WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

CYD-BWYLLGOR ADDYSG CYMRU

Tystysgrif Addysg Gyffredinol

Safon Uwch

MATHEMATICS A 1

Pure Mathematics

A.M. WEDNESDAY, 3 June 1992

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Find, correct to the nearest degree, all the values of θ between $0°$ and $360°$ satisfying the equation

$$8\cos^2\theta + 2\sin\theta = 7.$$ [4]

2. Use the trapezium rule with 5 ordinates and interval width 0·25 to find an approximate value of the integral

$$\int_1^2 \frac{e^x}{x}\,dx.$$

Give your answer correct to 2 decimal places. [4]

3. The cubic function f is given by

$$f(x) = x^3 + ax^2 - 4x + b$$

where a, b are constants.

Given that $(x - 2)$ is a factor of $f(x)$ and that a remainder of 6 is obtained when $f(x)$ is divided by $(x + 1)$, find the values of a and b. [4]

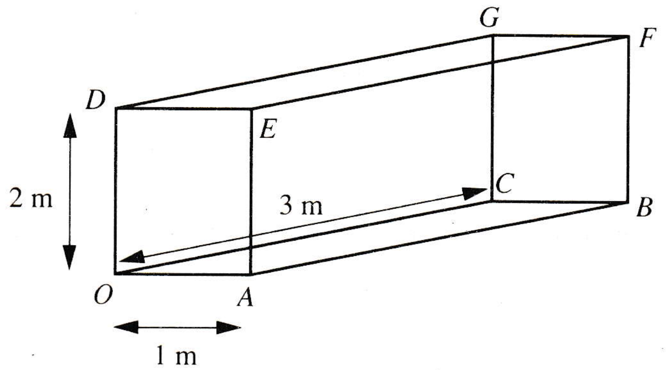
4. Show that the point $P\left(p^2, \frac{1}{p}\right)$ lies on the curve $y^2x = 1$ for all non-zero values of the parameter p.

Show further that, at P,

$$\frac{dy}{dx} = -\frac{1}{2p^3}.$$

Hence find the equation of the tangent to the curve at P. [5]

5.



The figure shows a cuboid in which $OA = 1$ m, $OC = 3$ m and $OD = 2$ m. Taking O as origin and unit vectors **i**, **j**, **k** in the directions OA, OC, OD respectively, express in terms of **i**, **j**, **k** the vectors

(i) **OF**

and (ii) **AG**.

By considering an appropriate scalar product, find the acute angle between the diagonals OF and AG. [5]

6. Given that

$$y = e^{-x^2}$$

find expressions for

(i) $\frac{dy}{dx}$,

and (ii) $\frac{d^2y}{dx^2}$.

Hence find the x coordinates of the two points on the graph of y for which $\frac{d^2y}{dx^2}$ is equal to zero. Show that these are both points of inflection. [5]

7. The complex numbers u, v and w are related by the equation

$$\frac{1}{u} = \frac{1}{v} + \frac{1}{w}.$$

Given that $v = 1 - 3\mathrm{i}$ and $w = 2 + \mathrm{i}$, show that

$$u = \tfrac{5}{13}(5 - \mathrm{i}).$$

Calculate, correct to 3 significant figures,

(i) the modulus of u,

(ii) the argument, in radians, of u.

[6]

8. An arithmetic progression has first term a and common difference d, where $d > 0$. Given that the 1st, 2nd and 4th terms are consecutive terms of a geometric progression, show that

$$d = a.$$

Given further that the sum of the first 5 terms of the arithmetic progression is 45, find the common value of a and d.

The sum of the first n terms of the arithmetic progression is 108. Find the value of n. [7]

Section B

Answer **four** *questions from this section.*

9. The points (5, 5) and $(-3, -1)$ are the ends of a diameter of the circle C with centre A. Write down the coordinates of A and show that the equation of C is

$$x^2 + y^2 - 2x - 4y - 20 = 0. \quad [3]$$

The line L with equation $y = 3x - 16$ meets C at the points P and Q. Show that the x coordinates of P and Q satisfy the equation

$$x^2 - 11x + 30 = 0.$$

Hence find the coordinates of P and Q. [3]

Write down the gradients of AP and AQ. Hence calculate the area of

(i) the triangle APQ,

(ii) the sector APQ.

[6]

Calculate the gradient of the tangent to C at P. Find the acute angle between L and this tangent. [3]

10. The points A, C have position vectors $\mathbf{a}$, $\mathbf{c}$ with respect to an origin O. D is a point on AC which divides AC in the ratio $\lambda : 1 - \lambda$, where $0 < \lambda < 1$. Show from first principles that D has position vector

$$(1 - \lambda)\,\mathbf{a} + \lambda\mathbf{c}. \quad [2]$$

B is a further point such that OB is a diagonal of the parallelogram $OABC$. The point E divides AB in the ratio $1 : 3$ and the line ED meets OA at F.

(i) Show that E has position vector

$$\mathbf{a} + \tfrac{1}{4}\mathbf{c}. \quad [2]$$

Denoting the position vector of F by $\mu\,\mathbf{a}$, show that

(ii) $\mathbf{FD} = (1 - \lambda - \mu)\,\mathbf{a} + \lambda\,\mathbf{c}$, [2]

(iii) $\mathbf{FE} = (1 - \mu)\,\mathbf{a} + \tfrac{1}{4}\mathbf{c}$. [2]

By noting that $\mathbf{FD}$ is a scalar multiple of $\mathbf{FE}$, show that

$$\mu = \frac{1 - 5\lambda}{1 - 4\lambda}. \quad [3]$$

(iv) Find the value of λ such that F is the mid-point of OA. [2]

(v) Find the range of values of λ such that F lies between O and A. [2]

11. By expanding both sides of the identity

$$5\cos\left(\theta - \frac{\pi}{3}\right) - 3\cos\left(\theta + \frac{\pi}{3}\right) \equiv r\cos(\theta - \alpha)$$

show that $r = 7$ and find the acute angle α in radians correct to 3 decimal places. [6]

(i) Find all the values of θ in the range $[0, 2\pi]$ satisfying the equation

$$5\cos\left(\theta - \frac{\pi}{3}\right) - 3\cos\left(\theta + \frac{\pi}{3}\right) = 4.$$

[4]

(ii) Show that

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{5\cos\left(\theta - \frac{\pi}{3}\right) - 3\cos\left(\theta + \frac{\pi}{3}\right)} = \frac{1}{7}\ln\left[\frac{\operatorname{cosec}\alpha + \cot\alpha}{\sec\alpha - \tan\alpha}\right].$$

[5]

12. (*a*) In the binomial expansion of $(1 + mx)^n$, the coefficients of x and x^2 are equal and the coefficient of x^3 is twice the coefficient of x^2.

Find the non-zero values of m and n. [5]

(*b*) Express

$$f(x) = \frac{1 + 3x^2}{(1 + x)^2(1 + 3x)}$$

in partial fractions. [5]

Hence, or otherwise, expand $f(x)$ in the form

$$f(x) = a + bx + cx^2 + \ldots$$

where the values of a, b, c should be stated. For what range of values of x is this expansion valid? [5]

13. (i) The function f is defined by

$$f(x) = \frac{1}{1 - x} \qquad (x \neq 0, x \neq 1)$$

Given that $h = f \circ f$, obtain expressions for

(*a*) $h(x)$

and (*b*) $h \circ f(x)$.

Hence, or otherwise, obtain an expression for $f^{-1}(x)$. [5]

(ii) The function g is defined by

$$g(x) = |x - 2| \qquad (-\infty < x < \infty)$$

Sketch the graphs of f and g on the same axes.

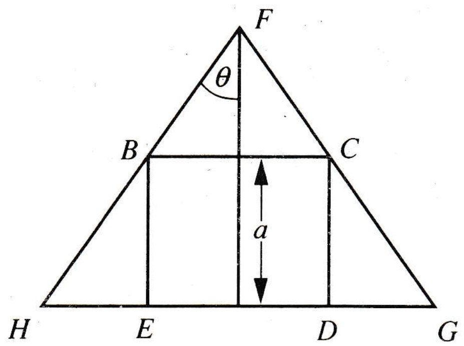
Find the x coordinate of the single point of intersection of the two graphs. [7]

(iii) Solve the equation

$$g \circ g(x) = 0.$$

[3]

14.



The figure shows a square $BCDE$ of side a inscribed in an isosceles triangle FGH in which $FG = FH$ and $H\hat{F}G = 2\theta$.

Show that the area A of the triangle FGH is given by

$$A = \frac{1}{4}a^2\left(4t + 1 + \frac{1}{t}\right) \quad \text{where } t = \tan\theta. \qquad [4]$$

(i) Show that the minimum value of A as t varies is $2a^2$. State the corresponding value of θ. [5]

(ii) The value of θ is increased by a small amount $\delta\theta$. Show that this results in an increase δA in A given by

$$\delta A \approx \frac{a^2}{4t^2}(4t^2 - 1)(t^2 + 1)\,\delta\theta.$$

Hence find the approximate increase in A when θ is increased from 45° to 45·5°. Give your answer in the form λa^2 where λ is a constant which should be given correct to 2 significant figures. [6]

15. The function f is defined on the domain $(-\infty, \infty)$ by

$$f(x) = \frac{x}{1 + x^2}.$$

Obtain an expression for $f'(x)$.

