YSGOL CREUDDYN
DAWN DYSG DAIONI

Unit 3, Workbook 18

13

Further Integration 2

$$\int f'\big(g(x)\big)g'(x)\,dx = f\big(g(x)\big) + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Name:

Background

What is the work?	Reversing the chain rule and the product rule to **integrate by substitution** and **integrate by parts**.
What is required before starting?	**A Level Unit 1:** Differentiation and integration. **A Level Unit 3:** Further differentiation and integration.
Where does this lead to?	Integration by parts can be extended to produce reduction formulae than can integrate expressions such as $\cos^n x$.

Theory

In cases where we are integrating something that includes **a function and its derivative**, we can try to find the answer through **integration by substitution**. We show how this works using the following example.

Example 1

$$\int (24x^2+9)e^{8x^3+9x}\,dx$$

We recognise that the integral contains the function $f(x) = 8x^3 + 9x$ and its derivative $f'(x) = 24x^2 + 9$. We choose the substitution based on this function: $u = 8x^3 + 9x$.

By differentiating u, we obtain

$$\frac{du}{dx} = 24x^2 + 9$$

We can re-arrange the above to make dx the subject:

$$du = (24x^2+9)\,dx$$
$$\frac{du}{24x^2+9} = dx$$

We now substitute u instead of $8x^3 + 9x$ and $\frac{du}{24x^2+9}$ instead of dx in the original integral:

$$\int (24x^2+9)e^{8x^3+9x}\,dx$$
$$= \int \cancel{(24x^2+9)}\,e^u\,\frac{du}{\cancel{24x^2+9}}$$
$$= \int e^u du$$

e^u integrates (with respect to u) to give e^u. Therefore

$$\int e^u\,du = e^u + c$$

Theory

To finish, we substitute back for u:

$$\int e^u \, du = e^{8x^3+9x} + c$$

Therefore

$$\int (24x^2 + 9)e^{8x^3+9x} \, dx = e^{8x^3+9x} + c$$

Exercise 1

Use integration by substitution to show that

$$\int 12x \cos(6x^2 - 4) \, dx = \sin(6x^2 - 4) + c$$

In general, we can **integrate by substitution** when we integrate an expression of the form $f'\big(g(x)\big)g'(x)$.

$$\int f'\big(g(x)\big)g'(x) \, dx = f\big(g(x)\big) + c$$

To show why the above is true, we can use the chain rule to differentiate $y = f\big(g(x)\big) + c$.

Let $u = g(x)$ so that $y = f(u) + c$.

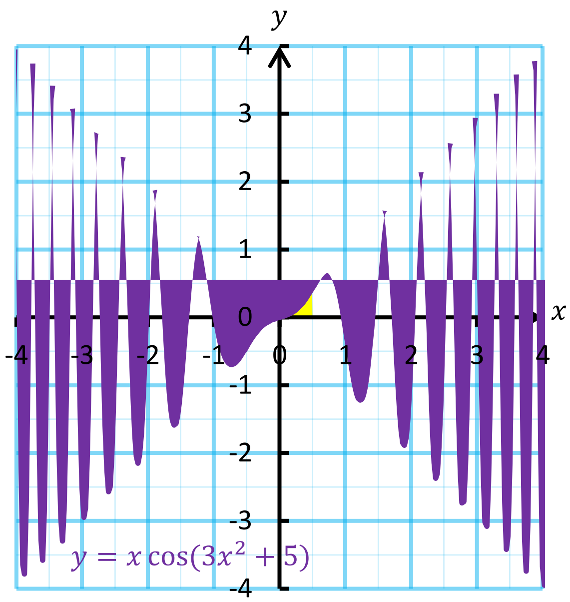
By differentiating, $\frac{du}{dx} = g'(x)$ and $\frac{dy}{du} = f'(u)$.

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = f'(u) \times g'(x)$$
$$\frac{dy}{dx} = f'\big(g(x)\big)g'(x)$$

So, $\int f'\big(g(x)\big)g'(x) \, dx = f\big(g(x)\big) + c$, and we can think of integration by substitution as the **reverse** of the chain rule.

