WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

CYD-BWYLLGOR ADDYSG CYMRU

Tystysgrif Addysg Gyffredinol

Safon Uwch

MATHEMATICS A1

Pure Mathematics

A.M. THURSDAY, 6 June 1991

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Using integration by parts, show that

$$\int_0^1 xe^x \, dx = 1.$$

[4]

2. Sketch, on the same axes, the graphs of

(i) $y = 2e^x$

and (ii) $y = 4 - x^2$

stating the coordinates of the points where the graphs cross the coordinate axes.

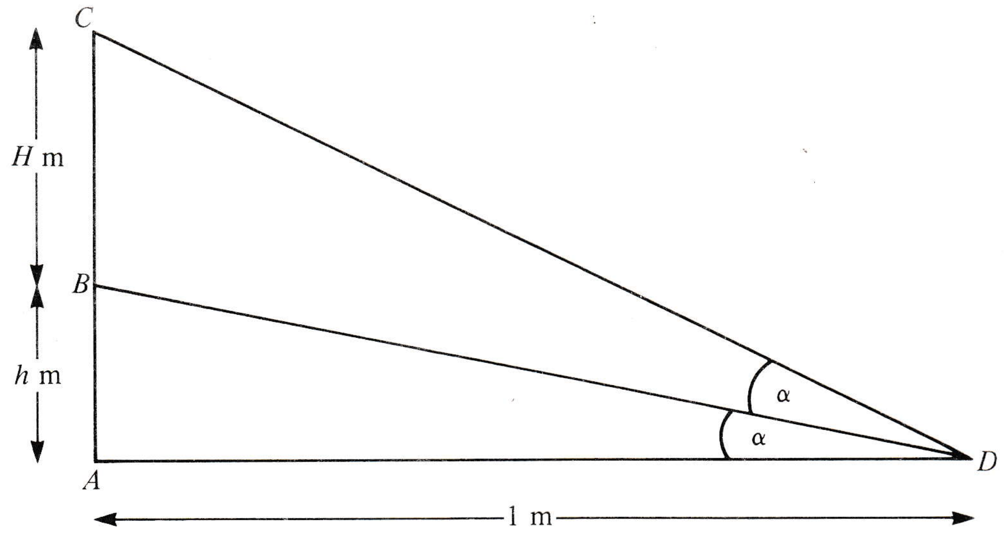
State the number of real roots of the equation

$$2e^x + x^2 = 4.$$

$2e^x = 4 - x^2$

Show that one of the roots lies between $0{\cdot}55$ and $0{\cdot}65$. [5]

3.



The above figure shows a cross-section of a vertical wall AB of height hm surmounted by a vertical flagpole BC of height Hm. D is a point on the same horizontal level as A and $AD = 1$ m. The wall and flagpole subtend equal angles α at D.

Write down expressions for $\tan\alpha$ and $\tan 2\alpha$ in terms of h and H. By using an appropriate double angle formula, show that

$$H = \frac{h(1 + h^2)}{1 - h^2}.$$ [4]

4. Solve the simultaneous equations

$$\begin{aligned} 4x^2 + y^2 &= 25 \\ xy &= 6 \end{aligned}$$

giving all possible pairs of values of x and y. [5]

5. For the geometric series

$$a + ar + ar^2 + \ldots.$$

the sum of the first two terms is 24 and the sum to infinity is 27.

(i) Show that $r = \pm\frac{1}{3}$.

(ii) Find the two possible values of a. [5]

6. The function f is defined on the domain $x > 0$ by

$$f(x) = 1 + \frac{1}{x}.$$

State the range of f.

Explain why the inverse function f^{-1} exists and obtain an expression for $f^{-1}(x)$ in terms of x.

The composite function g is defined as $f \circ f$.

(i) Show that

$$g(x) = 2 - \frac{1}{1 + x}.$$

(ii) State the range of g. [6]

7. Use the quadratic formula to find the two roots z_1, z_2 of the equation

$$z^2 - (2 - 8\mathrm{i})z - (11 + 8\mathrm{i}) = 0$$

giving your answers in the form $x + \mathrm{i}y$.

The complex number z_3 is given by

$$z_3 = \bar{z}_1^{\,2} + \bar{z}_2^{\,2}.$$

Verify that

$$z_3 = -38 + 16\mathrm{i}.$$

Find the modulus and argument of z_3, giving your answers correct to 3 significant figures. [6]

8. The pressure P and volume V of a gas are related by

$$PV^n = k$$

where k and n are positive constants.

(i) Show that

$$\frac{\mathrm{d}P}{\mathrm{d}V} = -\frac{nP}{V}.$$

(ii) Find, in terms of n, the approximate percentage decrease in P if V is increased by 1%. [5]

Section B

Answer **four** *questions from this section.*

9. (*a*) Find the equation of the circle C which passes through the points $O(0, 0)$, $A(3,3)$ and $B(3, 1)$.
Show that the radius of C is $\sqrt{5}$ and find the coordinates of the centre.

Verify that the point $P(2, 4)$ lies on C.

Show that the gradient of the tangent at P to C is $-\frac{1}{2}$ and hence find the equation of this tangent. [8]

(*b*) Show that the equation of the tangent at the point (t, t^2) to the parabola $y = x^2$ is

$$y = 2tx - t^2.$$ [3]

Given that the tangent passes through the point $Q(3, 8)$, find the two possible values of t.

Hence find the acute angle between the two tangents from Q to the parabola.

Write down the coordinates of the points of contact of the two tangents with the parabola. [4]

10. (i) Using the binomial theorem, show that

$$\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots.$$

and state the range of values of x for which this expansion is valid.

Hence show that, neglecting h^4 and higher powers of h,

$$\sqrt{1 + h + h^2} = 1 + \frac{h}{2} + \frac{3h^2}{8} - \frac{3h^3}{16}.$$ [6]

(ii) Show that the roots of the quadratic equation

$$x^2 - (4h + 2)x - 3 = 0$$

are real for all values of the real constant h.

Show that the larger root is equal to

$$2h + 1 + 2\sqrt{1 + h + h^2}.$$

Given that h is small enough to neglect h^4 and higher powers of h, use your previous result to obtain a cubic approximation in h to this root. [5]

Let $L(h)$ denote this larger root. Show that

$$\frac{L(h) - L(0)}{h} = 3 + \lambda h + \mu h^2 + \dots.$$

where the values of the constants λ, μ are to be stated.

Deduce the value of the derivative $L'(0)$. [4]

11. (*a*) Let

$$t = \tan \tfrac{1}{2}x.$$

Show that

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

and derive an expression for $\sin x$ in terms of t. [3]

Given that

$$3 \sin x + 2 \cos x = 1 \qquad \text{(A)}$$

show that

$$3t^2 - 6t - 1 = 0.$$

Hence, or otherwise, find all the values of x between $0°$ and $360°$ satisfying equation (A), giving your answers correct to the nearest degree. [5]

(*b*) Assuming the formula for $\tan(A + B)$, show that

$$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$ [3]

Hence show that

(i) if A, B, C are acute angles, whose sum is $90°$, then

$$\tan B \tan C + \tan C \tan A + \tan A \tan B = 1,$$

(ii) if A, B, C are the angles of an acute-angled triangle, then

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$ [4]

12. (i) The cubic

$$f(x) = x^3 + ax^2 + bx + 3$$

leaves a remainder 9 when divided by $x + 2$ and has a stationary value where $x = 1$. Find the values of a and b, and express the cubic as a product of linear factors. [7]

(ii) Express

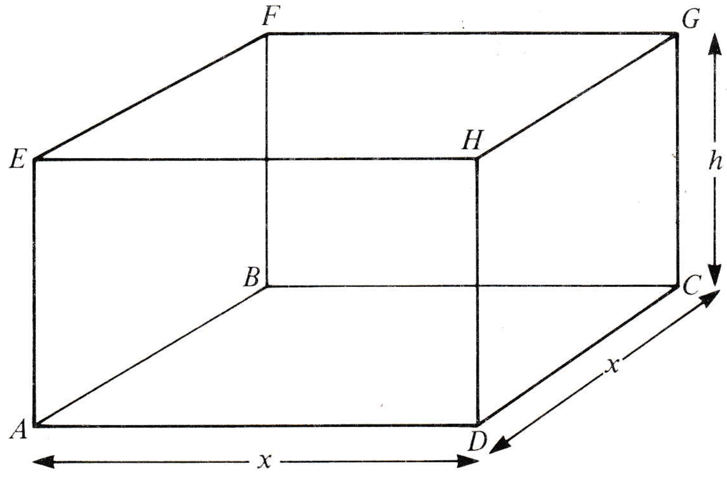
$$g(x) = \frac{16x^2}{(x - 1)^2 (x + 3)}$$

in partial fractions. [4]

(iii) Show that

$$\int_2^3 g(x)\, dx = 2 + 7 \ln 2 + 9 \ln \left(\tfrac{6}{5}\right).$$ [4]

13.



The above figure shows a water tank in the shape of a cuboid with a square base $ABCD$ and open top $EFGH$. Given that $AD = x$ and $CG = h$, write down expressions for the volume V of the tank and its total internal surface area S.

Show that

$$S = x^2 + \frac{4V}{x}.$$

[3]

Given that $V = 32$,

(i) find the value of x which minimises S,

(ii) find the minimum value of S, and show that the corresponding value of h is 2.

For a tank of these dimensions, calculate the length of DF. [7]

Find

(iii) the angle between DF and the base $ABCD$,

(iv) the angle between DF and DA.

Deduce the angle between DF and AB. [5]

14. (*a*) The vector equations of two lines are

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

and $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + t\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are perpendicular unit vectors and t is a constant.

(i) Given that the two lines intersect, calculate t and find the position vector of the point of intersection.

