4 O 227

WELSH JOINT EDUCATION COMMITTEE — CYD-BWYLLGOR ADDYSG CYMRU

General Certificate of Education — Tystysgrif Addysg Gyffredinol

Advanced Level — Safon Uwch

MATHEMATICS A 1

(3 Hours) A.M. 7 June 1978

Answer **seven** *questions*

1. (*a*) By summing the identity

$$(r+1)^3-r^3 = 3r^2+3r+1$$

from $r = 1$ to n, obtain a formula for $\sum_{r=1}^{n} r^2$, [4]

and check this formula by mathematical induction. [4]

(*b*) Solve the equation

$$12x^4+28x^3-9x^2-28x+12 = 0$$

by writing it in the form

$$12\left(x-\frac{1}{x}\right)^2+a\left(x-\frac{1}{x}\right)+b = 0$$

for suitable values of a and b. [7]

2. (*a*) Using the formula for $\tan(A+B)$, and any other basic formulae that you require, show that

$$\frac{\tan 3\theta}{\tan\theta} = \frac{2\cos 2\theta+1}{2\cos 2\theta-1}$$

(except for certain values of θ, which you need not specify, when the various expressions are not defined). [4]

(*b*) Using tables, solve the equation

$$\cos\theta+3\sin\theta = 2\sqrt{2}$$

for $0 < \theta < 360°$. [7]

(*c*) The quadratic equation $x^2-x-k = 0$, where k is positive, has real roots α and β. Find the quadratic equation whose roots are $\alpha-\beta$ and $\beta-\alpha$. [4]

Turn over.

3. (*a*) The coefficient of x^3 in the binomial expansion of $(1+4x)^n$, where n is a positive integer, equals the coefficient of x^4 in the expansion of $(1+2x)^n$; find the value of n. [5]

(*b*) Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -2 & 3 \\ 1 & 1 & -2 \end{pmatrix}$$

by row reduction, or otherwise. Hence solve the simultaneous equations

$$\begin{aligned} 2x-y+z &= 1 \\ 2x-2y+3z &= -1 \\ x+y-2z &= 3. \end{aligned}$$

[3, 2]

Calculate the matrix

$$B = A^{-1}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}A.$$

Without calculating B^2 directly, show that $B^2 = I_3$ (the 3×3 identity matrix). [2, 3]

4. (*a*) The transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

represent reflections of the plane; find the axes of these two reflections. [2, 2]

Denote the matrices of these reflections by R and S respectively; what is the transformation of the plane determined by the matrix SR? [4]

(*b*) Let x and y be real numbers such that $x+y = a$ and $xy = b$. Express x^3+y^3 in terms of a and b. [2]

Solve the simultaneous equations

$$\begin{aligned} x+y &= 2 \\ x^3+y^3 &= 98. \end{aligned}$$

[5]

5. Let P and Q be points on the rectangular hyperbola $xy = a^2$, with co-ordinates $(ap, a/p)$ and $(aq, a/q)$. Obtain the equation of the line PQ, and deduce, or obtain otherwise, the equation of the tangent at P to the hyperbola. [3, 2]

Show that, if the line PQ meets the co-ordinate axes at A and B, then the mid-point M of AB is the mid-point of PQ also. [5]

Obtain the co-ordinates of the point T where the tangents at P and Q meet, and show that the line MT passes through the origin. [5]

6. Find the centres and radii of the circles C_1 and C_2 whose equations are

$$x^2+y^2-2x = 0$$

and

$$x^2+y^2-2y = 0$$

respectively. Draw a figure to illustrate the circles. [5]

The line $y = mx$ through the origin O meets C_1 and C_2 again at P and Q. Find the co-ordinates of P and Q. [5]

Let A denote the second point of intersection of the two circles. Show that AP and AQ are perpendicular. [5]

7. (*a*) Find the local maxima, minima, and points of inflection of the function f defined by

$$f(x) = \frac{1}{x} - \frac{3}{x^3}. \qquad [5]$$

Sketch the graph of this function, and explain why the graph is symmetrical about the origin. [4, 2]

(*b*) The function g is defined by

$$g(x) = \frac{x+1}{x^2}.$$

Find the volume of the solid formed when that part of the graph of g lying between $x = 1$ and $x = 2$ is rotated about the x-axis. [4]

