WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

CYD-BWYLLGOR ADDYSG CYMRU

Tystysgrif Addysg Gyffredinol

Safon Uwch

MATHEMATICS A 1

A.M. WEDNESDAY, 5 June 1985

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

SECTION A

Answer **all** *questions in this Section.*

1. If α, β are the roots of the quadratic equation

$$x^2+px+q=0$$

find the quadratic equation whose roots are $\alpha-\beta$ and $\beta-\alpha$. [4]

2. Find all the roots of the equation

$$\cos 2\theta = 2\cos\theta$$

lying between 0° and 360°, giving your answers to the nearest degree. [4]

3. The complex numbers u, v and w are related by

$$\frac{1}{u}=\frac{1}{v}+\frac{1}{w}.$$

Given that $v=3+4\text{i}$ and $w=4-3\text{i}$, find u in the form $x+\text{i}y$. [5]

4. Use the binomial theorem to show that, for small x,

$$\sqrt{\frac{1-x}{1+x}}\approx 1-x+\tfrac{1}{2}x^2.$$

By putting $x=\frac{1}{9}$, show that

$$\sqrt{5}\approx\tfrac{324}{145}.$$ [5]

5. The sum of the first two terms of a geometric series is 20 and the third term is 1. Calculate

(i) the two possible values of the common ratio,

(ii) the sum to infinity of the series with the positive common ratio. [5]

6. Find the values of x which satisfy each of the following inequalities:

(i) $|2x+1|<3$, [2]

(ii) $\dfrac{x^2+x-1}{x^2+1}\leqslant\dfrac{1}{2}$. [3]

7. Differentiate the following functions with respect to x, simplifying your answers as much as possible:

(i) $\log_e(\sin x)$, [2]

(ii) x^3e^{2x}, [2]

(iii) $\dfrac{\sin x}{\sqrt{x}}$. [2]

8. Use integration by parts to show that

$$I = \int_0^1 x \tan^{-1} x \, dx = \frac{\pi}{8} - \frac{1}{2} \int_0^1 \frac{x^2 \, dx}{1+x^2}.$$

Hence, or otherwise, show that

$$I = \frac{1}{4}\pi - \frac{1}{2}.$$ [6]

SECTION B

Answer **four** *questions from this section.*

9. (*a*) Show that

$$\tan 4\theta = \frac{4 \tan \theta (1 - \tan^2\theta)}{\tan^4\theta - 6 \tan^2\theta + 1}.$$

Hence show that $\tan 22\tfrac{1}{2}^\circ$ is a root of the equation

$$t^4 - 6t^2 + 1 = 0.$$

Deduce that

$$\tan 22\tfrac{1}{2}^\circ = \sqrt{3 - \sqrt{8}}$$

and obtain a similar expression for $\tan 67\tfrac{1}{2}^\circ$.

Use your results to show that

$$\sec^2 22\tfrac{1}{2}^\circ + \sec^2 67\tfrac{1}{2}^\circ = 8.$$ [8]

(*b*) AB is a chord of a circle centre O which subtends an angle 2θ radians at O ($\theta < \frac{1}{2}\pi$). If AB divides the circle into two regions, one having twice the area of the other, show that θ satisfies the equation

$$6\theta - 3 \sin 2\theta - 2\pi = 0.$$

Hence show that θ lies between 1·30 and 1·31. [7]

10. The functions f and g are defined by

$$f(x) = x(x+1)$$

and

$$g(x) = e^x - 1$$

both with domain $[0, \infty)$. Explain why the inverse functions f^{-1} and g^{-1} exist. Find and simplify expressions for $f^{-1}(x)$ and $g^{-1}(x)$. [4]

The composite function h is defined by

$$h = f \circ g.$$

Show that

$$h(x) = e^{2x} - e^x$$

and sketch the graph of h. [3]

The graph of h passes through the points $O(0, 0)$ and $P(1, e^2 - e)$. Show that the region enclosed between the straight line OP and the graph of h between O and P has area $\frac{1}{2}(e - 1)$. [4]

Derive an expression for $h^{-1}(x)$ and state the geometrical relationship between the graphs of h and h^{-1}. [4]

11. (*a*) Calculate the coordinates of the centres C_1, C_2 and the radii of the two circles whose equations are

$$x^2 + y^2 + 2x - 4y - 5 = 0$$

and

$$x^2 + y^2 - 2x - 6y + 5 = 0.$$

Find the coordinates of the points of intersection A, B of the two circles, and show that AB is perpendicular to $C_1 C_2$. [6]

(*b*) Show that the point $P(ap^2, 2ap)$ lies on the curve having equation $y^2 = 4ax$ for all values of p.

The point $Q(aq^2, 2aq)$ is another point on the curve. Show that the equation of PQ is

$$(p+q)y - 2x = 2apq.$$

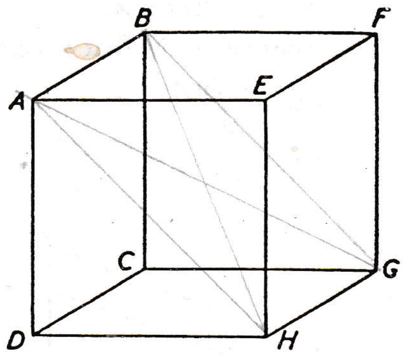
If PQ passes through the point $S(a, 0)$, show that

$$q = -1/p,$$

and in this case deduce the coordinates of R, the mid-point of PQ, in terms of p. Hence show that the equation of the locus of R as p varies is

$$y^2 = 2a(x-a).$$ [9]

12. (*a*)



The above figure shows a cube. Calculate, to the nearest degree, the angles between

(i) AG and EF, [2]

(ii) AG and BH, [2]

(iii) AG and the plane $ABFE$. [2]

(*b*) The points A, B, C have position vectors

$$\mathbf{OA} = -2\mathbf{i}+3\mathbf{j}-7\mathbf{k}$$
$$\mathbf{OB} = \mathbf{i}+7\mathbf{j}+5\mathbf{k}$$
$$\mathbf{OC} = 4\mathbf{i}+3\mathbf{j}+\mathbf{k}$$

relative to an origin O and a set of mutually perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Show that

$$\mathbf{AB} = 3\mathbf{i}+4\mathbf{j}+12\mathbf{k}$$

and obtain a similar expression for $\mathbf{AC}$. Calculate $|\mathbf{AB}|$, $|\mathbf{AC}|$ and the scalar product $\mathbf{AB}.\mathbf{AC}$ and deduce that

$$B\widehat{A}C = \cos^{-1}(\tfrac{57}{65}).$$

Write down an expression for the position vector of the point D which divides AC in the ratio $\lambda : 1-\lambda$. By equating the scalar product $\mathbf{AC}.\mathbf{OD}$ to zero, deduce that the position vector of the foot of the perpendicular from O to AC is given by

$$\tfrac{52}{25}\mathbf{i}+3\mathbf{j}-\tfrac{39}{25}\mathbf{k}.$$ [9]

