WELSH JOINT EDUCATION COMMITTEE | CYD-BWYLLGOR ADDYSG CYMRU

General Certificate of Education | Tystysgrif Addysg Gyffredinol

Advanced Level | Safon Uwch

MATHEMATICS A1

Pure Mathematics

P.M. WEDNESDAY, 7 June 1989

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Differentiate the following functions with respect to x:

(i) $x \sin x$,

(ii) $\tan^2 x$.

P2 ✓

[3]

2. Find all values of x lying between 0° and 360° satisfying the equation

$$3 \sin x + 4 \cos x = 1.$$

✗

Give your answers correct to the nearest degree. [5]

3. Express

$$\frac{x+1}{x^2(x-1)}$$

P2 ✓

in partial fractions. [4]

4. Use Simpson's Rule with 5 ordinates and an interval of 0·25 to find an approximate value of the integral

$$\int_0^1 \sqrt{x^3 + 1}\,dx.$$

✗

Give your answer correct to two decimal places. [5]

5. The complex numbers z_1, z_2 and z_3 are related by

$$z_3 = 1 + z_1^2 + \frac{5z_2}{z_1}.$$

X

In the particular case $z_1 = 2 + i$ and $z_2 = 3 + 2i$,

(i) express z_3 in the form $x + iy$,

(ii) calculate the modulus and argument of z_3. [5]

6. Determine the range of values of k for which the quadratic equation

$$x^2 + 2kx + 4k - 3 = 0$$

P1

has two real and distinct roots. [5]

7. The polynomial

$$x^3 + ax^2 + bx + 6$$

P2

is exactly divisible by $(x - 2)$ and has remainder 4 when divided by $(x - 1)$.

Calculate the values of the constants a, b and express the polynomial as a product of three linear factors. [6]

8. The vertices A, B, C of a triangle have position vectors **a**, **b**, **c** respectively. The points D, E lie on BC, AC respectively and $AE : EC = CD : DB = 2 : 1$. The lines AD and BE intersect at the point F. Show that the position vector of F is

X

$$\tfrac{1}{7}\mathbf{a} + \tfrac{4}{7}\mathbf{b} + \tfrac{2}{7}\mathbf{c}.$$ [7]

SECTION B

Answer **four** *questions from this section.*

9. (*a*) A man invests £A at the beginning of **each** year for n years, where n is a positive integer. Compound interest accumulates at the rate of R% per annum and is calculated at the end of each year.

Show that the total value, £S_n, of the investment n years after the initial deposit is given by

$$S_n = A\left(\frac{100}{R} + 1\right)\left[\left(1 + \frac{R}{100}\right)^n - 1\right].$$

P1/P2

Suppose that $R = 8$.

(i) The man requires his investment to be worth £50 000 after 10 years. Find, correct to the nearest pound, the value of A required to achieve this.

(ii) His wife decides that they can afford to invest £1000 per year. In how many years will the value of this investment reach £50 000?

[8]

(*b*) An arithmetic series has a positive common difference and the sum and product of the first 5 terms are respectively 40 and 12 320.

Show that there are two series which satisfy these conditions and find the two possible values of the first term and of the common difference. [7]

P1

10. The point $P(x, y)$ moves in such a way that its distance from the point $C(-g, -f)$ is always equal to r. Show that the equation of the circle described by P is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{where} \quad c = g^2 + f^2 - r^2. \qquad [3]$$

The line $y = mx$ cuts this circle at the points R and S. Show that the x co-ordinates of R and S satisfy the quadratic equation

$$(1 + m^2)x^2 + 2(g + fm)x + c = 0.$$

Deduce that the line is a tangent to the circle if

$$(f^2 - c)m^2 + 2fgm + g^2 - c = 0. \qquad [5]$$

Hence find the equations of the two tangents from the origin to the circle having the equation

$$x^2 + y^2 + 6x + 2y + 2 = 0$$

P2

and calculate

(i) the acute angle between the two tangents,

(ii) the co-ordinates of the points of contact of the two tangents. [7]

11. (a) Assuming the formula for $\tan(A + B)$, show that for $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant 1$,

$$\tan^{-1}a + \tan^{-1}b = \tan^{-1}\left(\frac{a + b}{1 - ab}\right).$$

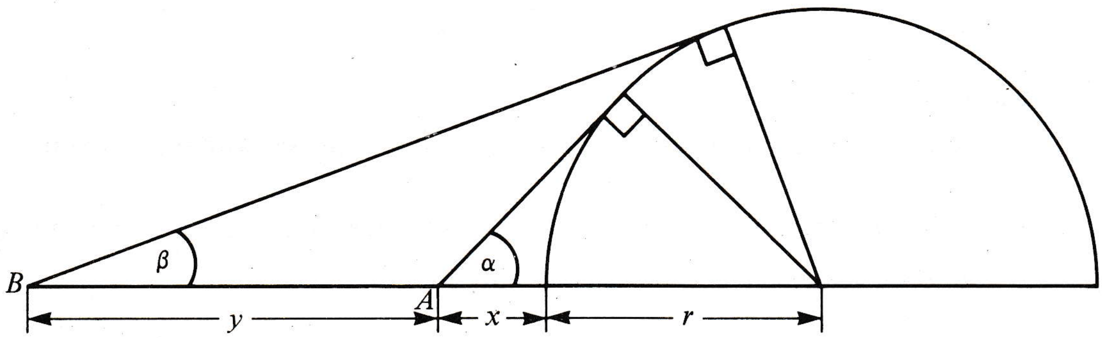
?

Hence find the positive values of x satisfying the equations

(i) $\tan^{-1}2x + \tan^{-1}3x = \dfrac{\pi}{4}$,

(ii) $\tan^{-1}2x + \tan^{-1}3x = \dfrac{\pi}{2}$. [8]

(b)



The figure shows the vertical cross-section through the centre of a hemispherical dome of radius r. From the point A, the elevation of the highest visible point on the dome is α. From the point B, a distance y further from the dome, the elevation of the highest visible point on the dome is β.

By considering $\sin\alpha$ and $\sin\beta$, show that

X

$$r = \frac{y}{\operatorname{cosec}\beta - \operatorname{cosec}\alpha}.$$

Suppose that α and r are fixed. Show that if y is increased by a small amount δy, then β is decreased by approximately

$$\frac{\delta y \sin\beta \tan\beta}{r}. \qquad [7]$$

12. A pyramid has a square base $ABCD$ of side a and its vertex is E. Each sloping face makes an angle $\frac{\pi}{3}$ with the base.

(i) Find the height of the pyramid.

(ii) Find the length of each sloping edge.

(iii) Find the angle between each sloping edge and the base.

(iv) Show that

$$\sin E\hat{A}B = \frac{2}{\sqrt{5}}.$$ [7]

P is a point on BE such that AP and CP are both perpendicular to BE.

(v) Calculate the length of AP.