8. (*a*) Differentiate $\frac{x}{\sqrt{(1+x^2)}}$, $\log \tan x$, $e^{\sin x}$,

simplifying your answers when possible. [3, 2, 2]

(*b*) The equal sides AB and AC of an isosceles triangle have length a, and the angle at A is denoted by 2θ. Show that the radius of the inscribed circle of the triangle is given by

$$r = \frac{a \sin\theta \cos\theta}{1+\sin\theta}. \qquad [3]$$

Show that, if a is constant and θ varies between 0° and 90°, the maximum value of r occurs when $\sin\theta = (\sqrt{5}-1)/2$. [You need not verify that this value of θ gives a maximum rather than a minimum.] [5]

9. Define the function $\sin^{-1}$, and state the domain and range of this function. Sketch the graph of the function. [2, 2, 2]

Show that $\cos(\sin^{-1}x) = \sqrt{(1-x^2)}$, and justify the choice of the positive square root. [4]

Evaluate $$\int_0^1 \sin^{-1}x \, dx. \qquad [5]$$

10. (*a*) Express $\frac{4}{x^2(x+2)}$ in partial fractions, and hence find $\int \frac{4}{x^2(x+2)} dx$. [6]

(*b*) Find $$\int \frac{x+4}{x^2+4} dx. \qquad [4]$$

(*c*) Evaluate $$\int_0^1 x^2 e^{2x} \, dx. \qquad [5]$$

4 O 221

WELSH JOINT EDUCATION COMMITTEE — **CYD-BWYLLGOR ADDYSG CYMRU**

General Certificate of Education — **Tystysgrif Addysg Gyffredinol**

Advanced Level — **Safon Uwch**

PURE MATHEMATICS A 1

(3 Hours) A.M. 7 June 1978

Answer **seven** *questions.*

1. (*a*) Find a quartic equation, with integer coefficients, whose roots are the four values of $1 \pm \sqrt{2} \pm \sqrt{3}$. [6]

(*b*) Show that if the equation $x^3+ax^2+bx+c=0$
has two roots whose sum is zero, then $ab=c$. Show also that if $ab=c$ the equation has two roots whose sum is zero. Hence, or otherwise, solve the equation

$$4x^3+8x^2-x-2=0.$$ [9]

2. Prove that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [4]

Evaluate $\sum_{r=1}^{2n+1} (-1)^{r+1}r^2$, giving your answer in its simplest form. [5]

Show that the sum of all possible products of two different numbers from the set $\{1, 2, 3, \ldots, n\}$ is

$$n(n+1)(n-1)(3n+2)/24.$$ [6]

3. (*a*) Solve the system of linear equations

$$x-y+kz=2$$
$$2x+3y-4z=3k$$
$$3x-8y-6z=16$$

in each of the cases (i) $k=1$, (ii) $k=-2$. [8]

(*b*) Show that the system of linear equations

$$x+3y-8z=3$$
$$2x+y-3z=2$$
$$3x-y+2z=4$$

represents three planes which form a triangular prism. [7]

Turn over.

4. A and B are 3×3 matrices such that $A+B = AB$, and A is non-singular. Prove that B is non-singular, and show that

$$A^{-1}+B^{-1} = I,$$

where I is the unit 3×3 matrix. [7]

Given
$$A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

find B so that A and B satisfy the above conditions. [8]

5. Prove that the transformation R of the plane consisting of an anti-clockwise rotation through an angle α about the origin O is given by the equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \qquad [3]$$

If R is followed by a translation S in which O maps to the point (h, k), show that the resultant transformation SR is given by the equation

$$\begin{pmatrix} x'' \\ y'' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & h \\ \sin\alpha & \cos\alpha & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \qquad [2]$$

Now let R be the rotation about O through the obtuse angle $\tan^{-1}(-\frac{3}{4})$ and S the translation in which O maps to the point $(3, 4)$. Find the 3×3 matrix which represents the transformation T consisting of S followed by R. (Note that here the translation occurs first.) [6]

Show that T has a fixed point, (i.e. a point which maps to itself), and find its co-ordinates. [4]

