9

12

Trigonometry

Part 1



Name:

Background

What is the work?	Considering the meaning of the trigonometric ratios sin, cos and tan, and their interconnections. Solving trigonometric equations.
What is required before starting?	**GCSE Work:** Trigonometry, solving quadratic equations.
Where does this lead to?	**Unit 3:** Solving equations containing the trigonometric ratios sec, cosec and cot. **Applications:** Finding the equation of path of a projectile; simple harmonic motion.

Theory

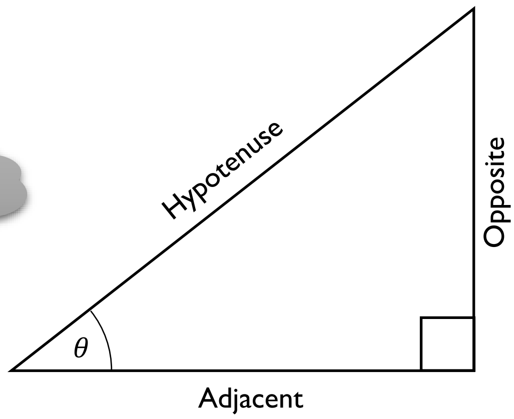
Trigonometry is the branch of mathematics studying the relationship between lengths and angles in triangles. The word derives from the Greek for "triangle" and "measure".

Here are the trigonometric ratios for a right-angled triangle.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



Definitions

Consider the unit circle shown on the right.

Considering the triangle ABC,

- The length of the hypotenuse is 1 unit (because the radius of the circle is 1 unit).
- $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$

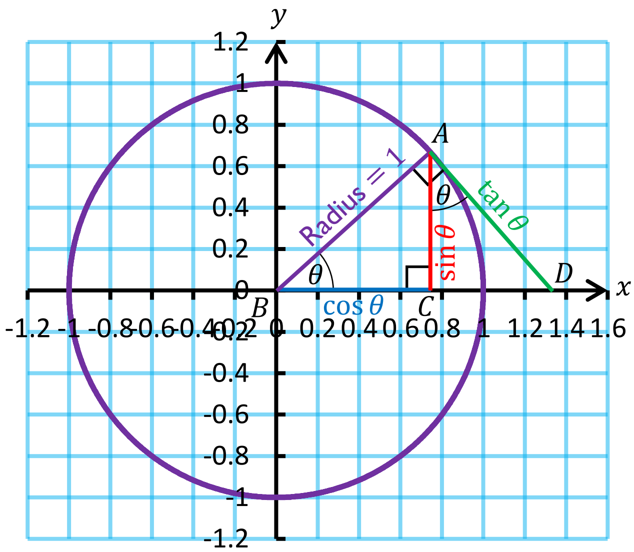
  $\sin\theta = \frac{\text{height of the triangle}}{1}$

  height of the triangle $= \sin\theta$
- $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

  $\cos\theta = \frac{\text{base of the triangle}}{1}$

  base of the triangle $= \cos\theta$
- $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

  $\tan\theta = \frac{\sin\theta}{\cos\theta}$



So, we can think of $\sin\theta$ as the height of a right-angled triangle in a unit circle, and we can think of $\cos\theta$ as the base of a right-angled triangle in a unit circle.

Considering the right-angled triangle ACD,

- $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$
  $\cos\theta = \frac{\sin\theta}{\text{hypotenuse}}$ (From the triangle ABC)
  $\text{hypotenuse} \times \cos\theta = \sin\theta$
  $\text{hypotenuse} = \frac{\sin\theta}{\cos\theta}$
  $\text{hypotenuse} = \tan\theta$ (From the fourth bullet point above)

So, we can think of $\tan\theta$ as the length of the tangent, measuring from the point A to the x–axis.

Using Pythagoras' Theorem on the right-angled triangle ABC,

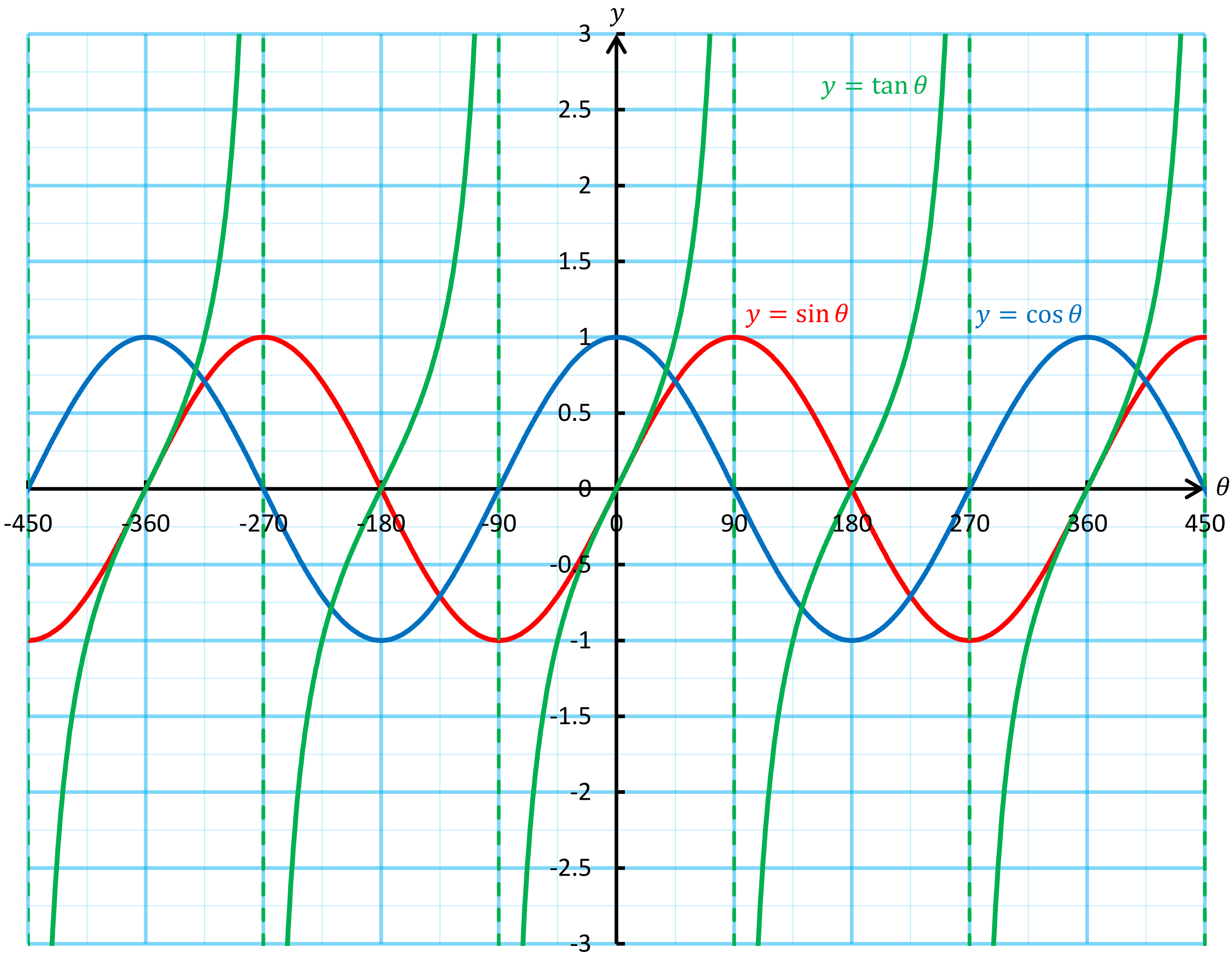
- $BC^2 + AC^2 = AB^2$
  $\sin^2\theta + \cos^2\theta = 1^2$
  $\sin^2\theta + \cos^2\theta = 1$

To summarise, here are the two identities that will be useful to us in the work that follows.

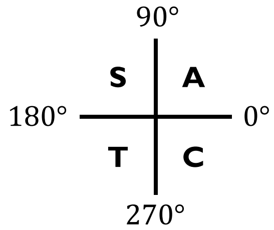
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \sin^2\theta + \cos^2\theta = 1$$

Trigonometric Graphs

Look again at the unit circle on the previous page. When we drag the point A anticlockwise around the circle, the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ will follow these paths:



The CAST Diagram



The **CAST** diagram summarises the sign of $\sin\theta$, $\cos\theta$ and $\tan\theta$ as the angle θ varies between $0°$ and $360°$.

- Between $0°$ and $90°$, **all** the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive.
- Between $90°$ and $180°$, only the value of $\mathbf{\sin\theta}$ is positive. ($\cos\theta$ and $\tan\theta$ are negative.)
- Between $180°$ and $270°$, only the value of $\mathbf{\tan\theta}$ is positive. ($\sin\theta$ and $\cos\theta$ are negative.)
- Between $270°$ and $360°$, only the value of $\mathbf{\cos\theta}$ is positive. ($\sin\theta$ and $\tan\theta$ are negative.)

We can use the CAST diagram to find all the angles satisfying a trigonometric equation.

Example 1

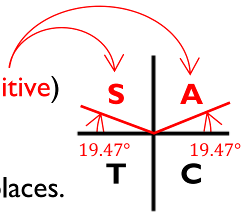
Solve $3\sin\theta = 1$ between $0°$ and $360°$.

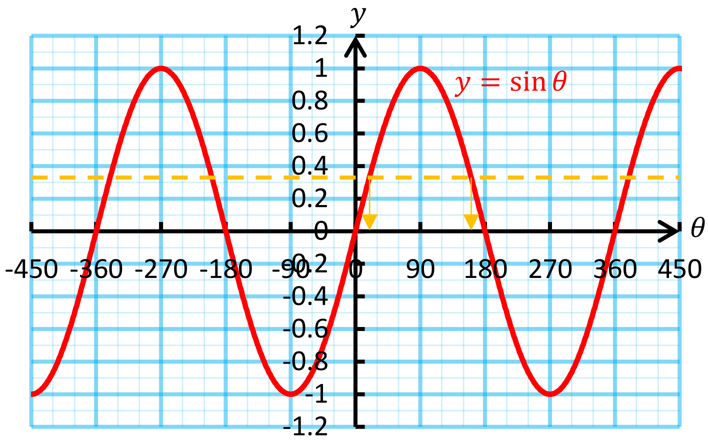
$$3\sin\theta = 1$$

$$\sin\theta = \frac{1}{3}$$ (sin is positive)

$$\theta = \sin^{-1}\left(\frac{1}{3}\right)$$

$\theta = 19.47°$ to 2 decimal places.





Also, considering the graph of $\sin\theta$ or the CAST diagram (where the vertical axis is a symmetry line),

$$\theta = 180° - 19.47°$$

$\theta = 160.53°$ to 2 decimal places.

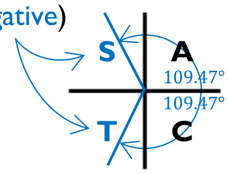
Example 2

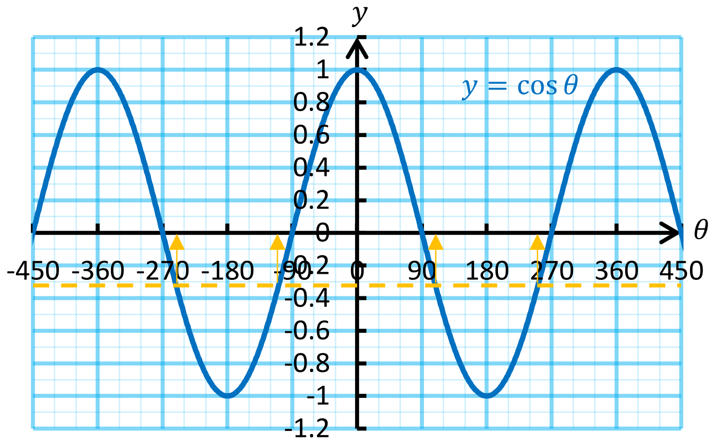
Solve $3\cos 2\theta = -1$ between $-180°$ and $180°$.

$$\cos 2\theta = -\frac{1}{3}$$ (cos is negative)

$$2\theta = \cos^{-1}\left(-\frac{1}{3}\right)$$

$2\theta = 109.47°$ to 2 decimal places.





Also, considering the graph of $\cos\theta$ or the CAST diagram (where the horizontal axis is a symmetry line),

$2\theta = -250.53, -109.47°, 109.47°, 250.53°$ to 2 decimal places.

$\theta = -125.26°, -54.74°, 54.74°, 125.26°$ to 2 decimal places.

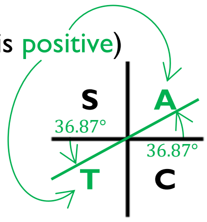
Because θ is between $-180°$ and $180°$, we must consider values of 2θ between $-360°$ and $360°$.

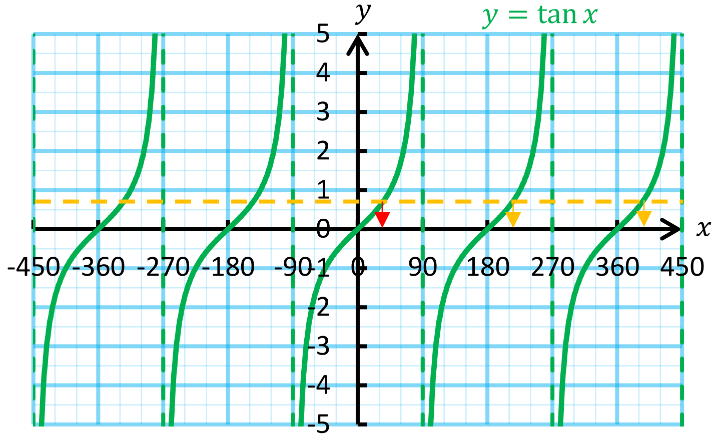
Note: Your calculator, when calculating the value of $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, will find the angle closest to zero, positive or negative.

Example 3

Solve $4\tan(2x + 40°) = 3$ between $0°$ and $180°$.

$\tan(2x + 40°) = 0.75$ (tan is positive)
$2x + 40° = \tan^{-1}(0.75)$
$2x + 40° = 36.87°$
to 2 decimal places.





Also, considering the graph of $\tan x$ or the CAST diagram (where we need to reflect in the vertical and horizontal axes, or rotate through $180°$),

$2x + 40° = 36.87°, 216.87°, 396.87°$
to 2 decimal places.

$2x = \sout{-3.13°}, 176.87°, 356.87°$ to 2 d.p.

$x = 88.43°, 178.43°$ to 2 decimal places.

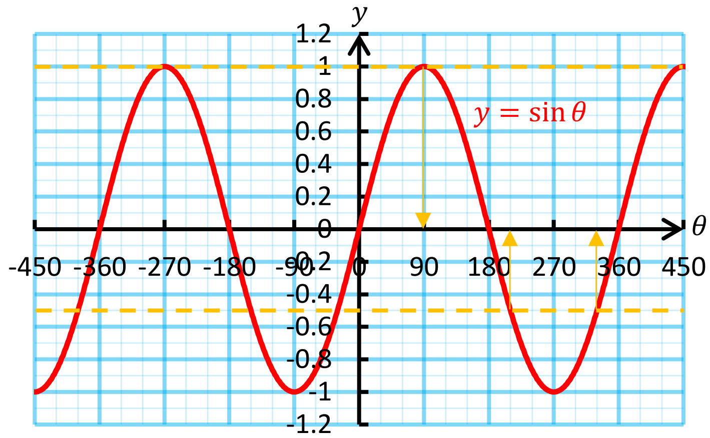
> Because x is between $0°$ and $180°$, we must consider values of $2x + 40°$ between $40°$ and $400°$.

Example 4

Solve $2\cos^2\theta + \sin\theta - 1 = 0$ between $0°$ and $360°$.

Because $\sin^2\theta + \cos^2\theta = 1$ we can use $\cos^2\theta = 1 - \sin^2\theta$ to change the equation to be in $\sin\theta$ only.

$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$
$2 - 2\sin^2\theta + \sin\theta - 1 = 0$
$-2\sin^2\theta + \sin\theta + 1 = 0$
$2\sin^2\theta - \sin\theta - 1 = 0$



Factorise: $2 \times -1 = -2$
$2\sin^2\theta - 2\sin\theta + \sin\theta - 1 = 0$
$2\sin\theta(\sin\theta - 1) + 1(\sin\theta - 1) = 0$
$(\sin\theta - 1)(2\sin\theta + 1) = 0$

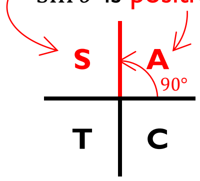
Either $\sin\theta - 1 = 0$ or $2\sin\theta + 1 = 0$

$\sin\theta - 1 = 0$	$2\sin\theta + 1 = 0$
$\sin\theta = 1$	$2\sin\theta = -1$
$\theta = \sin^{-1}(1)$	$\sin\theta = -\frac{1}{2}$
$\theta = 90°$	$\theta = \sin^{-1}\left(-\frac{1}{2}\right)$
	$\theta = -30°$

> If you like, you can substitute $x = \sin\theta$ to change this to $2x^2 - x - 1 = 0$; factorise as $(x - 1)(2x + 1) = 0$; and then change back to $(\sin\theta - 1)(2\sin\theta + 1) = 0$

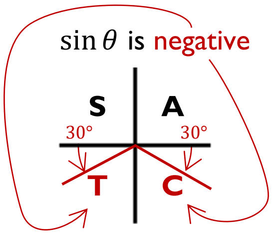
Also, considering the graph of $\sin\theta$ or the CAST diagram (where the vertical axis is a symmetry line),

sin θ is positive



No more solutions.

sin θ is negative



$\theta = 180° + 30°$ or $\theta = 360° - 30°$
$\theta = 210°, 330°$

Final solutions: $\theta = 90°, 210°, 330°$.

Top 10

(C2 Summer 2006)

2. *(a)* Find all values of x between 0° and 360° satisfying

$$\tan x = -0{\cdot}4.$$ [2]

(b) Find all values of x between 0° and 180° satisfying

$$\cos 3x = \frac{1}{2}.$$ [4]

(c) Find all values of θ between 0° and 360° satisfying

$$2\cos^2\theta + 3\sin\theta = 0.$$ [5]

(C2 Summer 2009)

2. *(a)* Find all values of θ between 0° and 360° satisfying

$$5\cos^2\theta + 2 = 3\sin^2\theta - 2\cos\theta. \quad [6]$$

(b) Find all values of x between 0° and 180° satisfying

$$\sin(2x + 12°) = -0{\cdot}53. \quad [3]$$

(C2 Winter 2007)

2. *(a)* Find the values of x in the range $0° \leqslant x \leqslant 360°$ satisfying

$$10 \sin^2 x - 3 \sin x = 4 \cos^2 x + 1.$$ [6]

(b) Find the values of x in the range $0° \leqslant x \leqslant 180°$ satisfying

$$\tan (2x + 30°) = \sqrt{3}.$$ [3]

(C2 Summer 2008)

2. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$2\sin\theta = 3\cos\theta.$$ [3]

(b) Find all values of x in the range $0° \leqslant x \leqslant 180°$ satisfying

$$\cos 3x = 0{\cdot}9.$$ [4]

(c) Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$\sin^2\theta - 4\cos^2\theta = 8\sin\theta.$$ [5]

(C2 Winter 2010)

2. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$3 - 7\cos\theta = 6\sin^2\theta.$$ [5]

(b) Find all values of x in the range $0° \leqslant x \leqslant 180°$ satisfying

$$\tan(2x + 45°) = 0{\cdot}7.$$ [3]

(c) Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$4\tan\theta\cos\theta + 1 = 0.$$ [3]

(C2 Winter 2012)

2. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$10\sin^2\theta + 7\cos\theta = 5\cos^2\theta + 8.$$ [6]

(b) Find all values of x in the range $0° \leqslant x \leqslant 360°$ satisfying

$$\sin(x - 50°) = -0{\cdot}682.$$ [3]

(c) Without carrying out any calculations, explain why there are no values of ϕ which satisfy the equation

$$\sin\phi + \cos\phi = 3.$$ [1]

(C2 Summer 2014)

2. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$4\cos^2\theta + 1 = 4\sin^2\theta - 2\cos\theta.$$ [6]

(b) The angle α satisfies

$$\sin(\alpha + 40°) = \frac{1}{\sqrt{2}}$$

and $$\sin(\alpha - 35°) = \frac{\sqrt{3}}{2}.$$

Given that $0° < \alpha < 180°$, find the value of α. [3]

(c) Find all values of ϕ in the range $0° \leqslant \phi \leqslant 360°$ satisfying

$$\frac{7}{\cos\phi} - \frac{10}{\sin\phi} = 0.$$ [3]

(C2 Winter 2014)

2. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$8\cos^2\theta - 7\sin^2\theta = 4\cos\theta - 3.$$ [6]

(b) The angles *X*, *Y* and *Z* are the three angles of a triangle. Given that $\tan X = -2{\cdot}246$ and that $\tan(Y - Z) = 0{\cdot}364$, find the values of *X*, *Y* and *Z*. Give each angle correct to the nearest degree. [4]

(C2 Summer 2017)

2. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$\sin^2\theta + 6\cos^2\theta + 13\sin\theta = 0.$$ [5]

(b) The angles A, B and C are the three angles of a triangle. Given that $\cos A = -0{\cdot}342$ and that $\tan(B - C) = 0{\cdot}404$, find the values of A, B and C. Give each angle correct to the nearest degree. [4]

(C2 Summer 2018)

2. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$10\sin^2\theta + 3\sin\theta = 4\cos^2\theta - 2.$$ [6]

(b) Find all values of ϕ in the range $0° \leqslant \phi \leqslant 360°$ satisfying

$$\frac{3}{\cos\phi} - \frac{5}{\sin\phi} = 0.$$ [3]

Revision Questions

(Unit 1 Summer 2019)

0 1 Solve the following equation for values of θ between 0° and 360°.

$$3\tan\theta + 2\cos\theta = 0$$

[6]

(C2 Summer 2012)

2. *(a)* Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$10\cos^2\theta + 3\cos\theta = 4\sin^2\theta - 2.$$ [6]

(b) Find all values of x in the range $0° \leqslant x \leqslant 180°$ satisfying

$$\sin(3x - 21°) = -0{\cdot}809.$$ [3]

(c) Find all values of ϕ in the range $0° \leqslant \phi \leqslant 360°$ satisfying

$$\cos\phi - 5\sin\phi = 0.$$ [3]

Unit 1, Workbook 10

12

Trigonometry

Part 2

The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Area of a Triangle = $\frac{1}{2}ab\sin C$

	sin	cos	tan
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	Not defined

Name:

Background

What is the work?	Introduce the exact values of sin, cos and tan for certain angles. Use the sine rule, the cosine rule and the formula for the area of a triangle to solve problems.
What is required before starting?	**GCSE Work:** Trigonometry; solving quadratic equations. **A Level Unit 1:** Surds; Trigonometry Part 1.
Where does this lead to?	**Applications:** Navigation; engineering; programming computer games.

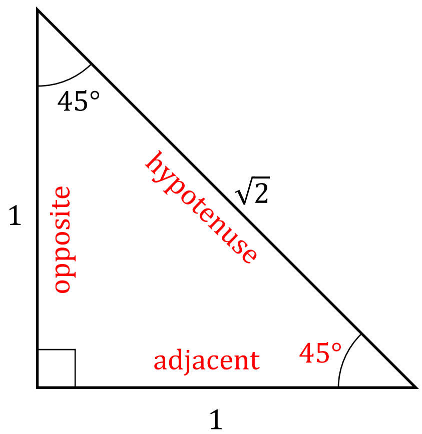
Theory

Exact Values

In a question containing a right-angled triangle and angles $30°$, $45°$ or $60°$, you are expected to use the exact values of $\sin$, $\cos$ and $\tan$ for these angles.

To begin, consider a right-angled triangle where the length of the shortest sides are 1 unit.

It is possible to use this triangle to calculate the exact values of $\sin$, $\cos$ and $\tan$ for the angle $45°$.

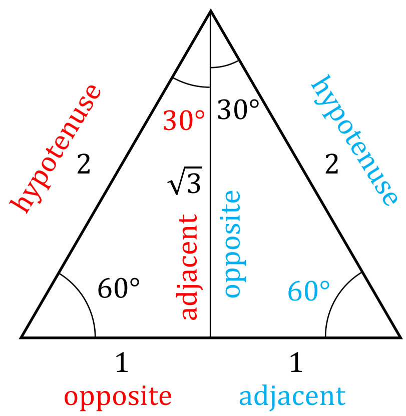


$$\sin(45°) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(45°) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(45°) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sin(45°) = \frac{1}{\sqrt{2}} \qquad \cos(45°) = \frac{1}{\sqrt{2}} \qquad \tan(45°) = 1$$

Next, consider an equilateral triangle where the length of the sides are 2 units.

