WELSH JOINT EDUCATION COMMITTEE | **CYD-BWYLLGOR ADDYSG CYMRU**

General Certificate of Education | **Tystysgrif Addysg Gyffredinol**

Advanced Level | **Safon Uwch**

MATHEMATICS **A 1**

Pure Mathematics

A.M. WEDNESDAY, 8 June 1988

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. The roots of the quadratic equation

$$ax^2 + bx + c = 0$$

are α and 2α. By considering the sum and the product of the roots, show that $2b^2 = 9ac$. [4]

2. Write down the binomial expansion of $(1 + 2x)^n$ in ascending powers of x as far as the x^3 term. P1

Given that the coefficient of x^3 is twice the coefficient of x^2 and that both coefficients are positive, find the value of n. [4]

3. The equations of two circles are

$$x^2 + y^2 - 4x - 2y - 4 = 0$$

and

$$x^2 + y^2 + 2x - 10y + 25 = 0.$$

P2

Calculate

(i) the radii of the two circles,

(ii) the distance between the centres of the two circles.

Deduce that the circles do not intersect. [6]

Turn over.

4. Show that

$$\tan\left(\frac{\pi}{4}+\theta\right)=\frac{1+\tan\theta}{1-\tan\theta}$$

and write down a similar expression for $\tan\left(\frac{\pi}{4}-\theta\right)$.

P2

Hence show that

$$\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)=2\tan 2\theta.$$

[5]

5. If

$$f(x)=\frac{1}{x^2}, \qquad x\neq 0,$$

show that

P1

$$\frac{f(x+h)-f(x)}{h}=\frac{-2x-h}{x^2(x+h)^2}.$$

Deduce an expression for $f'(x)$, the derivative of $f(x)$. [5]

6. Using the substitution $y=x^2-1$, or otherwise, evaluate the integral

$$\int_2^3 x\left(\frac{x^2+1}{x^2-1}\right)\mathrm{d}x$$

P2

[5]

7. The first term of a geometric series is equal to 12 and the sum of the first three terms is equal to 21.

P1

(i) Find the two possible values of the common ratio.

(ii) Calculate the sum to infinity of the series having the positive common ratio.

[5]

8. The real, non zero constants a and b satisfy the equation

$$(a+ib)^3=1.$$

By considering the real and imaginary parts, show that

X

$$a(a^2-3b^2)=1$$

and

$$b^2=3a^2.$$

Solve these simultaneous equations for a and b, and hence write down the two complex roots of the equation

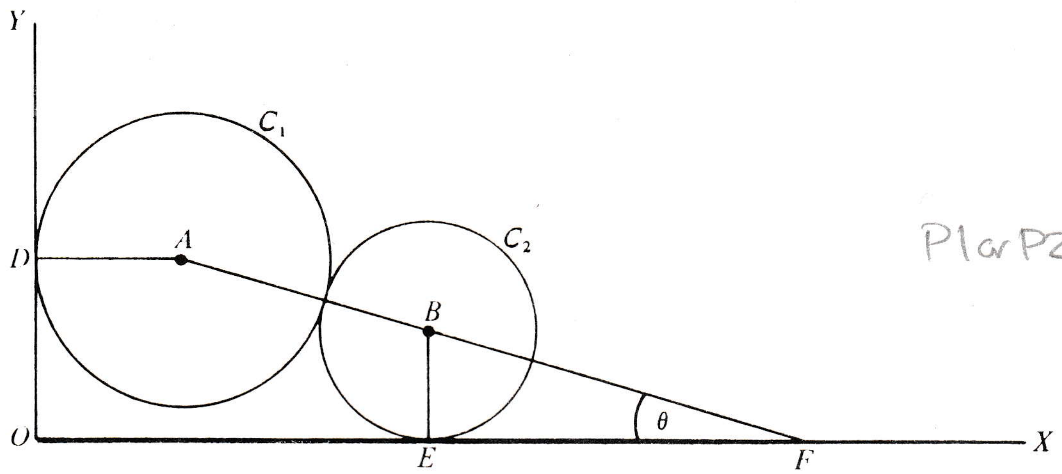
$$z^3=1.$$

[6]

Section B

Answer **four** *questions from this section.*

9.



P1 or P2

The figure shows two perpendicular axes OX and OY. Also shown are two circles C_1 and C_2 which touch each other and which both lie in the first quadrant (i.e. above OX and to the right of OY).

C_1 has radius 4 and touches OY at D; C_2 has radius 3 and touches OX at E. The line AB, joining the centres of C_1 and C_2, meets OX at F and $B\widehat{F}O = \theta$.

(i) Find expressions for OD and OE in terms of θ and show that

$$DE^2 = 74 + 42 \sin \theta + 56 \cos \theta.$$

Hence express DE^2 in the form

$$74 + r \cos (\theta - \alpha)$$

where the values of r and α are to be found. [6]

(ii) By considering the extreme positions in which

(*a*) both circles touch OX

and (*b*) both circles touch OY,

show that, correct to 1 decimal place,

$$8{\cdot}2^\circ \leqslant \theta \leqslant 98{\cdot}2^\circ.$$ [4]

(iii) Find the greatest and least possible lengths of DE and state the corresponding values of θ. [5]

Turn over.

P2

10. Show that the equation of the normal to the parabola $y^2 = 4x$ at the point $P(p^2, 2p)$ is

$$y + px = 2p + p^3. \qquad [3]$$

Given that $p \neq 0$ and that the normal at P cuts the x-axis at the point $B(b, 0)$, obtain an expression for p^2 in terms of b. Deduce that $b > 2$. [3]

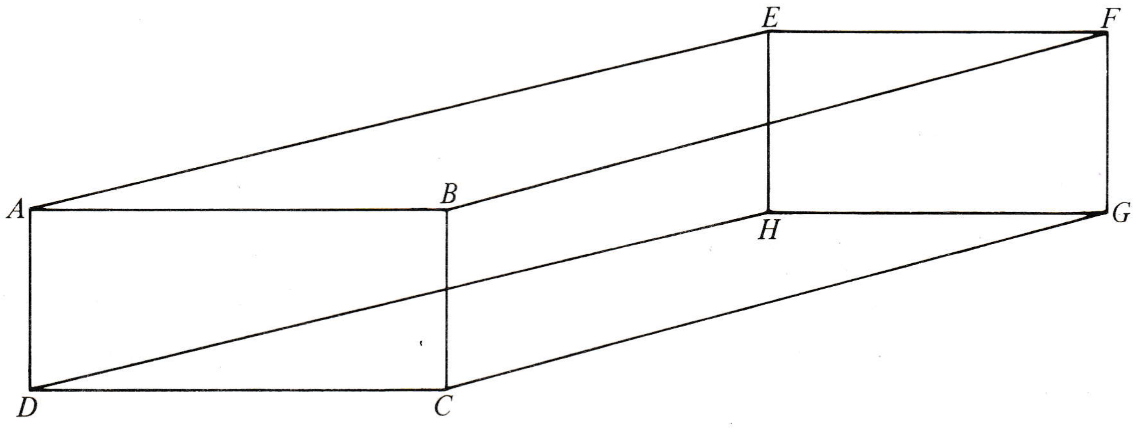
The normal at P meets the normal at $Q\ (q^2, 2q)$ at the point R. Show that the coordinates of R are

$$(2 + p^2 + pq + q^2,\ -pq(p + q)). \qquad [3]$$

Given that $pq = 2$, show that these coordinates can be written in the form $(r^2, 2r)$ where r is a function of p. What does this result tell you about the point R? [3]

The normal at R meets the parabola again at the point $S(s^2, 2s)$. Find s in terms of r. [3]

11.



