WELSH JOINT EDUCATION COMMITTEE | **CYD-BWYLLGOR ADDYSG CYMRU**

General Certificate of Education | **Tystysgrif Addysg Gyffredinol**

Advanced Level | **Safon Uwch**

MATHEMATICS A 1

PURE MATHEMATICS

P.M. TUESDAY, 2 June 1987

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

SECTION A

Answer **all** *questions in this section.*

1. The two roots of the quadratic equation

$$x^2-3x+1=0$$

are denoted by α, β. Without finding numerical values for α, β, determine the quadratic equation whose roots are $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$. [4]

2. The co-ordinates of A, B, C, D are respectively $(-1,-3)$, $(1,-2)$, $(3,5)$, $(2,4)$. Show that the equation of AC is $y=2x-1$ and find the equation of BD in a similar form. Hence find the co-ordinates of E, the point of intersection of AC and BD.

Calculate, to the nearest degree, the acute angle between the lines AC and BD. [6]

3. Use the trapezium rule with 5 ordinates and an interval of 0·25 to find an approximate value for the integral

$$\int_2^3 \sqrt{x-\frac{1}{x}}\,dx$$

giving your answer correct to 2 decimal places. [4]

4. Given that

$$(5+12i)^{\frac{1}{2}}=a+bi$$

where a, b are real, square both sides of the equation to obtain a pair of simultaneous equations for a and b. Hence find, in Cartesian form, the two square roots of $5+12i$. [4]

5. The power P developed in a resistance R carrying a current I is given by

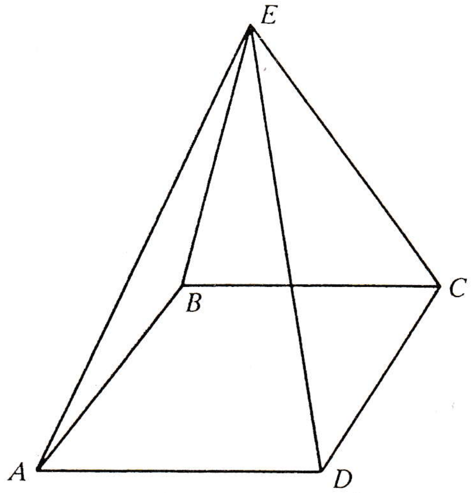
$$P = I^2 R.$$

Given that R is constant, show that

$$\frac{dP}{dI} = \frac{2P}{I}.$$

Find the approximate percentage increase in P if I increases by 1%. [4]

6.



The figure shows a square-based pyramid in which

$$AB = BC = CD = DA = AE = BE = CE = DE.$$

Find the angles between

(i) the lines AE and CD,

(ii) the line AE and the plane $ABCD$,

(iii) the planes ABE and $ABCD$. [6]

7. Given that A, B, C are acute angles, show that

$$\sin(A+B+C) = \cos A \cos B \cos C(\tan A + \tan B + \tan C - \tan A \tan B \tan C).$$

Deduce that if A, B, C are the angles of an acute-angled triangle, then

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$ [5]

8. The function f is defined by

$$f(x) = x + \frac{1}{x}$$

with domain $x > 1$. Show that f is an increasing function of x, and state its range.

Find an expression for $f^{-1}(x)$ and sketch the graphs of f and f^{-1} on the same axes. [7]

Section B

Answer **four** *questions from this section.*

9. Sketch the curve whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (a, b>0). \qquad [1]$$

Find the gradient of the curve at the point $P\ (a\cos\theta, b\sin\theta)$.

Hence show that the equation of the tangent at P is

$$bx\cos\theta+ay\sin\theta=ab$$

and that the equation of the normal at P is

$$ax\sin\theta-by\cos\theta=(a^2-b^2)\sin\theta\cos\theta. \qquad [4]$$

The tangent at P meets the x-axis at T and the y-axis at U; the normal at P meets the x-axis at M and the y-axis at N. Show that NT is perpendicular to MU. [5]

Write down the co-ordinates of the mid-point Q of UT and hence find the equation of the locus of Q as θ varies. [5]

10. Show by completing the square that x^2-x+1 is positive for all values of x.

The function f is defined by

$$f(x)=\frac{x^2+x+1}{x^2-x+1}.$$

(i) Obtain an expression for $f'(x)$ and show that

$$f''(x)=\frac{4(x^3-3x+1)}{(x^2-x+1)^3}. \qquad [5]$$

(ii) Find the co-ordinates of the two stationary points on the graph of f. Show that one is a maximum and the other a minimum. [3]

(iii) Determine the signs of $f''(x)$ when $x=-2, -1, 0, 1, 2$. Explain what your results tell you about the number and approximate positions of the points of inflection on the graph of f. [3]

(iv) State the limiting values of $f(x)$ as $x\to\pm\infty$. [1]

Sketch the graph of f for $-\infty<x<\infty$. [3]

11. The points A, B, C, D have position vectors $\mathbf{i}-2\mathbf{j}+5\mathbf{k}$, $\mathbf{i}+3\mathbf{j}$, $10\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $-2\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ respectively with respect to an origin O and a set of perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. The points P, Q lie on AB, CD respectively and are such that

$$\frac{AP}{PB}=\frac{\lambda}{1-\lambda} \quad \text{and} \quad \frac{CQ}{QD}=\frac{\mu}{1-\mu}.$$

Show that

$$\mathbf{OP}=\mathbf{i}+(5\lambda-2)\mathbf{j}+5(1-\lambda)\mathbf{k}$$

and obtain a similar expression for $\mathbf{OQ}$. [5]

Given that PQ is perpendicular to both AB and CD, obtain a pair of simultaneous equations for λ and μ. By solving these equations, show that

$$\mathbf{PQ}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$$

and find the length of PQ. [5]

Explain what is meant by the statement that $\mathbf{AB}$, $\mathbf{CD}$ and $\mathbf{PQ}$ form a set of base vectors. Given that

$$\mathbf{AD}=\alpha\mathbf{AB}+\beta\mathbf{CD}+\gamma\mathbf{PQ}$$

calculate the values of the constants α, β, γ. [5]

12. (*a*) Starting with the result

$$\sin x = 2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x$$

show that

$$\sin x = \frac{2t}{1+t^2}, \qquad \text{where } t = \tan\tfrac{1}{2}x.$$

Given that

$$2\sin x + 3\cos x = 1 \qquad \ldots\ldots\ldots\ldots\text{(i)}$$

express both $\sin x$ and $\cos x$ in terms of t, and show that

$$2t^2 - 2t - 1 = 0.$$

Hence find all values of x between $0°$ and $360°$ satisfying equation (i). [7]

(*b*) Show that

$$\sin 5\alpha - \sin\alpha \equiv 2\sin\alpha(\cos 4\alpha + \cos 2\alpha).$$

The triangle ABC has $B\widehat{A}C = \alpha$, $A\widehat{B}C = 5\alpha$ and $AC = 4BC$. Show that

$$\sin 5\alpha = 4\sin\alpha$$

and hence deduce that

$$2\cos 4\alpha + 2\cos 2\alpha = 3.$$

By solving a quadratic equation in $\cos 2\alpha$, or otherwise, find the value of α correct to the nearest degree. [8]

13. (*a*) Given that

$$\left(\frac{x+2}{x}\right)^{-\frac{1}{2}} = a + \frac{b}{x} + \frac{c}{x^2} + \ldots$$

use the binomial theorem to find a, b and c.

For what range of values of x is this expansion valid?

Putting $x = 7$, show that

$$\sqrt{7} \approx \tfrac{261}{98}.$$

Find another rational approximation to $\sqrt{7}$ by multiplying both sides of the above result by $\sqrt{7}$. Verify that the arithmetic mean of these two approximations differs from $\sqrt{7}$ by less than 6×10^{-5}. [7]

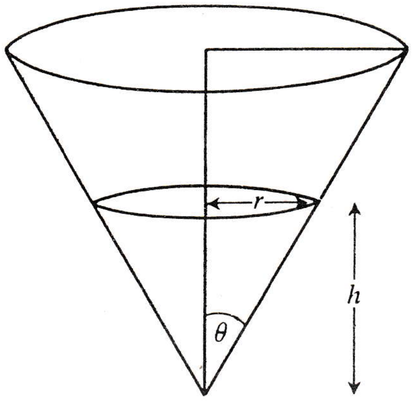
(*b*) The first, second and third terms of a geometric series are p, p^2, q respectively, where $p < 0$. The first, second and third terms of an arithmetic series are p, q, p^2 respectively.

