WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

CYD-BWYLLGOR ADDYSG CYMRU

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Safon Uwch

# MATHEMATICS A 1

## PURE MATHEMATICS

A.M. WEDNESDAY, 4 June 1986

(3 Hours)

*Answer* **all** *questions in Section A and* **four** *questions from Section B.*

## SECTION A

*Answer* **all** *questions in this section.*

1. Use the approximations $\sin x \approx x$; $\tan x \approx x$; $\cos x \approx 1-\frac{1}{2}x^2$ to find the smallest positive root of the equation

$$\sin x + \cos x + \tan x = 1{\cdot}5$$

giving your answer to 2 decimal places. [3]

2. The function $f$ is defined by

$$f(x) = \frac{x}{x+1}$$

with domain $x > -1$. Find its range.

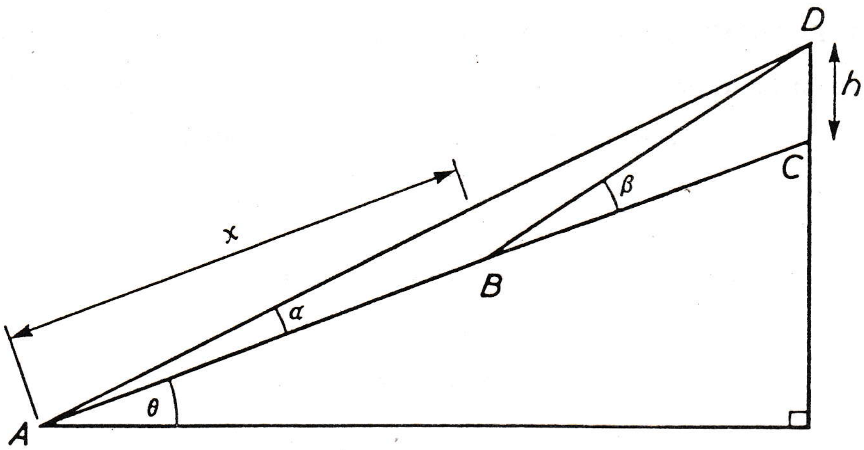
Obtain an expression for $f^{-1}(x)$, where $f^{-1}$ denotes the inverse function of $f$. Write down the domain of $f^{-1}$. [4]

3. Using the substitution $x = 3+\sin\theta$, show that

$$\int_2^4 \sqrt{(4-x)(x-2)}\,dx = \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta.$$

Hence evaluate the integral. [4]

**4.**




$A$, $B$ and $C$ are three points on a straight path inclined at an angle $\theta$ to the horizontal. $CD$ is a vertical flagpole of height $h$, $AB = x$, $D\hat{A}C = \alpha$ and $D\hat{B}C = \beta$. By equating expressions for $BD$ obtained by using the sine rule in triangles $ABD$ and $BCD$, or otherwise, show that

$$h = \frac{x \sin\alpha \sin\beta}{\sin(\beta - \alpha)\cos\theta}$$ [5]

5. Given the identity

$$\frac{1}{(x+2)(2x+3)} \equiv \frac{A}{x+2} + \frac{B}{2x+3}$$

find the values of the constants $A$ and $B$.

Hence, or otherwise, express

$$\frac{1}{(x+2)^2(2x+3)^2}$$

in partial fractions. [5]

6. The points $A$, $B$ have coordinates $(1, -2)$ and $(4, 2)$ respectively and $P$ is the point $(x, y)$. Show that

$$AP^2 = x^2 + y^2 - 2x + 4y + 5$$

and derive a similar expression for $BP^2$. Given that $P$ moves in such a way that $AP = 2BP$, show that the locus of $P$ is a circle.

Calculate the radius and the coordinates of the centre of this circle. [6]

7. (*a*) The variables $x$, $y$ are related by the equation

$$x^3y^2 = 16.$$

When $x = 4$ and $y = -\frac{1}{2}$, show that

$$\frac{dy}{dx} = \frac{3}{16}.$$ [3]

(*b*) The variables $x$, $y$ and $t$ are related by the equations

$$x = \sin^3 t$$
$$y = \cos^3 t.$$

Show that

$$\frac{dy}{dx} = -\cot t.$$ [3]

**8.** Write down the coefficient $c_r$ of $x^r$ in the binomial expansion of $(1+ax)^n$ and show that

$$\frac{c_{r+1}}{c_r} = \frac{a(n-r)}{r+1}$$

Given that the ratio of the coefficient of $x^5$ to that of $x^4$ is $8:1$ and that the ratio of the coefficient of $x^{20}$ to that of $x^{19}$ is $5:1$, find the values of $a$ and $n$.

For what values of $x$ is the binomial expansion valid? [7]

## SECTION B

*Answer* **four** *questions from this section.*

**9.** (*a*) Given that

$$a\cos\theta + b\sin\theta \equiv r\cos(\theta-\alpha)$$

where $r > 0$, show that

$$r = \sqrt{a^2+b^2}$$

and obtain an expression for $\tan\alpha$ in terms of $a$ and $b$.

The angle $x$ satisfies the equation

$$\sin(x-\tfrac{1}{3}\pi) = \sin x - \sin\tfrac{1}{3}\pi.$$

Show that

$$\sqrt{3}\cos x + \sin x = \sqrt{3}$$

and hence find all possible values of $x$ lying between $-\pi$ and $\pi$. [8]

(*b*) Show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

The variable $x$ satisfies the equation

$$x^3 - 3x + 1 = 0.$$

By putting $x = 2\cos\theta$ and using the above result, show that

$$\cos 3\theta = -\tfrac{1}{2}.$$

Write down the three values of $\theta$ between 0 and $\pi$ which satisfy this equation and hence calculate, correct to 2 decimal places, the three roots of the above cubic equation in $x$. [7]

**10.** $C$ is the curve with equation $y = e^x$.

(i) Find the equation of the normal at the point $(t, e^t)$ to $C$ and show that the normal at the point $(0, 1)$ has equation

$$x + y = 1.$$ [3]

(ii) Show that the equation of the tangent at the point $(t, e^t)$ to $C$ is

$$ye^{-t} = x + 1 - t.$$

Given that this tangent passes through the point $P(2, 3)$, show that $t$ satisfies

$$e^{-t} = 1 - \tfrac{1}{3}t.$$ [4]

Show graphically that this equation has 2 roots. State the value of the smaller root and show that the larger root lies between 2·82 and 2·83. Deduce the gradients of the two tangents from $P$ to $C$ and show that the acute angle between them is approximately 42°. [8]

**11.** (*a*) Show that the sum of the arithmetic series

$$\log_e a + \log_e ar + \log_e ar^2 + \ldots + \log_e ar^{n-1}$$

is equal to $\frac{1}{2}n \log_e(a^2 r^{n-1})$. In the case $a = 2$, $r = 3$, use this result together with trial and error to find the minimum value of $n$ such that the sum exceeds 20. [6]

(*b*) By expanding the left-hand sides of the inequalities

$$(1-x)^2 \geqslant 0$$

and

$$(1+x)^2 \geqslant 0$$

show that

$$\left|\frac{2x}{1+x^2}\right| \leqslant 1$$

for all values of $x$ and state the two values of $x$ for which equality holds. [3]

By using the formula for the sum to infinity of a geometric series, show that

$$\sum_{r=0}^{\infty} \left(\frac{2x}{1+x^2}\right)^r = \frac{1+x^2}{(1-x)^2}$$

for all values of $x$ except the two values stated above.

Deduce, by putting $x = \tan\theta$, that

$$\sum_{r=0}^{\infty} \sin^r 2\theta = (\cos\theta - \sin\theta)^{-2}.$$

For what values of $\theta$ does this result not hold? [6]

**12.** (*a*) Find the two complex roots of the equation

$$z^2 + z + 1 = 0$$

and express them in the form $re^{i\theta}$.

Show that:

(i) each root is the square of the other,

(ii) the cube of each root is equal to unity. [6]

(*b*) By first eliminating $z_2$, or otherwise, solve the simultaneous equations

$$(3-i)z_1 - iz_2 = 14 + 2i$$
$$(1+i)z_1 + 2z_2 = 8 + 10i$$

giving your answers in the form $x + iy$. [6]

Plot the four points which represent the complex numbers $z_1$, $z_2$, $2+i$ and $5+4i$ on an Argand diagram and show that the four points are the vertices of a rhombus. [3]

**13.** The points $A$, $B$, $C$ have position vectors **i**, **j**, **k** relative to an origin $O$, where **i**, **j**, **k** form a set of mutually perpendicular unit vectors. The points $P$, $Q$ lie on $AB$, $BC$ respectively and are such that

$$\frac{AP}{PB} = \frac{BQ}{QC} = \frac{\lambda}{1-\lambda}$$

where $0 \leqslant \lambda \leqslant 1$.