The figure shows a cuboid in which $BC = 1$ m, $AB = 2$ m and $AE = 3$ m.

Calculate the angles between

(i) the plane $CDEF$ and the plane $CDHG$,

(ii) the line EC and the plane $CDHG$,

(iii) the lines EC and EF,

(iv) the lines EC and DF. [8]

Calculate the smallest angle of the triangle DEG, and hence calculate

(v) the perpendicular distance from E to DG,

(vi) the area of the triangle DEG. [7]

12. The function f is defined by

$$f(x) = e^{x^2+1}$$

with domain $x < 0$.

(i) Find expressions for

(*a*) $f'(x)$, the derivative of $f(x)$,

(*b*) $f^{-1}(x)$, where f^{-1} denotes the inverse function of f.

P1

Sketch the graphs of f and f^{-1} on the same axis and state the domain of f^{-1}. [6]

(ii) Show graphically that the equation

$$e^{x^2+1} = 5 - x^2$$

P2

has a single negative root and show that its value lies between $-0{\cdot}8$ and $-0{\cdot}7$. [4]

(iii) The function g is defined on the domain $x > 1$ and

$$f \circ g(x) = e^x.$$

P1

Obtain an expression for $g(x)$ in terms of x. State, with a reason, whether or not the function $g \circ f$ can be formed. [5]

13. The function f is defined by

$$f(x) = \frac{a + bx}{x^2 - 1} \qquad (x^2 \neq 1).$$

Show that

$$f'(x) = \frac{-(bx^2 + 2ax + b)}{(x^2 - 1)^2}.$$ [2]

Given further that the graph of f has a turning point at $(3, -\frac{1}{2})$, show that $a = 5$ and $b = -3$. Hence find the coordinates of the other turning point.

P2

Express $f(x)$ in partial fractions and show that

$$f''(x) = \frac{2}{(x-1)^3} - \frac{8}{(x+1)^3}.$$ [8]

Hence

(i) identify the two turning points as maxima or minima, ✓ [2]

(ii) given that the graph of f has a single point of inflection, find its x coordinate. ✗ [3]

TURN OVER.

14. (i) The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively with respect to an origin O. The point R lies on PQ and has position vector $\lambda\mathbf{p} + \mu\mathbf{q}$. By considering the vectors $\mathbf{PQ}$ and $\mathbf{PR}$, or otherwise, show that $\lambda + \mu = 1$.

Show further that if R lies between P and Q, then both λ and μ lie between 0 and 1. [6]

(ii) The vertices A, B, C, D of a parallelogram $ABCD$ have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ respectively.

Show that

X

$$\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}.$$ [2]

The point E lies on AC and $AE : EC = 1 : 3$. The line DE meets AB in F. Show that

$$AF : FB = 1 : 2$$

and calculate the ratio $DE : EF$. [7]

15. (*a*) Using integration by parts, show that

$$\int_1^2 x^2 \log_e x \, dx = \frac{8}{3}\log_e 2 - \frac{7}{9}.$$ [4]

P2

(*b*) Sketch, for $0 \leqslant x \leqslant 2\pi$, the curve

$$y = \cos x.$$

Hence sketch on different axes, for $0 \leqslant x \leqslant 2\pi$, the curves

(i) $y = \sec x$

(ii) $y = \sec x - \cos x$. [3]

Write down an integral equal to the area of the region R enclosed by the curve $y = \sec x - \cos x$, the x axis and the line $x = \dfrac{\pi}{4}$. Hence find the area of R. [3]

X

If R is rotated about the x-axis through four right angles, find the volume generated in terms of π. [5]

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Advanced Level | Safon Uwch

MATHEMATICS A 2

Mechanics

A.M. WEDNESDAY, 15 June 1988

(3 Hours)

Answer **all** *questions in Section A and* **four** *from Section B.*

A calculator may be used except where it is expressly forbidden.

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

SECTION A

Answer **all** *questions in this section.*

1. A body moves along a straight line. At time t its velocity v satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} + 4v = \mathrm{e}^{-2t}.$$

Use the integrating factor method to find the general solution of this equation. [4]

2. A locomotive working at the rate of 240 kW pulls a train of total mass 56 000 kg (including the locomotive) up a straight track inclined to the horizontal at an angle of $\sin^{-1}(\frac{1}{50})$. When the speed is 5 m s^{-1}, the acceleration is $\frac{1}{4} \text{ m s}^{-2}$. Find the total resistance at this speed. [4]

3. The foot of a uniform ladder of length l m and mass M kg rests on rough horizontal ground and the top of the ladder rests against a smooth vertical wall. The ladder is inclined at 30° to the horizontal and the ground is sufficiently rough to prevent slipping. Find the frictional force and hence the magnitude of the total force exerted by the ground on the ladder. [4]

Turn over.

4. A particle P of mass 2 kg is attached to the end of a light inextensible string OP of length 1·5 m. The end O is fixed and P describes a horizontal circle with uniform angular speed ω radians/second, with OP inclined at an angle of 60° to the vertical. Show that

(i) the tension in the string is $4g$ N,

(ii) $\omega = 2\sqrt{\frac{g}{3}}$. [5]

5. Two particles A and B are of mass $3M$ kg and $2M$ kg respectively. They are connected by a light inextensible string passing over a smooth pulley. Particle A rests vertically below the pulley on a horizontal plane. Particle B is lifted so that the string is slack, and then allowed to fall from rest. The string becomes taut when particle B has fallen through a distance of 2·5 m.

Find the speed of particle B

(i) immediately before the string becomes taut,

(ii) immediately after the string becomes taut.

Find the impulsive tension in the string when it becomes taut. [5]

6. A body B of mass 2 kg moves such that its position vector at time t seconds is

$$\mathbf{r} = (\sin \pi t + \alpha t)\mathbf{i} + (\cos \pi t + \alpha t)\mathbf{j},$$

where α is a constant and distances are measured in metres. Write down the velocity and acceleration vectors of B at time t seconds. Given that at $t = 1$ the force acting on B is perpendicular to $\mathbf{r}$, show that $\alpha = 1$. Find the kinetic energy of the body when $t = \frac{1}{4}$. [5]

7. A particle of mass 2 kg is hung vertically on a light elastic string and is found to stretch it by 0·3 m. Find the value of $\frac{\lambda}{l}$ where λ N and l m are the modulus of elasticity and natural length of the string respectively.

The particle is pulled down a further distance x m. Write down the decrease of potential energy of the particle. Show that the total potential energy of the particle and string is increased by $\frac{10gx^2}{3}$ J.