(i) Find the values of p and q.

(ii) Find expressions for the sum of the first n terms of both the geometric series and the arithmetic series.

(iii) The sum of the first N terms of the geometric series is $-171/512$. Find N. [8]

14. (*a*)



The figure shows a hollow inverted right circular cone with semi-vertical angle θ. Water is poured into the cone and at time t, the solid cone of water has volume V, height h, base radius r and base area A. Given that the volume of water increases at a constant rate k per unit time, show that

$$\frac{dh}{dt} = \frac{k}{\pi r^2} \quad \text{and} \quad \frac{dA}{dt} = \frac{2k}{h}. \qquad [7]$$

(*b*) A sector of a circle has radius r and the angle between the two radii is θ. Obtain expressions for the area A and perimeter P of the sector in terms of r and θ and show that

$$A = \tfrac{1}{2}r(P-2r).$$

Hence show that the maximum area of a sector with a given perimeter P is $\frac{1}{16}P^2$.

Find the minimum perimeter of a sector with a given area A. [8]

15. (*a*) Express $\dfrac{1}{x^2(x+1)}$ in partial fractions. Hence evaluate the indefinite integral

$$\int \frac{dx}{x^2(x+1)}. \qquad [5]$$

(*b*) Writing $x = a\cos^2\theta + b\sin^2\theta$, show that for $b > a$,

$$\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} = \pi. \qquad [5]$$

(*c*) Show that the area enclosed by the curve $y = \sin^{-1}x$, the x-axis and the ordinate $x = 1$ is equal to $\frac{1}{2}\pi - 1$. [5]

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General Certificate of Education — Tystysgrif Addysg Gyffredinol

Advanced Level — Safon Uwch

MATHEMATICS A 2

MECHANICS

A.M. WEDNESDAY, 10 June 1987

(3 Hours)

Answer **all** *questions in Section A and* **four** *from Section B.*

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

SECTION A

Answer **all** *questions in this section.*

1. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y = 9x. \qquad [4]$$

2. A catapult consists of two lengths of elastic, each of modulus 20 N and natural length 0·2 m. The catapult is stretched so that the length of each elastic is increased by 0·08 m. Ignoring the effect of gravity and using energy considerations, find the speed the catapult will give to a stone of mass 0·02 kg. [3]

3. A car of mass 1500 kg is moving along a horizontal road. The resistance to the motion of the car is 750 N. Assuming that the car's engine works at 15 kW, find

(i) the maximum constant speed at which the car can travel,

(ii) the acceleration of the car when its speed is 8 m s^{-1}. [5]

4. Two particles of mass M and $4M$ respectively, are connected by a light inextensible string passing over a smooth fixed pulley. The particles are released from rest with the hanging parts of the string vertical. Find

(i) the acceleration of the particles,

(ii) the force exerted by the string on the pulley during the subsequent motion. [4]

5. A thin hemispherical cup of radius 4 cm is made of thin sheet metal of surface density ρ kg/cm^2. It is filled with liquid of density 3ρ kg/cm^3. Show that the centre of mass of the full cup is at a distance of $\frac{8}{5}$ cm from the liquid surface.

[The surface area of a hemisphere of radius a is $2\pi a^2$ and its volume is $\frac{2}{3}\pi a^3$.] [5]

6. Particles A and B move with velocities $3\mathbf{i}+4\mathbf{j}$ and $\mathbf{i}+3\mathbf{j}$ respectively. Write down the velocity of B relative to A.

When $t=0$, particle A is at the origin and particle B is at the point with position vector $\mathbf{i}-\mathbf{j}$. Find the displacement of B relative to A at time t, and show that the shortest distance between A and B in the subsequent motion is $\dfrac{3\sqrt{5}}{5}$. [5]

7. $OABC$ is a square of side a m. Forces of magnitude 1 N, 2 N, 3 N, 4 N and $\sqrt{2}$ N act along OA, AB, BC, CO and AC respectively. This system of forces can be reduced to a single resultant force which cuts OC produced at D.

Find

(i) the magnitude of the resultant,

(ii) the length of OD in terms of a. [7]

8. Two particles A and B of equal mass m are joined by a taut light inextensible string on a smooth horizontal table. The particles are at rest when B is acted upon by a horizontal impulse of magnitude I at an angle of 45° to AB away from A. Find the initial speed of B and the impulsive tension in the string. [7]

Section B

Answer **four** *questions from this section.*

9. A lift when empty is of mass 1200 kg.

(i) The lift carries a man of mass 80 kg who is holding a parcel of mass 2 kg.

(*a*) The man and lift are descending with an acceleration 1 m s^{-2}. Find the tension in the lift cable, the vertical force exerted on the man by the lift, and the vertical force exerted on the parcel by the man. [3]

(*b*) The man drops the parcel from a height of $1\frac{1}{2}$ m. Given that the man takes 1 s to react and catch the parcel, find the least downwards acceleration of the lift in order that the man may catch the parcel before it hits the floor of the lift. [4]

(ii) In the following, assume that the mass of any passenger is 80 kg. Safety regulations do not allow the lift cable to bear a tension greater than T newtons or an acceleration greater than 1 m s^{-2}. The tension in the cable is $\frac{1}{2}T$ newtons when N people are accelerating upwards in the lift at 1 m s^{-2}. The same tension occurs when $N+5$ people are accelerating downwards at 1 m s^{-2}. Find N and T. [8]

10. (i) The natural length and modulus of a light elastic spring AB are l_0 m and λ N respectively. The end A of the spring is fixed. When a particle of mass M kg is attached to the spring at B and hangs freely under gravity, the extension of the spring is 0·2 m. The mass is pulled down through a further small distance so that the spring is extended, and then released from rest. Show that the subsequent motion is simple harmonic of period $2\pi/7$ s. [4]

(ii) A particle moves with simple harmonic motion along the x-axis about the origin O, of period $\dfrac{2\pi}{\omega}$ and amplitude a.

(*a*) Write down expressions giving the acceleration and speed at a point distance x from O.

At a certain time, $x = 3$ m and the speed and the acceleration of the particle are equal in magnitude. Given further that the maximum speed in the motion is 2 m s^{-1}, show that the period and amplitude of the motion are $2\sqrt{3}\pi$ s and $2\sqrt{3}$ m respectively. [7]

(*b*) Two particles A and B move in simple harmonic motion about O with amplitude $2\sqrt{3}$ m and period $2\sqrt{3}\pi$ s. A is released from rest at time $t = 0$ from the extreme point P where $x = 2\sqrt{3}$. Particle B is released from P at time $t = \dfrac{\sqrt{3}\pi}{2}$ s. Show that the particles will collide $\dfrac{3\sqrt{3}\pi}{4}$ s after the release of B. Find how far from O the collision will occur. [4]

11. (i) An economic theory suggests that the relationship between the national debt £D and national income £I can be represented by the following differential equations:

$$\frac{dD}{dt} = aI \quad \text{and} \quad \frac{dI}{dt} = bI,$$

where a and b are positive constants. If I_0, D_0 are the initial values of I and D respectively, show that

$$D = \frac{aI_0}{b}(e^{bt} - 1) + D_0. \qquad [6]$$

(ii) A body of mass 49 kg is released from rest at $t = 0$ and travels in a horizontal straight line in a medium which exerts a resistance of $5gv$ N, where v m s^{-1} is the velocity of the body. There is also a propulsive force of $196\,e^{-t/2}$ N acting on the body. Show that v satisfies the differential equation

$$\frac{dv}{dt} + v = 4e^{-t/2}.$$

Find v in terms of t and show that the acceleration vanishes just once during the motion. Sketch the graph of v against t ($t \geqslant 0$). [9]

12. (i) A constant force of $(5\mathbf{i}+3\mathbf{j})$ N is applied to a ring which is threaded on a smooth rod. The rod is fixed in a horizontal plane and connects the origin to the point P of position vector $(3\mathbf{i}+\mathbf{j})$ m. Find the work done by the force in moving the ring from the origin to P.

