Unit 3, Workbook 16

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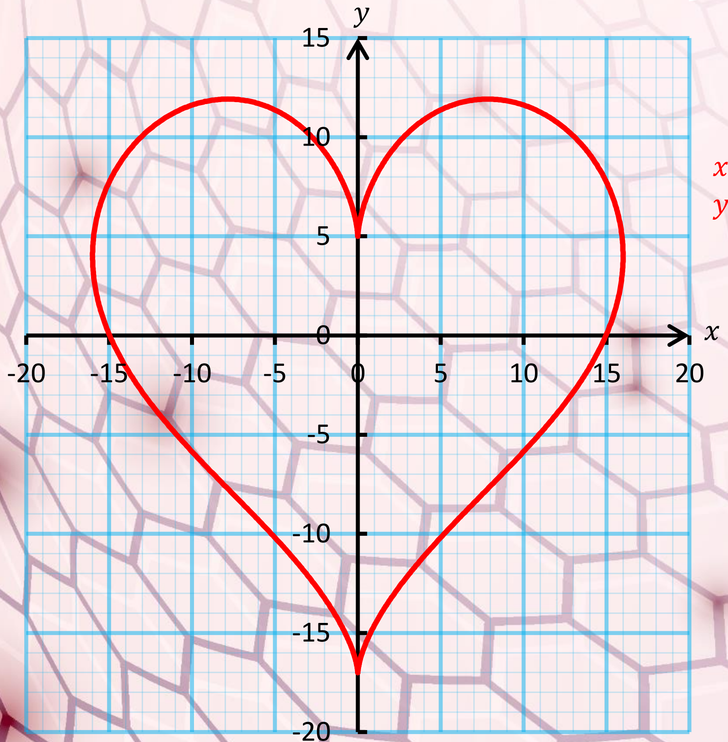
# Parametric and Implicit Equations




$x = 16\sin^3(t)$

$y = 13\cos(t) - 5\cos(2t) - 2\cos(3t) - \cos(4t)$

Name:

## Background

| | |
|---|---|
| **What is the work?** | Finding the equation of tangents and normals to curves that are defined **parametrically** or **implicitly**. Converting between Cartesian and parametric forms. |
| **What is required before starting?** | **GCSE Work:** Substitution, solving equations, changing the subject.<br>**A Level Unit 3:** Differentiation. |
| **Where does this lead to?** | Parametric equations can model movement that is circular in nature, or movement that returns to the same location. |

## Theory

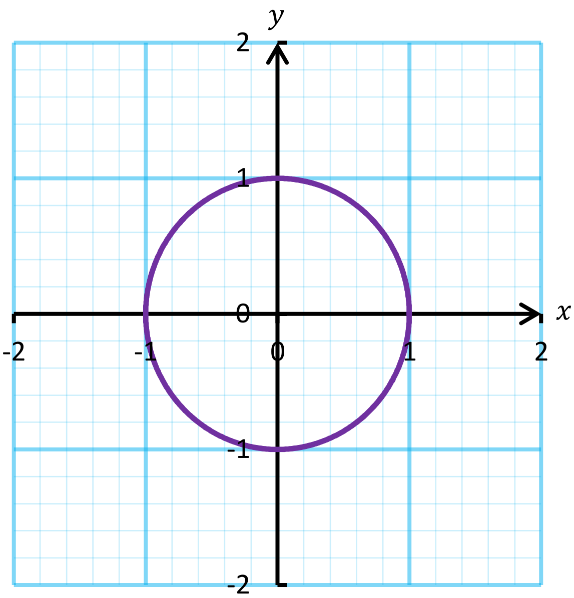
Thus far, we have been differentiating functions of the form $y = f(x)$, where there is one value of $y$ for each value of $x$. Not every curve is of this type however; consider the equation $x^2 + y^2 = 1$ that gives the unit circle. To try to differentiate this function, we could attempt to re-arrange it to the form $y = f(x)$:

$$y^2 = 1 - x^2$$
$$y = \pm\sqrt{1 - x^2}$$

This is not an **explicit** function, as there are two values of $y$ for each value of $x$. For example, if $x = 0$, we have

$$y = \pm\sqrt{1 - 0^2}$$
$$y = \pm 1$$

This can be seen on the graph as the points $(0, 1)$ and $(0, -1)$ lie on the circle. Because it is not explicit what the value of $y$ is for each value of $x$, we say that $x^2 + y^2 = 1$ is an **implicit** equation.