[5]

8. $ABCDEF$ is a regular hexagonal lamina of side 2 m with centre at O. Forces of 1 N, 2 N, 3 N, 4 N, 5 N and 6 N act along the directions AB, CB, CD, ED, EF and FA respectively.

(i) Show that the magnitude of the sum of the moments of the forces about O is $9\sqrt{3}$ Nm.

(ii) Find the components of the resultant force along and perpendicular to OC.

(iii) Find where the line of action of the single force equivalent to the force system cuts the line COF. [8]

Section B

Answer **four** *questions from this section.*

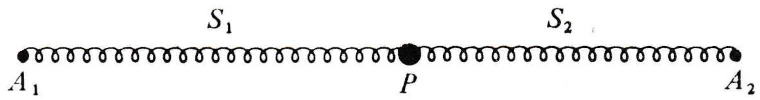
9. (*a*) A particle of mass 2 kg is acted upon by a variable force which makes it move in simple harmonic motion about O on the straight line Ox. Given that the maximum speed attained is $0{\cdot}5 \text{ m s}^{-1}$ and the maximum acceleration is $0{\cdot}1 \text{ m s}^{-2}$, find the period of the motion. Show that the amplitude of the motion is $2{\cdot}5$ m.

The particle passes through O at time $t = 0$.

(i) Write down the displacement of the particle from O at time t. Hence find the velocity and acceleration at time t.

(ii) Find the force acting on the particle at time t. Show that the force generates a power of $-0{\cdot}05 \sin(0{\cdot}4t)$ watts during the motion and find the values of t for which the power is a maximum. [8]

(*b*) A particle P of mass M kg is attached to two light elastic springs S_1 and S_2 each of natural length 1 m, and of moduli λ_1 N and λ_2 N respectively. The springs, together with the particle, are placed on a smooth flat table with the ends of the springs attached to two fixed points A_1 and A_2, 2 metres apart, A_1PA_2 being a straight line.



The system is disturbed slightly from equilibrium in the line A_1PA_2 and then allowed to move freely. Show that the particle P performs simple harmonic motion. Show also that the period of the motion may be written as $\dfrac{T_1T_2}{\sqrt{T_1^2 + T_2^2}}$, where T_1 is the period when S_1 only is attached to P, and T_2 is the period when S_2 only is attached to P. [7]

10. A body is projected at time $t = 0$ from a point O with speed V in a direction inclined at an angle θ to the horizontal. Write down expressions for the horizontal and vertical components x and y of its displacement from O at time t.

(i) Show that the range R on a horizontal plane through the point of projection and the time of flight T satisfy

$$gT^2 = 2R \tan\theta.$$

Show that the time of flight corresponding to a maximum range of 392 m is $8{\cdot}9$ s, correct to one decimal place. Find the speed of projection and the maximum height attained, correct to one decimal place. [10]

(ii) Show that x and y satisfy

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta.$$

A is a point on the plane at a distance of 196 m from O. It is required to hit a target directly above A at a height of 98 m. Given that $V = \sqrt{490\,g}$ find the corresponding values of θ. [5]

Turn over.

11. ACB is the diameter of a uniform semicircular lamina of radius a m and centre C. Show that the centre of mass of the lamina is at a distance of $\dfrac{4a}{3\pi}$ m from ACB.

[4]

The semicircle having CB as diameter is removed from the lamina.

(i) Show that the centre of mass of the lamina remaining is at a distance of $\dfrac{14a}{9\pi}$ m from ACB.

[5]

(ii) The remaining lamina is suspended freely from A and lies in a vertical plane. Show that ACB makes an angle $\tan^{-1}\left(\dfrac{28}{15\pi}\right)$ with the vertical. [6]

12. (*a*) A body of mass 4 kg is moving along the Ox axis. The motion is subject to a resisting force of $(28v + 24x)$ N, where v m s^{-1} is the velocity of the body and x m is its displacement from O. Show that the displacement x satisfies

$$\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 6x = 0, \text{ where } t \text{ denotes time.}$$

Given that $x = 4$, $v = 1$ when $t = 0$, find x in terms of t. Show that the body passes through O at time $\frac{1}{5}\log_e\left(\frac{1}{5}\right)$ s. [8]

(*b*) A manufacturer estimates that the rate at which the value of machinery at time t depreciates is proportional to the value of the machinery at that time. Write down a differential equation for P, the value of the machinery at time t. Find the general solution of this equation.

A machine which costs £8000 new is only worth £4000 in 3 years time. Find the value of the machine after 8 years, correct to the nearest pound. [7]

13. Two identical small smooth spheres A and B, each of mass M kg, are connected by a light inextensible string AB of length 1 m. They are placed on a smooth horizontal table with AB just taut. The line AB is at right angles to the plane of a vertical wall, B being the sphere nearer to the wall and distant 10 m from it. A is projected towards B with speed 2 m s^{-1}. The coefficient of restitution for collisions between the spheres, and for collisions between the spheres and the wall is e (e > 0).

(i) Show that the speeds of A and B after their first collision are $(1 - \text{e})$ m s^{-1} and $(1 + \text{e})$ m s^{-1} respectively. [3]

(ii) Show that if the string does not become taut before B collides with the wall then e $< \dfrac{1}{19}$.

[5]

(iii) Assuming that e $< \dfrac{1}{19}$, find the speed of B after its first collision with the wall. The kinetic energy lost as a result of the first collision between A and B is k times the kinetic energy lost due to the collision of B with the wall. Show that the value of k lies between $\dfrac{361}{200}$ and 2. [7]

14. At $t = 0$, a small boat is at the origin of a Cartesian coordinate system and a ship is at the point with position vector $10\mathbf{i}$, where distances are measured in nautical miles. The constant velocity of the boat is $(V \cos\theta\,\mathbf{i} + V\sin\theta\,\mathbf{j})$ knots (nautical miles/hour), where V and θ are constants and $V > 0$. The constant velocity of the ship is $(-6\mathbf{i} + 8\mathbf{j})$ knots. Write down the position vectors of the boat and ship after t hours. [1]

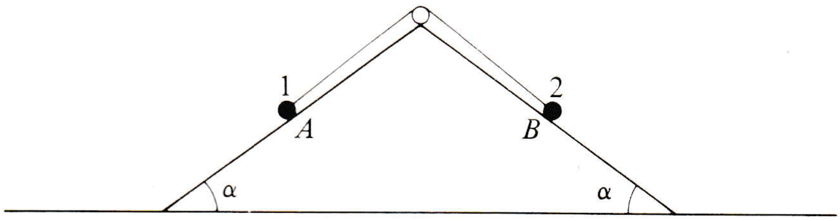
(i) The boat intercepts the ship after T hours. Write down two equations relating V, θ and T. Find the least value of V at which the boat can travel and still intercept the ship. State the corresponding value of θ and the time taken. [5]

(ii) The boat is 6 nautical miles from the ship after 2 hours. Show that in this case

$$8\sin\theta - \cos\theta = \frac{V^2 + 56}{2V}.$$

Hence show that $V^2 - 2\sqrt{65}\,V + 56 \leqslant 0$. [5, 4]

15.