By splitting this triangle in half, it is possible to calculate the exact values of $\sin$, $\cos$ and $\tan$ for the angles $30°$ and $60°$.



$$\sin(30°) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(30°) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(30°) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sin(30°) = \frac{1}{2} \qquad \cos(30°) = \frac{\sqrt{3}}{2} \qquad \tan(30°) = \frac{1}{\sqrt{3}}$$

$$\sin(60°) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(60°) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(60°) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sin(60°) = \frac{\sqrt{3}}{2} \qquad \cos(60°) = \frac{1}{2} \qquad \tan(60°) = \sqrt{3}$$

Summary

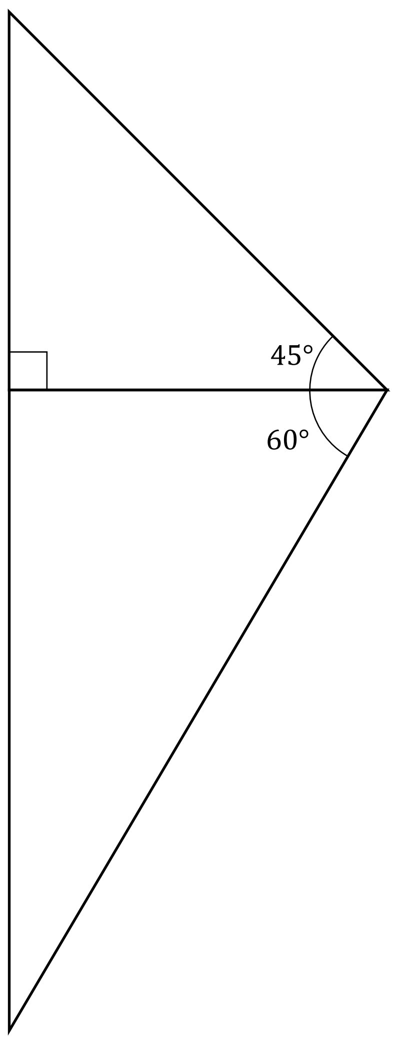
	sin	cos	tan
$0°$	0	1	0
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	1	0	Not defined

We can find the value of $\sin$, $\cos$ and $\tan$ for multiples of the above angles by using the **symmetry** in the graphs of $\sin$, $\cos$ and $\tan$.

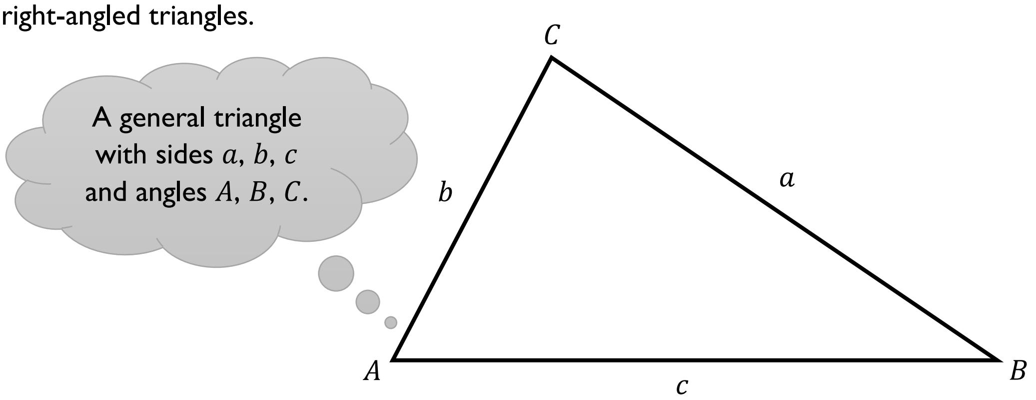
Note that the values in the yellow cells are different to the ones shown on your calculator. (Why?)

Exercise 1

The diagram on the right shows two connected triangles.
If the common length of the triangles is 10 cm, calculate the combined height of the two triangles. Write your answer as a surd.



The **Sine Rule** and the **Cosine Rule** are used to calculate the size of angles and sides in triangles that are not right-angled triangles.



Sine Rule for finding sides:

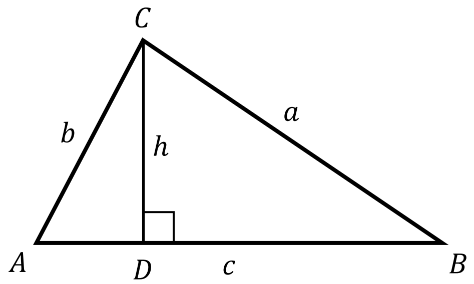
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Sine Rule for finding angles:

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ [Take the reciprocal]

Proof[1] (for triangles without an obtuse angle[2])

Draw the perpendicular from C to the base AB.



Using the triangle CDB, $\sin B = \frac{h}{a}$
so that $h = a\sin B$.

Using the triangle CAD, $\sin A = \frac{h}{b}$
so that $h = b\sin A$.

Using the two expressions for the height of the triangle h, we have $a\sin B = b\sin A$.

Therefore, $\frac{a}{\sin A} = \frac{b}{\sin B}$ [Divide by $\sin A$ and $\sin B$]

It would be possible to repeat the above ("Draw the perpendicular from A to the base BC...") to obtain $\frac{b}{\sin B} = \frac{c}{\sin C}$. We can combine the two formulae that use fractions to obtain the Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine Rule for finding sides:

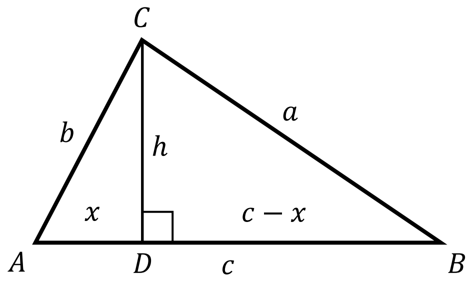
$a^2 = b^2 + c^2 - 2bc\cos A$

Cosine Rule for finding angles:

$\cos A = \frac{b^2+c^2-a^2}{2bc}$ [Re-arrange the formula]

Proof[1] (for triangles without an obtuse angle[2])

Draw the perpendicular from C to the base AB.



Pythagoras' Theorem for triangle BCD:
$a^2 = (c-x)^2 + h^2$ so that $h^2 = a^2 - (c-x)^2$.

Pythagoras' Theorem for triangle ACD:
$b^2 = x^2 + h^2$ so that $h^2 = b^2 - x^2$.

Using the two expressions for h^2:

$$a^2 - (c-x)^2 = b^2 - x^2$$
$$a^2 - (c-x)(c-x) = b^2 - x^2$$
$$a^2 - (c^2 - cx - cx + x^2) = b^2 - x^2$$
$$a^2 - c^2 + 2cx - \cancel{x^2} = b^2 - \cancel{x^2}$$
$$a^2 = b^2 + c^2 - 2cx$$

Using the triangle ACD, $\cos A = \frac{x}{b}$
so that $x = b\cos A$.

So, $a^2 = b^2 + c^2 - 2c(b\cos A)$ which gives the Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

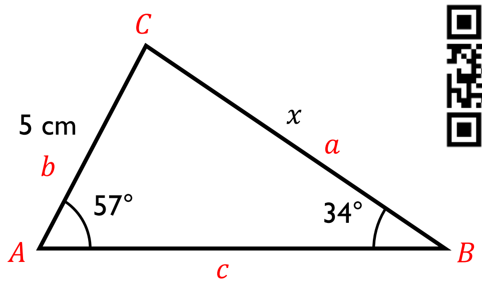
[1] You are not required to memorise this proof.
[2] Search on the internet for a proof in the case of a triangle that includes an obtuse angle.

The Sine Rule

Example 1

Calculate the missing side x or the missing angle θ. (The diagrams are not drawn to scale.)

(a)



Answer: To start, label the angles and then the corresponding sides. Becasuse we want to find the length of the side x, we write the Sine Rule for finding sides:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

We do not know the length of the side c nor the size of the angle C, so we cross out this fraction from the formula:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \cancel{\frac{c}{\sin C}}$$

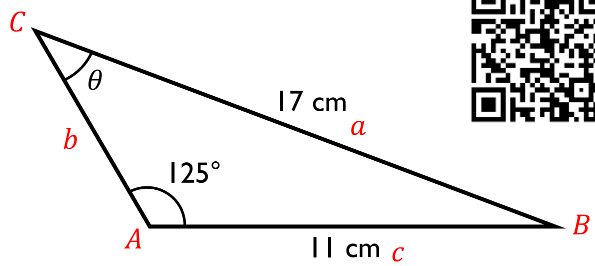
By substituting into the two fractions that ae left, we obtain

$$\frac{x}{\sin 57°} = \frac{5}{\sin 34°}$$

We can solve this equation by multiplying each side by $\sin 57°$:

$$x = \left(\frac{5}{\sin 34°}\right) \times \sin 57°$$
$x = 7.50$ cm to 2 decimal places.

(b)



Answer: To start, label the angles and then the corresponding sides. Because we want to find the size of the angle θ, we write the Sine Rule for finding angles:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

We do not know the length of the side b nor the size of the angle B, so we cross out this fraction from the formula:

$$\frac{\sin A}{a} = \cancel{\frac{\sin B}{b}} = \frac{\sin C}{c}$$

By substituting into the two fractions that ae left, we obtain

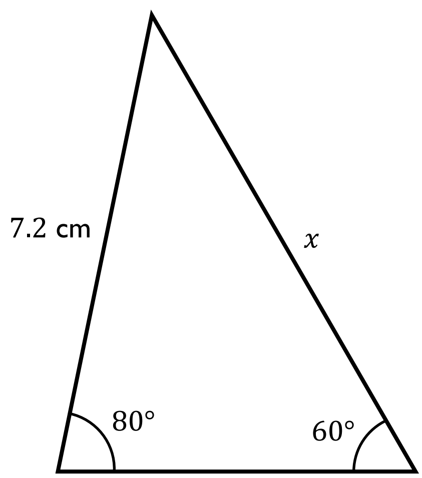
$$\frac{\sin 125°}{17} = \frac{\sin \theta}{11}$$

We can solve this equation by multiplying each side by 11:

$$\sin \theta = \left(\frac{\sin 125°}{17}\right) \times 11$$
$$\theta = \sin^{-1}\left(\left(\frac{\sin 125°}{17}\right) \times 11\right)$$
$\theta = 32.01°$ to 2 decimal places.

Exercise 2

Calculate the length x in the triangle shown on the right.

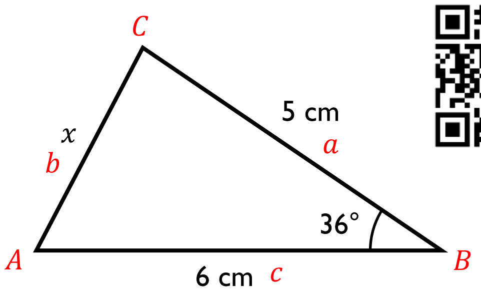


The Cosine Rule

Example 2

Calculate the missing side x or the missing angle θ. (The diagrams are not drawn to scale.)

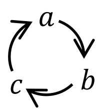
(a)



Answer: To start, label the angles and then the corresponding sides. Because we want to find the length of the side x, we write the Cosine Rule for finding sides:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

The formula does not fit the labels we have chosen, so we change the variables in the Cosine Rule by cycling around the following circle once.

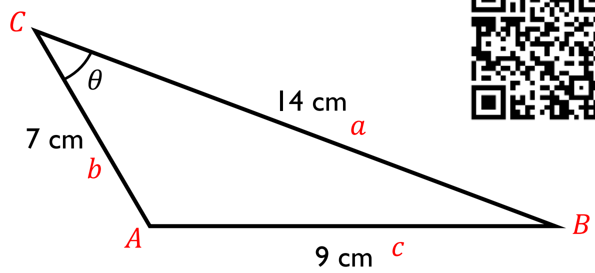


$$b^2 = c^2 + a^2 - 2ca\cos B$$

We can now substitute in the values from the triangle:

$$x^2 = 6^2 + 5^2 - 2 \times 6 \times 5 \times \cos 36°$$
$$x = \sqrt{12.45898034}$$
$x = 3.53$ cm to 2 decimal places.

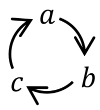
(b)



Answer: To start, label the angles and then the corresponding sides. Because we want to find the size of the angle θ, we write the Cosine Rule for finding angles:

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

The formula does not fit the labels we have chosen, so we change the variables in the Cosine Rule by cycling around the following circle twice.



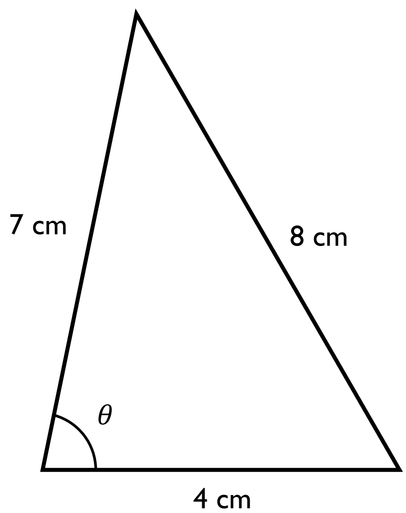
$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

We can now substitute in the values from the triangle:

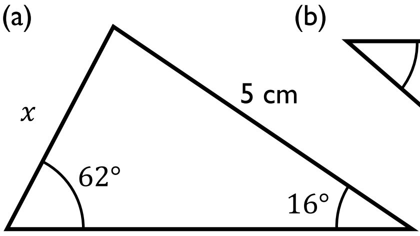
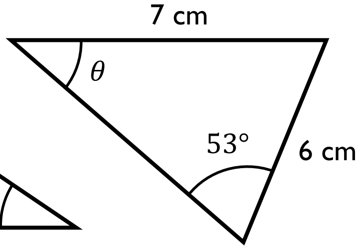
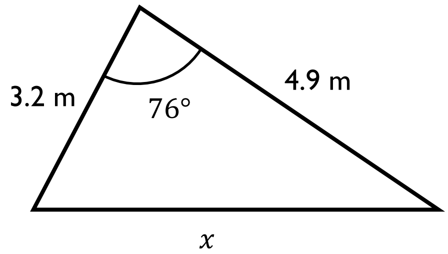
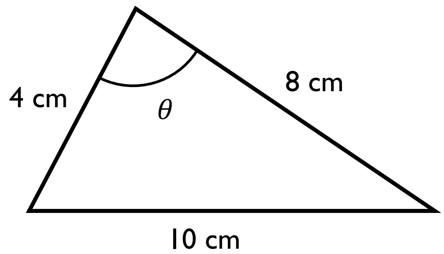
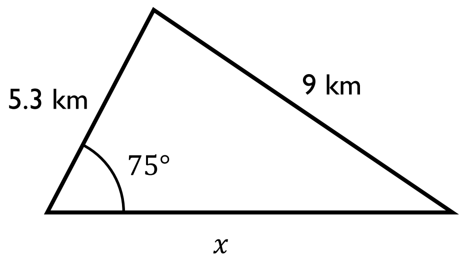
$$\cos\theta = \frac{14^2+7^2-9^2}{2\times14\times7}$$
$$\theta = \cos^{-1}\left(\frac{41}{49}\right)$$
$\theta = 33.20°$ to 2 decimal places.

Exercise 3

Calculate the size of the angle θ in the triangle on the right.



Sine Rule or Cosine Rule?

We know:	We want to find:	Sine Rule or Cosine Rule?	Examples
Three of the four values in two pairs of corresponding sides/angles.	The missing angle or side in the two pairs.	The Sine Rule.	(a)  (b) 
Two sides and the angle between the sides.	The side opposite the known angle.	The Cosine Rule.	(c) 
All the sides.	Any angle.	The Cosine Rule.	(d) 
Two sides and an angle that is not between the sides.	The other side.	The Sine Rule twice (easier), or the Cosine Rule (harder).	(e) 

Exercise 4

Calculate the missing side x or the missing angle θ in the above triangles.

The Sine Rule: the ambiguous case

Sometimes, it is possible to find two different angles for a missing angle using the Sine Rule.
This is because the $\sin$ ratio is positive for angles between 0° and 180°, so there are two different answers to the equation $\sin\theta = x$ when x is positive.

Example 3

In a triangle ABC, given that $AB = 5$ cm, $BC = 4$ cm and $B\hat{A}C = 38°$, what are the possible values of $A\hat{C}B$?

Looking at the diagram on the right, C lies somewhere on the horizontal line.

Using the Sine Rule,

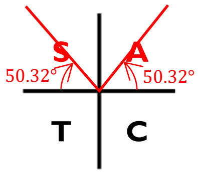
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

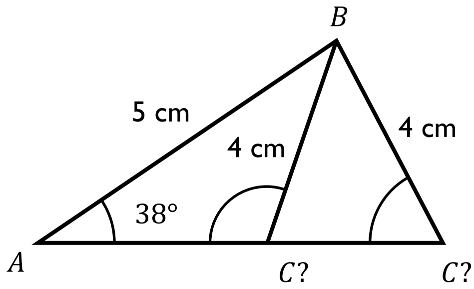
$$\frac{\sin 38°}{4} = \frac{\sin C}{5}$$

$$\sin C = 5 \times \frac{\sin 38°}{4}$$

$$C = \sin^{-1}\left(5 \times \frac{\sin 38°}{4}\right)$$

Either $C = 50.32°$ or $C = 129.68°$ (to 2 decimal places).



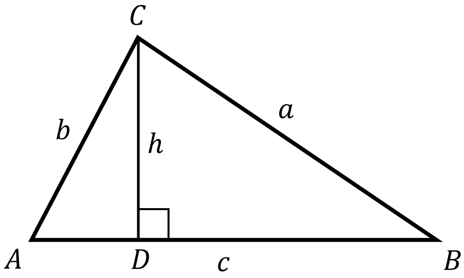


Exercise 5

The triangle ABC is such that $AB = 12$ cm, $BC = 10$ cm and $C\hat{A}B = 43°$.
Find the possible values of $B\hat{C}A$ and $A\hat{B}C$.

Area of a Triangle

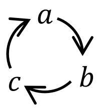
The following diagram shows a general triangle with sides a, b, c and angles A, B, C.



Let us draw the perpendicular from the vertex C to the base AB. Using the triangle CAD that is formed, we see that $\sin A = \frac{h}{b}$, and so $h = b \sin A$. Using the formula Area of a Triangle $= \frac{\text{base} \times \text{height}}{2}$,

$$\text{Area of a Triangle} = \frac{c \times b \sin A}{2}$$

$$\text{Area of a Triangle} = \frac{1}{2} bc \sin A$$

Or, changing the variables using 

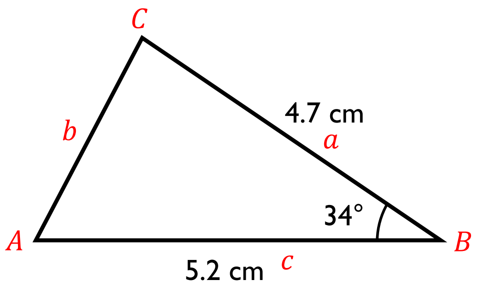
,

$$\text{Area of a Triangle} = \frac{1}{2} ac \sin B$$

$$\text{Area of a Triangle} = \frac{1}{2} ab \sin C$$

Example 4

(a) Calculate the area of the triangle below.



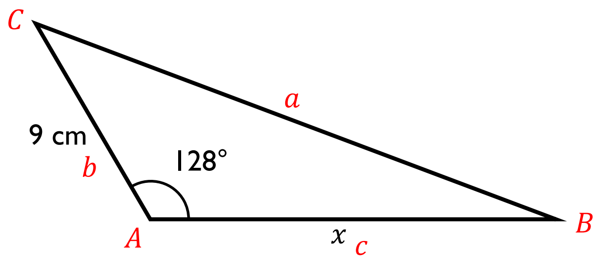
Answer: To start, we label the angles and then the corresponding sides. Using the formula

$$\text{Area of a Triangle} = \frac{1}{2} ac \sin B$$

$$\text{Area of a Triangle} = \frac{1}{2} \times 4.7 \times 5.2 \times \sin 34°$$

Area of a Triangle $= 6.83$ cm² to 2 decimal places.

(b) Given that the area of the triangle below is 27 cm², calculate the length x of the base of the triangle.



Answer: To start, we label the angles and then the corresponding sides. Using the formula

$$\text{Area of a Triangle} = \frac{1}{2} bc \sin A$$

$$27 = \frac{1}{2} \times 9 \times x \times \sin 128°$$

$$27 \times 2 = 9 \times x \times \sin 128°$$

$$\frac{54}{9 \times \sin 128°} = x$$

$x = 7.61$ cm to 2 decimal places.

We can use the formula

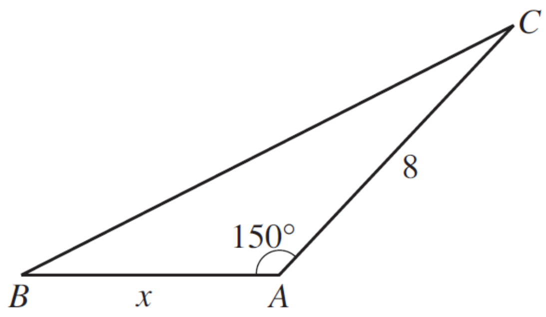
$$\text{Area of a Triangle} = \frac{1}{2} ab \sin C$$

if we know the length of two of the sides of a triangle, and the size of the angle **between** the sides.

Top 10

(C2 Summer 2006)

3. The diagram below shows the triangle ABC with $AB = x$ cm, $AC = 8$ cm and $B\hat{A}C = 150°$.



Given that the area of the triangle ABC is 10 cm^2,

(a) find x, [3]

(b) calculate the length of the longest side of the triangle ABC, giving your answer correct to two decimal places. [3]

(C2 Winter 2007)

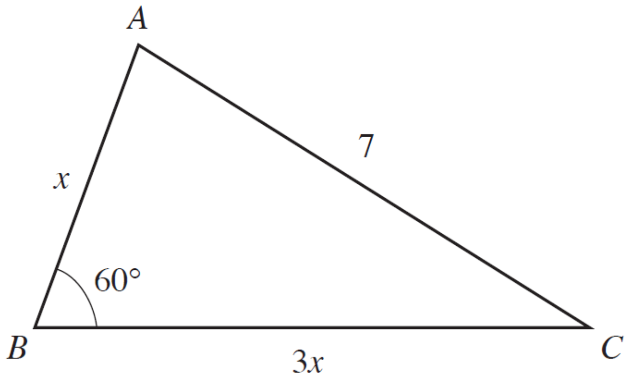
6. The triangle ABC is such that $AB = 6$ cm, $AC = 10$ cm and $B\hat{A}C$ is an **obtuse** angle. The area of triangle ABC is $15\sqrt{3}$ cm^2.

(a) Find the size of $B\hat{A}C$. [3]

(b) Calculate the length of BC. [3]

(C2 Summer 2007)

3. The diagram below shows the triangle ABC with $AB = x$ cm, $BC = 3x$ cm, $AC = 7$ cm and $A\hat{B}C = 60°$.



(a) Show that $x = \sqrt{7}$. [3]

(b) Find $A\hat{C}B$. [2]

(C2 Winter 2008)

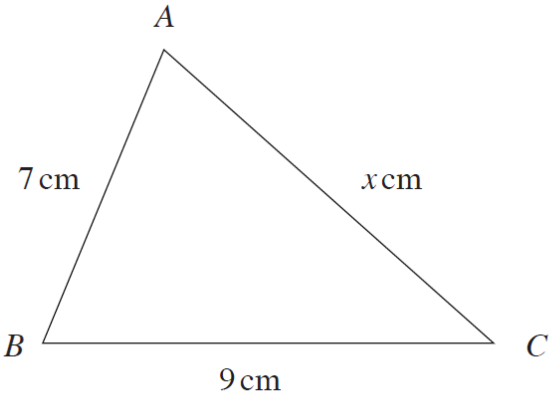
5. In triangle ABC, $AB = 6$ cm, $BC = 13$ cm and $CA = 9$ cm.

(a) Find the value of $\cos B\hat{A}C$ as a fraction in its lowest terms. [3]

(b) Show that the area of triangle ABC is $4\sqrt{35}\text{cm}^2$. [3]

(C2 Winter 2009)

3. The diagram below shows a sketch of the triangle ABC with $AB = 7$ cm, $AC = x$ cm, $BC = 9$ cm and $\cos B\hat{A}C = \frac{2}{7}$.



(a) Write down and simplify a quadratic equation satisfied by x. Hence evaluate x. [3]

(b) (i) Express $\sin B\hat{A}C$ in the form $\frac{\sqrt{m}}{n}$, where m, n are integers whose values are to be found.

