WELSH JOINT EDUCATION COMMITTEE — CYD-BWYLLGOR ADDYSG CYMRU

General Certificate of Education — Tystysgrif Addysg Gyffredinol

Advanced Level — Safon Uwch

MATHEMATICS A 1

A.M. MONDAY, 13 June 1983

(3 Hours)

Answer seven questions.

1. (*a*) Find the sum of the series

$$1\times 2\tfrac{1}{2}+2\times 3\tfrac{1}{4}+3\times 4\tfrac{1}{6}+\ldots+n\left(n+1+\frac{1}{2n}\right).$$ [4]

(*b*) Expand $(1+a)^5$ giving every term in its simplest form. Use mathematical induction to show that n^5-n+5 is divisible by 5 for all positive integers n. [6]

(*c*) Find an expression for $\log_{25}200$ in terms of $\log_{10}2$. Hence evaluate $\log_{25}200$. [5]

2. (*a*) Assuming that

$$\frac{\cot A}{1+\cot A}\times\frac{1}{1+\tan(45^\circ-A)}$$

is defined and is not equal to zero, show that its value is $\tfrac{1}{2}$. [3]

(*b*) Find all solutions of

$$\sin\theta-\sin 2\theta+\sin 3\theta-\sin 4\theta=0$$

lying between 0° and 180° inclusive. [7]

(*c*) In a triangle ABC, $AB = 9$ cm, $BC = 10$ cm, and $AC = 7$ cm. A point O inside the triangle is such that $O\widehat{B}C = 18^\circ$ and $O\widehat{C}B = 40^\circ$. Calculate the length of OB and the size of $A\widehat{B}O$. [5]

3. (*a*) Given the matrix $A=\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix}$, find A^{-1}. Hence, or otherwise, solve the simultaneous equations

$$x+y+z=1$$
$$x-y+z=0$$
$$x+y-z=-1.$$ [5]

(*b*) $C=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ and $Z=\begin{pmatrix}4 & 3\\ 3 & 2\end{pmatrix}$.

(i) Find the matrix B which satisfies $BZ = ZC$.

(ii) If D and Y are two matrices such that $YD = CY$ (and Y^{-1} exists) show that $D^2 = I$. [6]

(*c*) Write down the matrix representation of the following transformations of the plane:

(i) an anticlockwise rotation about the origin through an angle of $\pi/4$,

(ii) the reflection in the line through the origin at an angle $\pi/4$ to the x-axis.

The transformations are applied successively in the order given above. Find the single matrix representation for the composite transformation. [4]

4. (*a*) Given

$$4x^3+3x^2y+y^3=8,$$
$$2x^3-2x^2y+xy^2=1$$

and $y = mx$, show that m satisfies the equation

$$m^3-8m^2+19m-12=0.$$

Hence of otherwise, find the real values of x and y, leaving your answers in surd form. [6]

(*b*) Given that $x = 2$ is an approximate root of the equation

$$x^3 - 2x^2 + x - 1 = 0$$

find the value of this root correct to two significant figures. [4]

(*c*) Given that α and β are the roots of

$$4x^2 + 9x + 1 = 0,$$

find the values of $\alpha + \beta$ and $\alpha\beta$. Show that

$$\left(\sqrt{\frac{\alpha}{\beta}} - \sqrt{\frac{\beta}{\alpha}}\right)^2 = \frac{65}{4}.$$ [5]

5. (*a*) A is the point (2, 1) and C is the point (5, 2). Show that the line bisecting AC at right angles is given by $y = -3x + 12$. This line meets the y-axis at B. Find the area of the triangle ABC. If D is the point such that $ABCD$ is a parallelogram, find the coordinates of D and the angles of the parallelogram. [3, 2, 2, 3]