(vi) Show that the angle between adjacent sloping faces is $\cos^{-1}(-\frac{1}{4})$. [6]

Is it possible to construct a pyramid on a square base whose adjacent sloping faces are perpendicular? Explain your answer briefly. [2]

13. **(a) By writing $(x+h)$ in the form $x(1+\frac{h}{x})$, show that for h sufficiently small,**

$$(x+h)^n - x^n = nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots.$$ P1

Deduce the derivative of x^n. **[4]**

(b) The function f is defined on the domain [0, 1] by

$$f(x) = \frac{2x^2}{1+x^2}.$$

Obtain an expression for $f'(x)$ in its simplest form and show that P1

$$f''(x) = \frac{4(1-3x^2)}{(1+x^2)^3}.$$

Hence find

(i) the co-ordinates of the point at which $f''(x) = 0$ and show that this point is a point of inflection,

(ii) the largest and smallest values of $f'(x)$, stating the values of x for which these occur.

State the range of f and sketch its graph.

[8]

The composite function g is defined by $g = f \circ f$. State the domain and range of g and obtain an expression for $g(x)$. **[3]**

14. A hollow circular cylinder has height h and base radius r. Write down expressions for the volume V and surface area A (including both circular ends). Deduce that

$$A = 2\pi r^2 + \frac{2V}{r}.$$

Hence show that, for fixed V, the minimum value of A is given by

P1 or P2

$$A_{\text{min}} = 3(2\pi V^2)^{\frac{1}{3}}.$$

Find the ratio of $h:r$ which gives this minimum value. [7]

A particular cylinder of volume V is constructed with $h = r$. Calculate its surface area and verify that this is approximately 6% more than A_{min}. [3]

The dimensions of a further cylinder are such that

$$A^3 = 240\,V^2.$$

Writing $h = kr$, show that

$$k^3 + (3 - \frac{30}{\pi})k^2 + 3k + 1 = 0.$$

Given that this cubic equation has three real roots, show that one root is negative, and of the two positive roots, one is less than unity and the other greater than unity. ✗ [5]

15. Use the binomial expansion to show that for x sufficiently small,

$$\frac{x^2}{\sqrt{4 - x^2}} \approx \tfrac{1}{2}x^2 + \tfrac{1}{16}x^4.$$ ✗ [3]

Use this approximation to show that the value of the integral

$$I = \int_0^1 \frac{x^2\,\mathrm{d}x}{\sqrt{4 - x^2}}$$

is approximately $\frac{43}{240}$. [2]

Putting $x = 2\sin\theta$, show that

$$I = 4\int_0^{\pi/6} \sin^2\theta \,\mathrm{d}\theta.$$

P2

Hence show that the value of I is $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}$. [5]

Using integration by parts, or otherwise, evaluate the integral

P2

$$\int_0^1 \frac{x^4\,\mathrm{d}x}{(4 - x^2)^{\frac{3}{2}}}$$

giving your answer correct to three decimal places. [5]

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General Certificate of Education

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MATHEMATICS A2

Mechanics

A.M. WEDNESDAY, 14 June 1989

(3 Hours)

Answer **all** *questions in Section A and* **four** *from Section B.*

A calculator may be used except where it is expressly forbidden.

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

SECTION A

Answer **all** *questions in this section.*

1. A particle P describes simple harmonic oscillations of amplitude 7 cm and period 4 s. Find the maximum speed of P and its speed when at a distance of 3 cm from the centre of oscillation. [4]

2. Show that an appropriate integrating factor for the differential equation

$$\frac{dy}{dx} + \frac{3y}{x} = x^3, \quad x > 0,$$

is x^3. Hence, or otherwise, find the solution for which $y = 1$ when $x = 1$. [5]

3. A uniform square lamina $ABCD$ of side $5a$ is marked out into 25 equal small squares. The small square with one corner at C is removed. Find the distances from AB and AD of the centre of mass of the resulting lamina. [5]

4. At time $t = 0$ a particle is projected vertically upwards from a point O with speed $19{\cdot}6 \text{ m s}^{-1}$ and, two seconds later, a second particle is projected vertically upwards from O with the same speed. Assuming that the only force acting is that due to gravity, express the heights above O of both particles in terms of t and hence, or otherwise, find the value of t when they collide. [4]

Find the speeds of the particles at the instant of collision. [3]

5. $ABCD$ is a rectangle with sides AB and CD 3 m long and sides BC and AD 4 m long. Forces of magnitude 11 N, 3 N, 15 N and 7 N act along BA, BC, CA and CD respectively, the sense of the force being that indicated by the order of the letters. Find the moment of this system of forces about C and the components parallel to BA and BC of the single resultant force to which this system can be reduced. [1, 4]

Find the distance from C of the point at which this resultant intersects BC. [2]

6. At time t seconds the position vector $\mathbf{r}$, measured in metres, of a particle of mass 0·3 kg is given by

$$\mathbf{r} = \cos 3t\,\mathbf{i} + \sin 2t\,\mathbf{j}.$$

Find the force acting on the particle at time t seconds and the rate at which this force is working. [4, 2]

7. A smooth wire, on which a small bead B of mass m is threaded, is formed into a circle of radius a and fixed in a vertical plane. The bead is projected from the lowest point of the circle with speed $(8ag)^{\frac{1}{2}}$. Show that, when the direction of motion of B has turned through an acute angle θ, the square of its speed is $6ag + 2ag\cos\theta$. [4]

Find, in terms of m, g and θ, the reaction of the wire on the bead. [2]

SECTION B

Answer **four** *questions from this section.*

8. Two particles P and Q move with constant velocities $(5\mathbf{i} + k\mathbf{j})\,\mathrm{m\,s^{-1}}$ and $(4\mathbf{i} + 3\mathbf{j})\,\mathrm{m\,s^{-1}}$, respectively, where k is a constant. At time $t = 0$ the position vectors, in metres, of P and Q are $4\mathbf{i} + 2\mathbf{j}$ and $7\mathbf{i} + 14\mathbf{j}$, respectively. Find the position vector of Q relative to P at time t seconds. [3]

(i) Find the value of k such that P and Q will collide and the value of t when this collision occurs. [3]

Given that the mass of P is twice that of Q and that the particles coalesce on collision find their common velocity immediately after collision. [2]

(ii) In the particular case when $k = 4$ show that

$$PQ^2 = 2t^2 - 30t + 153$$

and find the value of t when P and Q are closest together and their distance apart at this time. Find also the length of time for which the distance between P and Q is less than or equal to 9 m. [4, 3]

9. At a point O, coordinate axes Ox, Oy are chosen with Ox horizontal and Oy vertically upwards. At time $t = 0$ a particle P is projected from O with velocity components $8\,\mathrm{m\,s^{-1}}$ and $24{\cdot}5\,\mathrm{m\,s^{-1}}$ parallel to Ox and Oy respectively. Assuming that the only force acting is that due to gravity, write down the horizontal and vertical displacements of P from O at any subsequent time t seconds. Find the time taken for P to reach its highest point above O and the horizontal displacement of this point from O. [5]

Let $\mathbf{g}\,\mathrm{m\,s^{-2}}$ denote the acceleration vector due to gravity and $\mathbf{v}\,\mathrm{m\,s^{-1}}$ be the velocity of P then, when air resistance is taken into account, the acceleration of P is $(\mathbf{g} - 2\mathbf{v})\,\mathrm{m\,s^{-2}}$. Write down the differential equations satisfied by the horizontal and vertical components of velocity. Show that, in this case, the time taken for P to reach its highest point is $\frac{1}{2}\ln 6$ s. [6]

Find the speed of P at this instant. [4]