(ii) Express $\sin A\hat{C}B$ in the form $\frac{\sqrt{p}}{3}$, where p is an integer whose value is to be found. [4]

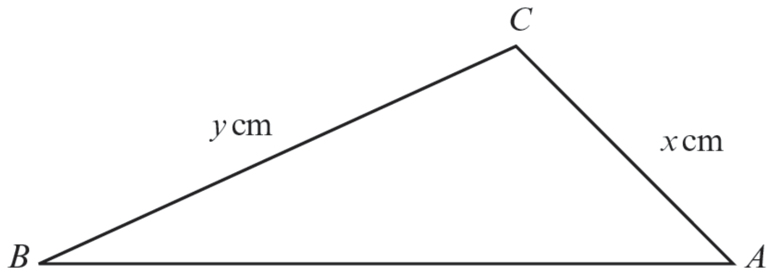
(C2 Summer 2010)

3. *(a)* The triangle ABC is such that $AB = 11$ cm and $B\hat{A}C = 110°$. Given that the area of the triangle ABC is 31 cm^2, find the length of BC. [4]

(b) The triangle XYZ is such that $XY = 2$ cm, $YZ = (2\sqrt{3} - 1)$ cm and $Y\hat{X}Z = 60°$. Find an expression for $\sin X\hat{Z}Y$ in the form $\frac{m + \sqrt{3}}{n}$, where m, n are integers whose values are to be found. [3]

(C2 Summer 2011)

3. The diagram below shows a sketch of the triangle ABC with $\sin A = \frac{3}{5}$, $\sin B = \frac{5}{13}$, $\sin C = \frac{56}{65}$, $AC = x$ cm and $BC = y$ cm.

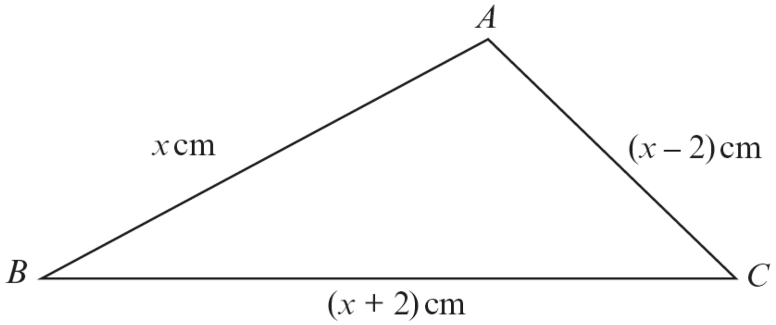


(a) Show that $y = 1{\cdot}56x$. [2]

(b) Given that the area of triangle ABC is $4{\cdot}2\,\text{cm}^2$, find the value of x and the value of y. [5]

(C2 Summer 2013)

3. The diagram below shows a sketch of the triangle ABC with $AB = x$ cm, $AC = (x - 2)$ cm and $BC = (x + 2)$ cm.



(a) Show that $\cos B\hat{A}C = \dfrac{x-8}{2x-4}$. [3]

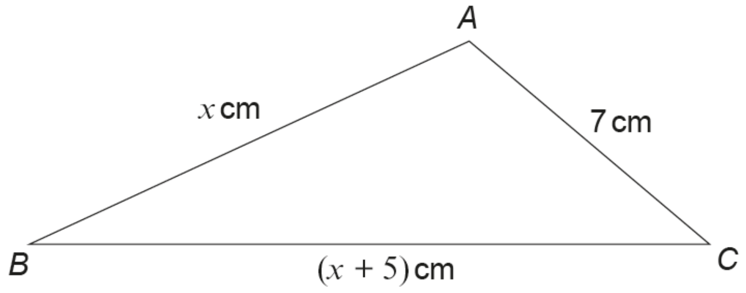
(b) Given that $B\hat{A}C = 120°$,

(i) find the value of x,

(ii) find the size of $A\hat{B}C$. [4]

(C2 Summer 2016)

3. The diagram below shows a sketch of the triangle ABC with $AB = x$ cm, $BC = (x + 5)$ cm, $AC = 7$ cm and $\cos B\hat{A}C = -\frac{3}{5}$.



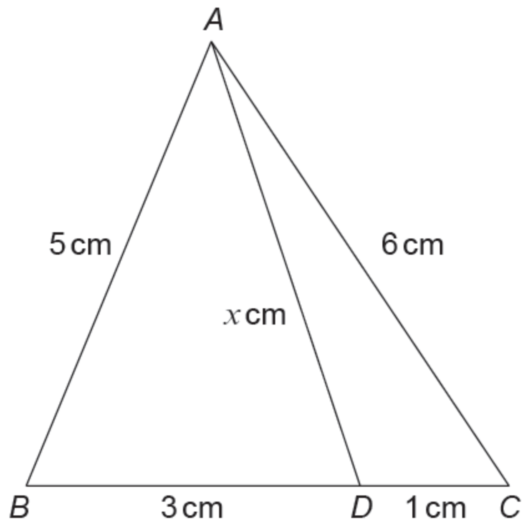
(a) Write down an equation satisfied by x. Hence show that $x = 15$. [3]

(b) Find the exact value of the area of triangle ABC. [3]

(c) The point D lies on BC and is such that AD is perpendicular to BC. Find the length of AD. [2]

(C2 Winter 2014)

5. The diagram below shows a sketch of the triangle *ABC* with *AB* = 5 cm and *AC* = 6 cm. The point *D* is on *BC* such that *BD* = 3 cm, *DC* = 1 cm and *AD* = x cm.



(a) (i) By applying the cosine rule in each of the triangles *ADB* and *ADC*, show that

$\cos A\hat{D}B = \dfrac{x^2 - 16}{6x}$ and find a similar expression for $\cos A\hat{D}C$.

(ii) Noting that $A\hat{D}B$ and $A\hat{D}C$ are angles on a straight line, use the expressions derived in part (i) to write down an equation satisfied by x. Hence show that $x = 5{\cdot}5$. [6]

(b) Find the area of triangle *ADB*. Give your answer correct to two decimal places. [3]

Revision Question

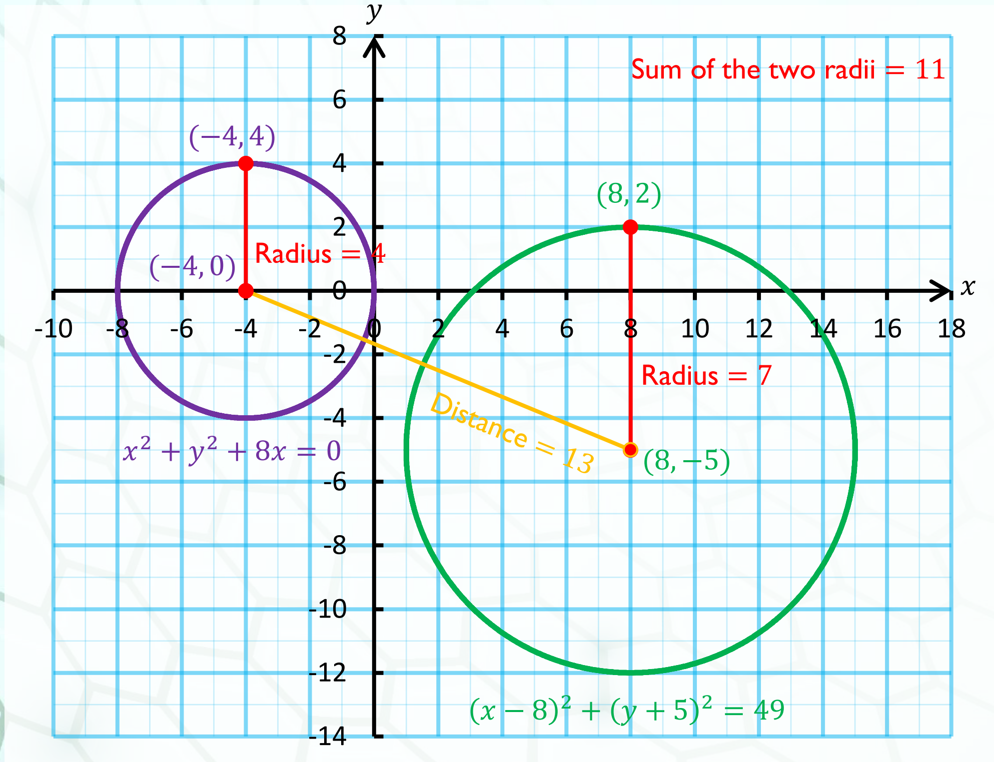
(Unit 1 Summer 2018)

0 9 The triangle ABC is such that $AC = 16$ cm, $AB = 25$ cm and $A\hat{B}C = 32°$. Find two possible values for the area of the triangle ABC. [5]

Unit 1, Workbook 11

12

Equation of a Circle



Name:

Background

What is the work?	Understand and use the two forms of the equation of a circle. Find the equation of a particular tangent to a circle.
What is required before starting?	**GCSE Work:** Pythagoras' theorem; trigonometry. **A Level Unit 1:** Cartesian coordinate geometry; completing the square.
Where does this lead to?	**Further Maths Unit 1:** Complex transformations. **Applications:** Mechanical engineering; astronomy; architecture.

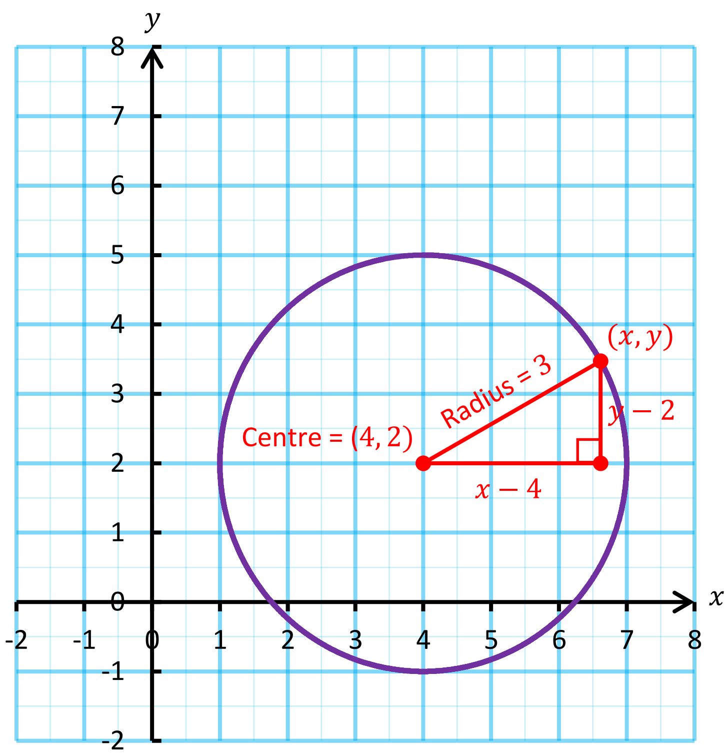
Theory

Equation of a circle

There are two ways of writing the equation of a circle.

(1) $(x - a)^2 + (y - b)^2 = r^2$
The equation of a circle with radius r and centre $(a,\ b)$.
This is a special case of Pythagoras' theorem.

(2) $x^2 + y^2 + 2gx + 2fy + c = 0$
The general equation of a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.



It is possible to show that these two ways are equivalent:

Start with method (2):
$x^2 + y^2 + 2gx + 2fy + c = 0$

Completing the square:
$(x + g)^2 - g^2 + (y + f)^2 - f^2 + c = 0$

Re-arrange:
$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$

Comparing with $(x - a)^2 + (y - b)^2 = r^2$:

$g = -a$
$-a = g$
$a = -g$

$f = -b$
$-b = f$
$b = -f$

$g^2 + f^2 - c = r^2$
$r^2 = g^2 + f^2 - c$
$r = \sqrt{g^2 + f^2 - c}$
(take the positive root as we have a radius here)

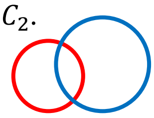
Exercise 1

Complete the following table to find the centre and the radius of each circle.

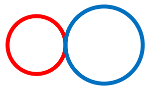
Circle	Centre	Radius
$(x-3)^2+(y+7)^2=25$		
$(x+8)^2+(y-4)^2=20$		
$(x-2)^2+(y+11)^2=49$		
$x^2+y^2=144$		
$x^2+y^2+6x+2y-6=0$		
$x^2+y^2+14x-4y-5=0$		
$2x^2+2y^2-24x+16y+6=0$		
$x^2+y^2+7y+3.25=0$		

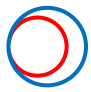
Intersecting circles

Consider the two circles C_1 and C_2 with radii r_1 and r_2, respectively.
Let p represent the distance between the centre of C_1 and the centre of C_2.

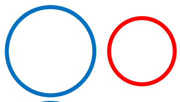


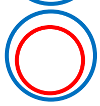
- If C_1 and C_2 intersect at two distinct points, then $p < r_1 + r_2$.





- If C_1 and C_2 meet at one point only, then either $p = r_1 + r_2$ (they touch externally) or $p = |r_1 - r_2|$ (they touch internally).
  ($|r_1 - r_2|$ is the positive difference between the radii.)





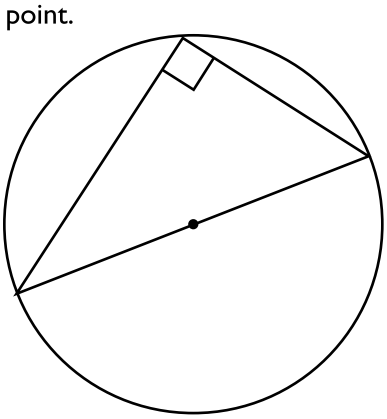
- If C_1 and C_2 do not intersect, then either $p > r_1 + r_2$ (separate circles) or $p < |r_1 - r_2|$ (one circle within another).

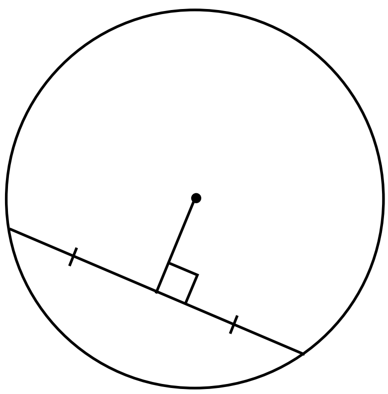
You may explore these relationships on the GeoGebra website.

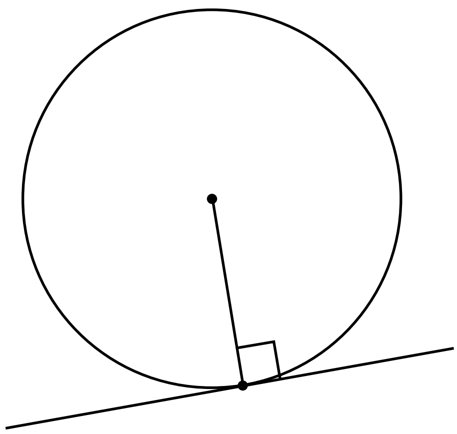
Circle Theorems

The following circle theorems (from GCSE work) can appear in the work on the equation of a circle.

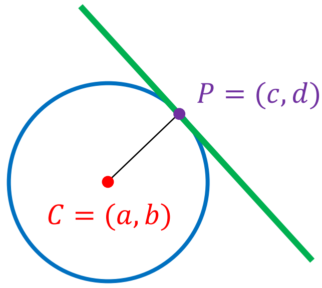
- The angle in a semicircle is a right angle.
- The perpendicular from the centre to a chord bisects the chord.
- The radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point.







Equation of the tangent



Let $P = (c, d)$ be a point on a circle with centre $C = (a, b)$.

- The gradient of the radius PC is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{b-d}{a-c}$.
- Radius and tangent meet at a right angle, so the gradient of the tangent to the circle at the point P is $-\frac{a-c}{b-d}$.
- The equation of the tangent to the circle at the point P is $y - y_1 = m(x - x_1)$
  $y - d = -\frac{a-c}{b-d}(x - c)$.

Example (C2 Summer 2005)

The circle C is given by the equation

$$x^2 + y^2 - 8x + 4y - 5 = 0.$$

(a) Find the radius and the coordinates of the centre of C.

(b) (i) Show that $P(1, -6)$ lies on C.

(ii) Find the equation of the tangent to C at P.

Answer: (a) **Method 1:** Completing the square.

$(x-4)^2 - 4^2 + (y+2)^2 - 2^2 - 5 = 0$
$(x-4)^2 - 16 + (y+2)^2 - 4 - 5 = 0$
$(x-4)^2 + (y+2)^2 = 4 + 5 + 16$
$(x-4)^2 + (y+2)^2 = 25$
Comparing to $(x-a)^2 + (y-b)^2 = r^2$,
Radius $C = \sqrt{25}$
$= 5$
Centre coordinate $C = (a, b)$
$= (4, -2)$

Method 2: Comparing to $x^2 + y^2 + 2gx + 2fy + c = 0$.

$2g = -8, \quad 2f = 4, \quad c = -5$
$g = -4 \quad f = 2$
Radius $C = \sqrt{g^2 + f^2 - c}$
$= \sqrt{(-4)^2 + 2^2 - -5}$
$= \sqrt{16 + 4 + 5}$
$= \sqrt{25}$
$= 5$
Centre coordinate $C = (-g, -f)$
$= (4, -2)$

(b) (i) Substituting $x = 1$, $y = -6$ into the left-hand side of the equation of the circle:

$$1^2 + (-6)^2 - 8(1) + 4(-6) - 5$$
$$= 1 + 36 - 8 - 24 - 5$$
$$= 0$$

So, $(1, -6)$ is on C.

(ii) The gradient of the radius from $(4, -2)$ to $(1, -6)$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-6 - -2}{1 - 4}$$
$$= \frac{-4}{-3}$$
$$= \frac{4}{3}$$

So, the gradient of the tangent is $-\frac{3}{4}$ (which is the negative of the reciprocal).

Equation of the tangent:

$$y - y_1 = m(x - x_1)$$
$$y - -6 = -\frac{3}{4}(x - 1)$$
$$y + 6 = -\frac{3}{4}x + \frac{3}{4}$$
$$y = -\frac{3}{4}x - \frac{21}{4}$$

Theory

Top 10

(C2 Winter 2005)

6. The circles C_1 and C_2 are given

by $(x+1)^2 + (y+2)^2 = 25$

and $x^2 + y^2 - 10x - 5y + 25 = 0$, respectively.

(a) Write down the radius and the coordinates of the centre of C_1. [2]

(b) Find the radius and the coordinates of the centre of C_2. [3]

(c) Show that C_1 and C_2 touch. [3]

(C2 Winter 2006)

8. *(a)* Find the centre and radius of the circle C given by

$$x^2 + y^2 - 8x + 4y + 11 = 0.$$ [3]

(b) Given that the circle

$$x^2 + y^2 = a^2 \ (a > 0)$$

touches C externally, find the value of a, giving your answer correct to two decimal places. [4]

(C2 Summer 2006)

9. A circle C has centre D and equation

$$x^2 + y^2 + 2x - 8y + 8 = 0.$$

(a) Find the coordinates of D and the radius of C. [3]

(b) A line is drawn through the point P (4, 6) so that it touches the circle C at the point T.

(i) Show that $PT = \sqrt{20}$.

(ii) Find the equation of the circle centre P which passes through the point T. [5]

(C2 Summer 2007)

8. The circle $x^2 + y^2 + 4x - 16y + 18 = 0$ has centre A and radius r.

(a) Find the coordinates of A and the value of r. [3]

(b) The line $y = x + 2$ and the circle $x^2 + y^2 + 4x - 16y + 18 = 0$ intersect at the points B and C. Find the coordinates of B and C. [4]

(C2 Winter 2008)

8. The circle C has centre A and equation

$$x^2 + y^2 - 4x + 6y - 12 = 0.$$

(a) Find the coordinates of A and the radius of C. [3]

(b) The point P has coordinates (5, 1) and lies on C. Find the equation of the tangent to C at P. [4]

(c) The line L has equation $y = x + 3$. Show that L and C do not intersect. [4]

(C2 Winter 2010)

8. The circle C has centre A and equation

$$x^2 + y^2 + 4x - 8y + 10 = 0.$$

(a) Find the coordinates of A and the radius of C. [3]

(b) The line L has equation

$$x - 3y + 4 = 0.$$

Show that L is a tangent to the circle C. [4]

(C2 Summer 2008)

8. The circle C has centre A and radius r. The points $P(1, -4)$ and $Q(9, 10)$ are at either end of a diameter of C.

(a) (i) Write down the coordinates of A.

(ii) Show that $r = \sqrt{65}$.

(iii) Write down the equation of C. [4]

(b) Verify that the point $R(4, 11)$ lies on C. [2]

(c) Find $Q\hat{P}R$. [3]

(C2 Summer 2009)

8. The circle C_1 has centre A and equation

$$x^2 + y^2 - 6x + 2y - 15 = 0.$$

(a) Find the coordinates of A and the radius of C_1. [3]

(b) The point P has coordinates (7, 2) and lies on C_1. Find the equation of the tangent to C_1 at P. [4]

(c) The circle C_2 has centre $B(11, 14)$ and radius 8. A point Q lies on C_1 and a point R lies on C_2. Find the shortest possible length of the line QR. [3]

(C2 Summer 2014)

8. *(a)* The circle C_1 has centre $A(-2, 9)$ and radius 5. The circle C_2 has centre $B(10, -7)$ and radius 15.

(i) Show that C_1 and C_2 touch, justifying your answer.

(ii) Given that the circles touch at the point $P(1, 5)$, find the equation of the common tangent at P. [7]

(b) Gareth, who has been asked by his teacher to investigate the properties of another circle C_3, claims that the equation of this circle C_3 is given by

$$x^2 + y^2 + 4x - 6y + 20 = 0.$$

Show that Gareth cannot possibly be correct. [3]

(Unit 1 Summer 2018)

1 8 The coordinates of three points A, B, C are (4, 6), (–3, 5) and (5, –1) respectively.

a) Show that $B\hat{A}C$ is a right angle. [3]

b) A circle passes through all three points A, B, C. Determine the equation of the circle. [5]

Revision Questions

(Unit 1 Summer 2019)

0 9 The points $A(-2, 4)$ and $B(6, 10)$ are such that AB is the diameter of a circle.

a) Show that the centre of the circle has coordinates (2, 7). [1]

b) The equation of the circle is $x^2 + y^2 + ax + by + c = 0$.
Determine the values of a, b, c. [3]

A straight line, with equation $y = x + 6$, passes through the point A and cuts the circle again at the point C.

c) Find the coordinates of C. [5]

d) Calculate the exact area of the triangle ABC. [3]

(Unit I Summer 2024)

18. (a) Canol cylch *C* yw (–3, –1) a'r radiws yw $\sqrt{5}$. Dangoswch ein bod ni'n gallu ysgrifennu hafaliad *C* fel $x^2 + y^2 + 6x + 2y + 5 = 0$. [2]

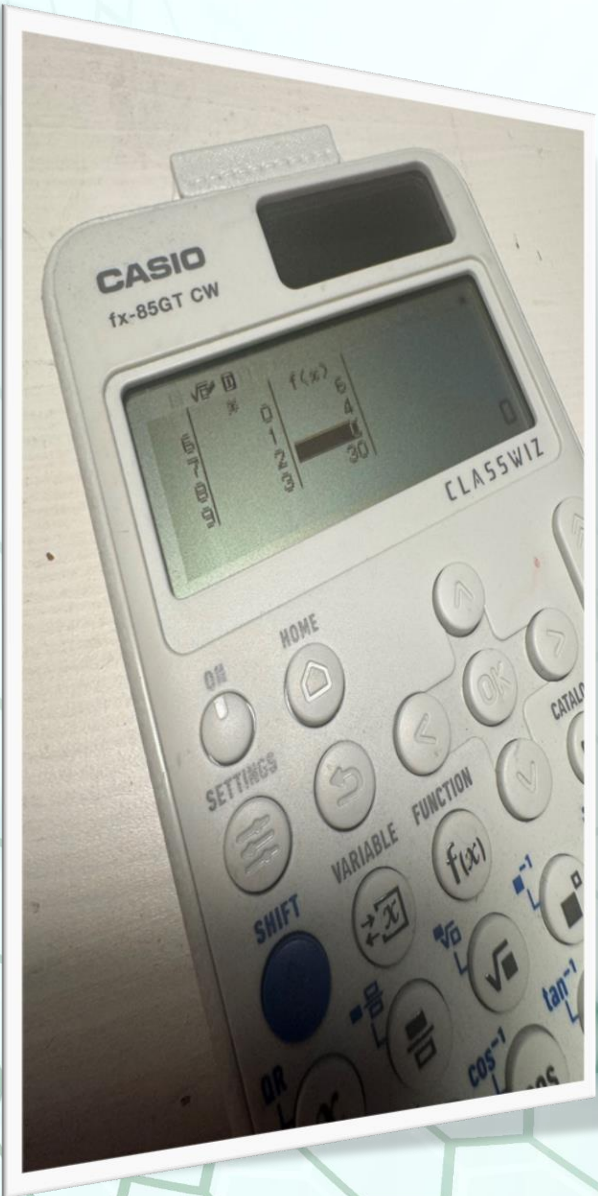
(b) (i) Find the equations of the tangents to *C* that pass through the origin *O*. [6]

(ii) Determine the coordinates of the points where the tangents touch the circle. [4]

Unit 1, Workbook 12

12

The Factor Theorem



Name:

Background

What is the work?	Factorising and solving cubic equations by dividing out a known factor.
What is required before starting?	**GCSE Work:** Substitution; solving quadratic equations.
Where does this lead to?	**Unit 3:** Simplifying rational expressions. **Applications:** Drawing shapes on a computer using Bezier curves.