Example 2



Evaluate $\int_0^{0.5} x\cos(3x^2+5)\,dx$.

We recognise that the integral contains the function $f(x) = 3x^2 + 5$ and a **multiple** of its derivative $f'(x) = 6x$. We choose the substitution based on this function: $u = 3x^2 + 5$.

By differentiating u, we obtain

$$\frac{du}{dx} = 6x$$

We can re-arrange the above to make dx the subject:

$$du = 6x\,dx$$
$$\frac{du}{6x} = dx$$

We now substitute u instead of $3x^2 + 5$ and $\frac{du}{6x}$ instead of dx in the original integral:

$$\int_0^{0.5} x\cos(3x^2+5)\,dx$$
$$= \int_0^{0.5} \cancel{x}\cos(u)\frac{du}{6\cancel{x}}$$
$$= \int_0^{0.5} \frac{1}{6}\cos(u)\,du$$

Before integrating with respect to u, we must ensure that the **limits** are also in terms of u, not x. Currently, we are integrating between $x = 0$ and $x = 0.5$. Remembering that $u = 3x^2 + 5$,

If $x = 0$ then $u = 3 \times 0^2 + 5$
$u = 5$

If $x = 0.5$ then $u = 3 \times 0.5^2 + 5$
$u = 5.75$

So, the integral in terms of u is

$$\int_5^{5.75} \frac{1}{6}\cos(u)\,du$$

$\cos(u)$ integrates (with respect to u) to give $\sin(u)$. Therefore

$$\int_5^{5.75} \frac{1}{6}\cos(u)\,du$$
$$= \left[\frac{1}{6}\sin(u)\right]_5^{5.75}$$
$$= \left(\frac{1}{6}\times\sin(5.75)\right) - \left(\frac{1}{6}\times\sin(5)\right)$$
$$= -0.08471317958 - -0.1598207124$$
$$= 0.07510753282$$
$= 0.075$ square units, to 3 decimal places

Exercise 2

Evaluate $\int_1^2 12x\, e^{3x^2-4}\, dx$.

Integrating $\sin^2(\theta)$ and $\cos^2(\theta)$

We remember from previous work that

$\cos(2\theta) = 2\cos^2(\theta) - 1$ or $\cos(2\theta) = 1 - 2\sin^2(\theta)$

By re-arranging,

$\frac{\cos(2\theta)+1}{2} = \cos^2(\theta)$ $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$

Therefore

$\int \cos^2(\theta)\, d\theta$
$= \int \frac{\cos(2\theta)+1}{2}\, d\theta$
$= \int \frac{1}{2}\cos(2\theta) + \frac{1}{2}\, d\theta$
$= \frac{1}{4}\sin(2\theta) + \frac{\theta}{2} + c$

$\int \sin^2(\theta)\, d\theta$
$= \int \frac{1-\cos(2\theta)}{2}\, d\theta$
$= \int \frac{1}{2} - \frac{1}{2}\cos(2\theta)\, d\theta$
$= \frac{\theta}{2} + \frac{1}{4}\cos(2\theta) + c$

Example 3

(a) Use the substitution $x = 4\sin(\theta)$ to show that

$$\int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{(16-x^2)}}\, dx = \int_0^a b\sin^2\theta\, d\theta$$

where a and b are constants whose values are to be determined.

(b) **Hence**, evaluate

$$\int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{(16-x^2)}}\, dx$$

Give your answer in the form $c\pi + d$, where c and d are integers whose values are to be determined.