10. Find the solution of

$$\frac{d^2x}{dt^2} = 16x$$

with $x = a$ and $\frac{dx}{dt} = -b$ at $t = 0$, where a and b are positive constants. [6]

Verify that, if $a > \frac{b}{4}$ then x will never become zero and $\frac{dx}{dt}$ will be zero for a positive value of t. [3]

It is assumed, in a simple war game, that there are two military forces, the 'A-force' and the 'B-force', engaged in combat. The numbers in the A-force and B-force at time t are denoted by x and y respectively. It is also assumed that the rate at which the number in the A-force is decreasing is equal to twice the number in the B-force at that time and that the number in the B-force is decreasing at a rate equal to eight times the number in the A-force. Show that x satisfies the above differential equation. [4]

The game is won when one of the forces is annihilated and the other is not. Given that x_0 and y_0 denote the numbers in the A-force and B-force respectively at time $t = 0$, find a condition on x_0 and y_0 which will ensure that the A-force wins the game [2]

11. Two small smooth spheres P and Q of equal radii but of masses m and $3m$ respectively are moving towards each other on a smooth horizontal table. Before collision the speeds of P and Q are $3u$ and $6u$ respectively and after collision the direction of motion of P is reversed and it moves with speed $5u$. Find the impulse on Q, the speed of Q after the collision and the coefficient of restitution between P and Q. [6]

After collision, P moves with constant speed $5u$ until it catches up, and collides directly, with a third sphere S which is identical to P and which is moving in the same direction with speed $2u$. The kinetic energy lost in this second collision is $2mu^2$. Find the speed of P immediately after colliding with S. [The coefficient of restitution between P and S is not the same as that between P and Q.] [9]

12. The magnitude of the air and road resistance on a vehicle of mass 1 000 kg is proportional to v^n where v denotes its speed and n is an integer. When the vehicle is ascending a hill of inclination $\sin^{-1}(\frac{1}{7})$ at a steady speed of 5 m s^{-1} the engine is working at a rate of 22 kW. When the engine is working at a steady rate of 74 kW the vehicle ascends the same hill at a steady speed of 10 m s^{-1}. Show that the only possible value of n is 1. [5]

When the vehicle moves on a level road it is working at a constant rate of 40 kW. At time t the kinetic energy is E joules and the speed is v m s^{-1}. Express $\frac{dE}{dt}$ in terms of v and $\frac{dv}{dt}$, and express $\frac{dv}{dt}$ in terms of v. Hence show that

$$\frac{dE}{dt} = a + bE,$$

where the numerical values of a and b are to be determined. [6]

[It is to be assumed that the air and road resistance is the same function of speed as it is on the hill.]

Given that the time taken for the kinetic energy to increase from 1000 J to 30 000 J is 1·89 s find the work done by the air and road resistance during this period. [4]

13. Two particles P and Q, of masses $4m$ and $5m$ respectively, are attached one to each end of a light inextensible string which passes over a small smooth pulley. The particles move in a vertical plane with both hanging parts of the string vertical and they are released from rest. Find, in terms of m and/or g as appropriate, the magnitudes of the accelerations of the particles and the tension in the string. [6]

When the particle P is moving upwards with speed V it picks up, from rest at a point A, an additional particle of mass $2m$ so as to form a composite particle R of mass $6m$. Find the initial speed of R. [2]

When the composite particle R descends back to A the particle of mass $2m$ is removed, without applying an impulse to the remaining particle P. The cycle is then repeated with the particle of mass $2m$ being picked up from rest and added to P on the upward journey through A and removed on the downward journey.

Find the time that elapses between

(i) P's first upward and first downward passages through A,

(ii) P's first and second upward passages through A,

(iii) P's second and third upward passages through A. [7]

14.

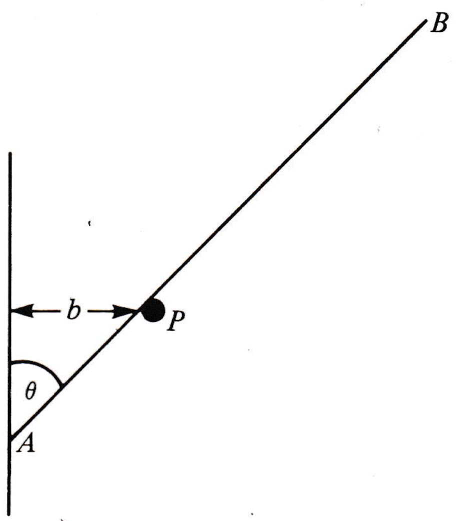

Fig. 1

Fig. 1 shows a uniform rod AB of length $2a$ and weight W, in equilibrium, resting on a rough peg P. The lower end A of the rod is in contact with a smooth vertical wall and the points A, P and B are in the same vertical plane perpendicular to the wall. The angle between AB and the wall is denoted by θ and b is the perpendicular distance of P from the wall. Find, in terms of W, a, b and θ, the reaction of the wall at A and the component normal to AB of the reaction of P. Deduce that equilibrium is only possible for $b \leqslant a \sin \theta$. [6]

Show that the frictional force at P along AB, in the sense from A to B, is equal to

$$\frac{W(b - a\sin^3\theta)}{b\cos\theta}.$$ [2]

Deduce that

$$a\sin^2\theta\,(\sin\theta - \mu\cos\theta) \leqslant b \leqslant a\sin^2\theta\,(\sin\theta + \mu\cos\theta),$$

where μ is the coefficient of friction. [4]

Give a physical interpretation of the situation when b takes the lower value in the above inequality. [1]

Find the range of μ for which equilibrium is possible when b takes the upper value in the above inequality. [2]

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MATHEMATICS A3

Probability and Statistics

P.M. MONDAY, 19 June 1989

(3 Hours)

Answer **all** *questions in Section A* and **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. The discrete random variable X has the following distribution.

$x =$	1	2	3
$P(X = x) =$	α	$0{\cdot}8 - \alpha$	$0{\cdot}2$

Given that the mean of the distribution is equal to $1{\cdot}7$, determine the value of α. [3]

2. Three cards are to be drawn at random without replacement from a pack of ten cards. Six of the cards in the pack are red and numbered from 1 to 6, respectively, while the other four cards are blue and numbered from 1 to 4, respectively. Calculate the probabilities that

(i) exactly two red cards will be drawn, [2]

(ii) exactly one 2 will be drawn. [2]

3. A doubleglazing salesperson canvasses customers to purchase replacement windows and patio doors. The salesperson receives a commission of 10% of the total order value for replacement windows, and a commission of 15% of the total order value for patio doors. In any month, the salesperson's total order value for replacement windows is a random variable having mean £2500 and standard deviation £500, and independently, the salesperson's total order value for patio doors is a random variable having mean £1200 and standard deviation £250. Find the mean and the standard deviation of this salesperson's total monthly commission. [5]

4. In the first trial of a random experiment the probability of a successful outcome is $\frac{2}{5}$. In the second trial the probability of a successful outcome will be $\frac{1}{5}$ if the outcome of the first trial was successful, and will be $\frac{4}{5}$ if the outcome of the first trial was not successful.