(i) Write down the position vectors of $P$ and $Q$ and deduce a vector expression for **PQ** in terms of **i**, **j**, **k** and $\lambda$. Show that

$$|\mathbf{PQ}| = \sqrt{6(\lambda - \tfrac{1}{2})^2 + \tfrac{1}{2}}$$

and hence find the minimum and maximum lengths of $PQ$ ad $\lambda$ varies. [5]

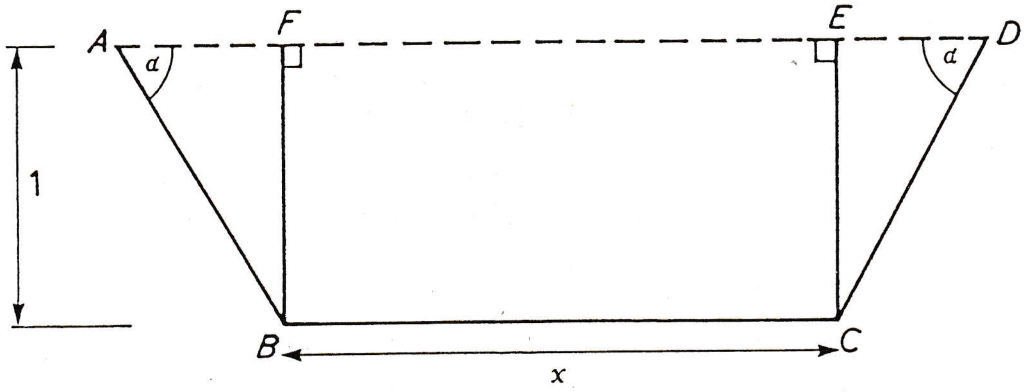
(ii) The angles between **PQ** and **OA**, **PQ** and **OB** and **PQ** and **OC** are denoted by $\alpha$, $\beta$, $\gamma$ respectively. Obtain expressions for $\cos\alpha$, $\cos\beta$, $\cos\gamma$ in terms of $\lambda$ and show that

$$\cos\alpha + \cos\beta + \cos\gamma = 0.$$ [5]

(iii) Given that $\lambda = \frac{2}{3}$, find the position vector of the point of intersection of $AQ$ and $CP$. [5]

**14.** Using the standard rules for differentiation and assuming the derivatives of $\sin\theta$ and $\cos\theta$, show that

$$\frac{d}{d\theta}(\cot\theta) = -\operatorname{cosec}^2\theta \quad \text{and} \quad \frac{d}{d\theta}(\operatorname{cosec}\theta) = -\operatorname{cosec}\theta\cot\theta. \qquad [3]$$




*All measurements are in metres.*

In the figure above, $ABCD$ represents the cross-section of a rainwater channel. $BC$ and $AD$ are horizontal and the total length of $AB$, $BC$ and $CD$ is 4. Given that $BF = CE = 1$, $BC = x$ and $D\widehat{A}B = A\widehat{D}C = \alpha$, show that

$$x + 2\operatorname{cosec}\alpha = 4$$

and deduce that $\alpha \geqslant \frac{1}{6}\pi$. [4]

Show further that the cross-sectional area $Y$ is given by

$$Y = 4 + \cot\alpha - 2\operatorname{cosec}\alpha.$$

Hence show that the maximum value of $Y$ occurs when $\alpha = \frac{1}{3}\pi$ and find this maximum value correct to 2 decimal places. [6]

Show that if $ABCD$ is bent into the shape of a semi-circle, an even larger cross-sectional area can be obtained. [2]

**15.** (*a*) Given that

$$f(x) = 3 + 2x - x^2$$

(i) find the set of values of $x$ for which $f(x) > 0$,

(ii) express $f(x)$ in the form

$$f(x) = a - (x-b)^2$$

where $a$, $b$ are constants whose values must be found. [4]

(*b*) Given that

$$g(x) = \frac{1}{\sqrt{3+2x-x^2}}$$

find

(i) the set of values of $x$ for which $g(x)$ exists,

(ii) the minimum value of $g(x)$. [2]

Express as an integral the area of the region enclosed between the graph of $g$, the $x$-axis and the ordinates at $x = 0$ and $x = 2$. Using the substitution $z = x - 1$, or otherwise, show that this area is $\frac{1}{3}\pi$. [4]

This region is rotated through four right-angles about the $x$-axis. Find the volume generated. [5]

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# MATHEMATICS A 2

## MECHANICS

A.M. MONDAY, 16 June 1986

(3 Hours)

*Answer* **all** *questions in Section A and* **four** *from Section B.*

$[g = 9{\cdot}8 \text{ m s}^{-2}]$

## SECTION A

*Answer* **all** *questions in this section.*

1. A particle moves along the $x$-axis such that its displacement at time $t$ satisfies the equation

$$\frac{dx}{dt} + x = 3e^{-t}.$$

Given that the particle passes through the point $x = 1$ at $t = 0$, find $x$ in terms of $t$. [4]

2. Forces of 5 N, 20 N act along the sides $AB$ and $BC$ respectively of a square of side 1 m. Find the forces acting along the sides $AD$, $DC$ and the direction $CA$ which together with the given forces will produce equilibrium. [4]

3. A particle $P$ is attached by a light inelastic string of length $a$ to a fixed point $A$. The particle is moving with constant speed in a horizontal circle whose centre $O$ is vertically below $A$. During the motion the string makes an angle of 60° with the downward vertical through $A$. Prove that the angular speed of $P$ about $O$ is $\sqrt{\dfrac{2g}{a}}$. [4]

4. The position vector of a particle at time $t$ is given by

$$\mathbf{r} = (a\cos\omega t)\mathbf{i} + (a\sin\omega t)\mathbf{j} + bt\mathbf{k},$$

where $a$, $b$ and $\omega$ are constants. Find the velocity and acceleration vectors. Show that the speed of the particle is constant and that the magnitude of the particle's acceleration is always non-zero.

Comment briefly on these results. [4, 1]

5. The end $A$ of a light elastic string $AB$ of natural length 1·2 m is fixed. When a particle of mass 2·4 kg is attached to the string at $B$ and hangs freely under gravity, the extension of the string is 0·09 m. Find the modulus of elasticity of the string.

The particle is now pulled down vertically a further 0·12 m and released from rest. By energy considerations, or otherwise, find the greatest height reached above the point of release in the subsequent motion. [5]

6. A particle of mass $m$ is moving with speed $u$ along a straight line on a smooth horizontal plane. It strikes a particle of mass $2m$ which is at rest before the collision. After the collision, the first particle travels in the same direction as before but its speed is now $\frac{1}{4}u$.

Calculate:

(*a*) the speed of the second particle after the collision,

(*b*) the coefficient of restitution,

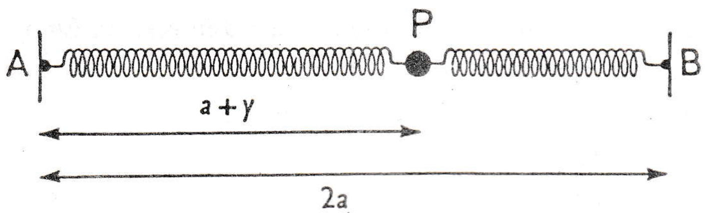
(*c*) the fraction of the initial kinetic energy lost due to the collision. [5]

7. Two particles of masses $m$ and $3m$ are attached to the ends $A$ and $B$ respectively of a light inelastic string. The string passes over a smooth peg $C$ fixed at the top of a rough plane inclined at an angle $\tan^{-1}(\frac{3}{4})$ to the horizontal. The portion $AC$ of the string lies along the line of greatest slope of the plane and the portion $CB$ hangs vertically. The system is released from rest. Given that the coefficient of friction between the particle and the plane is $\frac{1}{4}$, find

(i) the acceleration of the system,

(ii) the tension in the string. [6]

8.




A particle $P$ of mass $m$ rests on a smooth horizontal surface. Two light horizontal springs $AP$ and $PB$ are attached to $P$, $A$ and $B$ being fixed points, and $APB$ being a straight line. $AB$ is of length $2a$. Both springs are of natural length $a$ and modulus $ka$. The system is released from rest at $t = 0$ with $AP = 7a/6$ and $PB = 5a/6$. Given that at time $t$, $AP$ is of length $a + y$, derive the equation of motion of $P$. Hence show that the motion is simple harmonic of period $2\pi\sqrt{\frac{m}{2k}}$.