13. The function f is defined by

$$f(x) = \frac{\cos^2 x}{1+\sin^2 x}$$

with domain $[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]$. Show that

$$f'(x) = \frac{-2\sin 2x}{(1+\sin^2 x)^2}$$

and

$$f''(x) = \frac{-4(2\sin^4 x - 5\sin^2 x + 1)}{(1+\sin^2 x)^3}.$$ [5]

Show that f has a maximum at $(0, 1)$ and find, correct to 2 decimal places, the x-coordinates of the two points of inflection. Sketch the graph of f. [5]

Write down an integral which gives the area enclosed between the graph of f, the x-axis between $(0, 0)$ and $(\tfrac{1}{2}\pi, 0)$, and the y-axis between $(0, 0)$ and $(0, 1)$. Use Simpson's Rule with 5 ordinates and an interval of $\tfrac{1}{8}\pi$ to show that its value is approximately 0·65. [5]

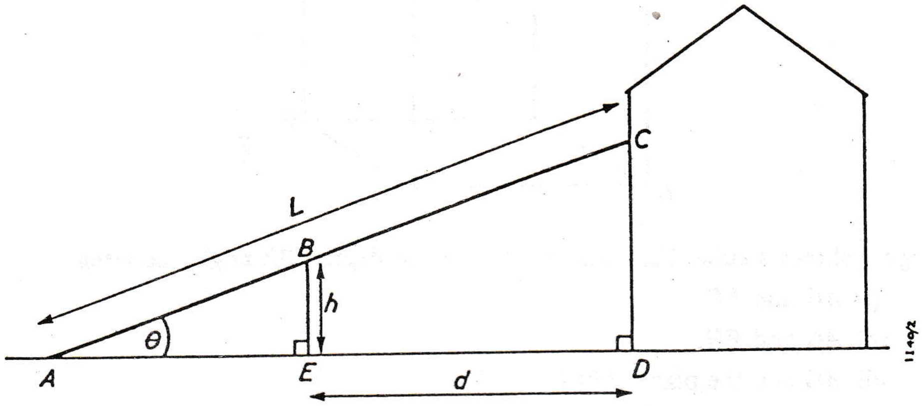
14. (*a*) The length a of the side BC of a triangle ABC is related to the lengths b, c of the sides CA, AB and the included angle $B\widehat{A}C$ (measured in radians) by the cosine formula

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

If A is increased by 1% and b, c held constant, show that a increases by $x\%$ where

$$x \approx \frac{Abc\sin A}{a^2}.$$ [6]

(*b*)



The above figure shows a ladder AC of length L leaning against the side of a house. The ladder, which is inclined at an angle θ to the horizontal AD, also rests on the top of a wall BE of height h situated at a distance d from the house. Obtain an expression for L in terms of h, d and θ and show that its minimum value as θ varies is

$$(h^{2/3} + d^{2/3})^{3/2}$$ [9]

15. (*a*) Show, using the substitution $y = x - 2$, or otherwise, that

$$\int_3^4 \left(\frac{x+2}{x-2}\right)^3 dx = 12\log_e 2 + 49.$$ [5]

(*b*) Evaluate

$$\int_0^1 \frac{(3x-1)\,dx}{(x+1)(x^2+1)}$$

giving your answer correct to 2 decimal places. [6]

(*c*) Sketch the curve having equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0).$$ [1]

Calculate the volume generated when the region enclosed by the curve and the x-axis between the points $(-a, 0)$ and $(a, 0)$ is rotated through 4 right angles about the x-axis. [3]

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MATHEMATICS A2

Mechanics

P.M. FRIDAY, 14 June 1985

(3 Hours)

Answer **all** *questions in Section A and* **four** *questions from Section B.*

A calculator may be used except where it is expressly forbidden.

$[g = 9{\cdot}8 \text{ ms}^{-2}.]$

SECTION A

Answer **all** *questions in this section.*

1. Find the particular solution of the differential equation

$$(y + 1)(x^2 + 1)\frac{dy}{dx} = 1 \qquad (y \neq -1)$$

for which $y = 0$ when $x = 1$, in the form $x = g(y)$. [5]

2. A body of mass 2 kg is raised by a vertical force such that after being raised through 10 m, the mass, which is initially at rest, has a speed of v ms^{-1}. If the work done by the force is 340 J, find v. [3]

3. A mass $2M$ resting on a rough horizontal table (coefficient of dynamic friction $\mu < \frac{1}{2}$) is connected by a light inextensible string passing over a smooth pulley at the edge of the table to another mass M hanging vertically. The system is released from rest. Find the acceleration of the system in terms of μ and g. [4]

4. A locomotive of mass M kg working at the rate of R kW ascends a straight track which rises at an angle α to the horizontal. The acceleration due to gravity is g ms^{-2}. When the speed is v ms^{-1}, the acceleration is a ms^{-2}. Find an expression for the frictional resistance at speed v, stating the units in which it is measured. [4]

5. A uniform ladder of length $2a$ and mass M rests against a smooth vertical wall with its lower end on rough horizontal ground. If the ladder makes an angle θ with the horizontal, show that the coefficient of friction μ satisfies $\mu > \frac{1}{2}\cot\theta$. [5]

6. A body moves in a straight line such that its distance x at time t from a fixed point O in the line is given by

$$x = 3\cos\pi t + \sin\pi t.$$

Show that the motion is simple harmonic. Find the period and the amplitude of the motion. [6]

7. A particle moves in a plane such that at time t its position vector relative to a set of Cartesian axes is

$$\mathbf{r} = (t - 2\sin t)\,\mathbf{i} + (1 - 2\cos t)\,\mathbf{j}$$

Find $\mathbf{v}$ and $\mathbf{a}$, its velocity and acceleration vectors, and the values of t for which the velocity and acceleration vectors are perpendicular. Find the values of the particle's speed at those times. Show that the acceleration never vanishes. [6]

8. A particle of mass M is attached to an end A of a light inextensible string of length a. The other end O of the string is fixed. Initially, with A hanging vertically below O, the mass is projected at right angles to OA with speed $\sqrt{3ag}$ so that it begins to move in a vertical circle of centre O. Find the tension in the string when it makes an angle θ with the downward vertical through O. Does the string ever become slack? Justify your answer. [7]

SECTION B

Answer **four** *questions from this section.*

9. A golf ball is driven from a point A with an initial speed of 42 ms^{-1} at an angle α to the horizontal. Neglecting air resistance, derive the horizontal component x and the vertical component y of the ball's displacement from A at time t. Show that

$$y = x \tan \alpha - \frac{x^2}{360 \cos^2 \alpha}$$

The golf ball just clears a tree at B, where B is on the same horizontal level as A and $AB = 150$ m. The tree is 5 m high. Show that one possible value of $\tan \alpha$ is $\frac{3}{4}$ and find the other possible value.

