WELSH JOINT EDUCATION COMMITTEE
General Certificate of Education
Advanced Level

CYD-BWYLLGOR ADDYSG CYMRU
Tystysgrif Addysg Gyffredinol
Safon Uwch

MATHEMATICS A1

Pure Mathematics

A.M. THURSDAY, 7 June 1990

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Prove the identity

$$\tan A + \cot A \equiv 2 \operatorname{cosec} 2A. \qquad [4]$$

2. Given that

$$\sqrt{z} = \frac{2}{1 - \mathrm{i}} + 1 - 2\mathrm{i}$$

express the complex number z in the form $x + \mathrm{i}y$.
Find the argument of z, giving your answer correct to the nearest degree. [4]

3. The two roots of the quadratic equation

$$x^2 + 2x + 3 = 0$$

are denoted by α, β. Without solving the equation, find the quadratic equation whose roots are $\alpha + \frac{1}{\beta}$, $\beta + \frac{1}{\alpha}$. [5]

4. Write down and simplify the binomial expansion of $(1 + 2x)^{-\frac{1}{2}}$ up to and including the term in x^3.
By putting $x = \frac{1}{8}$, use your expansion to obtain an approximation to $\sqrt{5}$, giving your answer as a ratio of two integers. [5]

5. The cubic

$$x^3 + ax^2 + bx + c$$

(i) has a stationary value where $x = 1$,

(ii) leaves a remainder of 5 when divided by $x - 3$.

The graph of this cubic has a point of inflection where $x = 2$.

Find the values of a, b and c. [5]

6. Using the substitution $y = 2 - x$, show that

$$\int_0^1 \left(\frac{x}{2-x}\right)^2 \mathrm{d}x = 3 - 4 \ln 2.$$ [5]

7. The function f is defined on the domain $x \geqslant 0$ by

$$f(x) = \ln (1 + 2x).$$

Obtain an expression for $f^{-1}(x)$, where f^{-1} denotes the inverse function of f. State the domain of f^{-1}.

Show that the equation

$$f^{-1}(x) = f(x)$$

has a root between 1·2 and 1·3. [5]

8. The three non-zero, distinct numbers a, b, c are such that a, b, c are the first three terms of an arithmetic series and a, c, b are the first three terms of a geometric series.

(i) Find the common ratio of the geometric series and show that its sum to infinity is $\frac{2a}{3}$.

(ii) Find the common difference of the arithmetic series in terms of a. [7]

SECTION B

Answer **four** *questions from this section.*

9. A pyramid of vertical height h has a square horizontal base of side x and the vertex is directly above the centre of the base.

If A denotes the total surface area (i.e. the area of the base plus the area of the four sloping faces) show that

$$A = x^2 + x\sqrt{x^2 + 4h^2}.$$ [2]

The volume V of the pyramid is given by $V = \frac{1}{3}x^2h$. Show that

$$V^2 = \frac{Ax^2(A - 2x^2)}{36}.$$ [3]

Hence show that, for fixed A, the maximum value of V^2 is $\dfrac{A^3}{288}$ and show that the corresponding value of $\dfrac{h}{x}$ is $\sqrt{2}$. [6]

For a pyramid having this shape, calculate

(i) the angle between a sloping face and the base,

(ii) the angle between a sloping edge and the base. [4]

10. (i) Determine the values of α, β such that

$$x^2 - 6x - 1 \equiv (x + \alpha)^2 + \beta.$$

Hence find the minimum value of $x^2 - 6x - 1$ and state the value of x for which this occurs. [4]

(ii) Sketch the graph of

$$y = x^2 - 6x - 1$$

marking the co-ordinates of the points where the graph meets the co-ordinate axes.

Find the set of values of x for which

$$-6 \leqslant x^2 - 6x - 1 \leqslant 6.$$ [8]

Hence find the set of values of x for which

$$-6 \leqslant e^{2x} - 6e^x - 1 \leqslant 6.$$ [3]

11. (*a*) The points A, B have position vectors $\mathbf{a}$, $\mathbf{b}$ respectively with respect to an origin O. The point C lies on AB and $AC/AB = \lambda$.

Show that the position vector of C is given by

$$\mathbf{c} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}. \qquad [3]$$

(*b*) $ABCD$ is a quadrilateral and X, Y denote respectively the mid-points of the diagonals AC and BD. Prove that

$$4\mathbf{YX} = \mathbf{BA} + \mathbf{BC} + \mathbf{DA} + \mathbf{DC}. \qquad [5]$$

(*c*) (i) $OABCDE$ is a regular hexagon and $\mathbf{a}$, $\mathbf{b}$ denote respectively the position vectors of A, B with respect to O. Using the fact that $\mathbf{OC} = 2\mathbf{AB}$, obtain an expression for $\mathbf{OC}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and obtain similar expressions for $\mathbf{OD}$ and $\mathbf{OE}$. [3]

(ii) The lines OB and AC meet in F. Find an expression for $\mathbf{OF}$ in terms of $\mathbf{b}$. [4]

12. The triangle OAB has vertices $O(0, 0)$, $A(1, 3)$ and $B(8, 4)$.

(i) Show that the equation of the perpendicular bisector of OB is

$$2x + y = 10$$

and find the equation of the perpendicular bisector of OA. These two perpendicular bisectors meet in Q. Find the co-ordinates of Q and verify that Q is equidistant from O, A and B. [5]

(ii) Find the equation of the circle C passing through O, A and B. Show that the point $P(5(1 + \cos\theta), 5\sin\theta)$ lies on C for all values of θ. [4]

(iii) Show that the gradient of the tangent to C at P is given by $\dfrac{dy}{dx} = -\cot\theta$ and hence show that the equation of the tangent to C at P is

$$y + x\cot\theta = 5\cot\tfrac{1}{2}\theta. \qquad [6]$$

13. (*a*) Given that

$$a\cos\theta + b\sin\theta \equiv r\cos(\theta - \alpha) \qquad \text{(where } r > 0)$$

show that

$$\tan\alpha = \frac{b}{a}$$

and obtain an expression for r in terms of a and b.

Hence (or otherwise) find, in terms of a and b, the range of values of c for which the equation

$$a\cos\theta + b\sin\theta = c$$

has real roots.