(ii) Find the angle between the lines, giving your answer correct to the nearest degree. [7]

(*b*) The vertices A, B, C of a triangle have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ relative to an origin O. The points D, E lie on BC, CA respectively such that $BD : DC = 2 : 1$ and $AE : EC = 3 : 1$.

Show that

$$\mathbf{DE} = \tfrac{1}{4}\mathbf{a} - \tfrac{1}{3}\mathbf{b} + \tfrac{1}{12}\mathbf{c}.$$

[3]

When extended, the lines ED and AB meet at F. Find the position vector of F in terms of $\mathbf{a}$ and $\mathbf{b}$ and deduce the value of the ratio $AB : BF$. [5]

15. The function f is defined on the domain $x \geqslant 1$ by

$$f(x) = \frac{\ln x}{x}.$$

(i) Show that the graph of f has a single stationary point and state its coordinates. Identify it as a maximum or minimum. [4]

(ii) Sketch the graph of f. [2]

Show that the area of the region R enclosed by the graph of f and the x-axis between $x = 1$ and $x = 5$ is equal to $\frac{1}{2}(\ln 5)^2$. [4]

The region R is rotated through four right-angles about the x-axis. Use the trapezium rule to estimate the volume of the solid of revolution generated, taking 5 ordinates and an interval equal to 1. Give your answer correct to 2 decimal places. [5]

WELSH JOINT EDUCATION COMMITTEE — CYD-BWYLLGOR ADDYSG CYMRU

General Certificate of Education — Tystysgrif Addysg Gyffredinol

Advanced Level — Safon Uwch

MATHEMATICS A2

Mechanics

A.M. WEDNESDAY, 12 June 1991

(3 Hours)

Answer **all** *questions in Section A and* **four** *from Section B.*

A calculator may be used except where it is expressly forbidden.

[$g = 9{\cdot}8\ \mathrm{m\,s^{-2}}$]

SECTION A

Answer **all** *questions in this section.*

1. Find the constant a such that e^{-at} is an integrating factor for the equation

$$\frac{dx}{dt} - 3x = e^{3t}\sin t.$$

Hence, or otherwise, find the solution of the equation such that $x = 1$ when $t = 0$. [4]

2. A uniform solid right circular cone is of height h and its base is of radius a. Find, by integration, the distance of the centre of mass of the cone from its vertex. [4]

3. When an impulse is applied to a particle of mass 0·3 kg moving with constant velocity $(11\mathbf{i} + 7\mathbf{j})\ \mathrm{m\,s^{-1}}$ on a smooth horizontal table, its velocity is changed to $(10\mathbf{i} + 14\mathbf{j})\ \mathrm{m\,s^{-1}}$. Find
   (i) the impulse,
   (ii) the change in the kinetic energy of the particle.

[4]

4. An elastic spring which obeys Hooke's law has natural length 0·5 m. When the extension is 0·05 m the tension in the spring is 50 N. Find the work done when the spring is extended from a length of 0·6 m to a length of 0·7 m. [4]

5. Particles of mass m and $4m$, respectively, are attached one to each end of a light inextensible string. The string passes over a light smooth peg and initially the particles are held at rest with the hanging parts of the string taut and vertical. Find the tension in the string when the system is released from rest. [4]

6. A particle of mass m is in equilibrium on a rough plane inclined at an angle α to the horizontal. When a force of magnitude mg, in the sense up a line of greatest slope, is applied to the particle it is just about to move up the plane. When a force of magnitude $mg/2$, in the sense down a line of greatest slope, is applied to the particle it is just about to move down the plane. Find $\sin\alpha$ and the coefficient of friction between the particle and the plane. [7]

7. A particle moves along the x-axis and describes simple harmonic motion about the origin O with period 6 seconds. When $t = 0$ seconds, $x = 1$ metre and the particle is approaching O with speed $(\pi/\sqrt{3})$ m s^{-1}. Given that the displacement is of the form $a\sin(\omega t + \varepsilon)$, find ω, ε, a and the maximum speed of the particle. [7]

8. A car of mass 1000 kg moves, with its engine working at its maximum rate, at a constant speed of 10 m s^{-1} up a hill inclined at an angle $\sin^{-1}(1/14)$ to the horizontal. The frictional resistance to the motion is R Newtons. Express the maximum rate of working of the engine in terms of R. [3]

With the engine still working at its maximum rate, the car descends the same hill at a constant speed of 20 m s^{-1}. Given that the frictional resistance is now $4R$ Newtons, find R. [3]

SECTION B

Answer **four** *questions from this section.*

9. Two small smooth spheres A and B of equal radius, but of masses m and km respectively, are at rest on a smooth horizontal floor. The centres of the spheres lie on a line perpendicular to a smooth vertical wall with B lying between A and the wall. Sphere B is at a distance of 3 m from the wall. The coefficient of restitution for all collisions is 0·2. Sphere A is then projected directly towards sphere B with speed $u\text{ m s}^{-1}$. Find, in terms of u and k, the velocities of the spheres immediately after their first collision. [6]

(*a*) Given that $k = 3$ find

(i) the distance between A and B when B hits the wall, [3]

(ii) the distance from the wall to the point at which the spheres next collide. [3]

(*b*) Given that $k \neq 3$ find the least value of k so that the spheres do not collide after B hits the wall. [3]

10. A helicopter flies at a constant height with constant speed $101V$ relative to the air, where V is a constant. A wind blows with steady speed $25V$ from the direction N $\alpha°$ W, where $\sin \alpha = \frac{4}{5}$. Find the distance that the helicopter travels in a given time T from a fixed point O

(i) downwind, [1]

(ii) upwind. [1]

Show that the distance travelled directly north from O in time T is $84VT$ and find the corresponding distance that the helicopter can travel due south of O in time T. [9]

Explain why, in the absence of wind, all the points that can be reached from O in time T lie on a circle, centre O, and state the radius of this circle. Given that, when the wind speed is $25V$, all the points that can be reached from O in time T lie on a circle whose centre is directly downwind of O, state the radius and the exact position of the centre of this circle. [4]

11. A particle is projected from a point O on a horizontal plane with speed u in a direction making an angle α above the horizontal. At a subsequent time t the horizontal and vertical displacements of the particle from O are denoted by x and y respectively; the particle is moving in a direction inclined at an angle β above the horizontal, and the upward vertical component of its velocity is v. Show that

(i) $\dfrac{2y}{t} = v + u \sin \alpha$, [4]

(ii) $2y = x(\tan \alpha + \tan \beta)$. [4]

When moving at an angle of 45° above the horizontal it just clears a wall of height 3 m and at a perpendicular distance of 2 m from O. The wall is perpendicular to the plane of motion of the particle. Subsequently, when moving at an angle of 45° below the horizontal, the particle just clears a wall parallel to the first and of height 3 m. Find

(iii) $\tan \alpha$, [2]

(iv) the distance between the walls, [2]

(v) the range on the horizontal plane through O, [1]

(vi) the maximum height reached above the plane. [2]

12. Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0. \qquad (A)$$

[4]

A particle of mass 0·25 kg moves along the x-axis and at time t seconds its displacement from O in the positive x direction is x metres. The force acting on the particle is $\left(ax + b\frac{dx}{dt}\right)$ Newtons in the positive x direction, where a and b are constants. When $x = 2$ and the particle is moving away from O with speed 3 m s^{-1} the force acting on it is 4 Newtons in the negative x direction. When, at the same point, the particle is moving towards O with speed 3 m s^{-1} the force acting on it is 1 Newton in the negative x direction. Show that $a = -5/4$ and find the numerical value of b. [4]

Hence show that x satisfies the above differential equation (A). At time $t = 0$, the particle is projected towards O from the point $x = 2$ with speed 2 m s^{-1}. Find x in terms of t. Give a sketch showing how x varies with t. [7]

13. Relative to a fixed origin O the position vector, $\mathbf{r}$, of a particle P is given by

$$\mathbf{r} = 3 \sin t\,\mathbf{i} + 5 \cos t\,\mathbf{j},$$

where $\mathbf{r}$ is measured in metres, and t denotes the time in seconds.

(i) Find the velocity of P at time t seconds and show that its speed is v m s^{-1}, where $v^2 = 17 - 8 \cos 2t$. [5]

Hence find the maximum and minimum values of v. [2]

(ii) Find the cosine of the angle between the velocity and acceleration of P when $t = \pi/4$. [4]

(iii) Given that the particle is of mass 0·3 kg find the rate of working, at time t seconds, of the force acting on P and the maximum rate of working of this force. [4]

14. A particle of mass m is attached to one end of a light inextensible string of length a whose other end is attached to a fixed point O. Initially the particle is held at rest at a point B, with the string taut and OB inclined at an angle $\pi/3$ to the downward vertical through O.

(*a*) The particle is projected horizontally from B with speed u so that it describes, with constant speed, a horizontal circle whose centre lies on the vertical through O. Find u and the time taken to describe one complete circle. [7]

(*b*) The particle is projected from B, perpendicular to OB in the vertical plane containing OB, with speed w so that it starts describing a vertical circle, centre O. Find the tension in the string when it is inclined at an angle θ to the downward vertical. Find also the least value of w so that the particle describes complete circles. [8]

15. (*a*) A uniform rod of weight W and length $2a$ is smoothly hinged to a vertical wall. It is kept in equilibrium in a horizontal position by a light inextensible string attached to the rod at a point a distance x from the hinge and fastened to the wall at a distance a above the hinge. Find

(i) the horizontal and vertical components of the reaction at the hinge, [4]

(ii) the value of x for which the magnitude of this reaction is a minimum, [2]

(iii) the tension in the string when x has the value found in (ii), [2]

(iv) the magnitude and direction of the reaction at the hinge when $x = 2a$. [3]

(*b*) The base BC of a triangle ABC lies on the x-axis and the vertex A is at $(a, 2a)$. Forces represented in magnitude and direction by **AB**, 2**BC**, **CA** act along the sides of the triangle. Determine the magnitude and direction of the resultant of the system and a point on its line of action. [4]

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General Certificate of Education

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Advanced Level

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MATHEMATICS A3

Probability and Statistics

P.M. TUESDAY, 18 June 1991

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Two events A and B are such that $P(A) = 0{\cdot}6$, $P(A \cap B) = 0{\cdot}2$ and $P(A \cup B) = 0{\cdot}7$. Calculate

(i) $P(B)$,

(ii) $P(B' \mid A)$.

