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WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

CYD-BWYLLGOR ADDYSG CYMRU

Tystysgrif Addysg Gyffredinol

Safon Uwch

MATHEMATICS A 1

A.M. FRIDAY, 12 June 1981

(3 Hours)

Answer **seven** *questions.*

1. (*a*) Prove by mathematical induction that

$$\frac{3}{4}+\frac{5}{36}+\ldots+\frac{2n+1}{n^2(n+1)^2}=1-\frac{1}{(n+1)^2},$$

for all positive integers n. [7]

(*b*) Evaluate $\sum_{r=1}^{n}(3r+1)^2$. [3]

(*c*) Solve the equations

$$xy=6$$
$$x-2y+y^2=5.$$ [5]

2. (*a*) Given that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, show that

$$\frac{\sin 3\theta}{\sin\theta}-\frac{\sin 6\theta}{\sin 2\theta}=2(\cos 2\theta-\cos 4\theta),$$

assuming that $\sin\theta \neq 0$ and $\sin 2\theta \neq 0$. [4]

(*b*) Solve the equation

$$3\cos\theta+2\sin\theta=1$$

for $0° < \theta < 360°$. [6]

(*c*) The quadratic equation

$$x^2-kx-1=0$$

has roots α and β. Find the quadratic equation whose roots are $\alpha^2+\beta^2$ and $\alpha\beta$. [5]

Turn over.

3. (*a*) Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

by row reduction, or otherwise. Hence solve the simultaneous equations

$$\begin{aligned} 2x+y-z &= 2 \\ y+z &= 1 \\ x-y \quad &= 1. \end{aligned}$$

[4, 2]

If a matrix Y is given in terms of a matrix Z by

$$AY = ZA\,,$$

find an expression for Y in terms of Z, A and A^{-1}. Hence find an expression for Y^2 in terms of Z^2, A and A^{-1}. For the case

$$Z = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

find Z^2 and hence Y^2. [3, 3]

(*b*) Find the possible values of a if the coefficient of x^2 in the expansion of $(1+ax)^4$ is the same as the coefficient of x^4 in the expansion of $(1+x)^6$. [3]

4. (*a*) A transformation T of the plane, given by the formula

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

transforms the point (x, y) to (x', y'). Show that only one point remains fixed under this transformation. Find the images of the points $(1, 0)$ and $(1, -3)$ and describe the transformation. [2, 2, 2]

(*b*) Write down the matrix representing the reflection that transforms the positive x-axis into the negative y-axis. [3]

(*c*) The equation $e^x+x = 2$ has a root between 0 and 1. Find this root correct to two places of decimals. [6]

5. (*a*) The points $(1, 2)$ and $(2, 6)$ are denoted by A and B respectively. Show that the equation of the line AB is $y = 4x-2$. [2]

The line AB meets the x-axis at C. D is the point $(0, \frac{13}{2})$. Show that CB is perpendicular to DB and find the angle between DC and DB. [3, 2]

(*b*) Show that the equation of the normal to the curve $y^2 = 4ax$ at the point P, having coordinates $(ap^2, 2ap)$, is $y = -px+ap^3+2ap$. [3]

The normal meets the line $y = 2x$ at the point Q. Given the point $S(0, a)$ and that SQ is parallel to the tangent at P show that p satisfies the equation

$$2p^3-p^2+3p-4 = 0.$$

[5]

6. (*a*) The points A and B have coordinates $(\frac{18}{5}, \frac{21}{5})$ and $(\frac{2}{5}, \frac{9}{5})$ respectively. Show that the circle having AB as diameter has equation

$$x^2+y^2-4x-6y+9 = 0.$$

Sketch the circle. [4, 1]

Show also that the distance of the centre C of the circle from the line $3x+2y-4 = 0$ is greater than the radius of the circle. [2]

(*b*) Given that the line $y = mx+c$ is a tangent to the curve $4x^2+3y^2 = 12$, show that $c^2 = 3m^2+4$. Given that a tangent to the curve passes through the point $(1, 2)$ show that $c = 2$ or 4. [4, 4]

7. (a) State the domains and ranges of the functions f and g defined by $f(x) = \sin x$, $g(x) = \sqrt{(1-4x^2)}$. [2]