(a) If $x = 4\sin(\theta)$ then $\frac{dx}{d\theta} = 4\cos(\theta)$ and $x^2 = 16\sin^2(\theta)$

$dx = 4\cos(\theta)\, d\theta$

Limits: If $x = 0$ then

$$0 = 4\sin(\theta)$$
$$0 = \sin(\theta)$$
$$\theta = \sin^{-1}(0)$$
$$\theta = 0$$

If $x = 2\sqrt{2}$ then

$$2\sqrt{2} = 4\sin(\theta)$$
$$\frac{\sqrt{2}}{2} = \sin(\theta)$$
$$\theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$
$$\theta = \frac{\pi}{4}$$

Therefore

$$\int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{(16 - x^2)}}\, dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{16\sin^2(\theta)}{\sqrt{(16 - 16\sin^2(\theta))}} \times 4\cos(\theta)\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{16\sin^2(\theta)}{\sqrt{16(1 - \sin^2(\theta))}} \times 4\cos(\theta)\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{16\sin^2(\theta)}{\sqrt{16\cos^2(\theta)}} \times 4\cos(\theta)\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{16\sin^2(\theta)}{\cancel{4\cos(\theta)}} \cancel{\times 4\cos(\theta)}\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} 16\sin^2(\theta)\, d\theta$$

So, $a = \frac{\pi}{4}$ and $b = 16$.

(b)

$$\int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{(16 - x^2)}}\, dx$$
$$= \int_0^{\frac{\pi}{4}} 16\sin^2(\theta)\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} 16\left(\frac{1 - \cos(2\theta)}{2}\right) d\theta$$
$$= \int_0^{\frac{\pi}{4}} 8(1 - \cos(2\theta))\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} 8 - 8\cos(2\theta)\, d\theta$$
$$= [8\theta - 4\sin(2\theta)]_0^{\frac{\pi}{4}}$$
$$= \left(8 \times \frac{\pi}{4} - 4 \times \sin\left(2 \times \frac{\pi}{4}\right)\right) - (8 \times 0 - 4 \times \sin(2 \times 0))$$
$$= (2\pi - 4 \times 1) - (0 - 4 \times 0)$$
$$= (2\pi - 4) - (0 - 0)$$
$$= 2\pi - 4$$

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$
$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

So, $c = 2$ and $d = -4$.

Exercise 3

Use an appropriate sustitution to show that

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{\pi}{12} - \frac{\sqrt{3}}{8}.$$

Example 4

Use the substitution $u = 3x - 2$ to evaluate

This time the x on top of the fraction will not cancel.

$$\int_1^2 \frac{x}{(3x-2)^4}\,dx.$$

If $u = 3x - 2$ then $\frac{du}{dx} = 3$

$du = 3\,dx$

$\frac{du}{3} = dx$

If $u = 3x - 2$ then $u + 2 = 3x$

$\frac{u+2}{3} = x$

Limits: If $x = 1$ then $u = 3 \times 1 - 2$

$u = 1$

If $x = 2$ then $u = 3 \times 2 - 2$

$u = 4$

Therefore

$$\begin{aligned}
&\int_1^2 \frac{x}{(3x-2)^4}\,dx \\
&= \int_1^4 \frac{\left(\frac{u+2}{3}\right)}{u^4}\frac{du}{3} \\
&= \frac{1}{9}\int_1^4 \frac{u+2}{u^4}\,du \\
&= \frac{1}{9}\int_1^4 u^{-3} + 2u^{-4}\,du \\
&= \frac{1}{9}\left[\frac{u^{-2}}{-2} + \frac{2u^{-3}}{-3}\right]_1^4 \\
&= \frac{1}{9}\left[\left(\frac{4^{-2}}{-2} + \frac{2 \times 4^{-3}}{-3}\right) - \left(\frac{1^{-2}}{-2} + \frac{2 \times 1^{-3}}{-3}\right)\right] \\
&= \frac{1}{9}\left[\left(\frac{1}{-32} + \frac{2}{-192}\right) - \left(\frac{1}{-2} + \frac{2}{-3}\right)\right] \\
&= \frac{1}{9}\left[-\frac{1}{24} - -\frac{7}{6}\right] \\
&= \frac{1}{9} \times \frac{9}{8} \\
&= \frac{1}{8}
\end{aligned}$$

Exercise 4

Use the substitution $u = 2x - 3$ to evaluate

$$\int_2^3 \frac{x}{(2x-3)^3}\,dx.$$

Choosing an appropriate substitution

The substitution is not always given in an examination question. Here are **some** appropriate substitutions.

