The Mathematics Department

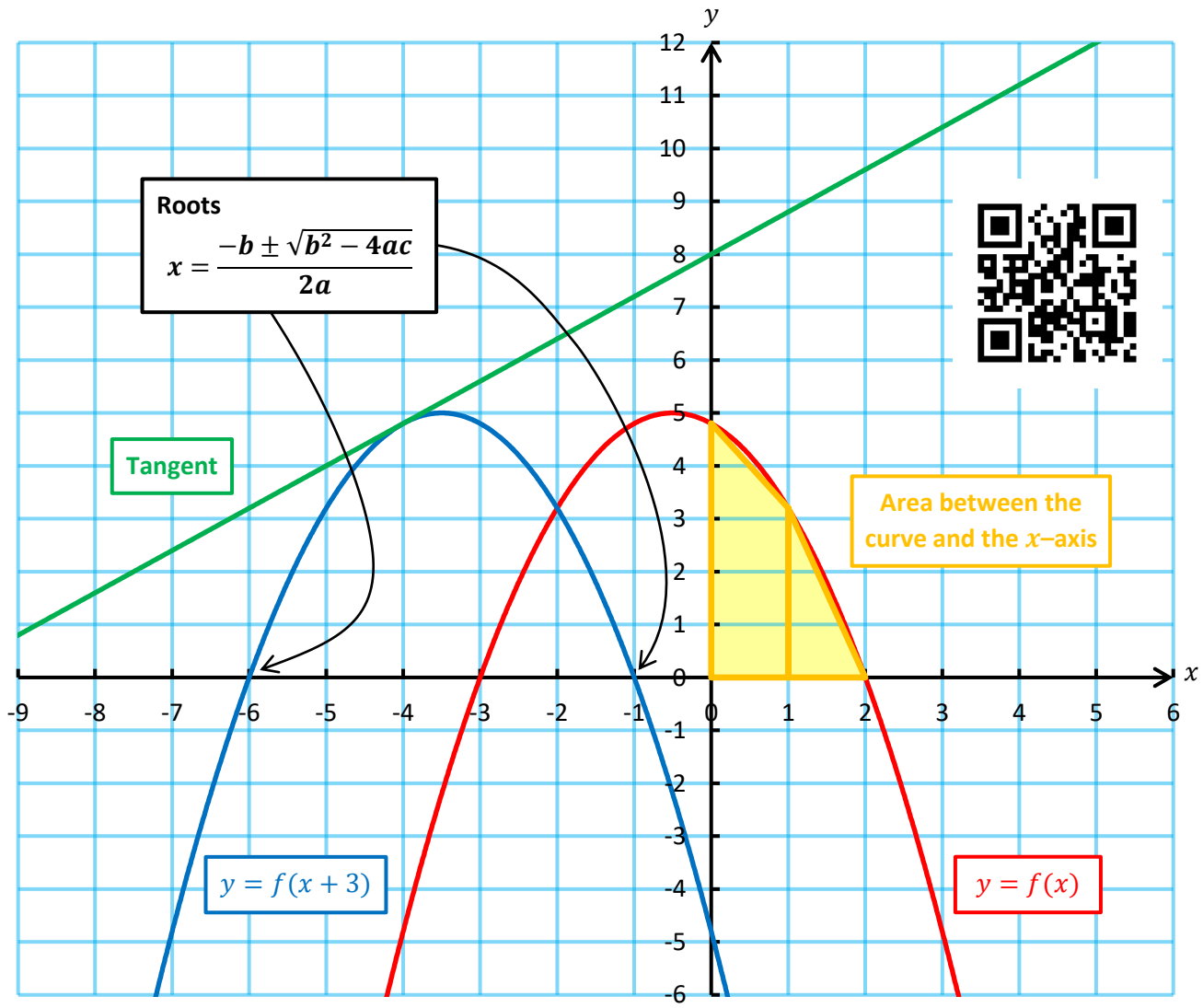
11

Developing Algebra 4

Name:

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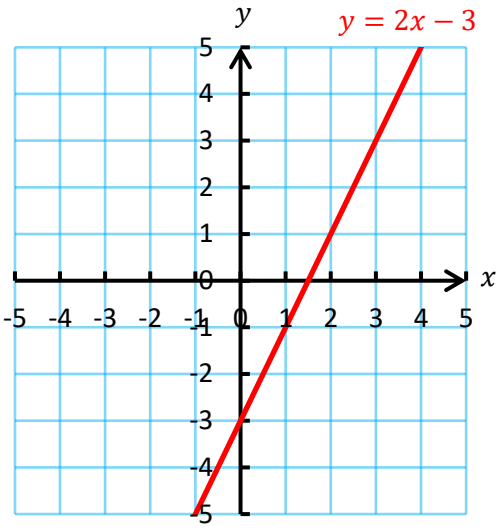
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Transformations of Functions

Function Notation

A graph for $y = 2x - 3$ is shown on the right.



We can see from the graph that the value of y for $x = 2$ is 1.
In mathematics, a special notation is used to refer to values such as these. We write

$$f(2) = 1,$$

where f represents the **function** $f(x) = 2x - 3$,
2 is the **input** of the function,
and 1 is the **output** of the function.
We say that 'f of two is one'.

Skill H

Exercise 1

Use the graph for $f(x) = 2x - 3$ to write
(a) $f(3)$ (b) $f(1)$ (c) $f(0)$

Example

If $f(x) = 5x - 2$, then it is possible to calculate $f(3)$ by substituting $x = 3$ into $5x - 2$:

$$f(3) = 5 \times 3 - 2$$
$$f(3) = 13$$

Exercise 2

H

(a) If $f(x) = 2x - 3$, calculate
(i) $f(5)$ (ii) $f(-2)$ (iii) $f(20)$

(b) If $f(x) = x^2 + 4x + 2$, calculate
(i) $f(2)$ (ii) $f(5)$ (iii) $f(0)$ (iv) $f(-2)$

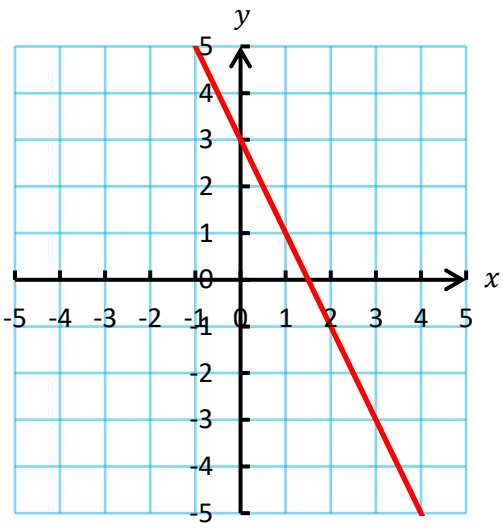
(c) If $f(x) = -4x + 15$, complete the following table.

x	−3	−2	−1	0	1	2	3
$f(x)$							

(d) If $f(x) = 3x - 2$, find the value of x so that $f(x) = 19$.

(e) If $f(x) = x^2$, find the values of x so that $f(x) = 25$.

(f) The graph on the right shows a function $f(x)$ plotted on graph paper. What is the function $f(x)$?



(g) For the function from part (f), what is the value of $f(5)$?

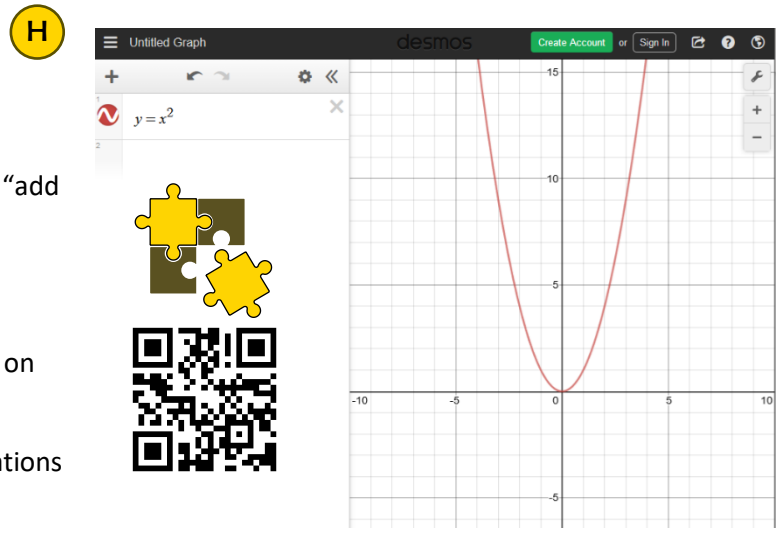
(h) If $g(x) = 2x^2 + 5$, calculate
(i) $g(4)$ (ii) $g(0)$ (iii) $g(-2)$

(i) If $h(x) = \frac{2}{x-3}$, calculate
(i) $h(11)$ (ii) $h(-7)$ (iii) $h\left(\frac{1}{2}\right)$

Transformations of Functions

Function notation is useful to describe the effect a transformation has on the graph of a function.

Exercise 3



Go to the website www.desmos.com/calculator.
Type $y = x^2$ into the first box.

(a) Type $y = x^2 + a$ into the second box. When the "add slider" option appears click on a. What effect does changing a have on the graph?

(b) Change box 2 to show $y = (x - a)^2$ instead of $y = x^2 + a$. What effect does changing a now have on the graph?

(c) What type of transformations are the transformations from parts (a) and (b) of this question?

(d) Change box 2 to show $y = ax^2$. What effect does changing a now have on the graph?

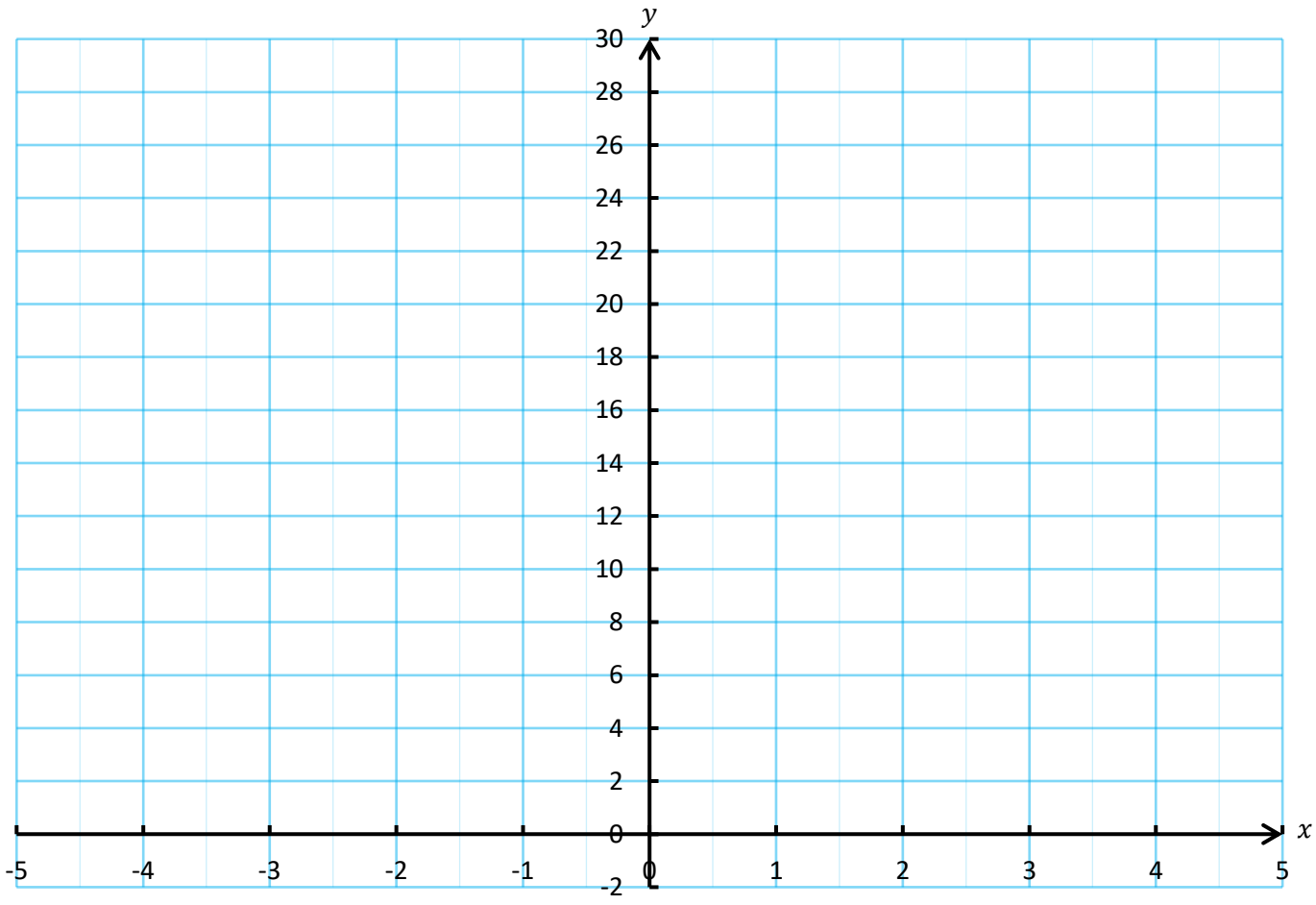
(e) Change box 2 to show $y = (ax)^2$. What effect does changing a now have on the graph?

(f) What is the difference between the way changing a changes the graph in parts (d) and (e) of this question?

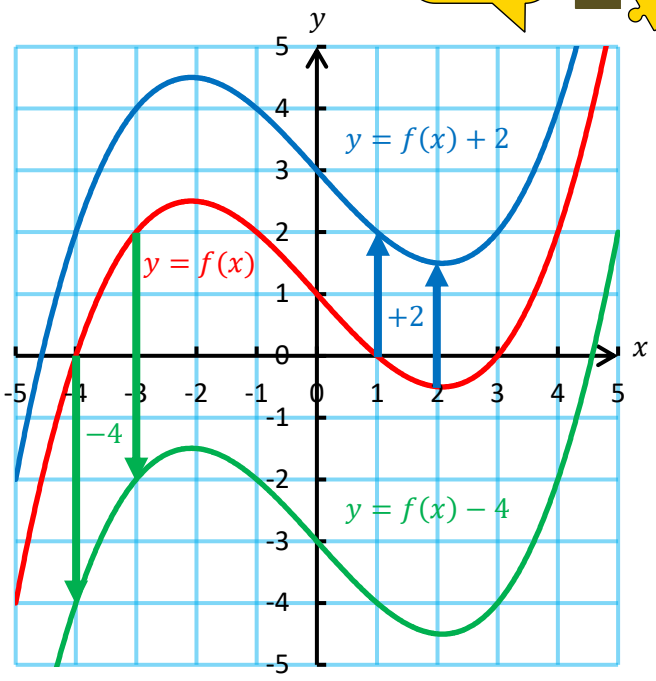
Exercise 4

Draw, on the axes shown below, graphs for the following.