Given that $a = 4$, $b = -2$ and $c = 3$, determine (correct to the nearest degree) all the roots of the equation lying between 0° and 360°. [8]

(*b*) Given that

$$7\tan 2x + 4\sin x = 0 \,\ldots\ldots\ldots\, (1)$$

show that

$$\sin x = 0 \qquad \text{or} \qquad \cos x = \tfrac{1}{4}.$$

Hence find the general solution (in radians) of equation (1). [7]

14. The function f is defined by $f(x) = \cos^{-1}x$ and the range of f is $[0, \pi]$. State the domain of f and sketch its graph. [3]

(i) Assuming the derivative of $\cos x$, show that if $y = \cos^{-1} x$, then

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$ [4]

(ii) Using integration by parts, find the indefinite integral

$$\int \cos^{-1} x \, dx.$$ [5]

(iii) For $y \approx 0$, $\cos y \approx 1 - \frac{1}{2}y^2$. Use this result to show that, for $x \approx 1$,

$$\cos^{-1} x \approx \sqrt{2(1 - x)}.$$ [3]

15. The integral I is defined by

$$I = \int_0^1 \frac{1 + x + x^2}{1 + x + x^2 + x^3}\, dx.$$

(i) Using Simpson's Rule with 3 ordinates and an interval of 0·5, show that

$$I \approx \frac{329}{360}.$$ [4]

(ii) Using the theory of partial fractions and noting that $1 + x + x^2 + x^3 \equiv (1 + x)(1 + x^2)$, show that

$$\frac{1 + x + x^2}{1 + x + x^2 + x^3} = \frac{1}{2}\left(\frac{1}{1 + x} + \frac{1 + x}{1 + x^2}\right).$$ [4]

(iii) Hence show that the exact value of I is $\frac{\pi}{8} + \frac{3}{4}\ln 2$. [5]

(iv) Using a suitable substitution, deduce the value of the integral

$$\int_0^1 \frac{x + x^3 + x^5}{1 + x^2 + x^4 + x^6}\, dx.$$ [2]

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MATHEMATICS A2

Mechanics

A.M. WEDNESDAY, 13 June 1990

(3 Hours)

Answer **all** *questions in Section A and* **four** *from Section B.*

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$[g = 9{\cdot}8\ \mathrm{m\,s^{-2}}]$

SECTION A

Answer **all** *questions in this section.*

1. Verify that an integrating factor for the equation
$$\frac{dy}{dx} - y\tan x = \sin x$$
is $\cos x$. Hence, or otherwise, find the solution of the differential equation for which $y = 1$ when $x = 0$. [4]

2. One end of a light inextensible string of length $2a$ is attached to a fixed point O. A particle of mass m is attached to the other end and moves in complete circles, centre O, in a vertical plane. Its speed at the lowest point of the circle is $6\sqrt{ga}$. Find
   (i) the square of its least speed,
   (ii) the greatest tension in the string. [4]

3. A particle P is placed on the inner surface of a fixed rough hollow sphere of centre O. Given that the coefficient of friction is $0{\cdot}5$ and that the particle rests in limiting equilibrium, find the tangent of the angle between the downward vertical and OP. [3]

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SECTION A

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$$\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = \sin x$$

is $\cos x$. Hence, or otherwise, find the solution of the differential equation for which $y = 1$ when $x = 0$. [4]

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3. A particle P is placed on the inner surface of a fixed rough hollow sphere of centre O. Given that the coefficient of friction is $0{\cdot}5$ and that the particle rests in limiting equilibrium, find the tangent of the angle between the downward vertical and OP. [3]

4. A particle of mass 0·2 kg moves on a smooth horizontal plane. Its position vector $\mathbf{r}$, at time t seconds, is given, in metres, by

$$\mathbf{r} = 4t^3\mathbf{i} + 4t^4\mathbf{j}.$$

Find

(i) the kinetic energy when $t = 0{\cdot}5$,

(ii) the cosine of the angle between the velocity when $t = 0{\cdot}5$ and the vector $4\mathbf{i} - 3\mathbf{j}$. [5]

5. Two small smooth spheres A and B of mass $2m$ and $5m$ respectively, moving along Ox, collide. The velocity of A immediately before collision is $5u$ in the positive x direction, and immediately after collision the velocities of A and B in the positive x direction are $2u$ and $4u$ respectively. Determine

(i) the velocity of B immediately before collision,

(ii) the magnitude of the impulse on B,

(iii) the value of the coefficient of restitution. [6]

6. A woman of mass 50 kg stands on the floor of a lift which is descending with a downwards acceleration of 0·8 m s^{-2}. Find the reaction of the floor of the lift on the woman. [2]

A small particle, initially at rest relative to the lift, is dislodged from the ceiling of the lift. Given that the distance from the floor to the ceiling of the lift is 4·5 m, find the time taken for the particle to reach the floor of the lift. [3]

7. A train starting from rest moves with constant acceleration during the first 50 seconds of its journey when it covers 500 m. It then travels at constant speed until it is brought to rest by applying a constant retardation. The distance travelled whilst the retardation is applied is 800 m. By drawing a speed-time graph, or otherwise, find

(i) the maximum speed attained by the train,

(ii) the magnitude of the retardation. [4]

Given that the total journey time was 4 minutes, determine the distance covered at constant speed. [2]

8. A vehicle of mass 4000 kg is moving up a hill inclined at an angle $\sin^{-1}(1/20)$ to the horizontal. Its initial speed is 2 m s^{-1}. Five seconds later it has travelled 15 m up the hill and its speed is then 6 m s^{-1}. Find the change in the kinetic energy and in the potential energy of the vehicle. Given that the engine is working at a constant rate of 44 kW, find the total work done against the resistive forces (which may not be assumed to be constant) during this 5 second period. [4, 3]

SECTION B

Answer **four** *questions from this section.*

9. The function y satisfies the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + (4 - k)y = k(k + 1)(6x + 4),$$

where k is a constant, and, when $x = 0$, $y = 5$ and $\frac{dy}{dx} = -1$.

Find y in the three cases

(*a*) $k = 1$, [6]

(*b*) $k = -1$, [5]

(*c*) $k = 0$. [4]

10. A particle P moving along the x axis describes simple harmonic motion with the origin O as centre so that its displacement x from O at time t satisfies the equation

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

where ω is a constant. Given that P is instantaneously at rest when $x = a$, show from the above equation that v, the speed of the particle at time t, satisfies

$$v^2 = \omega^2(a^2 - x^2).$$

Given that the particle is at O when $t = 0$, prove that

$$x = a \sin \omega t.$$ [5]

Given that the period of the motion is $2\pi/3$ seconds and that the maximum speed is 24 m s^{-1}, find

(i) the amplitude of the motion, [2]

(ii) the time taken for P to travel from O directly to a point 4 m from O. [2]

Given that the particle is of mass 0·25 kg, find

(iii) the rate at which the force acting on P is working when $t = \pi/9$, [3]

(iv) the maximum rate of working of this force. [3]

11. A particle moves with constant angular speed ω in a circle, centre O and radius a. Its position vector $\mathbf{r}$ at time t is given by

$$\mathbf{r} = a(\cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}).$$

Determine the acceleration of the particle, and hence deduce that this acceleration is directed towards O and is of magnitude $\omega^2 a$. [4]

A particle of mass m is suspended from a fixed point A by an elastic string of natural length b and modulus λ. The particle describes a horizontal circle with angular speed ω with the string being of constant length $l(>b)$, the centre of the circle being directly below A.