Show further that any tree at B of height less than $27\frac{1}{2}$ m could be cleared by the ball for a suitable choice of α, and give the value of $\tan \alpha$ for a tree of height $27\frac{1}{2}$ m. [8, 3, 4]

10. (*a*) A particle P of mass m moves in a plane such that its position vector at time t is given by

$$\mathbf{r} = (t^2 - t)\mathbf{i} + (t \sin \pi t)\mathbf{j}$$

Find

(i) the momentum of P when $t = 0$,

(ii) the kinetic energy of P when $t = 1$,

(iii) the force acting on P when $t = 1$,

(iv) the rate of working of the force acting on P when $t = 1$. [6]

(*b*) An ice-skater A moves on a flat ice-rink with constant velocity $10\mathbf{i} + 5\mathbf{j}$, passing through the point $9\mathbf{i} + 8\mathbf{j}$ at time $t = 0$. A second ice-skater B travelling with constant velocity $p\mathbf{i} + 6\mathbf{j}$ is at the point $3\mathbf{i} - \mathbf{j}$ at time $t = 0$.

(i) If the ice-skaters collide at some later time, find p.

(ii) If $p = 11$ show that the ice-skaters are closest together when $t = 7 \cdot 5$. [5, 4]

11. A light elastic string AB obeying Hooke's Law is of natural length 1 m and has the end A fixed. When a particle of mass 2 kg is attached to the string at B and is allowed to hang freely, the string is extended by a length of $0 \cdot 14$ m. Find the modulus of elasticity, stating the units in which it is measured. [3]

The particle is pulled down a further $0 \cdot 2$ m and is then released from rest at time $t = 0$. Show that until the string becomes slack, the motion of the particle is simple harmonic. Show also that the particle passes through the equilibrium position after approximately $0 \cdot 19$ seconds and find the speed of the particle at that time.

From energy considerations or otherwise, find how high above the point of release the particle reaches during the subsequent motion. [4, 2, 3, 3]

12. A body of mass 240 kg is dropped by parachute with negligible initial speed. Whilst the parachute is opening, the body is subject at time t to a resistance due to the atmosphere of $40v$N, where v is its speed at that time. Show that whilst the parachute is opening, the speed v satisfies

$$6\frac{dv}{dt} = 6g - v.$$

If the parachute is fully open after 6 seconds, prove that its speed will then be $6g(1 - e^{-1})$ ms^{-1}. [2, 5]

Find an expression for the distance fallen by the body at time t seconds ($t < 6$). Show that the body has fallen through a distance of $36ge^{-1}$ m whilst the parachute is opening. [4]

Find the distance fallen by the body when the speed is v ms^{-1} where $v < 6g(1 - e^{-1})$. [4]

13. Prove that the centre of mass of a uniform solid hemisphere of radius a is at a distance of $\frac{3a}{8}$ from its plane face. [5]

A toy consists of a uniform solid hemisphere of radius a surmounted by a uniform solid cone of the same material and base radius a, joined together at their rims. The height of the cone is h.

(i) Show that the centre of mass of the toy is at a depth of

$$\frac{3h^2 + 8ha + 3a^2}{4(h + 2a)}$$

below the vertex. [6]

(ii) The toy is placed with the hemisphere in contact with a smooth horizontal floor and the axis of the cone vertical. Show that the toy has a tendency to return to its equilibrium position when the axis is disturbed slightly from the vertical if

$$h < \sqrt{3}a$$ [4]

14. Two small smooth spheres A and B of equal mass moving in the same straight line (and in the same direction) with speeds $2u$ and u respectively, collide directly. After collision their directions of motion remain unchanged but their speeds are v and $1 \cdot 5v$ respectively. Show that the coefficient of restitution is $0 \cdot 6$. [4]

Sphere A is set to move on a smooth horizontal floor and collides directly with B which is at rest at a distance of 2 m from a smooth vertical wall, B being the nearer to the wall. The motions before and after all possible collisions are perpendicular to the wall, and the coefficient of restitution for all collisions is $0 \cdot 6$.

(i) Show that when B collides with the wall for the first time, A is at a distance of $1 \cdot 5$ m from the wall. [7]

(ii) Find the distance of the spheres from the wall when they collide for the second time. [4]

15. (*a*) A particle moves in a straight line such that at time t its displacement y in metres from a fixed point O satisfies the differential equation

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 12y = 12t + 20.$$

Find the general solution of the differential equation.
Given that when $t = 0$ the particle is at rest at O, find its displacement at time t. [4, 4]

(*b*) It is known from experiments that water escapes from a flat-bottomed tank through an orifice in its base at a rate of approximately $2 \cdot 66 A\sqrt{h}$ cubic metres per second, where A is the orifice area in square metres and h is the height in metres of the water surface above the orifice.

A tank 1 m deep has a rectangular cross-section of 2 m by 3 m. The tank is initially filled with water. At time $t = 0$, the water starts to escape through a circular orifice in its base, the orifice being of radius $0 \cdot 01$ m. Taking π to be $3 \cdot 14$ and calculating correct to 2 significant figures, show that at time t seconds the height of the water in the tank satisfies

$$\frac{dh}{dt} = -0 \cdot 00014\sqrt{h}$$

How long does the tank take to empty?

[3, 4]

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MATHEMATICS A3

Probability and Statistics

A.M. MONDAY, 17 June 1985

(3 Hours)

Answer **all** *questions in Section A and* **four** *from Section B.*

Mathematical and Statistical tables are provided.