(a) $y = x^2$ (b) $y = x^2 + 3$ (c) $y = (x - 2)^2$ (d) $y = 2x^2$

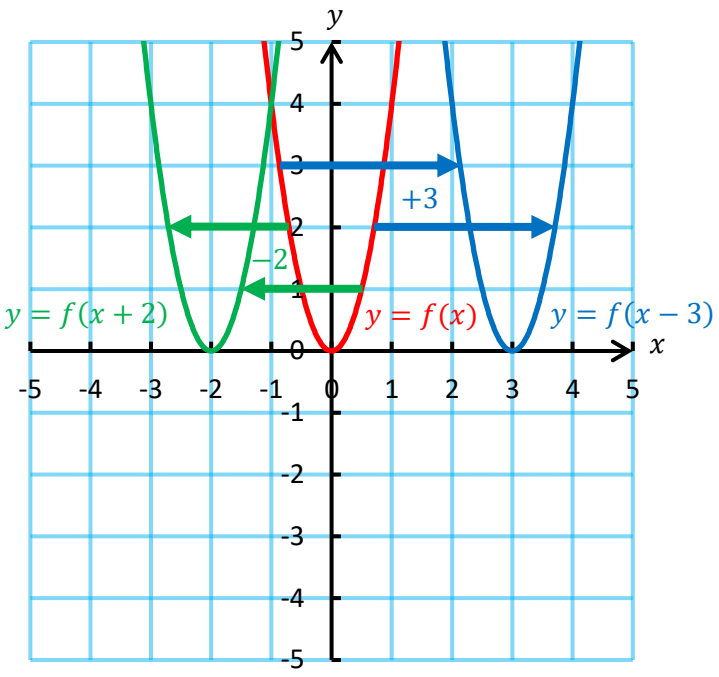


$y = f(x) + a$



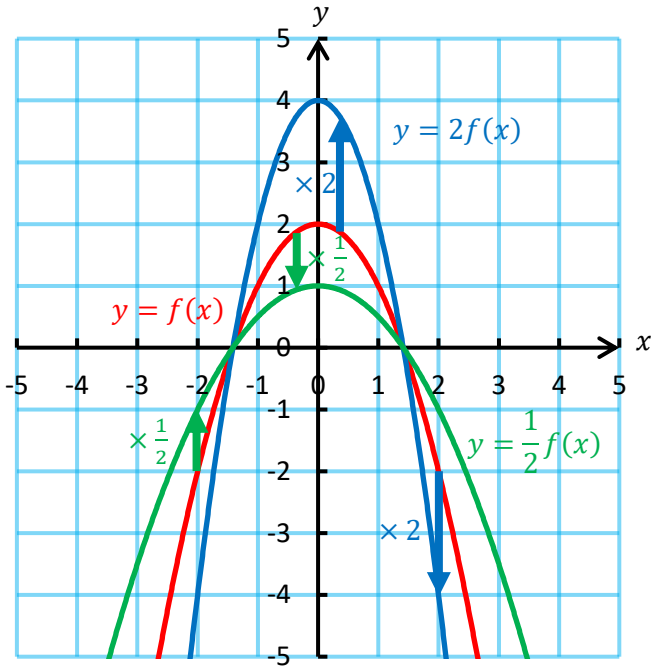
The transformation $y = f(x) + a$ translates the graph a units up (if a is positive) or a units down (if a is negative).

$y = f(x + a)$



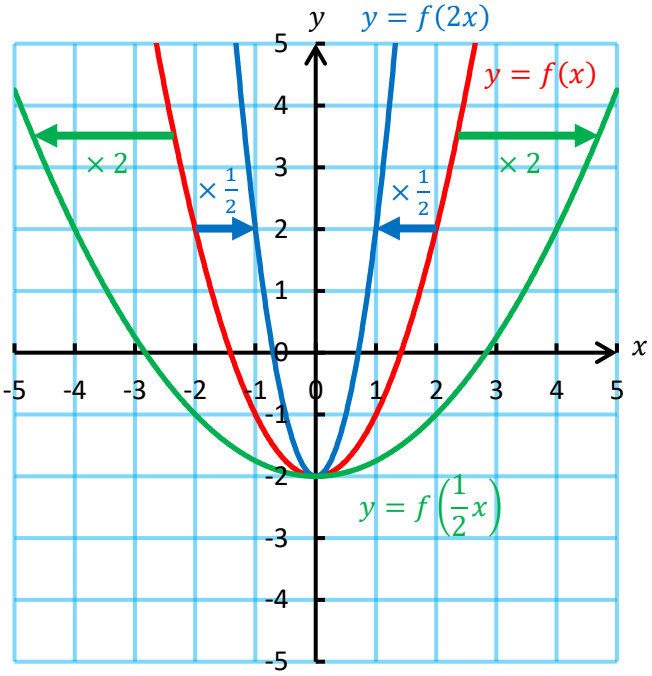
The transformation $y = f(x + a)$ translates the graph a units to the left (if a is positive) or a units to the right (if a is negative).

$y = af(x)$



The transformation $y = af(x)$ stretches the graph in the direction of the y–axis (if $a > 1$) or compresses the graph in the direction of the y–axis (if $0 < a < 1$).

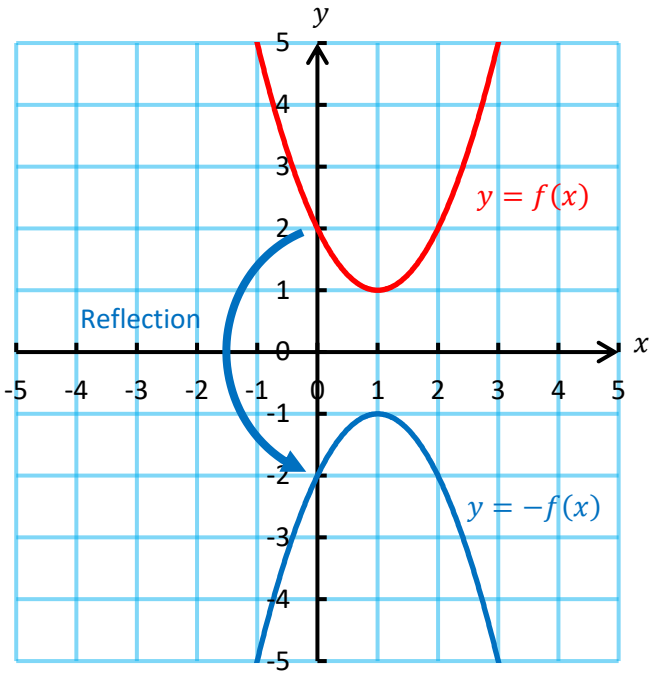
$y = f(ax)$



The transformation $y = f(ax)$ compresses the graph in the direction of the x–axis (if $a > 1$) or stretches the graph in the direction of the x–axis (if $0 < a < 1$).

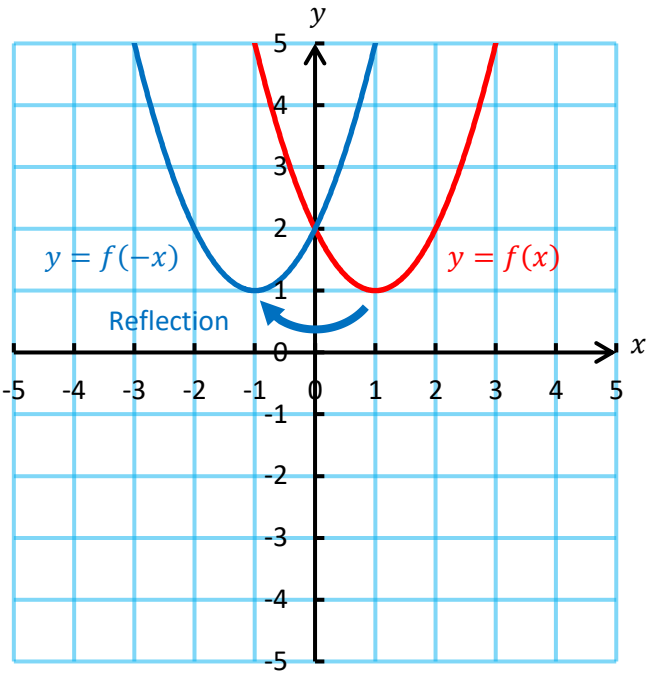
As general advice, we need to **follow** anything that takes place **outside** parentheses, and **undo** anything that takes place **inside** parentheses. So, for example, we follow $y = f(x) + 3$ and translate the graph 3 units up, but undo $y = f(x + 3)$ by translating the graph 3 units to the left.

$y = -f(x)$



The transformation $y = -f(x)$ reflects the graph $y = f(x)$ in the x–axis.

$y = f(-x)$



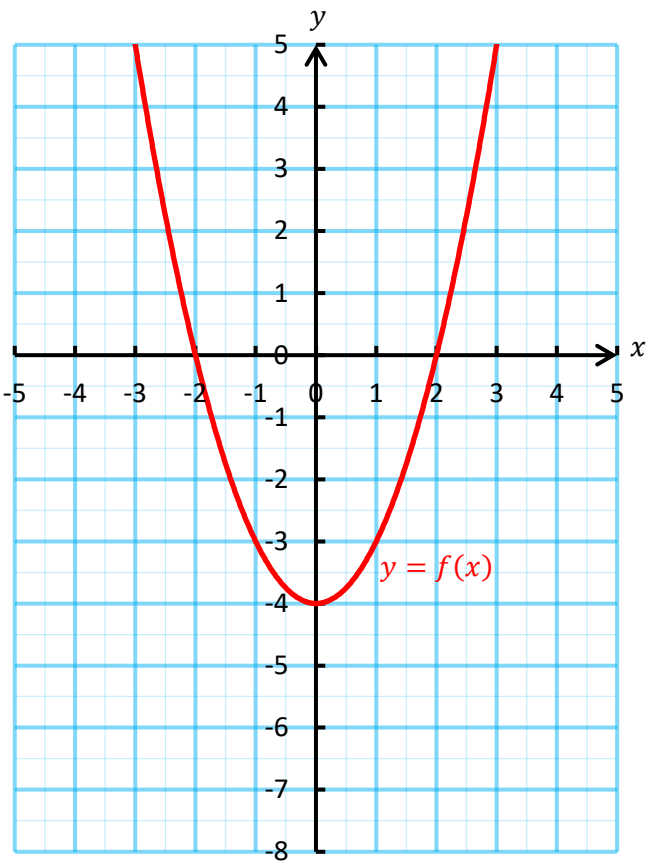
The transformation $y = f(-x)$ reflects the graph $y = f(x)$ in the y–axis.

Exercise 5

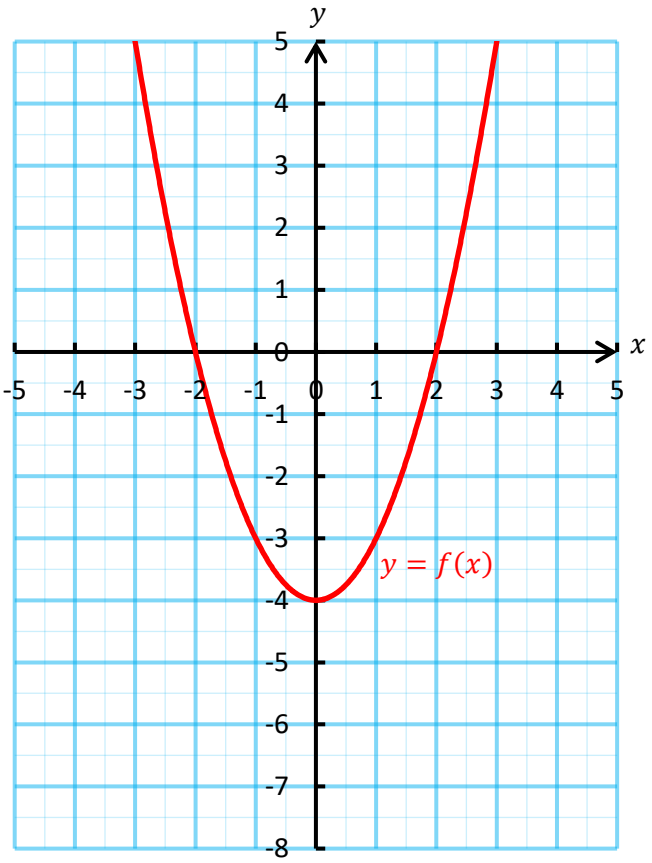


Draw, on the following graphs, the following transformations.