(*b*) O is the origin and A and B are the points $(a, 0)$ and $(-a, 0)$ respectively. The point P moves such that $PA \, . \, PB = OP^2$. Show that the coordinates (x, y) of P satisfy the equation $x^2 - y^2 = a^2/2$. Show also that the slope of this curve at the point $(a, a/\sqrt{2})$ is $\sqrt{2}$. [5]

6. (*a*) Sketch the circle $x^2 + y^2 - 2x - 6y + 8 = 0$. Show that the line $y = x + 4$ is a tangent to the circle and find the equation of the other tangent to the circle from the point $(-4, 0)$. [3]

(*b*) Find the equation of the tangent to the curve $y^2 = 4ax$ at the point $P(at^2, 2at)$.

The line through the origin O parallel to this tangent meets the curve again at Q. Find the coordinates of R, the midpoint of OQ. Show that the line through P parallel to the x-axis passes through R. [3, 4]

7. (*a*) A function f is defined by

$$f(x) = \sin x \quad \left(-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}\right).$$

State briefly why f has an inverse function f^{-1}, giving the domain and range of f^{-1}. Find $f^{-1}(\frac{1}{2})$. Given further that $g(x) = \cos x$ for all x, find $(g \circ f^{-1})\left(\frac{\sqrt{3}}{2}\right)$ and $(f^{-1} \circ g)\left(\frac{\pi}{3}\right)$.

[5]

(*b*) Examine the function given by

$$h(x) = \frac{(x-1)^2}{(x+1)^3} \qquad (x \neq -1)$$

for maximum and minimum points and sketch its graph. [6, 4]

8. (*a*) Differentiate with respect to x

(i) $3^{\log_e x}$,

(ii) $\sin^{-1}\sqrt{(1-x^2)}$. [3, 4]

(*b*) A point P has coordinates given parametrically by $x = \cos^{1/3}\theta$, $y = 4\sin^{1/3}\theta$ $(0 \leqslant \theta \leqslant \pi/2)$. Show that the maximum value of OP^2, where O is the origin, occurs when $\tan\theta = 8$. Find this maximum value. [8]

9. (*a*) Use integration by parts to find

$$\int (x+1)e^{2x}\,dx.$$ [3]

(*b*) Use the substitution $\tan\frac{1}{2}x = t$ or otherwise to find

$$\int \frac{dx}{3 + \cos x}.$$ [4]

(*c*) Express $1 - x^4$ as a product of linear and quadratic factors. Hence or otherwise find

$$\int \frac{x+2}{1-x^4}\,dx.$$ [8]

10. (*a*) Find a if

$$\int_{2a}^{3a} (x-a)^2\,dx = 63.$$ [3]

(*b*) Using the substitution $x = 2\sin\theta$, or otherwise, evaluate

$$\int_{-1}^{2} \sqrt{(4-x^2)}\,dx.$$ [7]

(*c*) The area enclosed by the curve $y = \cos x + \sin x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$, is rotated through four right angles about the x-axis. Find the volume generated. [5]

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WELSH JOINT EDUCATION COMMITTEE — CYD-BWYLLGOR ADDYSG CYMRU

General Certificate of Education — Tystysgrif Addysg Gyffredinol

Advanced Level — Safon Uwch

MATHEMATICS A 2

A.M. TUESDAY, 21 June 1983

(3 Hours)

Answer seven questions.

$[g = 9{\cdot}8 \text{ m s}^{-2}.]$

1. (*a*) Show that any complex number $z = x+iy$ can be expressed in polar form $z = r(\cos\theta + i\sin\theta)$. Hence prove that, for any two complex numbers z_1, z_2,

$$|z_1 z_2| = |z_1| . |z_2|.$$

Verify that $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ when $z_1 = -\sqrt{3}+i$ and $z_2 = 1+i\sqrt{3}$. [1, 4, 3]

(*b*) Find the cartesian equation for the locus of points satisfying

$$\text{Im}(z^2) = -2.$$ [2]