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Three balls are to be drawn at random without replacement from a bag which contains four red balls, three white balls, and two blue balls. Calculate the probabilities that

(i) a red ball will not be drawn,

(ii) one ball of each colour will be drawn,

(iii) the three balls drawn will be of the same colour. [5]

2. The continuous random variable X has the probability density function f, where

$$f(x) = \tfrac{2}{3}x, \quad 1 < x < 2,$$
$$f(x) = 0, \quad \text{otherwise.}$$

A square is constructed having each of its sides of length X. Evaluate $E[X^2]$ and $E[X^4]$. Deduce the mean and the variance of the area of the square. [6]

3. The two discrete random variables X and Y have the joint probability distribution displayed in the following table.

		x		
		0	1	2
y	0	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{11}{30}$
	1	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{2}{15}$

(i) Determine whether or not X and Y are independent.

(ii) Show that $E[XY] = E[X]E[Y]$. [7]

4. The random variable X has the Poisson distribution with mean $2{\cdot}5$.

(i) Show that $5P(X = 2) = 6P(X = 3)$.

(ii) Using tables, or otherwise, find the value of $P(X < 3)$ correct to four decimal places.

(iii) The random variable Y is independent of X and has the Poisson distribution with mean 6. Find the mean and the standard deviation of $3Y - 2X$. [7]

5. A random sample of eight observations from a normal distribution gave the values

$$1\cdot6,\ 1\cdot3,\ 2\cdot1,\ 1\cdot8,\ 1\cdot6,\ 1\cdot1,\ 1\cdot9,\ 2\cdot2.$$

(i) Calculate unbiased estimates of the mean and the variance of the distribution.

(ii) Determine a 95% confidence interval for the mean of the distribution. [7]

6. The score X that may be obtained in one play of a certain game is a discrete random variable whose probability distribution is shown in the following table.

x	0	1	2	3
$P(X = x)$	0·4	0·2	0·2	0·2

(i) Find the mean and the variance of X.

(ii) Let $\bar{X}$ denote the mean of the scores in 136 independent plays of the game. Write down the mean value of $\bar{X}$ and show that its standard deviation is 0·1. Using a normal approximation to the sampling distribution of $\bar{X}$, find, correct to three decimal places, the probability that $\bar{X}$ will be less than unity. [8]

SECTION B

Answer **four** *questions from this section.*

7. The continuous random variable X has the probability density function f, where

$$\begin{aligned} f(x) &= \tfrac{1}{2}, & 0 < x < 1,\\ f(x) &= \tfrac{1}{4}(3 - x), & 1 < x < 3,\\ f(x) &= 0, & \text{otherwise.} \end{aligned}$$

(i) Find the mean of the distribution. [3]

(ii) Obtain expressions for $F(x)$, where F is the cumulative distribution function of X. [3]

(iii) Find the value of a given that $P(X < a) = \tfrac{7}{8}$. [4]

(iv) Determine the probability density function of $Y = 3 - X$. [5]

8. The amount of a certain chemical in a type A cell is normally distributed with mean 10 and standard deviation 1, while that in a type B cell is normally distributed with mean 14 and standard deviation 2. To determine whether a cell is of type A or of type B, the amount of the chemical in the cell is measured and the cell is classified as being of type A if the amount is less than a specified value c, and as being of type B otherwise.

(i) If $c = 12$ calculate, correct to three decimal places, the probability that a type A cell will be misclassified and the probability that a type B cell will be misclassified. [4]

(ii) Find the value of c for which the two probabilities of misclassification will be equal. [4]

(iii) Calculate, correct to three decimal places, the probability that the sum of the amounts of the chemical in one type A cell and two type B cells will be greater than 41. [3]

(iv) One cell is chosen at random from a collection of cells of which 70% are of type A and 30% are of type B. Calculate, correct to three decimal places, the probability that the amount of the chemical in the chosen cell will be at least 12. [4]

9. The two events A and B are independent and their probabilities of occurring are $P(A) = \dfrac{2}{3}$ and $P(B) = \dfrac{2}{5}$.

(i) Show that $P(A \cup B) = \dfrac{4}{5}$. [2]

(ii) Calculate the probability that only one of the two events will occur. [3]

(iii) Calculate the conditional probability that A occurred given that only one of the two events occurred. [3]

Another event C is such that A and C are mutually exclusive, $P(C) = \dfrac{1}{5}$, and $P(B' \cap C') = \dfrac{7}{15}$.

Calculate

(iv) $P(B \cap C)$, [3]

(v) $P(B|A \cup C)$. [4]

10. Each trial of a random experiment must result in one and only one of the events A, B and C occurring, the probabilities of these events being $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$, respectively. Independent trials of the experiment are to be conducted until one of the three events occurs for the second time.

(i) Show that the probability that the trials will stop with A occurring for the second time

(a) in the second trial is $\frac{1}{4}$,

(b) in the third trial is $\frac{1}{4}$,

(c) in the fourth trial is $\frac{3}{32}$. [3]

(ii) Determine the corresponding probabilities that the trials will stop with B occurring for the second time. [4]

(iii) Let X denote the number of trials that will be conducted. Find the distribution of X. Hence determine the most probable value of X and the expected value of X. [8]

11. An analysis of the membership of a large organisation shows that 60% of the members are over 50 years old and 2·5% are under 20 years old.

(i) Using appropriate distributional approximations *when necessary*, calculate the probabilities, correct to three decimal places, that a random sample of 100 members will include

(a) 60 or more members over 50 years old, [3]

(b) 3 or fewer members under 20 years old, [3]

(c) exactly 40 members aged from 20 years to 50 years, inclusive. [4]

(ii) Use a normal approximation to find the smallest number of members that should be sampled at random in order that the probability is at least 0·9 that 55% or more of the sampled members will be over 50 years old. [5]

12. The continuous random variable X has the probability density function f, where

$$f(x) = \tfrac{1}{2}(1 + \theta x), \quad -1 < x < 1,$$
$$f(x) = 0, \quad \text{otherwise},$$

and θ is a constant whose value lies somewhere in the interval from -1 to $+1$.