Find $y$ in terms of $t$. [7]

12. (*a*) A particle is projected under gravity from a point $O$ with speed $u$ at an angle $\alpha$ to the horizontal. Write down expressions for the horizontal and vertical displacements $x$ and $y$ of the particle from $O$ at time $t$. [2]

(*b*) A boy throws a ball from the foot of a plane inclined at an angle $\beta$ to the horizontal, and later the ball falls on to the plane. The motion of the ball takes place in the vertical plane containing the line of greatest slope of the plane. The initial velocity of the ball has magnitude $u$ and is inclined at an angle $\alpha$ ($\alpha > \beta$) to the horizontal.

(i) By using the results given in (*a*), show that the ball meets the inclined plane after a time $\dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$.

[Hint: the equation of the line of greatest slope of the plane is $y = x\tan\beta$.] [5]

(ii) Find the range on the inclined plane. [3]

(iii) Given that the ball strikes the plane at right angles, show that

$$\tan\alpha = \cot\beta + 2\tan\beta.$$ [5]

13. A particle of mass $m$ hangs by a light inelastic string of length $a$ from a fixed point $O$. The particle is given an initial horizontal velocity of magnitude $u$. Find the speed of the particle and the tension in the string when the string makes an angle $\theta$ with the downward vertical through $O$. Given that the particle makes complete revolutions about $O$, show that $u \geqslant \sqrt{5ga}$.

The particle is now held at a horizontal distance of $\frac{3}{5}a$ from $O$ and then allowed to fall vertically. Show that immediately after the string becomes taut the speed of the particle is $\sqrt{72ga/125}$ and find the impulsive tension in the string. In the subsequent motion, the string catches against a small peg at a depth $h$ ($< a$) below $O$. Show that the particle will make complete revolutions about the peg if $h \geqslant 503a/625$. [6, 5, 4]

14. $AC$ is a fixed rough straight horizontal wire of length greater than $4a$. A straight uniform rod $AP$ of mass $4M$ and length $2a$ is smoothly jointed to the wire at $A$. The end $P$ of the rod is attached by means of a light inelastic thread $PB$ of length $2a$ to a bead $B$ of mass $M$ which can slide on the wire.

(*a*) The system is in equilibrium with $P\widehat{A}B$ equal to $\theta$. By taking moments about $A$ for the rod $AP$, or otherwise, show that the tension in the string is $Mg/\sin\theta$. Find the frictional force at $B$ and show that $\mu \geqslant \frac{1}{2}\cot\theta$, where $\mu$ is the coefficient of friction between the bead and the wire. [5]

(*b*) With $P\widehat{A}B = \theta$, suppose $\mu = \frac{1}{4}\cot\theta$ and that a horizontal force $S$ applied at $P$ in the plane $APB$ keeps the system in equilibrium. If $B$ is on the point of slipping towards $A$, show that the tension in the string is $Mg/(3\sin\theta)$ and find the value of $S$. [7]

(*c*) When $P\widehat{A}B = \theta$, $\mu = \frac{1}{4}\cot\theta$ and no horizontal force is applied at $P$, what is the minimum value of the mass of $B$ required to ensure equilibrium? [3]

15. Prove that the centre of mass of a uniform solid cone of height $h$ and base radius $b$ is at a height of $\frac{1}{4}h$ above its base.

A uniform solid cone $C_1$ has height $3a$ and base radius $2a$. A smaller cone $C_2$ of height $2a$ and base radius $a$ is contained symmetrically inside $C_1$. The bases of $C_2$ and $C_1$ have a common centre and the axis of $C_2$ is part of the axis of $C_1$. If $C_2$ is removed from $C_1$, show that the centre of mass of the solid remaining is at a distance of $11a/5$ from the vertex of $C_1$.

The remaining solid is suspended from a string which is attached to a point on the outer curved surface at a distance of $\frac{1}{3}\sqrt{13}a$ from the vertex of $C_1$. Given that the axis of symmetry is inclined at an angle $\alpha$ to the vertical, find $\tan\alpha$. [4, 5, 6]

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# MATHEMATICS A3

## Probability and Statistics

A.M. WEDNESDAY, 18 June 1986

(3 Hours)

*Answer* **all** *questions in Section A and* **four** *from Section B.*

*Mathematical and Statistical tables are provided.*

*A calculator may be used except where it is expressly forbidden.*

## SECTION A

*Answer* **all** *questions in this section.*

1. Given that $P(A) = 0{\cdot}4$ and $P(A \cup B) = 0{\cdot}7$, determine the value of $P(B)$ in each of the cases when
   (i) $A$ and $B$ are mutually exclusive,
   (ii) $A$ and $B$ are independent. [3]

2. The continuous random variables $X$ and $Y$ are independent, with $X$ having the uniform distribution over the interval from 2 to 6, and with $Y$ having the uniform distribution over the interval from 3 to 9. Assuming the results given in the information booklet for the mean and the variance of a uniform distribution, evaluate $E(XY)$ and $\text{Var}(3X - Y)$. [4]

3. Suppose that 2% of all animals of a certain species have a particular disease. A test can be applied to an animal to determine whether or not it has the disease. When applied to an animal which has the disease, the test will give a positive response (indicating the presence of the disease) with probability $0{\cdot}96$. When applied to an animal which does not have the disease, the test will give a positive response with probability $0{\cdot}01$.
   (i) Calculate the probability that when the test is applied to a randomly chosen animal, it will give a positive response.
   (ii) Given that a positive response is obtained on an animal calculate, to two decimal places, the probability that the animal has the disease. [6]

4. The continuous random variable $X$ has the probability density function $f$, where

$$f(x) = \tfrac{1}{2}(x - 2), \text{ for } 2 \leqslant x \leqslant 4,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

By first expanding $(X - c)^2$, or otherwise, find two values of $c$ such that $\mathrm{E}[(X - c)^2] = \tfrac{2}{3}$. [6]

5. The random variables $X_1$, $X_2$, and $X_3$ are independent. Their means are $\mu$, $2\mu$, and $3\mu$, respectively, and each has unit variance. Show that each of the statistics

$$T_1 = \frac{1}{6}(X_1 + X_2 + X_3) \text{ and } T_2 = \frac{1}{3}(X_1 + \frac{1}{2}X_2 + \frac{1}{3}X_3)$$

is an unbiased estimator of $\mu$. Determine which of $T_1$ and $T_2$ is the better unbiased estimator of $\mu$. [6]

6. **Calculators are not to be used in this question.**

The random variable $X$ has the binomial distribution $\mathrm{B}(n, p)$. Use the tables provided and, when necessary, a distributional approximation, to calculate $\mathrm{P}(X \geqslant np)$ to three decimal places in each of the cases when (i) $n = 20$, $p = 0{\cdot}3$; (ii) $n = 20$, $p = 0{\cdot}8$; (iii) $n = 10$, $p = 0{\cdot}25$; (iv) $n = 200$, $p = 0{\cdot}015$. [7]

7. The random variable $X$ is normally distributed with unknown mean $\mu$ cm and unknown standard deviation $\sigma$ cm.

(i) A random sample of six observations of $X$ had the values (in cm)

$$13{\cdot}2,\ 16{\cdot}6,\ 15{\cdot}7,\ 16{\cdot}1,\ 14{\cdot}3,\ 15{\cdot}9,$$

respectively. Calculate 90% confidence limits for $\mu$.

(ii) Assuming that $\sigma = 1{\cdot}6$ calculate, to four decimal places, the probability that the mean of a random sample of 16 observations of $X$ will be within 1 cm of the value of $\mu$. [8]

## SECTION B

*Answer* **four** *questions from this section.*

8. Of the ten cards in a pack, five are numbered 1, three are numbered 2, and the remaining two cards are numbered 3. Three cards are selected at random without replacement from the pack. Calculate the probabilities that

(i) all three selected cards have the same number, [2]

(ii) the numbers on the three selected cards are 1, 2 and 3 (in any order), [2]

(iii) the largest (or equal largest) of the numbers cn the selected cards is 2. [4]

Let $X$ denote the sum of the numbers on the three selected cards.

(iv) Find the probability distribution of $X$. [5]

(v) Deduce the most probable value of the sum of the numbers on the seven cards remaining in the pack. [2]

**9.** A tennis team takes part in a competition in which it plays four matches at home and four matches away. Independently for each match played the probability that the team will win a home match is 0·8 and the probability that it will win an away match is 0·3.