[3]

2. Independently for each seed of a particular variety of flower that is sown, the probability that the seed will germinate is 0·8.

(i) Twenty such seeds are sown. Use tables to find the probability that the number that will germinate lies between 14 and 18, inclusive.

(ii) Use a Normal approximation to find the probability that exactly 324 of 400 seeds sown will germinate, giving your answer correct to two decimal places.

[6]

3. A random sample of 16 observations is to be drawn from a Normal distribution having mean 11 and standard deviation 3. Let $\bar{X}$ denote the sample mean. Find, correct to three decimal places,

(i) the probability that $\bar{X}$ will have a value between 9·2 and 12·2,

(ii) the value of c for which $P(\bar{X} < c) = 0{\cdot}03$.

[6]

4. A random sample of 10 observations from a Normal distribution gave the following results:

Sum of the 10 values = 126.

Sum of the squares of the 10 values = 1620.

(i) Calculate unbiased estimates of the mean and the variance of the Normal distribution which was sampled.

(ii) Calculate 99% confidence limits for the mean of the distribution.

[6]

5. The discrete random variables X and Y have the joint probability distribution shown in the following table.

		x = 0	x = 1	x = 2
y	0	α	β	α
	1	β	0	β
	2	α	β	α

(i) Express β in terms of α.

(ii) Show that $E(XY) = 1$.

(iii) Given that Var $(XY) = 2$, find the values of α and β.

[6]

6. The continuous random variable X is distributed with probability density function f given by

$$f(x) = 12x^2(1-x), \quad \text{for } 0 \leqslant x \leqslant 1,$$
$$f(x) = 0, \quad \text{otherwise.}$$

Evaluate the mean and the variance of X. Deduce the mean and the variance of $Y = 6 - 5X$.

[6]

7. The variables x and y are known to be linearly related. Experiments were conducted with x having ten prespecified values, the corresponding values of y being measured. The results (x_i, y_i), $i = 1, 2, \ldots, 10$, of the experiments were summarised as follows:

$$\sum x_i = 80, \quad \sum x_i^2 = 3140, \quad \sum y_i = 422, \quad \sum x_i y_i = 12276.$$

The y-measurements were subject to independent Normally distributed errors having mean zero and standard deviation 2·4. Find

(i) the least squares estimate of the equation expressing y in terms of x,

(ii) 95% confidence limits for the increase that will occur in the value of y when x is increased by one unit.

[7]

Section B

Answer **four** *questions from this section.*

8. An insurance company offering comprehensive policies to car drivers classifies each applicant as being high-risk, average-risk, or low-risk. Independently for each year the probability that a high-risk driver will submit a claim is 0·4; the corresponding probabilities for an average-risk driver and a low-risk driver are 0·2 and 0·1, respectively. The proportions of the policy holders who have been classified as high-risk, average-risk, and low-risk are 0·3, 0·6, and 0·1, respectively.

(i) Two brothers have comprehensive policies with the company; one is classified as average-risk and the other as low-risk. Calculate the probability that exactly one of the two brothers will submit a claim in a year. [3]

(ii) Six teachers at a school have comprehensive policies with the company. Two of the teachers are classified as average-risk and four are classified as low-risk. Calculate the probability that exactly one of the six teachers will submit a claim in a year. [4]

(iii) Calculate the probability that a randomly chosen policy holder will submit a claim in a year. [2]

(iv) Given that a randomly chosen policy holder did make a claim in one year, calculate the conditional probability that the policy holder was a high-risk driver. [2]

(v) Calculate the conditional probability that a policy holder chosen at random from those who made a claim in one year will also make a claim in the following year. [4]

9. Independently for each page the number of typing errors per page in the first draft of a novel has a Poisson distribution with mean 0·4.

(*a*) Calculate, correct to five decimal places, the probabilities that

(i) a randomly chosen page will contain no error,

(ii) a randomly chosen page will contain 2 or more errors,

(iii) the third of three randomly chosen pages will be the first to contain an error. [5]

(*b*) Write down an expression for the probability that each of n randomly chosen pages will contain no error. Hence find the largest n for which there is a probability of at least 0·1 that each of the n pages contains no error. [3]

Independently for each page the number, Y, of typing errors per page in the first draft of a Mathematics textbook also has a Poisson distribution.

(*c*) Given that $P(Y = 2) = 2P(Y = 3)$

(i) find $E(Y)$,

(ii) show that $P(Y = 5) = 4P(Y = 6)$. [4]

(*d*) One page is chosen at random from the first draft of the novel and one page is chosen at random from the first draft of the Mathematics textbook. Calculate, correct to three decimal places, the probability that exactly one of the two chosen pages will contain no error. [3]

10. The continuous random variable X is distributed with probability density function f given by

$$f(x) = \frac{4}{(x+2)^2}, \qquad \text{for } 0 \leqslant x \leqslant 2,$$

$$f(x) = 0, \qquad \text{otherwise.}$$

(i) Find the cumulative distribution function, F, of X. Hence, or otherwise, evaluate the conditional probability

$$P(X \leqslant \tfrac{2}{3} \mid X > \tfrac{1}{3}).$$ [6]

(ii) Evaluate $E(X + 2)$ and $E[(X + 2)^2]$. Hence, or otherwise, find the mean and the variance of X. [4]

(iii) Determine the cumulative distribution function, G, and the probability density function, g, of $Y = (X + 2)^2$. [5]

11. A small housing estate consists of ten bungalows. Three of the bungalows have 2 bedrooms, six have 3 bedrooms, and the remaining one has 4 bedrooms.

(*a*) Calculate the mean, μ, and the standard deviation, σ, of the number of bedrooms per bungalow on the estate. [3]

(*b*) A sample of three of the ten bungalows is to be chosen at random (**without replacement**). Let $\bar{X}$ denote the mean number of bedrooms in the three chosen bungalows.

(i) There are seven possible combinations of the numbers of bedrooms in the three chosen bungalows. The following table shows three of these combinations together with their probabilities and the corresponding values of $\bar{X}$.

Combination	*Probability*	$\bar{X}$
{2, 2, 2}	$\frac{1}{120}$	2
{2, 2, 3}	$\frac{18}{120}$	$2\frac{1}{3}$
{2, 2, 4}	$\frac{3}{120}$	$2\frac{2}{3}$

Copy this table and insert the remaining possible combinations, their probabilities and the corresponding values of $\bar{X}$. Hence derive the sampling distribution of $\bar{X}$. [6]

(ii) Verify that $\bar{X}$ is an unbiased estimator of μ, and determine its standard error correct to four decimal places. [3]

(*c*) Suppose, instead, that the bungalows are to be sampled at random **with replacement**. Find the smallest sample size that will have to be taken for the standard error of the sample mean number of bedrooms per bungalow in this case to be less than the standard error of $\bar{X}$. [3]

12. In the triangle ABC, angle $ACB = 90°$, angle $BAC = \alpha°$, and angle $ABC = \beta° = (90 - \alpha)°$. Independent measurements are made of α and β. Let X and Y denote the measured values of α and β, respectively. It is known that X and Y are Normally distributed with standard deviation σ and means α and β, respectively.

(i) Calculate, correct to two decimal places, the probability that $X + Y$ will have a value which differs from 90 by less than σ. [3]

(ii) Show that $T_1 = X + \frac{1}{2}(90 - X - Y)$ is an unbiased estimator of α. Verify that T_1 is better than X as an estimator of α. [5]

(iii) Calculate, correct to three decimal places, the probability that $| T_1 - \alpha |$ will be less than σ. [3]

(iv) Let $T_2 = Y + \frac{1}{2}(90 - X - Y)$. Show that

$$\mathrm{Var}(T_1 - T_2) = 2\{\mathrm{Var}(T_1) + \mathrm{Var}(T_2)\}.$$ [4]

13 (*a*) Very large numbers of workers use one of two methods, A and B, to complete a particular task.

(i) The times taken by 125 workers using method A had mean 30·2 minutes and standard deviation 2·2 minutes. The times taken by 100 workers using method B had mean 29·8 minutes and standard deviation 2·9 minutes. Calculate approximate 95% confidence limits for the difference between the mean times of the two methods to complete the task.

State, with your reason, whether or not you would conclude that one method is faster, on average, than the other method. [4]

(ii) It is required to find a 95% confidence interval for the difference between the mean times of the two methods having a width of less than one minute. If n workers are assigned to method A and another n workers to method B, find the smallest value of n to meet the requirement. Assume that the times taken are Normally distributed and that the sample standard deviations given above are the true population values. [5]

(*b*) A computer file holds a very large number of TRUE/FALSE questions. In a random sample of 75 of the questions it was found that 60 were ones for which the correct answer is TRUE.

(i) Calculate an unbiased estimate of the proportion of all the questions for which the correct answer is TRUE.

(ii) Calculate an estimate of the standard error of your estimate in (i).

(iii) Calculate an unbiased estimate of the difference between the proportion of all the questions for which the correct answer is TRUE and the proportion for which the correct answer is FALSE.

(iv) Calculate an estimate of the standard error of your estimate in (iii). [6]

14. Five cards are dealt at random **without replacement** from a pack of ten cards, of which eight are red and two are white.

(i) Let X denote the number of white cards that are dealt. Find the probability distribution of X. [3]

The five cards that are dealt are placed in a bag and three are then drawn at random **without replacement** from the bag. Let Y denote the number of white cards that are drawn.