Given that the ring is of mass 4 kg and that it was initially at rest at the origin, find its speed when it reaches P. [3]

(ii) A particle of mass 3 kg moves in a plane, its position vector at time t being

$$\mathbf{r} = (2\cos t + \sin t)\mathbf{i} + (\sin t - 2\cos t)\mathbf{j}.$$

Find the values of t

(*a*) when its speed is a maximum,

(*b*) when the force acting on it is perpendicular to the velocity vector,

(*c*) when the power generated by the force acting on it is greatest. [7]

(iii) A force of magnitude 10 N acting in a direction parallel to the vector $3\mathbf{i}+4\mathbf{j}$ is the resultant of two forces which act parallel to the vectors $\mathbf{i}+2\mathbf{j}$ and $2\mathbf{i}+\mathbf{j}$ respectively. Find vectors representing these two forces. [5]

13. (i) A particle is projected from a point O on horizontal ground with speed V at an angle θ with the horizontal. Neglecting air resistance, find the time to reach the highest point of the trajectory and the time of flight. Show that the range R and the greatest height H reached are given by

$$R = \frac{V^2}{g}\sin 2\theta, \qquad H = \frac{V^2 \sin^2\theta}{2g}.$$

Given that $R = 4H$, find θ. [5]

(ii) A boy is standing on level ground at a horizontal distance a from the foot of a smooth vertical wall. He throws a ball at the wall, the ball's motion being in a vertical plane perpendicular to the wall. The speed of projection is $\sqrt{2ag\sec\theta}$, where θ is the angle of projection. Given that the ball hits the wall at the instant when its velocity is horizontal, show that $\theta = \frac{1}{6}\pi$.

The boy starts to run towards the wall with speed $\sqrt{\dfrac{ag}{48}}$ at the instant that the ball leaves his hand. He catches the ball at the same level as the point of projection after it rebounds from the wall. Given that the coefficient of restitution for the collision between the ball and the wall is e, write down the horizontal component of the ball's velocity after the rebound. Hence show that the ball is caught at a distance ea from the wall. Show further that the value of e is $1-\frac{1}{6}(3)^{\frac{1}{2}}$. [3, 7]

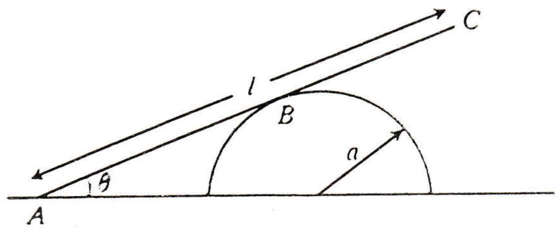
14. (i) A bead sliding on a fixed smooth vertical circular hoop of radius a has speed V at the lowest point. Prove that the bead makes complete revolutions if $V^2 > 4ag$ and the force exerted by the hoop on the bead is always radially inwards if $V^2 > 5ag$.

If the ratio of the greatest and lowest speeds is $\sqrt{7}:1$ show that $V = \sqrt{\dfrac{14ag}{3}}$. Find the position of the bead when the force between it and the hoop vanishes. [7]

(ii) A particle of mass m is attached by a light inextensible string of length l to the vertex of a cone of semivertical angle α. The cone is fixed with its axis vertical and the particle moves in a horizontal circle on the smooth outside surface of the cone with angular speed ω. Find expressions for the reaction R between the cone and the particle, and the tension T in the string. Find the greatest possible value of ω in terms of g, l and α if the particle is to remain on the cone.

Suppose now that the string is elastic, of natural length a and modulus $3mg$, $\alpha = \tan^{-1}(\frac{3}{4})$ and that $\omega^2 = \dfrac{g}{2a}$. Find the extension of the string. [4, 1, 3]

15.



A uniform plank AC of length l and mass M rests with one end A on smooth horizontal ground. The plank is also supported at a point B between A and C by a fixed semicircular ramp of radius a whose base rests on the ground. The plank is perpendicular to the axis of the ramp and makes an angle θ with the horizontal where $\tan\theta > a/l$. The coefficient of friction for the contact between plank and ramp is μ.

Draw a diagram showing the forces acting on the plank.

(i) Find the forces acting on the plank and deduce two further inequalities satisfied by $\tan\theta$.

(ii) When $\mu = \frac{1}{4}$, $l = 4a$ and $\theta = \tan^{-1}(\frac{5}{12})$, a horizontal force of magnitude P is applied at A to ensure equilibrium. Show that the minimum value of P is $20Mg/169$. [2, 7, 6]

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General Certificate of Education **Tystysgrif Addysg Gyffredinol**

Advanced Level **Safon Uwch**

MATHEMATICS **A 3**

Probability and Statistics

A.M. MONDAY, 15 June 1987

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

Mathematical and Statistical tables are provided.

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. The two events A and B are such that

$$P(A) = 0{\cdot}2, \qquad P(B) = 0{\cdot}3, \qquad \text{and } P(A \cup B) = 0{\cdot}4.$$

(i) Determine whether or not A and B are independent.

(ii) Evaluate $P(B \mid A')$. [4]

2. A soft-drinks machine is regulated so as to deliver an average of 200 ml per cup. The actual amount delivered per cup is a normally distributed random variable having mean 200 ml and standard deviation 8 ml.

(i) Given that the maximum capacity of a cup is 210 ml find, to three decimal places, the probability that a cup will overflow.

(ii) Find, to the nearest ml, what the maximum capacity of each cup should be for only 1% of the cups to overflow. [5]

3. The continuous random variable X has probability density function f given by

$$f(x) = 3x^2, \quad \text{for } 0 < x < 1,$$
$$f(x) = 0, \quad \text{otherwise.}$$

Find the mean and the variance of

(i) X,

(ii) X^{-1}. [5]

4. A pack of 52 playing cards consists of 4 aces, 12 picture cards, and 36 other cards. A hand of 13 cards is dealt at random without replacement from the pack.

(i) Write down expressions, but **do not** simplify them, for the probabilities that the hand will include

(*a*) exactly 2 aces and 11 picture cards,

(*b*) exactly 2 aces and exactly 8 picture cards.

(ii) Let p_1 denote the probability that the hand of 13 cards will include no ace and exactly 8 picture cards, and let p_2 denote the probability that the hand will include exactly 3 aces and exactly 6 picture cards. Show that $\frac{p_1}{p_2} = \frac{6}{7}$. [6]

5. The random variable X is normally distributed with unknown mean μ cm and standard deviation 2 cm.

(i) A random sample of 16 observations of X had values which summed to 118·4 cm. Calculate a 95% confidence interval for the value of μ.

(ii) Find the smallest sample size, n, of observations of X that should be taken for the width of the 95% confidence interval for μ calculated from the sample values to be less than 1 cm. [6]

6. The discrete random variable X has the distribution

$$P(X = 1) = 0{\cdot}4, \qquad P(X = 2) = 0{\cdot}2, \qquad P(X = 3) = 0{\cdot}4.$$

Calculate the variance of X.

Let X_1 and X_2 denote two independent observations of X. Derive the sampling distribution of $T = \frac{1}{2}(X_1 - X_2)^2$, and verify that T is an unbiased estimator of the variance of X. [7]

7. In an investigation of the relationship $y = \alpha + \beta x$ connecting the two variables x and y, five experiments were conducted with x having the values 20, 30, 40, 50, and 60, respectively, and the corresponding values of y were measured. The least-squares estimate of the relationship was calculated from the results to be

$$y = 3{\cdot}4 - 0{\cdot}65x.$$

Assuming that the errors in the y-measurements are independent and normally distributed with mean zero and standard deviation 0·2, calculate 90% confidence limits for the values of

(i) α,

(ii) β. [7]

SECTION B

Answer **four** *questions from this section.*

8. The demand per week for videos in a particular shop has the Poisson distribution with mean 4.

(i) Find, to four decimal places, the probabilities that in a week the demand for videos at this shop will be

(*a*) exactly 2,

(*b*) 5 or fewer. [3]

(ii) Assuming that the demands in successive weeks are independent, use an appropriate distributional approximation to calculate, to three decimal places, the probability that the demand for videos at the shop during a year of 52 weeks will be at least 220. [4]

Videos are delivered to the shop at the beginning of each week only, and the policy is to accept as many as are required for the stock level to be 5.

(iii) Show that the most probable number of videos that will be *sold* in a week is 5, and find, to two decimal places, the expected number of videos that will be *sold* in a week. [8]

9. A bag contains 2 white balls and 3 black balls. Two balls are drawn simultaneously and at random from the bag. Each black ball drawn (if any) is then replaced in the bag, and each white ball drawn (if any) is discarded. Two balls are then drawn simultaneously and at random from the balls that are now in the bag. Let X denote the number of white balls obtained in the first draw, and let Y denote the number of white balls obtained in the second draw.