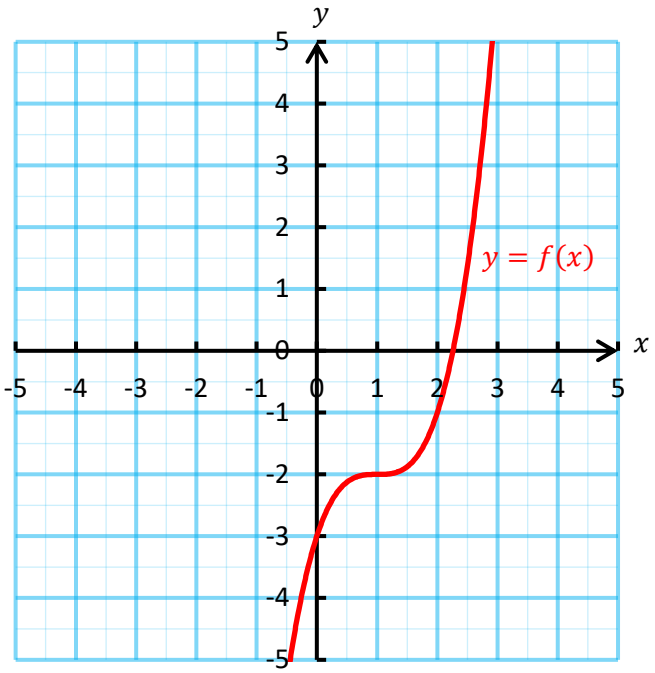
(a) $y = f(x) + 2$



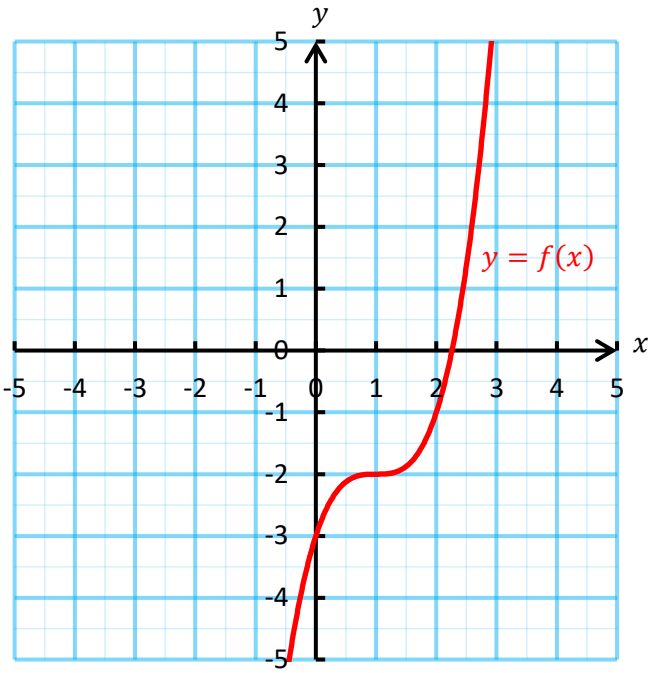
(b) $y = f(x + 2)$



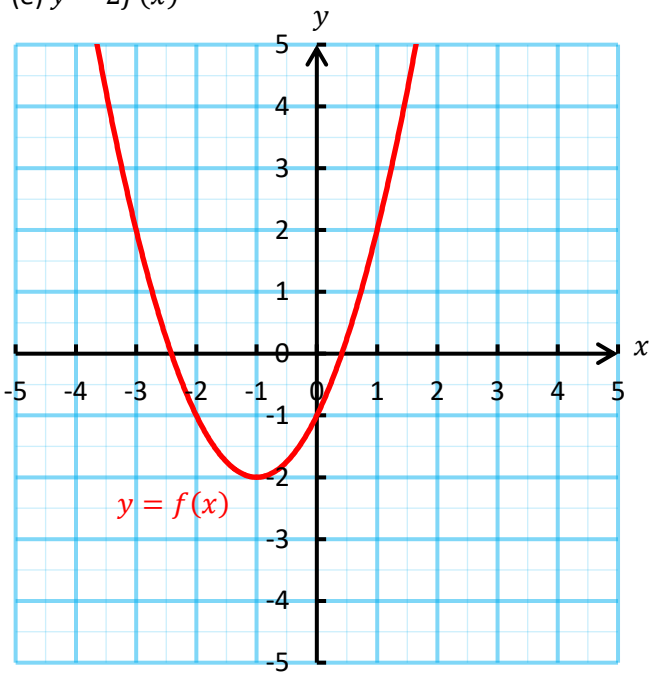
(c) $y = f(-x)$



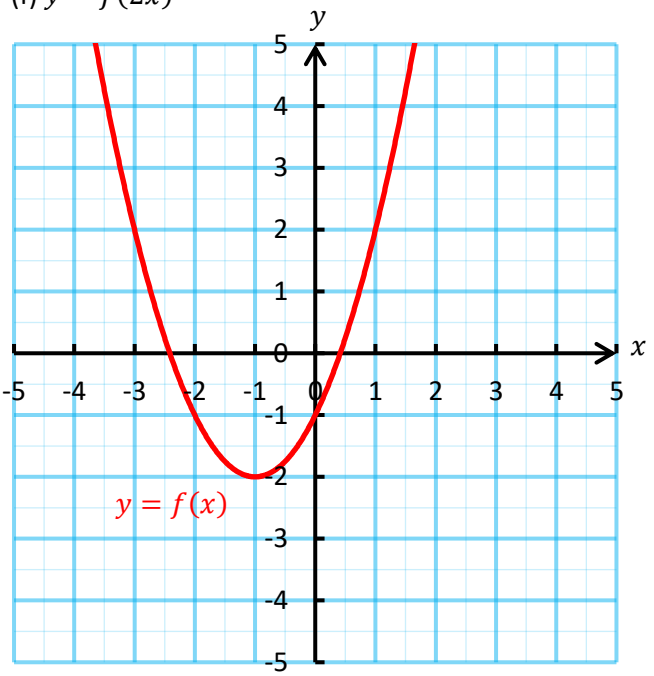
(d) $y = -f(x)$



(e) $y = 2f(x)$



(f) $y = f(2x)$



Exercise 6

H

(a) Draw a set of axes with x and y values going from –10 to 10.
Plot the graph of $y = 2^x$ on the axes, before using transformations of functions to plot the following graphs.
(i) $y = 2^x - 4$ (ii) $y = 2^{x-3}$ (iii) $y = 2^{-x}$

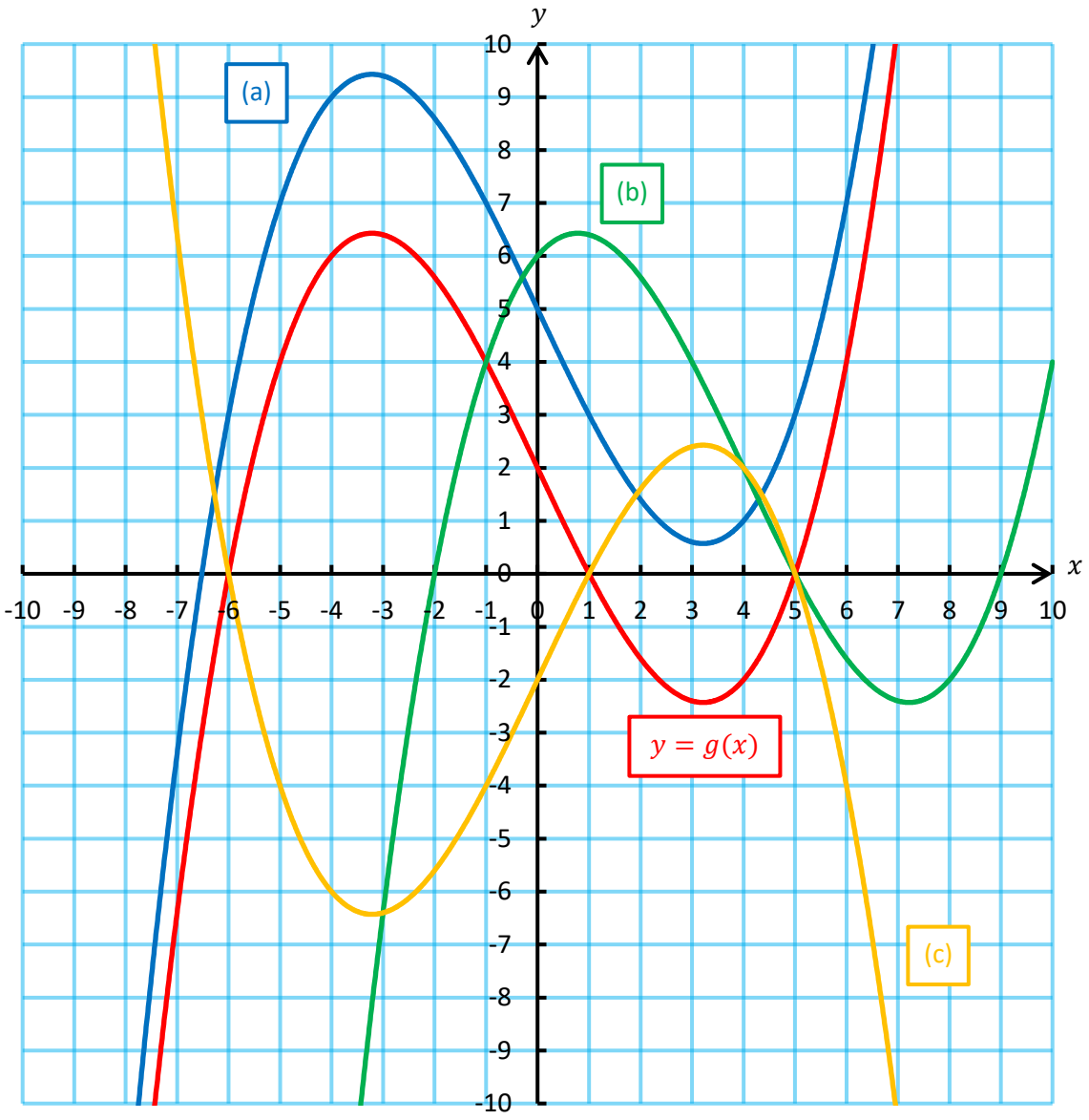
(b) Draw a set of axes with x and y values going from –10 to 10.
Plot the graph of $y = \frac{1}{x}$ on the axes, before using transformations of functions to plot the following graphs.
(i) $y = -\frac{1}{x}$ (ii) $y = \frac{1}{x} - 5$ (iii) $y = \frac{1}{x+4} - 5$

(c) Draw a set of axes with x and y values going from –10 to 10.
Plot the graph of $y = x^3$ on the axes, before using transformations of functions to plot the following graphs.
(i) $y = (x+4)^3$ (ii) $y = \frac{1}{4}x^3$ (iii) $y = \left(\frac{1}{4}x\right)^3$

Exercise 7

H

The following graph shows the function $y = g(x)$.
Write, in terms of $g(x)$, a function for each of the other graphs that are shown.



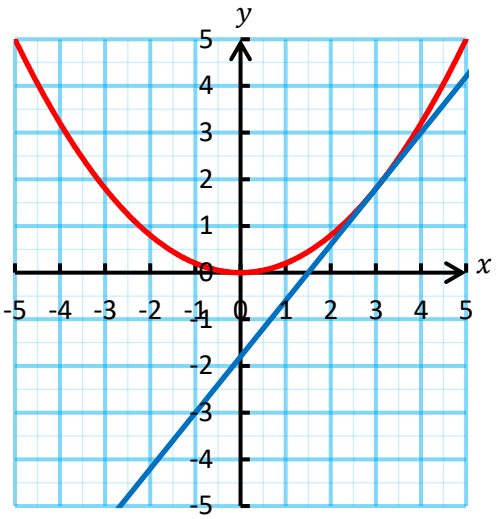
Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Pre-Calculus

Tangent to a Function

Given a **non-linear function** (a function that is not a straight line), a **tangent** to a specific point on the function is a straight line that meets the point so that the gradient of the tangent is equal to the gradient of the function at that point.



Example

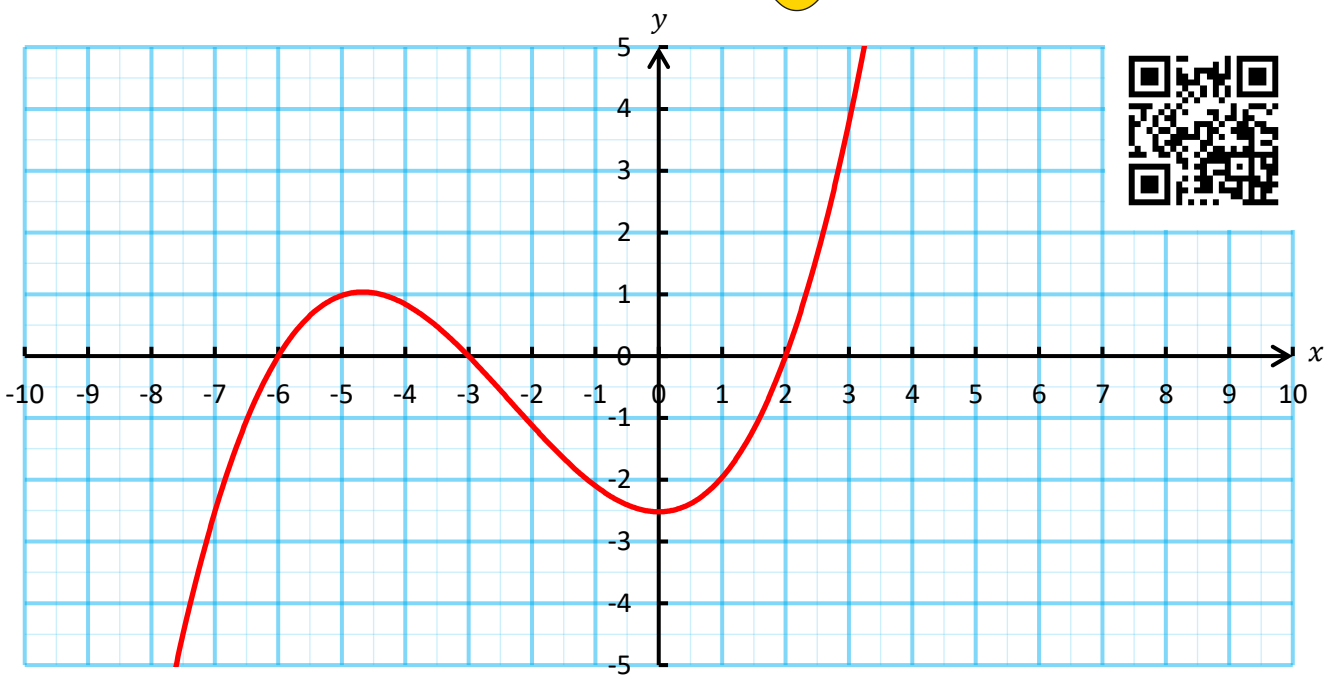
The graph on the right shows a tangent to the red curve at the point where $x = 3$. This tangent has a positive gradient.