Determine the probability distribution and the mean value of the number of successful outcomes that will be obtained in the first two trials of the random experiment. [5]

5. The random variable X is normally distributed with mean μ (> 0) and standard deviation $0{\cdot}1\,\mu$.

(i) If $\mu = 5$ find the probability, correct to three decimal places, that a randomly observed value of X will be greater than $5{\cdot}5$. [2]

(ii) Find the value of μ, correct to two decimal places, if the probability of a randomly observed value of X being greater than 30 is equal to $0{\cdot}4$. [3]

6. In each trial of a random experiment the probability that the event A will occur is $0{\cdot}6$ and the probability that the event B will occur is $0{\cdot}5$. It is known that A and B are independent.

(i) Calculate the probability that at least one of A and B will occur in a single trial. [2]

(ii) Using the tables provided, or otherwise, find the probability that in twenty independent trials the event A will occur exactly twelve times; give your answer correct to three decimal places. [3]

7. A soil scientist wishing to estimate the mean pH (a measure of acidity) of the soil in a particular area randomly selected 6 samples of soil from the area. The sample pH measurements were:

$$6{\cdot}7,\ 6{\cdot}5,\ 6{\cdot}6,\ 6{\cdot}7,\ 6{\cdot}3,\ 6{\cdot}5.$$

Assuming that the pH measurements are normally distributed, calculate a 90% confidence interval for the mean pH of the soil in the area. [6]

8. The continuous random variable X is distributed with probability density function f, where

$$f(x) = \frac{3x^2}{\alpha^3}, \qquad \text{for } 0 \leqslant x \leqslant \alpha,$$

$$f(x) = 0, \qquad \text{otherwise.}$$

(i) Find the mean and the variance of X in terms of α. [3]

(ii) Let $\bar{X}$ denote the mean of a random sample of 15 observations of X.

Show that $T = \frac{4}{3}\bar{X}$ is an unbiased estimator of α and find its standard error in terms of α. [4]

SECTION B

Answer **four** *questions from this section.*

9. (*a*) A committee of 9 persons is to be selected from a group of 13 persons, of whom 7 are males and 6 are females. Calculate the probability that more females than males will be selected in **each** of the cases when

(i) the selection is made randomly,

(ii) the selection is made randomly subject to there being at least 4 females on the committee. [7]

(*b*) The three events A, B and C, are such that

$$P(A) = \frac{1}{5}, \quad P(B) = \frac{1}{10}, \quad P(A \cup C) = \frac{7}{15}, \quad P(B \cup C) = \frac{23}{60}$$

and the events A and C are independent.

(i) Show that $P(C) = \dfrac{1}{3}$.

(ii) Determine whether or not B and C are independent.

(iii) Given further that A and B are mutually exclusive, evaluate $P(A \cup B \mid C)$. [8]

10. The numbers, X and Y, of two types of bacteria A and B, respectively, in a given volume of a solution are independent random variables having Poisson distributions. The mean number of type A bacteria per litre of the solution is 3·6, while the mean number of type B bacteria per litre of the solution is 1·24.

(*a*) Find, correct to three decimal places, the probabilities that a sample of 1 litre of the solution will contain

(i) exactly 4 bacteria of type A,

(ii) 3 or fewer bacteria of type B,

(iii) exactly 2 bacteria of type A and exactly 2 bacteria of type B. [7]

(*b*) Let $W = XY$.

(i) Find, correct to three decimal places, the probability that for a sample of 1 litre of the solution the value of W is zero. [4]

(ii) Evaluate the mean and the variance of W. [4]

11. The continuous random variable X has probability density function f given by

$$f(x) = a(x - b), \qquad \text{for } 2 \leqslant x \leqslant 5,$$
$$f(x) = 0, \qquad \text{otherwise},$$

where a and b are constants.

(i) Given that a is positive write down the largest possible value that b may have. [1]

(ii) By integrating $f(x)$ over an appropriate interval, show that

$$21a - 6ab = 2.$$ [2]

(iii) Given that the mean of the distribution is 3·8, find the values of a and b, and show that the variance of the distribution is equal to 0·66. [6]

(iv) Find the cumulative distribution function of

$$Y = \frac{X}{X - 1}.$$ [6]

12. Let X denote the number of successes that will be obtained in n independent trials in **each** of which the probability of a success is p.

(i) If $n = 20$ and $p = 0{\cdot}8$ find, correct to three decimal places, the probability that X will have a value from 12 to 16, inclusive. [3]

(ii) If $n = 200$ and $p = 0{\cdot}8$, find an approximate value for the probability that X will be 165 or less, giving your answer correct to three decimal places. [4]

(iii) If $n = 200$ and $p = 0{\cdot}01$, find an approximate value for the probability that $X = 1$, giving your answer correct to three decimal places. [3]

(iv) If $n \geqslant 2$ and $0 < p < 1$, show that one and only one of the two statistics

$$T_1 = \left(\frac{X}{n}\right)^2 \quad \text{and} \quad T_2 = \frac{X(X-1)}{n(n-1)}$$

is an unbiased estimator of p^2. [5]

13. A fixed length of steel wire was subjected to forces of 300, 400, 500, 600 and 700 Newtons, respectively. The measured lengths, in centimetres, of the stretched wire are shown in the following table. It may be assumed that the errors in the measured lengths are independent and normally distributed with mean zero and standard deviation 0·01 cm.

Force (x N)	300	400	500	600	700
Length (y cm)	10·35	10·46	10·58	10·71	10·80

You are given that

$$\sum y = 52{\cdot}90 \quad \text{and} \quad \sum xy = 26565.$$

According to Hooke's Law, when wire of natural length α cm is subjected to a force of x Newtons its stretched length, y cm, is given by $y = \alpha + \beta x$, where β is a constant dependent upon some properties of the wire.

(*a*) Calculate the least squares estimates of α and β. [5]

(*b*) Calculate 95% confidence limits for

(i) the natural length α of the wire used in the experiment, [5]

(ii) the stretched length of the wire when it is subjected to a force of 550 Newtons. [5]

14. A garage stocks two brands, A and B, of car batteries. The lifetimes, X months, of brand A batteries may be assumed to be distributed as N (μ, σ^2).

(i) Given that 1% of brand A batteries last less than 40 months and that 25% last more than 46 months, find the values of μ and σ; give your answers correct to two decimal places. [5]

The lifetimes, Y months, of brand B batteries may be assumed to be normally distributed with mean 40 and standard deviation 4. A particular person always buys a replacement car battery at the garage. Given the choice, this person will buy a brand A battery but will accept a brand B battery if only brand B batteries are in stock at the time. Suppose that at any point in time the probability of the garage having only brand B batteries in stock is 0·6.