Sketch the graph of f. [4]

(i) The region R is bounded by the graph of f, the x-axis and the ordinates $x = 1$ and $x = 3$. Show that the area of R is $\frac{1}{2}\ln 5$. [4]

(ii) R is rotated through four right-angles about the x-axis. Using the substitution $x = \tan\theta$, show that the volume V of the solid generated is given by

$$V = \pi\int_{\frac{\pi}{4}}^{\alpha} \sin^2\theta \, d\theta$$

where α is an angle whose tangent should be stated.

Hence evaluate V, giving your answer correct to 3 decimal places. [7]

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MATHEMATICS A 2

Mechanics

A.M. THURSDAY, 11 June 1992

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

SECTION A

Answer **all** *questions in this section.*

1. A water pump raises 40 kg of water a second through a height of 20 m and ejects it with a speed of 45 m s^{-1}. Find the kinetic energy and potential energy per second given to the water and the effective rate at which the pump is working. [3]

2. The velocity $\mathbf{v}$ (in m s^{-1}) of a particle at time t seconds is given by

$$\mathbf{v} = e^{4t}\mathbf{i} + a e^{-t}\mathbf{j}$$

where a is a constant. Find the acceleration of the particle and determine the condition that has to be satisfied by a in order that the velocity and acceleration are perpendicular for some value of $t > 0$. [4]

3. A car travels at steady speed on a horizontal winding road.

(*a*) State whether the velocity is

(i) always zero,

(ii) always constant but not zero,

(iii) variable.

Give a reason for your choice of answer. [2]

(*b*) State whether the acceleration is

(i) always constant,

(ii) always parallel to the velocity,

(iii) variable but not always parallel to the velocity.

Give a reason for your choice of answer. [2]

4. A smooth loop of wire in the form of a circle, centre O and of radius $0{\cdot}3$ m, is fixed in a vertical plane. A bead of mass $0{\cdot}5$ kg is threaded on the wire and projected with speed u m s^{-1} from the lowest point of the wire so that it comes to instantaneous rest at a height of $0{\cdot}1$ m above the level of O. Find

(i) the value of u,

(ii) the reaction of the wire on the particle when the particle is level with O.

[5]

5. At time $t = 0$ seconds, the position vectors (in metres) of two particles A and B are $7\mathbf{i} + 9\mathbf{j}$ and $5\mathbf{i} + 3\mathbf{j}$ respectively, and the particles are moving at all times with constant velocities (in m s^{-1}) of $4\mathbf{i} - 7\mathbf{j}$ and $9\mathbf{i} + 8\mathbf{j}$ respectively.

(*a*) Write down the velocity of B relative to A.

(*b*) Find the position vector of B relative to A at time t seconds.

(*c*) Determine whether the particles will collide.

(*d*) Find the cosine of the angle between their velocities.

[7]

6.

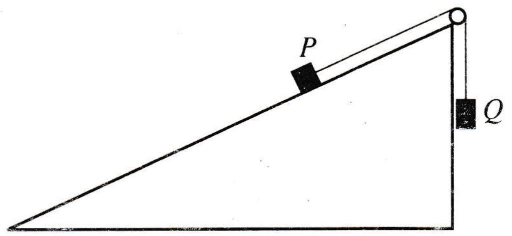

Fig. 1

Fig. 1 shows a particle P, of mass $5m$, on a rough plane, connected by a light inextensible string passing over a small smooth pulley at the top of the plane to a particle Q of mass $10m$ which is hanging freely. The string is parallel to a line of greatest slope of the plane. The coefficient of friction between P and the plane is $\frac{1}{4}$ and the plane is inclined at an angle $\sin^{-1}\frac{3}{5}$ to the horizontal. The particles are released from rest. Find the tension in the string during the subsequent motion.

[5]

After Q has dropped a distance $10d$ it hits an inelastic horizontal plane. Assuming that P does not reach the pulley, find the further distance travelled by P before first coming to instantaneous rest.

[4]

7. A particle of mass 0·2 kg moves in a horizontal straight line under the action of a resistive force directly proportional to its speed. The force is 2 N when the particle is moving with speed 10 m s^{-1}. Given that the speed is v m s^{-1} at time t seconds, show that

$$\frac{dv}{dt} = -v.$$

Find

(i) the time taken for the speed to decrease from 4 m s^{-1} to 2 m s^{-1},

(ii) the distance travelled during this period.

[8]

Section B

Answer **four** *questions from this section.*

8. A car of mass 1300 kg tows a trailer of mass m kg along a level road. The resistance on the car is 1000 N and that on the trailer is 1·5 m N.

(*a*) Find the power developed by the car's engine when $m = 600$ and the car and trailer are moving at a steady speed of 20 m s^{-1}. [2]

(*b*) Given that $m = 700$, the car's speed is 25 m s^{-1} and its engine is working at a steady rate of 75 kW, find

(i) the acceleration of the car and trailer,

(ii) the tension in the coupling between the car and trailer.

[6]

State whether or not the car and trailer can maintain a steady speed of 25 m s^{-1} with the engine working at a steady rate of 75 kW. [1]

(*c*) Given that the safest minimum speed for travel on a motorway is 18 m s^{-1} and that the car engine is working at a rate of 72 kW, find the range of possible values of m so that the car and trailer can travel at steady speeds which are no lower than the safest minimum speed. [3]

(*d*) For $m = 400$ find the work done in attaining a speed of 10 m s^{-1} in a distance of 100 m from rest. [3]

9. One end of a light elastic string of modulus 4·9 N and natural length 0·5 m is attached to a fixed point A and a particle of mass 0·1 kg is attached to the other end. The particle is held at A and released from rest. Its speed after it has dropped a distance of x metres is v m s^{-1}.

(i) Write down an expression for its speed when $x \leqslant 0{\cdot}5$.

(ii) Show, by use of energy, or otherwise, that for $x \geqslant 0{\cdot}5$, $v^2 = 117{\cdot}6x - 98x^2 - 24{\cdot}5$.

(iii) Show that the acceleration of the particle, for $x \geqslant 0{\cdot}5$, is $98(0{\cdot}6 - x)$.

[6]

Deduce that, for $x \geqslant 0{\cdot}5$, the motion is simple harmonic. Find the position of the centre of the simple harmonic motion and the time taken from the centre to the lowest point reached. [5]

(iv) Find the maximum speed and the maximum depth reached below A. [4]

10. Find the solution of

$$\frac{d^2p}{dt^2} + 2\frac{dp}{dt} + 17p = 34,$$

such that $p = 2{\cdot}5$ and $\dfrac{dp}{dt} = 0$ when $t = 0$. [7]

In an economic model the demand D (the amount required) for a particular item is given by

$$D = 6\frac{d^2p}{dt^2} + 7\frac{dp}{dt} + 6p + 16$$

where p is the price in pounds of the item at time t months. Similarly the supply S (the amount available) is given by

$$S = 7\frac{d^2p}{dt^2} + 9\frac{dp}{dt} + 23p - 18.$$

Given that $S = D$ and that, at $t = 0$, $p = 2{\cdot}5$ and $\dfrac{dp}{dt} = 0$ find

(i) the price after a long time, [3]

(ii) the lowest price at which the item can be bought and the time when this would be possible. [5]

11. (*a*) In a uniform triangular lamina ABC the sides AB and AC are equal, BC is of length $2h$ and the perpendicular distance of A from BC is $3h$. Show, by integration, that the centre of mass of the lamina is at a distance $2h$ from A. [4]

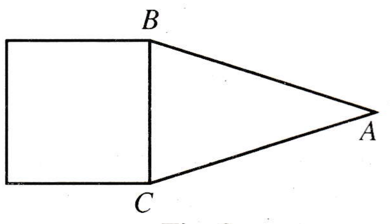

Fig. 2

Fig. 2 shows the composite lamina formed by joining the edge BC of the above lamina to one of the edges of a uniform square lamina, made of the same material, and of edge $2h$. Find the distance of the centre of mass of the composite lamina from the point A. [3]

(*b*) Two particles P_1 and P_2, of mass m and $2m$ respectively, are at rest on a smooth horizontal plane and joined together by a light inextensible string of length $3a$, with the string just taut. An impulse of magnitude $5mv$ is applied to P_2 at an angle $\cos^{-1}\frac{3}{5}$ to P_1P_2 in the sense which causes both particles to move immediately. Find

(i) the velocities of both particles immediately after the impulse is applied,

(ii) the impulsive tension in the string,

(iii) the kinetic energy created by the impulse.