Theory

Algebraic Division

What is $14 \div 5$? One way of writing the answer is $14 \div 5 = 2$ remainder 4, because $14 = (2 \times 5) + 4$.

We can express algebraic division sums in a similar fashion, with the division performed using a division frame.

Example 1

$(5x^3 + 8x^2 - 6x + 3) \div (x - 2)$

Explanation

$$
\begin{array}{r|l}
 & 5x^2 + 18x + 30 \\
x - 2 & 5x^3 + 8x^2 - 6x + 3 \\
 & 5x^3 - 10x^2 \\
 & 18x^2 - 6x + 3 \\
 & 18x^2 - 36x \\
 & 30x + 3 \\
 & 30x - 60 \\
 & 63
\end{array}
$$

So, $(5x^3 + 8x^2 - 6x + 3) \div (x - 2) = 5x^2 + 18x + 30$ remainder 63, or

$$5x^3 + 8x^2 - 6x + 3 = (5x^2 + 18x + 30)(x - 2) + 63$$

We can use the **Remainder Theorem** to find the remainder quickly.

> When a polynomial $f(x)$ is divided by an expression of the form $x - a$, where a is a constant, the remainder is $f(a)$.

So, in the above example, the remainder when $f(x) = 5x^3 + 8x^2 - 6x + 3$ is divided by $x - 2$ is $f(2)$, which is

$$5 \times 2^3 + 8 \times 2^2 - 6 \times 2 + 3 = 63$$

Exercise 1

Find the remainder when $4x^3 - 7x^2 + 9x - 1$ is divided by $x + 3$,
(a) using a division frame;
(b) using the Remainder Theorem.

The Factor Theorem

(a) The polynomial $x - a$ is a factor of the polynomial $f(x)$ if $f(a) = 0$.
(b) If $f(a) = 0$, then the polynomial $x - a$ is a factor of the polynomial $f(x)$.

Example 2

(a) The polynomial $x + 5$ is a factor of the polynomial $2x^2 - 8x - 10$ because

$$f(-5) = 2(-5)^2 - 8(-5) - 10$$
$$f(-5) = 0$$

(b) The polynomial $x - 3$ is not a factor of the polynomial $2x^3 + 9x^2 - 11x - 30$ because

$$f(3) = 2(3)^3 + 9(3)^2 - 11(3) - 30$$
$$f(3) = 72$$
$$f(3) \neq 0$$

Exercise 2

(a) Is the polynomial $x - 4$ a factor of the polynomial $3x^2 + 5x - 28$?
(b) Is the polynomial $x + 2$ a factor of the polynomial $x^3 + 5x^2 + 6x$?

Solving Cubic Equations

In order to solve a cubic equation such as $2x^3 + 3x^2 - 45x + 54 = 0$,

(a) Use the Factor Theorem to find one of the factors of the polynomial.
(The table mode on your calculator will be useful for this.)

(b) Divide out the factor using a division frame, so that the cubic expression may be written as a product of a quadratic expression by a linear expression.

(c) Solve the new equation by using techniques from the GCSE course.

Example 3

Solve the cubic equation $2x^3 + 3x^2 - 45x + 54 = 0$.

Answer: To begin, use the table mode on a calculator to substitute different values into $2x^3 + 3x^2 - 45x + 54$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$2x^3 + 3x^2 - 45x + 54$	104	154	162	140	100	54	14	-8	0	50	154

Because substituting $x = 3$ into the polynomial gives the answer 0, we can say that $x - 3$ is a factor of the polynomial. We can divide out this factor by using a division frame:

$$\begin{array}{r|l} & \quad 2x^2 + \;\; 9x - 18 \\ \hline x - 3 & 2x^3 + 3x^2 - 45x + 54 \\ & \underline{2x^3 - 6x^2} \\ & \qquad\;\; 9x^2 - 45x + 54 \\ & \qquad\;\; \underline{9x^2 - 27x} \\ & \qquad\qquad\quad - 18x + 54 \\ & \qquad\qquad\quad \underline{- 18x + 54} \\ & \qquad\qquad\qquad\qquad\;\; 0 \end{array}$$

So, it is possible to rewrite the original equation $2x^3 + 3x^2 - 45x + 54 = 0$ as

$$(x - 3)(2x^2 + 9x - 18) = 0.$$

We can use GCSE techniques to factorise the quadratic expression $2x^2 + 9x - 18$:

$$\begin{aligned} &2 \times -18 = -36 \\ &2x^2 + 12x - 3x - 18 \\ &= 2x(x + 6) - 3(x + 6) \\ &= (x + 6)(2x - 3) \end{aligned}$$

It is now possible to solve the original equation:

$$\begin{aligned} (x - 3)(2x^2 + 9x - 18) &= 0 \\ (x - 3)(x + 6)(2x - 3) &= 0 \end{aligned}$$

Either $x - 3 = 0$ or $x + 6 = 0$ or $2x - 3 = 0$

$$x = 3 \qquad x = -6 \qquad 2x = 3$$

$$x = \tfrac{3}{2}$$

Top 10

(Unit 1 Summer 2019)

0 3 Use an **algebraic method** to solve the equation $12x^3 - 29x^2 + 7x + 6 = 0$.
Show all your working. [6]

(Unit 1 Summer 2023)

0 4 **a)** Find the remainder when the polynomial $3x^3+2x^2+x-1$ is divided by $(x-3)$. [3]

b) The polynomial $f(x)=2x^3-3x^2+ax+6$ is divisible by $(x+2)$, where a is a real constant.

i) Find the value of a. [3]

ii) Showing all your working, solve the equation $f(x)=0$. [4]

(Unit 1 Summer 2018)

0 8 Given that $(x - 2)$ and $(x + 2)$ are factors of the polynomial $2x^3 + px^2 + qx - 12$,

a) find the values of p and q, [4]

b) determine the other factor of the polynomial. [1]

(C1 Winter 2007)

3. When $9x^3 + 6x^2 - 5x + p$ is divided by $x - 1$, the remainder is 8.

(a) Show that $p = -2$. [2]

(b) Factorise $9x^3 + 6x^2 - 5x - 2$. [5]

(C1 Winter 2009)

7. *(a)* Find the remainder when $x^3 - 17$ is divided by $x - 3$. [2]

(b) Solve the equation $6x^3 - 7x^2 - 14x + 8 = 0$. [6]

(C1 Winter 2010)

8. The polynomial $f(x)$ is defined by

$$f(x) = 2x^3 + 11x^2 + 4x - 5.$$

(a) (i) Evaluate $f(-2)$.

(ii) **Using your answer to part (i)**, write down **one** fact which you can deduce about $f(x)$. [2]

(b) Solve the equation $f(x) = 0$. [6]

(C1 Summer 2017)

7. *(a)* Given that $x - 2$ is a factor of $kx^3 + 2x^2 - 41x + 10$, write down an equation satisfied by k. Hence show that $k = 8$. [2]

(b) Factorise $8x^3 + 2x^2 - 41x + 10$. [3]

(c) Find the remainder when $8x^3 + 2x^2 - 41x + 10$ is divided by $2x + 1$. [2]

(C1 Summer 2010)

8. *(a)* Given that $x + 2$ is a factor of $12x^3 + kx^2 - 13x - 6$, write down an equation satisfied by k. Hence show that $k = 19$. [2]

(b) Factorise $12x^3 + 19x^2 - 13x - 6$. [3]

(c) Find the remainder when $12x^3 + 19x^2 - 13x - 6$ is divided by $2x - 1$. [2]

(C1 Summer 2011)

8. The polynomial $px^3 - x^2 - 31x + q$ has $x + 2$ as a factor. When the polynomial is divided by $x - 1$, the remainder is -36.

(a) Show that $p = 6$ and $q = -10$. [6]

(b) Factorise $6x^3 - x^2 - 31x - 10$. [3]

(Sample Assessment Materials)

4. The cubic polynomial $f(x)$ is given by $f(x) = 2x^3 + ax^2 + bx + c$, where a, b, c are constants. The graph of $f(x)$ intersects the x-axis at the points with coordinates $(-3, 0)$, $(2{\cdot}5, 0)$ and $(4, 0)$. Find the coordinates of the point where the graph of $f(x)$ intersects the y-axis. [5]

Revision Questions

(C1 Summer 2013)

8. Solve the equation $8x^3 - 2x^2 - 7x + 3 = 0$. [6]

(C1 Summer 2018)

8. *(a)* (i) Find one real root of the equation $8x^3 + 7x^2 - 13x + 10 = 0$.

(ii) Show that the root you have found is the only real root of the equation

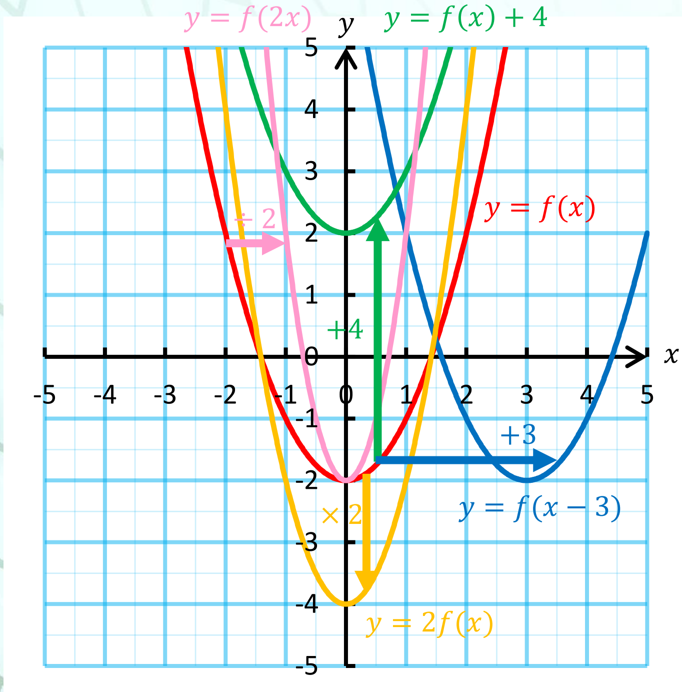
$$8x^3 + 7x^2 - 13x + 10 = 0.$$ [7]

(b) When $x^3 - 80$ is divided by $x - a$, the remainder is 45. Find the value of the constant a. [2]

Unit 1, Workbook 13

12

Transformations of Graphs



Name:

Background

What is the work?	Curve sketching, shading inequalities and plotting transformations.
What is required before starting?	**GCSE Work:** Graph plotting, transformations of graphs.
Where does this lead to?	**Unit 3:** Combining transformations of graphs. **Applications:** Unit conversion (e.g. temperature in Celsius, Fahrenheit and Kelvin); simple harmonic motion.

Theory

Curve Sketching

When sketching a graph of a function, you should show the key features of the function:

- The general shape, including any symmetry.
- The x and y intercepts.
- Any asymptotes (which are straight lines that do not intersect the function).

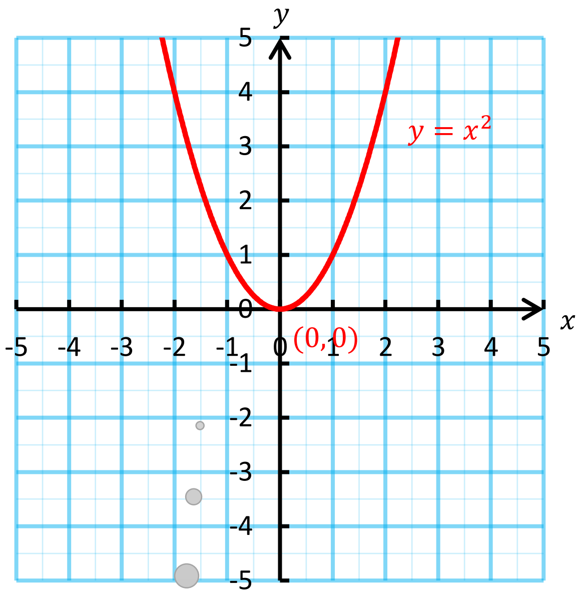
You can use the '*table mode*' on a scientific calculator; or a graphical calculator, to help with graph sketching.

- Factorising the function will often help you find the intercepts.
- It will be useful to remember the shape of some common functions.
- The solutions to a pair of simultaenous equations can be shown as the intersections of two graphs.

Common Functions

$y = x^2$

- A quadratic function.
- U shaped.
- Symmetric about the y–axis.



It is possible to transform the function $y = x^2$ to give any other quadratic function, of the form $y = ax^2 + bx + c$.

- If $a > 0$, the graph is U shaped.
- If $a < 0$, the graph is n shaped.
- The graph is symmetric about the line $x = -\frac{b}{2a}$.

By completing the square, it is possible to write a general quadratic function $y = ax^2 + bx + c$ as $y = a(x + p)^2 + q$.

- The stationary point (minimum or maximum) is $(-p, q)$.
- The graph is symmetric about the line $x = -p$.

The graph $y = x^4$ has a similar shape.

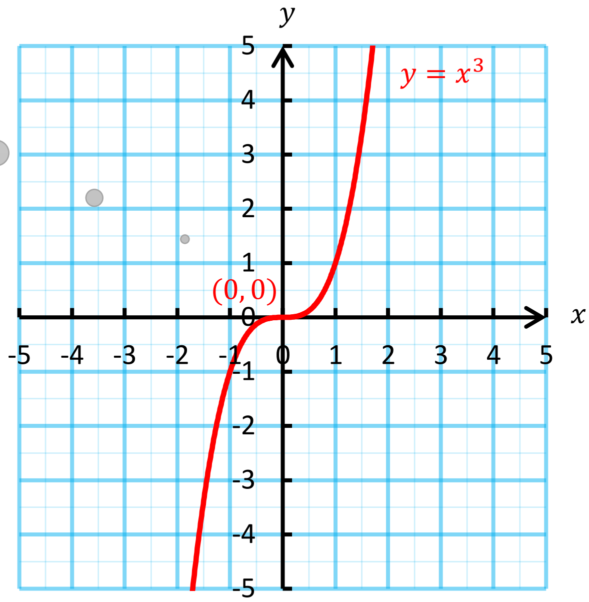
$y = x^3$

- A cubic function.
- S shaped.

The graph $y = x^5$ has a similar shape.

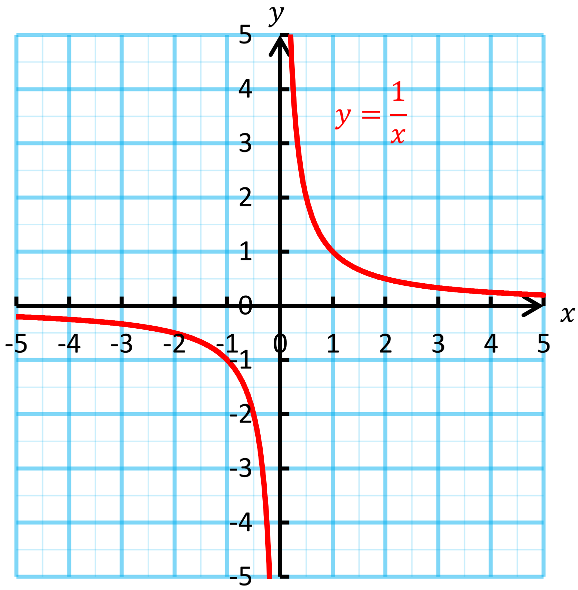
A general cubic function has the form
$y = ax^3 + bx^2 + cx + d$.

- The graph will have one point of inflection, or one minimum point and one maximum point.



$y = \frac{1}{x}$

- A reciprocal function.
- Not defined for $x = 0$.
- The x and y axes are asymptotes.



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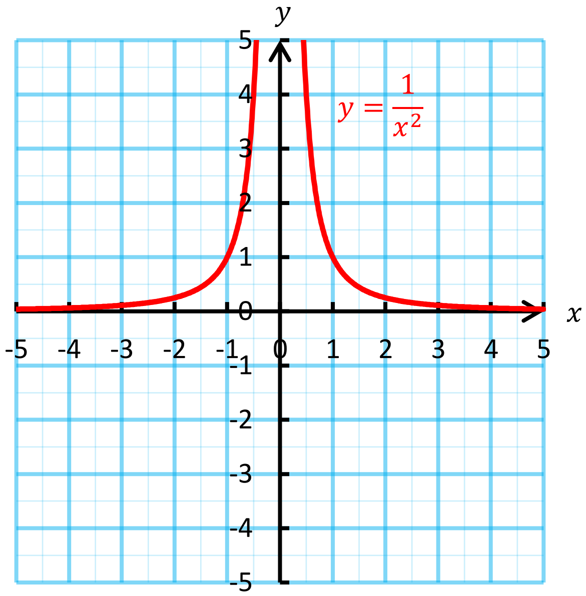
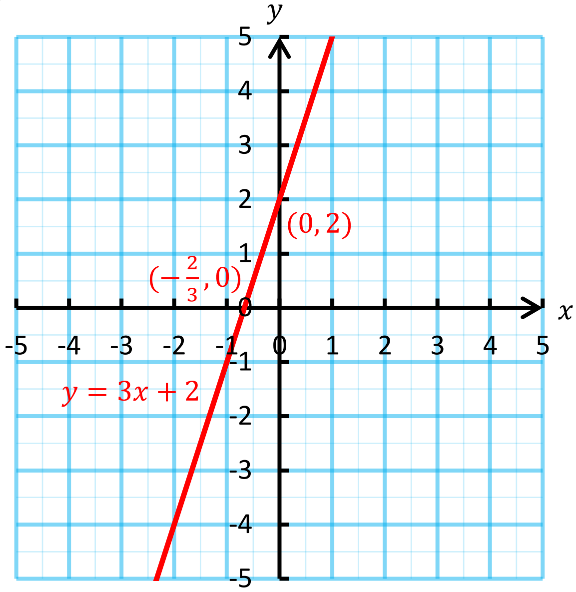
- The x and y axes are asymptotes.



I unit to the right,
m units up/down.

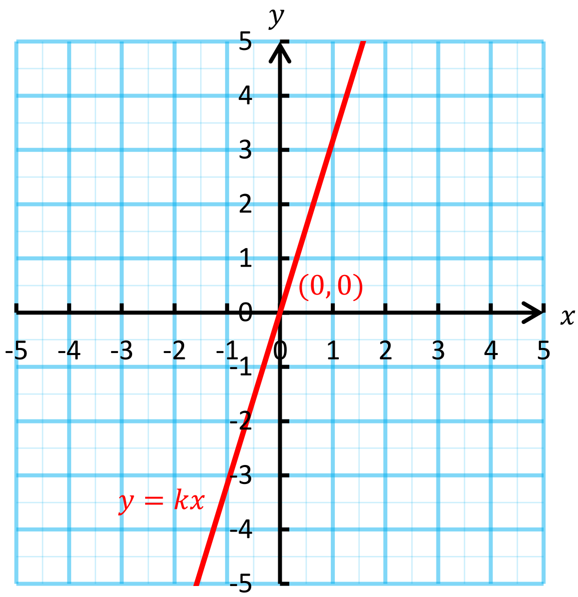
$y = mx + c$

- A linear function.
- The gradient of the line is m.
- The y–intercept of the line is $(0, c)$.



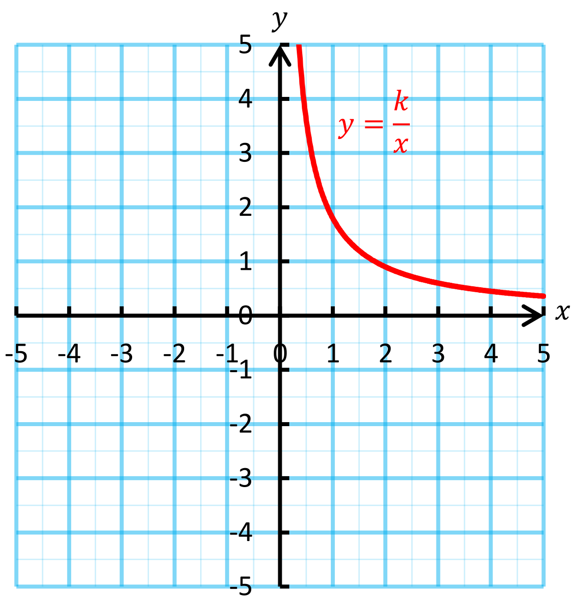
$y \propto x$
$y = kx$

- A linear function (direct proportion).
- The gradient of the line is k.
- The y–intercept of the line is $(0, 0)$.

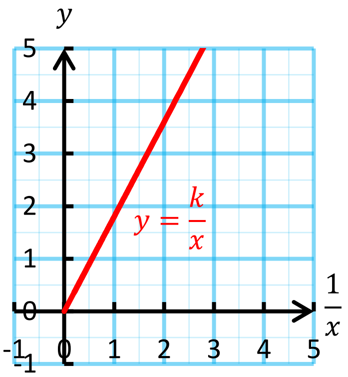


$y \propto \frac{1}{x}$
$y = \frac{k}{x}$

- A reciprocal function (inverse proportion).
- The x and y axes are asymptotes.



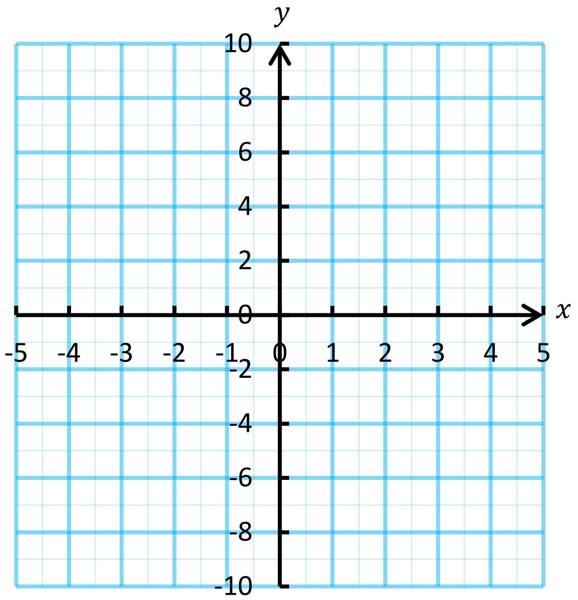
By plotting y against $\frac{1}{x}$, we obtain a linear function with gradient k.



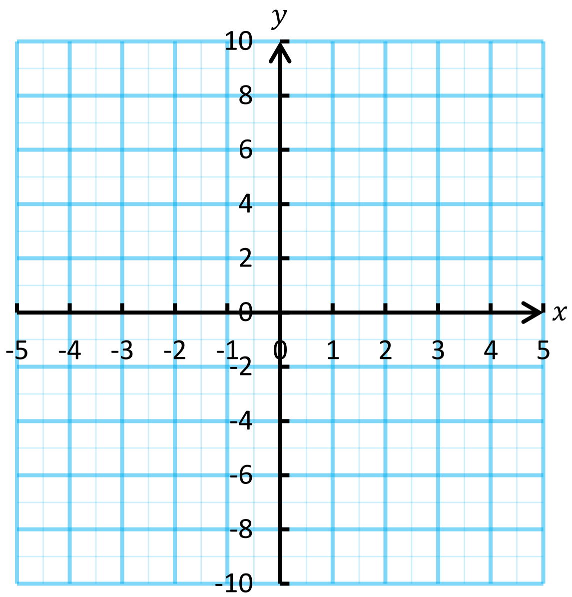
Exercise 1

Plot the following graphs.