(ii) Calculate, to three decimal places, the probability that a battery bought by the above person at the garage will last for more than 46 months. [4]

(iii) Let T_A denote the sum of the lifetimes of a random sample of five batteries of brand A, and let T_B denote the sum of the lifetimes of a random sample of six batteries of brand B. Taking the mean (μ) and the standard deviation (σ) of the lifetimes of brand A batteries to be 45 months and 2 months, respectively, calculate the probability that T_A and T_B will not differ by more than 10 months, giving your answer correct to three decimal places. [6]

15. The following table shows the values of the joint probability $P(X = x,\ Y = y)$ for two discrete random variables X and Y.

		x 1	2	3
y	1	0·1	0·15	0·25
	2	p	0·15	0·20
	4	0·05	p	0

(i) Find the value of p. [2]

(ii) Show that $P(X = 1,\ Y = 1) = P(X = 1)P(Y = 1)$. [2]

(iii) Determine whether or not X and Y are independent. [2]

(iv) Evaluate $P(XY > X + Y)$. [4]

(v) Given that (X_1, Y_1) and (X_2, Y_2) are two independent observations of (X, Y), evaluate $P(X_1Y_1 = 3X_2Y_2)$. [5]

WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

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Safon Uwch

MATHEMATICS A4

Further Pure Mathematics

A.M. FRIDAY, 23 June 1989

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Given that

$$y = x^{\frac{1}{x}} \quad (x > 0)$$

obtain an expression for $\dfrac{dy}{dx}$ in terms of x. [4]

2. Find the three real roots of the equation

$$|x^2 + x - 1| = |x^2 + 3x - 5|.$$ [4]

3. Find the co-ordinates of the point of intersection of the line

$$\frac{x-1}{2} = \frac{y+1}{3} = z - 2$$

and the plane

$$x + 2y - z = 11.$$ [4]

4. Show that

$$\int_0^1 \frac{dx}{x^2 + 4x + 1} = \frac{1}{2\sqrt{3}} \log_e(2 + \sqrt{3}).$$ [5]

5. The function f is defined for all real values of x by

$$f(x) = 2x \quad \text{for } x < 0$$

$$f(x) = ax^3 + bx^2 + cx + d \quad \text{for } 0 \leqslant x \leqslant 2$$

$$f(x) = x \quad \text{for } x > 2.$$

Calculate the values of the constants a, b, c, d if both f and its derivative are continuous for all values of x. [5]

6. Express the complex number $2 + 2\mathrm{i}$ in polar form. Hence find the three values of $(2 + 2\mathrm{i})^{\frac{2}{3}}$, expressing your answers in the form $x + y\mathrm{i}$. Verify that the sum of these three values is zero. [6]

7. Using integration by parts, or otherwise, show that

$$\int_0^{\pi/2} \sin x \sinh x \, \mathrm{d}x = \frac{1}{2} \cosh \frac{\pi}{2}. \qquad [6]$$

8. The roots of the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

are denoted by α, β, γ. Show that the cubic equation whose roots are $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ is given by

$$a^2x^3 - acx^2 + bdx - d^2 = 0. \qquad [6]$$

SECTION B

Answer **four** *questions from this section.*

9. **Show that the equation of the chord joining the points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ on the rectangular hyperbola $xy = c^2$ is given by**

$$pqy + x = c(p + q).$$

Hence obtain the equation of the tangent at P. [4]

The line PQ meets the co-ordinate axes at A and B. Show that the mid-point M of AB is also the mid-point of PQ. [3]

The tangents at P and Q meet at T. Show that MT passes through the origin. [4]

The normal to the rectangular hyperbola at P meets the curve again at R. Find the co-ordinates of R. [4]

10. **(i) By writing $a^x = \mathrm{e}^{x \log_e a}$ (where $a > 0$), show that**

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \log_e a. \qquad [2]$$

(ii) Sketch the curve $y = 2^x$. Given that the area of the region enclosed between the curve, the two co-ordinate axes and the line $x = b$ $(b > 0)$ is equal to unity, obtain an expression for b in terms of natural logarithms. Evaluate b correct to two decimal places. [6]

(iii) Show that the equation

$$2^x = 3 + x^2$$

has a root between 4 and 5. By drawing another curve on your sketch, show that this is the only real root. Taking 4·5 as the initial approximation, use the Newton-Raphson method to find the value of this root correct to two decimal places. [7]

11. (*a*) Express

$$\frac{1}{r(r+1)(r+2)}$$

in partial fractions and hence show that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}.$$

Write down the limiting value of this sum as $n \to \infty$. [7]

(*b*) Use the method of mathematical induction to prove that

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}.$$

Hence, or otherwise, find an expression in its simplest form for

$$\sum_{r=1}^{2n} (-1)^r r^3$$ [8]

12. (*a*) The function f is defined by

$$f(x) = \frac{px+q}{x+r}$$

where p, q, r are constants.

(i) Find the condition on p, q, r for f to be an even function and show that in this case $f(x)$ is equal to a constant for all x.

(ii) Find an expression for $f^{-1}(x)$ and obtain the condition on p, q, r which makes

$$f(x) = f^{-1}(x) \quad \text{for all } x.$$ [6]

(*b*) The function g is defined by

$$g(x) = \frac{3x+1}{x+2}.$$

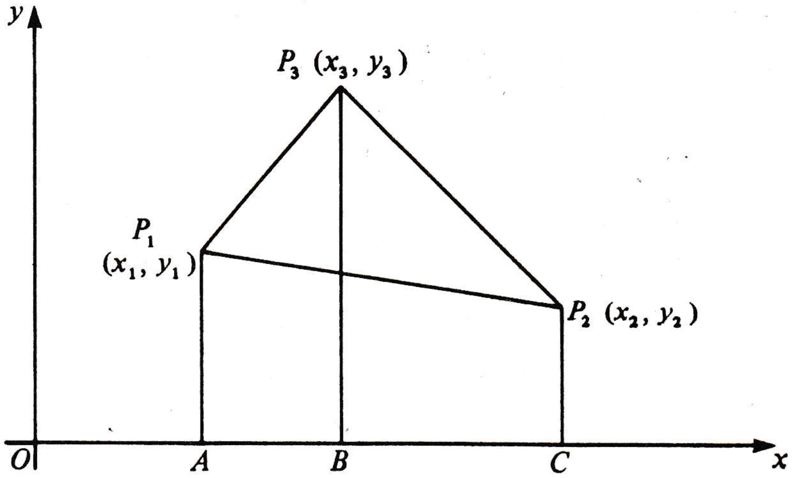
(i) Sketch the graph of g.

(ii) Obtain an expression for $g^{-1}(x)$ and sketch the graph of g^{-1} on the same axes as your sketch of g. Draw the line $y = x$ on your sketch.

(iii) Find the set of values of x for which

$$g(x) < g^{-1}(x).$$ [9]

13. (*a*)



The figure shows a triangle $P_1P_2P_3$ with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3) respectively. By evaluating the areas of the trapezia P_1P_3BA, P_3P_2CB and P_1P_2CA, show that the area of the triangle $P_1P_2P_3$ is equal to

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

[5]

(*b*) A transformation T of the plane in which (x, y) is transformed to (x', y') is defined by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{(where } ad > bc\text{).}$$

(i) The points P_1, P_2, P_3 above are transformed under T to the points P_1', P_2', P_3' respectively. Show that the area of the triangle $P_1'P_2'P_3'$ is equal to

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ ax_1 + by_1 & ax_2 + by_2 & ax_3 + by_3 \\ cx_1 + dy_1 & cx_2 + dy_2 & cx_3 + dy_3 \end{vmatrix}.$$

The first term in the expansion of this determinant about the first row is

$$(ax_2 + by_2)(cx_3 + dy_3) - (ax_3 + by_3)(cx_2 + dy_2).$$

Show that this term can be written in the form

$$(ad - bc)(x_2y_3 - x_3y_2).$$

Hence, by considering the other terms in the expansion, show that the area of the triangle $P_1'P_2'P_3'$ is λ times the area of the triangle $P_1P_2P_3$ where $\lambda = ad - bc$. [5]

(ii) The line L_1 passes through the origin and has slope m_1. L_1 is transformed under T to the line L_2 which also passes through the origin and has slope m_2. Show that

$$bm_1m_2 - dm_1 + am_2 - c = 0.$$

Given further that L_2 is transformed under T to L_1 and that L_1 is distinct from L_2, show that $a + d = 0$. [5]

Turn over.