(ii) Show that $P(Y = 1 \mid X = 1) = \frac{3}{5}$, and that $P(X = 1,\ Y = 1) = \frac{1}{3}$. [2]

(iii) The joint probability distribution of X and Y is displayed in the following table.

		x = 0	x = 1	x = 2
y	0	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{45}$
	1	0	$\frac{1}{3}$	$\frac{2}{15}$
	2	0	0	$\frac{1}{15}$

Verify the value given in the table for $P(X = 2,\ Y = 1)$. [2]

(iv) Evaluate $E(Y)$, $E(Y^2)$ and $E(XY)$. [3]

(v) Let $U = X - Y$ and $W = 3X + cY$, where c is a constant. Find the value of c for which

$$E(UW) = E(U)E(W).$$ [5]

WELSH JOINT EDUCATION COMMITTEE

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Advanced Level

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MATHEMATICS A4

Further Pure Mathematics

A.M. FRIDAY, 21 June 1991

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Find the values of x for which

$$\begin{vmatrix} 1 & 2x & -1 \\ 2 & 1 & 4 \\ 3 & 4 & x \end{vmatrix} = 0.$$

[4]

2. Differentiate the following functions with respect to x:

(i) $\displaystyle\int_0^x e^{\tan t}\,dt$;

(ii) $(\sin x)^x$.

[4]

3. Find the 4 fourth roots of -1, giving your answers in the form $x + iy$ where x, y are surds. [4]

4. The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are in arithmetic progression. By taking the roots to be $\alpha - \beta$, α and $\alpha + \beta$, show that

$$2p^3 + 27r = 9pq.$$

[5]

5. Show, using mathematical induction, that

$$\sum_{r=1}^{n} r(\tfrac{1}{2})^r = 2 - (n+2)(\tfrac{1}{2})^n.$$ [5]

6. The function f is defined by

$$f(x) = \frac{2 + x^2}{1 - 2x}. \qquad (x \neq \tfrac{1}{2})$$

Show that the two values of x (x_1 and x_2) corresponding to $f(x) = y$ are given by

$$-y \pm \sqrt{y^2 + y - 2}.$$

(i) Hence find the range of f.

(ii) By considering the values of y for which x_1 and x_2 are coincident, find the coordinates of the two stationary points on the graph of f.

[6]

7. The function F is defined on the domain $-a < x < a$. Show, by considering the identity

$$F(x) = \tfrac{1}{2}[F(x) + F(-x)] + \tfrac{1}{2}[F(x) - F(-x)]$$

that F can be expressed as the sum of an even function E and an odd function O.

In the particular case

$$F(x) = \ln(1 + \sin x) \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

(i) find an expression, in its simplest form, for $E(x)$,

(ii) show that

$$O(x) = \ln(\sec x + \tan x).$$

[6]

8. The equations of the curves C_1 and C_2 are

$$C_1: x^2 - 2y^2 = 1,$$
$$C_2: x^2 + 2y^2 = a^2. \quad (a > 1)$$

C_1 and C_2 intersect at right-angles at the point P which lies in the first quadrant.

(i) Find the coordinates of P in terms of a.

(ii) By considering the gradients of C_1 and C_2 at P, find the value of a.

[6]

SECTION B

Answer **four** *questions from this section.*

9. A point moves in such a way that the ratio of its distance from the point $S(a\sqrt{2}, 0)$ to its distance from the line L with equation $x = a/\sqrt{2}$ is equal to $\sqrt{2}$. Show that the equation of its locus is

$$x^2 - y^2 = a^2.$$

Sketch the locus, marking S, L and the asymptotes on your diagram. [5]

The perpendiculars from S to the lines $y = x$ and $y = -x$ meet these lines at P and Q respectively. Show that P and Q both lie on L. [3]

The line SP meets the locus at R. Find the coordinates of R and show that the tangent to the locus at R passes through Q. [7]

10. A transformation T under which the point (x, y) is transformed to the point (u, v) $((x, y) \to (u, v))$ is defined by

$$w = \frac{az + b}{z + c}$$

where $z = x + iy$, $w = u + iv$ and a, b, c are real constants.

Find the values of a, b, c given that, under T,

(i) $(0, 0) \to (-5, 0)$

and (ii) $(-1, 1) \to (-3, 2)$.

Verify that

$$z = -\left(\frac{w + 5}{w + 3}\right). \qquad [6]$$

Show that, under T, there are two points which are transformed to themselves and state their coordinates. [2]

Obtain expressions for x, y in terms of u, v and hence show that the line $y = x$ is transformed into a circle with radius $\sqrt{2}$. State the coordinates of its centre. [7]

11. (i) Let the McLaurin series of $\ln(1+x)$ be

$$a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots$$

Write down, in terms of $a_1, a_2, a_3 \ldots$, the series for

$$y = (1+x)\ln(1+x) - x.$$

State the coefficient of $x^r (r \geqslant 2)$ in your series.

Verify that

$$\frac{dy}{dx} = \ln(1+x).$$

Hence, by differentiating your series for y term by term and comparing the result with the assumed series for $\ln(1+x)$, show that

$$a_1 = 1$$

$$\text{and } a_r = -\frac{(r-1)\,a_{r-1}}{r} \quad (r \geqslant 2).$$

Hence find an expression for a_r $(r \geqslant 1)$. [9]

(ii) Show that

$$\ln(1+\alpha x)^{\beta/x} = \alpha\beta - \frac{\alpha^2\beta x}{2} + \frac{\alpha^3\beta x^2}{3} + \ldots.$$

Deduce the value of

$$\lim_{x\to 0}(1+\alpha x)^{\beta/x}.$$ [6]

12. (*a*) The point P has coordinates (x_0, y_0) and the line L has equation $y = x\tan\alpha$. Write down the equation of the line perpendicular to L passing through P and find the coordinates of Q, the point of intersection of this perpendicular and L. P' denotes the reflection of P in L. By noting that Q is the mid-point of PP', show that P' has coordinates

$$(x_0\cos 2\alpha + y_0\sin 2\alpha,\ x_0\sin 2\alpha - y_0\cos 2\alpha).$$

Deduce the 2×2 transformation matrix corresponding to a reflection in the line $y = x\tan\alpha$.

A reflection in the line $y = x\tan\alpha$ is followed by a reflection in the line $y = x\tan\beta$. Show that the overall effect is equivalent to a rotation about the origin and state the angle of the rotation.

Find the condition on α, β which ensures that the overall transformation is independent of the order in which the reflections occur. [10]

(*b*) A line has cartesian equations

$$x - 1 = y - 2 = \frac{z-3}{2}.$$

Find the coordinates of the point on the line which is closest to the point $(2, 3, 1)$. [5]

13. (*a*) Given that

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}$$

evaluate $\mathbf{AA}'$, where $\mathbf{A}'$ denotes the transpose of $\mathbf{A}$.

Hence find the inverse matrix $\mathbf{A}^{-1}$.

Using your results, solve the equations

$$\begin{aligned} 2x + 3y + 6z &= 1 \\ 6x + 2y - 3z &= 2 \\ 3x - 6y + 2z &= 3. \end{aligned}$$

[7]

(*b*) Given that $\mathbf{B}$ is a square matrix satisfying $\mathbf{B}^3 = \mathbf{O}$, show that

$$(\mathbf{I} - \mathbf{B})^{-1} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2$$

where $\mathbf{I}$ denotes the identity matrix and $\mathbf{O}$ is a matrix all of whose elements are zero.

Verify that

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^3 = \mathbf{O}.$$

Hence, using the above result, find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

[8]

14. (*a*) Show that the equation

$$e^{-x} = \sin\left(x - \frac{\pi}{6}\right)$$

has a root, α, between 0·9 and 1·0.

Two possible rearrangements of the above equation are

$$x = F(x) \text{ where } F(x) = -\ln\sin\left(x - \frac{\pi}{6}\right)$$

$$\text{and } x = G(x) \text{ where } G(x) = \frac{\pi}{6} + \sin^{-1}(e^{-x}).$$

Using an iterative method based on these rearrangements and taking $x_0 = 0{\cdot}9$, calculate x_1 and x_2 for each iterative process, giving your answers correct to 4 decimal places. Hence state which one converges and which diverges.

For the convergent process, determine x_3 and x_4 and deduce the value of α correct to 2 decimal places.

Evaluate the derivatives F' and G' at your approximation to α and explain the relevance of your results to the iterative processes above. [10]

(*b*) The equation has another root, β, close to $\frac{7\pi}{6}$. Use the Newton-Raphson method to find the value of β correct to 2 decimal places. [5]

Turn over.

15. (i) The integral I_n is defined by

$$I_n = \int_0^{\alpha} \sinh^n \theta \, d\theta$$

where n is a non-negative integer and $\sinh \alpha = 1$.

Show that, for $n \geqslant 2$,

$$n I_n = \sqrt{2} - (n-1) I_{n-2}.$$ [6]

(ii) Use a suitable hyperbolic substitution to show that

$$\int_0^1 x^{\frac{3}{2}} \sqrt{1+x} \, dx = 2(I_4 + I_6).$$

Hence, by evaluating I_0 and using the above reduction formula, show that the value of this integral is

$$\tfrac{1}{24}(3\alpha + 7\sqrt{2}).$$

Evaluate this expression correct to 3 decimal places. [9]

WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

Special Paper

CYD-BWYLLGOR ADDYSG CYMRU

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MATHEMATICS S

A.M. MONDAY, 24 June 1991

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from Section A and no more than* **four** *questions from either Section B or Section C.*

A calculator may be used except where it is expressly forbidden.

Section A

Pure Mathematics

No more than **four** *questions should be answered from this section.*

1. A couple borrows £P at the beginning of a particular month at a *monthly* interest of I%. The interest is calculated at the end of each month and added to the amount outstanding. The couple repays a fixed amount £R at the end of each month.