(i) Calculate the probability that the team's first home win will be the third home match it plays. [2]

(ii) Calculate, to four decimal places, the probability that in the competition the team will end up having won exactly two of its home matches and exactly two of its away matches. [3]

(iii) Calculate, to four decimal places, the probability that in the competition the team will win 3 more home matches than away matches. [3]

(iv) Let $T$ denote the total number of matches (home and away) that the team will win in the competition. Find the mean and the variance of $T$. Hence, or otherwise, show that $T$ is not binomially distributed. [7]

**10.** The continuous random variable $X$ has the probability density function $f$, where

$$f(x) = \frac{1}{8}, \quad \text{for } 0 \leqslant x < 2,$$

$$f(x) = \frac{1}{8}x, \quad \text{for } 2 \leqslant x \leqslant 4,$$

$$f(x) = 0, \quad \text{otherwise.}$$

(i) Obtain expressions for $F(x)$, where $F$ is the cumulative distribution function of $X$. [3]

(ii) Evaluate

(*a*) $P(1 \leqslant X \leqslant 3)$,

(*b*) $P(X \geqslant 1 | X \leqslant 3)$. [4]

(iii) Let $g$ denote the probability density function of $Y = |X - 3|$. Show that

$$g(y) = \frac{1}{8}, \quad \text{for } 1 \leqslant y \leqslant 3.$$

Find $g(y)$ for all other values of $y$ for which $g(y)$ is non-zero. Hence, or otherwise, evaluate $E(|X - 3|)$. [8]

**11.** The two discrete random variables $X$ and $Y$ have the joint probability distribution displayed in the following table.

| | | $x$ | | |
|---|---|---|---|---|
| | | 0 | 1 | 2 |
| | 1 | 0 | 0·05 | 0 |
| $y$ | 2 | 0·2 | 0 | 0·15 |
| | 3 | 0 | 0.6 | 0 |

(i) Show that $X$ and $Y$ are not independent. [1]

(ii) Evaluate $E(XY)$. [6]

(iii) Derive the joint probability distribution of $U = Y + X$ and $W = Y - X$. Show that $U$ and $W$ are independent. Express $Y - 3X$ in terms of $U$ and $W$ and hence evaluate $\text{Var}(Y - 3X)$. [8]

**12.** $X$ and $Y$ are independent random variables having Poisson distributions with means 2 and 4, respectively. Find, to four decimal places in each case, the probabilities that a random observation of $X$ and a random observation of $Y$ will be such that

(i) each has the value 2, [2]

(ii) at least one of them has a value of 2 or more, [3]

(iii) one of them has the value 2 and the other has a value less than 2. [4]

Evaluate

(iv) $E(2^X)$; [3]

(v) $E(2^{X-Y})$. [3]

**13.** The time, in hours, required to roast a chicken of weight $w$ kg is a normally distributed random variable having mean $(0{\cdot}4w + 2{\cdot}2)$ and standard deviation $0{\cdot}05w$.

(i) Find, to three decimal places, the probability that at least $3\frac{1}{4}$ hours will be required to roast a chicken weighing 2 kg. [2]

(ii) Find the weight in kg, to three decimal places, of a chicken for which there is a probability of $0{\cdot}95$ that it will require less than 4 hours to roast it. [3]

(iii) Find, to three decimal places, the probability that a chicken weighing 4 kg will require at least half an hour longer to roast than a chicken weighing 2 kg. [3]

Three chickens, two weighing 2 kg and one weighing 4 kg, are to be roasted successively in random order. Find, to three decimal places, the probabilities that

(iv) the first chicken roasted will require at least $3\frac{1}{4}$ hours,

(v) the total time required to roast all three chickens will be less than $9\frac{1}{2}$ hours. [7]

**14.** Measurements of the heart rate of a person are subject to independent random errors that are normally distributed with mean zero and standard deviation $0{\cdot}5$ beats per minute. It is also known that when a person is given $x$ grains of a certain drug, where $0 < x < 5$, the person's heart rate, $y$ beats per minute, is such that $y = \alpha + \beta x$, where $\alpha$ and $\beta$ may vary from one person to another. The following table shows the measured heart rates of a particular person after being given various doses of the drug.

| Dosage of drug ($x$ grains) | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Heart rate ($y$ beats per min.) | 52 | 63 | 75 | 89 | 101 |

(i) Determine the least squares estimates of $\alpha$ and $\beta$ for this person. [4]

(ii) Calculate a 90% confidence interval for this person's heart rate when no drug is administered. [4]

(iii) The same experiment was conducted independently on another person using the same dosages of the drug (as given in the above table). From the results of this experiment the least squares estimate of the value of $\beta$ for this second person was found to be $10{\cdot}9$. Calculate 95% confidence limits for $\beta_1 - \beta_2$, where $\beta_1$ and $\beta_2$ are the respective values of $\beta$ for the first person and the second person. [7]

WELSH JOINT EDUCATION COMMITTEE | CYD-BWYLLGOR ADDYSG CYMRU

**General Certificate of Education** | **Tystysgrif Addysg Gyffredinol**

**Advanced Level** | **Safon Uwch**

# MATHEMATICS A 4

## FURTHER PURE MATHEMATICS

A.M. WEDNESDAY, 11 June 1986

(3 Hours)

*Answer* **all** *questions in Section A and* **four** *from Section B.*

*A calculator may be used except where it is expressly forbidden.*

## SECTION A

*Answer* **all** *questions in this section.*

**1.** Show that $\int_0^2 |x^2-1|\,dx = 2.$ [3]

**2.** The rotation $R$ of the plane about the fixed point $P$ transforms the point $(x, y)$ to the point $(x', y')$ according to the equation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 6 \\ \frac{4}{5} & \frac{3}{5} & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Find the coordinates of $P$ and the angle of rotation. [4]

**3.** Write down the first three terms in the Taylor series expansion of the function given by $f(x) = \sin x$ about the value $x = \frac{1}{6}\pi$.

Hence evaluate $\sin 31°$ correct to three decimal places. [4]

**4.** Given that $2-\mathrm{i}$ is a root of the equation

$$x^3+px^2+9x+q=0$$

find the other roots and the values of the real constants $p$ and $q$. [5]

**5.** Find the sum $S_n$ of the first $n$ terms of the series

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\ldots+\frac{1}{n(n+2)}.$$

Write down the limit of $S_n$ as $n \to \infty$. [5]

6. Find the coordinates of the foot of the perpendicular from the point $P(3, 5, 0)$ to the plane with equation $x+2y-z=1$.

Hence, or otherwise, find the length of the perpendicular from $P$ to the plane. [5]

7. Show that $\dfrac{d}{dx}(2^{-x}) = -a2^{-x}$ where $a = \log_e 2$.

Deduce that

$$\int_0^{\pi} 2^{-x}\cos x \, dx = \frac{a(1+2^{-\pi})}{1+a^2}.$$ [7]

8. The function $\log_e$ is defined by

$$\log_e x = \int_1^x \frac{1}{t}\,dt \qquad (x > 0).$$

(*a*) Use this definition to show that

(i) $\log_e 1 = 0$,

(ii) $\log_e \dfrac{1}{a} = -\log_e a \qquad (a > 1)$.

(*b*) By considering the area under the curve $y = \dfrac{1}{x}$ from $x = 1$ to $x = b$, with $b > 1$, show that

$$1 - \frac{1}{b} < \log_e b < b - 1.$$ [7]

## SECTION B

*Answer* **four** *questions from this section.*

9. Find the stationary points of the function $f$ given by

$$f(x) = \frac{1}{x-1} - \frac{4}{x-2}$$

and sketch its graph.

State the range of $f$. [9]

Show, by reference to your graph, that the equation $x = f(x)$ has only one real root.