### Differentiating Implicit Equations

How do we differentiate $x^2 + y^2 = 1$ with respect to $x$? There is no problem in differentiating $x^2$ with respect to $x$: this would give us $2x$. We could also differentiate $1$ with respect to $x$ to give $0$. But how do we differentiate $y^2$ with respect to $x$? To see how, let us define $z$ as $z = y^2$. We can differentiate $z$ with respect to $y$:

$$\frac{dz}{dy} = 2y$$

Then, we can use the chain rule to differentiate $z$ with respect to $x$:

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$
$$\frac{dz}{dx} = 2y \times \frac{dy}{dx}$$

Theory

So, to differentiate $z$ with respect to $x$, we differentiate it with respect to $y$, and then multiply by $\frac{dy}{dx}$.

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx}$$

In general, to differentiate any function of $y$ with respect to $x$, we differentiate it with respect to $y$, and then multiply by $\frac{dy}{dx}$.

$$\frac{d}{dx}\big(f(y)\big) = \frac{d}{dy}\big(f(y)\big) \times \frac{dy}{dx}$$

Returning to the equation of a unit circle, $x^2 + y^2 = 1$, when we differentiate it with respect to $x$, we obtain

$$2x + 2y \times \frac{dy}{dx} = 0$$

We can re-arrange this to find $\frac{dy}{dx}$:

$$2y \times \frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{2y}$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Given a point that lies on the circle, we can substitute the point into $\frac{dy}{dx}$ to find the gradient of the circle at that point. For example, on the previous page we saw that the point $(0, 1)$ lies on the circle. Here we have

$$\frac{dy}{dx} = -\frac{0}{1}$$
$$\frac{dy}{dx} = 0$$

We can now (for example) find the equation of the tangent to the circle at the point $(0, 1)$:

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 0(x - 0)$$
$$y - 1 = 0$$
$$y = 1$$

This is a horizontal line that goes through the point $(0, 1)$.

**Example 1**

Find $\frac{dy}{dx}$ for the curve $3x^2 + 5y^2 = 7$.

*Answer:* By differentiating implicitly:

$$6x + 10y \times \frac{dy}{dx} = 0$$
$$10y \times \frac{dy}{dx} = -6x$$
$$\frac{dy}{dx} = \frac{-6x}{10y}$$
$$\frac{dy}{dx} = -\frac{3x}{5y}$$

**Exercise 1**

(a) Find $\frac{dy}{dx}$ for the curve $8x^2 + 2y^2 = 10$.

(b) The point $(-1, 1)$ lies on the curve $8x^2 + 2y^2 = 10$.
Find the equation of the tangent to the curve $8x^2 + 2y^2 = 10$ at the point $(-1, 1)$.

**Exercise 2**

Find the equation of the normal to the curve $4x^2 + 3y^2 - 7x + 2y = 13$ at the point $(1, 2)$.
Write your answer in the form $y = mx + c$.

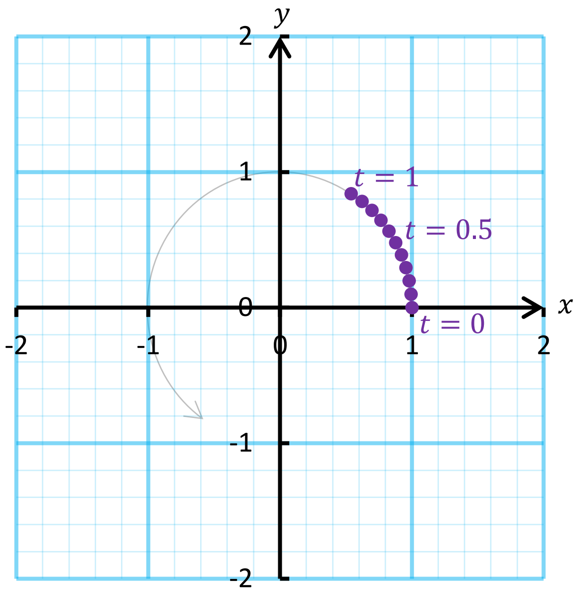
## Parametric Equations

The **parametric equations** for a curve define the coordinates of the curve using a third variable, usually $t$, which represents the **parameter**.

$$x = f(t)$$
$$y = g(t)$$

For example, we could represent the unit circle using the following parametric equations.

$$x = \cos(t)$$
$$y = \sin(t)$$




The following table shows the value of $x$ and $y$ for various values of $t$, starting at $t = 0$, and going up in increments of $0.1$. (The values are shown correct to four decimal places, with the angle $t$ in radians). These are plotted as the purple coordinates on the graph on the right. (To plot the entire circle, the value of $t$ would need to reach $2\pi$.)

| $t$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 1 | 0.9950 | 0.9801 | 0.9553 | 0.9211 | 0.8776 | 0.8253 | 0.7648 | 0.6967 | 0.6216 | 0.5403 |
| $y$ | 0 | 0.0998 | 0.1987 | 0.2955 | 0.3894 | 0.4794 | 0.5646 | 0.6442 | 0.7174 | 0.7833 | 0.8415 |

To find the gradient of the curve at any point, we can differentiate the functions $f(t)$ and $g(t)$ with respect to $t$:

$$\frac{dx}{dt} = f'(t)$$
$$\frac{dy}{dt} = g'(t)$$

We can then use the chain rule to find the gradient:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

where

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

The table uses values of $t$ from zero upwards, but it is important to note that $t$ could also take negative values.