A double inclined plane is in the form of a wedge fixed on the ground with its base horizontal and smooth faces inclined at an angle α to the horizontal. A particle A of mass 1 kg on one face is connected by a light inextensible string passing over a smooth pulley at the top of the wedge to a particle B of mass 2 kg on the other face.

Given that $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$, find

(i) the reactions of the wedge on A and B,

(ii) the common acceleration of the particles and the tension in the string. [8]

Given that the mass of the wedge is 10 kg, draw a diagram showing the forces on the wedge. Show that the vertical thrust of the ground on the wedge is $\dfrac{959}{75}g$. [7]

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General Certificate of Education | **Tystysgrif Addysg Gyffredinol**

Advanced Level | **Safon Uwch**

MATHEMATICS A 3

Probability and Statistics

A.M. TUESDAY, 21 June 1988

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. The two events A and B are such that

$$P(A) = 0{\cdot}2,\ P(B) = 0{\cdot}4, \text{ and } P(A|B) = 0{\cdot}3.$$

(i) State, with your reason, whether or not A and B are independent. [1]

(ii) Find the values of $P(A \cup B)$ and $P(B|A)$. [3]

2. In each trial of a random experiment, the probability that the event A will occur is 0·6. In eight independent trials of the experiment calculate, correct to four decimal places, the probabilities that

(i) A will occur in exactly five of the trials, [2]

(ii) A will occur in exactly five of the trials, these trials being consecutive. [2]

3. Four cards are numbered 2, 2, 4 and 6, respectively. Two of these cards are chosen at random without replacement. Let Y denote the larger of the numbers on the two chosen cards. (Take $Y = 2$ if the two chosen cards are both numbered 2.) Determine the sampling distribution of Y and hence find the expected value of Y. [5]

4. The length of each side of a cube is X cm, where X is a continuous random variable having probability density function f given by

$$f(x) = \frac{2}{x^2}, \qquad \text{for } 1 \leqslant x \leqslant 2,$$

$$f(x) = 0\,, \qquad \text{otherwise.}$$

Find the mean and the variance of the volumes of the cubes. [5]

5. Suppose that the heights of adult males are normally distributed with mean 174·5 cm and standard deviation 10 cm.

(i) Calculate, correct to three decimal places, the probability that the height of a randomly chosen adult male will be greater than 175 cm. [2]

(ii) Use a distributional approximation to calculate the probability that at least 40 of 80 randomly chosen adult males will have heights greater than 175 cm; give your answer correct to three decimal places. [4]

6. A planted tuber produces 0, 1, or 2 shoots with probabilities 0·2, 0·3 and 0·5, respectively. Independently for each shoot produced, the probability that it will flower is 0·4. Let X denote the number of shoots from a planted tuber that will flower.

(i) Show that $P(X = 0) = 0{\cdot}56$ and that $P(X = 1) = 0{\cdot}36$. [5]

(ii) Find the mean and the standard deviation of X. [3]

7. Independently on each day, the number of times that a machine will require attention has the Poisson distribution with mean 1·2. Find, correct to three decimal places, the probabilities that

(i) on any day, the machine will require attention exactly three times, [1]

(ii) on two successive days, the machine will require attention twice on one of the days and once on the other day. [3]

The cost of running the machine on a day when it requires attention X times is £$(12 + 10X^2)$.

(iii) Find the expected daily cost of running the machine. [4]

SECTION B

Answer **four** *questions from this section.*

8. The continuous random variable X, which is restricted to values in the interval from 2 to 4, inclusive, has cumulative distribution function F given by

$$F(x) = ax^2 + bx, \text{ for } 2 \leqslant x \leqslant 4.$$

(i) Giving a clear indication of your method, show that $a = \frac{1}{8}$ and $b = -\frac{1}{4}$. [3]

(ii) Find the value of c such that $P(X > c) = 0{\cdot}88$. [3]

(iii) Find the cumulative distribution function of $Y = X(X - 2)$. [5]

(iv) Let N denote the integer obtained by rounding an observed value of X to its nearest integer value. Find the probability distribution of N and, hence, evaluate $E[N]$. [4]

9. A box labelled A contains three white balls and five black balls. Three balls are drawn at random without replacement from the box and placed in an empty box labelled B.

(i) Find the probability distribution of the number of white balls drawn from box A. [5]

One ball is now drawn at random from the five balls remaining in box A, and one ball is drawn at random from the three in box B.

(ii) Calculate the probability that the ball drawn from B is white. [3]

(iii) Given that the ball drawn from A and the ball drawn from B are the same colour, calculate the probability that the three balls transferred from A to B consisted of one white ball and two black balls. [7]

10. When an object is weighed repeatedly using a certain type of balance, the observed masses are normally distributed with mean equal to the true mass of the object.

(i) Using a particular balance of the above type, it is known that the standard deviation of the observed masses is 0·5 mg.

(*a*) If an object of true mass 5·0 mg is weighed four times on the balance calculate, correct to three decimal places, the probability that the mean of the four observed weights will be greater than 5·1 mg. [3]

(*b*) Find the least number of times that an object should be weighed on this balance for the width of the 90% confidence interval for the object's true mass to be less than 0·5 mg. [3]

(*c*) An object was weighed 10 times on the balance and the mean of the observed masses was 6·8 mg. Another object was weighed 15 times on the balance and the mean of the observed masses was 4·6 mg. Calculate 95% confidence limits for the difference between the true masses of the two objects. [4]

(ii) Using another balance of the above type, the standard deviation of the observed masses is not known. Nine weighings of an object on this balance gave the observed values (in mg):

7·4, 7·5, 7·5, 7·4, 7·4, 7·6, 7·5, 7·5, 7·7,

respectively. Calculate 95% confidence limits for the true mass of the object. [5]

11. It is known that 5% of all women have a certain disease. The first stage in diagnosing the disease is based on a reading obtained from a blood test. For women having the disease, the blood test reading is normally distributed with mean 16·4 and standard deviation 6. For women not having the disease, the reading is normally distributed with mean 7·4 and standard deviation 3. A woman whose blood test reading is 14 or more is required to undergo further investigation.

(i) Calculate, correct to four decimal places, the proportion of those women having the disease who will have to undergo further investigation as a result of the blood test. [2]

(ii) Calculate, correct to four decimal places, the proportion of those women not having the disease who will have to undergo further investigation as a result of the blood test. [2]

(iii) Calculate, correct to three decimal places, the proportion of all women who will be required to undergo further investigation as a result of the blood test. [3]

(iv) If two women who have the disease are blood tested, show that the probability that their readings will differ by at most 2 is 0·186 correct to three decimal places. [4]

(v) If one woman who has the disease and one woman who does not have the disease are blood tested calculate, correct to three decimal places, the probability that their readings will differ by at most 2. [4]

12. The random variables X and Y are independent, with X having mean μ and variance σ^2, and Y having mean μ and variance σ^2/k, where k is a positive constant. Let $\bar{X}$ denote the mean of a random sample of 10 observations of X, and let $\bar{Y}$ denote the mean of a random sample of 15 observations of Y. Show that

$$T_1 = \frac{2\bar{X} + 3k\bar{Y}}{2 + 3k} \quad \text{and} \quad T_2 = \frac{2\bar{X} + 3\bar{Y}}{5}$$

are both unbiased estimators of μ. [3]

Find expressions for the variances of T_1 and T_2, and show that the variance of T_1 is less than or equal to the variance of T_2. [6]

Find, in terms of k, the value a for which

$$T = a\bar{X} + (1 - a)\bar{Y}$$

has smallest possible variance. [6]

13. The discrete random variables X and Y are independent with probability distributions as shown in the following tables.