By rearranging $x = f(x)$, show that this root is also a root of

$$x^3 - 3x^2 + 5x - 2 = 0.$$

By using the Newton-Raphson method, with a starting value $x_0 = \frac{1}{2}$, find the value of this root correct to two decimal places. [6]

**10.** The functions $f_1$ and $f_2$ are defined by

$$f_1(x) = \sqrt{-x}, \quad \text{for } x < 0; \qquad f_2(x) = -\frac{1}{x^2}, \quad \text{for } x \neq 0;$$

$$f_1(x) = -\sqrt{x}, \quad \text{for } x \geqslant 0; \qquad f_2(x) = 0, \quad \text{for } x = 0.$$

(i) Sketch the graphs of $f_1$ and $f_2$ on separate diagrams. [3]

(ii) Prove that $f_1$ is an odd function. [2]

(iii) The function $f$ is defined for all non-zero real values by

$$f(x) = f_1(x) \quad \text{for } x < 0,$$
$$f(x) = f_2(x) \quad \text{for } x > 0.$$

Sketch the graphs of $f$ and $f \circ f$ in separate diagrams. [4]

(iv) State why the inverse function $f^{-1}$ exists and sketch its graph. Write down the formula for $f^{-1}(x)$ and show that

$$f^{-1}(x) = \frac{1}{f(x)}.$$ [6]

**11.** The matrix equation $A\mathbf{x} = \mathbf{b}$ represents a system of simultaneous equations where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 4 & 1 \\ -4 & -11 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

(i) Show that the system has solutions only when $3p+2q+r = 0$. Solve the equations when $p = 1$, $q = -1$, $r = -1$ and interpret your answer geometrically. [6]

(ii) Calculate $A^2$ and $A^3$ and verify that $A^3 = 2A^2 - A$. Without calculating $A^4$, deduce that $A^4 = 3A^2 - 2A$. Suggest and prove a formula for $A^n$ in terms of $A^2$ and $A$ where $n$ is an integer ($n \geqslant 3$). [9]

**12.** The point $P$ moves such that its distance from the fixed point $A(3,0)$ is equal to its distance from the fixed line $l$ with equation $x = -3$. Show that the locus of $P$ is the curve $C$ with equation $y^2 = 12x$. [2]

The point $Q$ moves such that its distance from $l$ is twice its distance from $A$. Show that the locus of $Q$ is the curve $K$ with equation

$$\frac{(x-5)^2}{16} + \frac{y^2}{12} = 1.$$

Sketch the curves $C$ and $K$ on the same diagram. [5]

Find the equations of the two lines $l_1$ and $l_2$ which pass through the origin $O$ and are tangents to the curve $K$. The line $l_1$ intersects the curve $C$ at $O$ and $P_1$. The line $l_2$ intersects the curve $C$ at $O$ and $P_2$. Show that the curve $K$ is inscribed in the triangle $OP_1P_2$. [5]

The point $R$ moves such that its distance from $l$ is half its distance from $A$. Find and sketch the locus of $R$. [3]

**13.** Derive the Maclaurin series expansion of $e^x$.

Show that $\lim_{x \to \infty} xe^{-x} = 0$. [4]

By making the substitution $x = e^y$, or otherwise, find

$$\lim_{x \to \infty} \left(\frac{\log_e x}{x}\right).$$ [2]

Find and classify the turning point of the function $f$ defined by

$$f(x) = \frac{\log_e x}{x}.$$

Sketch the graph of $f$ and state its range. [5]

Use your graph to find all solutions of the equation $n^m = m^n$, where $m$ and $n$ are unequal positive integers. Explain why there is no other solution. [4]

**14.** (*a*) Find the four fourth roots of $8(1+i\sqrt{3})$ in the form $a+ib$, giving the values of $a$ and $b$ correct to two decimal places. [5]

(*b*) Sketch in the Argand diagram the circle $C$ and the line $L$ defined by the respective equations

$$|z-(1+2i)| = \sqrt{5},$$
$$|z-(1+2i)| = |z-(5+4i)|.$$

Show that $L$ is a tangent to $C$ and find the complex number representing the point of contact. [5]

Find the complex number corresponding to that point $z$ on $L$ for which $|z|$ is least. [5]

**15.** (i) Given that $f(t) = t - \coth t$, show that $f'(t) = \coth^2 t$. [2]

(ii) Show that $\coth(\sinh^{-1} t) = \dfrac{\sqrt{1+t^2}}{t}$. [2]

(iii) A region $D$ of the $xy$-plane is bounded by the curve with equation $xy = 1$, the $x$-axis and the lines $x = 1$, $x = 2$. A solid $S$ is formed by rotating $D$ about the $x$-axis through an angle of $2\pi$ radians. Show that the curved surface area $A$ of $S$ can be written as

$$A = 2\pi \int_1^2 \frac{\sqrt{1+x^4}}{x^3}\,dx.$$

Using the substitution $x^2 = \sinh t$, or otherwise, show that

$$A = \pi\left[\log_e\{(4+\sqrt{17})(\sqrt{2}-1)\} - \frac{\sqrt{17}}{4} + \sqrt{2}\right].$$ [11]

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Papur Arbennig

# MATHEMATICS S

A.M. MONDAY, 23 June 1986

(3 Hours)

*Answer* **seven** *questions.*

*No more than* **four** *questions may be answered from Section A and no more than* **four** *questions from either Section B or Section C.*

*A calculator may be used except where it is expressly forbidden.*

## SECTION A

### Pure Mathematics

*No more than* **four** *questions should be answered from this section.*

**1.** Show that, if $x \neq 0$,

$$x^4+6x^3-5x^2+6x+1 \equiv x^2\left[\left(x+\frac{1}{x}\right)^2+6\left(x+\frac{1}{x}\right)-7\right].$$

Hence or otherwise, solve completely the quartic equation

$$x^4+6x^3-5x^2+6x+1=0.$$ [6]

Now consider the quartic equation

$$x^4+6x^3+kx^2+6x+1=0$$

where $k$ is a real constant. Given that this equation has four distinct real roots, show that either

$$k < -14 \qquad \text{or} \qquad 10 < k < 11.$$ [9]

**2.** (i) Given that

$$I = \int_0^{\pi/b} e^{-ax} \sin bx \, dx$$

and

$$J = \int_0^{\pi/b} e^{-ax} \cos bx \, dx$$

where $a$, $b$ are positive constants, use integration by parts to show that

$$bI = 1 + e^{-(\pi a/b)} - aJ$$

and

$$bJ = aI.$$

Hence obtain expressions for $I$ and $J$ in terms of $a$ and $b$. [5]

(ii) Sketch the graph of the function

$$f(x) = e^{-ax}|\sin bx| \qquad \text{for } x \geqslant 0.$$

Show that for positive integral $m$,

$$\int_{m\pi/b}^{(m+1)\pi/b} f(x)\,dx = e^{-(m\pi a/b)}I$$

and hence show that the total area enclosed between the curve and the $x$-axis is

$$\frac{b}{(a^2+b^2)} \cdot \frac{(1+e^{-(\pi a/b)})}{(1-e^{-(\pi a/b)})}.$$ [10]

3. Show that the equation of the tangent at the point $P(\frac{1}{2}t^2, \frac{1}{3}t^3)$ to the curve $C_1$ having equation $9y^2 = 8x^3$ is

$$6y - 6tx + t^3 = 0.$$ [7]

Find the co-ordinates of the point $Q$ where this tangent cuts the curve again. Hence find the co-ordinates of the two points $R$ and $S$ on $C_1$ at which the normal is also a tangent to $C_1$.

Prove that the equation of the locus $C_2$ of the point of intersection of perpendicular tangents to $C_1$ is

$$36y^2 = 6x - 1.$$

Verify that $C_1$ and $C_2$ touch at the points $R$ and $S$. [8]

4. $OABC$ is a quadrilateral and $A'$, $B'$, $C'$ are points on $OA$, $OB$, $OC$ respectively. $BC$ meets $B'C'$ at $P$, $CA$ meets $C'A'$ at $Q$ and $AB$ meets $A'B'$ at $R$. Taking $O$ as origin and denoting the position vectors of $A$, $B$, $C$, $A'$, $B'$, $C'$ by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\alpha\mathbf{a}$, $\beta\mathbf{b}$, $\gamma\mathbf{c}$ respectively, use vector methods to show that $PQR$ is a straight line.

(It may be assumed that the values of the constants $\alpha$, $\beta$, $\gamma$ are all different.) [15]

5. (*a*) In a triangle $ABC$, the angles $B$ and $C$ are equal. Show that the maximum value of $\cos A + \cos B$ is $\frac{9}{8}$ and find, to 2 decimal places, the stationary values of

(i) $\sin A + \sin B$

(ii) $\tan A + \tan B$.

Indicate in each case whether your stationary values are maxima or minima. [9]

(*b*) A curve is defined parametrically by the equations

$$x = \cos\theta + \theta\sin\theta$$
$$y = \sin\theta - \theta\cos\theta$$

for $0 \leqslant \theta \leqslant 4\pi$. Obtain expressions for $dy/dx$ and $d^2y/dx^2$ in terms of $\theta$ and sketch the curve. Indicate the co-ordinates of all stationary points and show that there are no points of inflection. [6]

## Section B

### Mechanics

*No more than* **four** *questions should be answered from this section.*

6. A particle is suspended from a fixed point $A$ by a light inelastic string of length $a$. Find the speed with which it must be projected horizontally from its lowest point in order that it should pass through $A$. [15]

7. Two particles $A$ and $B$ of equal mass $m$ lie touching one another in a smooth horizontal circular groove of radius $a$. $A$ is projected away from $B$ with speed $V$. Given $e$ is the coefficient of restitution, find the speeds of the particles after 1, 2, 3 collisions. [3]

On the basis of the above derived results, suggest (but do not prove) expressions giving the speeds of the particles after $n$ collisions, checking that your expressions give correct answers for $n = 0$, 1, 2 and 3. [4]

Using your postulated expressions,

(i) calculate the speeds of the particles after the $(n+1)$th collision and comment briefly on your results,

(ii) calculate the time that elapses before the $n$th collision takes place,

(iii) calculate the energy lost after $n$ collisions and comment on your result as $n$ increases indefinitely. [8]

**8.** (*a*) A uniform rod of length $2a$ rests against a fixed rough cylinder whose axis is horizontal. The rod is perpendicular to the axis of the cylinder and inclined at an angle $\theta$ to the horizontal, and is held in this position by a horizontal string attached at the highest point of the rod. The string is also perpendicular to the axis of the cylinder. The coefficient of friction between the rod and the cylinder is $\mu$.