(i) Show that $P(X = 1, Y = 1) = 0{\cdot}3$, and that $P(X = 2, Y = 0) = 0{\cdot}1$. [4]

(ii) Construct a two-way table for the joint probability distribution of X and Y. [4]

(iii) Find the expected value of Y. [2]

(iv) Each white ball obtained in either draw scores 3 points, while each black ball obtained in either draw scores 1 point. Let T denote the sum of the scores after the two draws. Find the mean and the variance of T. [5]

10. A box contains ten coins. Four of the coins are fair, while each of the other six coins is such that when tossed the probability of obtaining a head is $\frac{1}{4}$.

(i) One coin is chosen at random from the box and is tossed twice. Let A denote the event that the first toss gives head, and let B denote the event that the second toss gives head. Show that A and B are not independent. [7]

(ii) Suppose, instead, that two coins are chosen at random from the ten coins in the box and that they are tossed together once.

(*a*) Calculate the probability that one head and one tail will be obtained.

(*b*) Given that one head and one tail were obtained, calculate the conditional probability that at least one of the two chosen coins was a fair coin. [8]

11. A self-service cafeteria offers only two main courses, A and B, at lunchtime. Experience has shown that 60% of the customers opt for A.

(i) Calculate the probabilities that on any day

(*a*) exactly 3 of the first 6 customers will opt for A, [2]

(*b*) the fourth customer will be the first to opt for B. [1]

(ii) Using the tables provided find, to four decimal places, the probability that of the first 10 customers on any day, 7 or fewer will opt for A. [2]

(iii) On a particular day only 30 servings of A and 25 servings of B are available. Use the tables provided to find, to four decimal places, the probabilities that of the first 50 customers on that day,

(*c*) every one will be able to have the course of his/her choice, [3]

(*d*) the number opting for A will be at least twice the number opting for B. [3]

(iv) On a day when the cafeteria had an unlimited supply of each course, there were 120 customers. Use a distributional approximation to calculate, to three decimal places, the probability that at least 80 of the 120 customers opted for A. [4]

12. A garage sells three grades of petrol, namely 2-star, 3-star, and 4-star. In a week the amounts in litres sold of 2-star, 3-star, and 4-star are independent and normally distributed with means 7000, 13 000, and 22 000, respectively, and standard deviations 320, 400, and 500, respectively.

(i) Calculate, to three decimal places in each case, the probabilities that in a week

(*a*) the combined total amount of petrol sold will exceed 41 000 litres, [3]

(*b*) the amount of 4-star sold will exceed the combined amounts of 2-star and 3-star sold. [4]

(ii) Find the value of c, to the nearest litre, if the combined total amount sold exceeds c litres in 90% of the weeks. [3]

(iii) The profits per litre sold are 1·5 pence for 2-star petrol and 2·5 pence for 4-star petrol. Calculate, to three decimal places, the probability that in a week the profit from the sale of 4-star petrol is at least five times that from the sale of 2-star petrol. [5]

13. The continuous random variable X has probability density function f, where

$$f(x) = \frac{25}{12(x+1)^3}, \qquad \text{for } 0 \leqslant x \leqslant 4,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(i) Evaluate $\mathrm{E}[X+1]$. Hence, or otherwise, find the mean of X. [4]

(ii) Find the value of $c > 0$ for which $P(X \leqslant c) = c$. [4]

(iii) Find the cumulative distribution function of

$$Y = (X+1)^{-2}.$$

Hence, or otherwise, find the probability density function of Y. [7]

14. (*a*) Five independent measurements of the diameter of a ball bearing were made using a certain instrument. The results obtained, in millimetres, were:

8·9, 9·1, 9·1, 8·9, 8·9.

(i) Given that the true diameter of the ball bearing is 9 mm, calculate unbiased estimates of the mean and the variance of the measurement error of the instrument. [3]

(ii) Assuming that the measurement errors are independent and normally distributed, calculate 90% confidence limits for the mean measurement error. [4]

(*b*) The weekly wages received by a random sample of 80 personnel employed in factory A had a mean of £86.40 and a standard deviation of £3.80. Use this information to calculate an approximate 99% confidence interval for the mean weekly wage of all personnel employed in factory A. [4]

The weekly wages received by a random sample of 100 personnel employed in factory B had a mean of £87.60 and a standard deviation of £5.20. Calculate an approximate 95% confidence interval for the difference between the mean weekly wages in the two factories. State, with your reason, whether or not your interval discredits the claim that the mean weekly wages in the two factories are equal. [4]

WELSH JOINT EDUCATION COMMITTEE — CYD-BWYLLGOR ADDYSG CYMRU

General Certificate of Education — Tystysgrif Addysg Gyffredinol

Advanced Level — Safon Uwch

MATHEMATICS A4

Further Pure Mathematics

P.M. WEDNESDAY, 17 June 1987

(3 Hours)

Answer **all** *questions in Section A and* **four** *from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Given that

$$S_n = 2.1 + 3.3 + 4.5 + \ldots + (n+1)(2n-1)$$

obtain an expression for S_n in terms of n, simplifying your answer as far as possible. [4]

2. Find the three cube roots of $-\mathrm{i}$, giving your answers in the form $x + \mathrm{i}y$. [4]

3. By writing $x = \sinh y$, show that

$$\tanh(\sinh^{-1} x) = \frac{x}{\sqrt{1+x^2}}.$$ [3]

4. Evaluate, correct to 4 decimal places, the integral

$$\int_1^2 \frac{x+1}{x^2+x-1}\,\mathrm{d}x.$$ [5]

5. Given that

$$y = \mathrm{e}^x \sin x$$

prove by induction that

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 2^{n/2}\mathrm{e}^x \sin(x + \tfrac{1}{4}n\pi)$$

for all positive integers n. [5]

6. The equations of two circles are

$$x^2 + y^2 - 2x + 4y - 4 = 0$$

and

$$x^2 + y^2 - 8x - 4y + 4 = 0.$$

Show that the circles intersect at right angles. [5]

7. Given that

$$I_n = \int_0^{\pi/2} \cos^n\theta \, d\theta$$

show that, for $n \geqslant 2$,

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}.$$

Hence evaluate

(i) $\int_0^{\pi/2} \cos^4\theta \, d\theta$,

(ii) $\int_0^{\pi/2} \cos^5\theta \, d\theta$. [7]

8. The matrix equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

defines a transformation in the plane in which (x, y) is transformed to (x', y'). Show that the points $(1, 0)$, $(0, 1)$ are transformed respectively to (a, c), (b, d). Hence determine the matrices corresponding to

(i) a rotation through an angle α about the origin,

(ii) a reflection in the line $y = x \tan \beta$.

Deduce that a reflection in the line $y = x \tan \gamma$ followed by a reflection in the line $y = x \tan \delta$ is equivalent to a rotation and state the angle of the rotation. [7]

SECTION B

Answer **four** *questions from this section.*

9. The cubic function f is defined by

$$f(x) = x^3 + ax^2 + bx + c$$

where a, b, c are real constants. Explain briefly with the aid of a graph why the condition that f is monotonic is *sufficient* for the equation $f(x) = 0$ to have exactly one real root. Show, by considering graphically the case where f is not monotonic, that this condition is not however *necessary* for the equation $f(x) = 0$ to have exactly one real root. [5]

By considering $f'(x)$, show that f is monotonic if $3b > a^2$. Hence deduce that the equation

$$x^3 - 2x^2 + 2x - 5 = 0$$

has exactly one real root. Use any suitable numerical method to find its value correct to 3 decimal places. [5]

Let α, β denote the complex roots and γ the real root of the above cubic equation. Find $\alpha + \beta$ and $\alpha\beta$ in terms of γ and deduce that α, β are the roots of the quadratic equation

$$y^2 + (\gamma - 2)y + \frac{5}{\gamma} = 0.$$

Hence, by substituting your value for γ, find α and β in the form $p \pm iq$, giving the values of p and q correct to 2 decimal places. [5]

10. Obtain an expression for the determinant of the matrix

$$\begin{pmatrix} 1 & 3 & 2\lambda \\ \lambda & \lambda+2 & 2 \\ 1 & 2\lambda+1 & 3\lambda-1 \end{pmatrix}$$

in terms of λ, expressing your answer in its simplest form. [3]

The unknowns x, y, z satisfy the equations

$$x+3y+2\lambda z = a$$
$$\lambda x+(\lambda+2)y+2z = b$$
$$x+(2\lambda+1)y+(3\lambda-1)z = c$$

where a, b, c, λ are constants.