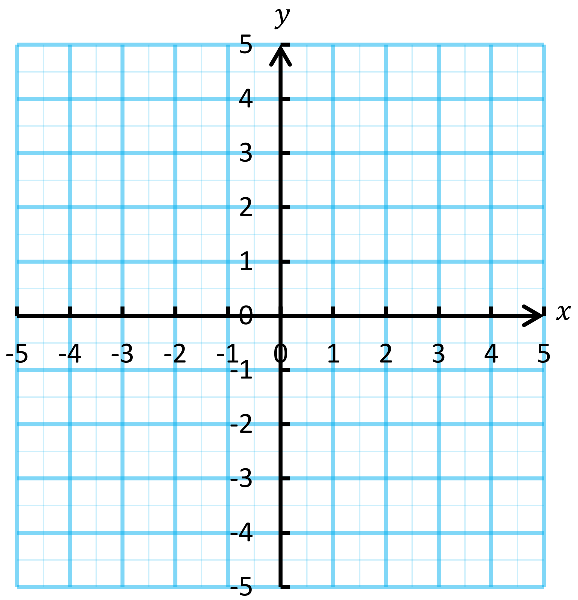
(a) $y = (x-2)^2 - 3$



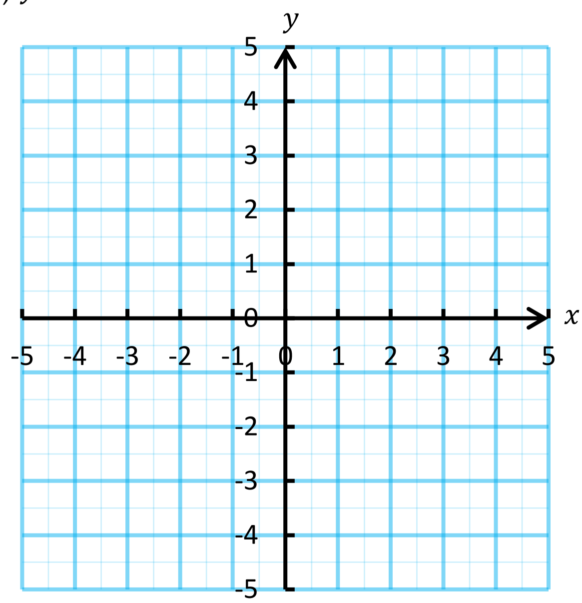
(b) $y = \frac{2}{x} + 4$



(c) $y = 3x - 4$



(d) $y = x^3 - 5x^2 + 7x - 3$

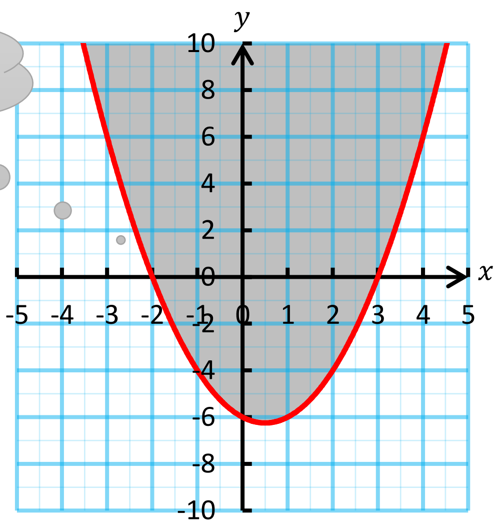


Plotting Inequalities

$\leq, \geq$: Solid line
$<, >$: Dotted line

To plot an inequality like $y \geq x^2 - x - 6$,

- Plot the equation $y = x^2 - x - 6$.
- Choose a point that does not lie on the curve (such as $(0,0)$ in the case of $y = x^2 - x - 6$). Substitute the values into the inequality, to see if it is true (we need to shade the point) or not true.
  Substituting $x = 0$, $y = 0$: $\quad 0 \geq 0^2 - 0 - 6$
  $0 \geq -6$
  This is true so we need to shade the region that includes the point $(0,0)$.



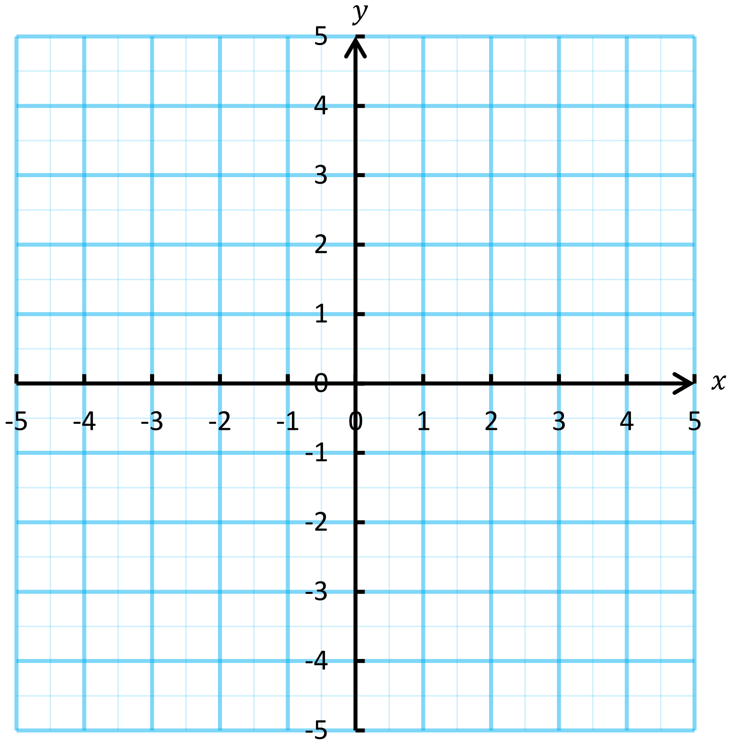
Exercise 2

(a) Plot $y = 4 - 2(x-1)^2$.

(b) Use your graph from part (a) to solve the equation $4 - 2(x-1)^2 = -4$.

(c) Shade the region that satisifes the following inequalities.

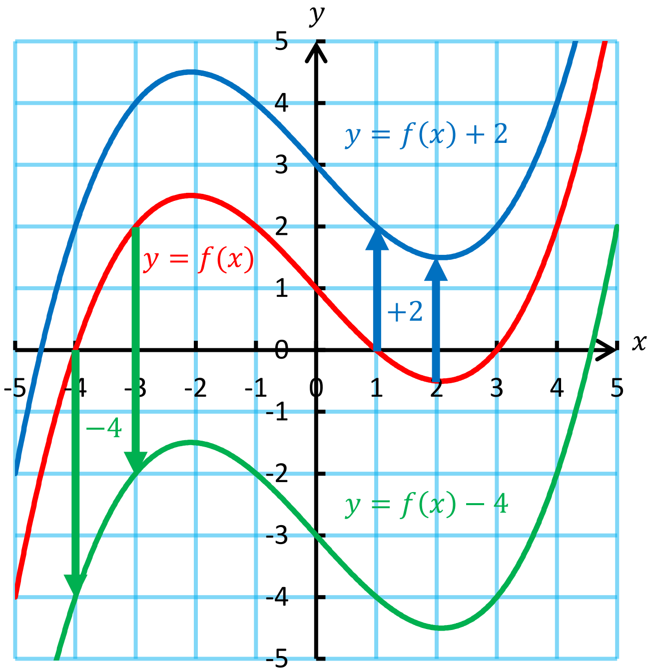
$$y \leq 4 - 2(x-1)^2$$
$$y \geq -4$$
$$y < 1$$



Transformations of Graphs

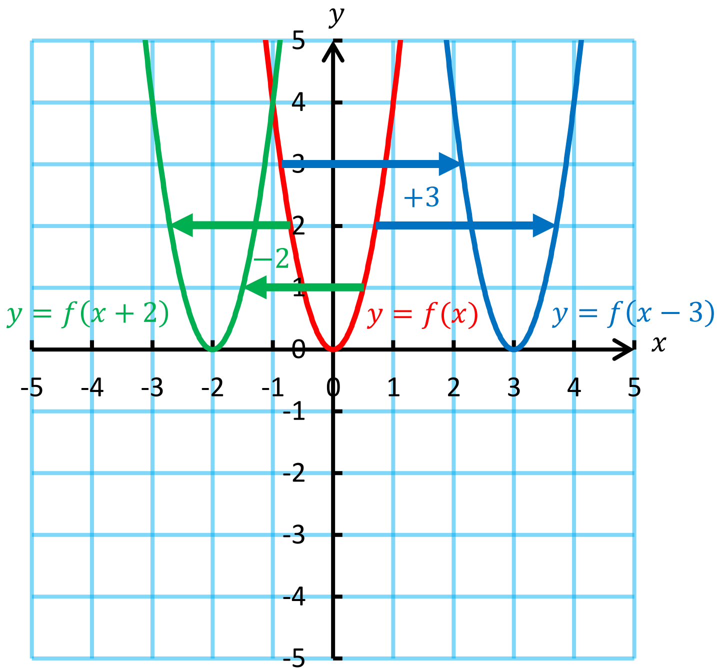
We can transform the function $y = f(x)$ using the following techniques.

$\boldsymbol{y = f(x) + a}$



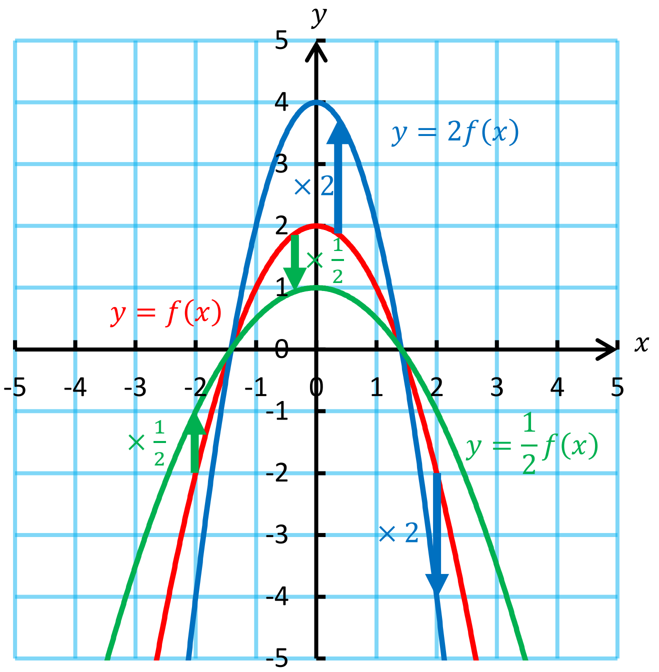
The transformation $y = f(x) + a$ moves the graph a units up (if a is positive) or a units down (if a is negative).

$\boldsymbol{y = f(x + a)}$

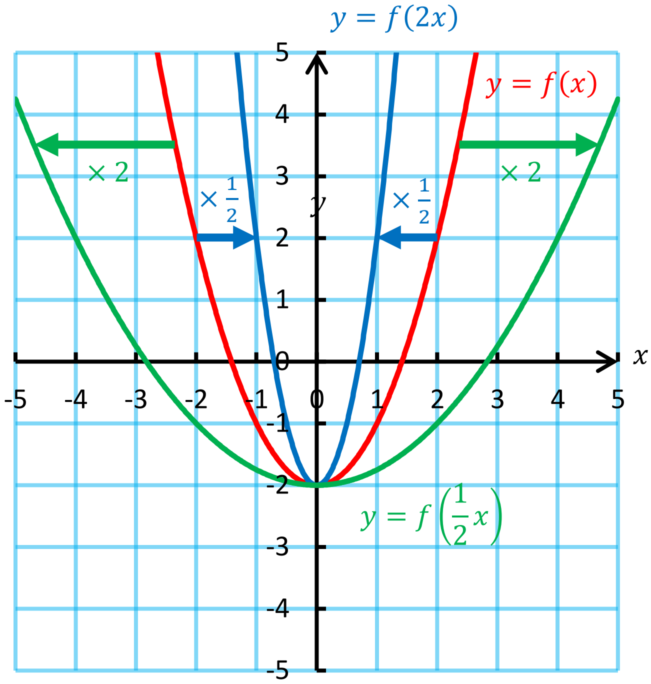


The transformation $y = f(x + a)$ moves the graph a units to the left (if a is positive) or a units to the right (if a is negative).

$y = af(x)$



$y = f(ax)$



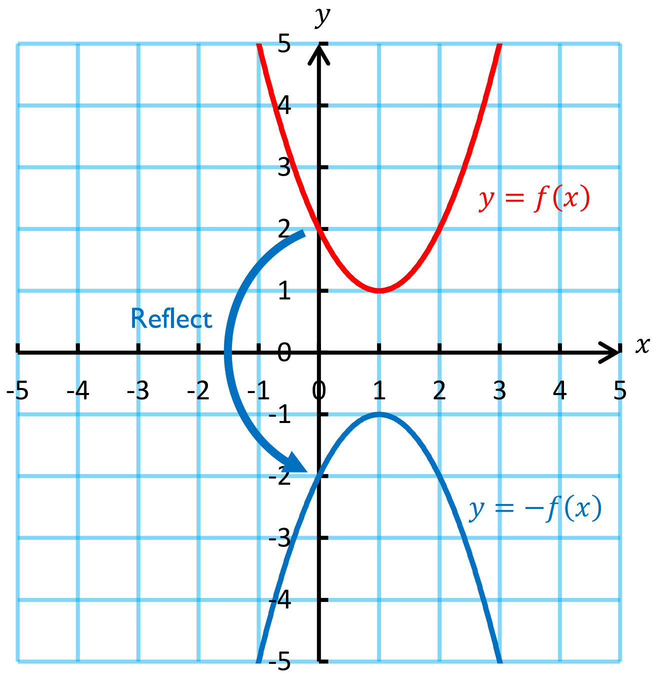
The transformation $y = af(x)$ stretches the graph in the direction of the y–axis (if $a > 1$) or compresses the graph in the direction of the y–axis (if $0 < a < 1$).

The transformation $y = f(ax)$ compresses the graph in the direction of the x–axis (if $a > 1$) or stretches the graph in the direction of the x–axis (if $0 < a < 1$).

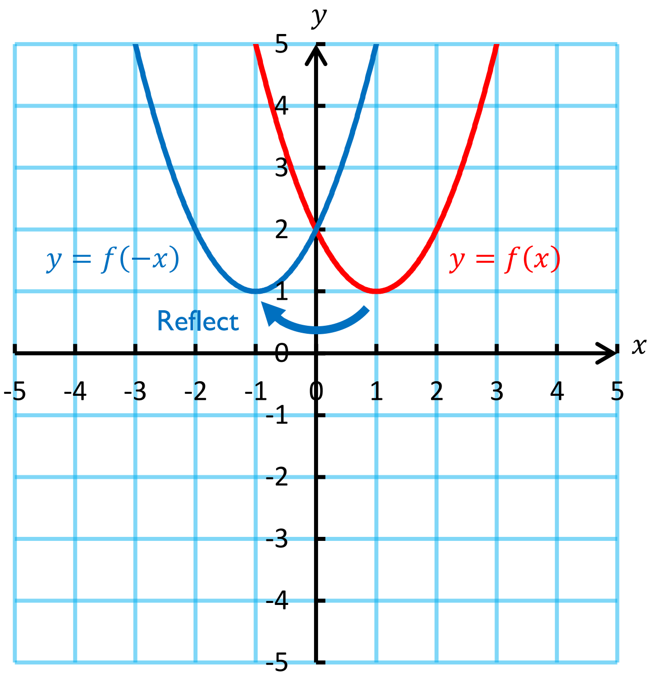
As general advice, we need to **follow** anything that takes place **outside** parentheses, and **undo** anything that takes place **inside** parentheses. So, for example, we follow $y = f(x) + 3$ and translate the graph 3 units up, but undo $y = f(x + 3)$ by translating the graph 3 units to the left.

Special Cases

$y = -f(x)$



$y = f(-x)$



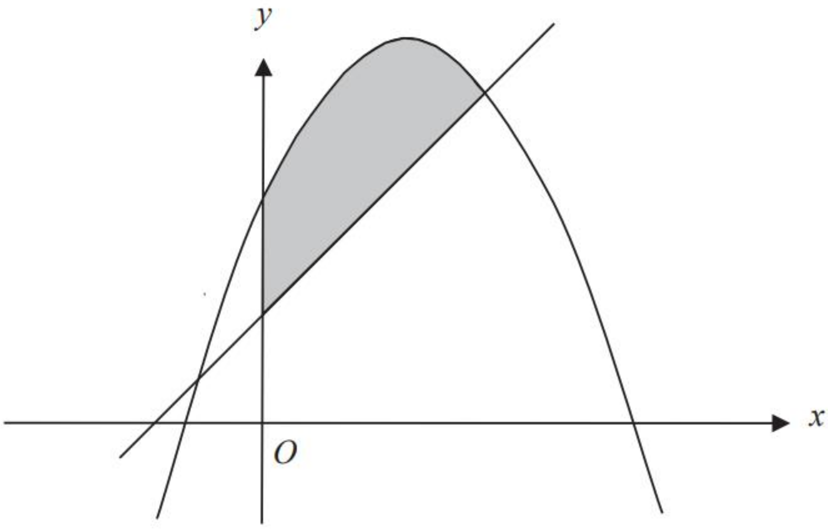
The transformation $y = -f(x)$ reflects the graph $y = f(x)$ in the x–axis.

The transformation $y = f(-x)$ reflects the graph $y = f(x)$ in the y–axis.

Top 10

(Sample Assessment Materials)

11.



The diagram shows a sketch of the curve $y = 6 + 4x - x^2$ and the line $y = x + 2$. The point P has coordinates (a, b). Write down the three inequalities involving a and b which are such that the point P will be strictly contained within the shaded area above, if and only if, all three inequalities are satisfied. [3]

(Unit 1 Summer 2022)

0 5 The curve C_1 has equation $y = -x^2 + 2x + 3$ and the curve C_2 has equation $y = x^2 - x - 6$. The two curves intersect at the points A and B.

a) Determine the coordinates of A and B. [4]

b) On the same set of axes, sketch the graphs of C_1 and C_2. Clearly label the points where the two curves intersect. [3]

c) In the diagram drawn in part (b), shade the region satisfying the following inequalities: [2]

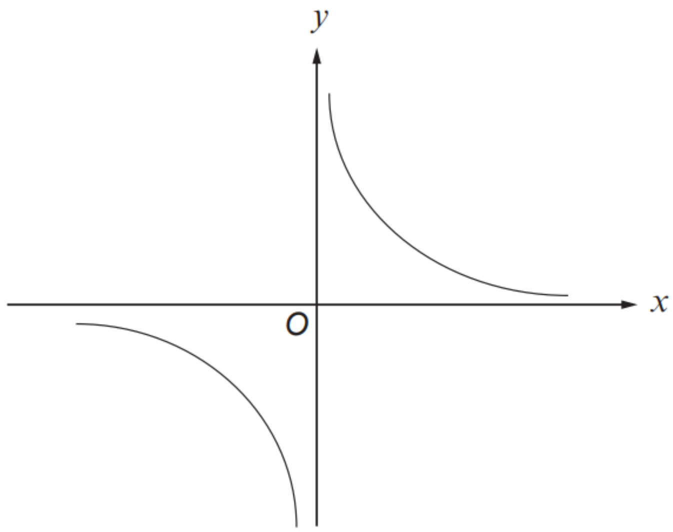
$$x > 0,$$

$$y < -x^2 + 2x + 3,$$

$$y > x^2 - x - 6.$$

(Unit 1 Summer 2018)

0 5 The diagram below shows a sketch of $y = f(x)$.



a) Sketch the graph of $y = 4 + f(x)$, clearly indicating any asymptotes. [2]

b) Sketch the graph of $y = f(x - 3)$, clearly indicating any asymptotes. [2]

(Unit 1 Summer 2023)

1 1 The function f is defined by $f(x) = \frac{8}{x^2}$.

a) Sketch the graph of $y = f(x)$. [2]

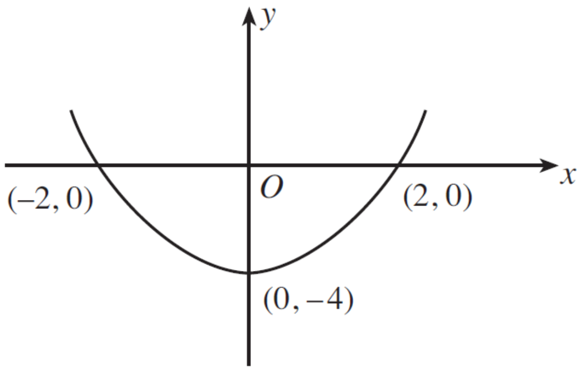
b) On a separate set of axes, sketch the graph of $y = f(x-2)$. Indicate the vertical asymptote and the point where the curve crosses the y-axis. [3]

c) Sketch the graphs of $y = \frac{8}{x}$ and $y = \frac{8}{(x-2)^2}$ on the same set of axes.

Hence state the number of roots of the equation $\frac{8}{(x-2)^2} = \frac{8}{x}$. [2]

(C1 Summer 2006)

9.



The diagram shows the graph of $y = f(x)$. The curve passes through the points (2, 0) and (–2, 0), and has a minimum point at $(0, -4)$.

Sketch on separate diagrams the graphs of

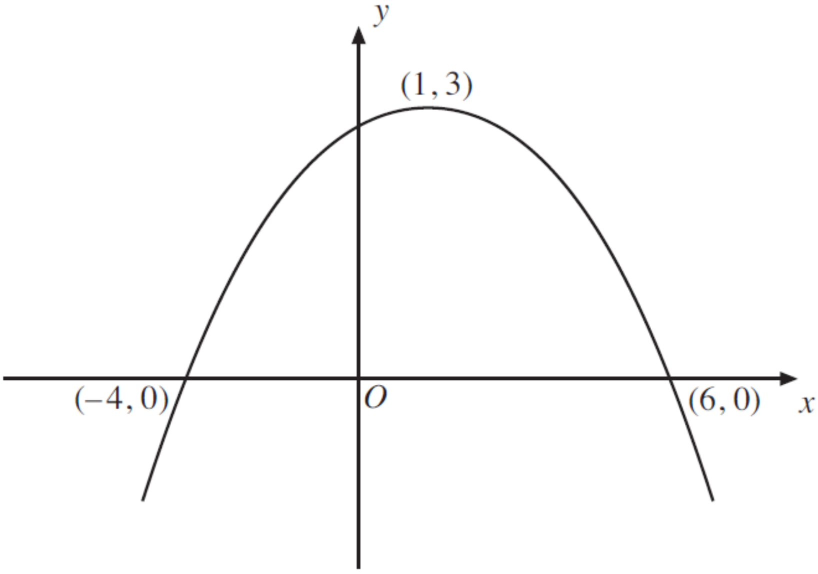
(a) $y = f(x) + 4$, [2]

(b) $y = f(x + 2)$, [3]

indicating the coordinates of the points of intersection with the x-axis and the coordinates of the stationary points.

(C1 Summer 2010)

9. The diagram shows a sketch of the graph of $y = f(x)$. The graph passes through the points $(-4, 0)$ and $(6, 0)$ and has a maximum point at $(1, 3)$.



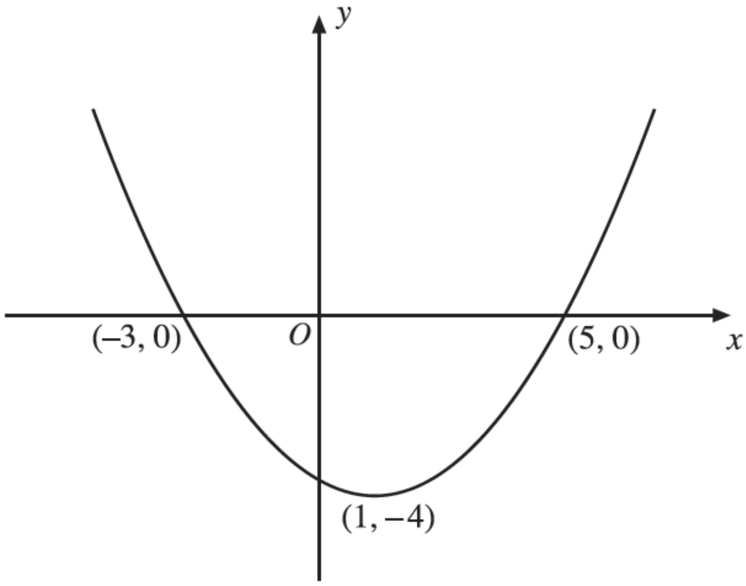
Sketch the following graphs, using a separate set of axes for each graph. In each case, you should indicate the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis.

(a) $y = 2f(x)$ [3]

(b) $y = f(-x)$ [3]

(C1 Winter 2011)

9. The diagram shows a sketch of the graph of $y = f(x)$. The graph passes through the points $(-3, 0)$ and $(5, 0)$ and has a minimum point at $(1, -4)$.