Given that £S_n denotes the amount outstanding immediately after the nth monthly payment has been made, explain why

$$S_n = \left(1 + \frac{I}{100}\right)S_{n-1} - R \quad (\text{for } n \geqslant 1 \text{ taking } S_0 = P).$$

Hence by summing a suitable geometric series, or otherwise, show that

$$S_n = P\left(1 + \frac{I}{100}\right)^n - \frac{100R}{I}\left[\left(1 + \frac{I}{100}\right)^n - 1\right].$$ [6]

The couple borrows £50 000 at a monthly interest rate of 1%. Calculate the monthly payment needed to repay the loan in 20 years, giving your answer correct to the nearest penny.

After one year, the couple has made 12 of these monthly payments. What percentage of their total monthly payments to date cover

(i) repayment of capital,

(ii) interest? [3]

After ten years, the interest rate is increased to 1·2% per month. The couple is given the choice,

either (A) to increase the monthly payment so that the loan is repaid in a further 10 years as planned,

or (B) to maintain the same monthly payment resulting in the loan being repaid later than planned.

Calculate

(iii) the new monthly payment corresponding to option (A),

(iv) the *extra* time (in months) needed to repay the loan under option (B). [6]

2. ABC is an acute-angled triangle. Show, with the usual notation, that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R denotes the radius of the circle passing through A, B and C. [4]

Hence, by considering

$$\frac{b-c}{b+c}$$

prove the 'tangent formula'

$$\tan \tfrac{1}{2}(B-C) = \left(\frac{b-c}{b+c}\right) \cot \tfrac{1}{2}A.$$ [4]

Show further that

$$\sin \tfrac{1}{2}(B-C) = \left(\frac{b-c}{a}\right) \cos \tfrac{1}{2}A$$

and deduce a similar expression for $\cos \frac{1}{2}(B-C)$. [4]

In a particular triangle, $b = 8$, $c = 10$ and $A = 44°$. Using one of the above results, and not the cosine rule, find (correct to the nearest degree) the values of B and C. [3]

3. The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ $(p > q > 0)$ lie on the parabola $y^2 = 4ax$ and S is the point $(a, 0)$.

(*a*) Given that α denotes the angle PSQ and β denotes the acute angle between the tangents at P and Q, show that

(i) $\tan \beta = \dfrac{p-q}{1+pq}$

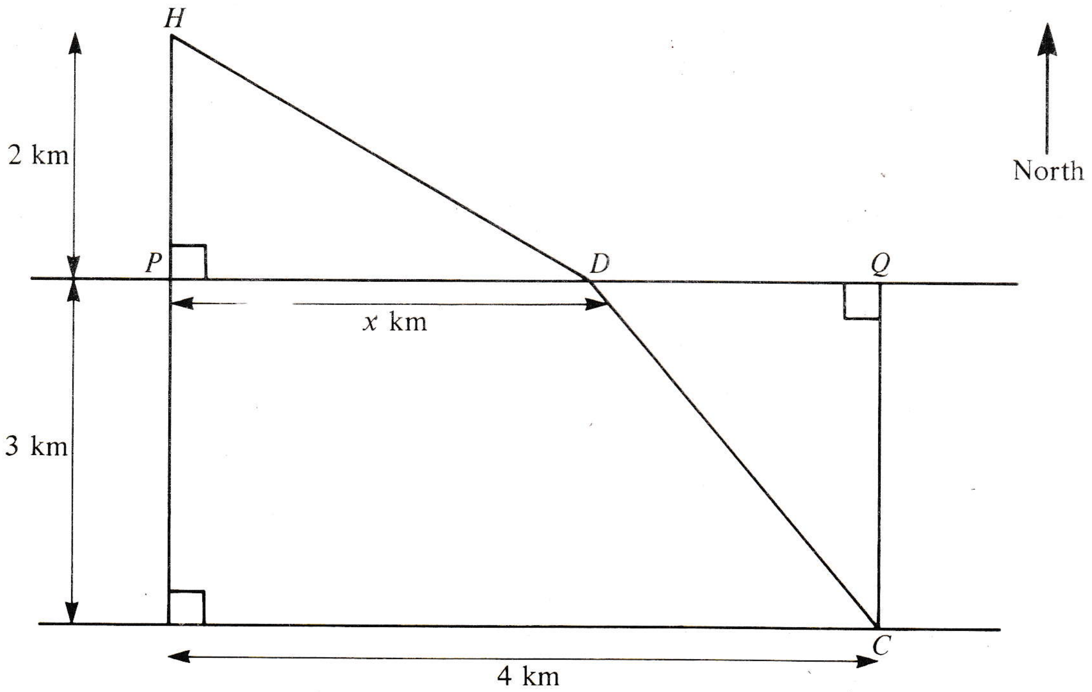
and (ii) $\tan \alpha = \dfrac{2(pq+1)(p-q)}{(p^2-1)(q^2-1)+4pq}$. [7]

Hence show that $\alpha = 2\beta$.

(*b*) The tangents at P and Q meet in R. Find the coordinates of R in terms of p and q. Given that $\alpha = \dfrac{\pi}{2}$, show that the equation of the locus of R as p and q vary is

$$(x + 3a)^2 - y^2 = 8a^2.$$ [8]

4.



A choirboy, Bill, lives in a house H. He sings in a church C which is situated 4 km East and 5 km South of H. A road PQ runs West–East and is 2 km South of H. The land North of PQ is smooth pastureland over which Bill cycles at 20 km/h. The land South of PQ is too rough to cycle over and Bill runs over it at 10 km/h.

To go to church, Bill moves in a straight line from H to C, cycling over the pastureland and running over the rougher ground. Calculate, to the nearest second, how long it takes Bill to go to church. [2]

Bill now learns about calculus in school and he realises that it might be quicker not to travel in a straight line from H to C. He therefore decides to cycle in a straight line from H to a point D, x km East and 2 km South of H, and then to run in a straight line from D to C. Write down an expression for the time T (hours) taken by Bill to go to church along this route and show that the value of x which minimises T satisfies the equation

$$\frac{x}{\sqrt{4+x^2}} = \frac{2(4-x)}{\sqrt{9+(4-x)^2}}. \qquad [3]$$

(i) Prove that the optimal route satisfies the 'law of refraction', namely that the ratio $\sin P\hat{H}D : \sin Q\hat{C}D$ is equal to the ratio of the speeds in the two regions. [3]

(ii) Show that the above equation can be rewritten as

$$3x^4 - 24x^3 + 55x^2 - 128x + 256 = 0.$$

Find the smallest positive root of this equation correct to 2 decimal places. Show that this value of x minimises T and confirm that Bill can save just over a minute by taking the optimal route. [7]

5. (*a*) Differentiate $\sec\theta\tan\theta$, and hence evaluate the indefinite integral

$$\int \sec^3\theta \, d\theta.$$ [3]

(*b*) Given that

$$I(a) = \int_0^1 \sqrt{x^2 + x + a}\, dx$$

(i) show that

$$I(\tfrac{1}{4}) = 1,$$ [3]

(ii) using a suitable trigonometric substitution, show that for $a > \frac{1}{4}$,

$$I(a) = \frac{3\sqrt{a+2} - \sqrt{a}}{4} - \frac{(1-4a)}{8}\ln\left(\frac{3 + 2\sqrt{a+2}}{1 + 2\sqrt{a}}\right).$$ [9]

Turn over.

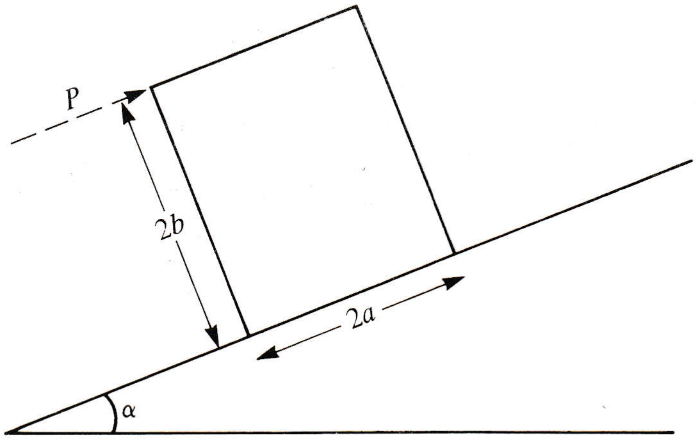
SECTION B

Mechanics

No more than **four** *questions should be answered from this section.*

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

6.



The diagram shows a uniform rectangular block of weight W resting on a rough plane which is inclined at an angle α to the horizontal. Four of the edges of the block are parallel to a line of greatest slope of the plane, these edges being of length $2a$. The edges of the block which are perpendicular to the plane are of length $2b$. There is a force of magnitude P applied, as shown in the diagram, to the middle of the lower horizontal edge of the face of the block parallel to, but not in contact with, the plane. This force acts parallel to a line of greatest slope in the sense which would tend to move the block up the plane. The coefficient of friction between the block and the plane is denoted by μ.