(*c*) Sketch the region in the Argand plane enclosed by parts of the following four loci:

$$|z| = 4, \quad |z+1| = 2, \quad \arg z = \pi, \quad \arg(z-i) = 0,$$

and whose points have a positive imaginary part. [5]

2. (*a*) Find the particular solution of the differential equation

$$\frac{dy}{dx} = (y^2+1)(x^4+1)$$

passing through the point $x = 5$, $y = 1$. [5]

(*b*) Translate into words the law of cooling expressed by the differential equation

$$\frac{dT}{dt} = -\lambda(T-A)$$

where t is time, T is the temperature of a hot body, A is the temperature of its surroundings and λ is a positive constant.

If A is constant, show that

$$T_0 - T = (T_0 - A)(1 - e^{-\lambda t})$$

where T_0 is the value of T when $t = 0$. [1, 6]

If the body, starting at time $t = 0$, cools from 80 °C to 60 °C in 15 minutes and from 60 °C to 50 °C in a further 15 minutes, find λ. [3]

3. (*a*) A stationary particle of mass $9m$ disintegrates into three particles without loss of mass. After the disintegration the resulting particles move freely with constant velocities in the Oxy plane. One second after the disintegration the particles are observed to have position vectors $3\mathbf{i}+4\mathbf{j}$, $4\mathbf{j}$ and $\mathbf{i}+\mathbf{j}$ and one second later the corresponding position vectors are $4\mathbf{i}+5\mathbf{j}$, $-2\mathbf{i}+5\mathbf{j}$ and $-\mathbf{j}$. Find the position vector of the original particle of mass $9m$ and the masses of the three particles resulting from the disintegration. Show that $15m$ units of kinetic energy are released. [5, 5, 1]

(*b*) A satellite S is orbiting a planet whose centre C may be assumed to be fixed in space. If the force attracting the satellite to the planet is inversely proportional to the square of the distance CS and the orbit is a circle, show that the periodic time of the satellite's motion is proportional to $(CS)^{3/2}$. [4]

Turn over.

4. A ball is projected with speed u at an angle α to the horizontal from a point A on horizontal ground. Obtain expressions for x and y, its horizontal and vertical displacements from A, as functions of time. Find a relation between x and y and deduce that if the ball passes through a point with coordinates (x_0, y_0) the two possible angles of projection are given by

$$\tan\alpha = \frac{R}{x_0} \pm \frac{1}{x_0}\sqrt{\{(R-y_0)^2-(x_0^2+y_0^2)\}}$$

where R is the maximum horizontal range of a ball projected with speed u. What is the condition on R for the ball to reach (x_0, y_0)? [10, 3]

When kicking a ball at rest on level ground, the greatest horizontal distance a player can kick the ball before it first bounces, is 55 m. Determine whether or not the player could kick the ball over a post 3 m high when the distance from the ball to the foot of the post is 50 m. [2]

5. One end A of a light elastic string of natural length a, is fixed. To the other end B is attached a particle of mass m which hangs freely in equilibrium at a depth $\frac{6a}{5}$ below A. The particle is pulled vertically downwards through a distance $a/5$, held at rest and then released. Write down Newton's law of motion for the particle when it is at a depth $\frac{6a}{5}+x$ below A during the subsequent motion. Find x as a function of time. What is the periodic time of the motion? [6, 7, 2]

6. An electric train of mass M kg moves from rest along a straight level track. The tractive force of the motors, initially P N, decreases uniformly with time to R N over a period of T s and then remains constant at R N. The total resistance to motion is R N. Show that the acceleration a of the train at time t seconds after it starts to move is given, for $t \leqslant T$, by

$$Ma = P+(R-P)t/T-R. \qquad [2]$$

Find the maximum speed achieved by the train, the distance it travels before reaching that speed and show that its average speed over that distance is $\frac{3}{4}$ of the maximum speed. [9]

Find the power developed by the motors (i) at time $T/2$, (ii) at time $3T/2$. [4]