Exercise 8

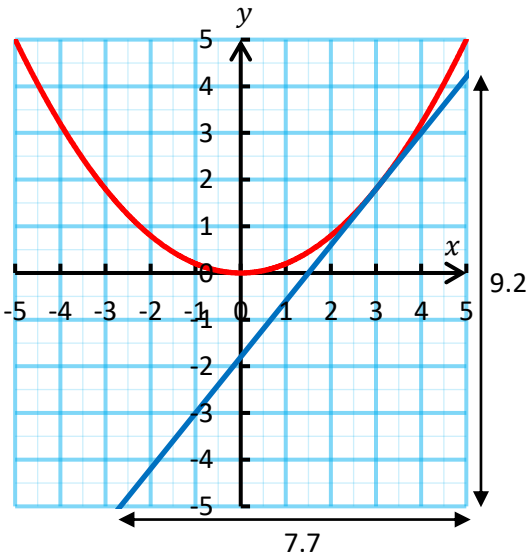
For the curve shown below, draw (by eye) a tangent for the points where

(a) $x = -5$ (b) $x = 2.5$ (c) $x = 0$ (d) $x = -2$





The gradient of a tangent gives the **rate of change** for the point under consideration. That is, it represents how much the variable on the vertical axis (y) changes with respect to one unit of the unit on the horizontal axis (x). The steeper the tangent, the greater the rate of change.



For the example on the right, the gradient of the tangent is the change in the vertical distance divided by the change in the horizontal distance.

$$\text{Gradient of the tangent} = \frac{9.2}{7.7}$$

$$\text{Gradient of the tangent} = 1.19 \text{ to 2 decimal places.}$$



Exercise 9



Calculate the gradient of your four tangents from Exercise 8.

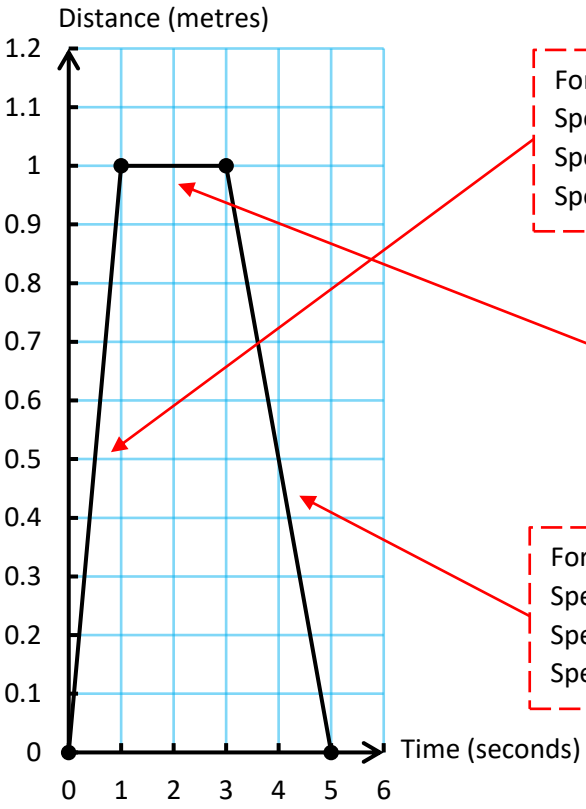
Travel Graphs

We saw travel graphs for the first time in the *Movement with Sphero* workbook in year 9.

Example

The following graph shows the journey of a sphero over 5 seconds.

Watch the journey of the sphero here:



For the **first** part of the journey,
Speed = Change in distance ÷ Change in time
Speed = 1 ÷ 1
Speed = 1 metre per second.

For the **second** part of the journey,
Speed = Change in distance ÷ Change in time
Speed = 0 ÷ 2
Speed = 0 metres per second.

For the **third** part of the journey,
Speed = Change in distance ÷ Change in time
Speed = 1 ÷ 2
Speed = 0.5 metres per second.

Speed is an example of a **scalar** measure, where the **direction of travel** is ignored. So, during the third part of the journey, where the sphero returns to its original position, and the gradient of the distance-time graph is negative, the speed remains positive. In order to consider the direction of travel, and therefore differentiate between positive and negative gradients in a distance-time graph, we need to consider new measures that are **vector** measures.

Displacement and Velocity

Distance and **displacement** are measures that measure how far an object is from a specific origin, but displacement takes into account the **direction of travel** as well as the distance from the origin. For example, at the end of the first part of the above journey, during which time the sphero moves away from the origin (the starting point), the distance and displacement are both 1 metre. For the final part of the journey however, where the sphero travels back to the origin, the distance travelled is 1 metre, but the displacement is –1 metre. The negative sign represents the direction of travel of the sphero, and also reflects the negative gradient of the distance-time graph for this part of the journey.

Velocity is the vector that corresponds to the scalar measure speed, and is calculated using the formula

$$\text{Velocity} = \frac{\text{Change in displacement}}{\text{Change in time}}$$

For the final part of the above journey, where the speed of the sphero is 1 ÷ 2 = 0.5 metres per second, the velocity is –1 ÷ 2 = –0.5 metres per second. Again, the velocity takes into account that the distance-time graph has a negative gradient during the third part of the journey.

Exercise 10

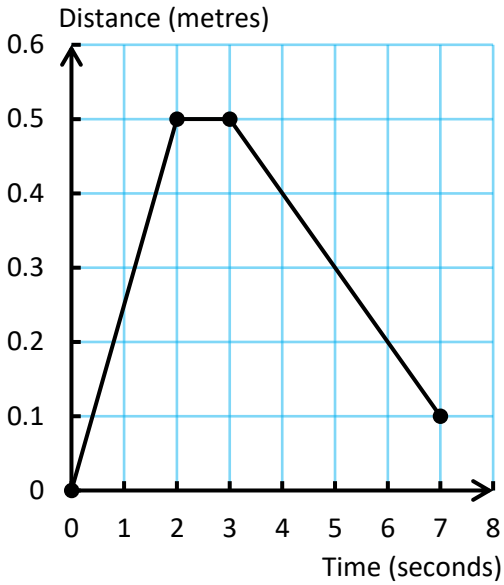
Applying

H

The graph on the right shows the journey of a sphero over 7 seconds. Complete the following table.

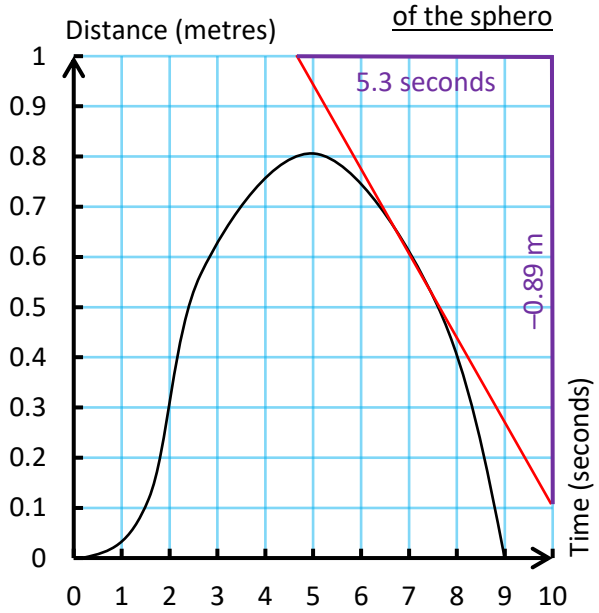
Part of the journey	Change in distance	Change in displacement	Change in time	Speed (scalar)	Velocity (vector)
1st					
2nd					
3rd					

Distance-time graph for Exercise 10



In any **non-linear** distance-time graph, it is possible to estimate the velocity at any point by calculating the gradient of the tangent at that point.

Distance-time graph to show the journey of the sphero



Example

The graph on the left shows the journey of a sphero over 9 seconds. Estimate the velocity of the sphero at time 7 seconds.

Answer: To begin, we draw (by eye) a tangent to the curve at 7 seconds. This tangent is shown in red on the graph.

Next, we complete a right-angled triangle around the tangent, in order to measure the change in displacement and change in time. From these we can calculate an estimate for the velocity:

$$\text{Velocity} = \frac{\text{Change in displacement}}{\text{Change in time}}$$

$$\text{Velocity} = \frac{-0.89}{5.3}$$

Velocity = –0.17 metres per second, to 2 decimal places.

Exercise 11

H

The graph on the right shows the journey of a sphero over 10 seconds.

(a) Estimate the velocity of the sphero at time 2 seconds.

(b) Estimate the velocity of the sphero at time 8 seconds.

(c) Estimate the speed of the sphero at time 2 seconds.

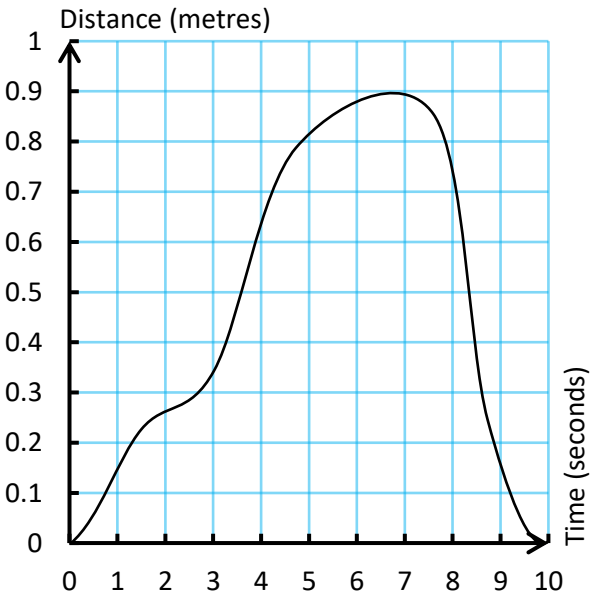
(d) Estimate the speed of the sphero at time 8 seconds.

(e) What is the total distance travelled by the sphero during its journey?

(f) What is the average speed for the entire journey?

(g) What is the average velocity for the entire journey?

Distance-time graph for Exercise 11



Acceleration

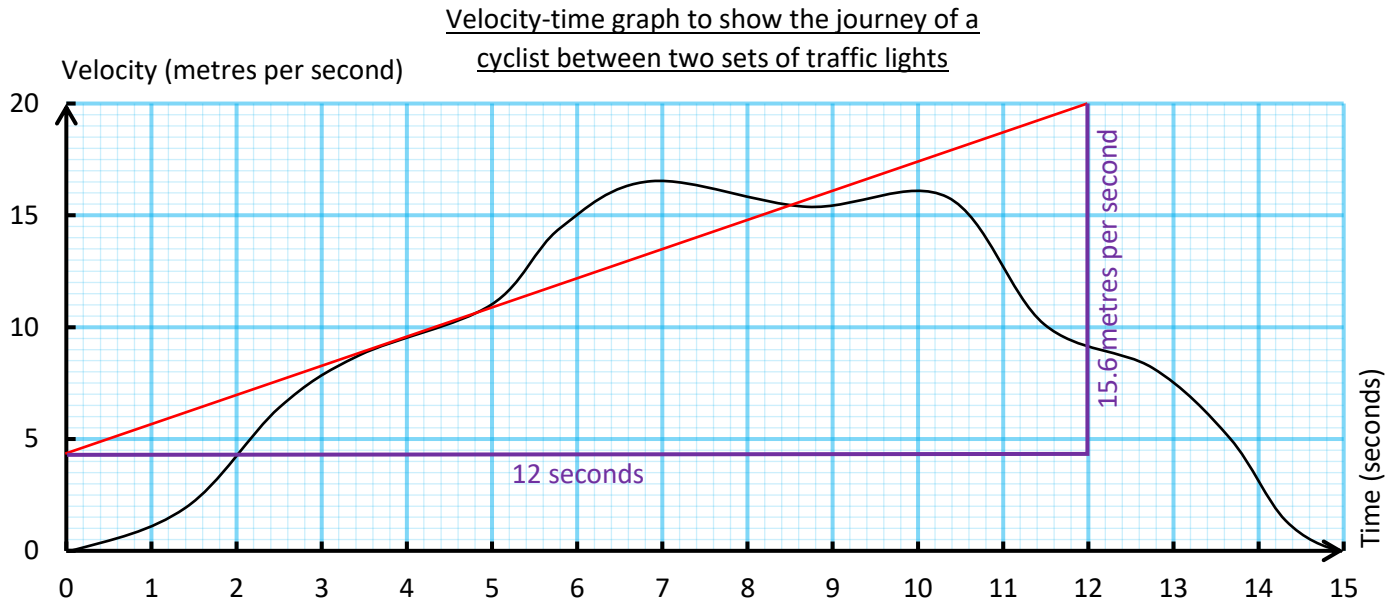
Whilst velocity is a vector that measures the rate of change of displacement with respect to time, **acceleration** is a vector that measures the rate of change of velocity with respect to time.
We can use the formula

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Change in time}}$$

to calculate acceleration, or we can estimate the gradient of a tangent to a point on a velocity-time graph.

Example

The following graph shows the velocity of a cyclist on a straight road between two sets of traffic lights. Calculate the acceleration of the cyclist at time 4 seconds. Give the units of your answer.



Answer: To begin, we draw (by eye) a tangent to the curve at time 4 seconds. This tangent is shown in red on the graph.

Next, we complete a right-angled triangle around the tangent, in order to measure the change in velocity and change in time. From these we can calculate an estimate for the acceleration:

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Change in time}}$$

$$\text{Acceleration} = \frac{15.6}{12}$$

Acceleration = 1.3 metres per second squared, or m/s².

Units of Acceleration

The following table shows some common units for measuring acceleration.