[8]

12. Two small smooth spheres P and Q of equal radius but of mass $2m$ and $4m$ respectively are moving directly towards each other on a smooth horizontal table with speeds $2u$ and $3u$, respectively. The collision is such that Q receives an impulse of magnitude $8amu$, where a is a constant.

(i) Find the speeds of the spheres immediately after collision. [6]

(ii) Find the coefficient of restitution and deduce the range of possible values of a. [6]

(iii) Find the value of a such that the total kinetic energy after collision is $6mu^2$. [3]

13. A point O is vertically above a fixed point A of a horizontal plane. A particle P is projected from O with speed $5V$ at an angle $\cos^{-1}\frac{3}{5}$ above the horizontal and hits the plane at a point B at a distance $\dfrac{48V^2}{g}$ from A.

(i) Show that the height of O above A is $\dfrac{64V^2}{g}$. [5]

(ii) Find the distance of P from O when it is directly level with it. [2]

A second particle is now projected with speed $24W$ from O at an angle of α above the horizontal and it also hits the plane at B. Find an equation involving V, W, and α. [3]

Given that one value of α is $45°$ find W in terms of V and show that the other value of α is such that

$$7\tan^2\alpha - 6\tan\alpha - 1 = 0.$$ [5]

14.

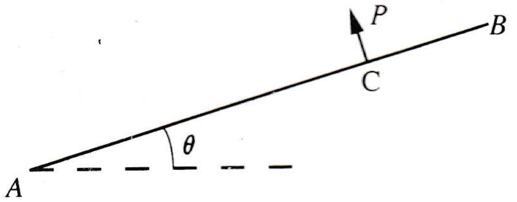

Fig. 3

Fig. 3 shows a uniform rod AB of weight W and length $2a$ freely hinged to a fixed point at A. When a weight $2W$ is attached at B, the rod is kept in equilibrium by a force of magnitude P acting at C perpendicular to AB, in the vertical plane containing AB. In the equilibrium position AB is inclined at angle $\theta(>0)$ to the horizontal and $AC = b(a < b < 2a)$. By taking moments about A, or otherwise, find P. [3]

Find also

(i) the vertical and horizontal components of the reaction at A, [5]

(ii) the maximum value of the horizontal component of the reaction at A as θ varies, [2]

(iii) the value of a/b when the reaction at A is along the rod, [2]

(iv) $\cos^2\theta$ when the reaction at A acts at an angle θ to the upward vertical. [3]

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MATHEMATICS A 3

Probability and Statistics

P.M. TUESDAY, 16 June 1992

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. A box contains 10 videos of which 6 are films and 4 are educational. If 4 of these videos are chosen at random calculate the probabilities that

(i) all four will be films,

(ii) two will be films and the other two educational.

[3]

2. Mass-produced pipes have internal diameters that are Normally distributed with mean 10 cm and standard deviation 0·4 cm.

(i) Calculate, correct to three decimal places, the probability that a randomly chosen pipe will have an internal diameter greater than 10·3 cm.

(ii) Find the value of d, correct to two decimal places, if 95% of the pipes have internal diameters greater than d cm.

[4]

3. A factory has three machines A, B, and C producing a particular type of item. In a day's output 50% of the items are produced on A, 30% on B, and 20% on C. A randomly chosen item from those produced on A has probability 0·01 of being defective, the corresponding probabilities for items produced on B and C being 0·02 and 0·03, respectively.

(i) If an item is chosen at random from a day's output calculate the probability that it will be defective.

(ii) Given that two items chosen at random from a day's output were produced on the same machine and were both defective, calculate the conditional probability that both items were produced on A.

[5]

4. Two variables x and y are known to be related linearly in the form $y = \alpha + \beta x$. To investigate this relationship four experiments were conducted with x having the values 10, 20, 30, and 40, respectively. The corresponding values of y were then measured. A measured value of y is a random observation from a Normal distribution having mean equal to the true value of y and standard deviation 1·2. On applying the method of least-squares to the experimental results the estimated relationship was

$$y = 8{\cdot}4 + 2{\cdot}6x\,.$$

(i) Find the mean of the four measured values of y.

(ii) Calculate a 95% confidence interval for the value of β.

[6]

5. A loaded die is such that when it is thrown the probability of obtaining a score of 6 is 0·25. In five independent throws of the die calculate, correct to three decimal places, the probabilities that

(i) a 6 will be obtained exactly once,

(ii) a 6 will be obtained for the first time on the fifth throw,

(iii) a 6 will be obtained for the second time on the fifth throw.

[7]

6. The number of accidents occurring in a factory over a period of t weeks has a Poisson distribution with mean $2{\cdot}25t$.

(i) Calculate, correct to three decimal places, the probability that there will be exactly 3 accidents during a particular week.

(ii) Use tables to find, correct to three decimal places, the probability that over a period of four weeks the total number of accidents will be 5, 6 or 7.

(iii) Use an appropriate distributional approximation to calculate the probability that over a period of 25 weeks the total number of accidents will be 50 or more; give your answer correct to three decimal places.

[7]

7. The continuous random variable X has probability density function f, where

$$f(x) = \frac{4x^3}{\alpha^4}, \qquad \text{for } 0 \leqslant x \leqslant \alpha,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(i) Find the mean and the variance of X in terms of α. Deduce the values of the mean and the variance of

$$Y = \frac{7\alpha - 5X}{\alpha}.$$

(ii) If $\bar{X}$ is the mean of a random sample of 24 observations of X show that

$$T = \frac{5}{4}\bar{X}$$

is an unbiased estimator of α. Express the standard error of T in terms of α.

[8]

SECTION B

Answer **four** *questions from this section.*

8. Mass produced steel ball bearings have diameters that are Normally distributed with mean μ mm and standard deviation σ mm. A random sample of 9 of the ball bearings had the following diameters (in mm):

5·01, 5·03, 4·96, 4·91, 4·96, 5·06, 5·02, 4·94, 4·93

You are given that the sum of these 9 diameters is 44·82 and that the sum of their squares is 223·2248.

(*a*) Calculate 95% confidence limits for μ in each of the cases when

(i) σ is known to be equal to 0·03,

(ii) σ is unknown.

[8]

(*b*) Given that 10% of all the ball bearings produced have diameters less than 4·95 mm and that 5% have diameters greater than 5·05 mm, determine the values of μ and σ, giving each answer correct to two decimal places.

[7]

9. The continuous random variable X has probability density function f given by

$$f(x) = 3c, \qquad \text{for } 0 \leqslant x < 1,$$
$$f(x) = c(4 - x), \qquad \text{for } 1 \leqslant x \leqslant 4,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(*a*) Show that $c = 2/15$. [2]

(*b*) Find the mean value of X. [3]

(*c*) Find expressions for $F(x)$, where F is the cumulative distribution function of X, indicating clearly the values of x for which each expression is valid. Hence, or otherwise,

(i) evaluate $P(0{\cdot}5 < X < 2{\cdot}5)$,

(ii) determine the cumulative distribution function of $Y = \sqrt{(4 - X)}$.

[10]

10. An envelope contains 10 stamps, including 7 of value 20 pence and 3 of value 25 pence.

(i) Find the mean, μ, and the variance, σ^2, of the values of these 10 stamps. [3]

Four of the 10 stamps are to be chosen at random. Let T_1 denote the total value of the 4 chosen stamps if they are to be chosen **without replacement**, and let T_2 denote their total value if they are to be chosen **with replacement**.

(ii) Find the sampling distribution of T_1 and evaluate its mean and variance.

Explain why $E(T_1) = 4\mu$ and why $\text{Var}(T_1) \neq 4\sigma^2$. [6]

(iii) Find the sampling distribution of T_2 and evaluate its mean and variance.

Explain how the values of $E(T_2)$ and $\text{Var}(T_2)$ could have been obtained without first finding the sampling distribution of T_2. [6]

11. In a certain game two fair cubical dice are thrown together. In one throw of the two dice let X denote the number of 6s that occur and let Y denote the number of even scores (including 6) that occur.

(*a*) Show that $P(X = 0,\ Y = 1) = 1/3$ and that $P(X = 1,\ Y = 2) = 1/9$. [3]

(*b*) Derive the joint probability distribution of X and Y, displaying it in the form of a two-way table. Use your table to evaluate the mean and the variance of X, verifying your answers using the formulae for the mean and the variance of a binomial distribution. [6]

(*c*) In each play of this game the thrower of the dice receives three pence for every 6 that occurs and one penny for every even score other than a 6 that occurs. Let W pence denote the amount that the player will receive in one throw. Find the mean and the variance of W. [6]

12. A newsagent stocks copies of two gardening magazines, *Gardening News* and *Today's Garden*, both of which are published monthly. In any month, the number of customers requiring the current issue of *Gardening News* has the Poisson distribution with mean 10 and, independently, the number requiring the current issue of *Today's Garden* has the Poisson distribution with mean 5. In each month the newsagent has only 10 copies of *Gardening News* and only 5 copies of *Today's Garden* available for sale.