(i) Determine, in terms of θ, the probability that a randomly observed value of X will be positive. [2]

(ii) Let R denote the number of positive values in a random sample of n observations of X. Name the distribution of R. Show that $T_1 = \dfrac{4R}{n} - 2$ is an unbiased estimator of θ and that the variance of T_1 is $(4 - \theta^2)/n$. [6]

(iii) Let $\bar{X}$ denote the mean of a random sample of n observations of X. Show that $T_2 = 3\bar{X}$ is an unbiased estimator of θ and that its variance is $(3 - \theta^2)/n$. [6]

(iv) State, with your reason, which of T_1 and T_2 you would prefer for estimating θ. [1]

13. It is known that the two variables u and y are related in the form

$$y = \alpha + \frac{\beta}{u}$$

for values of u in the interval $0 < u \leq 1$. An experiment was conducted in which the value of y was observed for each of four specific values of u. The values of u and the corresponding observed values of y are given in the following table.

u	0·1	0·2	0·5	1·0
y	4·5	9·6	14·2	15·1

(i) By introducing a new variable instead of u, show that the least-squares estimate of β is equal to $-1·2$, and find the least-squares estimate of α. [5]

(ii) The errors in the observed values of y are independent and normally distributed with mean zero and standard deviation 0·7. Calculate 95% confidence limits for

(a) the true value of y when $u = 0·4$, [5]

(b) the true increase in the value of y when the value of u is increased from 0·5 to 1·0. [5]

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MATHEMATICS A4

Further Pure Mathematics

A.M. WEDNESDAY, 12 June 1985

(3 Hours)

Answer **all** *questions in Section A and* **four** *from Section B.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Answer **all** *questions in this section.*

1. Given that the equation

$$x^3 + bx^2 + cx + d = 0$$

has three roots in geometric progression, show that $b^3d = c^3$. [3]

2. The rth term of the series

$$3 + 7 + 13 + \ldots$$

may be written in the form $ar^2 + br + c$ where a, b and c are constants. Find a, b and c and hence find an expression for the sum of the first n terms of the series. [4]

3. Find the cosine of the acute angle between the planes with equations $x - y + 2z = 1$ and $x + y - 3z = 0$. Find equations of the line of intersection of these planes in the form

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

[4]

4. Let $I_n = \int_0^{\frac{1}{2}\pi} \cos^n \theta \, d\theta$. By writing $\cos^n \theta$ as $\cos^{n-1} \theta \cos \theta$, or otherwise, prove that

$$I_n = \frac{(n-1)}{n} I_{n-2} \quad (n > 1)$$

[5]

5. Find the four roots of the equation $z^4 + 1 = 0$, giving each root in the form $a + ib$. Hence express $z^4 + 1$ as the product of two quadratic factors with real coefficients. [5]

6. Let $f(x) = \int_0^x e^{\sin t} \, dt$.

Find the first three non-zero terms in the Maclaurin series expansion of $f(x)$. [6]

7. Find the values of the constants p and q in terms of a and b given that

$$\frac{d}{dx}\{p \cos ax \sinh bx + q \sin ax \cosh bx\} = \cos ax \cosh bx$$

Hence find $\int \cos 2x \cosh 3x \, dx$. [6]

8. The transformation T of the plane is equivalent to a rotation about the origin followed by a translation. The transformation T maps $(0, 1)$ to $(-1, 3)$ and $(3, 0)$ to $(2, 4)$. Find the angle of the rotation and describe the translation in terms of a vector. [7]

SECTION B

Answer **four** *questions from this section.*

9. The functions f_n are given by

$$f_n(x) = \frac{x}{1 + x^n} \qquad n = 1, 2, 3, \ldots$$

Determine and classify the turning points (if any) of f_1 and f_2. Sketch their graphs in separate diagrams. State the domain and range of each function. [8]

The functions g_n are given by

$$g_1 = f_1$$
$$g_{n+1} = g_n \circ f_1 \qquad n = 1, 2, 3, \ldots$$

Find $g_2(x)$. Suggest and prove a formula for $g_n(x)$. [4]

Find the function ϕ such that $g_2 \circ \phi = f_1$. [3]

10. Reduce the following system of equations to echelon form

$$\begin{aligned} x + 2y + z &= k \\ 2x - y &= 2 \\ x + 7y + 3z &= 1. \end{aligned}$$ [3]

(i) When $k = 1$, deduce that the solutions can be written in the form

$(x, y, z) = (0, -2, 5) + t(1, 2, -5)$

where t is any real number. [3]

(ii) When $k = 7$, show that the equations are inconsistent and deduce that they then represent three planes (p_1, p_2 and p_3 respectively) which intersect to form a triangular prism. [2]

Show that p_1 and p_2 intersect at right angles and state the equations of the line of intersection l of the planes p_2 and p_3. [4]

Prove that the perpendicular distance from l to p_1 is $\sqrt{6}$. [3]

11. The point P $(a \sec\theta, b \tan\theta)$ lies on the hyperbola h_1 given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Show that the normal to h_1 at P has the equation

$$ax \cos\theta + by \cot\theta = a^2 + b^2$$ [4]

This normal meets the x-axis and y-axis at S and T respectively. M is the midpoint of ST. Show that M has coordinates $(c \sec\theta, d \tan\theta)$ where

$$c = \frac{a^2 + b^2}{2a} \quad \text{and } d = \frac{a^2 + b^2}{2b}$$

[3]

The locus of M is a hyperbola h_2.

Sketch h_1 and h_2 on the same diagram, in the case when $a > b$. What happens when $a = b$? [4]

By applying the above procedure to h_2, a new hyperbola h_3 can be found. Show that h_3 intersects the x-axis at the points

$$\left(\pm \frac{(a^2 + b^2)^2}{4ab^2}, 0\right)$$ [4]

12. (*a*) Show that $\sinh^{-1} x = \log_e\{x + \sqrt{(x^2 + 1)}\}$ [3]

(*b*) Show that

$$\int \sqrt{(x^2 + 1)}\, dx = \tfrac{1}{2}x\sqrt{(x^2 + 1)} + \tfrac{1}{2}\log_e\{x + \sqrt{(x^2 + 1)}\} + \text{constant.}$$ [4]

(*c*) Sketch the finite region D of the plane bounded by the lines $x = 1$ and $x = -1$ and the hyperbola

$$y^2 - cx^2 = 1 \qquad (c > 0)$$

The area A of the curved surface formed when D is rotated through π radians about the x-axis is given by

$$4\pi \int_0^1 \sqrt{(x^2 + 1)}\, dx.$$

Find the value of the constant c.