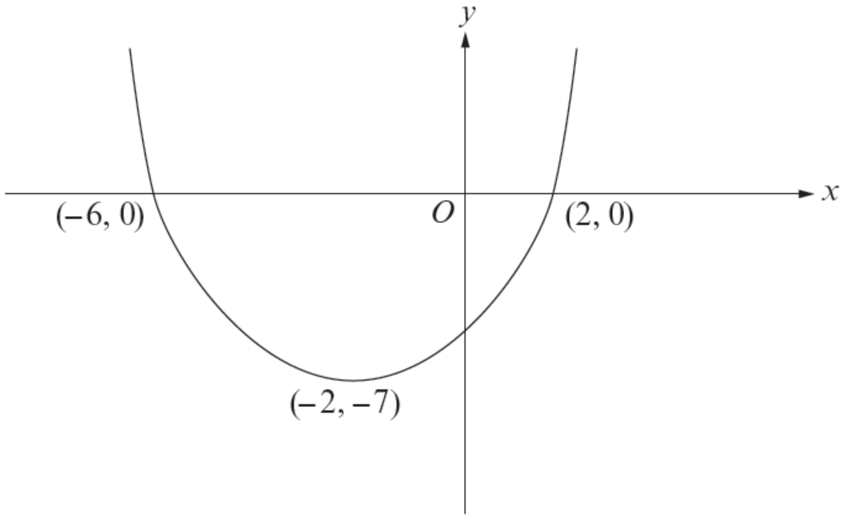
Sketch the following graphs, using a separate set of axes for each graph. In each case, you should indicate the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis.

(a) $y = f(x + 3)$, [3]

(b) $y = -f(x)$. [3]

(C1 Summer 2012)

9. The diagram shows a sketch of the graph of $y = f(x)$. The graph passes through the points $(-6, 0)$ and $(2, 0)$ and has a minimum point at $(-2, -7)$.



Sketch the following graphs, using a separate set of axes for each graph. In each case, you should indicate the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis.

(a) $y = f(x - 5)$ [3]

(b) $y = f\left(\frac{x}{2}\right)$ [3]

(C1 Winter 2014)

7. **Figure 1** shows a sketch of the graph of $y = f(x)$. The graph has a maximum point at (2, 6) and intersects the x-axis at the points (–4, 0) and (8, 0).

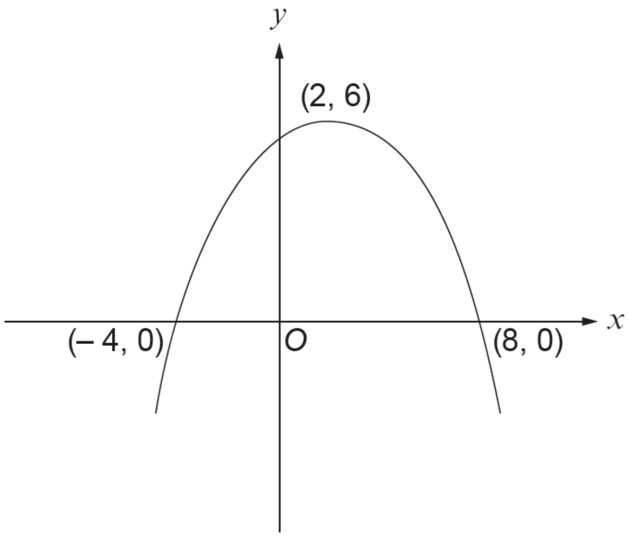


Figure 1

(a) Sketch the graph of $y = f(x - 3)$, indicating the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis. [3]

(b) **Figure 2** shows a sketch of the graph having **one** of the following equations with an appropriate value of p, q or r.

$y = f(x) + p$, where p is a constant
$y = f(qx)$, where q is a constant
$y = rf(x)$, where r is a constant

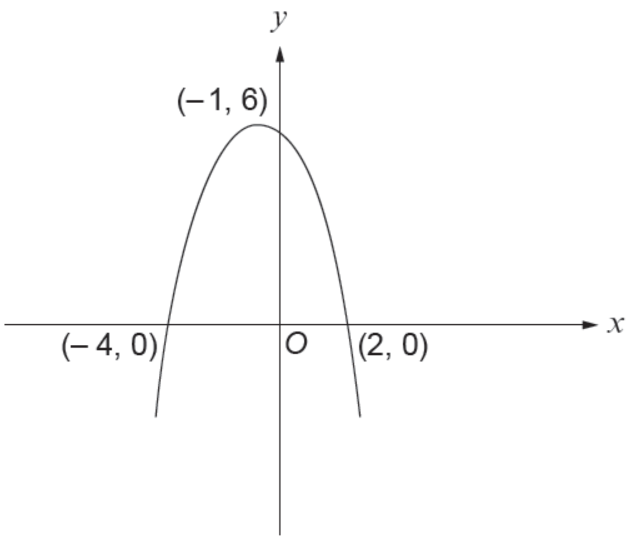
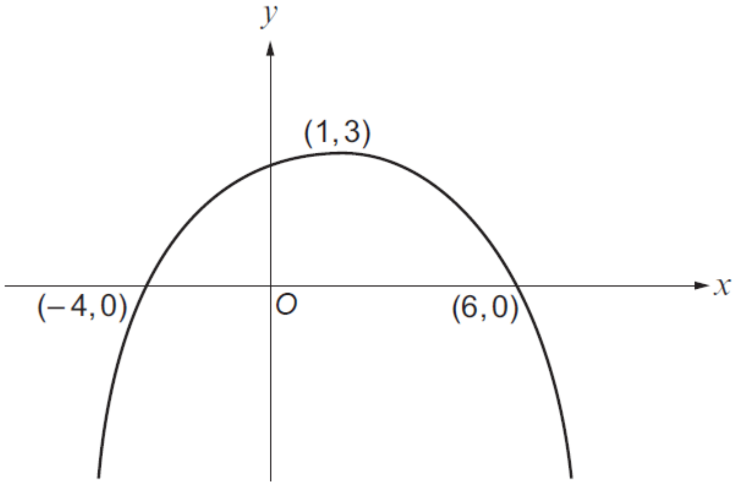


Figure 2

Write down the equation of the graph sketched in **Figure 2**, together with the value of the corresponding constant. [2]

(C1 Summer 2018)

9. The diagram shows a sketch of the graph of $y = f(x)$. The graph passes through the points $(-4, 0)$ and $(6, 0)$ and has a maximum point at $(1, 3)$.



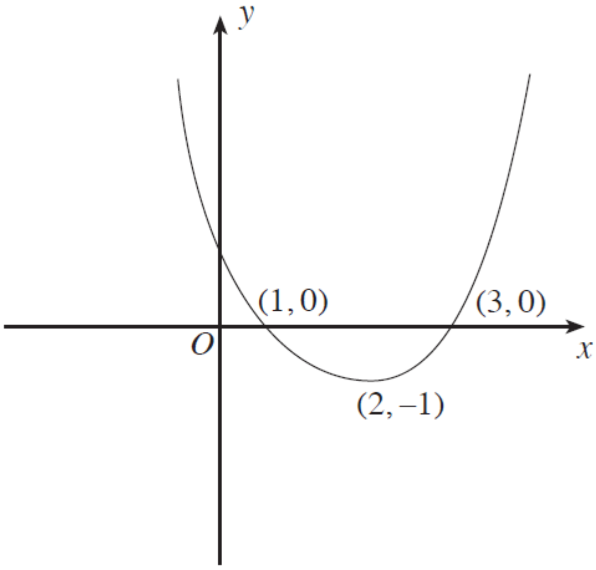
(a) Sketch the graph of $y = f(x + 3)$, indicating the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis. [3]

(b) Gwen is asked by her teacher to draw the graph of $y = f(ax)$ for various values of the constant a. Two of Gwen's graphs pass through the point $(2, 0)$.
Find the value of a corresponding to each of these two graphs. [2]

Revision Questions

(C1 Winter 2008)

9. The diagram shows the graph of $y = f(x)$. The graph has a minimum point at (2, –1) and intersects the x-axis at the points (1, 0) and (3, 0).

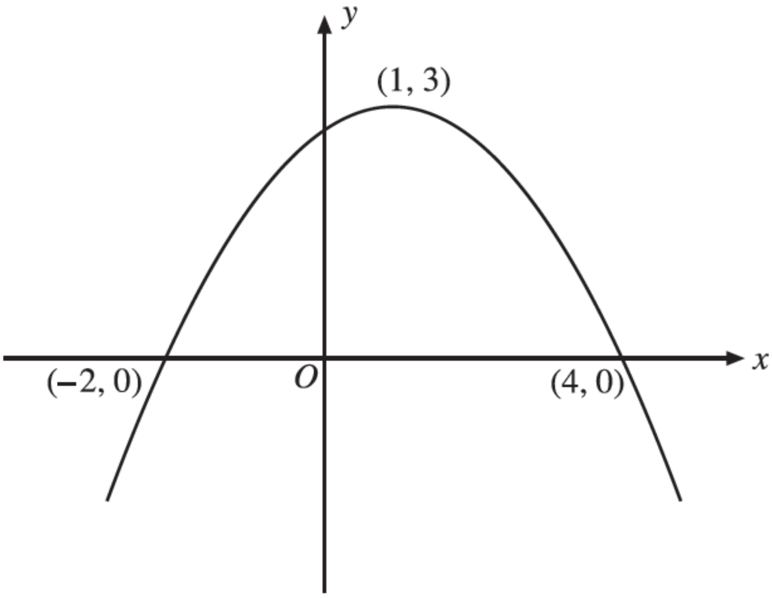


Sketch the following graphs, using a separate set of axes for each graph. In each case you should indicate the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis.

(a) $y = 3f(x)$ *(b)* $y = f(x+5)$ [3], [3]

(C1 Winter 2012)

9. The diagram shows a sketch of the graph of $y = f(x)$. The graph has a maximum point at (1, 3) and intersects the x-axis at the points (–2, 0) and (4, 0).



(a) Sketch the graph of $y = f(2x)$, indicating the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis. [3]

(b) (i) Sketch the graph of $y = f(x) - 5$, indicating the coordinates of the stationary point.

(ii) Given that f is a quadratic function, use the graph you have drawn in part (i) to write down the number of real roots of the equation

$$f(x) - 5 = 0.$$ [3]

Unit 1, Workbook 14

12

Introducing Logarithms

log😄 😅 = 💧

Name:

Background

What is the work?	Understanding the purpose of logarithms, and their use in solving equations.
What is required before starting?	**GCSE Work:** Solving equations; rules of indices.
Where does this lead to?	**Unit 1:** Curve fitting. **Applications:** Measuring the magnitude of an earthquake; calculating the time taken to repay a loan; finding population growth.

Theory

The meaning of a logarithm

Consider the sum $2^3 = 8$. To undo this sum, at GCSE level we have been taking the cube root of 8, to go back to the **base** 2. Now in the A Level course, we introduce another way of undoing the sum, to go back to the **power** 3. The technique to go back to the power 3 is to take a **logarithm base 2** of 8.

$$\log_2(8) = 3$$

To take a logarithm base 2 of 8, we consider the question "2 to which power gives the answer 8?"

$$2^? = 8$$

Only 2 to the power 3 gives the answer 8, so $\log_2(8) = 3$.

Theory

In general,

if $b^y = x$, then $y = \log_b(x)$.

We say that b is the base of the logarithm.

Important logarithms

- Base 10: For $\log_{10}(x)$, we do not usually write the 10, and write $\log(x)$ only.
- Base e: We will soon come across the important number $e = 2.71828$ ..., which appears so often in mathematics $\log_e(x)$ has its own special notation, namely $\ln(x)$.

Exercise 1

Complete the following table.

$\log_2(16)$	$\log_3(9)$	$\log(100)$	$\log_3(81)$	$\log_4(16)$	$\log_{16}(16)$	$\log_5(1)$	$\log_3\left(\frac{1}{9}\right)$	$\log_4(2)$	$\log_2\left(\frac{1}{64}\right)$

Solving Equations

It is possible to use logarithms to solve equations where the variable appears in the exponent.

Example 1

Solve the equation $5^{2x+1} = 8$.

Answer: Because the equation has the base 5 on the left hand side, we must take a logarithm base 5 of each side in order to work with the power $2x + 1$.

$$\log_5(5^{2x+1}) = \log_5(8)$$
$$2x + 1 = \log_5(8)$$
$$2x = \log_5(8) - 1$$
$$x = \frac{\log_5(8) - 1}{2}$$
$$x = 0.146 \text{ to 3 decimal places}$$

Exercise 2

Solve the equation $7^{3x-2} = 18$.

Example 2

Use the substitution $2^x = u$ to solve the equation $2^{2x} - 2^{x+3} + 16 = 0$.

Answer: If $2^x = u$, then $2^{2x} = (2^x)^2 = u^2$, and $2^{x+3} = 2^x \times 2^3 = 8 \times 2^x = 8u$.

So, we need to solve the equation

$$u^2 - 8u + 16 = 0$$
$$(u - 4)(u - 4) = 0$$
$$\text{Either } u - 4 = 0 \text{ or } u - 4 = 0$$
$$u = 4$$
$$2^x = 4$$
$$\log_2(2^x) = \log_2(4)$$
$$x = 2$$

Exercise 3

Use the substitution $5^x = u$ to solve the equation $5^{2x} - 5^{x+2} + 100 = 0$.

Rules of Logarithms

You need to learn the following logarithm rules, including learning the **proof** for each rule.

(1) The multiplication rule

Given that $x > 0$, $y > 0$,

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

Proof

Let $m = \log_a(x)$, $n = \log_a(y)$.

Then, by the definition of a logarithm, $a^m = x$, $a^n = y$.
So, $xy = a^m \times a^n$.
$xy = a^{m+n}$ (rules of indices).
$\log_a(xy) = m + n$.
$\log_a(xy) = \log_a(x) + \log_a(y)$. **QED**

This is the **minimum** that you need to learn. Do not delete any element from the proof!

(2) The division rule

Given that $x > 0$, $y > 0$,

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

Proof

Let $m = \log_a(x)$, $n = \log_a(y)$.

Then, by the definition of a logarithm, $a^m = x$, $a^n = y$.
So, $\frac{x}{y} = \frac{a^m}{a^n}$.
$\frac{x}{y} = a^{m-n}$ (rules of indices).
$\log_a\left(\frac{x}{y}\right) = m - n$.
$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$. **QED**

Notice how similar the proofs to rules (1) and (2) are.

(3) The power rule

Given that $x > 0$,

$$\log_a(x^n) = n\log_a(x)$$

Proof

Let $y = \log_a(x)$.

Then, by the definition of a logarithm, $a^y = x$.

So, $(a^y)^n = x^n$.
$a^{yn} = x^n$ (rules of indices).
$yn = \log_a(x^n)$.
$ny = \log_a(x^n)$.
So, $n\log_a(x) = \log_a(x^n)$
or $\log_a(x^n) = n\log_a(x)$. **QED**

For the proofs on this page, x and y have to be positive because (e.g.) if x is negative, there is no value for $\log_a x$. Why? Consider $a = 2$ as an example. There is no possible value for y so that 2^y can be negative.

It is possible to use the rules on the previous page to combine logarithms and solve more complex equations.

Example 3

Express $\log_4(2) + 2\log_4(18) - \frac{3}{2}\log_4(36)$ as a single logarithm, in its simplest form.

Answer: $\log_4(2) + 2\log_4(18) - \frac{3}{2}\log_4(36)$

$= \log_4(2) + \log_4(18^2) - \log_4\left(36^{\frac{3}{2}}\right)$ use the power rule

$= \log_4(2) + \log_4(324) - \log_4(216)$

$= \log_4(2 \times 324) - \log_4(216)$ use the multiplication rule

$= \log_4(648) - \log_4(216)$

$= \log_4\left(\frac{648}{216}\right)$ use the division rule

$= \log_4(3)$ simplify the fraction

Example 4

Solve the equation $\log_7(x^2 + 48) = \log_7(x) + 2\log_7(4)$.

Answer: $\log_7(x^2 + 48) = \log_7(x) + \log_7(4^2)$ use the power rule

$\log_7(x^2 + 48) = \log_7(x) + \log_7(16)$

$\log_7(x^2 + 48) = \log_7(x \times 16)$ use the multiplication rule

$\log_7(x^2 + 48) = \log_7(16x)$

$x^2 + 48 = 16x$ if $\log_n(A) = \log_n(B)$, then we must have $A = B$

$x^2 - 16x + 48 = 0$

$(x - 4)(x - 12) = 0$

Either $x - 4 = 0$ or $x - 12 = 0$

Either $x = 4$ or $x = 12$

Exercise 4

(a) Given that $x > \frac{4}{3}$, solve the equation $\log_8(x) + \log_8(3x + 4) = 2\log_8(3x - 4)$.

(b) Express $\log_9(36) + \frac{1}{2}\log_9(256) - 2\log_9(48)$ as a single logarithm, in its simplest form.

Top 10

(C2 Winter 2013)

7. *(a)* Given that $x > 0$, $y > 0$, show that

$$\log_a \frac{x}{y} = \log_a x - \log_a y.$$ [3]

(b) Solve the equation

$$6^{2x+5} = 7.$$

Show your working and give your answer correct to three decimal places. [3]

(C2 Summer 2008)

7. *(a)* Given that $x > 0$, show that

$$\log_a x^n = n \log_a x \,.$$ [3]

(b) Solve the equation

$$\log_a (3x + 4) - \log_a x = 3 \log_a 2 \,.$$ [4]

(c) Solve the equation

$$4^{3y+2} = 7 \,,$$

giving your answer correct to three decimal places. [3]

(C2 Winter 2012)

7. *(a)* Given that $x > 0, y > 0$, show that

$$\log_a xy = \log_a x + \log_a y.$$ [3]

(b) Solve the equation

$$2^{3-5x} = 12.$$

Show your working and give your answer correct to three decimal places. [3]

(c) (i) Express

$$\log_9(3x-1) + \log_9(x+4) - 2\log_9(x+1)$$

as a single logarithm.

(ii) Hence solve the equation

$$\log_9(3x-1) + \log_9(x+4) - 2\log_9(x+1) = \frac{1}{2}.$$ [5]

(C2 Summer 2010)

8. *(a)* Given that $x > 0$, show that

$$\log_a x^n = n \log_a x.$$

[3]

(b) Solve the equation

$$6^{2y-1} = 4.$$

Show your working and give your answer correct to three decimal places. [3]

(c) Given that $\log_a 4 = \frac{1}{2}$, find the value of a. [2]

(C2 Winter 2005)

2. Use the substitution $3^x = u$ to solve the equation

$$3^{2x} - 3^{x+2} + 14 = 0,$$

giving your answers correct to three decimal places. [6]

(C2 Winter 2006)

10. *(a)* Given that $x > 0$, $y > 0$, show that

$$\log_a(xy) = \log_a x + \log_a y.$$ [3]

(b) Given that $\int_1^3 \log_{10} x \, dx$ has an approximate value of $0{\cdot}5628$, find an approximate value for $\int_1^3 \log_{10}(10x) \, dx$. Give your answer correct to four decimal places. [4]

(C2 Winter 2010)

7. *(a)* Given that $x > 0$, show that

$$\log_a x^n = n \log_a x. \quad [3]$$

(b) Express $\frac{1}{2}\log_a 324 + \log_a 56 - 2\log_a 12$ in the form $\log_a b$, where b is a constant whose value is to be found. [4]

(c) (i) Rewrite the equation

$$3^x = 2^{x+1}$$

in the form

$$c^x = d,$$

where the values of the constants c and d are to be found.

(ii) Hence or otherwise, solve the equation

$$3^x = 2^{x+1},$$

giving your answer correct to two decimal places. [4]

(C2 Summer 2014)

7. *(a)* Solve the equation

$$3^{\frac{5x}{4}-2} = 7.$$

Show your working and give your answer correct to three decimal places. [3]

(b) The positive numbers a and b are such that

$$\log_a b = 5.$$

(i) Express b as a power of a.

(ii) **Using your answer to part (i)**, evaluate $\log_b a$. [3]

(Unit 1 Summer 2018)

1 7 a) Solve $2\log_{10}x = 1 + \log_{10}5 - \log_{10}2$. [4]

$\log_e(x) = \ln(x)$

b) Solve $3 = 2e^{0\cdot 5x}$. [2]

c) Express $4^x - 10 \times 2^x$ in terms of y, where $y = 2^x$.

Hence solve the equation $4^x - 10 \times 2^x = -16$. [5]

(Unit 1 Summer 2019)

10 **a)** Solve the following simultaneous equations.

$$3^{3x} \times 9^{y} = 27$$

$$2^{-3x} \times 8^{-y} = \frac{1}{64}$$

[6]

b) Find the value of x satisfying the equation

$$\log_a 3 + 2\log_a x - \log_a(x - 1) = \log_a(5x + 2).$$

[7]

Revision Questions

(C2 Winter 2014)

7. *(a)* Given that $x > 0$, show that

$$\log_a x^n = n\log_a x.$$ [3]

(b) Solve the equation

$$7^{5-4x} = 11.$$

Show your working and give your answer correct to three decimal places. [3]

(c) Solve the equation

$$\log_8 x = -\frac{1}{3}.$$ [2]

(Unit 1 Summer 2023)

10 Solve the following equations for values of x.

a) $\ln(2x+5)=3$ [2]

b) $5^{2x+1}=14$ [3]

c) $3\log_7(2x) - \log_7\left(8x^2\right) + \log_7 x = \log_3 81$ [6]

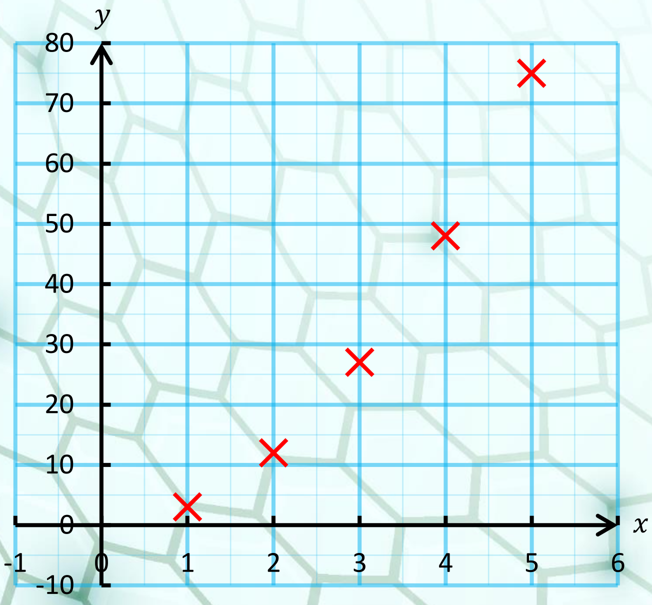
Unit 1, Workbook 15

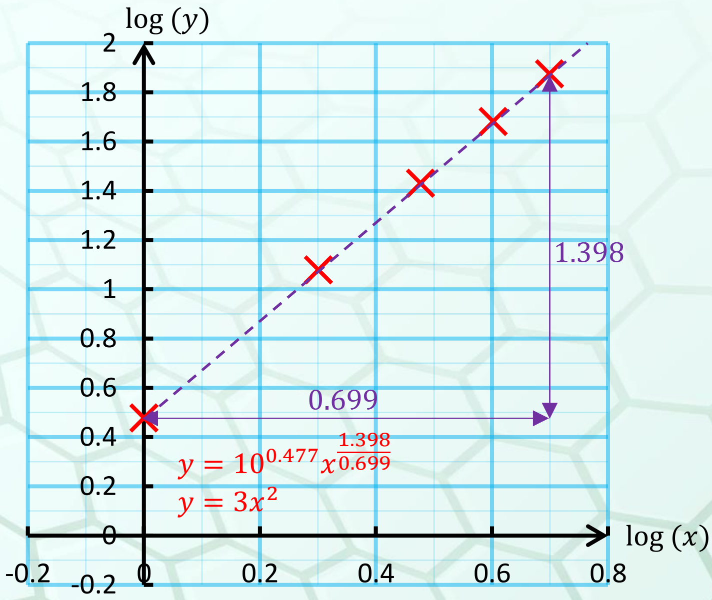
12

Curve Fitting

x	1	2	3	4	5
y	3	12	27	48	75

$\log(x)$	0	0.301	0.477	0.602	0.699
$\log(y)$	0.477	1.079	1.431	1.681	1.875





Name:

Background

What is the work?	Consider when and how to use exponential models. Fitting a polynomial or exponential curve to data.
What is required before starting?	**GCSE Work:** Nth term; graph plotting; rules of indices. **A Level Unit 1:** Introducing logarithms.
Where does this lead to?	**Applications:** Calculating the compound interest on a loan; radioactive decay; the decay of a drug in the body; modelling population growth.

Theory

Consider the following sequence of numbers.

$$4, 7, 10, 13, 16, 19, \ldots$$

We see that we **add three** each time to obtain the next number in the sequence.
From GCSE work, the nth term is

$$3n + 1$$

or

$$4 + 3(n - 1)$$

Next, consider the following sequence of numbers.

$$4, 12, 36, 108, 324, 972, \ldots$$

We see that we **multiply by three** each time to obtain the next number in the sequence. The nth term is

$$4 \times 3^{n-1}$$

This is an example of an **exponential function.** We use these in sequences where we multiply by the same number each time, to obtain the next number in the sequence.