Given that the angle between the string and the downward vertical is θ, show that

$$\cos \theta = \frac{g}{l\omega^2}. \qquad [6]$$

The breaking tension in the string is $10mg$ and it is found that this occurs when $b\omega^2 = 9g$. Find $\cos \theta$ when the string breaks and express λ in terms of m and g. [1, 4]

12. A particle P is projected from a point O on a horizontal plane with speed v in a direction making an angle α above the horizontal. It rises to a maximum height h above the plane and strikes the plane again at a point a distance r from O. Write down x and y, the horizontal and vertical displacements of P from O at a time t after projection, in terms of v, α, g and t. Hence find r and h in terms of v, g and α. [6]

The particle passes through the point A whose horizontal displacement from O is $pr(p < 0{\cdot}5)$. Find, in terms of p and α, the tangent of the angle β between the horizontal and the path of the particle at A. State the horizontal displacement from O of the point B where the path is inclined at an angle β below the horizontal. [5]

Given that, at A, the particle is at a height a above the horizontal plane, express a in terms of p and h. [4]

13.

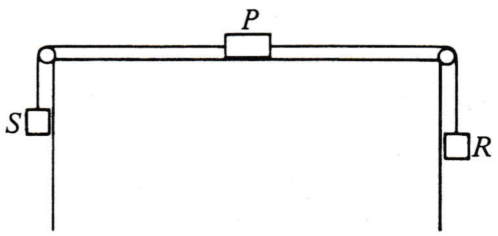

Fig. 1

Fig. 1 shows a particle P of mass $2m$ on a rough horizontal table and attached by light inextensible strings to particles R and S of mass $6m$ and $2m$ respectively. The coefficient of friction between P and the table is $0{\cdot}5$. The strings pass over light smooth pulleys on opposite sides of the table so that R and S can move freely with the strings perpendicular to the table edges. Given that the system is released from rest, find the magnitude of the common acceleration of the particles and the tension in the string joining P and S. [6]

After falling a distance d from rest, the particle R strikes an inelastic floor and is brought to rest. Find, for the period after R strikes the floor, the further distance that S rises. [4]

Find also, assuming that in the subsequent motion P remains on the table and S never reaches the table, the speed at which R moves when it is jerked off the floor. [5]

14. Show, by integration, that the centre of mass of a uniform solid hemisphere of base radius a is at a distance $3a/8$ from the centre of the base. [4]

A composite body is formed by joining at the rims of their circular bases, a uniform solid hemisphere of base radius a and a uniform solid right circular cone of base radius a and height $3a$. The density of the hemisphere is k times that of the cone, where $k > 3$. The centre of the base of the cone is denoted by O, and the vertex of the cone is denoted by V.

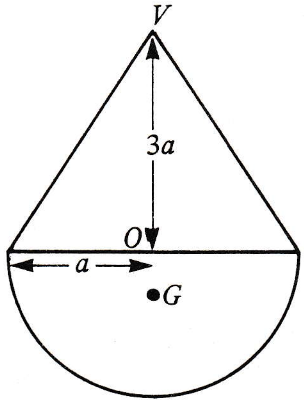

Fig. 2

(i) Let G denote the position of the centre of mass of the composite body as shown in Fig. 2. Show that

$$OG = \frac{3(k-3)a}{4(3+2k)}.$$ [4]

(ii) The body, when suspended from a point on the rim of the base of the cone, rests in equilibrium with the axis of the cone inclined at an angle α to the downward vertical and with O lower than V. Find $\tan\alpha$ in terms of k. State what would have occurred if k had been less than 3. [4]

(iii) The body is then removed from the point of suspension and placed in contact with a horizontal plane. Determine the range of values of k so that the body cannot rest in equilibrium with the slant surface of the cone in contact with the plane. [3]

15.

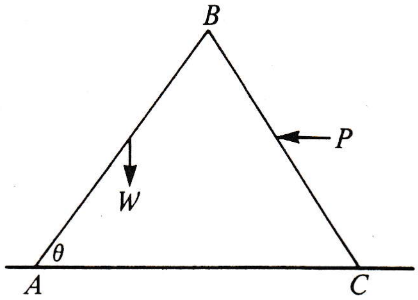

Fig. 3

Fig. 3 shows two identical light rods AB and BC in a vertical plane with the end A of the rod AB smoothly hinged to a fixed point on the surface of a rough horizontal table. The rods are also smoothly hinged at B. A weight of magnitude W is attached to the mid point of AB and the system is maintained in equilibrium by a horizontal force of magnitude P applied to the mid point of BC in the sense which would decrease the angle ABC. The angle BAC is denoted by θ, F denotes the value of the frictional force in the direction CA and R denotes the normal reaction at C.

By taking moments about A for the whole system, or otherwise, obtain an expression, in terms of W, P and θ, for the normal reaction R at C. [4]

Obtain also, by taking moments about B for BC, or otherwise, an equation relating P, R, θ and F. Hence, or otherwise, find F in terms of P, W and θ. [3, 2]

The coefficient of friction between the rod BC and the plane is μ.

(i) For the case $P = 0$ find, in terms of θ, the least possible value for μ in order that equilibrium is possible. [2]

(ii) Find, in terms of W, θ and μ, the maximum possible value of P. [4]

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MATHEMATICS A3

Probability and Statistics

P.M. MONDAY, 18 June 1990

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Two events A and B are such that

$$P(A) = 0{\cdot}5,\ P(B) = 0{\cdot}3, \text{ and } P(A \cap B) = 0{\cdot}1.$$

Calculate (i) $P(A \cup B)$, (ii) $P(A' \cap B)$, (iii) the probability that exactly one of the two events occurs. [5]

2. Independently for each page of a printed book the number of errors occurring has a Poisson distribution with mean 0·2. Find, correct to three decimal places, the probabilities that

(i) the first page will contain no error,

(ii) four of the first five pages will contain no error,

(iii) the first error will occur on the third page.

[5]

3. The continuous random variable X is normally distributed with mean 212·6 and standard deviation 2. Calculate, correct to three decimal places, the probabilities that

(i) a randomly observed value of X will lie between 212 and 213,

(ii) the mean of four randomly observed values of X will exceed 213.

[5]

4. The continuous random variable X is distributed with probability density function f, where

$$f(x) = 2(2 - x), \quad \text{for } 1 \leqslant x \leqslant 2,$$
$$f(x) = 0, \quad \text{otherwise.}$$

(i) Evaluate the mean of X.

(ii) Find the expected value of the area of a rectangle having adjacent sides of lengths X cm and $(5 - 2X)$ cm.

[5]

5. A random sample of 80 electrical elements produced by a manufacturer have resistances $x_1, x_2, \ldots, x_{80}$ ohms, where

$$\sum x_i = 790, \text{ and } \sum x_i^2 = 7821.$$

(i) Calculate unbiased estimates of the mean and the variance of the resistances of the elements produced by the manufacturer.

(ii) Use a normal distribution to calculate approximate 98% confidence limits for the mean resistance of the elements produced by the manufacturer.

[6]

6. The random variable X has the Poisson distribution with mean 0·4. Independently of X, the random variable Y has the distribution shown in the following table.

y	0	1	2
$P(Y = y)$	0·2	0·6	0·2

(i) State the variance of X and show that Y has the same variance.

(ii) Let $T = 2Y - X$ and let $U = Y + 2X$. Verify that

$$E[TU] = E[T]E[U].$$

[7]

7. A bag contains nine balls of which four are red, three are blue, and two are white. Three of these balls are to be selected at random without replacement.