7. A particle of mass m is connected by a light elastic string of natural length l, modulus of elasticity kmg, to a point A on a rough plane inclined at an angle α to the horizontal. The particle is free to move in contact with the plane along the line of greatest slope through A and the coefficient of friction between the particle and the plane is $\tan\lambda$, $\lambda < \alpha < \frac{1}{4}\pi$. Find the extension e_1 of the string when the particle is on the point of moving *up* the plane, and the extension e_2 when the particle is on the point of moving *down* the plane. Show that

$$e_1 : e_2 = \sin(\alpha+\lambda) : \sin(\alpha-\lambda). \qquad [8]$$

The particle is released from rest at the point below A, on the line of greatest slope through A, where the extension in the string is $x > e_1$. If the particle first comes to rest below A with the string still extended, use energy considerations to show that the particle will have travelled a distance $2(x-e_1)$ along the plane. [7]

8. A particle of mass m is projected vertically upwards in a medium in which at any instant the resistance to its motion is mk times the square of its speed at that instant. Show that the equation of motion is

$$\ddot{x} = -g-kv^2$$

where x is the height and v the speed of the particle at any instant.

Deduce that the greatest height H achieved by the particle is related to its initial speed u by

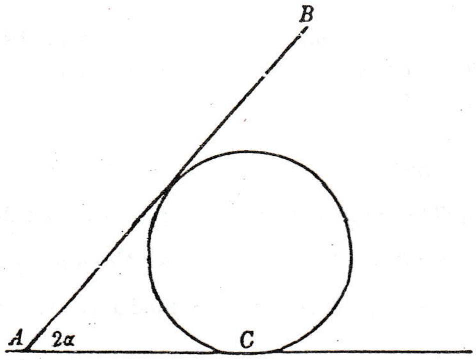
$$u^2 = \frac{g}{k}(e^{2kH}-1). \qquad [8]$$

Find the corresponding relation between H and the speed U of the particle when it returns to its starting point. Hence show that

$$U = ue^{-kH}. \qquad [5, 2]$$

9. A circular cylinder of radius a is fixed with its curved surface in contact with a horizontal plane. A uniform rod AB, of length $4a$, and weight W is laid across the cylinder in a plane perpendicular to the axis of the cylinder and has the end A on the horizontal plane. Contact between the rod and the cylinder is smooth and that between the rod and the plane is rough. The angle AB makes with the horizontal is 2α. If the rod is on the point of sliding down the cylinder, show that the coefficient of friction μ between the rod and the plane is given by

$$\mu = \frac{\sin 4\alpha}{\cot \alpha - \cos 4\alpha - 1}.$$ [11]



C is the point of contact between the cylinder and the plane that is nearest to A. A force of magnitude H applied to the end A of the rod in the direction AC causes the rod to be on the point of sliding up the cylinder. Show that

(i) F, the magnitude of the force of friction, is the same whether the force H is acting or not,

and

(ii) $H = 2F$.

Hence evaluate H. [4]

10. (*a*) Points A, B, C and D have position vectors $\mathbf{i}-\mathbf{j}$, $2\mathbf{i}+2\mathbf{j}$, $5\mathbf{i}+\mathbf{j}$ and $5\mathbf{i}-4\mathbf{j}$ respectively. Find, *by vector methods*, the position vector of P, the point of intersection of the lines AC and BD. [6]

(*b*) Prove that the centre of mass of a sector of a uniform circle of radius a, vertex angle 2α, is at a distance $\frac{2}{3}a\frac{\sin\alpha}{\alpha}$ from the centre of the circle along the axis of symmetry. [4]

The sector has its vertex at the origin of a rectangular coordinate system and its axis of symmetry lies along the positive x-axis. A circle of radius 1, centre (6, 1) is removed from the sector. If $a = 9$ and $\alpha = \frac{1}{6}\pi$, show that the centre of mass of the remainder will be at the point

$$\left(\frac{486}{25\pi} - \frac{12}{25}, -\frac{2}{25}\right).$$ [5]

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General Certificate of Education — Tystysgrif Addysg Gyffredinol

Advanced Level — Safon Uwch

MATHEMATICS A 3

A.M. TUESDAY, 21 June 1983

(3 Hours)

Answer seven questions.