Show that $A = 14{\cdot}42$ correct to two decimal places. [8]

13. (*a*) Define the derivative $f'(a)$ of the function f at $x = a$.
Use your definition to find $f'(a)$ when $f(x) = x^3 + x$. [4]

Deduce that f is strictly increasing and that the inverse function exists. Denote the inverse function by g. Find the value of $g(2)$ and the value of $g'(2)$. [5]

(*b*) Evaluate $\displaystyle\int_1^3 \sqrt{(6x - 5 - x^2)}\, dx$ [6]

14. The points P and Q in the Argand diagram represent the complex numbers z and w respectively, and

$$w = \alpha\bar{z}(1 + z) + \beta \qquad (\alpha, \beta \text{ real}; \alpha > 0)$$

where $\bar{z}$ is the complex conjugate of z.
P describes the circle given by $|z| = 1$. Show that the locus of Q is a circle whose centre represents $\alpha + \beta$ and whose radius is α. [6]

(i) Given that $\beta = 0$ and that one of the common tangents to the two circles touches the first circle at the point which represents $\frac{1}{5}(-3 + 4i)$, find the value of α. [5]

(ii) Given that $\alpha = 1$ and $\beta = -2$, show that the points of intersection of the two circles represent the two non-real cube roots of unity. [4]

15. Use the Intermediate Value theorem to show that the equation $x = 2 \sin x$ has a root α in the interval $I = \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$. [2]

For $f(x) = 2 \sin x$ show that $|f'(x)| < 1$ for all $x \in I$. Use the iterative formula $x_{n+1} = 2 \sin x_n$ to find α correct to one decimal place. [4]

The line $y = mx$ is a tangent to the curve with equation $y = \sin x$ at the point $(\beta, \sin \beta)$ where $\beta \in (2\pi, 3\pi)$. Draw a sketch and prove that $\beta = \tan \beta$. [2]

Show that when the Newton Raphson method is used to solve the equation

$$\sin x - x \cos x = 0$$

the iterative formula

$$x_{n+1} = x_n - \frac{1}{x_n} + \cot x_n$$ [2]

is obtained. Hence, or otherwise, find β correct to three decimal places and determine the range of positive values of m for which the equation $mx = \sin x$ has exactly three roots. [5]

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PURE MATHEMATICS S

A.M. MONDAY, 24 June 1985

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Pure Mathematics

No more than **four** *questions should be answered from this section.*

1. In the acute-angled triangle ABC, D is the foot of the perpendicular from B to AC. Show, in the usual notation, that

$$BD = c \sin A \quad \text{and} \quad CD = b - c \cos A$$

Hence derive the cosine rule in the form

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Deduce that

$$2bc(1 - \cos A) = (a + b - c)(a - b + c)$$

and obtain a similar expression for $2bc(1 + \cos A)$. [3]

Hence show that

(i) the area Δ of the triangle ABC is given by

$$\Delta = \frac{1}{4}\sqrt{p(p - 2a)(p - 2b)(p - 2c)}$$ [3]

(ii) $$\tan \tfrac{1}{2}A = \sqrt{\frac{(p - 2b)(p - 2c)}{p(p - 2a)}}$$ [3]

where $p = a + b + c$.

Using these results, show further that

(iii) $$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = 1 - \frac{2c}{p}$$

(iv) $$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C = \frac{4\Delta}{p^2}$$ [3]

The area of a certain triangle is 84, and the sum of the lengths of the three sides is 42. Given that one of the sides has length 14, find the lengths of the other two sides. [3]

2. (*a*) Show that the equation of the tangent to the curve $xy = c^2$ at the point $P\,(ct, c/t)$ is

$$t^2y + x = 2ct$$

The perpendicular from the origin to this tangent meets the curve at Q and R. The chords PQ and PR meet the x-axis at U and V. Prove that the mid-point of UV is the foot of the perpendicular from P to the x-axis. [8]

(*b*) An ellipse has equation $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b > 0)$$

The lines L_1, L_2 have equations $ex = a$ and $ex = -a$ respectively, where $e = \sqrt{a^2 - b^2}/a$, and the points S_1, S_2 have coordinates $(ae, 0)$ and $(-ae, 0)$ respectively. $P(a \cos\theta, b \sin\theta)$ is any point on the ellipse and the line through P parallel to the x-axis meets L_1 at Q and L_2 at R. Show that

(i) $$\frac{PS_1}{PQ} = \frac{PS_2}{PR} = e$$ (ii) $PS_1 + PS_2 = 2a$. [7]

3\. (*a*) The points P, Q, R have position vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ relative to an origin O. A further point S has position vector $\mathbf{s}$ such that

$$\mathbf{s} = \lambda\mathbf{p} + \mu\mathbf{q} + \nu\mathbf{r}$$

where $\lambda + \mu + \nu = 1$, and λ, μ, ν all lie between 0 and 1.

By writing $\mathbf{s}$ in the form

$$\mathbf{s} = \lambda\mathbf{p} + (\mu + \nu)\frac{(\mu\mathbf{q} + \nu\mathbf{r})}{(\mu + \nu)}$$

show that S lies inside the triangle PQR. [4]

(*b*) The vertices A, B, C of a triangle have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively relative to some origin. Write down the position vectors of D, E, the mid-points of BC, AC respectively, and show that the position vector of the point of intersection of AD and BE is given by

$$\tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b} + \tfrac{1}{3}\mathbf{c}$$

Deduce that the medians of a triangle are concurrent.

The point of intersection of the three medians of a triangle is called the centroid. Use a similar method to show that the four lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent. [11]

4\. (*a*) The function f is defined by

$$f(x) = \left(\frac{1}{x}\right)^{2x} \qquad (x > 0)$$

If $g(x) = \log_e f(x)$, show that $g(x)$ has a maximum value when $x = 1/e$. Deduce the maximum value of $f(x)$, giving your answer correct to 2 decimal places.

Sketch the graph of f, and determine the set of values of x for which $f(x) > 2$. [8]

(*b*) A cone is inscribed in a sphere of radius R. Prove that the volume of the cone has a maximum value of $\frac{32}{81}\pi R^3$. [7]

5\. A teacher asks the three pupils in his class to suggest a method for evaluating

$$I = \int_0^{\frac{1}{4}} \left(\frac{1 + x\sqrt{x}}{1 + \sqrt{x}}\right) dx$$

The first pupil suggests using the trapezium rule based on the ordinates at $x = 0$, $\frac{1}{8}$ and $\frac{1}{4}$.

The second suggests writing the integrand in the form

$$(1 + x\sqrt{x})(1 + \sqrt{x})^{-1},$$

using the binomial theorem to expand $(1 + \sqrt{x})^{-1}$ as far as the term involving $x\sqrt{x}$, multiplying out the factors and integrating term by term.

The third suggests using the substitution $x = (y - 1)^2$ and hence evaluating the integral exactly.