Given that the block is in equilibrium under the action of all the forces acting on it, show that the normal reaction of the plane acts through a point at a perpendicular distance of $\left(a - b\tan\alpha + \dfrac{2P}{W} b\sec\alpha\right)$ from the lower horizontal edge in contact with the plane. [4]

Deduce that, in the absence of the applied force, equilibrium is possible provided that $\mu \geqslant \tan\alpha$ and $a \geqslant b\tan\alpha$. [5]

Assuming that these conditions are satisfied and that the plane is such that equilibrium is broken by the block sliding, find P when sliding occurs. [2]

Given instead that the plane is such that equilibrium is broken by the block toppling, find P when toppling occurs. [2]

Hence find the condition that has to be satisfied if, as P gradually increases from zero, the block will slide rather than topple. [2]

7. A particle P of mass m is at rest on a rough horizontal plane, the coefficient of friction between the particle and the plane being denoted by μ. A light inextensible string is attached to the particle and passes over a small smooth pulley at a height h above the plane. The string is pulled with constant speed u and, at time t, the angle between the string and the table is denoted by θ. Assuming that P stays on the plane, show that

(i) $\dfrac{d\theta}{dt} = \dfrac{u \sin\theta \tan\theta}{h}$, [4]

(ii) the magnitude of the acceleration of P at time t is $\dfrac{u^2 \tan^3\theta}{h}$. [5]

Find the tension in the string. [6]

8. Two particles P and Q, of masses $3m$ and m respectively, are attached one to each end of a light inextensible string which passes over a smooth pulley. The particles move in a vertical plane with both the hanging parts of the string vertical. They are released from rest with P at a height h above a horizontal inelastic floor. Find the height to which P rises after n collisions with the floor. [8]

Show that the total time that the system is in motion is three times the time taken for P to reach the floor for the first time. [7]

9. A particle P of mass m is projected from a point O on a horizontal plane with speed u at an angle α to the horizontal. There is a resistive force acting on the particle so that, when its velocity is $\mathbf{v}$ the total force acting on it is $-mg\mathbf{j} - mk\mathbf{v}$ where k is a constant and $\mathbf{j}$ is the unit vector vertically upwards. The particle lands back on the plane after a time T and, at that time, the magnitude of the angle between $\mathbf{v}$ and the horizontal is denoted by θ. Show that

(i) $u \sin\alpha = \dfrac{g(kT + e^{-kT} - 1)}{k(1 - e^{-kT})}$, [7]

(ii) $\dfrac{\tan\theta}{\tan\alpha} = \dfrac{e^{kT} - 1 - kT}{e^{-kT} - 1 + kT}$. [8]

10. (*a*) The displacement x of a particle moving along the x-axis satisfies the equation

$$\frac{d^2x}{dt^2} + 2kn\frac{dx}{dt} + n^2x = 0,$$

where k and n are positive constants. At time $t = 0$ the particle is projected from the origin in the positive x direction and at a subsequent time it returns to the origin. Show that this motion would not have been possible for $k \geqslant 1$. [7]

(*b*) By making the transformation $v = e^t$, or otherwise, find the general solution of

$$v^2\frac{d^2x}{dv^2} + v\frac{dx}{dv} - x = 1 + \ln v.$$ [8]

SECTION C

Probability and Statistics

No more than **four** *questions should be answered from this section.*

11. (*a*) In a certain country a jury consists of 10 persons. A defendant will be convicted only if at least 6 of the jurors vote the defendant as being guilty. When a defendant is really guilty each juror, independently, will return a vote of guilty with probability 0·7. When a defendant is really innocent each juror, independently, will return a vote of guilty with probability 0·2. Given that 65% of the defendants are really guilty find, correct to three decimal places,

(i) the proportion of the defendants who will be convicted,

(ii) the probability that the decision made is the correct one,

(iii) the conditional probability that a convicted defendant is really guilty. [9]

(*b*) A manufacturer of fluorescent tubes claimed that at least 60% of the tubes produced would operate for more than 5000 hours. A large sample of tubes was tested. After the test had been running for 3000 hours it was found that 98·8% of the tubes were still operating, and after 4000 hours it was found that 90% of the tubes were still operating. Assuming that the operational lifetimes of the tubes are Normally distributed, determine whether or not the manufacturer's claim is justified. [6]

12. In a particular area there are N families each having at least one child.

(*a*) Suppose that 30% of the families have one child, 40% have two children, 20% have three children, and 10% have four children. One of the children in the area is chosen at random.

(i) If X denotes the number of children in the family of the chosen child, show that $P(X = 1) = 1/7$ and evaluate $P(X = r)$ for $r = 2, 3, 4$. [4]

(ii) Given that the chosen child is not an only child, calculate the conditional probability that the chosen child's family has three children. [1]

(*b*) [In this part of the question you may assume that

$$\sum_{r=1}^{k} r^2 = \tfrac{1}{6}k(k+1)(2k+1), \quad \sum_{r=1}^{k} r^3 = \tfrac{1}{4}k^2(k+1)^2\,]$$

Suppose, instead, that the N families in the area are such that N/k of them have r children for $r = 1, 2, 3, \ldots, k$, where $k \geqslant 2$.

(i) Show that μ, the mean number of children per family, is $\tfrac{1}{2}(k+1)$. [1]

One child is chosen at random from all the children in the area. Let Y denote the number of children in the family of the chosen child.

(ii) Show that $P(Y = r) = \dfrac{2r}{k(k+1)}, \quad r = 1, 2, 3, \ldots, k.$ [1]

(iii) Show that Y is a biased estimator of μ. [2]

(iv) Find a linear function of Y which is an unbiased estimator of μ, and calculate its standard error if $k = 4$. [6]

13. Suppose that the number, X, of children in a randomly chosen family has a Poisson distribution with mean 2.

(i) Show that $E(|X-2|) = 8e^{-2}$. [6]

(ii) Suppose further that, independently for each birth, the probability of a boy is p. Given that a randomly chosen family has r boys, show that the conditional distribution of $X-r$ is Poisson. Hence find the mean number of children in families having r boys. [9]

14. A random sample of 25 observations is taken from a Normal distribution whose mean and standard deviation are equal to θ, where θ is unknown.

(*a*) Given that only one of the 25 observations had a value greater than 10, obtain a sensible estimate of θ. [5]

(*b*) Let $\bar{X}$ denote the mean of the 25 observations.

(i) Evaluate, correct to three decimal places,

$$P(\bar{X} > \tfrac{3}{5}\theta) \quad \text{and} \quad P(\bar{X} < \tfrac{4}{5}\theta \mid \bar{X} > \tfrac{3}{5}\theta).$$

(ii) Show that $P(\bar{X} > 1{\cdot}392\theta) = 0{\cdot}025$ and find k so that $P(\bar{X} < k\theta) = 0{\cdot}025$. Given that the observed value of $\bar{X}$ is 4·6, deduce 95% confidence limits for θ. [10]

15. The continuous random variable X is distributed with probability density function f given by

$$f(x) = \tfrac{1}{2}x, \quad \text{for } 0 \leqslant x < 1,$$
$$f(x) = \tfrac{1}{6}(4-x), \quad \text{for } 1 \leqslant x \leqslant 4,$$
$$f(x) = 0, \text{ otherwise.}$$

(*a*) Find the mean and the variance of the random variable Y defined as follows:

$$Y = X^2, \quad \text{for } X < 1,$$
$$Y = 1, \quad \text{for } X \geqslant 1.$$ [6]

(*b*) Denoting the cumulative distribution function of X by F, show that the cumulative distribution function, G, of $W = (X-2)^2$ is such that

$$G(w) = F(2+\sqrt{w}) - F(2-\sqrt{w}).$$

Hence show that the probability density function, g, of W is given by

$$g(w) = \frac{1}{3\sqrt{w}}, \quad \text{for } 0 \leqslant w \leqslant 1,$$

and determine the expression for $g(w)$ for all other possible values of w for which $g(w) \neq 0$. [9]

WELSH JOINT EDUCATION COMMITTEE | **CYD-BWYLLGOR ADDYSG CYMRU**

General Certificate of Education | **Tystysgrif Addysg Gyffredinol**

Advanced Level | **Safon Uwch**

Special Paper | **Papur Arbennig**

PURE MATHEMATICS S

A.M. MONDAY, 24 June 1991

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

Addendum to 'Information for the use of Candidates in Mathematics'.

The Taylor series expansion of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$$

1021 * 0023/7

Section A

Pure Mathematics

No more than **four** *questions should be answered from this section.*

1. A couple borrows £P at the beginning of a particular month at a *monthly* interest of I%. The interest is calculated at the end of each month and added to the amount outstanding. The couple repays a fixed amount £R at the end of each month.

Given that £S_n denotes the amount outstanding immediately after the nth monthly payment has been made, explain why

$$S_n = \left(1 + \frac{I}{100}\right)S_{n-1} - R \qquad (\text{for } n \geqslant 1 \text{ taking } S_0 = P).$$

Hence by summing a suitable geometric series, or otherwise, show that

$$S_n = P\left(1 + \frac{I}{100}\right)^n - \frac{100\,R}{I}\left[\left(1 + \frac{I}{100}\right)^n - 1\right]. \qquad [6]$$

The couple borrows £50 000 at a monthly interest rate of 1%. Calculate the monthly payment needed to repay the loan in 20 years, giving your answer correct to the nearest penny.

After one year, the couple has made 12 of these monthly payments. What percentage of their total monthly payments to date cover

(i) repayment of capital,

(ii) interest? [3]

After ten years, the interest rate is increased to 1·2% per month. The couple is given the choice,

either (A) to increase the monthly payment so that the loan is repaid in a further 10 years as planned,

or (B) to maintain the same monthly payment resulting in the loan being repaid later than planned.

Calculate

(iii) the new monthly payment corresponding to option (A),

(iv) the *extra* time (in months) needed to repay the loan under option (B). [6]

2. ABC is an acute-angled triangle. Show, with the usual notation, that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R denotes the radius of the circle passing through A, B and C. [4]

Hence, by considering

$$\frac{b-c}{b+c}$$

prove the 'tangent formula'

$$\tan \tfrac{1}{2}(B - C) = \left(\frac{b-c}{b+c}\right) \cot \tfrac{1}{2}A. \qquad [4]$$

Show further that

$$\sin \tfrac{1}{2}(B - C) = \left(\frac{b-c}{a}\right) \cos \tfrac{1}{2}A$$

and deduce a similar expression for $\cos \frac{1}{2}(B - C)$. [4]

In a particular triangle, $b = 8$, $c = 10$ and $A = 44°$. Using one of the above results, and not the cosine rule, find (correct to the nearest degree) the values of B and C. [3]

3. The points $P\,(ap^2, 2ap)$ and $Q\,(aq^2, 2aq)$ $(p > q > 0)$ lie on the parabola $y^2 = 4ax$ and S is the point $(a, 0)$.