14. Show that

$$\frac{\sin n\theta - \sin (n-2)\theta}{\sin \theta} = 2\cos (n-1)\theta.$$

If
$$\mathrm{I}_n = \int_0^{\pi} \frac{\sin n\theta}{\sin \theta}\, d\theta$$

show that, for $n \geqslant 2$,

$$\mathrm{I}_n = \mathrm{I}_{n-2}.$$

Hence show that $\mathrm{I}_n = \pi$ for n odd and find the value of I_n for n even. [5]

If
$$\mathrm{J}_n = \int_0^{\pi} \left(\frac{\sin n\theta}{\sin \theta}\right)^2 d\theta$$

show that, for $n \geqslant 1$,

$$\mathrm{J}_n - \mathrm{J}_{n-1} = \pi$$

and deduce an expression for J_n. [6]

If
$$\mathrm{K}_n = \int_0^{\pi} \frac{\sinh n\theta}{\sinh \theta}\, d\theta$$

obtain an expression for $\mathrm{K}_n - \mathrm{K}_{n-2}$ and deduce that, for even n,

$$\mathrm{K}_n = 2\sum_{i=1}^{n/2} \frac{\sinh (n+1-2i)\pi}{n+1-2i}.$$ [4]

15. Given that

$$y = \cos \log_e(1+x) \quad (|x| < 1)$$

show that

$$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 0.$$ [3]

Hence, or otherwise, show that the McLaurin series of y is given by

$$1 - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 - \tfrac{5}{12}x^4 + \dots.$$ [4]

Deduce that

$$\sin \log_e(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \dots$$ [4]

and hence find a series expansion for $\tan \log_e(1+x)$ as far as the x^3 term. [4]

WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

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Advanced Level

Special Paper

Safon Uwch

Papur Arbennig

PURE MATHEMATICS S

A.M. MONDAY, 26 June 1989

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Pure Mathematics

No more than **four** *questions should be answered from this section.*

1. (*a*) Find, as a function $f(a)$, the shortest distance between the point $(0, a)$ $(-\infty < a < \infty)$ and the curve C having equation $y = x^2$.

   Sketch the graph of f. [8]

   (*b*) Find, as a function $g(b)$, the shortest distance between the point $(1 - 2b,\ 1 + b)$ and C for $b \leqslant \frac{1}{2}$.

   (You are advised to consider separately the cases $b < \frac{1}{4}$ and $\frac{1}{4} \leqslant b \leqslant \frac{1}{2}$). [7]

2. (*a*) Show that

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \equiv x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2 - (\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta+\alpha\beta\gamma)x + \alpha\beta\gamma\delta.$$

Deduce that if $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

then

$$\alpha+\beta+\gamma+\delta = \frac{-b}{a} \quad \text{and} \quad \alpha\beta\gamma\delta = \frac{e}{a}.$$

[3]

(*b*) Given that $\sin\theta \neq 0$, express $\frac{\sin 4\theta}{\sin\theta}$ and $\frac{\sin 5\theta}{\sin\theta}$ as polynomials in $\cos\theta$.

Show that

(i) $$\left(\frac{\sin 5\theta - \sin 4\theta}{\sin\theta}\right)^2 = \frac{1+\cos 9\theta}{1+\cos\theta}$$

and (ii) $$\frac{\sin 5\theta - \sin 4\theta}{\sin\theta} = x^4 - x^3 - 3x^2 + 2x + 1, \text{ where } x = 2\cos\theta.$$

Deduce (in terms of π) the four roots of the equation

$$x^4 - x^3 - 3x^2 + 2x + 1 = 0.$$

[8]

Hence

(iii) show that

$$8\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = 1$$

and (iv) find the value of

$$\cos\frac{\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9}.$$

[4]

3. Simpson's Rule applied to the function f over the interval $[-3h, 3h]$ can be written in the form

$$\int_{-3h}^{3h} f(x)\,dx \approx \frac{h}{3}[f(-3h) + 4f(-2h) + 2f(-h) + 4f(0) + 2f(h) + 4f(2h) + f(3h)].$$

A similar result, Weddle's Rule, is as follows:

$$\int_{-3h}^{3h} f(x)\,dx \approx \frac{3h}{10}[f(-3h) + 5f(-2h) + f(-h) + 6f(0) + f(h) + 5f(2h) + f(3h)].$$

(i) Show that both rules give the exact value of the integral when $f(x)$ is a cubic polynomial. [5]

(ii) Show that Weddle's Rule gives the exact value of the integral when $f(x)$ is a quartic polynomial but Simpson's Rule does not. [2]

(iii) Show that when $f(x) = x^n$, where n is an even integer, the difference between the exact value of the integral and the value given by Simpson's Rule is

$$\left[2.3^{n-1}\left(\frac{8-n}{n+1}\right) - \frac{4}{3}(1+2^{n+1})\right]h^{n+1}$$

and obtain a similar expression for Weddle's Rule. [8]

4. (*a*) By considering the binomial expansion of $(1 - \frac{3}{4})^{-\frac{2}{3}}$, find the sum of the infinite series

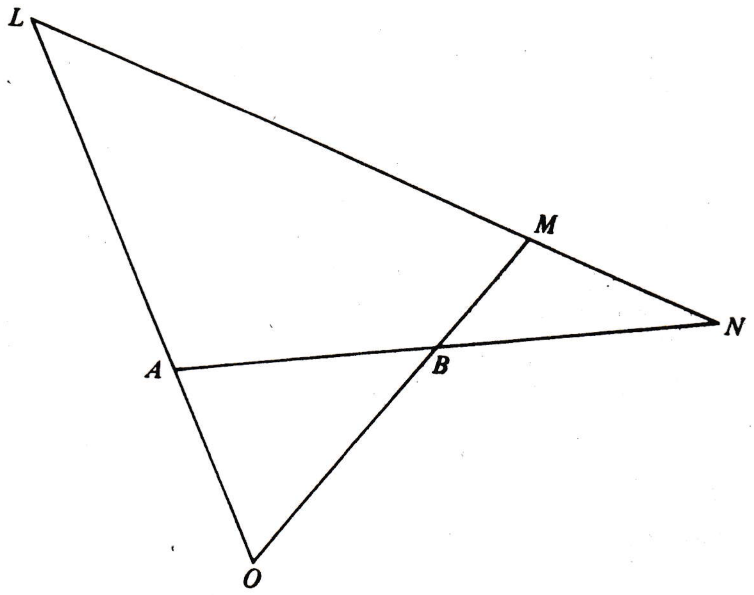
$$1 + \frac{5}{8} + \frac{5.8}{8.12} + \frac{5.8.11}{8.12.16} + \frac{5.8.11.14}{8.12.16.20} + \cdots\cdots$$ [6]

(*b*) Show that for small x and $n > 1$,

$$(1 + x)^{\frac{1}{n}} \approx \frac{2n + (n + 1)x}{2n + (n - 1)x}.$$

Deduce a rational approximation to $\sqrt[3]{7}$. [9]

5.