Integral	Substitution
$\int \sqrt{a^2 - x^2}\, dx$	$x = a\sin\theta$
$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$	$x = a\sin\theta$
$\int \frac{1}{a^2 + x^2}\, dx$	$x = a\tan\theta$

Otherwise, choose a substitution for u so that

- u and its *derivative* appears in the integral
- u appears in a *bracket*
- u appears in the *denominator*
- u is a function raised to the *highest exponent*

Integration by Parts

In a previous workbook on further differentiation, we saw the **product rule**, which was the technique for differentiating an expression of the form $y = uv$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

We can integrate the above with respect to x to give the following.

$$\int \frac{dy}{dx}dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$
$$y = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$
$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$
$$uv - \int v\frac{du}{dx}dx = \int u\frac{dv}{dx}dx$$

This is given in the formula booklet.

By swapping sides, we arrive at the formula for **integrating by parts**:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

We use this technique to integrate a product made up of two parts, u and $\frac{dv}{dx}$.
As a rule, choose u as the function that appears **highest** in the following 'LIATE' list.

L	**L**ogarithmic functions, e.g. $\ln(3x)$
I	**I**nverse trigonometric functions, e.g. $\tan^{-1}(2x)$
A	**A**lgebraic functions (simple polynomials), e.g. $3x^2 - 5x + 8$
T	**T**rigonometric functions, e.g. $\sin(5x)$
E	**E**xponential functions, e.g. e^{7x}

Example 5

Find

$$\int 5x\cos x\,dx.$$

Answer: The integral is a product of two expressions, so we can attempt to use integration by parts to find a solution.

Let $u = 5x$ $\quad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 5$ $\quad v = \sin x$

Then

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$
$$\int 5x\cos x\,dx = 5x \times \sin x - \int \sin x \times 5\,dx$$
$$\int 5x\cos x\,dx = 5x\sin x - \int 5\sin x\,dx$$
$$\int 5x\cos x\,dx = 5x\sin x - -5\cos x + c$$
$$\int 5x\cos x\,dx = 5x\sin x + 5\cos x + c$$
$$\int 5x\cos x\,dx = 5(x\sin x + \cos x) + c$$

Theory

Exercise 5

Find

$$\int 7xe^{3x}\,dx.$$

Example 6

Find

$$\int x^2\,e^x\,dx.$$

Answer: Let $u = x^2$ $\quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2x$ $\quad v = e^x$

Then

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$
$$\int x^2\,e^x\,dx = x^2 \times e^x - \int e^x \times 2x\,dx$$
$$\int x^2\,e^x\,dx = x^2e^x - \int 2xe^x\,dx$$

Now for $\int 2xe^x\,dx$ we must integrate by parts again.

Let $u = 2x$ $\quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2$ $\quad v = e^x$

Then

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$
$$\int 2x\,e^x\,dx = 2x \times e^x - \int e^x \times 2\,dx$$

$$\int 2x\, e^x\, dx = 2xe^x - \int 2e^x\, dx$$
$$\int 2x\, e^x\, dx = 2xe^x - 2e^x + c$$

We can continue to add the constant of integration (Why?)