Example 1

Consider investing £1,000 into a bank that pays yearly compound interest at a rate of 3%.
After one year, the amount of money we have will be

$$£1{,}000 \times 1.03 = £1{,}030$$

Or we can multiply by 103%

After two years:

$$£1{,}000 \times 1.03^2 = £1{,}060.90$$

After n years:

$$£1{,}000 \times 1.03^n$$

Plotting exponential graphs

You need to know how to plot a graph of the exponential function $y = a^x$, where a is positive.

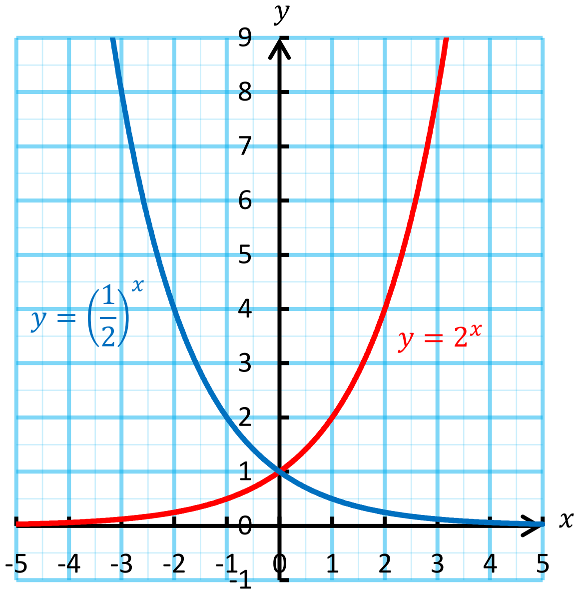
What happens if a is negative?

Example 2

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y = 2^x$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32
$y = \left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$

Note that $y = 2^x$ doubles each time, whilst $y = \left(\frac{1}{2}\right)^x$ halves each time.

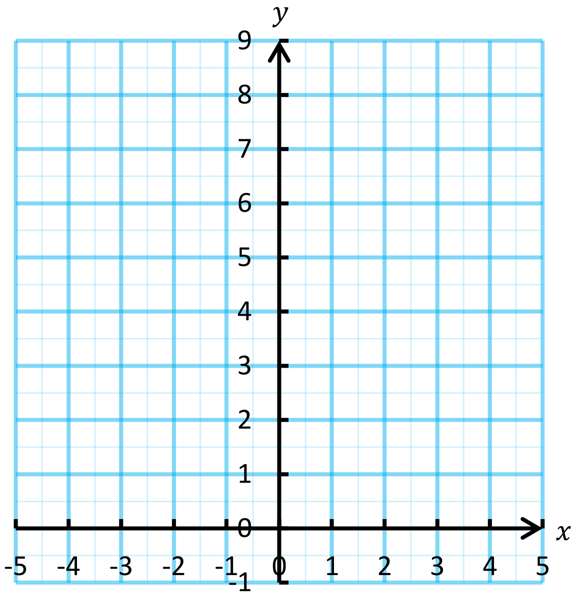
Note: $\left(\frac{1}{2}\right)^x = \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x}$. In general, $a^{-x} = \left(\frac{1}{a}\right)^x$.



Exercise 1

Complete the following table, before plotting the graphs for $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y = 3^x$											
$y = \left(\frac{1}{3}\right)^x$											



Experiment with a graphical calculator, or use the Desmos website, to investigate the shape of $y = a^x$ for different values of a. What happens if $a = 1$?

Euler's Number

Consider investing £1 into a bank that pays a yearly interest rate of 100%. How much money will you have at the end of the year? This depends on the number of instalments over the year.

Instalments	Money after one year
1 (100% after one year)	$1 \times 2^1 = £2$
2 (50% every six months)	$1 \times 1.5^2 = £2.25$
3 ($33.\dot{3}$% every four months)	$1 \times 1.\dot{3}^3 = £2.37$
4 (25% every three months)	$1 \times 1.25^4 = £2.44$
n	$\left(1 + \frac{1}{n}\right)^n$

As n increases, the money increases, but not forever. By modelling (for example using an Excel spreadsheet), we see that the money settles down to the value £2.71 ... This is a special number in mathematics, known as Euler's number, e. To five decimal places, $e = 2.71828$.

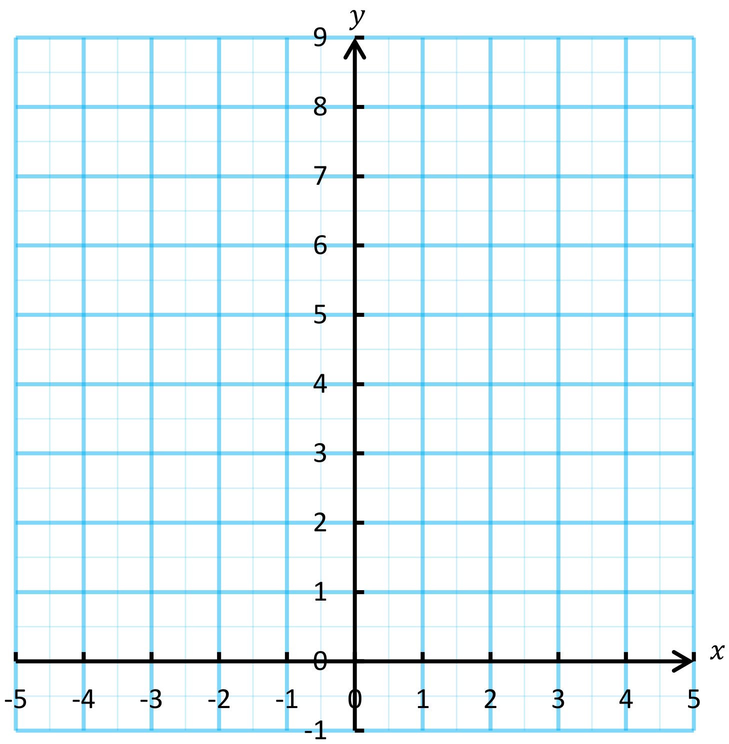
$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

The exponential function $y = e^x$ is special: its gradient is equal to the value of e^x at any point on the graph. In other words, $\frac{d}{dx}(e^x) = e^x$.

Exercise 2

Complete the following table, giving your answers correct to one decimal place, before plotting the graph $y = e^x$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y = e^x$											



Extension: Given that $\frac{d}{dx}(a^x) = \ln(a) \times a^x$, experiment with plotting the graphs $y = a^x$ and $y = \ln(a) \times a^x$ for different values of a.

Any function of the form $y = a^x$ is an exponential function, but the function $y = e^x$ is known as **the** exponential function.

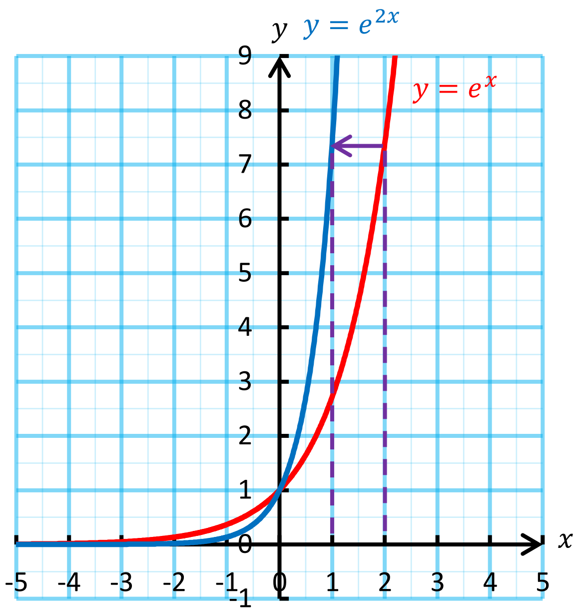
Exponential Models

Starting with the curve for the function $y = e^x$, it is possible to compress it along the x–axis, with scale factor $\frac{1}{k}$, to give the curve for the function $y = e^{kx}$.

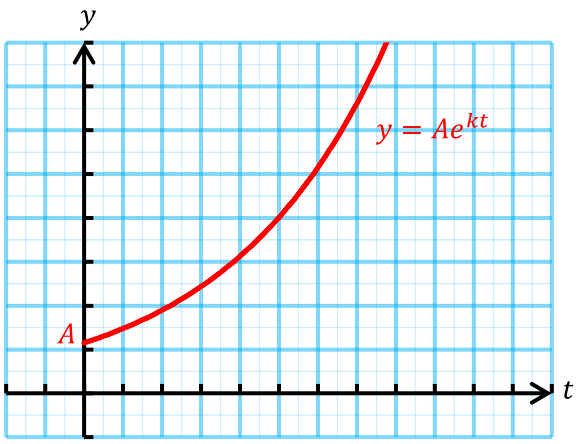
The gradient of the function $y = e^{kx}$ is ke^{kx}. That is, $\frac{dy}{dx} = ke^{kx}$ or $\frac{dy}{dx} \propto e^{kx}$. This is an excellent model for events in the natural world such as radioactive decay or population growth.

In general, an equation of the form $y = Ae^{kt}$, where t represents a specific time, will give an exponential function, where A and k are constants.

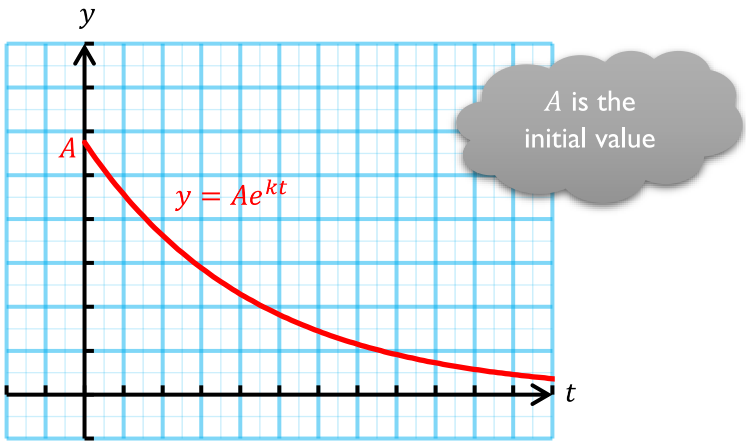
If $y = Ae^{kt}$, then $\frac{dy}{dt} = kAe^{kt}$.



Exponential growth: k is positive



Exponential decay: k is negative



Example 3

The population of a country at the start of a specific year, P million, grows exponentially such that $P = 13e^{0.07t}$, where t represents the time in years after 2020. Calculate

(a) The size of the population at the start of 2026.
(b) The rate of change of the population at the start of 2026.
(c) The average rate of change between the start of 2020 and the start of 2026.

Answer: (a) At the start of 2026, we have $t = 6$. So, $P = 13e^{0.07\times6} = 19.79$ million, to 2 decimal places.

(b) $\frac{dP}{dt} = 0.07 \times 13e^{0.07t} = 0.91e^{0.07t}$. When $t = 6$, we have $\frac{dP}{dt} = 0.91e^{0.07\times6} = 1.38$ million per year, to 2 decimal places.

(c) The change in population between 2020 and 2026 is $19.79 - 13 = 6.79$ million, to 2 decimal places. The average rate of change is $6.79 \div 6 = 1.13$ million per year, to 2 decimal places.

Exercise 3

A drug decays exponentially. Its concentration, C mg per ml, after t hours, is $C = 2 \times e^{-0.35t}$.

(a) What is its initial concentration?

(b) What is the concentration after 3 hours?

(c) What is the rate of change of the drug when (i) $t = 0$, (ii) $t = 3$?

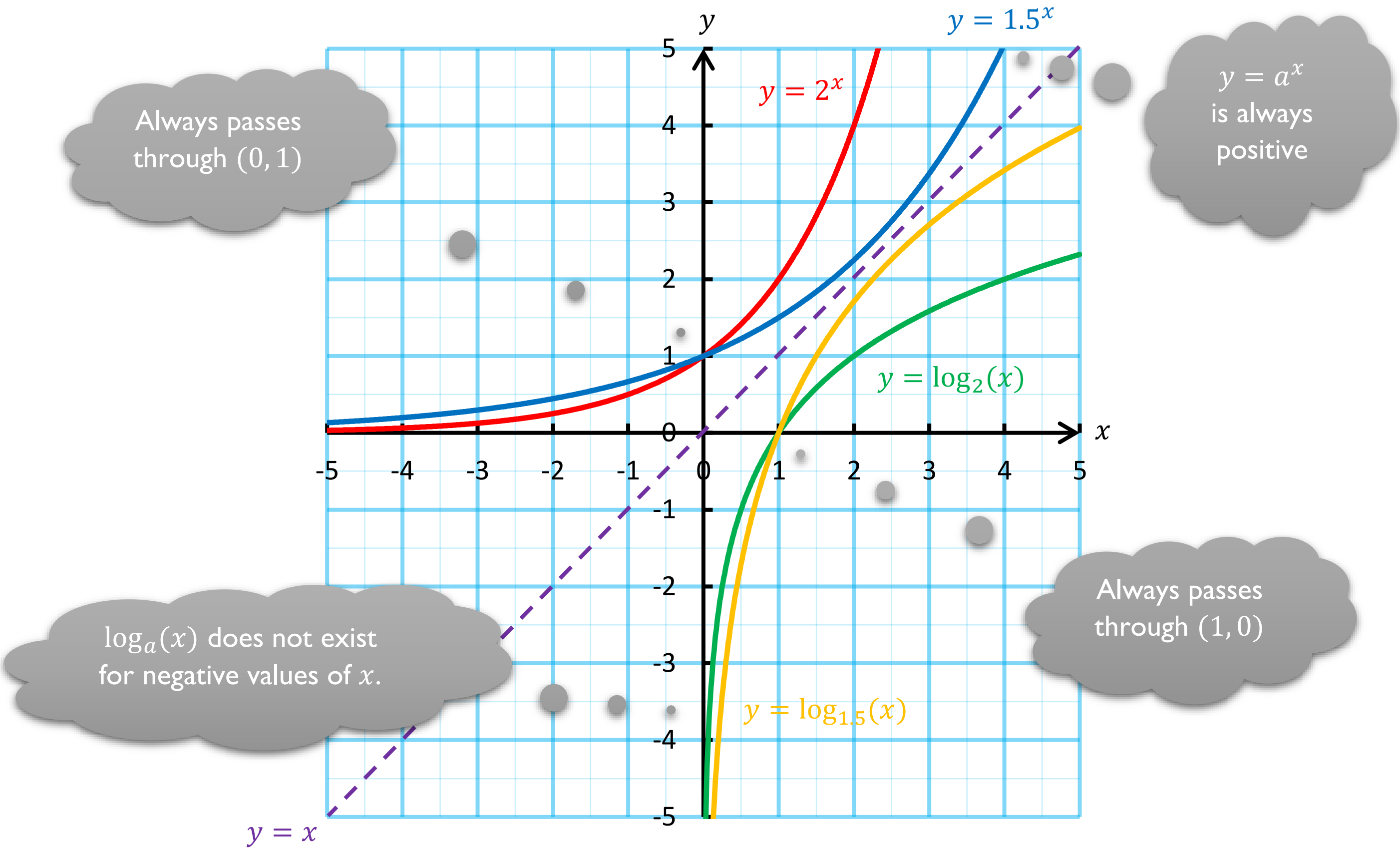
(d) What is the average rate of decrease during the first 3 hours?

Logarithmic Functions

The inverse of an exponential function is a logarithmic function.

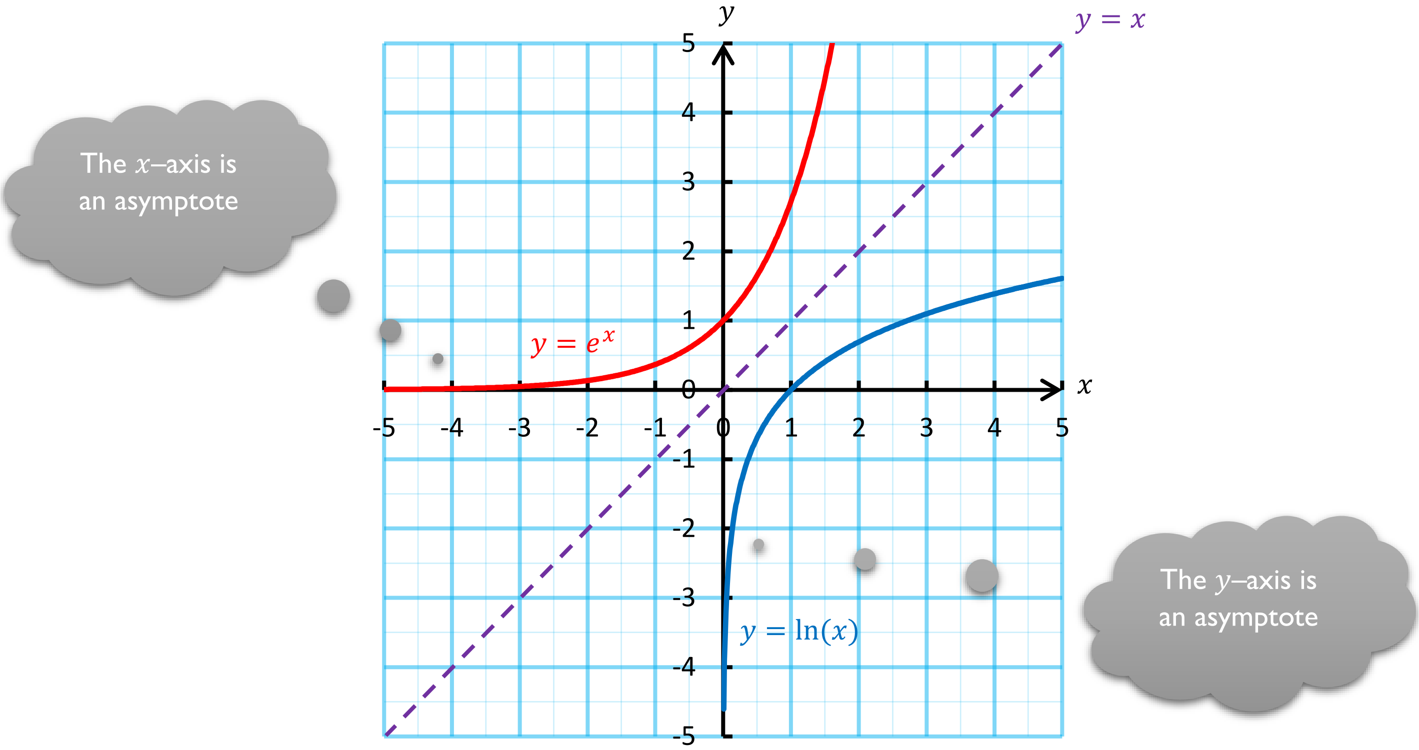
$y = a^x$	
$\log_a(y) = \log_a(a^x)$	Take logarithm base a of each side.
$\log_a(y) = x\log_a(a)$	The power rule.
$\log_a(y) = x$	The definition of a logarithm.

The inverse of the exponential function $y = a^x$ is the logarithmic function $y = \log_a(x)$.



Natural Logarithm

The inverse of $y = e^x$ is $y = \log_e(x)$ which is written as $y = \ln(x)$.
This is known as the **natural logarithm.**



Example 4

Solve the following equations.

(a) $\ln(3x - 1) = 5$

(b) $\log_4(x + 5) = 7$

Answer: (a) $\ln(3x - 1) = 5$

$3x - 1 = e^5$

$3x = e^5 + 1$

$x = \frac{e^5 + 1}{3}$

$x = 49.80$ to 2 decimal places

(b) $\log_4(x + 5) = 7$

$x + 5 = 4^7$

$x = 4^7 - 5$

$x = 16379$

The inverse of $\log_4(x)$ is 4^x

The inverse of $\ln(x)$ is e^x

Exercise 4

Solve the following equations.

(a) $\ln(2x + 5) = 3$

(b) $\log_8(x - 10) = 3$

Top 5

(Unit 1 Summer 2018)

15 The value of a car, £V, may be modelled as a continuous variable. At time t years, the value of the car is given by $V = Ae^{kt}$, where A and k are constants. When the car is new, it is worth £30 000. When the car is two years old, it is worth £20 000. Determine the value of the car when it is six years old, giving your answer correct to the nearest £100. [6]

(Sample Assessment Materials)

15. The size N of the population of a small island at time t years may be modelled by $N = Ae^{kt}$, where A and k are constants. It is known that N = 100 when t = 2 and that N = 160 when t = 12.

(*a*) Interpret the constant A in the context of the question. [1]

(*b*) Show that k = 0·047, correct to three decimal places. [4]

(*c*) Find the size of the population when t = 20. [3]

(Unit 1 Summer 2022)

0 1 Write down the inverse function of $y = e^x$. On the same set of axes, sketch the graphs of $y = e^x$ and its inverse function, clearly labelling the coordinates of the points where the graphs cross the x and y axes. [3]

(Unit 1 Summer 2023)

1	4

a) Two variables, x and y, are such that the rate of change of y with respect to x is proportional to y. State a model which may be appropriate for y in terms of x. [1]

b) The concentration, Y units, of a certain drug in a patient's body decreases exponentially with respect to time. At time t hours the concentration can be modelled by $Y = Ae^{-kt}$, where A and k are constants.

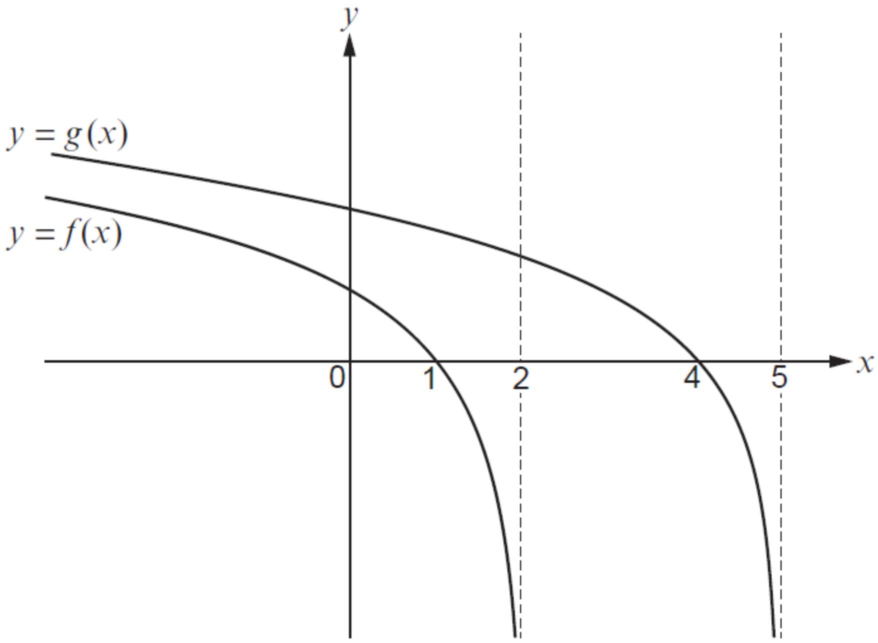
A patient was given a dose of the drug that resulted in an initial concentration of 5 units.

i) After 4 hours, the concentration had dropped to 1·25 units. Show that $k = 0{\cdot}3466$, correct to four decimal places. [2]

ii) The minimum effective concentration of the drug is 0·6 units. How much longer would it take for the drug concentration to drop to the minimum effective level? [3]

(Unit 1 Summer 2024)

17. A function f is defined by $f(x) = \log_{10}(2 - x)$. Another function g is defined by $g(x) = \log_{10}(5 - x)$. The diagram below shows a sketch of the graphs of $y = f(x)$ and $y = g(x)$.



(a) The point $(c, 1)$ lies on $y = f(x)$. Find the value of c. [2]

(b) A point P lies on $y = f(x)$ and has x-coordinate α. Another point Q lies on $y = g(x)$ and also has x-coordinate α. The distance between P and Q is 1·2 units. Find the value of α, giving your answer correct to three decimal places. [5]

Curve Fitting

Given values for x and y, perhaps from an experiment, it is possible to use logarithms to see if the data fits a relationship of the form $y = ax^n$ or $y = kb^x$.

(1) Is the data of the form $y = ax^n$ (a polynomial model)?

$y = ax^n$

$\log(y) = \log(ax^n)$ Take logarithm base 10 of each side

$\log(y) = \log(a) + \log(x^n)$ The multiplication rule

$\log(y) = \log(a) + n\log(x)$ The power rule

$\log(y) = n\log(x) + \log(a)$ Re-arranging

Comparing with $y = mx + c$, plotting $\log(x)$ against $\log(y)$ would give a straight line with gradient n and y–intercept $\log(a)$.

Example 5

Data was collected in an experiment about the variables x and y.
Use the values in the table to write the relationship between x and y in the form $y = ax^n$.

x	10	20	30	40	50
y	158	549	1140	1913	2858

We can form the following table to show the values of $\log(x)$ and $\log(y)$, and then plot them on a scatter diagram.