(*a*) Find

(i) the most probable number of copies of *Today's Garden* that will be **sold** in a month,

(ii) the mean value, correct to two decimal places, of the number of copies of *Today's Garden* **sold** per month.

[7]

(*b*) Find, correct to three decimal places, the probability that in a month

(i) the newsagent will sell exactly 5 copies of *Gardening News* and exactly 4 copies of *Today's Garden*,

(ii) the newsagent will sell all 10 copies of *Gardening News* but only at most 4 copies of *Today's Garden*,

(iii) the newsagent will not be able to supply the magazine of the choice of at least one customer.

[8]

13. The heights of the male students at a large college are Normally distributed with mean 180 cm and standard deviation 4 cm. Independently, the heights of the female students at the college are Normally distributed with mean 170 cm and standard deviation 5 cm.

(*a*) If one male student and one female student are chosen at random evaluate, correct to three decimal places, the probability that

(i) both will be taller than 175 cm,

(ii) the female student will be taller than the male student.

[7]

(*b*) If two male students and three female students are chosen at random evaluate, correct to three decimal places, the probability that

(i) the mean of the heights of the three female students will be greater than the mean of the heights of the two male students,

(ii) the sum of the heights of all five students will be greater than 850 cm.

[8]

14. A choconut is a sweet consisting of a nut coated with chocolate. The nut may be an almond or a cashew. Of the choconuts produced 60% have almond centres and 40% have cashew centres.

(*a*) If 10 choconuts are chosen at random from production **use tables** to find the probability, correct to three decimal places, that from 5 to 8, inclusive, of them will have almond centres. [2]

(*b*) If a jar contains 80 choconuts drawn at random from production use a Normal approximation to find the probability, correct to two decimal places, that at least 50 of the choconuts in the jar have almond centres. [4]

(*c*) Use a Normal approximation (ignoring the continuity correction) to find the number of choconuts that should be sampled at random from production for there to be a probability of about 0·95 that the sample will include more almond centres than cashew centres. [4]

(*d*) Suppose that the production process has been modified and that the proportion of choconuts having almond centres is θ. In a random sample of 100 choconuts from this production process it was found that 70 of them had almond centres. Write down an unbiased estimate, p, of θ and calculate an estimate of the standard error of p. Assuming that the sampling distribution of p is approximately Normal, calculate 95% confidence limits for θ. [5]

WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

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Safon Uwch

MATHEMATICS A 4

Further Pure Mathematics

A.M. FRIDAY, 19 June 1992

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 6 \\ 2 & 3 & 3 \end{bmatrix}.$$

[5]

2. Show that

$$\int_{\frac{1}{2}}^{2} \left| 1 - \frac{1}{x} \right| dx = \frac{1}{2}.$$ [4]

3. The roots of the cubic equation

$$x^3 - 3x^2 + 6x - 10 = 0$$

are denoted by α, β and γ.

Show that

$$\alpha^2 + \beta^2 + \gamma^2 = -3.$$

Explain why this result shows that only one of the roots is real. [5]

4. Given that $f(x) = x$, show from first principles that

$$f'(x) = 1.$$

Assuming the product rule for differentiation and using the identity $x^{n+1} \equiv x.x^n$, use mathematical induction to prove that

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

for all positive integral n. [5]

5. Given that

$$y = x^x$$

show that

$$\frac{d^2y}{dx^2} = x^x(1 + \ln x)^2 + x^{x-1}.$$

[5]

6. Show that the equation

$$\tan x = 2x$$

has a root in the interval (1·1, 1·2).

Use the Newton-Raphson method to find the value of this root correct to 3 decimal places. [5]

7. Given that

$$I_n = \int_0^{\pi} \theta^n \sin\theta \, d\theta$$

show that, for $n \geqslant 2$,

$$I_n = \pi^n - n(n-1)\,I_{n-2}.$$

Deduce that

$$I_4 = \pi^4 - 12\pi^2 + 48.$$

[6]

8. Given that

$$f(x) = \int_0^x \tan^{-1}(e^t)\, dt$$

show that

$$f''(x) = \tfrac{1}{2}\operatorname{sech} x.$$

Hence find the first two non-zero terms of the Maclaurin series for $f(x)$. [5]

Section B

Answer **four** *questions from this section.*

9. The line L_1 with direction ratios $1:3:\mu$ passing through the point $(\lambda, 14, 1)$ intersects at right-angles the line L_2 with equations

$$\frac{x-5}{7} = \frac{y-15}{-5} = \frac{z-9}{4}.$$

(i) Show that $\mu = 2$.

Find

(ii) the value of λ,

(iii) the coordinates of the point of intersection of L_1 and L_2.

[7]

The normal to the plane Π containing L_1 and L_2 has direction ratios $p:q:r$.

Write down the conditions for this normal to be perpendicular to both L_1 and L_2 and hence obtain values for p, q and r.

Deduce the cartesian equation of Π. [5]

The line L_3 has direction ratios $1:-3:2$ and passes through the point $(3, 4, 10)$.

Find the coordinates of the point of intersection of L_3 and Π. [3]

10. Show that the tangents at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ to the parabola $y^2 = 4ax$ intersect at the point $R(apq, a[p+q])$. [5]

The chord PQ passes through the point $(a, 0)$.

(i) Show that

$$pq = -1.$$ [3]

(ii) Write down the equation of the locus of R as p varies. [1]

(iii) Show that the coordinates of the mid-point M of PQ are

$$\left(\frac{1}{2}a\left[p^2 + \frac{1}{p^2}\right], a\left[p - \frac{1}{p}\right]\right).$$

Hence find the equation of the locus of M as p varies. [6]

11. Express

$$\frac{2(r+3)}{r(r+1)(r+2)}$$

in partial fractions. [5]

(i) Show that

$$\sum_{r=1}^{n} \frac{2(r+3)}{r(r+1)(r+2)} = \frac{5}{2} - \frac{(2n+5)}{(n+1)(n+2)}.$$

Deduce the value of

$$\lim_{n \to \infty} \sum_{r=1}^{n} \frac{2(r+3)}{r(r+1)(r+2)}.$$

[6]

(ii) Show that

$$\sum_{r=n+1}^{2n} \frac{2(r+3)}{r(r+1)(r+2)} = \frac{4n^2+11n}{2(n+1)(n+2)(2n+1)}.$$

[4]

12. Using the exponential definitions of the hyperbolic functions, show that

(i) $\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$

(ii) $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ where $-1 < x < 1$.

[6]

Given that

$$a \cosh x + b \sinh x \equiv r \cosh(x + \alpha)$$

where $a > b > 0$, show that

$$\alpha = \frac{1}{2} \ln\left(\frac{a+b}{a-b}\right)$$

and find an expression for r in terms of a and b. [4]

Find the two real roots of the equation

$$13 \cosh x + 5 \sinh x = 20$$

expressing your answers as natural logarithms. [5]

13. (i) Assuming the formula for $\tan 2\theta$ in terms of $\tan\theta$, show that

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

Hence show that

$$3\tan^{-1}\tfrac{1}{3} = \tan^{-1}\tfrac{13}{9}.$$ [5]

(ii) One of the cube roots of $18 + 26i$ is represented by a point lying in the first quadrant of the Argand diagram. This cube root is denoted by w. Using the result in (i), show that

$$\arg w = \tan^{-1}\tfrac{1}{3}.$$

Hence express w in cartesian form. [5]

(iii) Find, also in cartesian form, the other two cube roots of $18 + 26i$. [5]

14. R denotes an anticlockwise rotation of the plane through an angle α about the point $P(h, k)$. By considering R as a translation which takes P to the origin O, followed by an anticlockwise rotation of the plane through an angle α about O, followed by a translation which takes O to P, show that the matrix representing R is given by

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & -h\cos\alpha + k\sin\alpha + h \\ \sin\alpha & \cos\alpha & -h\sin\alpha - k\cos\alpha + k \\ 0 & 0 & 1 \end{bmatrix}.$$ [6]

R_1 is an anticlockwise rotation through $120°$ about the point $(-1, 0)$ and R_2 is an anticlockwise rotation through $120°$ about the point $(1, 0)$. Write down the matrices corresponding to R_1 and R_2. [2]

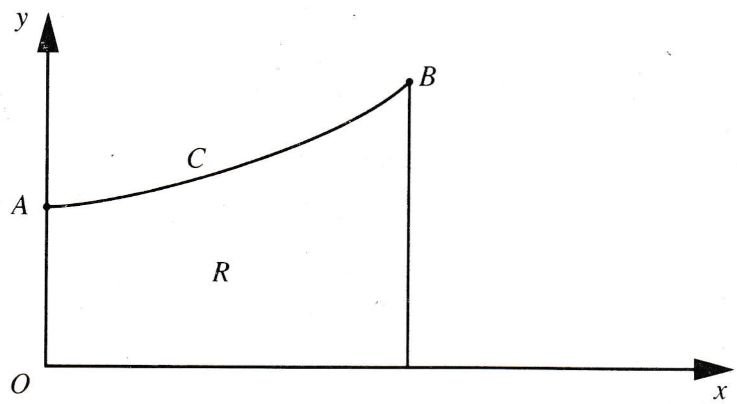
R_3 is the composite transformation 'R_1 followed by R_2'. Find the matrix representing R_3. Hence show that R_3 is a rotation and determine

(i) the angle of the rotation,

(ii) the centre of the rotation.