Distribution of X

x	1	2
$P(X = x)$	0·4	0·6

Distribution of Y

y	1	2	3
$P(Y = y)$	0·3	0·4	0·3

(i) Evaluate $E\left(\frac{Y}{X}\right)$. [3]

Let $U = XY$ and $W = 2X - Y$.

(ii) Show that $E[UW] = E[U]E[W]$. [6]

(iii) Evaluate $P(U = 2,\ W = 0)$. [2]

(iv) Determine whether or not U and W are independent. [4]

14. It is known that the true response, y, in a certain chemical process is linearly related to the operating temperature x°C. However, experimental determinations of y are subject to random errors, so that when the process is run at temperature x_i°C the observed response is given by

$$y_i = \alpha + \beta x_i + e_i,$$

where $\alpha + \beta x_i$ is the true response and e_i is the error. The following table gives the responses that were observed in nine runs, three at each of the temperatures 20°C, 30°C, and 40°C.

	Temperature (x_i) 20	30	40
Observed responses (y_i)	35	31	28
	33	32	27
	34	31	29

(i) Calculate the least squares estimates of α and β.
(You may assume that the sum of the products, $\sum x_i y_i$, of the 9 pairs of observed values (x_i, y_i) is equal to 8220.) [5]

(ii) The errors, e_i, are independent and normally distributed with mean zero and standard deviation 2·5. Calculate 95% confidence intervals for

(*a*) the value of β, [3]

(*b*) the true value of y when $x = 40$. [4]

(iii) Explain why the confidence interval you calculated in (ii)(*b*) is preferable to the one that could be obtained from the three observed responses when $x = 40$. [3]

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Advanced Level | Safon Uwch

MATHEMATICS A 4

Further Pure Mathematics

A.M. THURSDAY, 23 June 1988

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. The quadratic equations

$$x^2 + x - 1 = 0$$

and

$$x^2 - x + d = 0$$

have a common root. Show that

$$d = -2 \pm \sqrt{5}.$$

[4]

2. The function f is defined by

$$f(x) = \frac{1 + x + x^2}{1 - x^2}, \qquad x \neq \pm 1.$$

By obtaining a quadratic equation in x, find the range of f. [5]

Turn over.

3. The transformation T of the plane transforms the point (x, y) to the point (x', y') given by

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & a \\ 1 & 0 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

where a, b are constants. Determine the set of fixed points

(i) in the case $a + b = 0$,

(ii) in the case $a + b \neq 0$.

Given that $a + b = 0$, explain as simply as possible the effect of T followed by T. [5]

4. Show that

$$\int_0^4 \sqrt{x^2 + 9}\, dx = 10 + \frac{9}{2} \log_e 3.$$ [6]

5. The sum of the first n terms of a series is equal to $3n^2$.

Show that

(i) the nth term, T_n, is given by $T_n = 3(2n - 1)$,

(ii) $\sum_{r=1}^{n} T_r^2 = 3n(4n^2 - 1)$. [5]

6. Sketch the curves $y = x^{-n}$ for $x > 0$ and $n = \frac{1}{2}$, 1 and $\frac{3}{2}$ on the same axes.

By considering appropriate areas, show that for $a > 1$,

$$\frac{2}{\sqrt{a}}(\sqrt{a} - 1) < \log_e a < 2(\sqrt{a} - 1).$$ [5]

7. Use the method of induction to show that $5^n - 4n + 3$ is divisible by 4 for all positive integral values of n. [5]

8. The function f is defined on the domain $(-\infty, +\infty)$ by

$$f(x) = |x^2 + 4x + 3|.$$

Sketch the graph of f. Write down expressions for the derivative $f'(x)$ for values of x for which $f(x) \neq 0$. [5]

SECTION B

Answer **four** *questions from this section.*

9. The function f is defined by

$$f(x) = \frac{8x^2 + 1}{x(x-1)}.$$

Express $f(x)$ in the form

$$f(x) = A + \frac{B}{x} + \frac{C}{x-1}$$

where A, B, C are constants to be determined. [4]

Locate and classify the stationary values of f and sketch the graph of f. [6]

Determine, with the aid of your graph,

(i) $f(S_1)$, where S_1 is the interval $[2, 3]$,

(ii) $f^{-1}(S_2)$, where S_2 is the interval $[4, 8]$. [5]

10. (*a*) The non-singular square matrix $\mathbf{A}$ satisfies the equation

$$\mathbf{A}^r = \mathbf{I} \,.........\, (1)$$

where r is a positive integer greater than 1 and $\mathbf{I}$ is the identity matrix.

Show that

(i) $\mathbf{A}^{-1} = \mathbf{A}^{r-1}$

(ii) $(\mathbf{A}^{-1})^r = \mathbf{I}$.

In the case

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

find the smallest value of r satisfying equation (1) and hence write down the matrix $\mathbf{A}^{-1}$. [6]

(*b*) The variables x, y, z satisfy the equations

$$\begin{aligned} x + 2y + z &= k \\ 2x + y + 4z &= 6 \\ -x - 5y + kz &= 3 \end{aligned}$$

where k is a constant. Using reduction to echelon form, solve these equations for

(i) $k \neq 1$,

(ii) $k = 1$.

Interpret your results geometrically in each case. [9]

Turn over.

11. (*a*) Show by applying the Newton-Raphson method to the equation

$$x^m - a = 0$$

that the positive *m*th root of the positive number *a* may be found using the iterative process

$$x_{n+1} = \frac{(m-1)\,x_n^m + a}{m x_n^{m-1}}.$$

Hence calculate $\sqrt[5]{16}$ correct to 2 decimal places. [6]

(*b*) Show that the equation

$$1 - x^4 = \tan x - \sin x$$

has a root lying between 0·8 and 0·9.

Rewrite the equation in the form

$$x = F(x)$$

and use the iterative sequence

$$x_{n+1} = F(x_n)$$

to find the value of the root correct to 2 decimal places.

Illustrate graphically the convergence of the sequence. [9]

12. The points A, B, C, D, E have coordinates $(2, 2, 3)$, $(1, 1, 1)$, $(1, 2, 6)$, $(5, 4, 4)$ and $(-2, 7, 5)$ respectively. Show that the lines BC and AE do not intersect. [4]

Write down the direction ratios of AB and AC and deduce the acute angle between these two lines.