Given that $\mu = \tan\lambda$ and $\lambda < \theta$, show that the length $x$ of the rod above the point of contact with the cylinder must be such that

$$\frac{a\cos\theta}{\cos\lambda}\cos(\theta+\lambda) \leqslant x \leqslant \frac{a\cos\theta}{\cos\lambda}\cos(\theta-\lambda).$$ [9]

(*b*) A motor cycle which is too long to go on a weighing machine is weighed by placing first one wheel and then the other wheel on the scales. The platform of the scales is above the level of the surrounding ground.

During the weighings, the lower wheel is in contact with the ground, being clamped so that the motor cycle remains upright. The platform of the scales is horizontal at all times.

The distance between the centres $A$, $B$ of the wheels (which are of equal diameter) is $c$, and $h$ is the height of the centre of mass of the motor cycle above $AB$ when the motor cycle is on level ground. The inclination of $AB$ to the horizontal when either wheel is on the platform is $\theta$. Show that the sum of the two weighings is less than the actual weight $W$ by $\dfrac{2Wh}{c}\tan\theta$. [6]

**9.** The position vector relative to a fixed origin $O$ of a particle $P$ at any time $t$ is

$$\mathbf{r} = r\cos\omega t\mathbf{i} + r\sin\omega t\mathbf{j},$$

where $\omega$ is a constant and $r$ is a function of time. Show that the speed $v$ of the particle is given by

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\omega^2.$$ [2]

$P$ moves so that

(i) $OP$ increases with time, and

(ii) $r$ is always equal to $\beta$ times the speed of $P$, where $\beta$ is a constant.

Given that $\beta = \dfrac{\sin\alpha}{\omega}$ and $r = a$ when $t = 0$, show that $r = ae^{(\omega\cot\alpha)t}$. [6]

Show further that the direction of the acceleration of $P$ is always inclined at an angle $2\alpha$ with the direction of **OP**. [7]

**10.** (*a*) The spread of a non-fatal disease through a constant large population of $M$ people can be studied theoretically as follows. The rate at which people currently healthy are being infected is 0·01 times the product of the number of people who are healthy and the proportion $P$ of the total population who are infected. The rate of recovery of infected people is 0·009 times the number of people who are infected. Noting that the number of people infected at time $t$ is $MP$, show that the process can be approximated by the differential equation

$$\frac{dP}{dt} = 0{\cdot}001(1-10P)P$$

explaining any necessary assumptions. [3]

Find $P$ in terms of $t$ given that $P = 0{\cdot}01$ when $t = 0$. Estimate the proportion infected when $t$ is very large. [6]

(*b*) Given that $y = \dfrac{1}{u}$, write $\dfrac{dy}{dx}$ in terms of $u$ and $\dfrac{du}{dx}$.

Hence, or otherwise, find the general solution of

$$\frac{dy}{dx} + y = xy^2.$$ [6]

## SECTION C

### Probability and Statistics

*No more than* **four** *questions should be answered from this section.*

11. A cubical die has two faces numbered 1, two faces numbered 2, and two faces numbered 3. It is known that when this die is rolled, a score of 1 and a score of 2 are equally probable; denote this common probability by $p$.

(i) The die is rolled twice. Let $A$ denote the event that the same score is obtained in the two rolls, and let $B$ denote the event that the sum of the two scores is 5 or more. Find the value of $p$ for which $A$ and $B$ are independent. [5]

(ii) Write down an expression for the probability that in 10 rolls of the die, a score of 1 will be obtained twice, a score of 2 will be obtained three times, and a score of 3 will be obtained five times. Find the value of $p$ for which this probability is a maximum. [4]

(iii) Let $X$ denote the number of times a score of 3 will be obtained in $n$ rolls of the die. Write down an unbiased estimator $\hat{p}$ of $p$ in terms of $X$ and $n$ and determine an expression for the standard error of $\hat{p}$. For the case when $n = 100$ and $X = 62$, calculate approximate 90% confidence limits for $p$. [6]

12. A box contains $n$ calculators of which 3 are faulty. A technician is asked to locate and repair the 3 faulty calculators. The technician chooses one calculator at random from the box, tests it and, if it is faulty, repairs it. He then sets this calculator to one side and chooses another calculator at random from the $(n-1)$ calculators remaining in the box, tests it and, if it is faulty, repairs it, and sets it to one side. The technician continues in this way until he has located and repaired all 3 faulty calculators.

(i) Let $X$ denote the number of calculators that the technician will have to test in order to locate all 3 faulty ones. Show that

$$P(X \leqslant r) = \frac{r(r-1)(r-2)}{n(n-1)(n-2)}; \qquad r = 3, 4, 5, \ldots$$

Hence, or otherwise, find an expression for $P(X = r)$. [6]

(ii) It takes the technician 4 minutes to remove a calculator from the box and to test it, and a further 15 minutes to repair a calculator that is faulty. Let $Y$ denote the total time in minutes taken by the technician to complete the task of locating and repairing all 3 faulty calculators. Show that

$$E(Y) = 3(n+16),$$

and find the most probable value of $Y$.

[You may assume that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.]$$ [9]

**13.** (*a*) An examination paper consists of 60 true/false questions. Of the 60 questions, 20 are very difficult, 30 are of moderate difficulty, and 10 are easy. Suppose that a student's probabilities of correctly answering a question that is very difficult, of moderate difficulty, and easy are 0·4, 0·5, and 0·8, respectively. Assuming that the total number of questions answered correctly is approximately normally distributed, calculate the probability that a student will answer 35 or more questions correctly. [8]

(*b*) The number of emissions in a minute from a radioactive source has a Poisson distribution. Given that the probability of no emission within a period of one minute is 0·8, find the mean number, $\lambda$, of emissions per minute.

During a one-minute period when there was at least one emission, show that the probability that there were exactly $k$ emissions ($k \geqslant 1$) is equal to $4\lambda^k/k!$ Hence, or otherwise, calculate, correct to three decimal places, the mean number of emissions in periods of one minute in each of which there is at least one emission. [7]

**14.** The operational lifetime $X$ hours of a certain component used in an electronic system is distributed with probability density function $f$ where

$$f(x) = \tfrac{1}{100}e^{-x/100} \qquad \text{for } x > 0$$
$$f(x) = 0 \qquad \text{for } x \leqslant 0.$$

(*a*) Obtain an expression for $P(X \leqslant x)$ for $x > 0$. Hence, or otherwise, find the probability density function of $Y = e^{-X/100}$. [6]

(*b*) The component is replaced as soon as it fails or when it has operated in the system for 200 hours, whichever occurs first.

(i) Given that the component has already operated for 100 hours, calculate, correct to three decimal places, the probability that the component will not have to be replaced within the next 100 hours. [3]

(ii) Denoting the time in hours that a component is used in the system by $T$, evaluate $E(T)$ to three decimal places. [6]

**15.** (*a*) As a reward for a good year's work, a teacher decides to give each of her 25 pupils an apple or an orange, each pupil being allowed to choose either. The teacher proposes to buy $r$ apples and $r$ oranges for this purpose. Suppose that each pupil, independently of the other pupils, is equally likely to choose an apple or an orange. Using a normal approximation to a binomial distribution, find the smallest value of $r$ if the individual choices of all 25 pupils are to be met with probability 0·95 at least. [7]

(*b*) Let $\bar{X}$ denote the mean of a random sample of $n$ observations from a normal distribution having mean $(\mu+\lambda)$ and variance $\sigma^2$, and let $\bar{Y}$ denote the mean of an independent random sample of $2n$ observations from a normal distribution having mean $(2\mu-\lambda)$ and variance $\sigma^2$. Find the values of the constants $a$ and $b$ for $T = a\bar{X} + b\bar{Y}$ to be an unbiased estimator of $\lambda$, and determine the standard error of this estimator in terms of $\sigma$ and $n$.