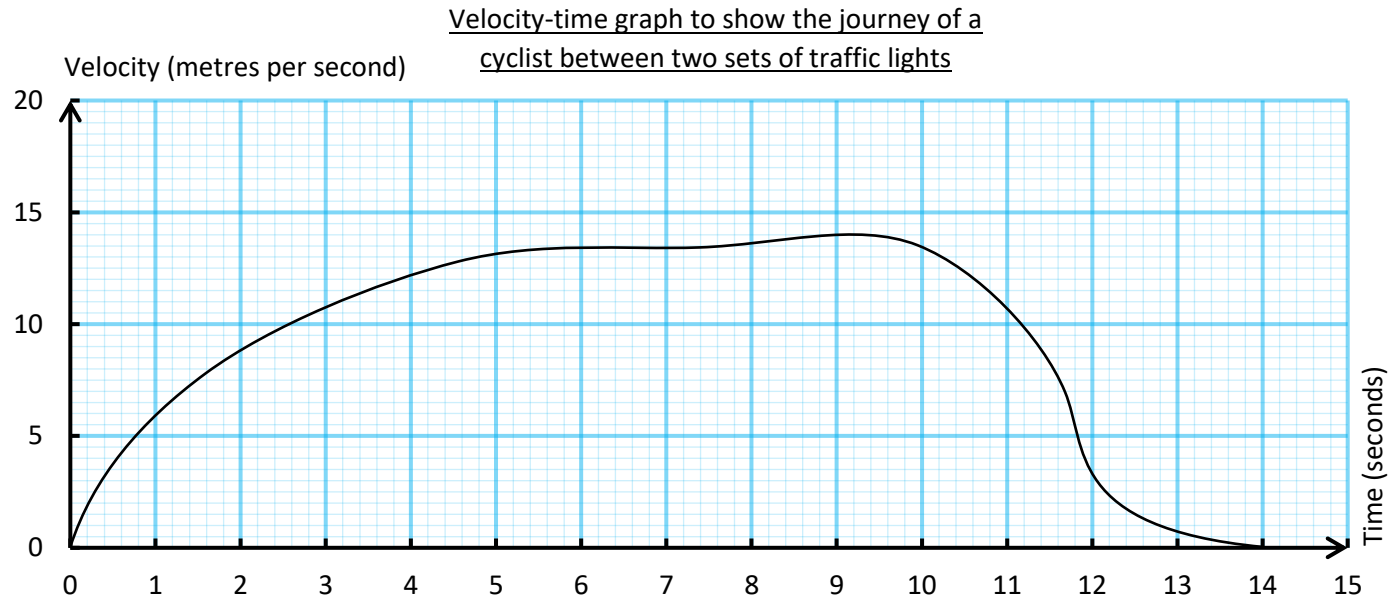
Displacement Unit	Time Unit	Velocity Unit	Acceleration Unit
Metres (m)	Seconds (s)	Metres per second (m/s)	Metres per second per second, or metres per second squared (m/s²)
Kilometres (km)	Hours (h)	Kilometres per hour (km/h)	Kilometres per hour squared (km/h²) or kilometres per hour per second
Miles (mi)	Hours (h)	Miles per hour (mph)	Miles per hour squared (mph²) or miles per hour per second

The acceleration unit can depend on the horizontal axis unit of a velocity-time graph

Exercise 12

H

The following graph shows the velocity of a cyclist on a straight road between two sets of traffic lights.



(a) Calculate the acceleration of the cyclist at time 2 seconds. Give the unit of your answer.

(b) Calculate the acceleration of the cyclist at time 11 seconds. Give the unit of your answer.

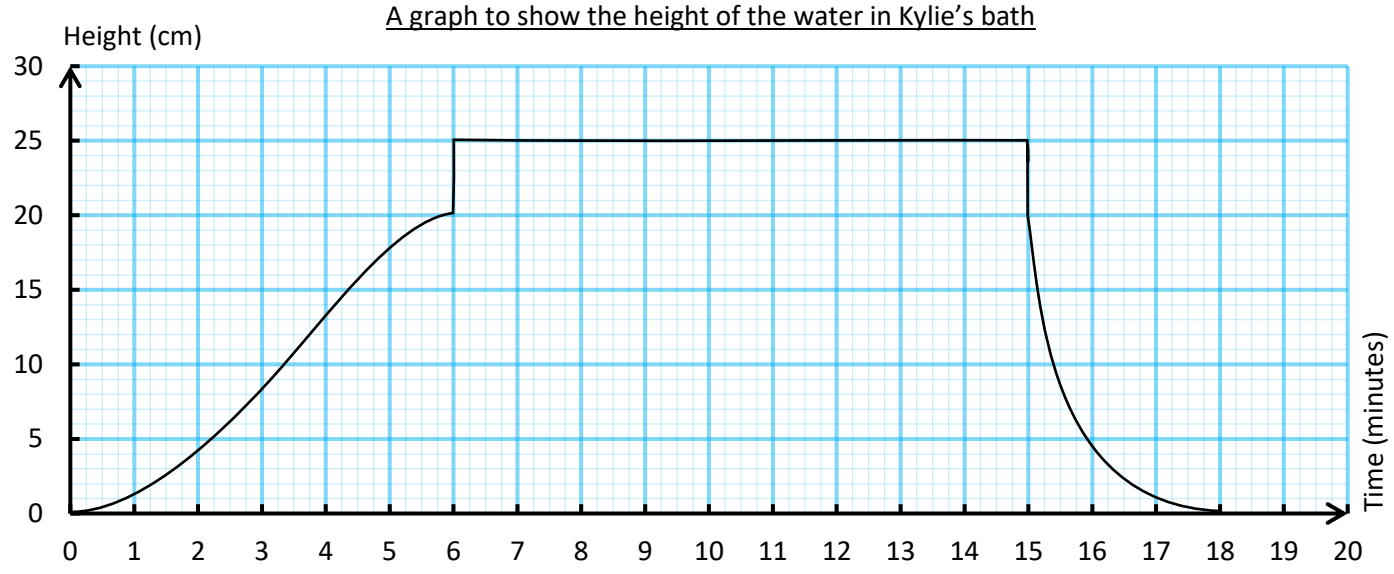
(c) What is the maximum velocity of the cyclist during the journey? At which time does this happen?

(d) Give the times at which the cyclist is travelling at a velocity of 5 metres per second.

Exercise 13

H

The following graph shows the height of the water in a bath as Kylie takes a bath one Sunday night.



(a) What is the height of the water in the bath at time 2 minutes?



(b) What happens at time 6 minutes?

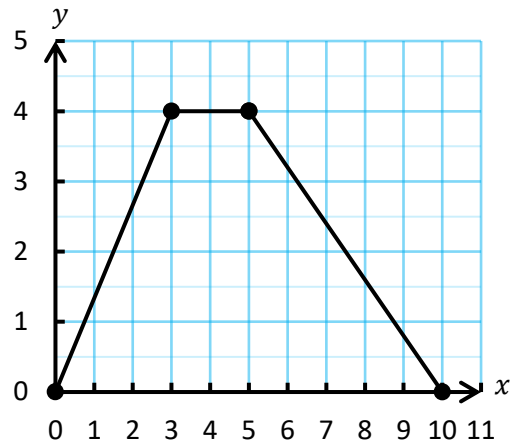
(c) How many minutes did Kylie spend in the bath?

(d) What is the rate of change of the height of the water in the bath at time 4 minutes?

(e) What is the rate of change of the height of the water in the bath at time 15 minutes 30 seconds?

The Area Between a Graph and the x–axis

If a graph uses straight lines only, then it is possible to find the area between the lines and the x–axis using formulae for the area of common 2-D shapes.



For example, considering the graph on the left, we can calculate the area between the lines of the graph and the x–axis by calculating the area of the trapezium:

Formula for the area of a trapezium = $\frac{1}{2}(a+b)h$

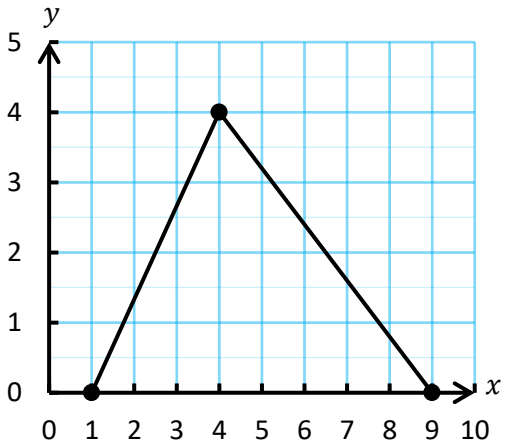
2 + 10 = 12	Calculating $a+b$
12 ÷ 2 = 6	Calculating $\frac{1}{2}(a+b)$
6 × 4 = 24	Calculating $\frac{1}{2}(a+b)h$

So, the required area is 24 square units.

Exercise 14

Skill H

For the graph shown on the right, find the area between the lines of the graph and the x-axis.



The Trapezium Rule

If we have a non-linear graph, then we cannot use a method similar to the above method to find the area between the graph and the x–axis – we must estimate the area using the **trapezium rule**.

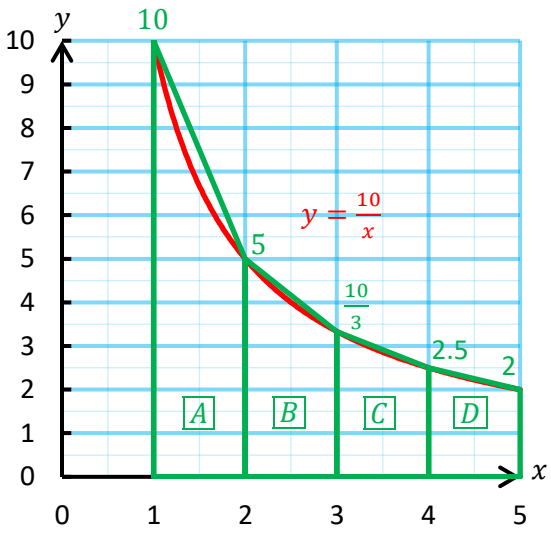
Example



The graph on the left shows the function $y = \frac{10}{x}$ between $x = 1$ and $x = 5$. Use the ordinates $x = 1$, $x = 2$, $x = 3$, $x = 4$ and $x = 5$ to estimate the area enclosed by the curve $y = \frac{10}{x}$ and the x–axis, between $x = 1$ and $x = 5$.

Answer: The first step is to add verical lines to the graph corresponding to the 5 ordinates, and then to complete the 4 trapezia formed by the ordinates. We can then read from the graph (or calculate) the values of the function for each of the five ordinates.

The estimate of the area between the curve and the x–axis is the total area of the four trapezia.



A	$5 + 10 = 15$	B	$5 + \frac{10}{3} = \frac{25}{3}$
	$15 \div 2 = 7.5$		$\frac{25}{3} \div 2 = \frac{25}{6}$
	$7.5 \times 1 = 7.5$		$\frac{25}{6} \times 1 = \frac{25}{6}$
C	$\frac{10}{3} + 2.5 = \frac{35}{6}$	D	$2.5 + 2 = 4.5$
	$\frac{35}{6} \div 2 = \frac{35}{12}$		$4.5 \div 2 = 2.25$
	$\frac{35}{12} \times 1 = \frac{35}{12}$		$2.25 \times 1 = 2.25$

Final answer: $7.5 + \frac{25}{6} + \frac{35}{12} + 2.25 = \frac{101}{6}$ square units (or $16.8\dot{3}$)

The Trapezium Rule Formula

Instead of calculating the area of all the individual trapezia, we can use the following formula to estimate the area between the curve and the x–axis:

$$\frac{1}{2}h\{y_0 + y_n + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\}$$

Here, h is the height of each individual trapezium, and y_0, y_1, y_2, ..., y_n are the values of the function for each of the ordinates x_0, x_1, x_2, ..., x_n.

For the example on the previous page, we have $h = 1$, and the following table summarises the values of x_n and y_n.

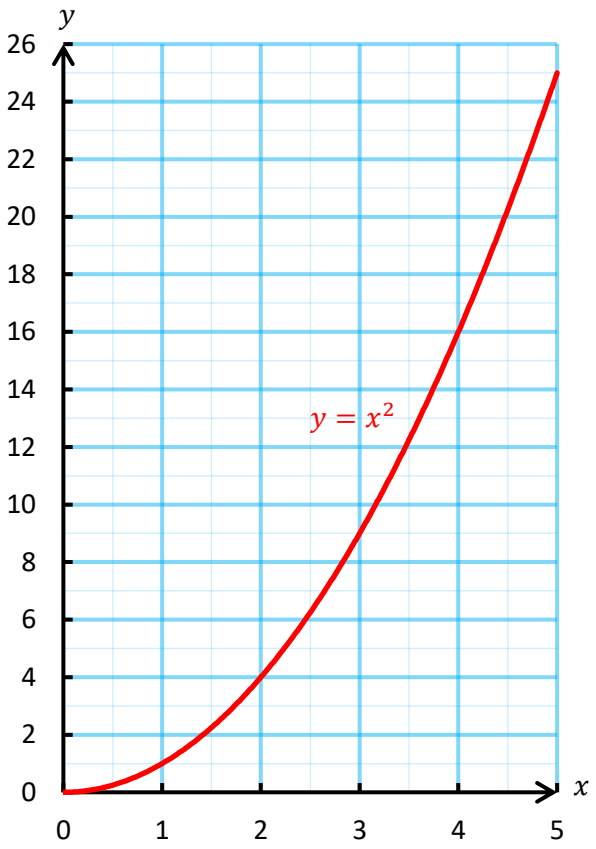
n	0	1	2	3	4
x_n	1	2	3	4	5
y_n	10	5	$\frac{10}{3}$	2.5	2

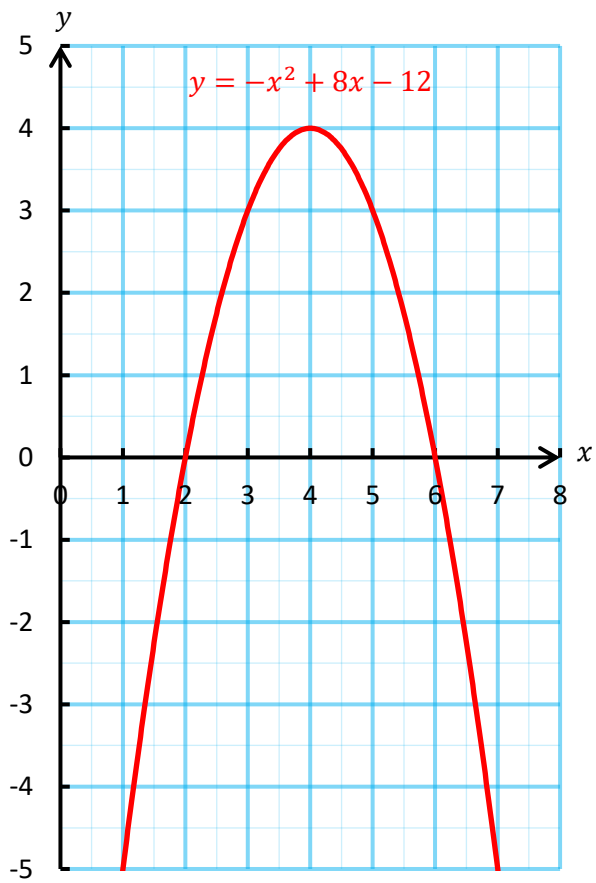
Using the formula for the trapezium rule, the estimate of the area between the curve and the x–axis is given by

$$\frac{1}{2} \times 1 \times \left\{10 + 2 + 2\left(5 + \frac{10}{3} + 2.5\right)\right\} = \frac{101}{6} \text{ square units (or } 16.8\dot{3})$$

Exercise 15 H

The graph on the right shows the function $y = x^2$ between $x = 0$ and $x = 5$. Use the trapezium rule and the ordinates $x = 1$, $x = 2$, $x = 3$, $x = 4$ and $x = 5$ to estimate the area enclosed by the curve $y = x^2$ and the x–axis, between $x = 1$ and $x = 5$.