Find the composite function $f \circ g$ and state its domain and range. Explain why $g \circ f$ cannot be formed. [2, 1]

(b) Investigate the function f defined by

$$f(x) = 3x^5 - 10x^3 + 10$$

for local maxima, minima and points of inflection. Sketch the graph of the function, showing clearly the stationary points and any points of inflection. [6, 4]

8. (a) Differentiate $\log_e \cos x$, $0 \leqslant x < \frac{1}{2}\pi$. [2]

(b) Find $\frac{d^2y}{dx^2}$ when $y = \sin^{-1}x - x\sqrt{(1-x^2)}$, expressing your answer as simply as possible. [5]

(c) A plane figure $ABCD$ is such that $A\widehat{B}C = A\widehat{C}D = 90°$ and $D\widehat{A}C = C\widehat{A}B = \theta$. Given that Y denotes $\frac{AC+CD}{CB}$, show that

$$Y = \sec\theta + \operatorname{cosec}\theta$$

d find its minimum value, showing clearly that the value you have found is the minimum value. [3, 5]

9. (a) The function f is defined by

$$f(x) = \frac{4x+5}{x^2-1}.$$

Show that $f(x)$ cannot take values between -4 and -1. Sketch the graph of the function showing clearly the behaviour of f as $x \to \pm 1$ and as $x \to \pm\infty$. [5, 5]

(b) The function g is defined by

$$g(x) = \sqrt{(1-x^2)}.$$

Find the area of the region bounded by the graph of g, the lines $x = 0$, $x = \frac{1}{2}$ and the x-axis, by using the substitution $x = \sin\theta$, or otherwise, to evaluate the integral. [5]

10. (a) Using integration by parts, or otherwise, find

$$\int xe^{-x}\,dx.$$ [3]

(b) Find

$$\int \frac{dx}{(x+1)^2(x^2+1)}.$$ [8]

(c) Using the change of variable $\tan\frac{1}{2}x = t$, or otherwise, show that

$$\int_0^{\pi/2} \frac{dx}{2+\cos x} = \frac{\pi}{3\sqrt{3}}.$$ [4]

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PURE MATHEMATICS A 1

A.M. FRIDAY 12 June 1981

(3 Hours)

Answer **seven** *questions*

1. (*a*) Given that α, β, γ are the roots of the equation

$$x^3+px+q = 0,$$

find the values of $\alpha+\beta+\gamma$, $\alpha^2+\beta^2+\gamma^2$ and $\alpha^3+\beta^3+\gamma^3$ in terms of p and q. [6]

Find an equation whose roots are $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta$. [4]

(*b*) Solve the simultaneous equations

$$x^3-y^3 = 98, \qquad x-y = 2.$$ [5]

2. (*a*) Find a formula for the sum S_n where

$$S_n = x+3x^2+5x^3+\ldots+(2n-1)x^n$$

and show that when $x = \frac{1}{2}$ the limit of S_n as $n \to \infty$ is 3. [7]

(*b*) The numbers $c_0, c_1, \ldots, c_n$ are the binomial coefficients such that

$$(1+x)^n = c_0+c_1x+c_2x^2+\ldots+c_nx^n.$$

Prove that

(i) $c_0+2c_1+\ldots+(n+1)c_n = 2^n+n.2^{n-1}$, [4]

(ii) $c_0-\frac{1}{2}c_1+\frac{1}{3}c_2-\frac{1}{4}c_3+\ldots+\dfrac{(-1)^n}{n+1}c_n = \dfrac{1}{n+1}$. [4]

Turn over.

3. (*a*) It is required to prove that if a and b are any real numbers then $a^2+b^2 \geqslant ab$. Criticise the following argument, and correct it if necessary:

'Consider $(a-b)^2 = a^2+b^2-2ab$. Since $(a-b)^2 \geqslant 0$ we have $a^2+b^2 \geqslant 2ab$. But $2ab \geqslant ab$. Hence $a^2+b^2 \geqslant ab$.' [5]

(*b*) A function f with domain the real numbers is given by

$$f(x) = \frac{x^2-3}{x^2+3x+3}.$$

Find the range of f, algebraically or otherwise. [5]

(*c*) Express as an interval of real numbers the set

$$\{x: |x+2| \leqslant |x+3| < |x-4|\}.$$ [5]