For the above example, we have

$$\frac{dx}{dt} = -\sin(t)$$
$$\frac{dy}{dt} = \cos(t)$$
$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$\frac{dy}{dx} = \cos(t) \times \frac{1}{-\sin(t)}$$
$$\frac{dy}{dx} = -\cot(t)$$

If (for example) $t = 1$, then $\frac{dy}{dx} = -\cot(1)$

$\frac{dy}{dx} = -0.6421$ to 4 decimal places.

Theory

To find the equation of the normal to the curve at the point where $t = 1$, we can use the following information from the previous page:

$$\text{If } t = 1, \text{ then (to 4 decimal places) } x = 0.5403,\ y = 0.8415,\ \frac{dy}{dx} = -0.6421$$

The gradient of the normal is $-\frac{1}{-0.6421} = 1.5574$ (which is the negative reciprocal of $\frac{dy}{dx}$).

The equation of the normal at the point where $t = 1$ is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0.8415 &= 1.5574(x - 0.5403) \\ y - 0.8415 &= 1.5574x - 0.8415 \\ y &= 1.5574x \end{aligned}$$

**Exercise 3**

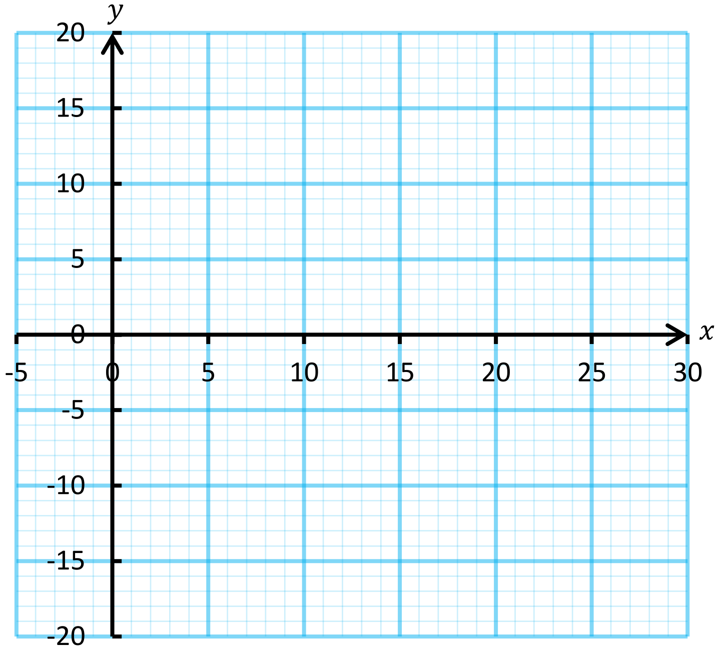
A curve is defined by the following parametric equations.

$$x = t^2$$
$$y = 2t + 3$$

(a) Complete the following table.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x$ | | | | | | |
| $y$ | | | | | | |

(b) Use the above table to plot a part of the curve on the graph paper on the right.




(c) Find the gradient $\frac{dy}{dx}$ of the curve, writing your answer in terms of $t$.

(d) Find the equation of the tangent to the curve at the point where $t = 2$.

(e) Find the equation of the normal to the curve at the point where $t = 4$.

**Writing Parametric Equations as a Cartesian Equation**

Sometimes, it is possible to transform a set of parametric equations to be one **Cartesian equation**, which avoids using the original parameter, and uses only $x$ and $y$.

**Example 2**

On page 5, the following parametric equations were used to define the unit circle.

$$x = \cos(t)$$
$$y = \sin(t)$$

To delete the parameter $t$ from these equations, we can use the identity $\sin^2(t) + \cos^2(t) = 1$ from Unit 1:

$$y^2 + x^2 = 1$$
$$x^2 + y^2 = 1$$

This is the equation for the unit circle from page 2.