(i) If $\lambda = -1$, solve the equations giving expressions for x, y, z in terms of a, b, c. [3]

(ii) State the two values of λ for which the equations cannot be solved uniquely. For each of these values, find the conditions on a, b, c necessary for solutions to exist. In each case, find the general solution and give a geometrical interpretation. [9]

11. The function f is defined by

$$f(x) = x^2+\frac{4}{x^2}-3.$$

Find the two stationary points on the graph of f and identify each as a maximum or minimum. State the domain and range of f and sketch its graph. [3]

Determine

(i) $f(A)$ where A is the interval $[\frac{1}{2}, 2]$,

(ii) $f^{-1}(B)$ where B is the interval $[1, 2]$. [5]

A new function g is defined by restricting the domain of f so that

$$g(x) = f(x) \qquad \text{for } x \in (0, k)$$

where $k > 0$ is chosen to be as large as possible consistent with g being monotonic. State the value of k.

Obtain an expression for $h(x)$, where h denotes the inverse function of g.

Evaluate $g'(1)$ and *deduce* the value of $h'(2)$. [7]

12. The points P, Q, R have co-ordinates $(0, 0, 2)$, $(0, 3, 1)$, $(-4, -6, 5)$ respectively relative to an origin O and a set of mutually perpendicular axes Ox, Oy, Oz. Find the equation of the plane PQR. Given that N is the foot of the perpendicular from O to this plane, calculate the direction of the line ON. Write down the equations of the line ON, and hence find

(i) the co-ordinates of N,

(ii) the length of ON. [8]

Find the direction cosines of the lines PQ, PR and use these to calculate $\cos Q\widehat{P}R$. Deduce that

$$\sin Q\widehat{P}R = \frac{13}{\sqrt{610}}.$$

Write down the lengths of PQ and PR. Given that the volume of a tetrahedron is one-third the base area times the height, show that the volume of the tetrahedron $OPQR$ is equal to 4. [7]

13. The complex numbers z, w are represented by the points P, Q respectively in the Argand diagram and $z = x+iy$, $w = u+iv$ where x, y, u and v are real.

(i) Given that

$$w = \frac{z}{z+1} \qquad (z \neq -1)$$

obtain expressions for u and v in terms of x and y.

(*a*) Show that w is real if and only if z is real.

(*b*) If $|z| = 1$, show that the real part of w is equal to $\frac{1}{2}$.

P traverses the semi-circle $|z| = 1$; $R(z) \geqslant 0$ starting at $z = -i$ and finishing at $z = i$. What path is traversed by Q? [10]

(ii) The relationship between w and z is now changed to

$$w = e^z.$$

Obtain new expressions for u and v in terms of x and y and hence, or otherwise, show that

$$x = \log_e|w| \qquad \text{and } y = \arg(w).$$

If Q traverses the unit circle $|w| = 1$, what path is traversed by P? [5]

14. (*a*) Given that

$$f(x) = \log_e(1+x)$$

obtain an expression for $f^n(x)$, the nth derivative of $f(x)$. Deduce an expression, in terms of n, for the coefficient of x^n in the Maclaurin series expansion of $f(x)$. Show by considering the Maclaurin series of $f(x)$ and $f(-x)$ that

$$\tfrac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) = x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\ldots$$

Hence calculate $\log_e 2$ correct to 4 decimal places. [6]

(*b*) The Taylor series expansion of $\tan^{-1}x$ about $x = 1$ is given by

$$\tan^{-1}x = a+b(x-1)+c(x-1)^2+d(x-1)^3+\ldots$$

Find the values of the constants a, b, c, d. Deduce the first three terms of the series expansion about $x = 1$ of the function

$$g(x) = \frac{\tan^{-1}x-\frac{1}{4}\pi}{x-1}$$

and hence evaluate $\lim_{x\to 1} g(x)$.

Using a similar method, evaluate

$$\lim_{x\to 1}\left[\frac{\tan^{-1}x-\frac{1}{4}\pi}{(x-1)^2} - \frac{1}{2(x-1)}\right].$$ [9]

15. The point P has co-ordinates $(\theta+\sin\theta, 1-\cos\theta)$ and the curve C is the locus of P as θ increases from 0 to 2π. Find an expression for $\dfrac{dy}{dx}$ in terms of θ and show that

$$\frac{d^2y}{dx^2} = \tfrac{1}{4}\sec^4\tfrac{1}{2}\theta.$$ [4]

Sketch C, giving particular attention to the neighbourhood of the point corresponding to $\theta = \pi$.
Show that

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2} = 2|\cos\tfrac{1}{2}\theta|.$$ [4]

Hence find

(i) the total length of C,

(ii) the surface area of the solid generated when C is rotated through 4 right angles about the x-axis. [7]

WELSH JOINT EDUCATION COMMITTEE | **CYD-BWYLLGOR ADDYSG CYMRU**

General Certificate of Education | **Tystysgrif Addysg Gyffredinol**

Advanced Level | **Safon Uwch**

Special Paper | **Papur Arbennig**

MATHEMATICS S

A.M. MONDAY, 22 June 1987

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from Section A and no more than* **four** *questions from either Section B or Section C.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Pure Mathematics

No more than **four** *questions should be answered from this section.*

1. In this question, it may be assumed that (*a*) the circumscribing circle of a triangle is the circle passing through its three vertices, (*b*) the inscribed circle of a triangle is the circle lying inside the triangle and touching all three sides.

The triangle ABC is isosceles with $AB = AC = a$ and $B\widehat{A}C = 2\theta$.

(i) Obtain an expression for the radius of the circumscribing circle of ABC in terms of a, θ. [2]

(ii) Show that the radius of the inscribed circle of ABC is equal to

$$\frac{a\sin\theta\cos\theta}{1+\sin\theta}$$

and show that its maximum value as θ varies is approximately $0{\cdot}3a$. [5]

(iii) Obtain an expression for the ratio R of the area of the circumscribing circle to the area of the inscribed circle and find its minimum value as θ varies.

Sketch the graph of R. [8]

2. Sketch, on the same axes, the curves with equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{and} \qquad xy=c^2$$

where a, b, c are positive constants. Determine the conditions on a, b, c for the curves to touch. In this case, find in terms of a and b,

(i) the co-ordinates of the two points of contact,

(ii) the distance between the points of contact. [6]

A point P moves on the line $y = x\tan\alpha$ and another point Q moves on the line $y = -x\tan\alpha$ $(0 < \alpha < \frac{1}{2}\pi)$. Given that P and Q move in such a way that the length PQ remains equal to a constant $2k$, find the equation of the locus of the mid-point of PQ.

Prove that this locus touches the curve having equation $2xy = k^2$ for all values of α and find the co-ordinates of the points of contact.

What value of α minimises the distance between the points of contact? [9]

3. If $a-b=h$ where h is small, show that

(i) $$\sqrt{\frac{a}{b}} = 1+\frac{h}{2b}-\frac{h^2}{8b^2}+\frac{h^3}{16b^3}-\frac{5h^4}{128b^4}+\dots$$

(ii) $$\frac{a}{a+b} = \tfrac{1}{2}+\frac{h}{4b}-\frac{h^2}{8b^2}+\frac{h^3}{16b^3}-\frac{h^4}{32b^4}.$$

Hence show that if a, b are approximately equal, then

$$\sqrt{\frac{a}{b}} \approx \frac{a}{a+b}+\frac{a+b}{4b}.$$

State the approximate error. [7]

Use a similar method to show that

$$\left(\frac{a}{b}\right)^{1/n} \approx \frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b}$$

with approximate error

$$\frac{(n^2-1)h^3}{12n^3b^3}.$$ [8]

4. (i) A student is asked to evaluate

$$I = \int_{-1}^{1} \frac{dx}{1+\alpha x+x^2}$$

where α is a constant. He makes the substitution $y = 1/x$, notes that $y = \pm 1$ when $x = \pm 1$ and writes

$$I = \int_{-1}^{1} \frac{-\frac{1}{y^2}}{1+\frac{\alpha}{y}+\frac{1}{y^2}} dy = -\int_{-1}^{1} \frac{dy}{y^2+\alpha y+1} = -I.$$

He therefore deduces that $I = 0$. Locate the flaw in his argument.