Therefore

$$\int x^2\, e^x\, dx = x^2 e^x - (2xe^x - 2e^x + c)$$
$$\int x^2\, e^x\, dx = x^2 e^x - 2xe^x + 2e^x + c$$
$$\int x^2\, e^x\, dx = e^x(x^2 - 2x + 2) + c$$

Exercise 6

Find

$$\int e^x \sin x\, dx.$$

Top 10

(C4 Summer 2010)

7. *(a)* Find $\int x^3 \ln x \, dx$. [4]

(b) Use the substitution $u = 2x - 3$ to evaluate $\int_1^2 x(2x-3)^4 \, dx$. [5]

(C4 Summer 2005)

7. *(a)* Use the substitution $u = 2x - 1$ to evaluate

$$\int_0^1 x(2x-1)^9 \,\mathrm{d}x \ .$$ [5]

(b) (i) Find $\int x\cos 2x \,\mathrm{d}x$. [4]

(ii) Use the result of *(b)*(i) to find

$$\int x\cos^2 x \ \mathrm{d}x \ .$$ [3]

(C4 Summer 2011)

7. *(a)* Find $\int x\sin 2x\,\mathrm{d}x$. [4]

(b) Use the substitution $u = 5 - x^2$ to evaluate

$$\int_0^2 \frac{x}{(5-x^2)^3}\,\mathrm{d}x\ .$$ [4]

(C4 Summer 2012)

7. *(a)* Find $\int x\mathrm{e}^{-2x}\,\mathrm{d}x$. [4]

(b) Use the substitution $u = 1 + 3\ln x$ to evaluate

$$\int_1^{\mathrm{e}} \frac{1}{x(1+3\ln x)}\,\mathrm{d}x.$$

Give your answer correct to four decimal places. [4]

(C4 Summer 2016)

6. *(a)* Find $\int (2x+1)\mathrm{e}^{-3x}\,\mathrm{d}x$. [4]

(b) Use the substitution $u = 4 + 5\tan x$ to evaluate

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{4+5\tan x}}{\cos^2 x}\,\mathrm{d}x.$$ [4]

(C4 Summer 2014)

7. *(a)* Find $\int x^4 \ln 2x \, dx$. [4]

(b) Use the substitution $u = 10\cos x - 1$ to evaluate

$$\int_0^{\frac{\pi}{3}} \sqrt{(10\cos x - 1)}\, \sin x \, dx.$$ [4]

(C4 Summer 2008)

6. *(a)* Find $\int (3x+1)\,e^{2x}\,dx$. [4]

(b) Use the substitution $x = 3\sin\theta$ to show that

$$\int_{1\cdot 5}^{3} \sqrt{9 - x^2}\,dx = \int_{a}^{b} k\cos^2\theta\,d\theta \quad ,$$

where the values of the constants a, b and k are to be found.

Hence evaluate $\int_{1\cdot 5}^{3} \sqrt{9 - x^2}\,dx$. [8]

(C4 Summer 2007)

7. *(a)* Find $\int x^2 \ln x \, \mathrm{d}x$. [4]

(b) Use the substitution $x = 2\sin\theta$ to show that

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} \, \mathrm{d}x = \int_0^a k \sin^2\theta \, \mathrm{d}\theta ,$$

where the values of a and k are to be determined.

Hence, or otherwise, evaluate $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} \, \mathrm{d}x$. [8]

(Unit 3 Summer 2022)

18 **a)** Use a suitable substitution to find

$$\int \frac{x^2}{(x+3)^4}\,\mathrm{d}x\,.$$ [5]

b) Hence evaluate $\displaystyle\int_0^1 \frac{x^2}{(x+3)^4}\,\mathrm{d}x\,.$ [2]

(Unit 3 Summer 2018)

1 4 Evaluate

a) $\int_1^2 x^3 \ln x \, \mathrm{d}x$. [6]

b) $\int_0^1 \frac{2+x}{\sqrt{4-x^2}} \mathrm{d}x$. [6]

Revision Question

(Unit 3 Summer 2019)

14 **a)** Find $\int (e^{2x} + 6\sin 3x)\,dx$. [2]

b) Find $\int 7(x^2 + \sin x)^6(2x + \cos x)\,dx$. [1]

c) Find $\int \frac{1}{x^2}\ln x\,dx$. [4]

d) Use the substitution $u = 2\cos x + 1$ to evaluate

$$\int_0^{\frac{\pi}{3}} \frac{\sin x}{(2\cos x + 1)^2}\,dx \ .$$ [4]