[7]

15.



The figure shows the curve C with parametric equations

$$\begin{aligned} x &= 2\sin\theta + \sin 2\theta \\ y &= 2\cos\theta - \cos 2\theta \end{aligned} \quad \left(0 \leqslant \theta \leqslant \frac{\pi}{3}\right)$$

The values of θ at the points A, B are respectively $0, \dfrac{\pi}{3}$.

(i) Show that

$$\frac{dx}{d\theta} = 4\cos\frac{3\theta}{2}\cos\frac{\theta}{2}$$

and obtain a similar expression for $\dfrac{dy}{d\theta}$. [4]

(ii) Show that the length of the arc AB is $\frac{8}{3}$. [4]

(iii) Show that the area of the region R enclosed by C, the x and y axes and the ordinate through B can be expressed in the form

$$2\int_0^{\pi/3} (2\cos^2\theta + \cos\theta\cos 2\theta - \cos^2 2\theta)\, d\theta.$$

Hence find this area correct to 3 significant figures. [7]

WELSH JOINT EDUCATION COMMITTEE
General Certificate of Education

Advanced Level
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Safon Uwch
Papur Arbennig

MATHEMATICS S

A.M. MONDAY, 22 June 1992

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from Section A and no more than* **four** *questions from either Section B or Section C.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Pure Mathematics

No more than **four** *questions should be answered from this section.*

1. (*a*) By considering a suitable area under the curve $xy = 1$, show that for $r > 0$,

$$\frac{1}{r+1} < \ln\left(\frac{r+1}{r}\right) < \frac{1}{r}.$$

Hence show that

$$\ln(1+n) < \sum_{r=1}^{n} \frac{1}{r} < 1 + \ln n$$

where n is a positive integer.

Deduce an approximate value for

$$\sum_{r=1}^{10^6} \frac{1}{r}.$$ [6]

Let

$$U_n = \sum_{r=1}^{n} \frac{1}{r} - \ln n.$$

Show that, for all positive integral n,

(i) $0 < U_n < 1$,

(ii) $U_{n+1} < U_n$.

[4]

(*b*) Show that, for integral $r \geqslant 4$,

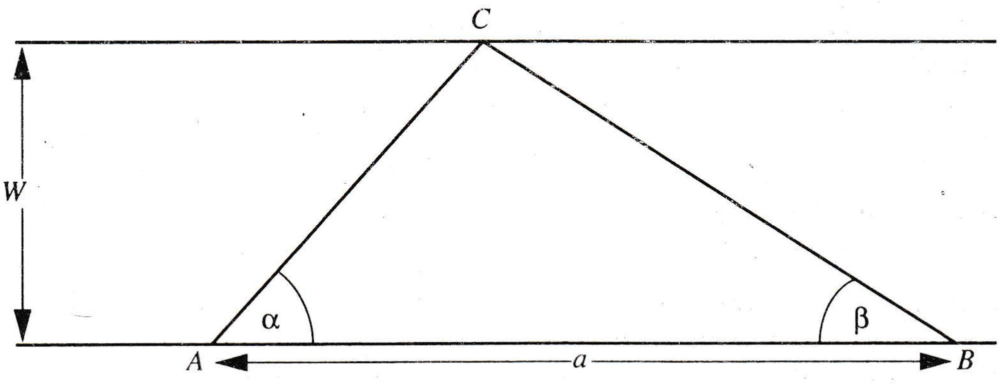
$$\frac{1}{r!} < \frac{1}{r-1} - \frac{1}{r}.$$

Use this result to show that

$$\sum_{r=1}^{n} \frac{1}{r!} < 2$$

for all positive integral n. [5]

2. (*a*)



The figure shows a straight river of constant width W. A, B, C are trees on the two river banks. A surveyor standing on the same river bank as A and B wishes to determine the width of the river. He therefore makes the following measurements:

$$AB = a,\ C\widehat{A}B = \alpha,\ C\widehat{B}A = \beta.$$

Find an expression for W in terms of a, α and β.

The surveyor realises that he made a small error $\delta\alpha$ in measuring $C\widehat{A}B$ although his other measurements were accurate. Show that the percentage error in W is given approximately by

$$\frac{100\,\delta\alpha \sin\beta}{\sin\alpha\sin(\alpha+\beta)}.$$ [7]

(*b*) Show that the equation

$$\sin x\,(\sin x + \cos x) = a$$

has real roots if and only if

$$\tfrac{1}{2}(1-\sqrt{2}) \leqslant a \leqslant \tfrac{1}{2}(1+\sqrt{2}).$$

Given that this condition is satisfied, find the general solution of the equation.

Write down the maximum and minimum values of the expression

$$\sin x\,(\sin x + \cos x).$$ [8]

 Turn over.

3. (*a*) The coplanar points O, A, B, C are such that OA is perpendicular to BC and OB is perpendicular to CA.

Writing $\mathbf{a} = \mathbf{OA}$, $\mathbf{b} = \mathbf{OB}$, and $\mathbf{c} = \mathbf{OC}$, show that

$$\mathbf{a}\,.\,\mathbf{b} = \mathbf{b}\,.\,\mathbf{c} = \mathbf{c}\,.\,\mathbf{a}\,.$$

Deduce that OC is perpendicular to AB. Use this result to show that the three altitudes of a triangle are concurrent. [4]

(*b*) $A_1A_2A_3A_4$ is a tetrahedron in which the perpendiculars from the four vertices to the opposite faces are concurrent at the point O. Writing $\mathbf{a}_1 = \mathbf{OA}_1$, $\mathbf{a}_2 = \mathbf{OA}_2$, $\mathbf{a}_3 = \mathbf{OA}_3$ and $\mathbf{a}_4 = \mathbf{OA}_4$, show that the value of $\mathbf{a}_r\,.\,\mathbf{a}_s$ is the same for all possible pairs r, s $(r \neq s)$.

Deduce that

(i) all pairs of edges which do not meet are perpendicular,

(ii) the foot of the perpendicular from any vertex to the opposite face is also the point of concurrence of the altitudes of that face. [11]

4. [You may assume in this question that the curved surface area of a right circular cone is πrl, where r is the base radius and l the slant height.]

It is required to construct a container in the shape of a right circular cone with no base. Given that the curved surface area is to be 100 cm^2, express the volume $V\text{ cm}^3$ of the cone in terms of its semi-vertical angle θ. Hence show that as θ varies, the maximum value of V is approximately 117. [6]

Suppose now that the container is to be closed and that the total surface area of the cone, that is the curved surface area plus the area of the circular base, is to remain 100 cm^2. Show that the volume is now given by

$$V^2 = \frac{10^6\cot^2\theta}{9\pi\,(1 + \operatorname{cosec}\theta)^3}.$$

Hence find the new maximum value of V. [9]

5. The curve C has equation

$$y = \frac{x}{\sqrt{3 + 2x - x^2}} \qquad (-1 < x < 3).$$

(i) Show that C has no stationary points. [2]

(ii) Given that C has one point of inflection find its x coordinate. [3]

(iii) Sketch C. [2]

(iv) Using the substitution $x = 1 + 2\sin\theta$, or otherwise, show that the area of the region R enclosed by C, the positive x-axis and the line $x = 2$ is equal to $\frac{\pi}{3}$. [3]

(v) The region R is rotated through four right-angles about the x-axis. Find the volume of the solid generated, giving your answer correct to 2 decimal places. [5]

SECTION B

Mechanics

No more than **four** *questions should be answered from this section.*

$$[g = 9{\cdot}8 \text{ m s}^{-2}]$$

6. A particle is projected from a point A on a horizontal plane with speed u at an angle α to the horizontal. The particle rises to a maximum height h above the plane at the point B. Given that, relative to a point O on the plane vertically below B, the particle has horizontal and vertical displacements X and Y, respectively, show that

$$Y = -pX^2 + h,$$

where p should be expressed in terms of u, g and α. [6]

The particle passes in succession through the points whose coordinates relative to O are $\left(-a, \frac{\sqrt{3}}{2}a\right), \left(-\frac{1}{2}a, \sqrt{3}\,a\right)$. Find h in terms of a and show that

$$OA = \frac{\sqrt{7}}{2}a \cdot$$ [9]

7. The vectors $\mathbf{a}$ and $\mathbf{b}$ vary with time t and satisfy the equations

$$\frac{d\mathbf{a}}{dt} = -\mathbf{b}, \qquad \frac{d\mathbf{b}}{dt} = \mathbf{a}.$$