4. (*a*) Given that no two of the numbers a, b, c, d are equal, solve for x, y, z the simultaneous equations

$$\begin{aligned} x+y+z &= 1 \\ ax+by+cz &= d \\ a^2x+b^2y+c^2z &= d^2. \end{aligned}$$

Express your solution in as simple a form as possible. You may use ideas of symmetry in completing your solution. [8]

(*b*) Show that when $k = 5$ the system of linear equations

$$\begin{aligned} x-5y+5z &= 1 \\ 2x-3y+4z &= 4 \\ 4x+\ y+2z &= k \end{aligned}$$

represents three planes which form a triangular prism. Find the value of k for which the equations represent three planes with a common line. [7]

5. (*a*) Write down the matrix representing the shear of the plane which has the x-axis as shear axis and transforms the point $(0, 1)$ to the point $(-1, 1)$. This shear transforms the parabola C with equation $y^2 = 4ax$ into a curve C'. Show that the equation of C' can be written in the form $(y-2a)^2 = 4a(x+a)$. [5]

Show the curves C, C' in one sketch, mark on C the points O, A, B given by $y = 0, y = 2a, y = -2a$ respectively, and mark the corresponding points on C'. [3]

(*b*) Find the adjugate matrix of the matrix

$$A = \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}.$$

Find A^{-1} and show that A^{-1} is in this case a scalar multiple of the transpose A^t. [7]

6. Find the equation of the circumcircle of the triangle formed by the points $A(-4, 2)$, $B(-3, 3)$ and $C(-1, 3)$. [7]

Show that the angle between the two tangents to this circle from the point $(2, 4)$ is $\tan^{-1}(\frac{4}{3})$. [8]

7. A parabola K and a cubic curve C are given respectively by the equations

$$ay = (x-a)^2, \qquad 27a^2y = 4x^3,$$

where a is a non-zero constant. Show that the curves have two and only two common points, and that the curves touch each other at one of these points. [8]

Find the equation of the common tangent at the point where the curves touch, and find the coordinates of the point where it meets C again. [7]

8. Find the equation of the chord joining the points $P(cp, c/p)$, $Q(cq, c/q)$ on the rectangular hyperbola $xy = c^2$. [4]

$R\ (cr,\ c/r)$ is a fixed point on the curve and the chord PQ varies in such a way that $P\widehat{R}Q$ is a right angle. Show that the mid-point of the chord PQ lies on the line L with equation $y = -r^2x$. [6]

Show conversely that if M is any point of the line L then there is a chord PQ which has mid-point M and subtends a right angle at R. [5]

9. (*a*) Find the general solution, in radians, of the equation

$$\sin 2\theta + \cos 2\theta = \sin\theta - \cos\theta + 1.$$

[6]

(*b*) From a point A on level ground, due south of a tower, the angle of elevation of the top of the tower is α. From a point B, a distance x due west of A, the angle of elevation of the top of the tower is β. Find the height h of the tower in terms of α, β and x. Evaluate h when $\alpha = 24°$, $\beta = 19°$ and $x = 100$ m. [6, 3]

10. (*a*) Given that z is a complex number such that

$$z^4 - z^3 + z^2 - z + 1 = 0,$$

show that $z^5 = -1$. [2]

Find a quadratic equation with roots $z - z^4$ and $z^3 - z^2$. [4]

(*b*) Points P and Q in the Argand diagram represent the complex numbers z and w respectively, where $w = 2z^2 + 3$. Show that, as P describes the circle given by $|z| = 1$, the point Q lies on a second circle. [5]

Show the two circles in a sketch and mark the possible positions of P such that Q is at the common point of the two circles. [4]

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PURE MATHEMATICS A 2

A.M. WEDNESDAY 17 June 1981

(3 Hours)

Answer **seven** *questions.*

1. (*a*) Find the stationary points of the function f given by

$$f(x) = \frac{x(x+3)}{x-1}.$$

Sketch the graph of f and find the domain and range of f. [8]

(*b*) The functions g and h each have domain the positive real numbers and are defined by

$$g(x) = \frac{1}{x}, \qquad h(x) = x+1.$$

Express $f_1(x)$ by a formula, where $f_1 = h \circ g \circ h$. Express in terms of g and h the functions f_2 and f_3 where

$$f_2(x) = \frac{1}{x+2}, \qquad f_3(x) = \frac{x}{x+1}.$$ [7]