Given that $n = 10$, $\sigma^2 = 0{\cdot}05$, $\bar{X} = 20{\cdot}8$, and $\bar{Y} = 16{\cdot}4$, determine 95% confidence limits for $\lambda$. [8]

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General Certificate of Education | Tystysgrif Addysg Gyffredinol

Advanced Level | Safon Uwch

Special Paper | Papur Arbennig

# APPLIED MATHEMATICS S

A.M. MONDAY, 23 June 1986

(3 Hours)

*Answer* **seven** *questions.*

*No more than* **four** *questions may be answered from either section.*

*A calculator may be used except where it is expressly forbidden.*

## SECTION A

## Mechanics

*No more than* **four** *questions should be answered from this section.*

**1.** A particle is suspended from a fixed point $A$ by a light inelastic string of length $a$. Find the speed with which it must be projected horizontally from its lowest point in order that it should pass through $A$. [15]

**2.** Two particles $A$ and $B$ of equal mass $m$ lie touching one another in a smooth horizontal circular groove of radius $a$. $A$ is projected away from $B$ with speed $V$. Given $e$ is the coefficient of restitution, find the speeds of the particles after 1, 2, 3 collisions. [3]

On the basis of the above derived results, suggest (but do not prove) expressions giving the speeds of the particles after $n$ collisions, checking that your expressions give correct answers for $n = 0$, 1, 2 and 3. [4]

Using your postulated expressions,

(i) calculate the speeds of the particles after the $(n+1)$th collision and comment briefly on your results,

(ii) calculate the time that elapses before the $n$th collision takes place,

(iii) calculate the energy lost after $n$ collisions and comment on your result as $n$ increases indefinitely. [8]

**3.** (*a*) A uniform rod of length $2a$ rests against a fixed rough cylinder whose axis is horizontal. The rod is perpendicular to the axis of the cylinder and inclined at an angle $\theta$ to the horizontal, and is held in this position by a horizontal string attached at the highest point of the rod. The string is also perpendicular to the axis of the cylinder. The coefficient of friction between the rod and the cylinder is $\mu$.

Given that $\mu = \tan\lambda$ and $\lambda < \theta$, show that the length $x$ of the rod above the point of contact with the cylinder must be such that

$$\frac{a\cos\theta}{\cos\lambda}\cos(\theta+\lambda) \leqslant x \leqslant \frac{a\cos\theta}{\cos\lambda}\cos(\theta-\lambda). \qquad [9]$$

(*b*) A motor cycle which is too long to go on a weighing machine is weighed by placing first one wheel and then the other wheel on the scales. The platform of the scales is above the level of the surrounding ground.

During the weighings, the lower wheel is in contact with the ground, being clamped so that the motor cycle remains upright. The platform of the scales is horizontal at all times.

The distance between the centres $A$, $B$ of the wheels (which are of equal diameter) is $c$, and $h$ is the height of the centre of mass of the motor cycle above $AB$ when the motor cycle is on level ground. The inclination of $AB$ to the horizontal when either wheel is on the platform is $\theta$. Show that the sum of the two weighings is less than the actual weight $W$ by $\dfrac{2Wh}{c}\tan\theta$. [6]

4. The position vector relative to a fixed origin $O$ of a particle $P$ at any time $t$ is

$$\mathbf{r} = r\cos\omega t\mathbf{i} + r\sin\omega t\mathbf{j},$$

where $\omega$ is a constant and $r$ is a function of time. Show that the speed $v$ of the particle is given by

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\omega^2. \qquad [2]$$

$P$ moves so that

(i) $OP$ increases with time, and

(ii) $r$ is always equal to $\beta$ times the speed of $P$, where $\beta$ is a constant.

Given that $\beta = \dfrac{\sin\alpha}{\omega}$ and $r = a$ when $t = 0$, show that $r = ae^{(\omega\cot\alpha)t}$. [6]

Show further that the direction of the acceleration of $P$ is always inclined at an angle $2\alpha$ with the direction of **OP**. [7]

5. (*a*) The spread of a non-fatal disease through a constant large population of $M$ people can be studied theoretically as follows. The rate at which people currently healthy are being infected is 0·01 times the product of the number of people who are healthy and the proportion $P$ of the total population who are infected. The rate of recovery of infected people is 0·009 times the number of people who are infected. Noting that the number of people infected at time $t$ is $MP$, show that the process can be approximated by the differential equation

$$\frac{dP}{dt} = 0{\cdot}001(1-10P)P$$

explaining any necessary assumptions. [3]

Find $P$ in terms of $t$ given that $P = 0{\cdot}01$ when $t = 0$. Estimate the proportion infected when $t$ is very large. [6]

(*b*) Given that $y = \dfrac{1}{u}$, write $\dfrac{dy}{dx}$ in terms of $u$ and $\dfrac{du}{dx}$.

Hence, or otherwise, find the general solution of

$$\frac{dy}{dx} + y = xy^2. \qquad [6]$$

## Section B

### Probability and Statistics

*No more than* **four** *questions should be answered from this section.*

6. A cubical die has two faces numbered 1, two faces numbered 2, and two faces numbered 3. It is known that when this die is rolled, a score of 1 and a score of 2 are equally probable; denote this common probability by $p$.

(i) The die is rolled twice. Let $A$ denote the event that the same score is obtained in the two rolls, and let $B$ denote the event that the sum of the two scores is 5 or more. Find the value of $p$ for which $A$ and $B$ are independent. [5]

(ii) Write down an expression for the probability that in 10 rolls of the die, a score of 1 will be obtained twice, a score of 2 will be obtained three times, and a score of 3 will be obtained five times. Find the value of $p$ for which this probability is a maximum. [4]

(iii) Let $X$ denote the number of times a score of 3 will be obtained in $n$ rolls of the die. Write down an unbiased estimator $\hat{p}$ of $p$ in terms of $X$ and $n$ and determine an expression for the standard error of $\hat{p}$. For the case when $n = 100$ and $X = 62$, calculate approximate 90% confidence limits for $p$. [6]

7. A box contains $n$ calculators of which 3 are faulty. A technician is asked to locate and repair the 3 faulty calculators. The technician chooses one calculator at random from the box, tests it and, if it is faulty, repairs it. He then sets this calculator to one side and chooses another calculator at random from the $(n-1)$ calculators remaining in the box, tests it and, if it is faulty, repairs it, and sets it to one side. The technician continues in this way until he has located and repaired all 3 faulty calculators.

(i) Let $X$ denote the number of calculators that the technician will have to test in order to locate all 3 faulty ones. Show that

$$P(X \leqslant r) = \frac{r(r-1)(r-2)}{n(n-1)(n-2)}; \qquad r = 3, 4, 5, \ldots$$

Hence, or otherwise, find an expression for $P(X = r)$. [6]

(ii) It takes the technician 4 minutes to remove a calculator from the box and to test it, and a further 15 minutes to repair a calculator that is faulty. Let $Y$ denote the total time in minutes taken by the technician to complete the task of locating and repairing all 3 faulty calculators. Show that

$$E(Y) = 3(n+16),$$

and find the most probable value of $Y$.

[You may assume that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.]$$ [9]

**8.** (*a*) An examination paper consists of 60 true/false questions. Of the 60 questions, 20 are very difficult, 30 are of moderate difficulty, and 10 are easy. Suppose that a student's probabilities of correctly answering a question that is very difficult, of moderate difficulty, and easy are 0·4, 0·5, and 0·8, respectively. Assuming that the total number of questions answered correctly is approximately normally distributed, calculate the probability that a student will answer 35 or more questions correctly. [8]

(*b*) The number of emissions in a minute from a radioactive source has a Poisson distribution. Given that the probability of no emission within a period of one minute is 0·8, find the mean number, $\lambda$, of emissions per minute.

During a one-minute period when there was at least one emission, show that the probability that there were exactly $k$ emissions ($k \geqslant 1$) is equal to $4\lambda^k/k!$ Hence, or otherwise, calculate, correct to three decimal places, the mean number of emissions in periods of one minute in each of which there is at least one emission. [7]

**9.** The operational lifetime $X$ hours of a certain component used in an electronic system is distributed with probability density function $f$ where

$$f(x) = \tfrac{1}{100}e^{-x/100} \qquad \text{for } x > 0$$
$$f(x) = 0 \qquad \text{for } x \leqslant 0.$$

(*a*) Obtain an expression for $P(X \leqslant x)$ for $x > 0$. Hence, or otherwise, find the probability density function of $Y = e^{-X/100}$. [6]

(*b*) The component is replaced as soon as it fails or when it has operated in the system for 200 hours, whichever occurs first.