Find, assuming that $\mathbf{a}$ and $\mathbf{b}$ are not parallel, the scalar constants p and q so that $p\,\mathbf{a}\cos t + \mathbf{b}\sin t$ and $q\,\mathbf{a}\sin t + \mathbf{b}\cos t$ are both constant vectors. [9]

Given that, when $t = 0$, $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$ where $\mathbf{c}$ and $\mathbf{d}$ are constant vectors, express $\mathbf{a}$ in terms of $\mathbf{c}$, $\mathbf{d}$ and t. [3]

Find the vector differential equation satisfied by $\mathbf{a}$ and verify that your form for $\mathbf{a}$ satisfies it. [3]

8. A particle P is threaded on a rough wire in the form of a circle of radius a and centre O. The wire is fixed in a vertical plane and P is released from rest from a point level with O. The coefficient of friction between the wire and P is denoted by μ. Show that when OP makes an angle θ with the horizontal, the angular speed ω of P about O satisfies

$$a\left\{\frac{d(\omega^2)}{d\theta} + 2\mu\omega^2\right\} = 2g(\cos\theta - \mu\sin\theta).$$ [7]

Find ω^2 in terms of a, g, μ and θ and show that, if the particle comes to rest on first reaching the lowest point of the wire, $(1 - 2\mu^2)e^{\mu\pi} = 3\mu$. [8]

[You may assume that

$$\int \cos\theta e^{2\mu\theta}\, d\theta = \left(\frac{2\mu\cos\theta + \sin\theta}{4\mu^2 + 1}\right)e^{2\mu\theta},$$

$$\int \sin\theta e^{2\mu\theta}\, d\theta = \left(\frac{2\mu\sin\theta - \cos\theta}{4\mu^2 + 1}\right)e^{2\mu\theta}.]$$

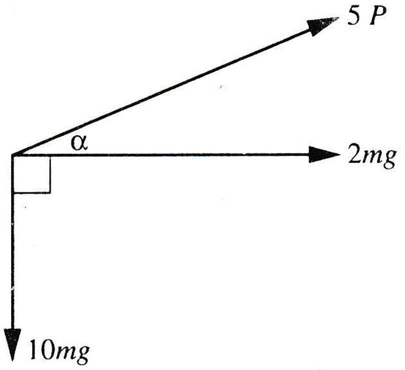
9. Two boats A and B are, at time $t = 0$, at the points P and Q, respectively. Q is a distance a directly East of P. Boat B sets off with speed u directly North and boat A sets off, with the same speed, on a bearing of $2\theta°$ ($\theta < 45$).

Find AB^2 at any subsequent time t in terms of a, u, t and θ. [4]

Show that the shortest distance apart of the boats is $a \sin \theta$. [6]

Given that at the instant when the boats are first at a distance of qa apart, ($q < 1$), boat A is moving directly towards boat B, show that $\tan \theta = q$. [5]

10.



A particle of mass $20m$ rests in equilibrium on a rough horizontal plane. In addition to the force of gravity, the normal reaction and the force of friction, there are three horizontal forces, of magnitudes $10mg$, $2mg$, $5P$, acting in the directions shown in the figure above.

The coefficient of friction between the plane and the particle is $\frac{1}{2}$.

(i) Given that $\alpha = \sin^{-1}(\frac{3}{5})$, show that, for equilibrium,

$$\left(\frac{22 - 8\sqrt{6}}{25}\right)mg \leqslant P \leqslant \left(\frac{22 + 8\sqrt{6}}{25}\right)mg. \qquad [8]$$

(ii) When the particle is in equilibrium with $25P = (22 + 8\sqrt{6})mg$, the value of P is instantaneously increased to $2mg$. Find the distance that the particle travels in time T. [3]

(iii) Assuming that α varies in the range $0 \leqslant \alpha \leqslant 180°$, find the value of α which gives the smallest value of P consistent with equilibrium and find this smallest value. [4]

SECTION C

Probability and Statistics

No more than **four** *questions should be answered from this section.*

11. Patients in a hospital ward have one or both of two diseases A and B. Medical records show that 64% of all such patients have disease A and that $12\frac{1}{2}$% of those patients who have disease A also have disease B.

(*a*) Calculate the probability that such a patient chosen at random has disease B only. [2]

The treatment for such patients has probability 0·8 of curing a patient who has disease A only, probability 0·5 of curing a patient who has disease B only, and probability 0·2 of curing a patient who has both diseases.

(*b*) Calculate the probability that a randomly chosen patient will be cured by the treatment. [4]

(*c*) Two such patients are chosen at random. Given that neither has both diseases, calculate the probability that the treatment will cure exactly one of the two patients. [9]

12. The discrete random variable X has the probability distribution given by

$$P(X = r) = \frac{\ln(r + 1) - \ln r}{\ln(n + 1)}, \quad r = 1, 2, 3, \ldots, n.$$

(*a*) Verify that the sum of these probabilities is unity. [1]

(*b*) Show that

$$E(X) = n - \frac{\ln(n!)}{\ln(n + 1)}. \qquad [4]$$

(*c*) For the particular case when $n = 3$, the continuous random variable Y is such that, for $r = 1, 2, 3$,

$$P(Y \leqslant y \mid X = r) = y^{r+1}, \quad 0 \leqslant y \leqslant 1.$$

Find a polynomial expression for $P(Y \leqslant y)$ and carry out appropriate checks on your result. Deduce the probability density function of Y and hence evaluate $E(Y)$ correct to four decimal places. [10]

13. The random variables X and Y are independent and each has the Poisson distribution with mean μ.

(*a*) Let $W = XY$. Find, in terms of μ,

(i) the mean of W,

(ii) the variance of W,

(iii) $P(W = 0)$. [6]

(*b*) Find the value of

$$P(XY \geqslant 3 \mid X + Y = 4). \qquad [4]$$

(*c*) Show that

$$E\left(\frac{Y}{X + 1}\right) = 1 - e^{-\mu}. \qquad [5]$$

14. An instrument used for measuring the diameter of a circle whose true diameter is a cm will give a reading X cm which is a continuous random variable having probability density function f, where

$$f(x) = 750[0{\cdot}01 - (x - a)^2], \text{ for } a - 0{\cdot}1 \leqslant x \leqslant a + 0{\cdot}1,$$

$$f(x) = 0, \text{ otherwise.}$$

Show that $E(X - a) = 0$ and hence, or otherwise, find the mean and the variance of X. [4]

The instrument is used to obtain five independent measurements X_1, X_2, X_3, X_4, X_5 of the diameter of a circle whose true diameter is a cm. Let $\bar{X}$ denote the mean of these five measurements. It is required to estimate the true area $(\pi a^2/4)$ of the circle. One person suggested the estimator

$$Z_1 = (\pi \sum X_i^2) / 20,$$

which is the average of the areas calculated from the individual measurements. Another person suggested the estimator

$$Z_2 = (\pi \bar{X}^2) / 4,$$

which is the area calculated from the mean of the measurements.

Show that both Z_1 and Z_2 are biased estimators of the area of the circle.

By suitably modifying Z_1 and Z_2, deduce two unbiased estimators of the area of the circle. [11]

15. Two variables, x and y, are known to be related in the form $y = \lambda(x - 1)$, where λ is an unknown constant. When experiments are conducted with x having specific values the measured values of y are known to be subject to independent errors which are Normally distributed with mean zero and standard deviation σ. Experiments were conducted with x having the values $x_1, x_2, \ldots, x_n$ and the measured values of y were $y_1, y_2, \ldots, y_n$, respectively. Use the method of least squares to derive an estimate of λ in terms of the x_i and the y_i $(i = 1, 2, \ldots, n)$.

Verify that the least squares estimator of λ is unbiased and show that its variance is $\sigma^2 / \sum (x_i - 1)^2$.

The results of five such experiments were as follows:

x	3	5	7	9	11
y	4·9	13·6	24·8	34·6	42·1

Given that $\sigma = 0{\cdot}5$, calculate a 98% confidence interval for λ.

Comment on the claim that y has the value 29·5 when $x = 8$. [15]

WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

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Advanced Level

Special Paper

Safon Uwch

Papur Arbennig

PURE MATHEMATICS S

A.M. MONDAY, 22 June 1992

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

Section A

Pure Mathematics

No more than **four** *questions should be answered from this section.*

1. (*a*) By considering a suitable area under the curve $xy = 1$, show that for $r > 0$,

$$\frac{1}{r+1} < \ln\left(\frac{r+1}{r}\right) < \frac{1}{r}.$$

Hence show that

$$\ln(1+n) < \sum_{r=1}^{n} \frac{1}{r} < 1 + \ln n$$

where n is a positive integer.