Show that in fact, for $|\alpha| < 2$,

$$I = \frac{\pi}{2\sqrt{1-\frac{1}{4}\alpha^2}}.$$ [8]

(ii) The integral J is defined by

$$J = \int_{0}^{1} \frac{dx}{1+\beta\sqrt{x}+x}$$

where β is a constant. Obtain an expression for J in terms of β given that $|\beta| < 2$. [7]

5. Use vector methods to show that the three altitudes of a triangle are concurrent. [15]

SECTION B

Mechanics

No more than **four** *questions should be answered from this section.*

6. A new man-made elastic material does not obey Hooke's Law. The tension in an extended light elastic string of this material is given by $T = 2mg\left[\frac{x}{a}+\frac{x^2}{4a^2}\right]$ where x is the extension, a is the natural length and m is a constant. Show that the work done in giving the string an extension a is $7mga/6$.

A ring of mass m is threaded on a smooth hoop of radius a which is fixed in a vertical plane. A light elastic string of the new material has natural length a. One end of the string is fixed to the highest point P of the hoop, the other end to the ring. The ring is held at a 'straight line' distance a from P and released.

Show that

(i) the speed of the ring at the lowest point of the hoop is $\sqrt{\frac{2ga}{3}}$,

(ii) when the string makes an angle θ with the downward vertical, the ring's velocity is given by

$$v^2 = \frac{2ga}{3}[9\cos\theta - 4\cos^3\theta - 4],$$

(iii) the maximum speed of the ring occurs when $\theta = 30°$. [15]

7. A light elastic string has natural length l and modulus mg. One end is attached to a particle P of mass m which is free to move on a smooth fixed horizontal table. The other end of the string is attached to a fixed point A which is at a distance of $5l/4$ vertically above a point O on the table. The string passes through a small smooth ring which is fixed at a point vertically below and distance l from A. Show that however the particle moves on the table the reaction of the table on it is always $3mg/4$.

Show that if P is projected along the table from O with speed v, the subsequent motion is simple harmonic motion and write down the period. P collides with a stationary particle Q of mass $2m$ at a distance l from O. The result of the collision is that P and Q adhere. Given that $v = 2\sqrt{gl}$ find the time elapsing between the initial projection from O and the return of the composite particle to O.

[15]

8. Particles of mass $3m$, $2m$ are connected by a light inextensible string which passes over a smooth fixed peg, the string hanging vertically on each side. The particles are released from rest at time $t = 0$. The air resistance on each particle is kv^2, where v is the velocity of each particle after it has moved a distance x from rest. Prove that

$$\frac{d}{dx}(v^2)+\frac{4k}{5m}v^2 = \frac{2g}{5}.$$

Show that

(i) the velocity of the particles is always less than $\sqrt{\frac{mg}{2k}}$,

(ii) the particles reach a velocity $\frac{1}{2}\sqrt{\frac{mg}{2k}}$ after each has moved a distance $\frac{5m}{4k}\log_e \frac{4}{3}$ from rest,

(iii) the tension in the string always lies between $\frac{12mg}{5}$ and $\frac{5mg}{2}$. [15]

9. A light aeroplane has a speed v and a flying range (outward and return) of R in calm weather. Neglecting petrol consumption at take off and landing, show that in the presence of a wind of speed w the aeroplane's range is

$$\frac{R(v^2-w^2)}{v\sqrt{(v^2-w^2\sin^2\varphi)}}$$

in a ground direction whose bearing is φ to the wind direction. What is the maximum possible range in the presence of the wind?

[15]

10. A uniform smooth rod AB of length a and weight $2W$ has small smooth rings of weights $5W$, $3W$ attached at A and B respectively. A light string passes through both rings (so that part of the string runs alongside the rod) and has both ends fastened to a fixed support O. The system hangs in equilibrium with AB in a non-vertical position. The angle between OA and the vertical is φ.

(i) Show that the strings are equally inclined to the vertical.

(ii) If $OA = 2x$ prove $OB = 3x$.

(iii) Show that $\cos\varphi = \sqrt{\frac{25x^2-a^2}{24x^2}}$.

(iv) Find the tension T in terms of W, x and a. [15]

Section C

Probability and Statistics

No more than **four** *questions should be answered from this section.*

11. For a particular species of animal, the probability that a pregnant female will have a litter of n offspring is p_n where

$$p_n = \frac{k\lambda^n}{n!}, \qquad \text{for } n = 1, 2, 3, \ldots$$
$$p_0 = 0.$$

(i) Show that $k = (e^{\lambda} - 1)^{-1}$ and find the mean litter size in terms of λ. [5]

Independently of the sex of any other offspring, the probability that an offspring will be female is p.

(ii) In a litter of n offspring write down an expression for the probability that each offspring in the litter is female. Deduce the proportion of all litters that consist of females only. [4]

(iii) In a litter of n ($\geqslant 2$) offspring write down an expression for the probability that the litter includes at least one female and at least one male. Hence show that the proportion of all litters that include offspring of both sexes is

$$\frac{(e^{p\lambda} - 1)(e^{q\lambda} - 1)}{e^{\lambda} - 1},$$

where $q = 1 - p$. [6]

12. Each of n (> 3) boxes $B_1, B_2, \ldots, B_n$ contains five cards. In box B_r ($r = 1, 2, \ldots, n$) four of the five cards are numbered zero and the other card is numbered r.

(i) Two of the boxes are chosen at random and one card is selected at random from each of the chosen boxes. Calculate the probabilities that the numbers on the two chosen cards are such that

(*a*) each is zero, [1]

(*b*) exactly one is zero, [1]

(*c*) their sum is equal to 3. [3]

(ii) The two selected cards are returned to their respective boxes. One card is then chosen at random from each of the n boxes. Let T denote the sum of the numbers on the n chosen cards.

(*a*) Find an expression for $P(T \leqslant 3)$ and evaluate it to four decimal places when $n = 10$. [5]

(*b*) For the case when $n = 50$ use a normal approximation to the distribution of T to evaluate $P(T \geqslant 100)$ to three decimal places. [5]

13. AB is a fixed diameter of a circle centred at O and having radius 4 cm. A straight line is drawn in a random direction from O to cut the circle at P, so that the angle AOP, in radians, is uniformly distributed over the interval $(0, \pi)$. Let Y cm denote the length of the chord AP and let Z cm^2 denote the area of the triangle AOP.

(i) Evaluate $P(Z < 4)$ and the expected value of Y. [6]

(ii) Show that Y and Z have identical distributions. [9]

14. (a) The continuous random variable X is distributed with probability density function f, where

$$f(x) = \tfrac{1}{50}x, \qquad \text{for } 0 < x < 10,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(i) If Y denotes the integer part of a randomly observed value of X, show that

$$P(Y = r) = \tfrac{1}{100}(2r+1), \qquad r = 0, 1, 2, \ldots, 9.$$ [2]

(ii) Find the probability that the integer parts of two randomly observed values of X will differ by 1 or less. [5]

(b) Let $\bar{X}$ denote the mean of a random sample of 16 observations from a normal distribution whose mean and standard deviation are both equal to λ, where λ is an unknown positive constant. Find a and b, in terms of λ, such that

$$P(\bar{X} < a) = 0{\cdot}05 \qquad \text{and} \qquad P(\bar{X} > b) = 0{\cdot}05.$$

Given that the observed value of $\bar{X}$ was 28·5, deduce 90% confidence limits for λ, giving each limit to one decimal place. [8]

15. The two variables x and y are known to be such that the ratio $\dfrac{y-10}{x}$ has a constant value β which is unknown. In n experiments, the variable x was controlled to have the values $x_1, x_2, \ldots, x_n$, respectively, and the corresponding y-values were measured to be $y_1, y_2, \ldots, y_n$, respectively. The y-measurements are subject to independent normally distributed errors having mean zero and standard deviation σ. Show that the method of least-squares leads to an estimate of b of β given by

$$b = \frac{\Sigma x_i y_i - 10\Sigma x_i}{\Sigma x_i^2},$$

each summation being over $i = 1, 2, \ldots, n$. Verify that b is an unbiased estimate of β and derive an expression for its variance. [8]

Given that $n = 10$, $\sigma = 1{\cdot}2$, $\Sigma x_i = 52$, $\Sigma x_i^2 = 400$, $\Sigma y_i = 82{\cdot}4$, and $\Sigma x_i y_i = 560$, obtain 95% confidence limits for (i) β, and (ii) the true value of y when $x = 5$. [7]

WELSH JOINT EDUCATION COMMITTEE | **CYD-BWYLLGOR ADDYSG CYMRU**

General Certificate of Education | **Tystysgrif Addysg Gyffredinol**

Advanced Level | **Safon Uwch**

Special Paper | **Papur Arbennig**

PURE MATHEMATICS S

A.M. MONDAY, 22 June 1987

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

PURE MATHEMATICS

No more than **four** *questions should be answered from this section.*

1. In this question, it may be assumed that (*a*) the circumscribing circle of a triangle is the circle passing through its three vertices, (*b*) the inscribed circle of a triangle is the circle lying inside the triangle and touching all three sides.