(i) Calculate the probability that one ball of each colour will be selected.

(ii) Calculate the probability that the three balls selected will be of the same colour.

(iii) Given that the three selected balls were not all of the same colour calculate the conditional probability that two of the selected balls were red.

[7]

SECTION B

Answer **four** *questions from this section.*

8. A random sample of ten quartz watches of a particular make were tested for accuracy over a period of four weeks. The times, in seconds, gained by the ten watches were:

$$-3,\ +7,\ +2,\ +6,\ +8,\ +2,\ -3,\ +6,\ +11,\ +8.$$

(i) Calculate unbiased estimates of the mean, μ, and the variance, σ^2, of the times gained by such watches over a period of four weeks. [2]

(ii) Stating any assumptions you make, find a 95% confidence interval for the mean time gained by such watches over a period of four weeks. [5]

(iii) Another random sample of ten of the watches was taken and the times, in seconds, they gained over a period of four weeks gave unbiased estimates of μ and σ^2 equal to 3·8 and 23·4, respectively. Use the combined set of twenty observations to determine a 95% confidence interval for the mean time gained by such watches over a period of four weeks. [6]

(iv) Comment on the manufacturer's claim that, on average, the watches will not gain or lose more than 7 seconds over a period of four weeks. [2]

9. The proportion X of a certain ingredient in a sample of a chemical compound has probability density function f, where

$$f(x) = 2x, \qquad \text{for } 0 < x < 1,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(i) Calculate the probability that the proportion of the ingredient in a sample of the compound is at least 0·5. [2]

(ii) Calculate, correct to three decimal places, the probability that the proportion of the ingredient is at least 0·5 in exactly 6 of 10 random samples of the compound. [2]

(iii) Show that when $0 < y < 0{\cdot}25$

$$P(0{\cdot}25 - y < X < 0{\cdot}25 + y) = y,$$

and obtain an expression for this probability when $0{\cdot}25 < y < 0{\cdot}75$. Using these results, or otherwise, determine the probability density function of

$$Y = |\,X - 0{\cdot}25\,|,$$

the amount by which the proportion of the ingredient in a sample deviates from 0·25. Hence, or otherwise, find the expected value of Y. [11]

10. A machine is used to bag coal, the mass of coal delivered per bag being normally distributed with mean 55 kg and standard deviation 1·25 kg.

(*a*) Given 100 filled bags chosen at random, use a distributional approximation to calculate the probability that at least 60 of them contain 55 kg or more; give your answer correct to three decimal places. [3]

(*b*) Given two filled bags chosen at random calculate, correct to three decimal places, the probabilities that

(i) each bag contains at least 56 kg, [2]

(ii) the combined mass of coal in the two bags is less than 112 kg, [3]

(iii) one bag contains at least 1 kg more than the other bag, [4]

(iv) the heavier of the two bags contains more than 57 kg. [3]

Turn over.

11. The continuous random variable X is distributed with probability density function f given by

$$f(x) = \frac{1}{2}x^2 + \frac{1}{3}\theta x + \frac{1}{3}, \qquad \text{for } -1 < x < 1,$$

$$f(x) = 0, \qquad \text{otherwise,}$$

where $0 < \theta < 2$.

(i) Show that $E(X) = \frac{2}{9}\theta$. [2]

(ii) Express $P(X > 0)$ in terms of θ. [2]

(iii) For a random sample of n observations of X, let $\bar{X}$ denote the sample mean and let Y denote the number of observations that are positive. Show that

$$T_1 = \frac{9}{2}\bar{X} \text{ and } T_2 = \frac{6}{n}Y - 3$$

are both unbiased estimators of θ. [5]

(iv) Determine which of T_1 and T_2 is the better estimator of θ. [6]

12. Two machines, A and B, are used to produce identical items. Independently for each item produced on machine A the probability that it is defective is 0·01, while for an item produced on machine B the corresponding probability is 0·03.

(*a*) Three items are chosen at random from those produced on machine A and three items are chosen at random from those produced on machine B.

(i) Calculate the probability that exactly one of the three items produced on machine A is defective.

(ii) Calculate, correct to three decimal places, the probability that exactly one of the six chosen items is defective.

[4]

(*b*) Of the total output of items produced in a day, 60% are produced on machine A and 40% are produced on machine B.

(i) If one item is chosen at random from a day's output show that the probability of it being nondefective is 0·982. [2]

(ii) Two items are chosen at random from a day's output and both are found to be nondefective. Calculate, correct to three decimal places, the conditional probability that one of the items was produced on machine A and the other was produced on machine B. [5]

(iii) A box contains 80 items chosen at random from a day's output. Find an approximate value for the probability that the box contains at most two defective items. [4]

13. A group of nine adults consists of three married men, three married women, two bachelors and one spinster. Two of these are chosen at random without replacement. Let X denote the number of married men that are chosen and let Y denote the number of women that are chosen.

(i) Show that $P(X = 0, Y = 2) = P(X = 1, Y = 0) = \frac{1}{6}$. [2]

(ii) Display the joint probability distribution of X and Y in tabular form. [4]

(iii) Verify numerically that

$$\mathrm{Var}(X + Y) + \mathrm{Var}(X - Y) = 2[\mathrm{Var}(X) + \mathrm{Var}(Y)].$$ [5]

(iv) Prove that, for any two random variables U and W,

$$\mathrm{Var}(U + W) + \mathrm{Var}(U - W) = 2[\mathrm{Var}(U) + \mathrm{Var}(W)].$$ [4]

14. A physical law states that the variables x and y are linearly related in the form $y = \alpha + \beta x$. To verify this relationship experiments were conducted with x having prespecified values and the corresponding values of y were measured. It is known that measured values of y are subject to independent random errors that are normally distributed with mean zero and standard deviation 0·06. The results of the experiments are shown in the following table.

x	2·0	2·5	3·0	3·5
Measured y	10·8	9·5	8·4	7·3

The experimenter used a calculator to determine the least-squares estimate of the relation and produced the equation

$$y = 2{\cdot}3 + 2{\cdot}4x.$$

(i) Without performing any calculations state why it is clear from the data that the experimenter's equation is incorrect. [2]

(ii) Calculate the correct least-squares estimate of the relation. [4]

(iii) Calculate 95% confidence limits for β. [3]

(iv) The law referred to above is actually valid only when the experiments are conducted at a fixed constant temperature. In another series of experiments to verify the relation the same values of x were used but the temperature was held constant at a value different from that in the first series of experiments. The results obtained in this second series of experiments led to the least-squares estimate of the relation being

$$y = 17{\cdot}6 - 1{\cdot}96x.$$

Assuming that the errors in the measured values of y from the second series of experiments have the same distribution as those in the first series, calculate 95% confidence limits for the difference, at the two temperature levels, between the true values of y when $x = 3$. [6]

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Advanced Level | Safon Uwch

MATHEMATICS A4

Further Pure Mathematics

A.M. FRIDAY, 22 June 1990

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

Addendum to 'Information for the use of Candidates in Mathematics'.