Deduce an approximate value for

$$\sum_{r=1}^{10^6} \frac{1}{r}.$$ [6]

Let

$$U_n = \sum_{r=1}^{n} \frac{1}{r} - \ln n.$$

Show that, for all positive integral n,

(i) $0 < U_n < 1$,

(ii) $U_{n+1} < U_n$. [4]

(*b*) Show that, for integral $r \geqslant 4$,

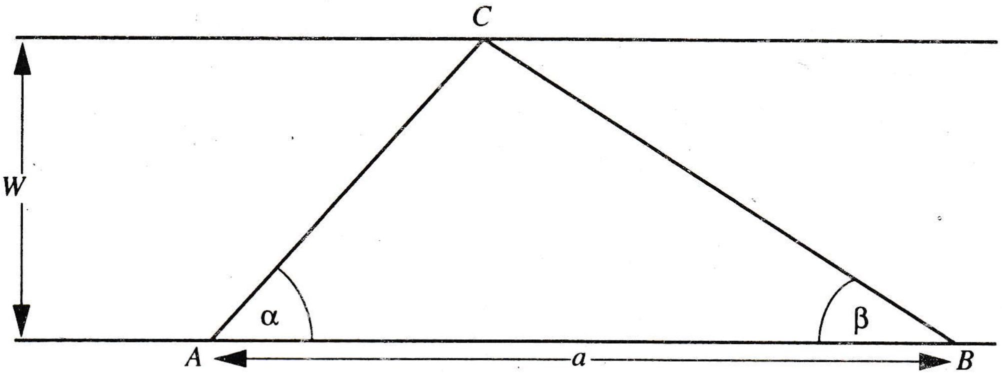
$$\frac{1}{r!} < \frac{1}{r-1} - \frac{1}{r}.$$

Use this result to show that

$$\sum_{r=1}^{n} \frac{1}{r!} < 2$$

for all positive integral n. [5]

2. (*a*)



The figure shows a straight river of constant width W. A, B, C are trees on the two river banks. A surveyor standing on the same river bank as A and B wishes to determine the width of the river. He therefore makes the following measurements:

$$AB = a,\ C\hat{A}B = \alpha,\ C\hat{B}A = \beta$$

Find an expression for W in terms of a, α and β.

The surveyor realises that he made a small error $\delta\alpha$ in measuring $C\hat{A}B$ although his other measurements were accurate. Show that the percentage error in W is given approximately by

$$\frac{100\,\delta\alpha \sin\beta}{\sin\alpha \sin(\alpha+\beta)}.$$ [7]

(*b*) Show that the equation

$$\sin x\,(\sin x + \cos x) = a$$

has real roots if and only if

$$\tfrac{1}{2}(1-\sqrt{2}) \leqslant a \leqslant \tfrac{1}{2}(1+\sqrt{2}).$$

Given that this condition is satisfied, find the general solution of the equation.

Write down the maximum and minimum values of the expression

$$\sin x\,(\sin x + \cos x).$$ [8]

Turn over.

3. (*a*) The coplanar points O, A, B, C are such that OA is perpendicular to BC and OB is perpendicular to CA.

Writing $\mathbf{a} = \mathbf{OA}$, $\mathbf{b} = \mathbf{OB}$ and $\mathbf{c} = \mathbf{OC}$, show that

$$\mathbf{a}\,.\,\mathbf{b} = \mathbf{b}\,.\,\mathbf{c} = \mathbf{c}\,.\,\mathbf{a}\,.$$

Deduce that OC is perpendicular to AB. Use this result to show that the three altitudes of a triangle are concurrent. [4]

(*b*) $A_1A_2A_3A_4$ is a tetrahedron in which the perpendiculars from the four vertices to the opposite faces are concurrent at the point O. Writing $\mathbf{a}_1 = \mathbf{OA}_1$, $\mathbf{a}_2 = \mathbf{OA}_2$, $\mathbf{a}_3 = \mathbf{OA}_3$ and $\mathbf{a}_4 = \mathbf{OA}_4$, show that the value of $\mathbf{a}_r\,.\,\mathbf{a}_s$ is the same for all possible pairs r, s $(r \neq s)$.

Deduce that

(i) all pairs of edges which do not meet are perpendicular,

(ii) the foot of the perpendicular from any vertex to the opposite face is also the point of concurrence of the altitudes of that face. [11]

4. [You may assume in this question that the curved surface area of a right circular cone is πrl, where r is the base radius and l the slant height.]

It is required to construct a container in the shape of a right circular cone with no base. Given that the curved surface area is to be 100 cm^2, express the volume $V\text{ cm}^3$ of the cone in terms of its semi-vertical angle θ. Hence show that as θ varies, the maximum value of V is approximately 117. [6]

Suppose now that the container is to be closed and that the total surface area of the cone, that is the curved surface area plus the area of the circular base, is to remain 100 cm^2. Show that the volume is now given by

$$V^2 = \frac{10^6 \cot^2\theta}{9\pi\,(1 + \operatorname{cosec}\theta)^3}.$$

Hence find the new maximum value of V. [9]

5. The curve C has equation

$$y = \frac{x}{\sqrt{3 + 2x - x^2}} \qquad (-1 < x < 3).$$

(i) Show that C has no stationary points. [2]

(ii) Given that C has one point of inflection find its x coordinate. [3]

(iii) Sketch C. [2]

(iv) Using the substitution $x = 1 + 2\sin\theta$, or otherwise, show that the area of the region R enclosed by C, the positive x-axis and the line $x = 2$ is equal to $\frac{\pi}{3}$. [3]

(v) The region R is rotated through four right-angles about the x-axis. Find the volume of the solid generated, giving your answer correct to 2 decimal places. [5]

SECTION B

Further Pure Mathematics

No more than **four** *questions should be answered from this section.*

6. Sketch the curve C defined parametrically by

$$x = 1 + t^3, y = t^2 \qquad (-\infty < t < \infty). \qquad [2]$$

(i) The points on C with parametric values t_1, t_2 and t_3 are collinear.

Show that

$$t_2t_3 + t_3t_1 + t_1t_2 = 0. \qquad [4]$$

(ii) Find the coordinates of the point P of intersection of the tangents to C at the points with parameters t_1 and t_2.

Given that these two tangents are perpendicular, find the equation of the locus of P as t_1, t_2 vary. [9]

7. (*a*) Use mathematical induction to prove that $2^{4n+3} + 3^{3n+1}$ is divisible by 11 for every positive integer n. [5]

(*b*) Two different numbers are selected from the positive integers

$$\{1, 2, 3, \ldots n\}.$$

Prove that the number of selections S_n for which the sum of the two numbers is at least $n + 2$ is given for $n \geqslant 3$ by

$$S_n = \tfrac{1}{4}n(n-2) \text{ for even } n,$$
$$S_n = \tfrac{1}{4}(n-1)^2 \text{ for odd } n. \qquad [10]$$

8. The integral $I(a)$ is defined by

$$I(a) = \int_0^{\pi/2} \frac{dx}{a + \cos x}.$$

(i) Given that $a > 1$, show that

$$I(a) = \frac{1}{\sqrt{a^2 - 1}} \sec^{-1} a. \qquad [8]$$

(ii) Given that $0 < a < 1$, obtain a similar expression for $I(a)$ in terms of an inverse hyperbolic function. [7]

9. (i) The complex numbers $w = u + iv$ and $z = x + iy$ are related by

$$w = \sin z$$

where x, y, u and v are real. Show that

$$u = \sin x \cosh y$$

and find a similar expression for v. [4]

(ii) Sketch, in the x–y plane for $0 < x < \pi$, the two curves

$$u = 1 \text{ and } v = 1.$$

Find the coordinates of the point of intersection of these two curves for $0 < x < \frac{\pi}{2}$. [7]

(iii) Show that at all points of intersection in the x–y plane of the curves

$$u = \lambda \text{ and } v = \mu$$

where λ, μ are real constants, the curves intersect at right-angles. [4]

10. The equation $f(x) = 0$ has a root r approximately equal to α. By considering the following approximation based on the first two terms of the Taylor series,

$$f(r) \approx f(\alpha) + (r - \alpha) f'(\alpha),$$

show that a second approximation to r is given by the Newton-Raphson formula

$$r \approx \alpha - \frac{f(\alpha)}{f'(\alpha)}.$$ [2]

A student decides to modify this method by using an approximation based on the first three terms of the Taylor series. Show that, in this case, the second approximation is given by

$$r \approx \alpha - \frac{f'(\alpha) \pm \sqrt{\{f'(\alpha)\}^2 - 2f(\alpha)f''(\alpha)}}{f''(\alpha)}.$$ [4]

You may assume that the ambiguity introduced by the $\pm$ sign can be resolved by choosing the sign opposite to that of $f'(\alpha)$.

The equation

$$2 \sin x - x = 0$$

has a root equal to 1·89549, correct to 5 decimal places.