A line L has direction ratios $p : q : r$. Write down the conditions for L to be perpendicular to both AB and AC and deduce direction ratios of the normal to the plane ABC. Hence obtain the equation of the plane ABC and show that D lies in this plane. [7]

Write down the equations of the perpendicular from E to the plane ABC. Hence find the coordinates of the foot of the perpendicular from E to the plane and deduce the length of the perpendicular. [4]

13. Given that $(2 + 3\mathrm{i})$ is a root of the equation

$$z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$$

find the other three roots. [4]

Plot the four roots on an Argand diagram.

Given that a circle with centre on the real axis passes through the four plotted points, find its equation in the form

$$|z - \alpha| = \beta.$$ [5]

The four roots are denoted by z_1, z_2, z_3, z_4 where

$$\mathrm{Im}(z_1) > \mathrm{Im}(z_2) > \mathrm{Im}(z_3) > \mathrm{Im}(z_4).$$

Given that

$$Z = \frac{z_1 - z_2}{z_3 - z_4}$$

calculate, correct to 3 significant figures,

(i) $\arg(Z)$,

(ii) $\mathrm{Re}(\mathrm{e}^Z)$. [6]

14. (i) Let the Maclaurin series of $\tan x$ be

$$\tan x = a_1 x + a_3 x^3 + a_5 x^5 + \ldots$$

where a_1, a_3, a_5 are constants. Deduce the series expansion of $\sec^2 x$ as far as the x^4 term

(*a*) by differentiation,

(*b*) using the relationship $\sec^2 x = 1 + \tan^2 x$.

Hence by comparing the two series, show that $a_1 = 1$, $a_3 = \frac{1}{3}$ and $a_5 = \frac{2}{15}$. [5]

(ii) Assuming the series for $\tan x$, obtain the Maclaurin series of $x \cot x$ as far as the term in x^4. Write down the value of

$$\lim_{x \to 0} x \cot x.$$ [4]

(iii) The Maclaurin series of $\tan^{-1} x$ is denoted by

$$\tan^{-1} x = b_1 x + b_3 x^3 + b_5 x^5 + \ldots$$

where b_1, b_3, b_5 are constants. By considering the tangents of both sides and using the series for $\tan x$, obtain numerical values for b_1, b_3 and b_5. [5]

Deduce the Maclaurin series of $\cot^{-1} x$ for $x \geqslant 0$. [1]

15. The point $P(x, y)$ moves in such a way that its distance from the point $(2a, 0)$ is equal to its distance from the y-axis. Show that the equation of the locus C of P is

$$y^2 = 4a(x - a).$$

Verify that the point $T(a \cosh^2 t, 2a \sinh t)$ lies on C for all t. [4]

Show that

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{x}{x - a}.$$ [2]

That part of C lying between the points $V(a, 0)$ and T is rotated through four right angles about the x-axis. Find in terms of t the area of the curved surface generated. [4]

Show further using the substitution $x = a \cosh^2 z$, or otherwise, that the length of arc of C between V and T is equal to

$$a(t + \tfrac{1}{2} \sinh 2t).$$ [5]

IMPORTANT NOTE

Please note that the Mathematics S paper is not included separately. It contains three sections, Section A which is identical to Section A of the Pure Mathematics S paper, Section B which is is identical to Section A of the Applied Mathematics S paper and Section C which is is identical to Section B of the Applied Mathematics S paper. Candidates are required to answer not more than four questions from Section A and not more than four questions from either Section B or not more than four questions from Section C.

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Advanced Level | **Safon Uwch**

Special Paper | **Papur Arbennig**

PURE MATHEMATICS S

A.M. MONDAY, 27 June 1988

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Pure Mathematics

No more than **four** *questions should be answered from this section.*

1. (*a*) Given that, for $0 < \theta < \frac{\pi}{2}$,

$$x = \sec\theta - \tan\theta$$

show that

(i) $0 < x < 1$,

(ii) $\tan\frac{1}{2}\theta = \dfrac{1-x}{1+x}$

and obtain an expression for $\dfrac{dx}{d\theta}$ in terms of x only. [6]

(*b*) The points A, B lie on the circumference of a circle of unit radius and the chord AB cuts the circle into two parts whose areas are in the ratio 2 : 1.

Calculate the length AB correct to 2 decimal places. [9]

2. The curve C_1 has equation $y^2 = 4ax$. $P(ap^2, 2ap)$ is a point on C_1 and the normal at P meets C_1 again at Q. The tangents at P and Q meet at R. Show that R has coordinates $\left(-ap^2 - 2a, \dfrac{-2a}{p}\right)$. The locus of R as p varies is the curve C_2. Find the cartesian equation of C_2.

Show that the normal to C_2 at R cuts C_2 in two further points if $p^2 > 2$. [15]

3. The function ϕ is defined on the domain $x \geqslant 0$ and $\phi(0) = 0$. The derivative $\phi'(x)$ is positive for $x \geqslant 0$. Sketch the graph of ϕ and indicate the areas represented by the integrals

$$\int_0^a \phi(x)\,\mathrm{d}x \quad \text{and} \quad \int_0^b \phi^{-1}(y)\,\mathrm{d}y \quad (a, b > 0)$$

where ϕ^{-1} denotes the inverse function of ϕ.

Hence show that

$$ab \leqslant \int_0^a \phi(x)\,\mathrm{d}x + \int_0^b \phi^{-1}(y)\,\mathrm{d}y$$

and state the condition for equality to hold. [5]

By taking $\phi(x) = \log_e (1 + x)$, show that for $a > 1$, $b > 0$,

$$a(b + 1) \leqslant a \log_e a + e^b$$

and state the relationship between a, b for equality. [10]

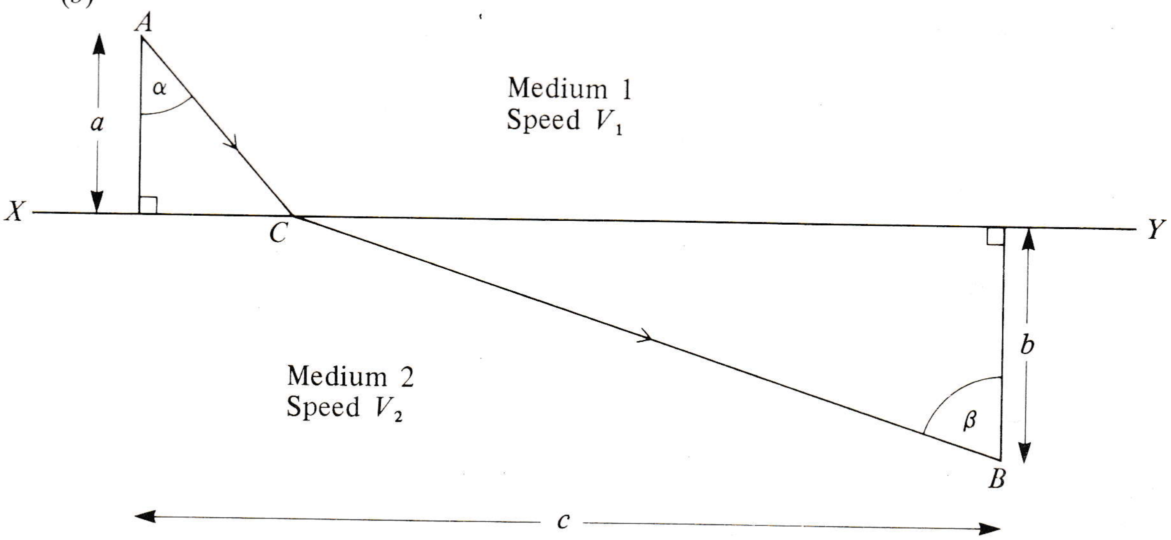
4. (*a*) The function f defined on the domain $x \geqslant 0$ is given by

$$f(x) = e^x - x^e.$$

Find the coordinates of the two turning points on the graph of f and hence sketch this graph.