Mathematical and Statistical tables are provided.

A calculator may be used except where it is expressly forbidden.

1. (*a*) In each of the following two games show that the probability that the player concerned will win is $\frac{1}{2}$.

Game 1: The player throws two fair dice together and wins if he throws at least one 5 or if the sum of the two scores is equal to 6 or equal to 7.

Game 2: The player tosses a fair coin four times in succession and wins if he tosses at least two heads successively. [7]

(*b*) The three events A, B and C are such that A and B are mutually exclusive, A and C are independent, and

$$P(A) = 0{\cdot}2, \quad P(B) = 0{\cdot}1, \quad P(A \cup C) = 0{\cdot}5, \quad P(B \cup C) = 0{\cdot}4.$$

(i) Evaluate $P(C)$.

(ii) Determine whether B and C are independent, mutually exclusive, or neither. [8]

2. Each of three boxes A, B and C contains four balls. Each of the four balls in A is red. Two of the four balls in B are red and the remaining two balls are white. Three of the four balls in C are white and the remaining ball is yellow. Two fair coins are tossed together. If two heads are tossed, one ball is drawn at random from box A; if two tails are tossed, one ball is drawn at random from box B; otherwise, one ball is drawn at random from box C.

(i) Show that the probability of the drawn ball being white is four times that of it being yellow.

(ii) Given that the drawn ball is red, find the conditional probability that it came from box B. [8]

The box from which the ball was drawn is then set aside, and a ball is drawn at random from one of the other two boxes, the choice of box being determined by the outcome of one toss of a fair coin.

(iii) Calculate the probability that the two balls drawn are of the same colour. [7]

3. Alec and Betty compete as a pair in a quiz. Alec's probability of answering a question correctly is 0·6, and independently, Betty's probability is 0·8.

(*a*) In the first round of the quiz each of Alec and Betty has to answer questions as individuals without collaboration.

(i) Calculate the probability that Betty will correctly answer exactly three of her first five questions.

(ii) Use tables to find the probability that Alec will correctly answer at least fifteen of his first twenty questions. [4]

(*b*) In the second round of the quiz either Alec or Betty may answer any question and they may collaborate. Use appropriate distributional approximations and tables to find probabilities, to three decimal places, that of the 60 questions put to them

(i) at least 35 of the questions have answers which will be known by both Alec and Betty.

(ii) Alec and Betty between them will know the correct answers to at least 50 of the questions. [11]

Turn over.

4. (*a*) Two independent random variables, X and Y, have Poisson distributions with means μ and 2μ, respectively. Show that for every non-negative integer r,

$$P(X = r, Y = r+2) = 4P(X = r+2, Y = r).$$ [4]

(*b*) During its first year in operation the number of occasions that a television set will need to be serviced has a Poisson distribution with mean 2·4. Independently of what happened in the first year, the number of occasions the set will need to be serviced during its second year in operation has a Poisson distribution with mean 4·6. During its third year in operation the number of occasions the set will need to be serviced is exactly twice the number of occasions it was serviced in its second year.

(i) Without assuming any result regarding Poisson distributions that is not given in the information booklet, find the probability, to three decimal places, that the set will need to be serviced a total of 2 or more occasions during its first two years in operation. [3]

(ii) Find, correct to three decimal places, the probability that the set will need to be serviced a total of 4 or more occasions during its second and third years in operation. [2]

Determine the mean and the variance of the total number of occasions that the set will need to be serviced

(iii) during its first two years in operation, [2]

(iv) during its first three years in operation. [4]

5. When a market gardener takes n cuttings from a shrub and plants them, the number X that will root successfully is a discrete random variable with

$$P(X = r) = \frac{2r}{n(n+1)}, \qquad r = 1, 2, \ldots, n.$$

The total cost in pence to the gardener of taking and planting n cuttings is equal to $20 + 0{\cdot}8n^2$. Cuttings that root successfully are sold by the gardener for 60 pence each.