The Taylor series expansion of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + (x - a)\,f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots.$$

SECTION A

Answer **all** *questions in this section.*

1. Assuming the summation formulae in the information booklet, show that

$$1.2.3 + 2.3.5 + \ldots. + n(n+1)(2n+1) = \frac{n(n+1)^2(n+2)}{2}.$$ [4]

2. The equation

$$x^3 - 3x - 4 = 0$$

has one real root. Use the Newton-Raphson method to find its value correct to three decimal places. [5]

3. Express $\tanh x$ in terms of exponential functions. Use your definition to show that

$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}.$$ [4]

4. Use the method of mathematical induction to prove that, for positive integral n,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Determine whether or not the result is true for $n = -1$. [5]

5. Using the substitution $t = \tan \frac{1}{2}x$, or otherwise, show that

$$\int_0^{\pi/2} \frac{dx}{2 + \cos x} = \frac{\pi}{3\sqrt{3}}.$$ [5]

6. The function f is defined on the domain $(0, \infty)$ by

$$f(x) = x^4 + x^2 + 1.$$

Let g denote the inverse function of f. By considering $f'(x)$ for a suitable value of x, or otherwise, find the value of $g'(3)$.

Write down the value of

$$\lim_{x \to \infty} g'(x).$$ [5]

7. The three roots of the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

are in geometric progression. Assuming that $b \neq 0$, show that

$$ac^3 = b^3d.$$ [6]

8. Given that

$$y = \sinh (m \sinh^{-1}x)$$

show that

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0.$$ [6]

SECTION B

Answer **four** *questions from this section.*

9. The points A, B, C, D have co-ordinates $(0, 2, 4)$, $(3, 2, 1)$, $(-1, -2, 4)$, $(5, 0, 4)$ respectively. Write down the co-ordinates of the point P which divides AB in the ratio $\lambda : 1 - \lambda$ and of the point Q which divides CD in the ratio $\mu : 1 - \mu$. [2]

Hence, or otherwise, show that the lines AB and CD are skew. [4]

Given that PQ is perpendicular to both AB and CD, show that $\lambda = \frac{1}{3}$ and $\mu = \frac{1}{2}$. Deduce the shortest distance between AB and CD. [5]

Find the equation of the plane containing the lines AB and PQ. [4]

10. R denotes a rotation of the plane through an angle θ about the point $P(h, k)$. By considering R as a translation which takes P to the origin O, followed by a rotation of the plane through an angle θ about O, followed by a translation which takes O to P, show that R is represented by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & -h\cos\theta + k\sin\theta + h \\ \sin\theta & \cos\theta & -h\sin\theta - k\cos\theta + k \\ 0 & 0 & 1 \end{bmatrix}.$$

[6]

The following matrix represents such a rotation.

$$\mathbf{M} = \begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} & 2 \\ \frac{3}{5} & -\frac{4}{5} & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the corresponding values of h, k and θ. [4]

The rotation corresponding to $\mathbf{M}$ is applied to the parabola $y^2 = x$. Show that its equation becomes

$$9x^2 + 16y^2 + 24xy - 88x - 159y + 329 = 0.$$

[5]

11. (*a*) Show that

$$\begin{vmatrix} 1 & 3 & \alpha - 1 \\ 2 & 2\alpha & 3\alpha - 5 \\ \alpha - 1 & \alpha + 3 & 4 \end{vmatrix} = -2\alpha(\alpha - 3)^2.$$

[3]

(*b*) The unknowns x, y, z satisfy the equations

$$\begin{bmatrix} 1 & 3 & \alpha - 1 \\ 2 & 2\alpha & 3\alpha - 5 \\ \alpha - 1 & \alpha + 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4p \\ 4q \\ 4r \end{bmatrix}$$

where p, q, r, α are constants.

(i) In the case $\alpha = 2$, solve the equations for x, y, z in terms of p, q, r. [4]

(ii) In the case $\alpha = 0$, find the condition on p, q, r necessary for the equations to be consistent and find the general solution. [3]

(iii) For what other value of α do the equations not have a unique solution? For this value of α, find the conditions on p, q, r necessary for the equations to be consistent and find the general solution. [3]

Give a geometrical interpretation to solution (ii) and to solution (iii). [2]

12. The point $P(x, y)$ moves in such a way that the ratio of its distance from the point $F(ae, 0)$ to its distance from the line $x = \frac{a}{e}$ is $e : 1$, where $e > 1$. Show that the equation of the locus C of P is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{where } b^2 = a^2(e^2 - 1). \qquad [4]$$

The point $P_1(x_1, y_1)$ lies on C and the line joining P_1 to the origin O meets the line $x = \frac{a}{e}$ at K. Show that FK is perpendicular to the tangent to C at P_1. [4]

Write down the equation of the line of gradient m passing through the point $Q\ (\alpha, \beta)$ and show that the x co-ordinates of the points of intersection of this line and C satisfy the equation

$$(b^2 - a^2m^2)\,x^2 + 2ma^2(m\alpha - \beta)x - a^2(\beta^2 + b^2 + m^2\alpha^2 - 2m\alpha\beta) = 0.$$

Hence show that the gradients of the tangents from Q to C satisfy

$$(\alpha^2 - a^2)m^2 - 2\alpha\beta m + \beta^2 + b^2 = 0.$$

Deduce the equation of the locus of Q given that these two tangents are perpendicular. Describe this locus in words, considering all possible values of e. [7]

13. The function f is defined on the domain $D = (-\infty, 0) \cup (0, 1) \cup (1, \infty)$ by

$$f(x) = \frac{1}{1 - x}.$$

State the range of f. The functions $g = f \circ f$ and $h = f \circ g$ are both defined on the domain D. State the ranges of g and h. Obtain an expression for $g(x)$ and show that $h(x) = x$.
Hence obtain an expression for $f^{-1}(x)$ where f^{-1} denotes the inverse function of f. [5]

It is required to approximate $f(x)$ in the neighbourhood of $x = 2$ by a quadratic function. Show that the approximation obtained by taking the first three terms of the appropriate Taylor series is

$$f(x) \approx -7 + 5x - x^2. \qquad [3]$$

Using this result, obtain a cubic approximation to $\log_e(x - 1)$ in the neighbourhood of $x = 2$ and deduce that for y close to 1,

$$\log_e y \approx -\frac{11}{6} + 3y - \frac{3}{2}y^2 + \frac{1}{3}y^3.$$

Integrate this result to show that a second approximation for y close to 1 is

$$\log_e y \approx -\frac{5}{6} + \frac{3}{2}y - \frac{1}{2}y^2 + \frac{1}{12}y^3 - \frac{1}{4y}. \qquad [7]$$

14. Given that α, β denote the two complex cube roots of unity, show that

(*a*) $\alpha = \beta^2$ and $\beta = \alpha^2$,

(*b*) $1 + \alpha + \alpha^2 = 1 + \beta + \beta^2 = 0$. [3]

The function F is defined for real x by

$$F(x) = \frac{1}{3}(e^x + e^{\alpha x} + e^{\beta x}).$$

Show that

(*c*) $F(x) + F'(x) + F''(x) = e^x$,

(*d*) $F'''(x) = F(x)$.