Taking O as the origin, use vector methods to show that, in the above figure,

$$\frac{OL}{AL} \cdot \frac{AN}{BN} \cdot \frac{BM}{OM} = 1.$$ [15]

SECTION B

Further Pure Mathematics

No more than **four** *questions should be answered from this section.*

6. The three sides of a triangle ABC have equations

$$3y - x - 9 = 0$$
$$2y + x - 1 = 0$$
and
$$y - 7x + 37 = 0$$

Explain why the curve with equation

$$\lambda(3y - x - 9)(2y + x - 1) + \mu(2y + x - 1)(y - 7x + 37) = (y - 7x + 37)(3y - x - 9)$$

passes through the points A, B and C for all values of the constants λ, μ. [3]

(*a*) By considering the coefficients of x^2, y^2 and xy, show that the equation of the circle passing through A, B and C is

$$(x - 2)^2 + (y - 2)^2 = 25.$$ [6]

(*b*) By considering the coefficients of x^2 and xy find the equation of the parabola which passes through A, B and C and whose axis is parallel to the x-axis.

Write down the co-ordinates of its vertex. [6]

7. Show, using de Moivre's theorem or otherwise, that

$$\tan 5\theta = \frac{5\cot^4\theta - 10\cot^2\theta + 1}{\cot^5\theta - 10\cot^3\theta + 5\cot\theta}. \qquad [3]$$

By considering the roots of the equation

$$\tan 5\theta = \tan 5\beta$$

show that

$$\sum_{k=0}^{4} \cot\left(\beta + \frac{k\pi}{5}\right) = 5\cot 5\beta. \qquad [5]$$

Deduce that

$$\sum_{k=1}^{4} \operatorname{cosec}^2\left(\beta + \frac{k\pi}{5}\right) = \frac{25}{\sin^2 5\beta} - \frac{1}{\sin^2\beta}. \qquad [3]$$

Find the first non-zero term in the series expansion of the right-hand side and deduce the value of

$$\sum_{k=1}^{4} \operatorname{cosec}^2\left(\frac{k\pi}{5}\right). \qquad [4]$$

8. Given that

$$I_n = \int_0^\infty x^n e^{-x} \cos x \, dx$$

and

$$J_n = \int_0^\infty x^n e^{-x} \sin x \, dx$$

show that for $n \geqslant 2$,

(i) $I_n = -\frac{1}{2}n(n-1)J_{n-2}$

(ii) $J_n = \frac{1}{2}n(n-1)I_{n-2}$.

[4]

Hence show that

$$I_n = \begin{cases} \frac{1}{2}(-\frac{1}{4})^r n! & \text{for } n = 4r \\ 0 & \text{for } n = 4r+1 \\ (-\frac{1}{4})^{r+1} n! & \text{for } n = 4r+2 \\ & \text{and } n = 4r+3 \end{cases}$$

where r is a positive integer. [11]

[You may assume that

$$\int e^{-x} \sin x \, dx = -\tfrac{1}{2}e^{-x}(\sin x + \cos x)$$

and

$$\int e^{-x} \cos x \, dx = \tfrac{1}{2}e^{-x}(\sin x - \cos x).]$$

9. (*a*) Show that α, β and $\alpha - \beta$ are all factors of the quintic polynomial represented by the determinant

$$\begin{vmatrix} 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \\ 1 & \alpha^3 & \beta^3 \end{vmatrix}.$$

Express the polynomial as the product of five linear factors. [6]

(*b*) (i) A square matrix **A** is called nilpotent if $\mathbf{A}^k = \mathbf{O}$ for some positive integer k. Does this imply that **A** is singular? Explain your answer briefly.

(ii) Give an example to show that the sum of nilpotent matrices is not necessarily nilpotent.

(iii) Show, however, that the sum of two nilpotent matrices which commute under multiplication is always nilpotent. [9]

[It may be assumed that matrices **B**, **C** satisfying **BC** = **CB** also satisfy

$$(\mathbf{B} + \mathbf{C})^n = \mathbf{B}^n + \binom{n}{1}\mathbf{B}^{n-1}\mathbf{C} + \binom{n}{2}\mathbf{B}^{n-2}\mathbf{C}^2 + \cdots + \binom{n}{n-1}\mathbf{B}\mathbf{C}^{n-1} + \mathbf{C}^n$$

where n is a positive integer.]

10. Given that

$$w = \frac{z-1}{z+1} \qquad (z \neq -1)$$

find, in Cartesian form, the equation of the locus of z if

(i) w is real,

(ii) w is imaginary,

(iii) $\arg w = \dfrac{\pi}{4}$. [6]

(*a*) If the locus of w is the circle

$$|w - (1 + \mathrm{i})| = 2$$

show that the equation of the locus of z is

$$|z + 1 - 2\mathrm{i}| = 2|z + 1|.$$

Show that this locus is a circle and find its radius and the co-ordinates of its centre. [5]

(*b*) Show that, in general, if the locus of w is a circle, then so is the locus of z. [4]

WELSH JOINT EDUCATION COMMITTEE | CYD-BWYLLGOR ADDYSG CYMRU

General Certificate of Education | Tystysgrif Addysg Gyffredinol

Advanced Level | Safon Uwch

Special Paper | Papur Arbennig

APPLIED MATHEMATICS S

A.M. MONDAY, 26 June 1989

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Mechanics

No more than **four** *questions should be answered from this section.*

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

1. (*a*) The position vector, at time t, of a particle P is $\mathbf{a}t + \mathbf{b}t^2$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors inclined at an angle θ to each other. The velocity of P is perpendicular to $\mathbf{a}$ when $t = t_1$ and perpendicular to $\mathbf{b}$ when $t = t_2$. Express the ratio $\dfrac{t_1}{t_2}$ in terms of θ. [5]

   (*b*) The variable two dimensional **unit** vectors $\mathbf{s}$ and $\mathbf{n}$ are such that

$$\mathbf{n} = p\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t},$$

   where p can vary with time. Show that $\mathbf{n}$ is perpendicular to $\mathbf{s}$. By expressing $\mathbf{s}$ in the form $l\mathbf{i} + m\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are constant perpendicular unit vectors, or otherwise, express $\dfrac{\mathrm{d}\mathbf{n}}{\mathrm{d}t}$ in terms of p and $\mathbf{s}$. [10]

2. A particle B of mass m is joined by light inextensible strings to two other particles A and C, also each of mass m. The particles lie on a smooth horizontal table with the strings just taut and with BC making an acute angle α with AB produced. An impulse of magnitude I is applied to A in the direction BA. Express the initial speed of C in terms of I, m and α. [9]

   Show also that the kinetic energy generated by the impulse is $\frac{1}{2}IV$, where V denotes the initial speed of A, and find, as α varies, the least possible value of this kinetic energy. [6]

3. A particle P of mass m rests on a rough horizontal table and is connected by a light elastic spring of natural length $16a$ to a fixed point O on the table with $OP = 16a$. The modulus of the spring is $16mn^2a$, where n is a constant, and the coefficient of friction between the particle and the table is $\dfrac{qan^2}{g}$, where q is a numerical constant such that $q < \frac{1}{4}$.