$\log(x)$	1	1.3010	1.4771	1.6021	1.6990
$\log(y)$	2.1987	2.7396	3.0569	3.2817	3.4561

The gradient of the graph is approximately $3.6 \div 2 = 1.8$.

The y–intercept of the graph is approximately 0.4.

Comparing with

$$\log(y) = n\log(x) + \log(a)$$

we have

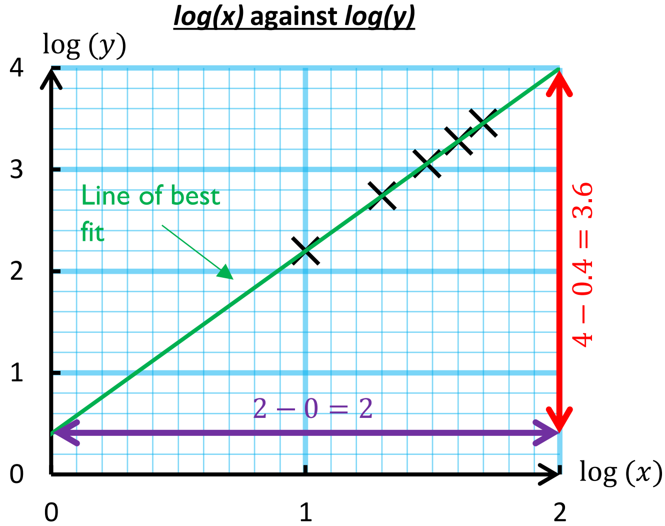
$$\log(y) = 1.8\log(x) + 0.4.$$

So, $a = 10^{0.4}$
$a = 2.51$ to 2 decimal places.

Comparing with $y = ax^n$, our model is

$$y = 2.51x^{1.8}$$

Scatter Diagram for *log(x)* against *log(y)*



(2) Is the data of the form $y = kb^x$ (an exponential model)?

$y = kb^x$

$\log(y) = \log(kb^x)$ Take logarithm base 10 of each side

$\log(y) = \log(k) + \log(b^x)$ The multiplication rule

$\log(y) = \log(k) + x\log(b)$ The power rule

$\log(y) = \log(b)\,x + \log(k)$ Re-arranging

Compring with $y = mx + c$, plotting x against $\log(y)$ would give a straight line with gradient $\log(b)$ and y–intercept $\log(k)$.

Example 6

Data was collected in an experiment about the variables x and y.
Use the values in the table to write the relationship between x and y in the form $y = kb^x$.

x	2	4	6	8	10
y	9	53	306	1,761	10,145

We can form the following table to show the values of x and $\log(y)$, and then plot them on a scatter diagram.

x	2	4	6	8	10
$\log(y)$	0.9542	1.7243	2.4857	3.2458	4.0063

The gradient of the graph is approximately $3.8 \div 10 = 0.38$.

The y– intercept of the graph is approximately 0.2.

Comparing with

$\log(y) = \log(b)\,x + \log(k)$

we have

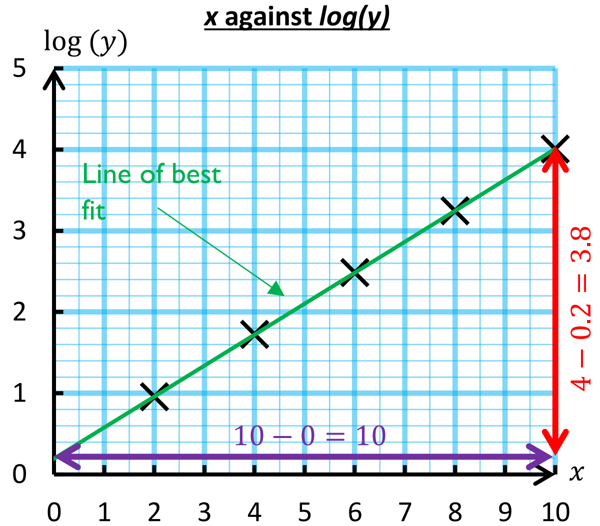
$\log(y) = 0.38x + 0.2$.

So, $b = 10^{0.38}$ $k = 10^{0.2}$
$b = 2.40$ to 2 d.p. $k = 1.58$ to 2 d.p.

Comaparing with $y = kb^x$, our model is

$$y = 1.58 \times 2.4^x$$

Scatter Diagram for *x* against *log(y)*



Exercise 5

A pipe empties water into a lake. The table shows the volume of water, y m³, that leaves the pipe each second, for different depths of water, x m, in the pipe. Engineers expect a relationship of the form $y = ax^n$ between x and y.

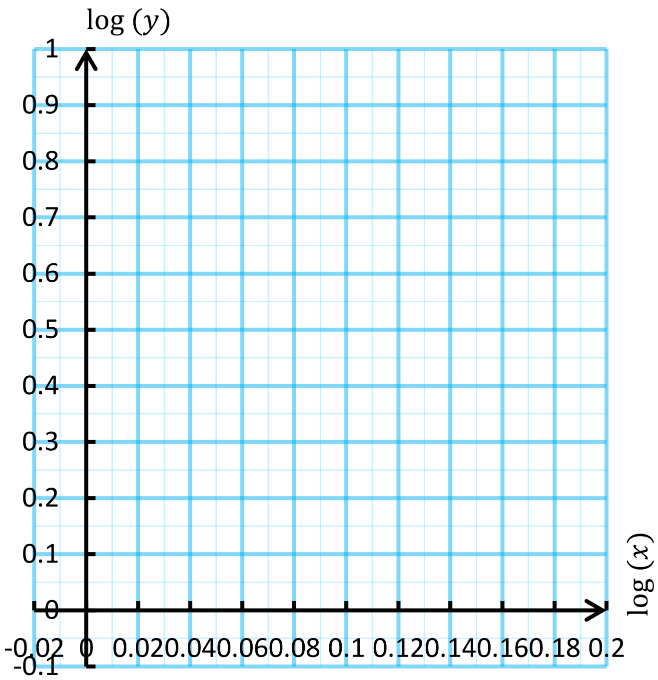
x	1.1	1.2	1.3	1.4	1.5
y	4.73	5.78	6.95	8.24	9.66

(a) Show, by drawing a line of best fit on suitable axes, that the engineers are correct.
Find the values of the constants a and n.

(b) What volume of water (in m³) would you expect to leave the pipe each second when the depth of water is (i) 1.45 metres; (ii) 3.5 metres?

(c) Which one of your answers to part (b) is most reliable? Explain your answer.

log (x)					
log (y)					



Exercise 6

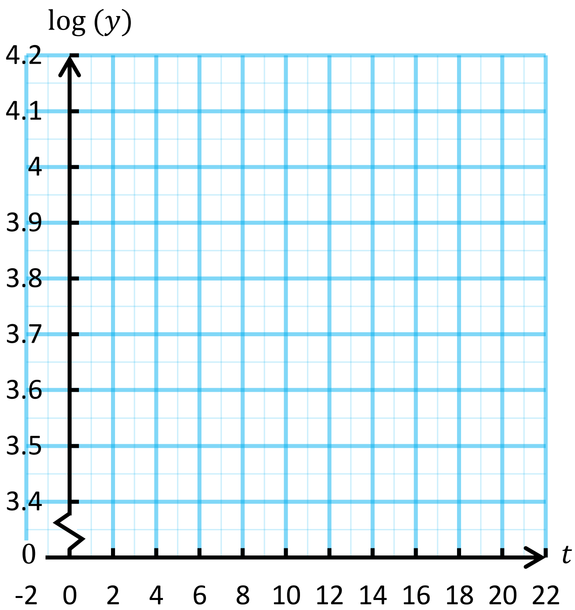
The population of Mathdref has been increasing since 1990.

Year	1990	1995	2005	2010
Time, t	0	5	15	20
Population, p	3170	4242	7597	10167

(a) Show that the population since 1990 can be modelled using $p = k \times b^t$, where you are required to find the value of the constants k and b.

(b) Estimate the population of Mathdref in 2000 and in 2020. Which one of your answers gives the best estimate? Explain why.

t				
log (y)				



Exercises

(Edexcel Paper 1 Summer 2023 [9MA0/01])

11.

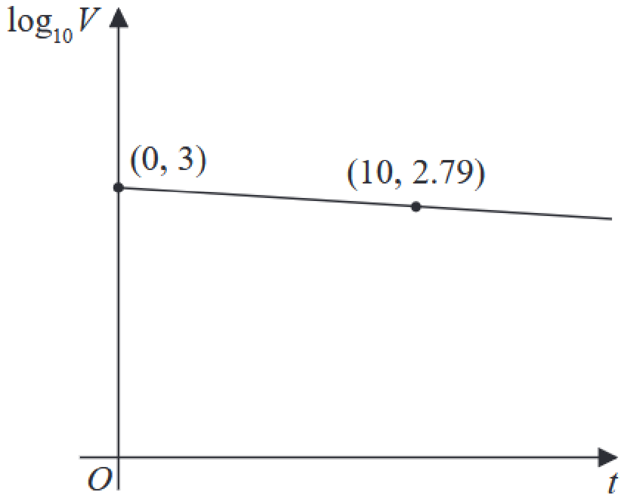


Figure 2

The value, V pounds, of a mobile phone, t months after it was bought, is modelled by

$$V = ab^t$$

where a and b are constants.

Figure 2 shows the linear relationship between $\log_{10} V$ and t.

The line passes through the points (0, 3) and (10, 2.79) as shown.

Using these points,

(a) find the initial value of the phone,

(2)

(b) find a complete equation for V in terms of t, giving the exact value of a and giving the value of b to 3 significant figures.

(3)

(Unit 1 Summer 2019)

1 1 Two quantities are related by the equation $Q = 1{\cdot}25P^3$. Explain why the graph of $\log_{10}Q$ against $\log_{10}P$ is a straight line. State the gradient of the straight line and the intercept on the $\log_{10}Q$ axis of the graph. [4]

Unit 1, Workbook 16

12

The Binomial Theorem

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Name:

Background

What is the work?	Expanding expressions of the form $(a+bx)^n$ efficiently.
What is required before starting?	**GCSE Work:** Expanding brackets; collecting like terms. **A Level Unit 1:** Introducing logarithms. **A Level Unit 2:** Binomial distribution probabilities.
Where does this lead to?	**A Level Unit 3:** Expanding $(a+bx)^n$ in cases where n is not a positive integer.

Theory

It would be possible to expand $(1+x)^5$ using GCSE techniques…

Theory

$$
\begin{aligned}
(1+x)^5 &= (1+x)(1+x)(1+x)(1+x)(1+x)\\
(1+x)^5 &= (1+x)(1+x)(1+x)(1+x+x+x^2)\\
(1+x)^5 &= (1+x)(1+x)(1+x)(1+2x+x^2)\\
(1+x)^5 &= (1+x)(1+x)(1+2x+x^2+x+2x^2+x^3)\\
(1+x)^5 &= (1+x)(1+x)(1+3x+3x^2+x^3)\\
(1+x)^5 &= (1+x)(1+3x+3x^2+x^3+x+3x^2+3x^3+x^4)\\
(1+x)^5 &= (1+x)(1+4x+6x^2+4x^3+x^4)\\
(1+x)^5 &= 1+4x+6x^2+4x^3+x^4+x+4x^2+6x^3+4x^4+x^5\\
(1+x)^5 &= 1+5x+10x^2+10x^3+5x^4+x^5
\end{aligned}
$$

…but it would be tedious! A better method would be to use the pattern seen in the above coefficients.

Pascal's Triangle

In Pascal's triangle, each number is the sum of the above two numbers.

Named after the French mathematician Blaise Pascal.

Exercise 1

Complete the following Pascal's triangle, up to the row starting 1, 10, …

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

There is a connection between the numbers appearing in Pascal's triangle and the expansion for $(1+x)^n$.
For example, note the connection between the example $(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$ on the previous page and the row 1, 5, 10, 10, 5, 1 in Pascal's triangle.

Exercise 2

Expand $(1+x)^8$ using your Pascal's triangle from Exercise 1.

Formally, we call the top row in Pascal's triangle (the one containing only 1) row 0, and then row 1, row 2, row 3 and so on follow. So, to expand $(1+x)^5$, we use the **fifth** row in the triangle, the one starting with 1 and 5.

In the nth row of Pascal's triangle, the rth number is written as $\binom{n}{r}$. Again, we start counting from 0, so that the first number in the 4th row is $\binom{4}{0}$. The rule that we use to form the triangle is

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

or

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

In the formula booklet

We can think of $\binom{n}{r}$ as how many ways there are of choosing r items from n items. For example, $\binom{5}{2} = 10$ because there are 10 ways of choosing 2 items out of 5 items:

$$1,2 \quad 1,3 \quad 1,4 \quad 1,5 \quad 2,3 \quad 2,4 \quad 2,5 \quad 3,4 \quad 3,5 \quad 4,5$$

The following formula gives a method for calculating $\binom{n}{r}$ without having to form Pascal's triangle each time.

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

In the formula booklet

$n!$ represents 'n factorial' and is calculated as follows:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

So, for example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Notes

- It is possible to write $\binom{n}{r}$ as ${}^nC_r$. Probably, this is what appears on your calculator in order to calculate $\binom{n}{r}$. (C stands for '*choose*' or '*combination*'.)
- $\binom{n}{r}$ appears in the formula $P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$ for calculating probabilities in the Binomial probability distribution.

Exercise 3

(a) Calculate $6!$

(b) Calculate $\binom{7}{3}$

(c) Calculate $\binom{9}{5}$

Expanding $(a+bx)^n$

> In the formula booklet, with b instead of x

If n is a positive integer, then

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + x^n$$

$$(a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \cdots + x^n$$

$$(a+bx)^n = a^n + \binom{n}{1}a^{n-1}(bx) + \binom{n}{2}a^{n-2}(bx)^2 + \binom{n}{3}a^{n-3}(bx)^3 + \cdots + (bx)^n$$

Example 1

Expand $\left(3+\frac{x}{4}\right)^5$.

Answer: Comparing with $(a+bx)^n$, we have $a=3$, $b=\frac{1}{4}$, $n=5$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$\left(3+\frac{x}{4}\right)^5 = 3^5 + 5(3^4)\left(\frac{x}{4}\right) + 10(3^3)\left(\frac{x}{4}\right)^2 + 10(3^2)\left(\frac{x}{4}\right)^3 + 5(3^1)\left(\frac{x}{4}\right)^4 + \left(\frac{x}{4}\right)^5$$

$$\left(3+\frac{x}{4}\right)^5 = 243 + 5(81)\left(\frac{x}{4}\right) + 10(27)\left(\frac{x^2}{16}\right) + 10(9)\left(\frac{x^3}{64}\right) + 5(3)\left(\frac{x^4}{256}\right) + \left(\frac{x^5}{1024}\right)$$

$$\left(3+\frac{x}{4}\right)^5 = 243 + \frac{405x}{4} + \frac{135x^2}{8} + \frac{45x^3}{32} + \frac{15x^4}{256} + \frac{x^5}{1024}$$

Exercise 4

Expand $(2+5x)^4$.

Eliminating Combinations

We can use the general defintion

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

to eliminate the combinations from the binomial expansion on the previous page. For example,

$$\binom{n}{4} = \frac{n!}{4!\,(n-4)!}$$

$$\binom{n}{4} = \frac{n \times (n-1) \times (n-2) \times (n-3) \times \cancel{(n-4)} \times \cancel{(n-5)} \times \cdots \times \cancel{3} \times \cancel{2} \times \cancel{1}}{[4 \times 3 \times 2 \times 1] \times [\cancel{(n-4)} \times \cancel{(n-5)} \times \cdots \times \cancel{3} \times \cancel{2} \times \cancel{1}]}$$

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{4 \times 3 \times 2 \times 1}$$

So, if n is a positive integer, then

$$(a+bx)^n = a^n + na^{n-1}(bx) + \frac{n(n-1)}{2 \times 1}a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3 \times 2 \times 1}a^{n-3}(bx)^3 + \cdots + (bx)^n$$

If $a = 1$, $b = 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2 \times 1}x^2 + \frac{n(n-1)(n-2)}{3 \times 2 \times 1}x^3 + \cdots + x^n$$

A version of the above is given in the formula booklet:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \cdots \qquad (|x| < 1, n \in \mathbb{R})$$

A full stop represents multiplication here

Example 2

In the binomial expansion of $(1+2x)^n$, the coefficient of x^2 is double the coefficient of x.
Given that $n > 0$, find the value of n.

Answer: $(1+2x)^n = 1^n + n(1^{n-1})(2x) + \frac{n(n-1)}{2\times1}(2x)^2 + \cdots$

$(1+2x)^n = 1 + n(1)(2x) + \frac{n(n-1)}{2}(4x^2) + \cdots$

$(1+2x)^n = 1 + 2nx + 2n(n-1)x^2 + \cdots$

If the coefficient of x^2 is double the coefficient of x, then

$2n(n-1) = 2(2n)$
$2n^2 - 2n = 4n$
$2n^2 - 6n = 0$
$2n(n-3) = 0$
Either $2n = 0$ or $n - 3 = 0$
$n = 0$ $\quad$ $n = 3$
But $n > 0$, so $n = 3$.

Top 10

(C1 Summer 2008)

6. Use the binomial theorem to expand $(5 + 2x)^3$, simplifying each term of your expansion. [3]

(C1 Winter 2009)

6. *(a)* Expand $(a + b)^5$. [2]

(b) Use your answer to part *(a)* to find the coefficient of x^3 in the expansion of $\left(\frac{1}{4} + 2x\right)^5$.

Simplify your answer. [2]

(Unit 1 Summer 2019)

1 2 In the binomial expansion of $(2 - 5x)^8$, find

a) the number of terms, [1]

b) the 4th term, when the expansion is in ascending powers of x, [2]

c) the greatest positive coefficient. [3]

(Unit 1 Summer 2022)

1 3 Find the term which is independent of x in the expansion of $\dfrac{(2-3x)^5}{x^3}$. [4]

(C1 Summer 2010)

4. *(a)* Write down the expansion of $(1 + x)^6$ in ascending powers of x up to and including the term in x^3. [2]

(b) By substituting an appropriate value for x in your expansion in *(a)*, find an approximate value for $0{\cdot}99^6$. **Show all your working** and give your answer correct to four decimal places. [3]

(Unit 1 Summer 2023)

0 1 **a)** Using the binomial theorem, write down and simplify the first three terms in the expansion of $(1 - 3x)^9$ in ascending powers of x. [3]

b) Hence, by writing $x = 0{\cdot}001$ in your expansion in part (a), find an approximate value for $(0{\cdot}997)^9$. Show all your working and give your answer correct to three decimal places. [3]

(C1 Winter 2010)

7. In the binomial expansion of $(a + 3x)^5$, the coefficient of the term in x^2 is eight times the coefficient of the term in x. Find the value of the constant a. [4]

(C1 Summer 2009)

7. *(a)* Expand $\left(x+\dfrac{2}{x}\right)^4$, simplifying each term of the expansion. [4]

(b) The coefficient of x^2 in the expansion of $(1 + x)^n$ is 55. Given that n is a positive integer, find the value of n. [3]

(C1 Summer 2015)

6. *(a)* Using the binomial theorem, write down and simplify the first four terms in the expansion of $\left(1-\frac{x}{2}\right)^8$ in ascending powers of x. [4]

(b) The first two terms in the expansion of $(2 + ax)^n$ in ascending powers of x are 32 and $-240x$ respectively. Find the value of n and the value of a. [4]

(Unit 1 Summer 2018)

10 **a)** Use the binomial theorem to expand $\left(a+\sqrt{b}\right)^4$. [2]

b) Hence, deduce an expression in terms of a and b for $\left(a+\sqrt{b}\right)^4+\left(a-\sqrt{b}\right)^4$. [2]

Revision Question

(C1 Winter 2007)

4. *(a)* Expand $(a+b)^4$, simplifying your coefficients as much as possible. [2]

(b) Solve $(2+x)^4 = 14 + 33x + 25x^2 + 8x^3 + x^4$. [4]

Unit 1, Workbook 17

12

Three Types of Proof

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= b(a-b) \\ a+b &= b \\ 2b &= b \\ 2 &= 1 \end{aligned}$$

Name:

Background

What is the work?	Introducing how to prove propositions in mathematics.
What is required before starting?	**GCSE Work:** Expanding brackets; collecting like terms; changing the subject.
Where does this lead to?	**A Level Unit 3:** Proof by contradiction. **Applications:** Proving e.g. Pythagoras' theorem.

Theory

You need to be able to write three types of proof in Unit 1.

1) Proof by deduction.
2) Proof by exhaustion.
3) Disproof by counter-example.

Theory

(1) Proof by deduction

This type of proof uses reasoning (usually algebraic skills) to show that a statement is true.

Example 1

Prove that the sum of three consecutive whole numbers is a multiple of 3.

Proof

Let the first whole number be represented by the variable n.
It follows that the second whole number is $n+1$ and the third whole number is $n+2$.

The sum of the three consecutive whole numbers is

$$\begin{aligned} n + (n+1) + (n+2) &= 3n + 3 \\ &= 3(n+1) \end{aligned}$$

This sum has been written as a multiple of three (3 multipled by $n+1$) so the statement is true.

QED.

Quod Erat Demonstrandum:
From the Latin: "I have shown what needed to be shown".

(2) Proof by exhaustion

This type of proof looks at every possible case, in turn, and proves each one.
It only works if there are a small number of cases to consider.

Example 2

Prove, if n is a whole number between 1 and 5, that $n^2 + 3n + 1$ is always a prime number.

Proof

Let us form a table to consider each case, in turn.

$\boldsymbol{n}$	$\boldsymbol{n^2 + 3n + 1}$
1	$1^2 + 3 \times 1 + 1 = 5$
2	$2^2 + 3 \times 2 + 1 = 11$
3	$3^2 + 3 \times 3 + 1 = 19$
4	$4^2 + 3 \times 4 + 1 = 29$
5	$5^2 + 3 \times 5 + 1 = 41$

The numbers 5, 11, 19, 29 and 41 are all prime numbers (their only factors are 1 and themselves), so we have proven the statement.

QED.

(3) Disproof by counter-example

This type of proof proves that something is false by giving an example where it is false.

Example 3

Let the variable n represent a whole number. Twm thinks that $n^2 + 3n + 1$ is always a prime number.
Prove, by giving an appropriate counter-example, that this is false.

Proof

Let us consider the case $n = 6$.

$6^2 + 3 \times 6 + 1 = 55$.

55 is not a prime number (it has more than two factors, namely 1, 5, 11 and 55), so the statement is false

QED.

Top 5

(Unit I Summer 2019)

0 5 Given that n is an integer such that $1 \leqslant n \leqslant 4$, prove that $2n^2 + 5$ is a prime number. [3]

(Unit I Summer 2018)

0 7 Prove that

$$\frac{\sin^3\theta + \sin\theta\cos^2\theta}{\cos\theta} \equiv \tan\theta.$$ [3]

(Sample Assessment Materials)

6. In each of the two statements below, c and d are real numbers. One of the statements is true while the other is false.

A Given that $(2c + 1)^2 = (2d + 1)^2$, then $c = d$.
B Given that $(2c + 1)^3 = (2d + 1)^3$, then $c = d$.

(*a*) Identify the statement which is false. Find a counter example to show that this statement is in fact false.

(*b*) Identify the statement which is true. Give a proof to show that this statement is in fact true. [5]

(Unit 1 Summer 2018)

1 4 In each of the two statements below, c and d are real numbers. One of the statements is true, while the other is false.

A : $(2c - d)^2 = 4c^2 - d^2$, for all values of c and d.

B : $8c^3 - d^3 = (2c - d)(4c^2 + 2cd + d^2)$, for all values of c and d.

a) Identify the statement which is false. Show, by counter example, that this statement is in fact false. [2]

b) Identify the statement which is true. Give a proof to show that this statement is in fact true. [2]

(Unit 1 Summer 2022)

0 6 In each of the two statements below, x and y are real numbers. One of the statements is true while the other is false.

A : $x^2 + y^2 \geqslant 2xy$, for all real values of x and y.

B : $x + y \geqslant 2\sqrt{xy}$, for all real values of x and y.

a) Identify the statement which is false. Find a counter example to show that this statement is in fact false. [3]

b) Identify the statement which is true. Give a proof to show that this statement is in fact true. [2]

Revision Questions

(Unit 1 Summer 2023)

0 8 Show, by counter example, that the following statement is false.

"For all positive integer values of n, n^2+1 is a prime number." [3]

(Unit 1 Summer 2024)

4. Given that n is an integer such that $1 \leqslant n \leqslant 6$, use proof by exhaustion to show that n^2-2 is not divisible by 3. [3]