The triangle ABC is isosceles with $AB = AC = a$ and $B\widehat{A}C = 2\theta$.

(i) Obtain an expression for the radius of the circumscribing circle of ABC in terms of a, θ. [2]

(ii) Show that the radius of the inscribed circle of ABC is equal to

$$\frac{a \sin\theta \cos\theta}{1+\sin\theta}$$

and show that its maximum value as θ varies is approximately $0{\cdot}3\,a$. [5]

(iii) Obtain an expression for the ratio R of the area of the circumscribing circle to the area of the inscribed circle and find its minimum value as θ varies.

Sketch the graph of R. [8]

2. Sketch, on the same axes, the curves with equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{and} \qquad xy=c^2$$

where a, b, c are positive constants. Determine the conditions on a, b, c for the curves to touch. In this case, find in terms of a and b,

(i) the co-ordinates of the two points of contact,

(ii) the distance between the points of contact. [6]

A point P moves on the line $y = x\tan\alpha$ and another point Q moves on the line $y = -x\tan\alpha$ $(0 < \alpha < \frac{1}{2}\pi)$. Given that P and Q move in such a way that the length PQ remains equal to a constant $2k$, find the equation of the locus of the mid-point of PQ.

Prove that this locus touches the curve having equation $2xy = k^2$ for all values of α and find the co-ordinates of the points of contact.

What value of α minimises the distance between the points of contact? [9]

3. If $a-b=h$ where h is small, show that

(i) $\sqrt{\dfrac{a}{b}} = 1+\dfrac{h}{2b}-\dfrac{h^2}{8b^2}+\dfrac{h^3}{16b^3}-\dfrac{5h^4}{128b^4}+\dots$

(ii) $\dfrac{a}{a+b} = \frac{1}{2}+\dfrac{h}{4b}-\dfrac{h^2}{8b^2}+\dfrac{h^3}{16b^3}-\dfrac{h^4}{32b^4}$.

Hence show that if a, b are approximately equal, then

$$\sqrt{\frac{a}{b}} \approx \frac{a}{a+b}+\frac{a+b}{4b}.$$

State the approximate error. [7]

Use a similar method to show that

$$\left(\frac{a}{b}\right)^{1/n} \approx \frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b}$$

with approximate error

$$\frac{(n^2-1)h^3}{12n^3b^3}.$$ [8]

4. (i) A student is asked to evaluate

$$I = \int_{-1}^{1} \frac{dx}{1+\alpha x+x^2}$$

where α is a constant. She makes the substitution $y = 1/x$, notes that $y = \pm 1$ when $x = \pm 1$ and writes

$$I = \int_{-1}^{1} \frac{-\frac{1}{y^2}}{1+\frac{\alpha}{y}+\frac{1}{y^2}} dy = -\int_{-1}^{1} \frac{dy}{y^2+\alpha y+1} = -I.$$

She therefore deduces that $I = 0$. Locate the flaw in her argument.

Show that in fact, for $|\alpha| < 2$,

$$I = \frac{\pi}{2\sqrt{1-\frac{1}{4}\alpha^2}}.$$ [8]

(ii) The integral J is defined by

$$J = \int_{0}^{1} \frac{dx}{1+\beta\sqrt{x}+x}$$

where β is a constant. Obtain an expression for J in terms of β given that $|\beta| < 2$. [7]

5. Use vector methods to show that the three altitudes of a triangle are concurrent. [15]

SECTION B

No more than **four** *questions should be answered from this section.*

6. You are given that

$$\cosh 2n\theta = \sum_{r=0}^{n} a_{2r} \cosh^{2r}\theta$$

where n is a positive integer and $a_0, a_2, \ldots, a_{2n}$ are constants. Show that

$$\sum_{r=0}^{n} a_{2r} = 1.$$

By differentiating twice and equating appropriate terms, show that

$$a_{2r+2} = \frac{2(r^2-n^2)}{(r+1)(2r+1)} a_{2r} \qquad (r = 0, 1, \ldots, n-1).$$ [7]

Use these results to obtain expressions for $\cosh 8\theta$ and $\dfrac{\sinh 8\theta}{\sinh \theta}$ in terms of powers of $\cosh \theta$.

[5]

Hence find an expression for $\cosh 9\theta$ in terms of powers of $\cosh \theta$. [3]

7. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

The proposition P states that

$$\mathbf{A}^n = \tfrac{1}{6}[5^n - (-1)^n]\mathbf{A} + \tfrac{5}{6}[5^{n-1} + (-1)^n]\mathbf{I}.$$

(i) Verify that P is true for $n = 1$ and prove by induction that P is true for all positive integers. [7]

(ii) Show that P is true for $n = -1$ and prove by induction that P is true for all negative integers n. [8]

[N.B. For negative n, $\mathbf{A}^n$ means $(\mathbf{A}^{-1})^{-n}$.]

8. (i) The equation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \mathbf{M}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

where $\mathbf{M}$ is a 3×3 matrix, defines a transformation in the plane in which the point (x, y) is transformed to the point (x', y'). Two such $\mathbf{M}$ matrices are given by

$$\mathbf{M}_1 = \begin{bmatrix} -\frac{1}{2} & \sqrt{3}/2 & 6+2\sqrt{3} \\ \sqrt{3}/2 & \frac{1}{2} & -2-2\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_2 = \begin{bmatrix} \sqrt{3}/2 & \frac{1}{2} & 5-2\sqrt{3} \\ -\frac{1}{2} & \sqrt{3}/2 & \sqrt{3}-4 \\ 0 & 0 & 1 \end{bmatrix}.$$

One of these matrices corresponds to a rotation about the fixed point (x_0, y_0) and the other corresponds to a reflection in the line $y = mx + c$. Identify which is which and find the values of x_0, y_0, m and c. [9]

(ii) The functions $f_1, f_2, f_3, \ldots$ satisfy

$$f_n = f_{n-1} \circ f_1 \qquad (n = 2, 3, 4, \ldots).$$

Express f_n in terms of f_1 alone and show that

$$f_n = f_1 \circ f_{n-1} \qquad (n = 2, 3, 4, \ldots).$$

Given further that $f_n(1) = 1$ $(n = 1, 2, 3, \ldots)$ and letting μ_n denote $f'_n(1)$, express μ_n in terms of μ_1. [6]

9. If u, v are functions of x, prove by induction that

$$\frac{\mathrm{d}^n(uv)}{\mathrm{d}x^n} = \sum_{r=0}^{n} {}^nC_r \frac{\mathrm{d}^r u}{\mathrm{d}x^r} \cdot \frac{\mathrm{d}^{n-r} v}{\mathrm{d}x^{n-r}}$$

where $\dfrac{\mathrm{d}^0 u}{\mathrm{d}x^0}$ and $\dfrac{\mathrm{d}^0 v}{\mathrm{d}x^0}$ denote u and v respectively. [5]

Given that $y = \sin(m \sin^{-1} x)$, show that

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2 y = 0.$$

Using the above result, differentiate both sides of this equation n times to show that

$$y_{n+2} = (n^2 - m^2) y_n$$

where y_i denotes the value of $\dfrac{\mathrm{d}^i y}{\mathrm{d}x^i}$ when $x = 0$. [6]

Hence derive an expression for the coefficient of x^n in the Maclaurin series expansion of $\sin(m \sin^{-1} x)$, distinguishing between the cases (i) n even, (ii) n odd. [4]

10. Given that

$$3y^2 = x(1-x)^2$$

show that

$$1+\left(\frac{dy}{dx}\right)^2 = \frac{(3x+1)^2}{12x}.$$

and obtain an expression for $\dfrac{d^2y}{dx^2}$ in terms of x. Sketch the graph of y. [4]

(i) Find the length of the loop which passes through the origin and the point $(1,0)$. [3]

(ii) The region enclosed by this loop is rotated completely about the x-axis to form a solid S. Show that the surface area of S is equal to $\frac{1}{3}\pi$. [3]

(iii) The co-ordinates of the centre of curvature of the curve at the point (x, y) are given by

$$\left(x - \frac{y_1(1+y_1^2)}{y_2},\, y + \frac{(1+y_1^2)}{y_2}\right)$$

where y_i $(i = 1, 2)$ denotes the ith order derivative of y with respect to x. Find the co-ordinates of the two centres of curvature corresponding to the point $(1,0)$. [5]

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General Certificate of Education — Tystysgrif Addysg Gyffredinol

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Special Paper — Papur Arbennig

APPLIED MATHEMATICS S

A.M. MONDAY, 22 June 1987

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Mechanics

No more than **four** *questions should be answered from this section.*

1. A new man-made elastic material does not obey Hooke's Law. The tension in an extended light elastic string of this material is given by $T = 2mg\left[\frac{x}{a}+\frac{x^2}{4a^2}\right]$ where x is the extension, a is the natural length and m is a constant. Show that the work done in giving the string an extension a is $7mga/6$.