Exercise 16 H

The graph on the left shows the function $y = -x^2 + 8x - 12$ between $x = 1$ and $x = 7$. Use the trapezium rule and the ordinates $x = 2$, $x = 3$, $x = 4$, $x = 5$ and $x = 6$ to estimate the area enclosed by the curve $y = -x^2 + 8x - 12$ and the x–axis, between $x = 2$ and $x = 6$.

Applications

Sometimes, the area between a graph and the x–axis has a special meaning.

- In a **velocity-time graph**, the area between the graph and the x–axis gives the **distance travelled**.
- In a graph that plots **time** (in years) against **yearly salary**, the area between the graph and the x–axis gives the **total earnings**.
- In a graph that plots **time** (in minutes) against the **rate of filling a petrol tank** (in litres per minute), the area between the graph and the x–axis gives the **volume of petrol** that has been added to the petrol tank.

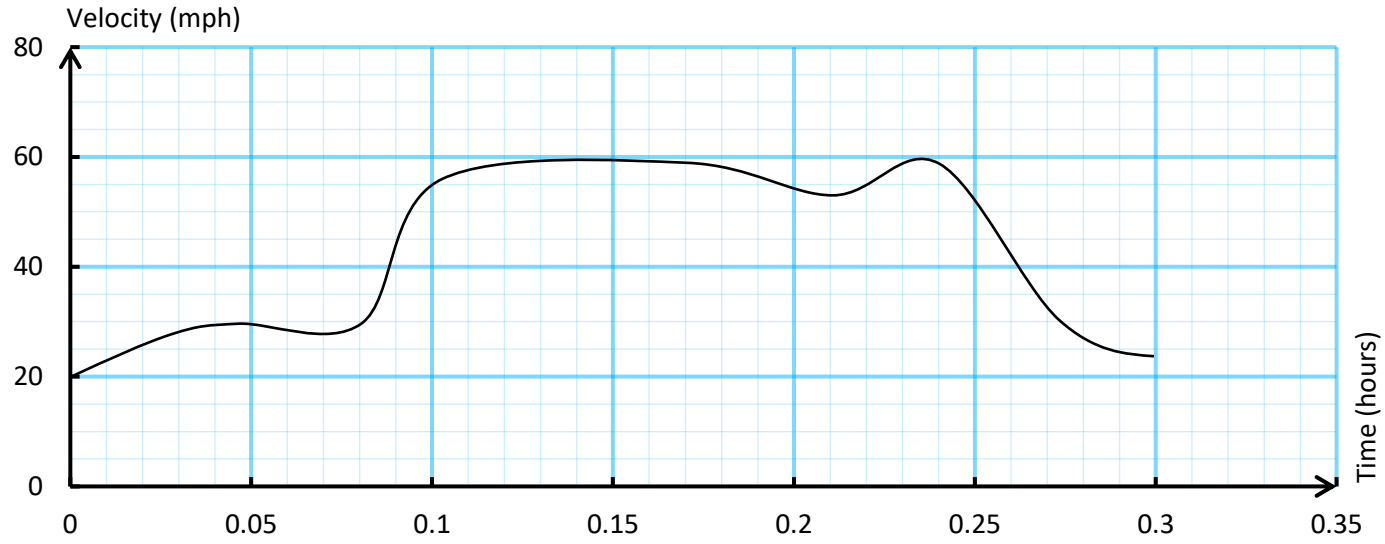
In general, the area between the graph and the x–axis represents the measure that has unit given by multiplying the unit on the vertical axis by the unit on the horizontal axis.

Exercise 17

Applying H

The following graph shows the velocity of a car as it travels between Bangor and Caernarfon.

Velocity-time graph to show the journey of a car travelling between Bangor and Caernarfon



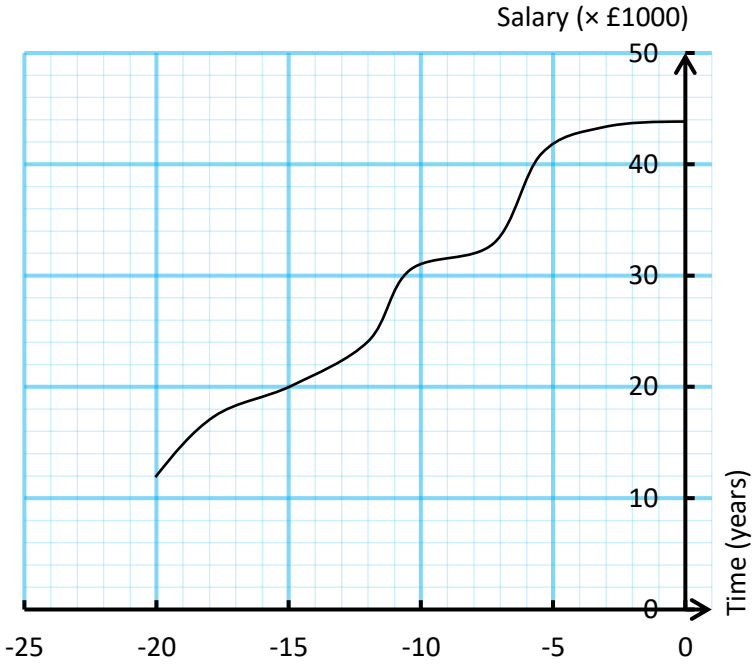
Use the trapezium rule and the ordinates $x = 0$, $x = 0.1$, $x = 0.2$ and $x = 0.3$ to estimate the distance that the car has travelled during this journey.

Exercise 18 H

The graph on the right shows how Trefor's yearly salary has increased during the past 20 years.

Use the trapezium rule and the ordinates $x = -20$, $x = -15$, $x = -10$, $x = -5$ and $x = 0$ to estimate Trefor's total earnings over the past 20 years.

A graph to show Trefor's salary during the past 20 years





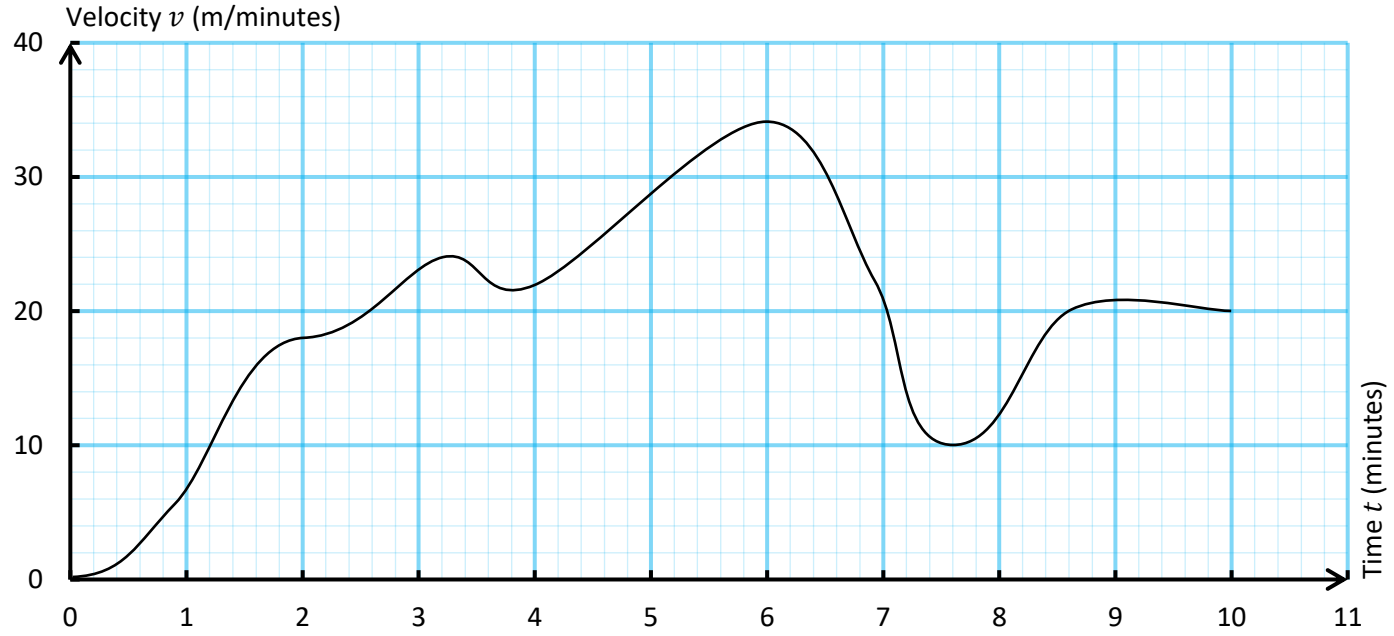
Exercise 19 (Revision)

H

Non conducted an experiment. She used equipment to measure the velocity, v, of an object during the first 10 minutes of the experiment.

The velocity-time graph for the experiment is shown below.

A graph to show the velocity of an object during an experiment



(a) Write down the gradient of the curve when the time is 7.6 minutes.

(b) Find an estimate for the acceleration of the object when the time is 4.5 minutes.

(c) Use the trapezium rule and the ordinates $t = 0$, $t = 2$, $t = 4$, $t = 6$, $t = 8$ and $t = 10$ to estimate the area enclosed by the curve, the positive time axis and the line $t = 10$.

(d) Calculate an estimate of the distance travelled by the object during the first 10 minutes of Non's experiment, giving your answer in kilometres.

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Further Changing the Subject

The purpose of **changing the subject** is to re-arrange a formula so that a particular **variable** appears on its own on the left-hand side of the formula.

Revision I

Exercise 20

(a) Make x the subject of the formula $2y = 3z + 5x$

(b) Make s the subject of the formula $t = \frac{s}{3}$

(c) Make a the subject of the formula $F = c(a - b)$

(d) Make z the subject of the formula $\frac{x}{z} = y$

(e) Make r the subject of the formula $C = \frac{4}{3}\pi r^3$

(f) Make e the subject of the formula $g = \frac{f}{e} + c$

Changing the subject where the subject appears more than once

Example

(a) Make x the subject of the formula $3z + 4x = yx + 6y$.

Answer: $3z + 4x = yx + 6y$

$4x - yx = 6y - 3z$ [Subtract $3z$ from both sides; subtract yx from both sides]

$x(4 - y) = 6y - 3z$ [Factorise x on the left-hand side]

$x = \frac{6y-3z}{4-y}$ [Divide both sides by $4 - y$]



(b) Make x the subject of the formula $\frac{2x+y}{3-5x} = 2$.

Answer: $\frac{2x+y}{3-5x} = 2$

$2x + y = 2(3 - 5x)$ [Multiply both sides by $3 - 5x$]

$2x + y = 6 - 10x$ [Expand the bracket]

$2x + 10x = 6 - y$ [Subtract y from both sides; add $10x$ to both sides]

$12x = 6 - y$ [Collect like terms]

$x = \frac{6-y}{12}$ [Divide both sides by 12]

Exercise 21

Skill H

(a) Make x the subject of the formula $5z + 3x = xz + 3y$

(b) Make y the subject of the formula $5y - 3x = 2y + 3z$

(c) Make z the subject of the formula $\frac{4z-5y}{6-3z} = 6$

(d) Make f the subject of the formula $11f - 1 = 4g(3f + e)$

(e) Make f the subject of the formula $7f - 5 = 3g(2f + h)$

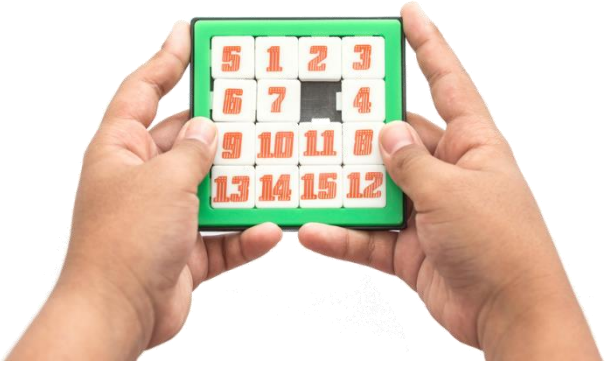
(f) Make k the subject of the formula $5(2k - m) = ck + 5$

(g) Make w the subject of the formula $8(w - 3y) = 3(w + 2y)$

(h) Make u the subject of the formula $\frac{8u+3y}{3-5u} = 3z$

(i) Make r the subject of the formula $5(r + 3t) = 7(2 - 6r)$

(j) Make v the subject of the formula $\frac{2}{3}v + 5w = \frac{1}{3}(3 - 2v)$



Example

(a) Make x the subject of the formula $z = \frac{1}{x} + \frac{1}{y}$.

Answer: $z = \frac{1}{x} + \frac{1}{y}$

$\frac{1}{x} + \frac{1}{y} = z$ [Swap sides]

$\frac{1}{x} = z - \frac{1}{y}$ [Subtract $\frac{1}{y}$ from both sides]

$x = \frac{1}{z - \frac{1}{y}}$ [Take reciprocal of both sides]

(b) Make x the subject of the formula $z = y + \sqrt{\frac{w}{x}}$.