(*a*) Given that α denotes the angle PSQ and β denotes the acute angle between the tangents at P and Q, show that

(i) $\tan \beta = \dfrac{p-q}{1+pq}$

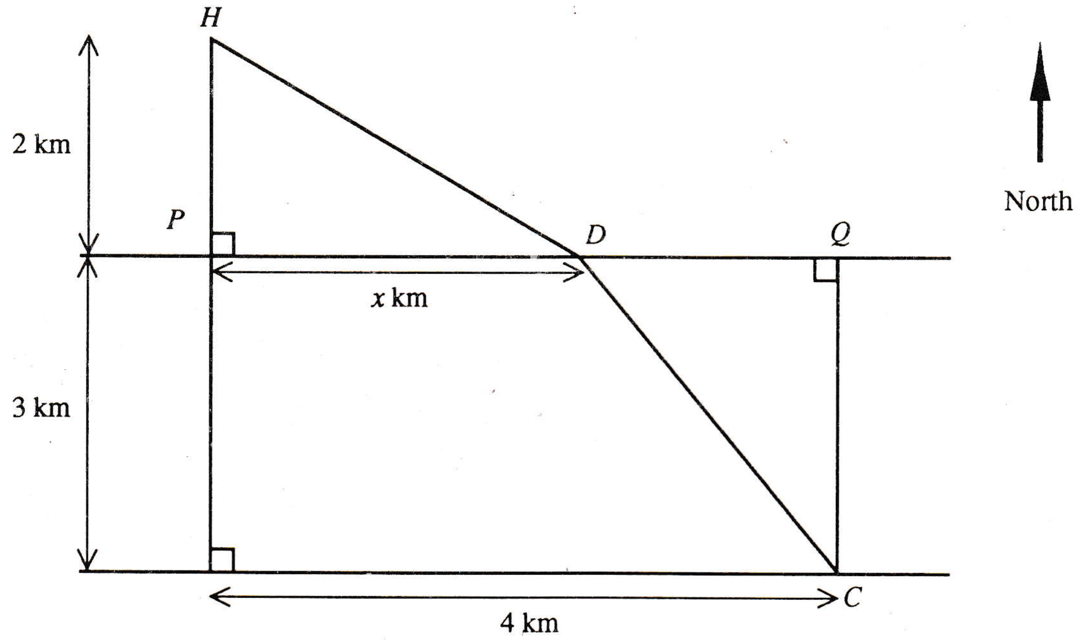
and (ii) $\tan \alpha = \dfrac{2(pq+1)(p-q)}{(p^2-1)(q^2-1)+4pq}$.

Hence show that $\alpha = 2\beta$. [7]

(*b*) The tangents at P and Q meet in R. Find the coordinates of R in terms of p and q. Given that $\alpha = \dfrac{\pi}{2}$, show that the equation of the locus of R as p and q vary is

$$(x + 3a)^2 - y^2 = 8a^2. \qquad [8]$$

4.



A choirboy, Bill, lives in a house H. He sings in a church C which is situated 4 km East and 5 km South of H. A road PQ runs West–East and is 2 km South of H. The land North of PQ is smooth pastureland over which Bill cycles at 20 km/h. The land South of PQ is too rough to cycle over and Bill runs over it at 10 km/h.

To go to church, Bill moves in a straight line from H to C, cycling over the pastureland and running over the rougher ground. Calculate, to the nearest second, how long it takes Bill to go to church. [2]

Bill now learns about calculus in school and he realises that it might be quicker not to travel in a straight line from H to C. He therefore decides to cycle in a straight line from H to a point D, x km East and 2 km South of H, and then to run in a straight line from D to C. Write down an expression for the time T (hours) taken by Bill to go to church along this route and show that the value of x which minimises T satisfies the equation

$$\frac{x}{\sqrt{4+x^2}} = \frac{2(4-x)}{\sqrt{9+(4-x)^2}}.$$ [3]

(i) Prove that the optimal route satisfies the 'law of refraction', namely that the ratio $\sin P\hat{H}D : \sin Q\hat{C}D$ is equal to the ratio of the speeds in the two regions. [3]

(ii) Show that the above equation can be rewritten as

$$3x^4 - 24x^3 + 55x^2 - 128x + 256 = 0.$$

Find the smallest positive root of this equation correct to 2 decimal places. Show that this value of x minimises T and confirm that Bill can save just over a minute by taking the optimal route. [7]

5. (*a*) Differentiate $\sec\theta \tan\theta$, and hence evaluate the indefinite integral

$$\int \sec^3\theta \, d\theta.$$ [3]

(*b*) Given that

$$I(a) = \int_0^1 \sqrt{x^2 + x + a} \quad dx$$

(i) show that

$$I(\tfrac{1}{4}) = 1,$$ [3]

(ii) using a suitable trigonometric substitution, show that for $a > \frac{1}{4}$,

$$I(a) = \frac{3\sqrt{a+2} - \sqrt{a}}{4} - \frac{(1-4a)}{8} \ln\left(\frac{3 + 2\sqrt{a+2}}{1 + 2\sqrt{a}}\right).$$ [9]

SECTION B

Further Pure Mathematics

No more than **four** *questions should be answered from this section.*

6. (*a*) Using de Moivre's Theorem, show that for even n,

$$\cos n\theta = \sum_{r=0}^{n/2} (-1)^r \binom{n}{2r} \cos^{n-2r}\theta (1 - \cos^2\theta)^r.$$

Obtain a similar result for odd n. [3]

(*b*) Writing $\theta = \cos^{-1}x$, show that for $-1 \leqslant x \leqslant 1$,

$$T_n(x) = \cos(n \cos^{-1}x)$$

is an nth degree polynomial in x.

Find expressions for $T_4(x)$ and $T_5(x)$. [4]

(*c*) Show that the equation

$$T_n(x) = 0$$

has n roots in the interval $(-1, 1)$ and write down their values.

State the maximum and minimum values of $T_n(x)$ for $-1 \leqslant x \leqslant 1$ and give the corresponding values of x. [4]

(*d*) By putting $x = \cos\theta$, show that for $m \neq n$,

$$\int_{-1}^{1} \frac{T_m(x)\, T_n(x)}{\sqrt{1 - x^2}}\, dx = 0.$$ [4]

7. (*a*) Show, by considering the Taylor series of $f(x)$ about $c = \frac{1}{2}(a + b)$, that

$$\int_a^b f(x)\, dx = \sum_{r=0}^{\infty} \frac{(b - a)^{2r+1} f^{(2r)}(c)}{2^{2r}(2r + 1)!}.$$

[You may assume that the Taylor series is valid throughout the interval [a, b] and you need not consider questions of convergence.] [5]

(*b*) Use the first two terms on the right-hand side to find an approximate value of the integral

$$\int_0^{\ln 2} \operatorname{sech} x\, dx.$$

Round your answer to 3 decimal places. [5]

(*c*) By putting $u = \sinh x$, show that

$$\int_0^{\ln 2} \operatorname{sech} x\, dx = \tan^{-1}(\tfrac{3}{4}).$$

Deduce that your answer to (*b*) narrowly fails to be correct to 3 decimal places. [5]

8. The curve C_1 has equation

$$288x^2 + 337y^2 + 168xy - 912x - 1516y + 1347 = 0.$$

C_1 is now subjected to the translation vector (h, k), i.e. it is translated h units in the direction of the x-axis and k units in the direction of the y-axis. Find the equation of the translated curve C_2. Determine the values of h and k which reduce the coefficients of the x and y terms to zero and show that, in this case, the equation of C_2 is

$$288x^2 + 337y^2 + 168xy = 625.$$ [5]

C_2 is now rotated anticlockwise through an acute angle θ about the origin. Find the equation of the rotated curve. Determine the value of θ which reduces the coefficient of xy to zero and show that in this case the equation becomes

$$9x^2 + 16y^2 = 25.$$ [7]

Hence sketch C_1, prior to translation and rotation. [3]

9. The numbers

$$u_1, u_2, u_3, \ldots$$

satisfy the equation

$$au_{n+2} + bu_{n+1} + cu_n = 0 \quad \text{for } n \geqslant 1 \qquad (1)$$

where a, b, c are given constants ($a, c \neq 0$).

The distinct roots of the equation

$$ax^2 + bx + c = 0$$

are denoted by α, β. Verify that

$$u_n = A\alpha^n + B\beta^n \qquad (2)$$

satisfies equation (1), where A, B are any constants. [4]

The Fibonacci numbers

$$1, 2, 3, 5, 8, 13, \ldots$$

are such that, from the third number onward, each number is the sum of the two preceding numbers. Using the above analysis, show that

$$\alpha, \beta = \tfrac{1}{2}(1 \pm \sqrt{5}).$$

Hence, by using equation (2) and choosing A, B to fit the first two Fibonacci numbers, show that the nth Fibonacci number is given by

$$v_n = \frac{1}{2^{n+1}\sqrt{5}}\{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\}.$$ [9]

Deduce the value of

$$\lim_{n \to \infty} \frac{v_{n+1}}{v_n}.$$ [2]

10. (*a*) Given that

$$I_n = \int_0^{\pi/4} \tan^n x \, dx$$

show that

(i) $I_m < I_n$ for $m > n$,

and (ii) $I_n = \dfrac{1}{n-1} - I_{n-2}$ for $n \geqslant 2$. [4]

(*b*) Prove by induction that, for even n,

$$I_n = \sum_{r=1}^{n/2} \frac{(-1)^{r+\frac{n}{2}}}{2r-1} + (-1)^{\frac{n}{2}} \frac{\pi}{4}.$$ [5]

(*c*) Show, by considering the relative magnitudes of I_{4n-2}, I_{4n} and I_{4n+2}, that

$$2\sum_{r=1}^{2n} \frac{(-1)^{r+1}}{2r-1} + \frac{1}{4n+1} < \frac{\pi}{2} < 2\sum_{r=1}^{2n} \frac{(-1)^{r+1}}{2r-1} + \frac{1}{4n-1}.$$

Deduce an infinite series for π. [6]

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APPLIED MATHEMATICS S

A.M. MONDAY, 24 June 1991

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

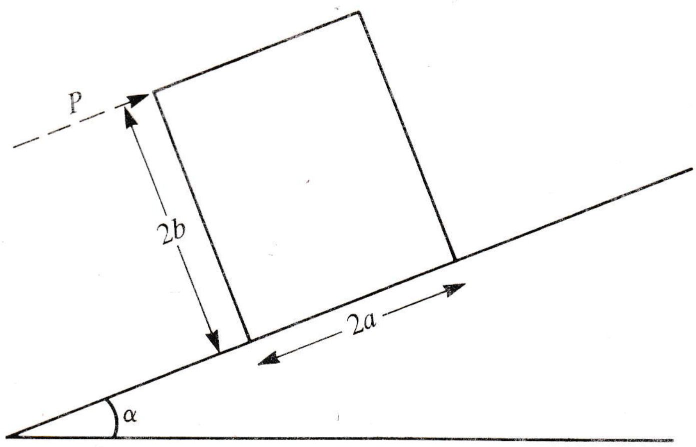
Section A

Mechanics

No more than **four** *questions should be answered from this section.*

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

1.