Deduce, *without using tables or a calculator*, which is the larger of e^π and π^e. [7]

(*b*)



The figure shows a light ray travelling along the path ACB through two media separated by the straight line XY. The speeds of light in the two media are V_1 and V_2. If T denotes the time taken for the light ray to traverse this path, show that

$$T = \frac{a \sec \alpha}{V_1} + \frac{b \sec \beta}{V_2}$$

and $c = a \tan \alpha + b \tan \beta$.

Given that the path of the light ray minimises T, show that

$$\frac{\sin \alpha}{\sin \beta} = \frac{V_1}{V_2}.$$

[Any stationary value found may be assumed to be a minimum]. [8]

5. Let

$$I_n = \int_0^{\pi/2} \frac{d\theta}{(a + b\cos\theta)^n}$$

(i) Using the substitution $t = \tan\frac{1}{2}\theta$, show that for $b > a > 0$,

$$I_1 = \frac{1}{\sqrt{b^2 - a^2}} \log_e \left[\frac{\sqrt{b+a} + \sqrt{b-a}}{\sqrt{b+a} - \sqrt{b-a}}\right].$$

Obtain an expression for I_1 in terms of a and b in the case $a > b > 0$. [9]

(ii) Show that

$$\frac{d}{d\theta}\left[\frac{\sin\theta}{a + b\cos\theta}\right] = \frac{a}{b}\cdot\frac{1}{(a + b\cos\theta)} + \frac{(b^2 - a^2)}{b}\cdot\frac{1}{(a + b\cos\theta)^2}$$

Hence obtain expressions for I_2 in the two cases

(a) $b > a > 0$,

(b) $a > b > 0$. [6]

Section B

Further Pure Mathematics

No more than **four** *questions should be answered from this section.*

The following Taylor series expansion may be assumed where necessary:

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2} f''(a) + \ldots$$

6. Given that $u_i = v_i - v_{i-1}$, show that

$$\sum_{i=1}^{n} u_i = v_n - v_0$$

Let

$$\sum_{i=1}^{n} i^r = S_r(n).$$

Putting $v_i = i^r(i + 1)^r$ and considering the binomial expansion of u_i, show that

$$\tfrac{1}{2}n^r(n + 1)^r = \binom{r}{1}S_{2r-1}(n) + \binom{r}{3}S_{2r-3}(n) + \binom{r}{5}S_{2r-5}(n) + \ldots$$ [6]

State the last term on the right-hand side, distinguishing between the cases

(a) r even, (b) r odd.

Use this result to obtain an expression for $S_5(n)$ in terms of n. [9]

7. (*a*) The members of a School Computer Society build a small computer in which the arithmetic unit can perform only the operations add, subtract and multiply; it cannot divide. They decide to program the computer to divide a by b $(b > 0)$ by multiplying a by the reciprocal (b^{-1}) of b. They consider the possibility of using an iterative sequence involving only the available operations to find b^{-1}.

Use the Newton-Raphson method to derive the following iterative sequence for b^{-1}, where x_0 is a suitably chosen initial approximation:

$$x_{n+1} = x_n (2 - bx_n)$$

Find the range of values of x_0 for which the sequence converges to b^{-1}. [7]

(*b*) Consider the equation

$$a \sinh x + b \cosh x = c$$

where a, b, c are positive quantities. Find the condition on a, b, c for this equation to have

(i) no real roots,

(ii) 1 real root,

(iii) 2 real roots. [8]

8. The characteristic polynomial of the square matrix $\mathbf{A}$ is defined as

$$f(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}).$$

The eigenvalues of $\mathbf{A}$ are the values of λ for which $f(\lambda) = 0$.

(i) Find in its simplest form, the characteristic polynomial of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}.$$

Hence find all the eigenvalues of $\mathbf{A}$. [6]

(ii) Verify that

$$\mathbf{A}^3 = 4\mathbf{A}^2 + \mathbf{A}$$

and use induction to prove that any polynomial in $\mathbf{A}$ of degree exceeding two can be expressed as a quadratic in $\mathbf{A}$. [5]

(iii) Express

$$\mathbf{A}^4 + \mathbf{A}^3 + \mathbf{A}^2 + \mathbf{A} + \mathbf{I}$$

as a quadratic in $\mathbf{A}$. [4]

9. Show, by considering an appropriate Taylor series, that

$$\int_a^b f(x)\,\mathrm{d}x \approx (b-a)\,f(c) + \frac{(b-a)^3}{24} f''(c)$$

for suitably small $b - a$, where $c = \frac{1}{2}(a+b)$. [4]

By putting $a = 0$ and $b = \frac{\pi}{2}$, find an approximation to the integral

$$I = \int_0^{\pi/2} \frac{\mathrm{d}x}{\cos x + \tan x}$$

giving your answer correct to 2 decimal places. [5]

Show that the exact value of I is

$$\frac{1}{\sqrt{5}} \log_e\left(\frac{7+3\sqrt{5}}{2}\right).$$ [6]

10. (*a*) If z_1 and z_2 are two complex numbers, show using an Argand diagram that

$$|z_1| - |z_2| \leqslant |z_1 - z_2|$$

and

$$|z_1| + |z_2| \geqslant |z_1 - z_2|$$

State the condition for equality in each case. [5]

(*b*) The complex number $z + \mathrm{i}$ satisfies the equation

$$|z - \tfrac{1}{2}\mathrm{i}| = \tfrac{1}{2}$$

and

$$w = \frac{\mathrm{i}z}{\mathrm{i} - z}.$$

Show that

(i) w is real,

(ii) the points representing w, z and i on the Argand diagram are collinear.

Illustrate (i) and (ii) on an Argand diagram. [10]

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General Certificate of Education | **Tystysgrif Addysg Gyffredinol**

Advanced Level | **Safon Uwch**

Special Paper | **Papur Arbennig**

APPLIED MATHEMATICS S

A.M. MONDAY, 27 June 1988

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Mechanics

No more than **four** *questions should be answered from this section.*

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

1. A particle P of mass m is attached to two elastic strings each of natural length a and modulus $3mg$. The first string AP has its other end A attached to a fixed point. The other string PB has its other end attached to a fixed point B which is vertically below A and at a distance $3a$ from it. Show that when the system is disturbed from equilibrium, with both strings remaining taut and vertical, the resulting motion is simple harmonic of period $2\pi\sqrt{\dfrac{a}{6g}}$.