Answer: $z = y + \sqrt{\frac{w}{x}}$

$y + \sqrt{\frac{w}{x}} = z$ [Swap sides]

$\sqrt{\frac{w}{x}} = z - y$ [Subtract y from both sides]

$\frac{w}{x} = (z - y)^2$ [Square both sides]

$w = x(z - y)^2$ [Multiply both sides by x]

$x(z - y)^2 = w$ [Swap sides]

$x = \frac{w}{(z-y)^2}$ [Divide by $(z - y)^2$]



Exercise 22 H

(a) Make x the subject of the formula $z = \frac{1}{x} - \frac{1}{y}$

(b) Make x the subject of the formula $z = \sqrt{\frac{w}{x}} - y$

(c) Make x the subject of the formula $z = \frac{y-w}{x}$

(d) Make x the subject of the formula $z = y + \sqrt[3]{\frac{x}{w}}$

(e) Make f the subject of the formula $\frac{a}{2f+1} = \frac{b}{3f-1}$

(f) Make p the subject of the formula $y = \frac{q}{p} - x$

(g) Make p the subject of the formula $q = \frac{y}{x^2-p}$

(h) Make x the subject of the formula $w = \sqrt{\frac{x-y}{x+y}}$

(i) Make h the subject of the formula $A = \pi r\sqrt{h^2 + r^2}$

(j) Make l the subject of the formula $t = 2\pi\sqrt{\frac{l}{g}}$

(k) Make b the subject of the formula $m = \frac{ax+by}{a+b}$

(l) Make y the subject of the formula $s = \sqrt{\frac{x^2+y^2}{n}}$

(m) Make q the subject of the formula $y = \frac{x-np}{\sqrt{npq}}$

(n) Make x the subject of the formula $F = \frac{x^2}{1-x^2}$

(o) Make x the subject of the formula $y = \frac{2}{x+3} - 5$

(p) Make a the subject of the formula $a - b = \frac{a+2}{b}$



Key words	Corrections	I am happy with...	I need to revise...

The Quadratic Formula

It is possible to solve some quadratic equations by factorisation.

Factorise using the splitting method or the detective method

Difference between two squares

Example

Solve the following equations.

(a) $x^2 + 9x + 14 = 0$

$x^2 + 9x + 14 = 0$
$(x + 2)(x + 7) = 0$
Either $x + 2 = 0$ or $x + 7 = 0$
$x = -2$ $x = -7$

(b) $3x^2 + 13x + 4 = 0$

$3x^2 + 13x + 4 = 0$
$(3x + 1)(x + 4) = 0$
Either $3x + 1 = 0$ or $x + 4 = 0$
$3x = -1$ $x = -4$
$x = -\frac{1}{3}$

(c) $4x^2 - 9 = 0$

$4x^2 - 9 = 0$
$(2x + 3)(2x - 3) = 0$
Either $2x + 3 = 0$ or $2x - 3 = 0$
$2x = -3$ $2x = 3$
$x = -\frac{3}{2}$ $x = \frac{3}{2}$

Exercise 23

Revision H

Solve the following equations.

(a) $x^2 + 10x + 24 = 0$ (b) $2x^2 + 21x + 40 = 0$ (c) $x^2 - 16 = 0$

(d) $x^2 + 3x - 54 = 0$ (e) $2x^2 + 7x - 15 = 0$ (f) $25x^2 - 36 = 0$

(g) $x^2 - 8x + 16 = 0$ (h) $4x^2 - 14x + 6 = 0$ (i) $8y^2 - 98 = 0$

If we cannot solve a quadratic equation through factorisation, then we can attempt to solve the equation using the **quadratic formula**.

THE QUADRATIC FORMULA

The solutions of $ax^2 + bx + c = 0$ where $a \neq 0$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The symbol $\pm$ means 'plus or minus'.

If $b^2 - 4ac < 0$, there are no real solutions.

Round off your answers to 2 decimal places at the end.

@MATHEMATEG

Exercise 24

Skill

H

The general form of a quadratic equation is $ax^2 + bx + c = 0$.
Write a, b and c for each of the following quadratic equations.

(a) $x^2 + 3x + 7 = 0$ (b) $2x^2 - 8x + 11 = 0$ (c) $9x^2 - 4 = 0$

(d) $5x - 3x^2 + 17 = 0$ (e) $23 + 5x + 3x^2 = 0$ (f) $4x^2 - 3x = 0$

Example

Using the quadratic formula, solve the equation $5x^2 + 10x - 3 = 0$.

Answer: In this case, we have $a = 5$, $b = 10$ and $c = -3$.
Substituting these values into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

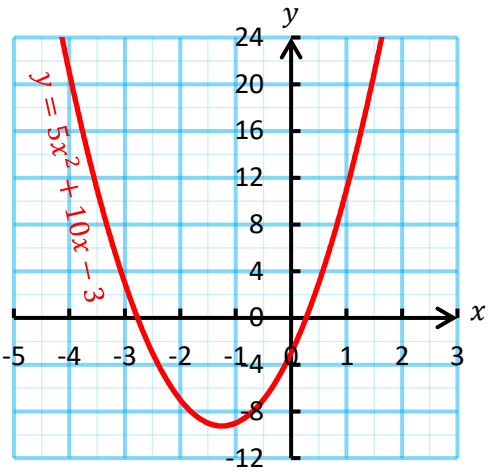
$$x = \frac{-10 \pm \sqrt{10^2 - 4 \times 5 \times -3}}{2 \times 5}$$

$$x = \frac{-10 \pm \sqrt{100 + 60}}{10}$$

$$x = \frac{-10 \pm \sqrt{160}}{10}$$

Either $x = \frac{-10+\sqrt{160}}{10}$ or $x = \frac{-10-\sqrt{160}}{10}$

Either $x = 0.26$ to 2 d.p., or $x = -2.26$ to 2 d.p.



H

Exercise 25

Using the quadratic formula, solve the following equations.
Round off your answers correct to two decimal places.
If there are no real solutions, state this.

(a) $x^2 + 8x + 5 = 0$ (b) $x^2 - 8x + 5 = 0$ (c) $x^2 + 8x - 5 = 0$

(d) $x^2 - 8x - 5 = 0$ (e) $x^2 + 8x + 25 = 0$ (f) $2x^2 + 10x + 5 = 0$

(g) $3x^2 + 8x + 1 = 0$ (h) $3x^2 - 6x - 4 = 0$ (i) $2x^2 - 9x + 8 = 0$

(j) $2x^2 + x - 20 = 0$ (k) $7x^2 + 3x - 1 = 0$ (l) $x^2 - 2x - 100 = 0$

(m) $5x^2 + 11x + 3 = 0$ (n) $4x^2 + 7x + 6 = 0$ (o) $4x^2 - 12 + 3x = 0$

Exercise 26

Applying

H

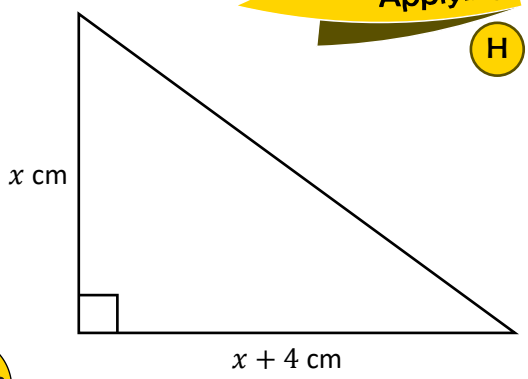
The area of the right-angled triangle shown on the right is 45 cm².

(a) Show that x satisfies the equation $x^2 + 4x - 90 = 0$.

(b) Use the quadratic formula to solve the equation $x^2 + 4x - 90 = 0$, giving your answers correct to two decimal places.

(c) Using your answer to (b), find the base and height of the triangle.

(d) Calculate the length of the hypoteuse of the triangle.
Give your answer correct to one decimal place.

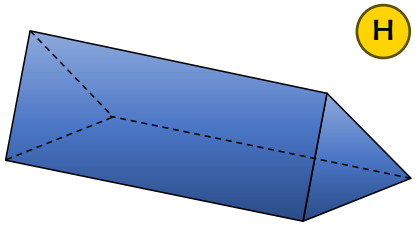


Challenge!

Solve the equation $5x^3 + 2x^2 - 11x = 0$

Exercise 27

H



The diagram shows a triangular prism.

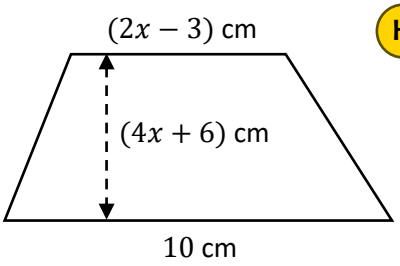
The cross-sectional area of the triangular prism is $3x$ cm² and its length is $(x + 4)$ cm. The volume of the prism is 89 cm³.

(a) Show that x satisfies the equation $3x^2 + 12x - 89 = 0$.

(b) Use the formula method to solve the equation $3x^2 + 12x - 89 = 0$, giving your answer correct to one decimal place. Hence, write the length of the prism correct to one decimal place.

Exercise 28

H



The diagram shows a trapezium.

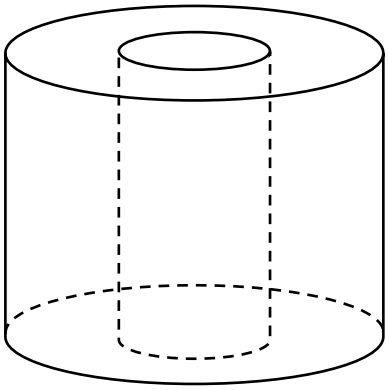
The lengths of the parallel sides of the trapezium are 10 cm and $(2x - 3)$ cm. The height of the trapezium is $(4x + 6)$ cm and its area is 70 cm².

(a) Show that $4x^2 + 20x - 49 = 0$.

(b) Use the quadratic formula to solve the equation $4x^2 + 20x - 49 = 0$.
Give your answers correct to two decimal places.
Hence, write the height of the trapezium correct to two decimal places.

Exercise 29

H



A length of plastic tube has a uniform circular cross-section.
The radius of the circular hole in the middle is x cm.
The thickness of the plastic is 3 cm and the length of the plastic tube is $5x$ cm.

(a) Show that the volume of the plastic used to make the tube is $(30\pi x^2 + 45\pi x)$ cm³.

(b) Given that the volume of plastic used to make the tube is 88π cm³, find the length of the tube correct to one decimal place.

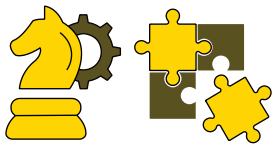
Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Algebraic Fractions

Numerical Fractions

In order to be able to work with fractions that include algbebraic expressions, it is a good idea to first revise how to work with numerical fractions.



Example

Here are two methods of calculating $\frac{3}{4}+\frac{1}{6}$.

The Traditional Method

Step 1: Find the **lowest common denominator** of the two fractions. Here, the lowest common multiple of 4 and 6 is 12.

Step 2: Write **equivalent fractions** for $\frac{3}{4}$ and $\frac{1}{6}$, using 12 as the common denominator.

$\frac{3}{4} \xrightarrow{\times 3} \frac{9}{12}$ $\qquad$ $\frac{1}{6} \xrightarrow{\times 2} \frac{2}{12}$

Step 3: **Add** the two new fractions.

$$\frac{9}{12}+\frac{2}{12}=\frac{11}{12}$$

Step 4: Check to see if it is possible to simplify the answer. (It is not possible in this case.)

*Step 1: You can find a common denominator by multiplying the two original denominators, however this might not be the **lowest** common denominator.*

The Peanut Method

Step 1: Draw the following template. (Note that the first fraction goes on the top and the second fraction goes on the left.)

	3	4
1	×	
6		

Step 2: Fill in the gaps in the table by **multiplying** the numbers.

	3	4
1	×	4
6	18	24

Step 3: **Add** the two numbers that form the peanut shape.

	3	4
1	×	4
6	18	24

18 + 4 = 22

The answer is $\frac{22}{24}$.

Step 4: Check to see if it is possible to simplify the answer. Here, $\frac{22}{24}$ simplifies to give the final answer $\frac{11}{12}$.

Revision F

Exercise 30

Calculate the following. Give your answers in their simplest form.

(a) $\frac{2}{5}+\frac{1}{3}$ (b) $\frac{1}{4}+\frac{3}{8}$ (c) $\frac{9}{10}-\frac{3}{5}$ (d) $\frac{3}{7}+\frac{1}{6}$

(e) $\frac{7}{8}-\frac{2}{5}$ (f) $\frac{2}{9}+\frac{2}{3}$ (g) $\frac{11}{12}-\frac{1}{2}$ (h) $2\frac{1}{3}+4\frac{3}{4}$

Algebraic Fractions

Algebraic fractions contain at least one numerator or denominator that is an algebraic expression.

Examples of numerical fractions	$\frac{3}{4}$	$\frac{4}{9}$	$\frac{12}{5}$	$-\frac{4}{5}$
Examples of algebraic fractions	$\frac{x+2}{3}$	$\frac{5}{x-2}$	$\frac{x+7}{x^2+x-1}$	$-\frac{1}{y}$

Example

Here are two methods of writing $\frac{2}{3x+4}+\frac{3}{x+2}$ as a single fraction in its simplest form.

The traditional method

Step 1: Find the **lowest common denominator** of $3x+4$ and $x+2$. Here, the smallest expression that is a multiple of both $3x+4$ and $x+2$ is the product $(3x+4)(x+2)$.

Step 2: Write **equivalent fractions** for $\frac{2}{3x+4}$ and $\frac{3}{x+2}$, with $(3x+4)(x+2)$ appearing in the denominator.

$$\frac{2}{3x+4} \xrightarrow{\times (x+2)} \frac{2(x+2)}{(3x+4)(x+2)}$$

$$\frac{3}{x+2} \xrightarrow{\times (3x+4)} \frac{3(3x+4)}{(3x+4)(x+2)}$$

Step 3: **Add** the two new fractions.

$$\begin{aligned}&\frac{2(x+2)}{(3x+4)(x+2)}+\frac{3(3x+4)}{(3x+4)(x+2)}\\ &=\frac{2(x+2)+3(3x+4)}{(3x+4)(x+2)}\\ &=\frac{2x+4+9x+12}{(3x+4)(x+2)}\\ &=\frac{11x+16}{(3x+4)(x+2)}\end{aligned}$$

Step 4: Check to see if it is possible to **simplify** the answer, e.g. through factorising. (This is not possible in this case.)