Obtain the McLaurin series for $F(x)$, stating the general term. [5]

By finding α, β, show that an alternative expression for $F(x)$ is

$$F(x) = \frac{1}{3}(e^x + 2e^{-\frac{1}{2}x} \cos \frac{\sqrt{3}}{2}x).$$

Show that the equation

$$F(x) = 0$$

has a root in the interval $[-2, -1{\cdot}5]$.

Denoting this root by γ, show that

$$\gamma = -\frac{2}{\sqrt{3}} \cos^{-1}\left(-\frac{1}{2}e^{\frac{3\gamma}{2}}\right).$$

Using an initial approximation of -2, use an iterative process based on this result to find the value of γ correct to three decimal places. [7]

TURN

OVER

FOR QUESTION

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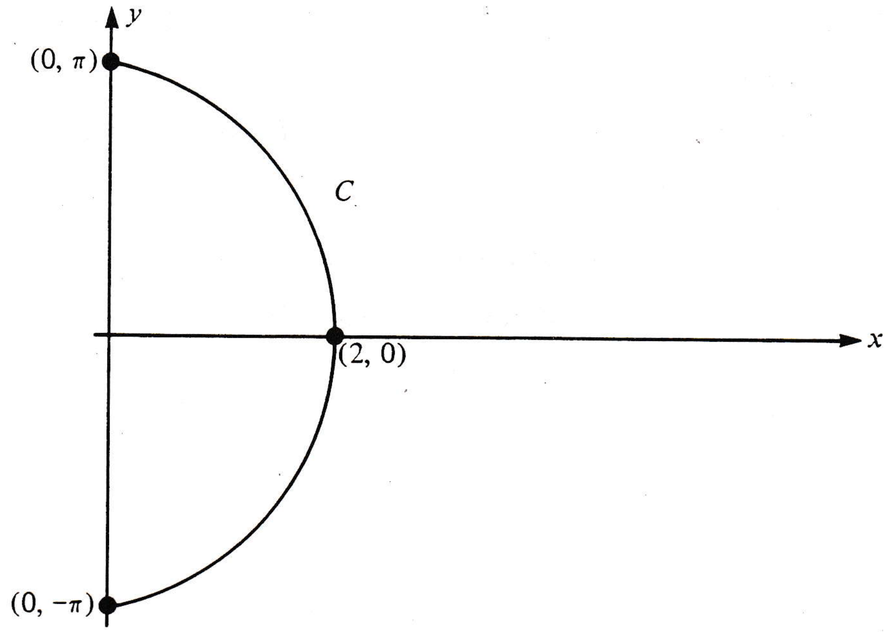
15.



The curve C, illustrated above, is the locus of the point $(1 + \cos t,\ t + \sin t)$ as t increases from $-\pi$ to π. Show that

$$\frac{dy}{dx} = -\cot \tfrac{1}{2}t$$

and obtain an expression for

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

in terms of a trigonometric function of $\frac{1}{2}t$. [6]

C is rotated through two right-angles about the x-axis. Show that the surface area of the solid generated is $\dfrac{8\pi}{3}(3\pi - 4)$. [5]

Calculate the surface area of the solid generated when C is rotated through four right-angles about the y-axis. [4]

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General Certificate of Education

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Advanced Level

Special Paper

Safon Uwch

Papur Arbennig

PURE MATHEMATICS S

A.M. MONDAY, 25 June 1990

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

IMPORTANT NOTE

Please note that the Mathematics S paper is not included separately. It contains three sections, Section A which is identical to Section A of the Pure Mathematics S paper, Section B which is identical to Section A of the Applied Mathematics S paper and Section C which is identical to Section B of the Applied Mathematics S paper. Candidates are required to answer not more than four questions from Section A and not more than four questions from either Section B or not more than four questions from Section C.

SECTION A

Pure Mathematics

No more than **four** *questions should be answered from this section.*

1. $f(x)$, $g(x)$ are defined for $x > 1$ by

$$f(x) = \left(1 + \frac{1}{x}\right)^x; \quad g(x) = \left(1 - \frac{1}{x}\right)^{-x}.$$

(*a*) (i) Show that

$$f(x) < g(x) \text{ for all } x > 1.$$

Using the binomial theorem, or otherwise, show that

(ii) $f(x)$ increases as x increases,

(iii) $g(x)$ decreases as x increases,

(iv)
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x)$$

and express the common limit, L, as an infinite series whose general term should be given.

(v) By considering the series

$$1 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots$$

show that $L < 3$. [7]

(*b*) Sketch the graphs of f and g on the same axes. [2]

(*c*) By considering the derivative of $\log_e f(x)$, show that

(vi) $\log_e x - \log_e(x-1) > \dfrac{1}{x}$ for $x > 2$

(vii) $e^x < \dfrac{1}{1-x}$ for $0 < x < \tfrac{1}{2}$. [6]

2. With the usual notation for triangles, prove the cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

$\left[\text{Your proof should consider separately the cases } (a)\ A < \frac{\pi}{2},\ (b)\ A > \frac{\pi}{2}.\right]$ [4]

Show that if a is increased by a small amount δa, the increase δA in A is given approximately by

$$\delta A \approx \frac{\delta a}{b \sin C}.$$

Obtain similar expressions for δA if

(i) b is increased by δb,

(ii) c is increased by δc.

[6]

Hence show that if a, b, c are increased simultaneously by small amounts δa, δb, δc respectively, then the increase in A is given approximately by

$$\delta A = \frac{\sin A}{\sin B \sin C}\cdot\frac{\delta a}{a} - \cot C\cdot\frac{\delta b}{b} - \cot B\cdot\frac{\delta c}{c}.$$

Verify that the right hand side of this equation reduces to zero if

$$\frac{\delta a}{a} = \frac{\delta b}{b} = \frac{\delta c}{c}$$

and explain in simple terms why this is so. [5]

3. Points D, E, F are taken on the sides OA, OB, AB respectively of the triangle OAB such that the lines AE, BD, OF are concurrent. Taking O as origin, use vector methods to show that

$$\frac{OD}{DA}\cdot\frac{AF}{FB}\cdot\frac{BE}{EO} = 1.$$ [10]

Assuming that the converse result is true, i.e. that if the above condition is satisfied then the three lines are concurrent, deduce that the perpendiculars from the three vertices of a triangle to the opposite sides are concurrent. [5]

4. (*a*) If $c = a^b (a, c > 0)$, then b is called the logarithm of c to base a (written $\log_a c$). Using this definition, show that

(i) $\log_a(bc) = \log_a b + \log_a c$

(ii) $\log_a b = \dfrac{1}{\log_b a}$

(iii) $\log_a b = \log_a c \times \log_c b$.

[6]

(*b*) The function f is defined on the domain $x > 1$ by

$$f(x) = \log_2 x + 8\log_x 2 + 2\log_x 4.$$

(iv) Find the two roots of the equation $f(x) = 7$.