**Example 3**

In Exercise 3, the following parametric equations were used to define a curve.

$$x = t^2$$
$$y = 2t + 3$$

To delete the parameter $t$ from these equations, we can re-arrange the second equation to make $t$ its subject:

$$y - 3 = 2t$$
$$\frac{y-3}{2} = t$$

We can then substitute for $t$ into the first equation:

$$x = \left(\frac{y-3}{2}\right)^2$$
$$x = \frac{(y-3)^2}{2^2}$$
$$x = \frac{(y-3)(y-3)}{4}$$
$$4x = y^2 - 3y - 3y + 9$$
$$4x = y^2 - 6y + 9$$
$$4x - y^2 + 6y - 9 = 0$$

**Exercise 4**

A curve is defined by the parametric equations

$$x = t^2 - 3$$
$$y = t + 2$$

Find the Cartesian equation that defines this curve.

Top 10

(C3 Winter 2010)

**3.** *(a)* The curve $C$ is defined by

$$y^3 + 2x^3y = 3x^2 + 4x - 3.$$

Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point (2, 1). [4]

(C3 Winter 2012)

**3.** *(a)* A function is defined parametrically by

$$x = 3t^2, \ y = t^6 - 4t^3.$$

(i) Find $\frac{dy}{dx}$ in terms of $t$.

(ii) Given that $\frac{dy}{dx} = \frac{7}{2}$, show that $2t^4 - 4t - 7 = 0$. [5]

(C3 Summer 2014)

**4.** Given that $x = 2e^t - 5$, $y = 8e^{-t} + 3e^t - 4$, find the value of $t$ when $\frac{dy}{dx} = -1$.

Give your answer correct to three decimal places. [7]

(C3 Summer 2006)

**3.** *(a)* Given that $x = \cos t$, $y = \sin 2t$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $t$. [4]

*(b)* Given that

$$x^4 + 2x^2y + y^2 = 21,$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$ and $y$. [4]

(C3 Winter 2008)

**3.** *(a)* Given that $x = t^4 + 1$, $y = e^{2t} + 5$, find $\frac{dy}{dx}$ in terms of $t$. [4]

*(b)* Given that $x^4 + \sin y + x^2y^3 = 9$, find $\frac{dy}{dx}$ in terms of $x$ and $y$. [3]

(C3 Summer 2008)

**3.** Given that

$$x^2 + x\sin y + y^3 = \pi^3 + 1,$$

find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1, \pi)$. [4]

(Unit 3 Summer 2024)

**11.** A curve is defined parametrically by

$$x = 2\theta + \sin 2\theta, \qquad y = 1 + \cos 2\theta.$$

(a) Show that the gradient of the curve at the point with parameter $\theta$ is $-\tan\theta$. [6]

(b) Find the equation of the tangent to the curve at the point where $\theta = \dfrac{\pi}{4}$. [4]

(Unit 3 Summer 2023)

**13** The curve $C_1$ has parametric equations $x = 3p + 1,\ y = 9p^2$.

The curve $C_2$ has parametric equations $x = 4q,\ y = 2q$.

Find the Cartesian coordinates of the points of intersection of $C_1$ and $C_2$. [7]

(Unit 3 Summer 2022)

**16** The parametric equations of the curve $C$ are

$$x = 3 - 4t + t^2, \qquad y = (4 - t)^2.$$

**a)** Find the coordinates of the points where $C$ meets the $y$-axis. [3]

**b)** Show that the $x$-axis is a tangent to the curve $C$. [5]

(Unit 3 Summer 2018)

**10** The equation of a curve $C$ is given by the parametric equations

$$x = \cos 2\theta, \quad y = \cos\theta.$$

**a)** Find the Cartesian equation of $C$. [2]

**b)** Show that the line $x - y + 1 = 0$ meets $C$ at the point $P$, where $\theta = \frac{\pi}{3}$, and at the point $Q$, where $\theta = \frac{\pi}{2}$. Write down the coordinates of $P$ and $Q$. [5]

**c)** Determine the equations of the tangents to $C$ at $P$ and $Q$. Write down the coordinates of the point of intersection of the two tangents. [7]

## Revision Questions

(Unit 3 Summer 2019)

**0 6** A curve $C$ has parametric equations $x = \sin\theta$, $y = \cos 2\theta$.

**a)** The equation of the tangent to the curve $C$ at the point $P$ where $\theta = \dfrac{\pi}{4}$ is $y = mx + c$. Find the exact values of $m$ and $c$. [6]

**b)** Find the coordinates of the points of intersection of the curve $C$ and the straight line $x + y = 1$. [5]

(C3 Winter 2009)

**3.** *(a)* Given that $x^2 + 3xy + 2y^2 - 2x = 13$, find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1, 2)$. [4]

*(b)* Given that $x = 2e^t + 6$, $y = 4e^{2t} + 3e^t + 1$, find the value of $t$ when $\frac{\mathrm{d}y}{\mathrm{d}x} = 6$, giving your answer correct to three decimal places. [7]