2. Given that the function f defined by

$$\begin{aligned} f(x) &= 2+x-x^2 && \text{if } x \in (-\infty, 1) \\ &= x^2+ax+b && \text{if } x \in [1, \infty) \end{aligned}$$

is continuous and has a continuous derivative for all values of x, show that $a = -3$ and $b = 4$. [5]

Find the stationary points of f and sketch the graphs of f and its derivative f' in one diagram. [6]

Evaluate $\int_0^3 f(x)\,dx$. [4]

Turn over.

3. (a) By considering a suitable graph, or otherwise, find the approximate position of each root of the equation

$$4x^2-\frac{1}{x}-4=0. \qquad [5]$$

By drawing an accurate graph of a suitable region, find to two decimal places the value of the root which is smallest in absolute value. Give a table of the values from which you plot your graph. [5]

(b) Assuming that $\lim_{x\to 0}\frac{\sin x}{x}=1$, evaluate the limits

(i) $\lim_{x\to 0}\frac{1-\cos 2x}{3x^2}$,

(ii) $\lim_{x\to \frac{1}{2}\pi}(x-\frac{1}{2}\pi)\sec x$. [5]

4. A function f has domain the closed interval $[-2, 2]$ and is defined by

$$f(x)=|x+1|+|x-1|.$$

Prove that f is an even function. [3]

A function g is defined on the interval $[-2, 2]$ by

$$g(x)=\int_{-2}^{x} f(t)\,dt,$$

where f is as above. Define the function g by formulae, and sketch carefully the graphs of f and g in one diagram. [7, 3]

Is it true that $g'(x)=f(x)$ for each value of x in the open interval $(-2, 2)$? Give a brief reason for your answer. [2]

5. Prove that $\sinh 3\theta = 3\sinh\theta + 4\sinh^3\theta$. [4]

Given that $f(x)=4x^3+12x^2+39x+13$, use the substitution $x=y-h$, with a suitable value of h, to transform the equation $f(x)=0$ into an equation in y with no term in y^2. [4]

Use the substitution $y=r\sinh\theta$, with a suitable value of r, to obtain an equation in $\sinh 3\theta$. Hence using tables, find the real root of the equation $f(x)=0$. [4, 3]

6. (a) By considering $f'(x)$, show that the function f given by

$$f(x)=x^3-2x^2+5x$$

is strictly increasing. If g is the inverse function of f, find the value of $g'(x)$ when $x=4$. [4, 5]

(b) If $x=1+t^2$, $y=\frac{t}{1+t}$, find $\frac{d^2y}{dx^2}$ in terms of t. [6]

7. (a) Prove that

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,$$

where n is any integer, positive, negative or zero. Express $\cos 6\theta$ as a polynomial in $\cos\theta$. [6, 5]

(b) Find $\int \sin^2 x\cos^2 x\,dx$. [4]

8. (a) Prove that the curves

$$y=e^x, \qquad y=\tfrac{5}{4}+\sinh x$$

cut in exactly two points and determine the tangent of the angle of intersection at each point. [4, 3]

(b) Show that the arc length of the curve $y^2=x^3$ from the origin to one of the points where $x=5$ is 335/27. [8]

9. (*a*) Prove that if $I_n = \int_0^{\frac{1}{4}\pi} \tan^n\theta \, d\theta$ where n is any integer $\geqslant 0$, then

$$I_n = \frac{1}{n-1} - I_{n-2} \qquad (n \geqslant 2).$$ [5]

Evaluate I_5 to two decimal places. [5]

(*b*) Evaluate $\int_1^2 \frac{\log_e x}{x^2} \, dx$. [5]

10. (*a*) Show that the area of the region between the curve $y = \frac{1-x}{x}$ and the lines $x = 1$, $y = 1$ is $-\log_e 2$. Find the volume of the solid formed when this region is rotated about the line $x = 1$. [4,5]

(*b*) Show that

$$\int_0^{\frac{1}{4}\pi} \frac{\tan x}{1+\sin 2x} \, dx = -\tfrac{1}{2}+\log_e 2.$$ [6]