The spring is extended slightly so that $OP = 17a$ and at time $t = 0$ the particle P is released from rest in this position. Show that, for $0 \leqslant t \leqslant \dfrac{\pi}{n}$, the extension x of the spring is given by

$$x = qa + a(1 - q)\cos nt. \qquad [7]$$

Find the corresponding expression for $\dfrac{\pi}{n} \leqslant t \leqslant \dfrac{2\pi}{n}$ and the least value of q so that the particle remains at rest for $t \geqslant \dfrac{2\pi}{n}$. [8]

4. The $n(> 1)$ identical uniform rods $A_1A_2, A_2A_3, \ldots, A_nA_{n+1}$ are smoothly jointed together at the points $A_2, A_3, \ldots, A_n$. Each rod is of weight W and length $2a$. The end A_1 is smoothly hinged to a fixed point and each rod, except A_1A_2, is supported at some point of its length so that the rods form a horizontal straight line. The points of support are such that the reaction at each of them is the same, being directly upwards and equal to R. The distance of the point of support of the rod A_rA_{r+1} from A_r is denoted by x_r. By considering the equilibrium of two adjacent rods, show that

$$R(x_{r+1} - x_r) = 2a(R - W) \qquad [6]$$

and hence determine x_r in terms of r, a, W and R. Also express R in terms of W and n. [9]

5. A particle is projected from a point O on a horizontal plane with speed u in a direction inclined at an angle θ above the horizontal. Assuming that the only force acting is that due to gravity, obtain the Cartesian equation of the path of the particle in a form which exhibits its symmetry about a vertical line at a horizontal distance $\dfrac{u^2 \sin\theta\cos\theta}{g}$ from O. [4]

A particle is projected with speed $(kga)^{\frac{1}{2}}$ so as just to clear two parallel walls of height $2a$ above the point of projection and at a distance $2a$ apart, the plane of the motion being perpendicular to the walls. Show that θ the angle of projection is such that

$$\tan^4\theta + (4k + 2 - k^2)\tan^2\theta + 1 + 4k = 0. \qquad [5]$$

Find the least possible value of k such that this motion is possible. [6]

Section B

Probability and Statistics

No more than **four** *questions should be answered from this section.*

6. Each trial of a random experiment will result in one and only one of the events A, B, and C occurring. In the r^{th} trial of a sequence of trials of the experiment, let A_r denote the event that A will occur, B_r the event that B will occur, and C_r the event that C will occur. It is known that for all $r \geqslant 1$,

$$P(A_{r+1} \mid A_r) = 0{\cdot}6, \quad P(B_{r+1} \mid A_r) = 0{\cdot}2,$$
$$P(A_{r+1} \mid B_r) = 0{\cdot}1, \quad P(B_{r+1} \mid B_r) = 0{\cdot}2,$$
$$P(A_{r+1} \mid C_r) = 0{\cdot}2, \quad P(B_{r+1} \mid C_r) = 0{\cdot}4,$$

and that $P(A_r) = p_A$, $P(B_r) = p_B$, $P(C_r) = p_C$.

(i) Show that $p_A = \frac{2}{7}$, and evaluate p_B and p_C. [7]

(ii) Given that A did not occur in a particular trial, calculate the conditional probability that A occurred in the immediately preceding trial. [4]

(iii) Given that C did not occur in a particular trial, calculate the conditional probability that A will occur in the next trial. [4]

7. The random variable X has the truncated Poisson distribution given by

$$P(X = x) = \frac{ke^{-\mu}\mu^x}{x!}, \quad x = 1, 2, 3, \ldots.$$

(i) Show that $k = \dfrac{e^{\mu}}{(e^{\mu} - 1)}$. [2]

(ii) If Y is a random variable having the Poisson distribution with mean μ, show that $E[X] = kE[Y]$, and $E[X^2] = kE[Y^2]$. [3]

Books borrowed from a certain library are due to be returned by a specified date, but are often returned later than that date. Let X denote the number of days *after* the specified date when an overdue book is returned. Suppose that X has the above truncated Poisson distribution with $\mu = 3$. Two schemes, A and B, for fining people who return books late are being considered by the Library Committee. Under these schemes the fines, F_A and F_B, to be levied for a book that is returned X days after the specified date are, respectively,

$$F_A = (1 + 2 + 3 + \ldots + X) \text{ pence},$$

and

$$F_B = (1 + 2X) \text{ pence}.$$

(iii) Show that the probability of a fine being 10 pence or less is the same for both schemes, and evaluate this probability correct to three decimal places. [5]

(iv) Determine which of the two schemes will yield the greater average income to the library. (You may assume that the behaviour of the borrowers will be unaltered by the introduction of either scheme.) [5]

8. The continuous random variable X has probability density function f given by

$$f(x) = \tfrac{1}{8}(2 - x), \quad \text{for } -2 \leqslant x \leqslant 2,$$
$$f(x) = 0, \quad \text{otherwise.}$$

(i) Find the cumulative distribution function F of X and show that $E[F(X)] = \frac{1}{2}$. [5]

(ii) Let $Y = |X - 1|$. Show that the probability density function g of Y is such that

$$g(y) = f(1 + y) + f(1 - y).$$

Hence, or otherwise, obtain expressions for $g(y)$. [5]

(iii) Find the values of the constants a and b if $W = aX + b$ is to have the probability density function h given by

$$h(w) = \tfrac{1}{2}w,$$

over an interval which is to be specified, and $h(w) = 0$ otherwise. [5]

9. An organism is treated in a certain way to stimulate its growth. It is known that when such an organism of initial size A is treated in this way its true size t days after the treatment is z, where

$$z = Ae^{\beta t},$$

and $\beta > 0$ is a constant dependent upon the organism. The size of the organism was measured $t_1, t_2, \ldots, t_n$ days after the treatment. If $y_i = \log_e (z_i/A)$, where z_i is the measured size of the organism t_i days after the treatment, it may be assumed that

$$y_i = \beta t_i + e_i, \quad i = 1, 2, 3, \ldots, n.$$

where the e_i are independent normally distributed random variables having mean zero and standard deviation σ.

By considering $\sum (y_i - \beta t_i)^2$, show that the least squares estimate, b, of β is given by

$$b = \frac{\sum t_i y_i}{\sum t_i^2}.$$ [4]

Verify that b is an unbiased estimate of β and that its standard error is $\dfrac{\sigma}{(\sum t_i^2)^{\frac{1}{2}}}$. [4]

Suppose that $\sigma = 0{\cdot}004$ and that the following data were obtained.

t	5	10	15	20
y	0·046	0·103	0·147	0·204

Given that the initial size of the organism was $A = 5$, calculate 95% confidence limits for the true size of the organism 12 days after treatment. [7]

10. The discrete random variable X has the probability distribution shown in the following table.

x	0	1	2
$P(X = x)$	θ	2θ	$1 - 3\theta$

Let $X_1, X_2, \ldots, X_n$ denote a random sample of n observations of X.

Show that

$$T_1 = \frac{1}{2} - \frac{1}{4n}\sum X_i \quad \text{and} \quad T_2 = \frac{2}{5} - \frac{1}{10n}\sum X_i^2$$

are both unbiased estimators of θ and determine which of them has the smaller variance. [10]

If the n observed values are found to consist of 0's and 1's only with the number having the value 0 being more than $\frac{1}{3}n$, show that each of T_1 and T_2 will give an estimated value of θ which cannot possibly be the true value of θ. [5]