The peanut method

Step 1: Draw the following template. (Note that the first fraction goes on the top and the second fraction goes on the left.)

	2	$3x+4$
3	$\times$	
$x+2$		

Step 2: Fill in the gaps in the table by **multiplying** the expressions.

	2	$3x+4$
3	$\times$	$3(3x+4)$
$x+2$	$2(x+2)$	$(x+2)(3x+4)$

Cam 3: **Add** the two expressions that form the peanut shape.

$$\begin{aligned}&2(x+2)+3(3x+4)\\ &=2x+4+9x+12\\ &=11x+16\end{aligned}$$

The answer is $\frac{11x+16}{(3x+4)(x+2)}$

Step 4: Check to see if it is possible to **simplify** the answer, e.g. through factorising. (This is not possible in this case.)

Exercise 31



Write each of the following as a single fraction. Give your answers in their simplest form.

(a) $\frac{5}{3x+2}+\frac{4}{x+1}$ (b) $\frac{7}{2x+3}+\frac{6}{x+4}$ (c) $\frac{1}{x+5}+\frac{2}{2x+7}$

(d) $\frac{5}{3x+2}+\frac{4}{x-1}$ (e) $\frac{7}{2x-3}+\frac{6}{x+4}$ (f) $\frac{1}{x+5}-\frac{2}{2x+7}$

(g) $\frac{5}{4x+2}-\frac{3}{x-8}$ (h) $\frac{3}{x-5}-\frac{4}{x+2}$ (i) $\frac{9}{x-3}+\frac{6}{2x-5}$

(j) $\frac{x+1}{2}+\frac{x+4}{3}$ (k) $\frac{x+3}{5}-\frac{x+1}{2}$ (l) $\frac{x-2}{4}+\frac{x-5}{3}$

(m) $\frac{3x+1}{4}+\frac{2x-5}{2}$ (n) $\frac{x+2}{3}+\frac{2}{x-1}$ (o) $\frac{1}{x-2}+\frac{x+3}{4}$

(p) $\frac{2x+5}{3}-\frac{2}{x-5}$ (q) $\frac{3x+2}{4x-1}+\frac{2}{7}$ (r) $\frac{5+3x}{2x-3}+\frac{1}{4}$

(s) $\frac{2}{x+3}+\frac{x+1}{x}$ (t) $\frac{x+2}{3x}+\frac{x}{x+3}$ (u) $\frac{x+3}{x-4}+\frac{x-3}{x+4}$

(v) $\frac{2x+3}{x-3}-\frac{x-2}{x-5}$ (w) $\frac{x+2}{3x-4}+\frac{x-3}{x+2}$ (x) $\frac{2}{2x+1}+\frac{3x+5}{x+2}$

(y) $\frac{4x+17}{x+3}-\frac{2x-15}{x-3}$ (z) $\frac{2x+3}{x-1}+\frac{4-x}{3x-5}$ (α) $\frac{3x-4}{x+1}-\frac{x+2}{5x+3}$

Solving Equations involving Algebraic Fractions

One fraction equal to another fraction

Technique: Multiply by each denominator in turn.

Example

Solve $\frac{4x}{3x+1} = \frac{3}{x+5}$.

Answer: $\frac{4x}{3x+1} = \frac{3}{x+5}$

$\frac{4x(x+5)}{3x+1} = 3$ [Multiply both sides by $x+5$]

$4x(x+5) = 3(3x+1)$ [Multiply both sides by $3x+1$]

$4x^2 + 20x = 9x + 3$ [Expand the brackets]

$4x^2 + 11x - 3 = 0$ [Subtract $9x$ from both sides; subtract 3 from both sides]

$(4x-1)(x+3) = 0$ [Factorise]

Either $4x - 1 = 0$ or $x + 3 = 0$ [Solve]

$4x = 1$ $x = -3$

$x = \frac{1}{4}$

Exercise 32 H

Solve the following equations.

(a) $\frac{x+4}{x-1} = \frac{x}{x-3}$ (b) $\frac{6}{x-4} = \frac{5}{x-3}$ (c) $\frac{1}{2x+3} = \frac{1}{3x-2}$

(d) $\frac{3}{2(2x-1)} = \frac{4}{3x+2}$ (e) $\frac{x}{4x+3} = \frac{1}{2x+9}$ (f) $\frac{3x}{4-x} = \frac{2}{x-4}$

(g) $\frac{3x}{x+1} - \frac{x+4}{3} = 0$ (h) $\frac{1}{3x-2} - \frac{2x+3}{x-4} = 0$ (i) $\frac{2}{3x+1} - \frac{5}{x+3} = 0$

Fractions where all the denominators are numbers

Technique: Multiply by the lowest common denominator of the fractions.

Example

Solve $\frac{2x+3}{6} + \frac{x-1}{3} = \frac{7}{12}$.

Answer: $\frac{2x+3}{6} + \frac{x-1}{3} = \frac{7}{12}$

$12\left(\frac{2x+3}{6}\right) + 12\left(\frac{x-1}{3}\right) = 12\left(\frac{7}{12}\right)$ [Multiply each term by 12]

$2(2x+3) + 4(x-1) = 7$ [Simplify]

$4x + 6 + 4x - 4 = 7$ [Expand the brackets]

$8x + 2 = 7$ [Collect like terms]

$8x = 5$ [Subtract 2 from both sides]

$x = \frac{5}{8}$ [Divide both sides by 8]

Exercise 33 H

Solve the following equations.

(a) $\frac{x+3}{6} + \frac{2x-5}{3} = \frac{2}{9}$ (b) $\frac{x-2}{5} - \frac{2x+5}{4} = \frac{1}{4}$ (c) $\frac{3-x}{4} + \frac{2x+5}{3} = 1$

(d) $\frac{2x-1}{3} + \frac{x-2}{6} = \frac{3x}{4}$ (e) $\frac{x+2}{3} - \frac{x-3}{2} = 2$ (f) $\frac{2x-3}{4} - \frac{x+1}{3} = \frac{3}{4}$

(g) $\frac{2x+5}{4} + \frac{x-4}{6} = \frac{x}{3}$ (h) $\frac{3-x}{8} + \frac{3}{16} = \frac{3x+2}{4}$ (i) $\frac{3x+5}{10} - \frac{2-x}{25} = \frac{3}{5}$

Example

Solve the equation $\frac{2x}{x+3}+\frac{3x+1}{2x-1}=3$.

Technique: Multiply by the lowest common denominator of the fractions.

Answer: $\frac{2x}{x+3}+\frac{3x+1}{2x-1}=3$

$(x+3)(2x-1)\times\frac{2x}{x+3}+(x+3)(2x-1)\times\frac{3x+1}{2x-1}=3(x+3)(2x-1)$ [Multiply by $(x+3)(2x-1)$]

$(2x-1)2x+(x+3)(3x+1)=3(x+3)(2x-1)$ [Simplify]

$4x^2-2x+(3x^2+x+9x+3)=3(2x^2-x+6x-3)$ [Expand the brackets]

$4x^2-2x+(3x^2+10x+3)=3(2x^2+5x-3)$ [Collect like terms]

$7x^2+8x+3=6x^2+15x-9$ [Collect like terms; expand]

$x^2-7x+12=0$ [Subtract $6x^2+15x-9$]

$(x-3)(x-4)=0$ [Factorise]

Either $x-3=0$ or $x-4=0$ [Solve]

$x=3$ $x=4$

Exercise 34 H

Solve the following equations.

(a) $\frac{2}{2x+3}+\frac{1}{x+2}=3$

(b) $\frac{3x}{x+4}+\frac{2x}{5x-2}=\frac{3}{2}$

(c) $\frac{2x}{x-3}-\frac{x}{x-2}=6$

(d) $\frac{6x}{3x-1}+\frac{15}{2x+3}=5$

(e) $\frac{4x}{5x-2}+\frac{3}{3x+1}=3$

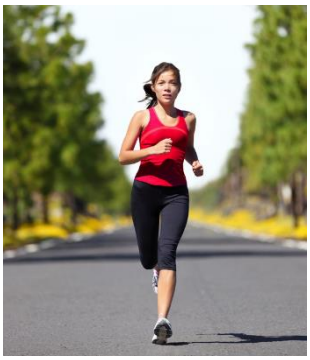
(f) $\frac{x+3}{x+1}+\frac{3}{x-3}=2$

(g) $\frac{8x}{4x-3}+\frac{20}{3x+2}=10$

(h) $\frac{2x}{x-5}+\frac{x-1}{3x}=2$

(i) $\frac{2}{x}+\frac{1}{x+1}=5$

Challenge!



Ava runs a distance of 26 miles at an average speed of x mph.
Delyth runs the same distance at an average speed that is 2 mph slower than Ava.
The difference between their times is exactly 1 hour.

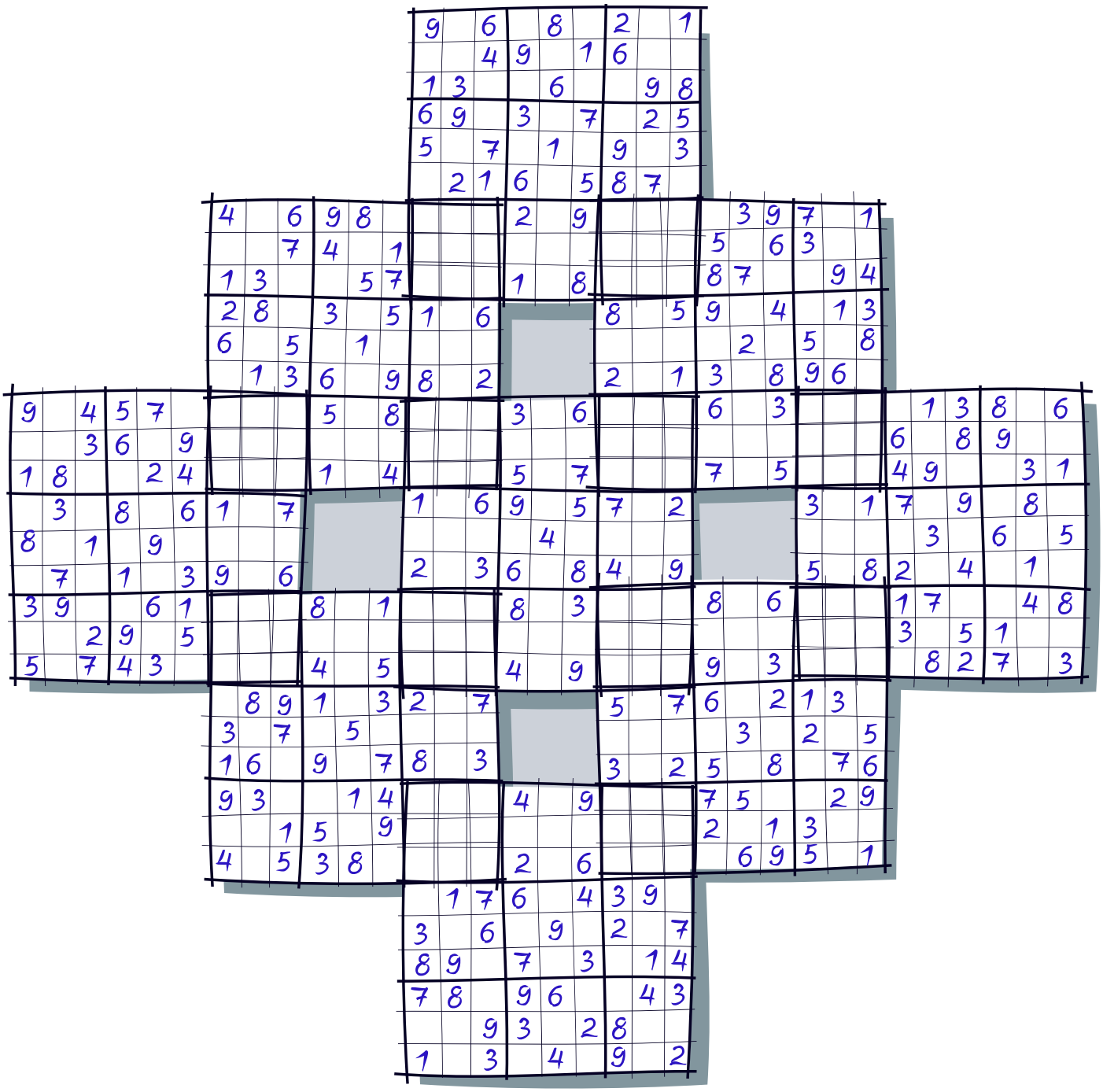
(a) Show that x satisfies the equation $x^2-2x-52=0$.

(b) Use the quadratic formula to find Ava's speed.
Give your answer correct to 2 decimal places.

Evaluation

Key words	Corrections	I am happy with...	I need to revise...

Giant Sudoku Puzzle



Developing Algebra 4

Reflection

Name: .....

Percentage in the test: .....

	I know this. 	I need to revise this. 	Question in the test:	Correct in the test?
I am familiar with using **function notation**, e.g. $f(x) = 3x^2 - 4x + 5$.			1	
I can **transform the graphs of functions** using the transformations $y = f(x) + a$, $y = f(x + a)$, $y = af(x)$, $y = f(ax)$, $y = -f(x)$ and $y = f(-x)$.			1	
I know how to draw a **tangent to a function** and **measure its gradient**.			2	
I know and can use the definitions for **velocity** and **acceleration**.			2	
I know how to find the **area between a graph and the x–axis** using the **trapezium rule**.			2	
I know when the area between a graph and the x–axis has a **special meaning**.			2	
I can change the subject of a formula when the **subject appears more than once** in the formula.			3	
I can change the subject of a formula in questions involving **fractions** and **roots**.			4, 5	
I can use the **quadratic formula** to solve quadratic equations.			6, 8	
I can **combine two algebraic fractions** to give a single fraction in its simplest form.			7	
I can **solve equations that include algebraic fractions**.			8, 9, 10	