A ring of mass m is threaded on a smooth hoop of radius a which is fixed in a vertical plane. A light elastic string of the new material has natural length a. One end of the string is fixed to the highest point P of the hoop, the other end to the ring. The ring is held at a 'straight line' distance a from P and released.

Show that

(i) the speed of the ring at the lowest point of the hoop is $\sqrt{\frac{2ga}{3}}$,

(ii) when the string makes an angle θ with the downward vertical, the ring's velocity is given by

$$v^2 = \frac{2ga}{3}[9\cos\theta - 4\cos^3\theta - 4],$$

(iii) the maximum speed of the ring occurs when $\theta = 30°$. [15]

2. A light elastic string has natural length l and modulus mg. One end is attached to a particle P of mass m which is free to move on a smooth fixed horizontal table. The other end of the string is attached to a fixed point A which is at a distance of $5l/4$ vertically above a point O on the table. The string passes through a small smooth ring which is fixed at a point vertically below and distance l from A. Show that however the particle moves on the table the reaction of the table on it is always $3mg/4$.

Show that if P is projected along the table from O with speed v, the subsequent motion is simple harmonic motion and write down the period. P collides with a stationary particle Q of mass $2m$ at a distance l from O. The result of the collision is that P and Q adhere. Given that $v = 2\sqrt{gl}$ find the time elapsing between the initial projection from O and the return of the composite particle to O. [15]

3. Particles of mass $3m$, $2m$ are connected by a light inextensible string which passes over a smooth fixed peg, the string hanging vertically on each side. The particles are released from rest at time $t = 0$. The air resistance on each particle is kv^2, where v is the velocity of each particle after it has moved a distance x from rest. Prove that

$$\frac{d}{dx}(v^2)+\frac{4k}{5m}v^2 = \frac{2g}{5}.$$

Show that

(i) the velocity of the particles is always less than $\sqrt{\dfrac{mg}{2k}}$,

(ii) the particles reach a velocity $\frac{1}{2}\sqrt{\dfrac{mg}{2k}}$ after each has moved a distance $\dfrac{5m}{4k}\log_e\frac{4}{3}$ from rest,

(iii) the tension in the string always lies between $\dfrac{12mg}{5}$ and $\dfrac{5mg}{2}$. [15]

4. A light aeroplane has a speed v and a flying range (outward and return) of R in calm weather. Neglecting petrol consumption at take off and landing, show that in the presence of a wind of speed w the aeroplane's range is

$$\frac{R(v^2-w^2)}{v\sqrt{(v^2-w^2\sin^2\varphi)}}$$

in a ground direction whose bearing is φ to the wind direction. What is the maximum possible range in the presence of the wind? [15]

5. A uniform smooth rod AB of length a and weight $2W$ has small smooth rings of weights $5W$, $3W$ attached at A and B respectively. A light string passes through both rings (so that part of the string runs alongside the rod) and has both ends fastened to a fixed support O. The system hangs in equilibrium with AB in a non-vertical position. The angle between OA and the vertical is φ.

(i) Show that the strings are equally inclined to the vertical.

(ii) If $OA = 2x$ prove $OB = 3x$.

(iii) Show that $\cos\varphi = \sqrt{\dfrac{25x^2-a^2}{24x^2}}$.

(iv) Find the tension T in terms of W, x and a. [15]

Section B

Probability and Statistics

No more than **four** *questions should be answered from this section.*

6. For a particular species of animal, the probability that a pregnant female will have a litter of n offspring is p_n where

$$p_n = \frac{k\lambda^n}{n!}, \qquad \text{for } n = 1, 2, 3, \ldots$$
$$p_0 = 0.$$

(i) Show that $k = (e^{\lambda} - 1)^{-1}$ and find the mean litter size in terms of λ. [5]

Independently of the sex of any other offspring, the probability that an offspring will be female is p.

(ii) In a litter of n offspring write down an expression for the probability that each offspring in the litter is female. Deduce the proportion of all litters that consist of females only. [4]

(iii) In a litter of n ($\geqslant 2$) offspring write down an expression for the probability that the litter includes at least one female and at least one male. Hence show that the proportion of all litters that include offspring of both sexes is

$$\frac{(e^{p\lambda} - 1)(e^{q\lambda} - 1)}{e^{\lambda} - 1},$$

where $q = 1 - p$. [6]

7. Each of n (> 3) boxes $B_1, B_2, \ldots, B_n$ contains five cards. In box B_r ($r = 1, 2, \ldots, n$) four of the five cards are numbered zero and the other card is numbered r.

(i) Two of the boxes are chosen at random and one card is selected at random from each of the chosen boxes. Calculate the probabilities that the numbers on the two chosen cards are such that

(*a*) each is zero, [1]

(*b*) exactly one is zero, [1]

(*c*) their sum is equal to 3. [3]

(ii) The two selected cards are returned to their respective boxes. One card is then chosen at random from each of the n boxes. Let T denote the sum of the numbers on the n chosen cards.

(*a*) Find an expression for $P(T \leqslant 3)$ and evaluate it to four decimal places when $n = 10$. [5]

(*b*) For the case when $n = 50$ use a normal approximation to the distribution of T to evaluate $P(T \geqslant 100)$ to three decimal places. [5]

8. AB is a fixed diameter of a circle centred at O and having radius 4 cm. A straight line is drawn in a random direction from O to cut the circle at P, so that the angle AOP, in radians, is uniformly distributed over the interval $(0, \pi)$. Let Y cm denote the length of the chord AP and let Z cm^2 denote the area of the triangle AOP.

(i) Evaluate $P(Z < 4)$ and the expected value of Y. [6]

(ii) Show that Y and Z have identical distributions. [9]

9. (*a*) The continuous random variable X is distributed with probability density function f, where

$$f(x) = \tfrac{1}{50}x, \qquad \text{for } 0 < x < 10,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(i) If Y denotes the integer part of a randomly observed value of X, show that

$$P(Y = r) = \tfrac{1}{100}(2r+1), \qquad r = 0, 1, 2, \ldots, 9. \qquad [2]$$

(ii) Find the probability that the integer parts of two randomly observed values of X will differ by 1 or less. [5]

(*b*) Let $\bar{X}$ denote the mean of a random sample of 16 observations from a normal distribution whose mean and standard deviation are both equal to λ, where λ is an unknown positive constant. Find a and b, in terms of λ, such that

$$P(\bar{X} < a) = 0{\cdot}05 \qquad \text{and} \qquad P(\bar{X} > b) = 0{\cdot}05.$$

Given that the observed value of $\bar{X}$ was 28·5, deduce 90% confidence limits for λ, giving each limit to one decimal place. [8]

10. The two variables x and y are known to be such that the ratio $\dfrac{y-10}{x}$ has a constant value β which is unknown. In n experiments, the variable x was controlled to have the values $x_1, x_2, \ldots, x_n$, respectively, and the corresponding y-values were measured to be $y_1, y_2, \ldots, y_n$, respectively. The y-measurements are subject to independent normally distributed errors having mean zero and standard deviation σ. Show that the method of least-squares leads to an estimate b of β given by

$$b = \frac{\Sigma x_i y_i - 10\Sigma x_i}{\Sigma x_i^2},$$

each summation being over $i = 1, 2, \ldots, n$. Verify that b is an unbiased estimate of β and derive an expression for its variance. [8]

Given that $n = 10$, $\sigma = 1{\cdot}2$, $\Sigma x_i = 52$, $\Sigma x_i^2 = 400$, $\Sigma y_i = 82{\cdot}4$, and $\Sigma x_i y_i = 560$, obtain 95% confidence limits for (i) β, and (ii) the true value of y when $x = 5$. [7]