Unit 3, Workbook 19

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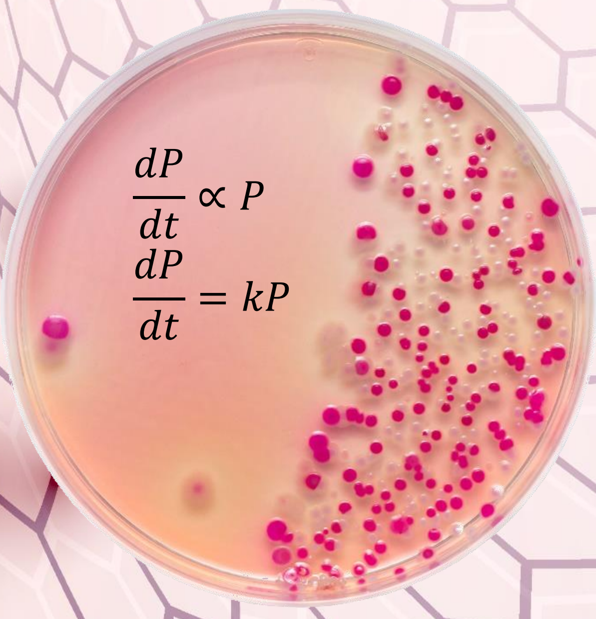
Differential Equations



$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = kP$$

Name:

Background

What is the work?	Solving **first order differential equations**, which are equations containing a function and its derivative.
What is required before starting?	**A Level Unit 1:** Differentiation and integration; logarithms. **A Level Unit 3:** Further integration.
Where does this lead to?	**Further Mathematics:** solving second order differential equations. **Applications:** Modelling electrical circuits; forces on beams.

Theory

A **differential equation** connects a function y to its derivative $\frac{dy}{dx}$. By separating the variables and integrating, we can find an equation that connects x and y.

Example (C4 Summer 2005)

The size P of a population of bacteria at time t days is to be modelled as a continuous variable such that the rate of increase of P is directly proportional to P.

(a) Write down a differential equation that is satisfied by P.

(b) Given that the initial size of the population is P_0, show that $P = P_0 e^{kt}$, where k is a positive constant.

(c) Two days after the start, the population is $1.2P_0$. Find when the population will be $2P_0$.

Answer: (a) If the rate of increase of P at time t days is directly proportional to P, then

$$\frac{dP}{dt} \propto P$$
$$\frac{dP}{dt} = kP$$

(b) Separating the variables in the differential equation above,

$$dP = kP\,dt$$
$$\frac{1}{P}dP = k\,dt$$

Integrating,

$$\int \frac{1}{P}dP = \int k\,dt$$
$$\ln|P| = kt + c$$

The initial size of the population is P_0, so $P = P_0$ when $t = 0$.

$$\ln|P_0| = k(0) + c$$
$$\ln|P_0| = c$$

Theory

Therefore

$$\ln|P| = kt + \ln|P_0|$$
$$\ln|P| - \ln|P_0| = kt$$
$$\ln\left|\frac{P}{P_0}\right| = kt$$
$$\frac{P}{P_0} = e^{kt}$$
$$P = P_0 e^{kt}$$

(c) If $t = 2$, $P = 1.2P_0$. Therefore

$$1.2P_0 = P_0 e^{k(2)}$$
$$1.2 = e^{2k}$$
$$\ln(1.2) = 2k$$
$$\frac{1}{2}\ln(1.2) = k$$

Therefore

$$P = P_0 e^{\frac{1}{2}\ln(1.2)t}$$

If $P = 2P_0$,

$$2P_0 = P_0 e^{\frac{1}{2}\ln(1.2)t}$$
$$2 = e^{\frac{1}{2}\ln(1.2)t}$$
$$\ln(2) = \frac{1}{2}\ln(1.2)\,t$$
$$\frac{2\ln(2)}{\ln(1.2)} = t$$

$t = 7.60$ days, to 2 decimal places.

Establishing a Differential Equation

Exercise 1

Complete the following table.

Description	Proportional Relationship	Differential Equation
The rate of increase of P at time t is directly proportional to P^2.	$\frac{dP}{dt} \propto P^2$	$\frac{dP}{dt} = kP^2$
The rate of increase of P at time t is inversely proportional to P^2.	$\frac{dP}{dt} \propto \frac{1}{P^2}$	$\frac{dP}{dt} = \frac{k}{P^2}$
The rate of decrease of P at time t is directly proportional to P^2.	$\frac{dP}{dt} \propto -P^2$	$\frac{dP}{dt} = -kP^2$
The rate of decrease of P at time t is inversely proportional to P^2.	$\frac{dP}{dt} \propto -\frac{1}{P^2}$	$\frac{dP}{dt} = -\frac{k}{P^2}$
The rate of increase of P at time t is directly proportional to P^3.		
The rate of decrease of P at time t is inversely proportional to P.		
The rate of increase of P at time t is inversely proportional to P^4.		
The rate of decrease of P at time t is directly proportional to $\sqrt{P}$.		