(i) Given that the component has already operated for 100 hours, calculate, correct to three decimal places, the probability that the component will not have to be replaced within the next 100 hours. [3]

(ii) Denoting the time in hours that a component is used in the system by $T$, evaluate $E(T)$ to three decimal places. [6]

**10.** (*a*) As a reward for a good year's work, a teacher decides to give each of her 25 pupils an apple or an orange, each pupil being allowed to choose either. The teacher proposes to buy $r$ apples and $r$ oranges for this purpose. Suppose that each pupil, independently of the other pupils, is equally likely to choose an apple or an orange. Using a normal approximation to a binomial distribution, find the smallest value of $r$ if the individual choices of all 25 pupils are to be met with probability 0·95 at least. [7]

(*b*) Let $\bar{X}$ denote the mean of a random sample of $n$ observations from a normal distribution having mean $(\mu+\lambda)$ and variance $\sigma^2$, and let $\bar{Y}$ denote the mean of an independent random sample of $2n$ observations from a normal distribution having mean $(2\mu-\lambda)$ and variance $\sigma^2$. Find the values of the constants $a$ and $b$ for $T = a\bar{X} + b\bar{Y}$ to be an unbiased estimator of $\lambda$, and determine the standard error of this estimator in terms of $\sigma$ and $n$.

Given that $n = 10$, $\sigma^2 = 0{\cdot}05$, $\bar{X} = 20{\cdot}8$, and $\bar{Y} = 16{\cdot}4$, determine 95% confidence limits for $\lambda$. [8]

WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

Special Paper

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# PURE MATHEMATICS S

A.M. MONDAY, 23 June 1986

(3 Hours)

*Answer* **seven** *questions.*

*No more than* **four** *questions may be answered from either section.*

*A calculator may be used except where it is expressly forbidden.*

## SECTION A

## PURE MATHEMATICS

*No more than* **four** *questions should be answered from this section.*

**1.** Show that, if $x \neq 0$,

$$x^4+6x^3-5x^2+6x+1 \equiv x^2\left[\left(x+\frac{1}{x}\right)^2+6\left(x+\frac{1}{x}\right)-7\right].$$

Hence or otherwise, solve completely the quartic equation

$$x^4+6x^3-5x^2+6x+1 = 0.$$ [6]

Now consider the quartic equation

$$x^4+6x^3+kx^2+6x+1 = 0$$

where $k$ is a real constant. Given that this equation has four distinct real roots, show that either

$$k < -14 \qquad \text{or} \qquad 10 < k < 11.$$ [9]

**2.** (i) Given that

$$I = \int_0^{\pi/b} e^{-ax} \sin bx \, dx$$

and

$$J = \int_0^{\pi/b} e^{-ax} \cos bx \, dx$$

where $a$, $b$ are positive constants, use integration by parts to show that

$$bI = 1+e^{-(\pi a/b)} - aJ$$

and

$$bJ = aI.$$

Hence obtain expressions for $I$ and $J$ in terms of $a$ and $b$. [5]

(ii) Sketch the graph of the function

$$f(x) = e^{-ax}|\sin bx| \qquad \text{for } x \geqslant 0.$$

Show that for positive integral $m$,

$$\int_{m\pi/b}^{(m+1)\pi/b} f(x)\, dx = e^{-(m\pi a/b)} I$$

and hence show that the total area enclosed between the curve and the $x$-axis is

$$\frac{b}{(a^2+b^2)} \cdot \frac{(1+e^{-(\pi a/b)})}{(1-e^{-(\pi a/b)})}.$$ [10]

**3.** Show that the equation of the tangent at the point $P(\frac{1}{2}t^2, \frac{1}{3}t^3)$ to the curve $C_1$ having equation $9y^2 = 8x^3$ is

$$6y - 6tx + t^3 = 0.$$

Find the co-ordinates of the point $Q$ where this tangent cuts the curve again. Hence find the co-ordinates of the two points $R$ and $S$ on $C_1$ at which the normal is also a tangent to $C_1$. [7]

Prove that the equation of the locus $C_2$ of the point of intersection of perpendicular tangents to $C_1$ is

$$36y^2 = 6x - 1.$$

Verify that $C_1$ and $C_2$ touch at the points $R$ and $S$. [8]

**4.** $OABC$ is a quadrilateral and $A'$, $B'$, $C'$ are points on $OA$, $OB$, $OC$ respectively. $BC$ meets $B'C'$ at $P$, $CA$ meets $C'A'$ at $Q$ and $AB$ meets $A'B'$ at $R$. Taking $O$ as origin and denoting the position vectors of $A$, $B$, $C$, $A'$, $B'$, $C'$ by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\alpha\mathbf{a}$, $\beta\mathbf{b}$, $\gamma\mathbf{c}$ respectively, use vector methods to show that $PQR$ is a straight line.

(It may be assumed that the values of the constants $\alpha$, $\beta$, $\gamma$ are all different.) [15]

**5.** (*a*) In a triangle $ABC$, the angles $B$ and $C$ are equal. Show that the maximum value of $\cos A + \cos B$ is $\frac{9}{8}$ and find, to 2 decimal places, the stationary values of

(i) $\sin A + \sin B$

(ii) $\tan A + \tan B$.

Indicate in each case whether your stationary values are maxima or minima. [9]

(*b*) A curve is defined parametrically by the equations

$$x = \cos\theta + \theta\sin\theta$$
$$y = \sin\theta - \theta\cos\theta$$

for $0 \leqslant \theta \leqslant 4\pi$. Obtain expressions for $dy/dx$ and $d^2y/dx^2$ in terms of $\theta$ and sketch the curve. Indicate the co-ordinates of all stationary points and show that there are no points of inflection. [6]

## SECTION B

### FURTHER PURE MATHEMATICS

*No more than* **four** *questions should be answered from this section.*

**6.** The roots of the equation

$$z^4 + 4az^3 + 6bz^2 + 4cz + d = 0$$

where $a$, $b$, $c$, $d$ are real, are represented by four points on an Argand diagram. Given that the four points are the vertices of a square, prove that $b = a^2$ and $c = a^3$. [15]

**7.** (i) Show that $\cos 3\theta = 4c^3 - 3c$, where $c = \cos\theta$, and express $\cos 6\theta$ as a polynomial in $c$ of degree 6. Hence show that the equation

$$32x^3 - 48x^2 + 18x - 1 = 0$$

has roots $\cos^2\left(\dfrac{r\pi}{12}\right)$ for $r = 1, 3, 5$. [6]

(ii) Show that

$$\cos(n+1)\theta = 2\cos n\theta\cos\theta - \cos(n-1)\theta.$$

Deduce that $\cos n\theta$ is a polynominal in $c$ of degree $n$. Conjecture and prove a formula for the coefficient of $c^n$ in the polynominal. State for which positive integers $n$ can $\sin n\theta$ be expressed as a polynominal in $\sin\theta$. [9]

8. $A$ is a $2 \times 2$ matrix with real coefficients such that $A^2 = A$.

Find all the solutions of this equation. [10]

Prove that any $2 \times 2$ matrix satisfying the equation $A^2 = A$ also satisfies the equation $A^3 = A$. By finding a suitable counter-example, show that the converse statement is false. [5]

9. Sketch the curve $y^2 = x(x-1)^2$, whose parametric equations are $x = t^2$, $y = t^3 - t$. [3]

(i) The points with parameters $t_1$, $t_2$, $t_3$ are collinear. Show that

$$t_2 t_3 + t_3 t_1 + t_1 t_2 = -1.$$ [3]

(ii) Deduce that the tangent at the point with parameter $t$ meets the curve again at the point $P$ with parameter

$$-\frac{(1+t^2)}{2t}.$$

Determine the locus of $P$ as $t$ varies. [6]

(iii) Show that the tangents at the points with parameters $t_1$ and $t_2$, $(t_1 \neq t_2)$, meet on the curve if and only if $t_1 t_2 = 1$. [3]

10. The function $f$ is defined over the interval $[0, 1]$.

Find the equation $y = p(x)$ of the tangent to the graph of $f$ at the point $(t, f(t))$ where $t \in (0, 1)$. [2]

Evaluate, in terms of $t$, the integral

$$A(t) = \int_0^1 p(x)\,dx.$$ [3]

Given that $f''(x) \neq 0$ for any value of $x$, find the value of $t$ for which $A(t)$ is stationary. Show that this point is a maximum if $f''(x) > 0$ for $x \in [0, 1]$ and is a minimum if $f''(x) < 0$ for $x \in [0, 1]$. [5]

For $f(x) = e^{x^2}$, show that $f''(x) > 0$ for $x \in [0, 1]$, and deduce that

$$\int_0^1 e^{x^2}\,dx > e^{1/4}.$$ [5]