Carry out these suggestions, giving your answers correct to 3 decimal places. [3, 4, 4]

The teacher subsequently reveals that the denominator of the integrand is a factor of the numerator. Show that this fact enables the integral to be evaluated very much more quickly than any of the other methods. [4]

SECTION B

Further Pure Mathematics

No more than **four** *questions should be answered from this section.*

6\. (*a*) What general result is suggested by the number pattern

$$1 = 1$$
$$3 + 5 = 8$$
$$7 + 9 + 11 = 27$$
$$13 + 15 + 17 + 19 = 64$$

.... = ...?

Prove your conjecture. [9]

(*b*) A number is said to be rational if it can be expressed in the form $\frac{p}{q}$ where p and q are integers with $q \neq 0$.

a and b are fixed rational numbers ($b \neq 0$). Given that $\sqrt{2}$ is not rational, prove by contradiction that $a + b\sqrt{2}$ is not rational. [6]

7. Prove that the equation

$$(2 - z)^5 = z^5$$

has roots $1 - i \tan \frac{1}{5} p\pi$, $p = 0, 1, 2, 3, 4$. [5]

By considering the product of these roots show that

$$4 \cos \frac{\pi}{5} \cos \frac{2\pi}{5} = 1$$

Deduce that $8c^3 - 4c - 1 = 0$ where $c = \cos \frac{\pi}{5}$ and hence show that $c = \frac{1}{4}(1 + \sqrt{5})$. [3, 3]

Write $\frac{1}{2}[x^5 - (2 - x)^5]$ as a product of one linear and two quadratic factors, each with real coefficients. [4]

8. Show that the locus of the point $P(2 \cos \theta, \sin \theta)$ where $\theta \in [0, 2\pi)$ is an ellipse.
The distance between the point $Q(q, 0)$ and the point on the ellipse nearest Q is denoted by d.
Show that

$$d = \big||q| - 2\big| \text{ when } |q| \geqslant \tfrac{3}{2}.$$ [8]

Find an expression for d when $|q| < \frac{3}{2}$. Sketch the graph of d as a function of q. [7]

9. (*a*) Given that $\sinh y = \tan x$, where $x \in [0, \frac{1}{2}\pi]$, show that

(i) $\cos x \cosh y = 1$

(ii) $y = \log_e (\sec x + \tan x)$

(iii) $\frac{dy}{dx} = \cosh y = \sec x$

(iv) $\int_0^\infty \operatorname{sech}^4 y \, dy = \frac{2}{3}$. [8]

(*b*) Given the iterative formula

$$x_{n+1} = \tan^{-1}(\sinh x_n) + m\pi$$

show that, with a suitable choice of m, the sequence $x_1, x_2, x_3, \ldots\ldots$ converges to the smallest positive root of the equation $\sinh x = \tan x$.
Find this root correct to three decimal places. [7]

10. The functions f_n where $n = 1, 2, 3, \ldots\ldots$ are defined on the interval $[0, \frac{1}{2}\pi]$ by

$$f_n(x) = \frac{\sin nx}{\sin x}, \qquad x \in (0, \tfrac{1}{2}\pi]$$

$$= n, \qquad x = 0$$

(i) Sketch the graphs of f_1, f_2 and f_3. [3]

(ii) Show that $\lim_{x \to 0} f_n(x) = n$. [4]

(iii) Let $I_n = \int_0^{\frac{1}{2}\pi} f_n(x) dx$ and $J_n = \int_0^{\frac{1}{2}\pi} \{f_n(x)\}^2 dx$.

Prove that $I_{2n+1} = I_{2n-1}$ and that $J_n = J_{n-1} + \frac{1}{2}\pi$.
Write down the general formula for J_n. [8]

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APPLIED MATHEMATICS S

A.M. MONDAY, 24 June 1985

(3 Hours)

Answer **seven** *questions.*

No more than **four** *questions may be answered from either section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

Mechanics

No more than **four** *questions should be answered from this section.*

1. A ball of mass m is thrown from a point O on level ground at time $t = 0$ with velocity of magnitude V at an angle α to the horizontal. The resistance to the ball's motion at time t may be supposed to be acting in the horizontal direction, being of magnitude mku where u is the **horizontal** component of the ball's velocity at that time.

   P is the highest point that the ball reaches above the horizontal plane through O, and Q is the point where the ball strikes the plane. Given that the ball passes through P and Q at times T_1, T_2 respectively, U_1 and U_2 being the corresponding values of the horizontal components of velocity, show that

$$T_2 = 2T_1$$
$$U_2 = U_1 e^{-\beta/2} \text{ where } \beta = \frac{2kV\sin\alpha}{g}.$$

[10]

   Show also that the effect of the resistance is to decrease the horizontal range on the plane through O by

$$\frac{V\cos\alpha}{k}(\beta - 1 + e^{-\beta})$$

[5]

2. A mass of $5M$ kg rests on a smooth table and is attached by two light inextensible strings to two bodies A and B of masses $2M$ kg and M kg respectively, which hang over smooth pulleys at opposite edges of the table. The mass A, after falling a distance 1 m from rest, comes into contact with an inelastic plane. Show that B will continue to ascend through a distance of 0·75 m. [8]

Show that when A is jerked into motion again an amount $\frac{3Mg}{16}$ J of kinetic energy is lost, and give the impulsive tensions in the strings. [7]

3. A ring A of mass M_1 is threaded onto a light inextensible string of length $3a$. One end of the string is fastened to a fixed point O, the other end being tied to a ring B of mass M_2 (mass M_2 < mass M_1) which is threaded on a smooth vertical rod passing through O.

(i) A is fixed at the mid-point of the string and B is allowed to slide on the vertical rod. Show that if A moves in a horizontal circle its angular speed is greater than $\sqrt{\frac{2(M_1 + 2M_2)g}{3M_1 a}}$ radians per second.

Find the angular speed if the tension in the lower string is $M_1 g$. [9]

(ii) Suppose now that A is free to slide on the string but that B is fixed on the rod. Given that $OB = 2a$ and that A moves in a horizontal circle with BA horizontal and the string taut, show that $BA = \frac{5a}{6}$ and find the angular speed of A. [6]

4. (a) A body is moving in a straight line with simple harmonic motion. When its distances from the central point are 3 m and 4 m, its speed has the values 5 ms^{-1} and 3·75 ms^{-1} respectively. Find the amplitude and period of the motion. Show that the particle takes $2\pi/15$ seconds to move 2·5 m from the central point. At what time after passing through the central point is the speed of the body equal to half its maximum value? [8]

(b) A body of mass 2 kg lies on a horizontal platform.