Top 10

(C4 Summer 2006)

8. Water leaks from a hole at the bottom of a large water tank. The depth of the water at time t minutes is x metres. The rate of decrease of x is directly proportional to $\sqrt{x}$.

(a) Write down a differential equation that is satisfied by x. [1]

(b) Given that the depth of water in the tank when $t = 0$ is 9 metres, show that

$$kt = 6 - 2\sqrt{x} \ ,$$

where k is a positive constant. [4]

(c) Given that the depth of water in the tank is 4 metres when $t = 20$, find the time taken for the tank to empty. [3]

(C4 Summer 2007)

8. The price £P of an item at time t years is to be modelled as a continuous variable such that the rate of increase of P is directly proportional to P.

(a) Write down a differential equation that is satisfied by P. [1]

(b) Given that the price of the item at $t = 0$ is £50, show that $P = 50e^{kt}$, where k is a positive constant. [5]

(c) After seven years the price of the item is £65. Find the price of the item after sixteen years. [4]

(C4 Summer 2009)

7. The value of an electronic component may be modelled as a continuous variable. The value of the component at time t years is £P. The rate of decrease of P is directly proportional to P^3.

(a) Write down a differential equation that is satisfied by P. [1]

(b) The value of the component when $t = 0$ is £20. Show that

$$\frac{1}{P^2} = \frac{1}{400} + At,$$

where A is a positive constant. [5]

(c) Given that the value of the component when $t = 1$ is £10, find the time when the value is £5. [4]

(C4 Summer 2010)

8. The value, £V, of a car may be modelled as a continuous variable. At time t years, the rate of decrease of V is directly proportional to V^2.

(a) Write down a differential equation satisfied by V. [1]

(b) Given that $V = 12000$ when $t = 0$, show that

$$V = \frac{12000}{at + 1},$$

where a is a constant. [4]

(c) The value of the car at the end of two years is £9000. Find the value of the car at the end of four years. [4]

(C4 Summer 2013)

8. Part of the surface of a small lake is covered by green algae. The area of the lake covered by the algae at time t years is $A\,\mathrm{m}^2$. The rate of increase of A is directly proportional to $\sqrt{A}$.

(a) Write down a differential equation satisfied by A. [1]

(b) The area of the lake covered by the algae at time $t = 3$ is $64\,\mathrm{m}^2$ and the area covered at time $t = 5{\cdot}5$ is $196\,\mathrm{m}^2$. Find an expression for A in terms of t. [6]

(C4 Summer 2015)

9. A bookseller values a rare book at £A on August 1st 2010. The value, £P, of the book t years after this date may be modelled as a continuous variable. The rate of increase of P may be assumed to be directly proportional to P^2.

(a) Write down a differential equation satisfied by P. [1]

(b) Show that

$$\frac{1}{k}\left(\frac{P-A}{PA}\right)=t,$$

where k is a constant. [4]

(c) The value of the book is £800 on August 1st 2013 and £900 on August 1st 2014. Find the value of A. [3]

(C4 Summer 2016)

7. The value, £V, of a particular car may be modelled as a continuous variable. At time t years, the rate of decrease of V is directly proportional to V^3.

(a) Write down a differential equation satisfied by V. [1]

(b) Given that the initial value of the car is £A, show that

$$V^2 = \frac{A^2}{bt+1},$$

where b is a constant. [4]

(c) When $t = 2$, the value of the car has fallen to a half of its initial value. Find the value of t when the value of the car will have fallen to a quarter of its initial value. [4]

(C4 Summer 2018)

8. The value of a painting on January 1st 2000 was £900. The value, £V, of the painting t years after this date may be modelled as a continuous variable. The rate of increase of V may be assumed to be directly proportional to $V^{\frac{3}{2}}$.

(a) Write down a differential equation satisfied by V. [1]

(b) The value of the painting on January 1st 2003 was £1600.
Find its value on January 1st 2008. [8]

(Unit 3 Summer 2018)

1 5 The variable y satisfies the differential equation

$$2\frac{\mathrm{d}y}{\mathrm{d}x} = 5 - 2y, \qquad \text{where } x \geqslant 0.$$

Given that $y = 1$ when $x = 0$, find an expression for y in terms of x. [5]

(Unit 3 Summer 2019)

13 Wildflowers grow on the grass verge by the side of a motorway. The area populated by wildflowers at time t years is A m^2. The rate of increase of A is directly proportional to A.

a) Write down a differential equation that is satisfied by A. [1]

b) At time $t = 0$, the area populated by wildflowers is $0{\cdot}2\,\text{m}^2$. One year later, the area has increased to $1{\cdot}48\,\text{m}^2$. Find an expression for A in terms of t in the form pq^t, where p and q are rational numbers to be determined. [7]

Revision Question

(Unit 3 Summer 2023)

12 The rate of change of a variable y with respect to x is directly proportional to y.

a) Write down a differential equation satisfied by y. [1]

b) When $x = 1$ and $y = 0{\cdot}5$, the rate of change of y with respect to x is 2.
Find y when $x = 3$. [6]