(v) Show that the minimum value of $f(x)$ is $2\sqrt{12}$ and calculate the value of x which gives this minimum.

[9]

5. (*a*) Show that, for $|\alpha| \leqslant 1$,

$$\int_{-1}^{1} \frac{dx}{\sqrt{1 - 2\alpha x + \alpha^2}} = 2\,.$$

Evaluate the integral for $|\alpha| > 1$. [8]

(*b*) Use the substitution $y = \pi - x$ to evaluate the integral

$$\int_{0}^{\pi} \frac{x \sin^3 x \, dx}{4 - \cos^2 x}\,.$$ [7]

SECTION B

Further Pure Mathematics

No more than **four** *questions should be answered from this section.*

6. (*a*) It is given that $f(x)$ is a polynomial of degree n, where $n \geqslant 2$. The arithmetic mean of the n roots of the equation $f(x) = 0$ is denoted by m. If $f^{(r)}(x)$ denotes the rth derivative of $f(x)$, show that for $1 \leqslant r \leqslant n-1$, the arithmetic mean of the $(n-r)$ roots of the equation $f^{(r)}(x) = 0$ is also equal to m. [6]

(*b*) The quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

has two roots whose sum is zero. Show that

$$bcd - ad^2 = eb^2.$$

The following is such an equation.

$$x^4 + 2x^3 + x^2 - 6x - 12 = 0.$$

Find all its roots. [9]

7. A stochastic matrix is a square matrix with the following properties:

(*a*) All elements of the matrix are non-negative.

(*b*) The sum of the elements in each row is unity.

The 3×3 matrices $\mathbf{P}$, $\mathbf{Q}$ are stochastic, where $\mathbf{P} = (p_{ij})$ and $\mathbf{Q} = (q_{ij})$. Given that $\mathbf{R} = \mathbf{PQ}$ and $\mathbf{R} = (r_{ij})$, show that

$$\sum_{i=1}^{3} r_{1i} = \sum_{j=1}^{3} p_{1j}\left(\sum_{k=1}^{3} q_{jk}\right).$$

Deduce that $\mathbf{R}$ is stochastic. [4]

In the particular case

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

show by induction that, for positive integral n,

$$\mathbf{P}^n = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{1}{3}\left(\frac{1}{4}\right)^n \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Deduce that $\mathbf{P}^n$ is stochastic for positive integral n. [8]

The above expression for $\mathbf{P}^n$ is also valid for negative integral n. Use this result to investigate whether or not $\mathbf{P}^n$ is stochastic for negative integral n. [3]

8. The curve C has equation $y^2 = 4ax$.

(*a*) Two points A, B are taken on C such that $A\hat{O}B$ is a right-angle, where O is the origin.

(i) Show that AB passes through a fixed point on the x-axis.

(ii) Show that the locus of the mid-point of AB has equation

$$y^2 = 2a(x - 4a).$$

[8]

(*b*) Show that there are at most three normals from a general point Q to C. Find the equation of the locus of Q if two of these normals are perpendicular. [7]

9. Given that

$$\mathrm{I}_{m,n} = \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta \quad (m, n \geqslant 0)$$

show that

(i) $\mathrm{I}_{m,n} = \mathrm{I}_{n,m}$

(ii) $\mathrm{I}_{m,n} = \left(\dfrac{n-1}{m+n}\right)\mathrm{I}_{m,n-2}$ (for $n \geqslant 2$).

[5]

Hence show that if m, n are both positive odd integers,

$$\mathrm{I}_{m,n} = \frac{\left(\frac{n-1}{2}\right)!\left(\frac{m-1}{2}\right)!}{2\left(\frac{m+n}{2}\right)!}.$$

[6]

Given that p, q are positive integers, deduce that

$$\int_{-\infty}^{0} e^{py}(1 - e^y)^q \, dy = \frac{(p-1)!\,q!}{(p+q)!}.$$

[4]

10. Assuming that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

derive expressions for $\cos\theta$, $\sin\theta$ in terms of complex exponential functions and hence show that, for real θ,

(*a*) $\cos i\theta = \cosh\theta$

(*b*) $\cosh i\theta = \cos\theta$

(*c*) $\sin i\theta = i\sinh\theta$

(*d*) $\sinh i\theta = i\sin\theta$.

[3]

(i) Show that for real $x \in (-1, 1)$ and real θ,

$$\sum_{r=0}^{\infty} x^r \cos r\theta = \frac{1 - x\cos\theta}{1 + x^2 - 2x\cos\theta}.$$ [4]

(ii) Show that

$$\sum_{r=0}^{\infty} \frac{\sinh\{(2r+1)\theta\}}{(2r+1)!} = \cosh(\cosh\theta)\sinh(\sinh\theta)$$

and deduce a similar result for

$$\sum_{r=0}^{\infty} \frac{\sin\{(2r+1)\theta\}}{(2r+1)!}.$$

[You may assume the McLaurin series for $\sinh x$, i.e.

$$\sinh x = \sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!}.$$] [8]

WELSH JOINT EDUCATION COMMITTEE

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Advanced Level

Special Paper

Safon Uwch

Papur Arbennig

APPLIED MATHEMATICS S

A.M. MONDAY, 25 June 1990

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Mechanics

No more than **four** *questions should be answered from this section.*

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

1. Relative to axes Ox and Oy points P, Q, R have coordinates $(2a, a)$, $(3a, -4a)$, $(6a, 0)$ respectively. A system of forces S has moments $64aF$, $11aF$, $42aF$, in the anti-clockwise sense, about P, Q and R respectively, where F is a constant.

(*a*) Find the x- and y- components of the force acting at O, which, together with a couple, is equivalent to S. Find also the magnitude of the couple. [8]

(*b*) Find the magnitude of the forces acting along the lines PO, OR, QR which, together, are equivalent to S. [7]

2. At time $t = 0$ four men M_1, M_2, M_3, M_4 are at the points (a, a), $(-a, a)$, $(-a, -a)$, $(a, -a)$ respectively. The men set off at the same instant with a constant speed u so that, at any subsequent time, the direction of motion of man M_i is towards man M_{i+1}, with man M_4 moving towards man M_1. Given that the coordinates of M_1 are (x, y) express the coordinates of M_2 in terms of x and y and hence show that

$$\frac{dy}{dx} = \frac{y - x}{y + x}. \qquad [2, 3]$$

By making the substitution $y = vx$, or otherwise, find the relationship between x and y. [8]

Find, in terms of u, the magnitude of the velocity of M_2 relative to M_1. [2]

3. Two small perfectly elastic smooth spheres S_1 and S_2 of equal radii, but of masses m_1 and m_2 respectively, are moving directly towards each other on a smooth horizontal plane with speeds u and v respectively. Show that, after collision, the speed of S_2 will not exceed $2u + v$ and that the sphere S_1 will be brought to rest by the collision provided that $2m_2v = (m_1 - m_2)u$. [5, 2]