The diagram shows a uniform rectangular block of weight W resting on a rough plane which is inclined at an angle α to the horizontal. Four of the edges of the block are parallel to a line of greatest slope of the plane, these edges being of length $2a$. The edges of the block which are perpendicular to the plane are of length $2b$. There is a force of magnitude P applied, as shown in the diagram, to the middle of the lower horizontal edge of the face of the block parallel to, but not in contact with, the plane. This force acts parallel to a line of greatest slope in the sense which would tend to move the block up the plane. The coefficient of friction between the block and the plane is denoted by μ.

Given that the block is in equilibrium under the action of all the forces acting on it, show that the normal reaction of the plane acts through a point at a perpendicular distance of $\left(a - b\tan\alpha + \dfrac{2P}{W} b\sec\alpha\right)$ from the lower horizontal edge in contact with the plane. [4]

Deduce that, in the absence of the applied force, equilibrium is possible provided that $\mu \geqslant \tan\alpha$ and $a \geqslant b\tan\alpha$. [5]

Assuming that these conditions are satisfied and that the plane is such that equilibrium is broken by the block sliding, find P when sliding occurs. [2]

Given instead that the plane is such that equilibrium is broken by the block toppling, find P when toppling occurs. [2]

Hence find the condition that has to be satisfied if, as P gradually increases from zero, the block will slide rather than topple. [2]

2. A particle P of mass m is at rest on a rough horizontal plane, the coefficient of friction between the particle and the plane being denoted by μ. A light inextensible string is attached to the particle and passes over a small smooth pulley at a height h above the plane. The string is pulled with constant speed u and, at time t, the angle between the string and the table is denoted by θ. Assuming that P stays on the plane, show that

(i) $\dfrac{d\theta}{dt} = \dfrac{u \sin\theta \tan\theta}{h}$, [4]

(ii) the magnitude of the acceleration of P at time t is $\dfrac{u^2 \tan^3\theta}{h}$. [5]

Find the tension in the string. [6]

3. Two particles P and Q, of masses $3m$ and m, respectively, are attached one to each end of a light inextensible string which passes over a smooth pulley. The particles move in a vertical plane with both the hanging parts of the string vertical. They are released from rest with P at a height h above a horizontal inelastic floor. Find the height to which P rises after n collisions with the floor. [8]

Show that the total time that the system is in motion is three times the time taken for P to reach the floor for the first time. [7]

4. A particle P of mass m is projected from a point O on a horizontal plane with speed u at an angle α to the horizontal. There is a resistive force acting on the particle so that, when its velocity is $\mathbf{v}$, the total force acting on it is $-mg\mathbf{j} - mk\mathbf{v}$ where k is a constant and $\mathbf{j}$ is the unit vector vertically upwards. The particle lands back on the plane after a time T and, at that time, the magnitude of the angle between $\mathbf{v}$ and the horizontal is denoted by θ. Show that

(i) $u \sin\alpha = \dfrac{g(kT + e^{-kT} - 1)}{k(1 - e^{-kT})}$, [7]

(ii) $\dfrac{\tan\theta}{\tan\alpha} = \dfrac{e^{kT} - 1 - kT}{e^{-kT} - 1 + kT}$. [8]

5. (*a*) The displacement x of a particle moving along the x-axis satisfies the equation

$$\frac{d^2x}{dt^2} + 2kn\frac{dx}{dt} + n^2x = 0,$$

where k and n are positive constants. At time $t = 0$ the particle is projected from the origin in the positive x direction and at a subsequent time it returns to the origin. Show that this motion would not have been possible for $k \geqslant 1$. [7]

(*b*) By making the transformation $v = e^t$, or otherwise, find the general solution of

$$v^2\frac{d^2x}{dv^2} + v\frac{dx}{dv} - x = 1 + \ln v.$$ [8]

Section B

Probability and Statistics

No more than **four** *questions should be answered from this section.*

6. (*a*) In a certain country a jury consists of 10 persons. A defendant will be convicted only if at least 6 of the jurors vote the defendant as being guilty. When a defendant is really guilty each juror, independently, will return a vote of guilty with probability 0·7. When a defendant is really innocent each juror, independently, will return a vote of guilty with probability 0·2. Given that 65% of the defendants are really guilty find, correct to three decimal places,

(i) the proportion of the defendants who will be convicted,

(ii) the probability that the decision made is the correct one,

(iii) the conditional probability that a convicted defendant is really guilty. [9]

(*b*) A manufacturer of fluorescent tubes claimed that at least 60% of the tubes produced would operate for more than 5000 hours. A large sample of tubes was tested. After the test had been running for 3000 hours it was found that 98·8% of the tubes were still operating, and after 4000 hours it was found that 90% of the tubes were still operating. Assuming that the operational lifetimes of the tubes are Normally distributed, determine whether or not the manufacturer's claim is justified. [6]

7. In a particular area there are N families each having at least one child.

(*a*) Suppose that 30% of the families have one child, 40% have two children, 20% have three children, and 10% have four children. One of the children in the area is chosen at random.

(i) If X denotes the number of children in the family of the chosen child, show that $P(X = 1) = 1/7$ and evaluate $P(X = r)$ for $r = 2, 3, 4$. [4]

(ii) Given that the chosen child is not an only child, calculate the conditional probability that the chosen child's family has three children. [1]

(*b*) [In this part of the question you may assume that

$$\sum_{r=1}^{k} r^2 = \tfrac{1}{6}k(k+1)(2k+1), \quad \sum_{r=1}^{k} r^3 = \tfrac{1}{4}k^2(k+1)^2]$$

Suppose, instead, that the N families in the area are such that N/k of them have r children for $r = 1, 2, 3, \ldots, k$, where $k \geqslant 2$.

(i) Show that μ, the mean number of children per family, is $\tfrac{1}{2}(k + 1)$. [1]

One child is chosen at random from all the children in the area. Let Y denote the number of children in the family of the chosen child.

(ii) Show that $P(Y = r) = \dfrac{2r}{k(k+1)}, \quad r = 1, 2, 3, \ldots, k.$ [1]

(iii) Show that Y is a biased estimator of μ. [2]

(iv) Find a linear function of Y which is an unbiased estimator of μ, and calculate its standard error if $k = 4$. [6]

8. Suppose that the number, X, of children in a randomly chosen family has a Poisson distribution with mean 2.

(i) Show that $E(|X-2|) = 8e^{-2}$. [6]

(ii) Suppose further that, independently for each birth, the probability of a boy is p. Given that a randomly chosen family has r boys, show that the conditional distribution of $X - r$ is Poisson. Hence find the mean number of children in families having r boys. [9]

9. A random sample of 25 observations is taken from a Normal distribution whose mean and standard deviation are equal to θ, where θ is unknown.

(*a*) Given that only one of the 25 observations had a value greater than 10, obtain a sensible estimate of θ. [5]

(*b*) Let $\bar{X}$ denote the mean of the 25 observations.

(i) Evaluate, correct to three decimal places,

$$P(\bar{X} > \tfrac{3}{5}\theta) \quad \text{and} \quad P(\bar{X} < \tfrac{4}{5}\theta \mid \bar{X} > \tfrac{3}{5}\theta).$$

(ii) Show that $P(\bar{X} > 1{\cdot}392\theta) = 0{\cdot}025$ and find k so that $P(\bar{X} < k\theta) = 0{\cdot}025$. Given that the observed value of $\bar{X}$ is 4·6, deduce 95% confidence limits for θ. [10]

10. The continuous random variable X is distributed with probability density function f given by

$$f(x) = \tfrac{1}{2}x, \quad \text{for } 0 \leqslant x < 1,$$

$$f(x) = \tfrac{1}{6}(4 - x), \quad \text{for } 1 \leqslant x \leqslant 4,$$

$$f(x) = 0, \text{ otherwise.}$$

(*a*) Find the mean and the variance of the random variable Y defined as follows:

$$Y = X^2, \quad \text{for } X < 1,$$
$$Y = 1, \quad \text{for } X \geqslant 1.$$ [6]

(*b*) Denoting the cumulative distribution function of X by F, show that the cumulative distribution function, G, of $W = (X-2)^2$ is such that

$$G(w) = F(2 + \sqrt{w}) - F(2 - \sqrt{w}).$$

Hence show that the probability density function, g, of W is given by

$$g(w) = \frac{1}{3\sqrt{w}}, \quad \text{for } 0 \leqslant w \leqslant 1,$$

and determine the expression for $g(w)$ for all other possible values of w for which $g(w) \neq 0$. [9]