Starting with the initial approximation 2, determine the number of iterations required to obtain the above value of this root using

(i) the Newton-Raphson formula, [3]

(ii) the above formula based on the first three terms of the Taylor series. [5]

State briefly whether or not you believe that the student's method is an improvement on the standard Newton-Raphson formula. [1]

WELSH JOINT EDUCATION COMMITTEE
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Advanced Level
Special Paper

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APPLIED MATHEMATICS S

A.M. MONDAY, 22 June 1992

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

2019 *0023/6

Section A

Mechanics

No more than **four** *questions should be answered from this section.*

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

1. A particle is projected from a point A on a horizonal plane with speed u at an angle α to the horizontal. The particle rises to a maximum height h above the plane at the point B. Given that, relative to a point O on the plane vertically below B, the particle has horizontal and vertical displacements X and Y, respectively, show that

$$Y = -pX^2 + h,$$

where p should be expressed in terms of u, g and α. [6]

The particle passes in succession through the points whose coordinates relative to O are $\left(-a, \frac{\sqrt{3}}{2}a\right), \left(-\frac{1}{2}a, \sqrt{3}\,a\right)$. Find h in terms of a and show that

$$OA = \frac{\sqrt{7}}{2}a. \quad [9]$$

2. The vectors **a** and **b** vary with time t and satisfy the equations

$$\frac{d\mathbf{a}}{dt} = -\mathbf{b}, \qquad \frac{d\mathbf{b}}{dt} = \mathbf{a}.$$

Find, assuming that **a** and **b** are not parallel, the scalar constants p and q so that $p\mathbf{a}\cos t + \mathbf{b}\sin t$ and $q\mathbf{a}\sin t + \mathbf{b}\cos t$ are both constant vectors. [9]

Given that, when $t = 0$, $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$ where **c** and **d** are constant vectors, express **a** in terms of **c**, **d** and t. [3]

Find the vector differential equation satisfied by **a** and verify that your form for **a** satisfies it. [3]

3. A particle P is threaded on a rough wire in the form of a circle of radius a and centre O. The wire is fixed in a vertical plane and P is released from rest from a point level with O. The coefficient of friction between the wire and P is denoted by μ. Show that when OP makes an angle θ with the horizontal, the angular speed ω of P about O satisfies

$$a\left\{\frac{d(\omega^2)}{d\theta} + 2\mu\omega^2\right\} = 2g(\cos\theta - \mu\sin\theta). \quad [7]$$

Find ω^2 in terms of a, g, μ and θ and show that, if the particle comes to rest on first reaching the lowest point of the wire, $(1 - 2\mu^2)e^{\mu\pi} = 3\mu$. [8]

[You may assume that

$$\int \cos\theta e^{2\mu\theta}\, d\theta = \left(\frac{2\mu\cos\theta + \sin\theta}{4\mu^2 + 1}\right)e^{2\mu\theta},$$

$$\int \sin\theta e^{2\mu\theta}\, d\theta = \left(\frac{2\mu\sin\theta - \cos\theta}{4\mu^2 + 1}\right)e^{2\mu\theta}.]$$

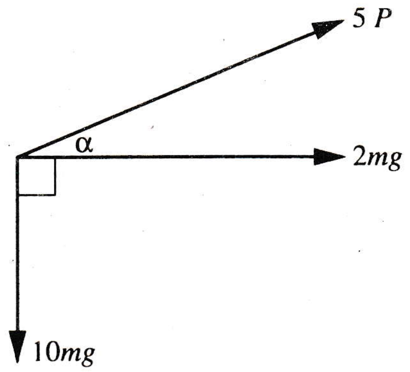
4. Two boats A and B are, at time $t = 0$, at the points P and Q, respectively. Q is a distance a directly East of P. Boat B sets off with speed u directly North and boat A sets off, with the same speed, on a bearing of $2\theta°$ ($\theta < 45$).

Find AB^2 at any subsequent time t in terms of a, u, t and θ. [4]

Show that the shortest distance apart of the boats is $a \sin \theta$. [6]

Given that at the instant when the boats are first at a distance qa apart, ($q < 1$), boat A is moving directly towards boat B, show that $\tan \theta = q$. [5]

5.



A particle of mass $20m$ rests in equilibrium on a rough horizontal plane. In addition to the force of gravity, the normal reaction and the force of friction, there are three horizontal forces, of magnitudes $10mg$, $2mg$, $5P$, acting in the directions shown in the above figure.

The coefficient of friction between the plane and the particle is $\frac{1}{2}$.

(i) Given that $\alpha = \sin^{-1}(\frac{3}{5})$, show that, for equilibrium,

$$\left(\frac{22 - 8\sqrt{6}}{25}\right)mg \leqslant P \leqslant \left(\frac{22 + 8\sqrt{6}}{25}\right)mg \cdot \quad [8]$$

(ii) When the particle is in equilibrium with $25P = (22 + 8\sqrt{6})mg$, the value of P is instantaneously increased to $2mg$. Find the distance that the particle travels in time T. [3]

(iii) Assuming that α varies in the range $0 \leqslant \alpha \leqslant 180°$, find the value of α which gives the smallest value of P consistent with equilibrium and find this smallest value. [4]

SECTION B

Probability and Statistics

No more than **four** *questions should be answered from this section.*

6. Patients in a hospital ward have one or both of two diseases A and B. Medical records show that 64% of all such patients have disease A and that $12\frac{1}{2}$% of those patients who have disease A also have disease B.

(*a*) Calculate the probability that such a patient chosen at random has disease B only. [2]

The treatment of such patients has probability 0·8 of curing a patient who has disease A only, probability 0·5 of curing a patient who has disease B only, and probability 0·2 of curing a patient who has both diseases.

(*b*) Calculate the probability that a randomly chosen patient will be cured by the treatment. [4]

(*c*) Two such patients are chosen at random. Given that neither has both diseases, calculate the probability that the treatment will cure exactly one of the two patients. [9]

7. The discrete random variable X has the probability distribution given by

$$P(X = r) = \frac{\ln(r+1) - \ln r}{\ln(n+1)}, \quad r = 1, 2, 3, \ldots, n.$$

(*a*) Verify that the sum of these probabilities is unity. [1]

(*b*) Show that

$$E(X) = n - \frac{\ln(n!)}{\ln(n+1)}.$$ [4]

(*c*) For the particular case when $n = 3$, the continuous random variable Y is such that, for $r = 1, 2, 3$,

$$P(Y \leqslant y \mid X = r) = y^{r+1}, \quad 0 \leqslant y \leqslant 1.$$

Find a polynomial expression for $P(Y \leqslant y)$ and carry out appropriate checks on your result. Deduce the probability density function of Y and hence evaluate $E(Y)$ correct to four decimal places. [10]

8. The random variables X and Y are independent and each has the Poisson distribution with mean μ.

(*a*) Let $W = XY$. Find, in terms of μ,

(i) the mean of W,

(ii) the variance of W,

(iii) $P(W = 0)$. [6]

(*b*) Find the value of

$$P(XY \geqslant 3 \mid X + Y = 4).$$ [4]

(*c*) Show that

$$E\left(\frac{Y}{X+1}\right) = 1 - e^{-\mu}.$$ [5]

9. An instrument used for measuring the diameter of a circle whose true diameter is a cm will give a reading X cm which is a continuous random variable having probability density function f, where

$$f(x) = 750[0{\cdot}01 - (x-a)^2], \text{ for } a - 0{\cdot}1 \leqslant x \leqslant a + 0{\cdot}1,$$
$$f(x) = 0, \text{ otherwise.}$$

Show that $E(X - a) = 0$ and hence, or otherwise, find the mean and the variance of X. [4]

The instrument is used to obtain five independent measurements X_1, X_2, X_3, X_4, X_5 of the diameter of a circle whose true diameter is a cm. Let $\bar{X}$ denote the mean of these five measurements. It is required to estimate the true area $(\pi a^2/4)$ of the circle. One person suggested the estimator

$$Z_1 = (\pi \sum X_i^2) / 20,$$

which is the average of the areas calculated from the individual measurements. Another person suggested the estimator

$$Z_2 = (\pi \bar{X}^2)/4,$$

which is the area calculated from the mean of the measurements.

Show that both Z_1 and Z_2 are biased estimators of the area of the circle. By suitably modifying Z_1 and Z_2, deduce two unbiased estimators of the area of the circle. [11]

10. Two variables, x and y, are known to be related in the form $y = \lambda(x - 1)$, where λ is an unknown constant. When experiments are conducted with x having specific values the measured values of y are known to be subject to independent errors which are Normally distributed with mean zero and standard deviation σ. Experiments were conducted with x having the values $x_1, x_2, \ldots, x_n$ and the measured values of y were $y_1, y_2, \ldots, y_n$, respectively. Use the method of least squares to derive an estimate of λ in terms of the x_i and the y_i $(i = 1, 2, \ldots, n)$.

Verify that the least squares estimator of λ is unbiased and show that its variance is $\sigma^2 / \sum(x_i - 1)^2$.

The result of five such experiments were as follows:

x	3	5	7	9	11
y	4·9	13·6	24·8	34·6	42·1

Given that $\sigma = 0{\cdot}5$, calculate a 98% confidence interval for λ.

Comment on the claim that y has the value 29·5 when $x = 8$. [15]