(i) The platform describes a simple harmonic motion vertically of amplitude 0·3 m and period 2 s. Find the greatest and least forces on the platform due to the body in terms of g, the acceleration due to gravity. [3]

(ii) The platform describes a simple harmonic motion horizontally of period 10 s. Given that the greatest speed of the body is $0{\cdot}12\pi$ ms^{-1} and also that it does not slip on the platform, show that the coefficient of friction is greater than $0{\cdot}024\pi^2/g$. [4]

5. (a) A uniform triangular lamina of mass 1 kg has $BC = 6$ m and $CA = AB = 5$ m. The lamina is in equilibrium in a vertical plane with AB horizontal and C above AB. A force of magnitude 9·8 N acts vertically upwards at A and three forces P, Q, R act along AB, BC, and CA respectively. Find P, Q, R if the system is in equilibrium. (Take g to be 9·8 ms^{-2}). [8]

(b) A body of mass M kg is placed on a fixed plane inclined at an angle α to the horizontal. The plane is rough with coefficient of static friction μ. The body is subjected to a force of magnitude P newtons acting in the vertical plane containing the line of greatest slope, θ being the angle between the direction of the force and the line of greatest slope. The sense of P is to pull the body up the plane. Given that the body is on the point of slipping **up** the plane, find P. Find the value of θ so that P is a minimum and find this minimum value of P.

Assuming that $\mu < \tan \alpha$, write down, but do **not** solve, equations that would enable you to find P in terms of θ when the body is on the point of slipping **down** the plane. [5, 2]

SECTION B

Probability and Statistics

No more than **four** *questions should be answered from this section.*

6. In a sequence of trials of a certain random experiment, the probability that the first trial will yield a success is $\frac{1}{2}$. Independently for each subsequent trial the probability that it will yield a success is α if the preceding trial yielded a success and is β if the preceding trial yielded a failure.

   (i) Find in terms of α and β the probability that the second trial will yield a success. [2]

   (ii) Show that the probability that the third trial will yield a success is $\beta + \frac{1}{2}(\alpha^2 - \beta^2)$. [3]

   (iii) Let A denote the event that the first trial will yield a success and let B denote the event that the third trial will yield a success. Find an expression for $P(B|A) - P(B|A')$ and hence, or otherwise, show that A and B are independent only if $\alpha = \beta$. [5]

   (iv) Let X denote the number of successes that will be obtained in the first three trials. For the case when $\alpha = \beta = \frac{2}{3}$, calculate the mean and the standard deviation of X. [5]

7. A counter is used to record the combined number of emissions from two sources of radiation A and B. The numbers of emissions from A and B in a period of t minutes are independent and have Poisson distributions with means μt and λt, respectively. You may assume that the combined number of emissions from the two sources in a period of t minutes has a Poisson distribution with mean $(\mu + \lambda)t$.

   (i) In a period of two minutes the recorded number of emissions was 5.

   Assuming that the numbers of emissions in successive one minute periods are independent, show that the probability that there were exactly two emissions during the first minute is $\frac{5}{16}$. [5]

   (ii) In a period of t minutes the recorded number of emissions was n. Show that the number of emissions from source A in that period has a binomial distribution which depends upon n and $\frac{\mu}{\lambda}$, but does not depend upon t.

   For the case when $\mu = \lambda$ and $n = 100$, find an approximate value, to three decimal places, for the probability that there were exactly 50 emissions from each of the two sources. [10]

8. A commuter travels each workday morning from a suburban railway station to the main city centre station on either of two trains which depart from the suburban station at 08.05 and 08.20 respectively. The commuter's time of arrival at the suburban station on any morning is uniformly distributed over the interval from 08.00 to 08.20. The commuter takes the first train after arriving at the station. Let W minutes denote the length of time that the commuter will have to wait for a train on any morning.

   (i) Calculate $P(W \leqslant 4)$ and $P(W \leqslant 12)$. [4]

   (ii) Verify that for any w in the interval from 0 to 5,

   $$P(W \leqslant w) = \frac{1}{10}w .$$

   Find an expression for $P(W \leqslant w)$ for values of w in the interval from 5 to 15. Deduce the probability density function of W. Hence, or otherwise, show that the commuter's average waiting time for a train is $6\frac{1}{4}$ minutes. [7]

   (iii) The time in minutes taken by a train to reach the main city centre station is uniformly distributed over the interval from 30 to 34. Calculate the commuter's average arrival time at the city centre station. [4]

9. The life time of a new car battery is normally distributed with mean 20 months and standard deviation 4 months. A person buys a new car with the intention of keeping it for six years, during which time any battery that fails will be replaced by a new one. Let N denote the number of batteries that the person will have to buy during the six years (not counting the new battery that is in the car when bought).

(i) Show that, correct to three decimal places, $P(N > 3) = 0{\cdot}159$. [3]

(ii) Find an expression for $P(N > n)$ in terms of n and Φ, where Φ is the cumulative distribution function of the standard normal distribution. Hence, or otherwise, show that $P(N \leqslant 1)$ and $P(N \geqslant 6)$ are negligibly small. [5]

(iii) Evaluate $P(N = n)$ for $n = 2, 3, 4$ and 5. Deduce the most probable value of N and calculate the expected value of N correct to two decimal places. [7]

10. An investigator carried out an experiment with a variable x fixed at the values 1, 2, 3, 4 respectively, and measured the corresponding values of another variable y. The investigator then applied the method of least squares to the results and produced the equation $y = 13{\cdot}8 - 1{\cdot}6x$. Records of the measurements obtained in the experiment showed that the observed values of y when $x = 1$ and when $x = 2$ were $12{\cdot}4$ and $10{\cdot}6$ respectively. Unfortunately, the records of the observed values of y when $x = 3$ and when $x = 4$ have been lost.

(i) Find the observed values of y when $x = 3$ and when $x = 4$. [6]

It is known that $y = \alpha + \beta x$ and that the observed values of y are subject to independent random errors that are normally distributed with mean zero and standard deviation $0{\cdot}5$.

(ii) Calculate a 95% confidence interval for the value of β. [3]

(iii) It was suggested to the investigator that instead of using the method of least squares, β could be estimated by b_1, where $b_1 = \frac{1}{3}(y_4 - y_1)$, y_1 and y_4 being the observed values of y when $x = 1$ and when $x = 4$ respectively. Show that b_1 is an unbiased estimate of β and that its standard error is greater than that of the least squares estimator of β. [6]