The system rests in equilibrium when the mass is given a velocity of magnitude u in the direction PA. Show that the string PA remains taut if $u^2 \leqslant \dfrac{8ag}{3}$. If $u^2 = \dfrac{25ag}{6}$, find the height of the point above B where the particle first comes to rest. [15]

2. A block of mass 1 kg rests in equilibrium on a rough horizontal plane under the action of four horizontal forces $3\mathbf{i}$ N, $(\mathbf{i} - \mathbf{j})$ N, $-(4\mathbf{i} + 4\mathbf{j})$ N and $(P\cos\alpha\,\mathbf{i} + P\sin\alpha\,\mathbf{j})$ N. The coefficient of friction between the block and plane is $0{\cdot}5$.

(i) Find the range of possible values of P when $\cos\alpha = \frac{4}{5}$.

(ii) Given that the block is on the point of moving, find the least value of P when α may be varied in the interval $0 \leqslant \alpha \leqslant 90°$. [15]

Turn over.

3. A particle of mass m is projected under gravity with a velocity whose horizontal and vertical components are u and v. It moves in a medium which exerts a resistance mk times the velocity. Show that

(i) the resulting force on the particle is constant in direction,

(ii) when the horizontal displacement of the particle is x, the magnitude of the force acting on it is $\left(1 - \frac{kx}{u}\right)$ times the initial value of the force,

(iii) the greatest height of the particle above its point of projection occurs when its horizontal displacement is $\frac{uv}{kv + g}$. [15]

4. A uniform hoop of mass M, threaded by two small smooth rings each of mass m, is standing in a vertical plane on a horizontal table. Initially the system is at rest with the rings held at the top of the hoop. The rings are released so that they fall down opposite sides of the hoop. Show that if $m \geqslant \frac{3M}{2}$, the hoop will leave the table, and find the position of the rings at the instant when this occurs. [15]

5. A straight non-uniform rod AD is of length a. The rod consists of three straight uniform parts AB, BC and CD, each of length $\frac{a}{3}$ and of masses M, kM and lM respectively. The constants k and l are each greater than 1. The rod is in equilibrium with its ends on the inside surface of a smooth fixed spherical bowl of radius a, D being lower than A. The inclination θ of DA to the horizontal is given by $\tan\theta = \frac{2\sqrt{3}}{27}$. Given further that the reaction at A is $\frac{42Mg}{\sqrt{247}}$, find the values of k and l. [15]

Section B

Probability and Statistics

No more than **four** *questions should be answered from this section.*

6. (*a*) John and Jane each throw two fair cubical dice, each die having its six faces numbered 1, 2, 3, 4, 5 and 6 respectively. Calculate the probability that they will obtain the same pair of numbers. [5]

(*b*) A lady has lost one of her ear-rings. She knows that the ear-ring must have slipped off her ear either in the bedroom or in the lounge. She assesses that the probabilities of the missing ear-ring being in the bedroom and the lounge are 0·6 and 0·4, respectively. The lady has her husband and two children to help in the search for the missing ear-ring. She decided to search the bedroom and tells her husband and two children to search the lounge. Assume that, independently for each person searching the room in which the missing ear-ring is, the probability is 0·7 that the person will not find the ear-ring.

(i) Calculate the probability that the missing ear-ring will be found. [4]

(ii) Determine how the four persons should have been allocated to the two rooms in order that the probability of finding the missing ear-ring is maximised. [6]

7. (*a*) A building contractor employs n labourers. The probability that a labourer will turn up for work on any day is p, independently for each labourer. On a particular day when a job is due to be completed, the contractor knows that if r of the n labourers turn up then the probability of completing the job that day is cr, where $0 < c \leqslant \frac{1}{n}$.

(i) Name the distribution of X, the number of labourers that will turn up for work on that day. Write down, in terms of n and p, expressions for $E[X]$ and $E[X^2]$. [2]

(ii) Find, in terms of c, n and p, the probability that the job will be completed that day. [2]

(iii) Given that the job was completed that day, find the expected number of labourers that turned up. [4]

(*b*) X and Y are independent random variables, each having the Poisson distribution with mean μ. Show that $Z = X + Y$ has the Poisson distribution with mean 2μ.

[You may assume the result

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.]$$

[3]

Suppose that X and Y are the numbers of two types of bacteria that may be present in a sample of liquid. Given that there is at least one bacterium present in a sample, show that the probability of all the bacteria present being of one type is $\frac{2}{(1 + e^{\mu})}$. [4]

8. For a randomly chosen jar containing honey, let X denote the mass in grams of the jar itself (including the lid), Y the mass in grams of honey in the jar, and T the combined mass in grams of the honey and the jar. Which two of these three random variables may reasonably be assumed independent? [1]

Given that X is normally distributed with mean 250 and standard deviation 2·5, and that T is normally distributed with mean 700 and standard deviation 6·5, what is the distribution of Y? [3]

Calculate, correct to three decimal places, the probability that the total mass of the honey contained in five filled jars is less than 64% of the total combined mass of the five filled jars. [7]

The filled jars are dispatched to shops in containers of 25 jars each. Given that the combined mass in grams of a container and its contents is normally distributed with mean 18 000 and standard deviation 40, what is the distribution of the masses of the empty cartons? [4]

9. A machine is set to dispense 250 ml of liquid into bottles. The actual amount dispensed per bottle is X ml, where X is normally distributed with mean 250 and standard deviation 1, so that $U = X - 250$ has the standard normal distribution whose probability density function ϕ is given by

$$\phi(u) = \frac{1}{\sqrt{(2\pi)}} e^{-u^2/2}, \qquad -\infty < u < \infty.$$

(i) Find, correct to three decimal places, the expected value of $|X - 250|$, the absolute difference between X and 250. [6]

(ii) Only bottles containing 250 ml or more are accepted. Let Y ml denote the liquid content of an accepted bottle. Show that the cumulative distribution function, G, of Y is such that

$$G(y) = 2\Phi(y - 250) - 1, \text{ for } y \geqslant 250,$$

where Φ is the cumulative distribution function of the standard normal distribution. Hence, or otherwise, calculate, correct to two decimal places, the mean liquid content of accepted bottles. [9]

Turn over.

10. The discrete random variable X has the distribution:

$$P(X = 1) = 0{\cdot}8 - \alpha,\ P(X = 2) = 0{\cdot}2 - \alpha,\ P(X = 3) = 2\alpha.$$

(i) Determine the interval of possible values for α. [2]

(ii) Find an unbiased estimator of α in terms of $\bar{X}$, the mean of a random sample of n observations of X. By means of an example with $n \geqslant 2$, or otherwise, show that this estimator could give an estimate of α which is outside the interval of possible values for α. [6]

In a random sample of 100 observations of X, the calculated value of $\bar{X}$ was $1{\cdot}5$. Explaining your method carefully, calculate approximate 95% confidence limits for α. [7]