The two spheres are now set in motion in a straight line, each with constant speed v, towards a smooth perfectly elastic vertical wall, with S_1 being ahead of S_2. The wall is perpendicular to the direction of motion of the two spheres. After colliding with the wall the sphere S_1 collides with the sphere S_2, whose direction of motion is reversed and whose speed after collision is U. Show that $U < 3v$. [1]

Given further that the masses of the spheres are such that the sphere S_1 is brought to rest by its collision with S_2 find, by energy considerations, or otherwise, U in terms of v. [3]

In this latter case S_2 then collides directly with a perfectly elastic sphere S_3 of the same radius, but of different mass, and moving towards it with speed v. In this latter collision S_2 is brought to rest. Find the speed of S_3 immediately after this collision. [4]

4. At time $t = 0$ a ship A is at the point O and is moving northwards with constant speed u. At the same time a second ship B is at a point a distance a directly east of O and is moving with a constant velocity. At time $T/4$, a shell fired from ship A at time $t = 0$ passes directly over ship B. The velocities of the ships are such that, if they did not change course, they would collide when $t = T$. Given unit vectors $\mathbf{i}$ and $\mathbf{j}$, due east and due north respectively, find the horizontal component of the velocity of the shell relative to ship A. [5]

Suppose instead that the constant speed of ship A is $3u$ and that the ships are closest together when $t = T/2$. Find u in terms of a and T. [10]

5. A particle is projected from a fixed point with speed u at an angle α above the horizontal, and its subsequent horizontal and vertical displacements from the point of projection are denoted by x and y respectively. Show that

$$y = x \tan \alpha - \frac{gx^2 \sec^2 \alpha}{2u^2}. \qquad [2]$$

A particle P is attached to one end of a light inextensible string of length a, the other end of which is attached to a fixed point O. Initially the particle is in equilibrium, suspended from O, at the point B and it is then projected with a horizontal speed u. The particle initially moves in a circle but at a particular point of its path the string becomes slack and the particle then moves in a parabolic path which passes through B. Find the angle between the string and the upward vertical when the string becomes slack and also determine u. [8, 2]

Find the maximum height that the particle reaches above B after the string tightens. [3]

Section B

Probability and Statistics

No more than **four** *questions should be answered from this section.*

6. In a certain country 55% of the adults are female, 25% of the adults are senior citizens, and 40% of the adult females are senior citizens. Three adults are chosen at random from the adults in the country. Calculate the probability that they will include at least one female and at least one senior citizen. [7]

In the same country there are only two political parties, the *Reds* and the *Semidemocrats*. The country is divided into 600 constituencies each having 40 000 voters. At an election each party nominates one candidate per constituency. Assuming that in each constituency every eligible person will vote and that, independently, each person will vote for the *Red* candidate with probability 0·501, calculate the probability that the *Reds* will win at least 400 of the constituencies. [8]

7. The number of traffic accidents occurring in a month along a particular stretch of motorway may be assumed to have a Poisson distribution with mean μ. Independently for each such accident the probability that no person is killed is p.

(i) In a month when there were n accidents write down an expression for the probability that no person was killed in r of the accidents. Hence, or otherwise, show that in a month the number of accidents in which no person is killed has a Poisson distribution with mean $p\mu$. [7]

(ii) In a month when there were r accidents in which no person was killed find an expression for the probability that there were n accidents in that month. Hence find the expected number of accidents in a month when there were r accidents in which no person was killed. [8]

8. (*a*) Spherical ball bearings are produced on two machines, A and B. The bearings produced on machine A have diameters that are normally distributed with mean 27 mm and standard deviation 2 mm, while those produced on machine B have diameters that are normally distributed with mean 31 mm and standard deviation 2·5 mm. In any one shift, machine A produces twice as many ball bearings as machine B. All the bearings produced in a shift are then sorted by size into three batches, one consisting of bearings whose diameters are less than 27·5 mm, one consisting of bearings whose diameters are greater than 34·0 mm, and the remaining batch consisting of bearings whose diameters are in the interval from 27·5 mm to 34·0 mm. Show that the batch of bearings having diameters in the interval from 27·5 mm to 34·0 mm consists of almost equal numbers of bearings produced on the two machines. [7]

(*b*) The continuous random variable X is uniformly distributed over a finite interval which includes the value zero, and is such that its mean and standard deviation have equal values. Show that the probability that the product of four randomly observed values of X will be negative is equal to $\frac{4}{9}$. [8]

9. A rod of length 2 units is cut at a randomly chosen point, so that the distance, X, of the cutting point from one end of the rod is uniformly distributed over the interval $(0, 2)$. Let Y denote the ratio of the length of the shorter piece to the longer piece.

(i) Find the expected value of Y. [5]

(ii) Show that for $0 < y < 1$,

$$P[(Y \leqslant y) \cap (0 < X < 1)] = \frac{y}{1 + y},$$

and find the corresponding result for $P[(Y \leqslant y) \cap (1 < X < 2)]$. [4]

Hence, or otherwise, determine the cumulative distribution function of Y. Evaluate the probability that the median of three independently observed values of Y will be less than $\frac{1}{3}$. [6]

10. An engineer knows that the two variables x and y are linearly related in the form $y = \alpha + \beta x$, where α and β are unknown. The engineer also knows that it is possible to conduct experiments with x having prespecified values, the resulting measured values of y being subject to independent random errors that are normally distributed with mean zero and standard deviation σ. In order to estimate the relationship the engineer plans to conduct five experiments with x having the values 1, 2, 3, 4, 5, respectively, and to measure the corresponding values of y. Denoting the values of y that will be obtained by Y_1, Y_2, Y_3, Y_4, and Y_5, respectively, the engineer (being unaware of the method of least squares) reasons that a suitable estimator of the relationship is of the form

$$y = \bar{Y} + B(x - \bar{x}),$$

where $\bar{x} = 3$ is the mean of the x-values, $\bar{Y}$ is the mean of the measured y-values, and B is an appropriately chosen estimator of β. The engineer reasons that β can be sensibly estimated by any one of the differences $(Y_2 - Y_1)$, $(Y_3 - Y_2)$, $(Y_4 - Y_3)$, or $(Y_5 - Y_4)$. State why each of these is a sensible estimator of β.

After some further thought, the engineer decided to take the estimator of β to be the average of the above four estimators; that is to take

$$B = \tfrac{1}{4}\{(Y_2 - Y_1) + (Y_3 - Y_2) + (Y_4 - Y_3) + (Y_5 - Y_4)\}.$$

Verify that this B is an unbiased estimator of β and show that its variance is 25% more than that of the least squares estimator of β. [6]

The engineer is particularly interested in estimating the true value of y when $x = 2$. Express the engineer's estimator of this true value of y as a linear combination of Y_1, Y_2, Y_3, Y_4, and Y_5 and hence find the variance of the estimator. [5]

Given that $\sigma = 1{\cdot}2$ and that the engineer's calculated estimate of the relationship is

$$y = 25{\cdot}44 + 2{\cdot}32\,(x - 3),$$

determine 95% confidence limits for the true value of y when $x = 2$. [4]