6. (*a*) A line of slope m is drawn through the point $P(1, 2)$. Show that this line is a tangent to the ellipse $x^2+4y^2 = 4$ if m satisfies the equation $\quad 3m^2+4m-3 = 0.$

Deduce that there are two real tangents to the ellipse through P and that they are perpendicular. [8]

(*b*) Find the centre of the inscribed circle of the triangle formed by the lines with equations

$$x-1 = 0, \quad 3x-4y+4 = 0, \quad 4x+3y-21 = 0,$$

and show that the circle has radius $47/60$. [7]

7. Show that the parabola $y^2 = 4ax$ and the circle $4(x^2+y^2)-25ax-ay+3a^2 = 0$ touch at a point P, (i.e. have a common tangent there), and cut at two other points Q, R which lie on the line $y = x+\frac{3}{4}a$. [12]

Show that the common tangent at P and the chord QR are equally inclined to the x-axis. [3]

8. The point P in the plane has co-ordinates (x, y) given in terms of a parameter t by

$$x = \frac{t^2+1}{t}, \qquad y = \frac{t^2-1}{t} \quad \text{where } t \in (0, \infty).$$

By considering the value of x^2-y^2, show that the locus of P is part of a hyperbola. Sketch the hyperbola and indicate that part which is the locus of P. [6]

Show that the normal to the locus at the point $t = 1$ is the x-axis. If the normal to the locus at any other point meets the x-axis at the point $(x_0, 0)$, find x_0 in terms of t, and find the interval of values within which x_0 must lie. [9]

9. (*a*) Show that if a and b are constants and $\sin(a-b) \neq 0$, there exist constants A and B (independent of x) such that

$$\frac{\sin x}{\sin(x-a)\sin(x-b)} \equiv \frac{A}{\sin(x-a)} + \frac{B}{\sin(x-b)},$$

and give the values of A and B in terms of a and b. [9]

(*b*) Find the general solution of the equation

$$\sin\theta + \sin 2\theta + \sin 3\theta = 0.$$ [6]

10. Points P and Q in the Argand diagram represent the complex numbers z and w respectively, where $w = 4z^2+4+3i$. Show that, as P describes the circle given by $|z| = 1$, the locus of Q is a second circle. [5]

Prove that the two circles touch, and find the complex number corresponding to the point of contact A. Show that when P is at either of the points given by $z = \pm(1-3i)/\sqrt{10}$ then Q is at A. [6]

Show that, if B is the point representing $-1+3i$, then the line AB is the common tangent at A. [4]

4 O 222

WELSH JOINT EDUCATION COMMITTEE — **CYD-BWYLLGOR ADDYSG CYMRU**

General Certificate of Education — **Tystysgrif Addysg Gyffredinol**

Advanced Level — **Safon Uwch**

PURE MATHEMATICS A 2

(3 Hours) A.M. 14 June 1978

Answer **seven** *questions.*

1. Show that the function f given by

$$f(x) = x^2 + \frac{1}{x^2} - \frac{5}{2}$$

has two stationary points both of which are minima. [4]

Find

(i) the domain and range of f,

(ii) $f(A)$ where A is the interval $[\frac{1}{2}, 2]$,

(iii) $f^{-1}(B)$ where B is the interval $[-\frac{1}{2}, 0]$,

(iv) an interval of values of x on which f is strictly increasing. [6]

Show that the total area of the two regions enclosed between the graph of f and the x-axis is $\frac{1}{3}\sqrt{2}$. [5]

2. (*a*) A function f is defined on the interval $[0, 3]$ as follows:

$$\begin{aligned} f(x) &= 3-x \quad \text{if } x \in [0, 2), \\ &= x-2 \quad \text{if } x \in [2, 3]. \end{aligned}$$

Sketch the graph of f, indicating clearly the point of the graph at any discontinuity. State the range of f. [3]

Sketch the graphs of the composite function $f \circ f$ and the inverse function f^{-1}. [6]

(*b*) Show that the function ϕ defined by

$$\phi(x) = |x-3| - |x+3|$$

is an odd function, and sketch its graph. [6]

Turn over.

3. Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. [2]

Find the value of h for which the substitution $x = y+h$ transforms the equation

$$x^3 - 3x^2 + 3 = 0$$

into an equation in y with no term in y^2. Use the substitution $y = r\sin\theta$, with a suitable value of r, to obtain an equation for $\sin 3\theta$. Hence solve the original equation in x, giving each root correct to **three** decimal places. [4, 4, 5]

4. A chord of a circle divides the area of the circle in the ratio $3:1$. Show that if the chord subtends an angle θ (radians) at the centre then θ satisfies the equation

$$\sin\theta = \theta - \tfrac{1}{2}\pi. \qquad [4]$$

By considering the graphs of $y = \sin\theta$ and $y = \theta - \frac{1}{2}\pi$ show that this equation has one and only one real root. [3]

By a careful drawing of these graphs in a suitable region, estimate the root to **three** significant figures. (Use graph paper and a suitable large scale.) [8]

5. By using de Moivre's theorem, or otherwise, express $\cos 4\theta$ and $\sin 4\theta$ each in terms of $\cos\theta$ and $\sin\theta$. Hence, or otherwise, show that

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}. \qquad [6]$$

Find the value of $\tan 4\alpha$ where α is the acute angle $\tan^{-1}(\frac{1}{2})$. Hence, or otherwise, show that one root of the equation

$$z^4 = -7 + 24i$$

is $z = 2+i$, and find the other roots of this equation in the form $a+ib$. [9]

6. (*a*) An arc PQ of a circle subtends an acute angle θ (radians) at the centre O, and the tangent at P meets OQ produced at R. Show that $\sin\theta < \theta < \tan\theta$ and hence find the value of $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$. [5]

By using this value, or otherwise, evaluate the limits

(i) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$, (ii) $\lim\limits_{x\to\pi}(\pi - x)\cot x$. [5]

(*b*) Show that if the graph of $y = \dfrac{\sin x}{x}$ $(x \neq 0)$ has a stationary point at $x = x_0$ then the value of d^2y/dx^2 at that point is $-\cos x_0$. [5]

7. Prove that $\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$. [4]

Show that if $a > b > 0$ then $f(x) = a\cosh x + b\sinh x$

can be expressed in the form $r\cosh(x+k)$. [5]

Show that if $a = b+1$ then $k = \log r$. Find r and k when $a = 13$, $b = 12$, and show that the equation

$$13\cosh x + 12\sinh x = \tfrac{25}{4}$$

has roots $-\log 10$ and $-\log\frac{5}{2}$. [6]

8. (*a*) Define $\log x$ (where $x > 0$) as an integral. Let n be a positive integer. By considering the area under the curve $y = 1/t$, between the ordinates $t = n$ and $t = n+1$, show that

$$\frac{1}{n+1} < \log\left(\frac{n+1}{n}\right) < \frac{1}{n}. \qquad [5]$$

Deduce that
$$\sum_{r=2}^{n+1} \frac{1}{r} < \log(n+1) < \sum_{r=1}^{n} \frac{1}{r}.$$

Hence show that $\log 7 = 2$ correct to the nearest integer. [4]

(*b*) Differentiate the following functions and in each case state the greatest interval of values of x for which the result holds:

$$f_1(x) = \sin^{-1}\left(\frac{x+1}{x-1}\right), \qquad f_2(x) = \int_0^x \sqrt{(1+e^t)}\, dt. \qquad [6]$$

9. (*a*) By using the substitution $x = 4\sin^2\theta + 2\cos^2\theta$, or otherwise, evaluate

$$\int_2^3 \sqrt{\left(\frac{x-2}{4-x}\right)}\, dx. \qquad [6]$$

(*b*) Evaluate
$$\int_0^1 \frac{x^3+2}{x^2+2x+2}\, dx,$$

giving your answer correct to **three** significant figures. [9]

10. A curve is given by the parametric equations

$$x = a\cos^3 t, \qquad y = a\sin^3 t,$$

where $a > 0$ and $0 \leqslant t \leqslant \frac{1}{2}\pi$. Show that if P is any point of the curve with parameter t, where $0 < t < \frac{1}{2}\pi$, and the tangent at P meets the co-ordinate axes at Q and R, then $QR = \text{constant} = a$. [6]

Show that the normal at P has equation

$$x\cos t - y\sin t = a\cos 2t. \qquad [3]$$

Find the length of the curve. [6]