(*a*) Show that $E(X) = (2n+1)/3$. [4]

(*b*) For $n = 20$, calculate

(i) the probability that the gardener will make a loss,

(ii) the gardener's expected profit. [6]

(*c*) Determine the value of n which will maximise the gardener's expected profit and evaluate this maximum expected profit. [5]

6. The continuous random variable X has the probability density function

$$f(x) = \tfrac{3}{4}x^2, \qquad 0 < x \leqslant 1,$$
$$f(x) = \tfrac{3}{8}(3-x), \qquad 1 < x \leqslant 3,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

(i) Calculate the mean value of X. [3]

(ii) Find expressions for the cumulative distribution function of X. [4]

(iii) Evaluate the conditional probability $P(X > 2 \mid X > 1)$. [3]

(iv) The discrete random variable Y is defined to be such that it takes the value 1 when $X \leqslant 1$, the value 2 when $1 < X \leqslant 2$, and the value 3 when $X > 2$. Determine the probability distribution of Y and evaluate its mean. [5]

7. The weights of the contents of cans of fruit are normally distributed with mean 250·2 g and standard deviation 2 g.

(i) Calculate the proportion of the cans that contain less than 250 g. [2]

(ii) Given that 75% of the cans contain at least w g, find the value of w correct to one decimal place. [4]

(iii) Find the probability that the combined contents of four cans will weigh more than 1 kg. [4]

(iv) The weights of the filled cans are distributed with mean 274·5 g and standard deviation 2·5 g. Assuming that the weight of the contents of a can is independent of the weight of the can when empty, determine the mean and the standard deviation of the weights of the empty cans. [5]

8. A cubical die has two each of its faces numbered 1, 2, and 3, respectively, and is such that the probabilities of obtaining these scores in a single throw are 0·1, 0·8, and 0·1, respectively.

(i) If X is the score obtained in one throw of the die, determine the mean and the variance of X. [2]

(ii) Let M denote the median of the three scores obtained in three independent throws of the die. Show that $P(M = 1) = 0{\cdot}028$. Evaluate $P(M = 2)$ and $P(M = 3)$, and hence determine the mean and the variance of the sampling distribution of M. [9]

(iii) Let $\bar{X}$ denote the mean of the three scores obtained in three independent throws of the die. Write down the values of the mean and the variance of $\bar{X}$, and verify that the variance of M is 84% of the variance of $\bar{X}$. [4]

9. (*a*) The drained weights, in grammes, of a random sample of 8 cans of fruit of a particular brand were found to be

342, 344, 340, 339, 341, 338, 341, and 343,

respectively. Given that the drained weights are normally distributed, calculate a 90% confidence interval for the mean drained weight of fruit per can of this particular brand. State, with your reason, whether or not your result is consistent with the claim on the can that the average weight of fruit is 340 g. [6]

(*b*) When an object is weighed on a certain weighing scale its recorded weight, in grammes, is a random value from a normal distribution whose mean is the true weight of the object and whose standard deviation is 0·2 g.

(i) Given that the mean of nine independently recorded weights of a particular object was 7·1 g, calculate a 99% confidence interval for the true weight of the object. [3]

A second object was weighed sixteen times on the same scale and the mean of the recorded weights was found to be 8·4 g.

(ii) Determine a 95% confidence interval for the difference between the true weights of the two objects. [6]

10. A chemist set up an experiment to determine how a variable y varied with an associated variable x. In the experiment, x was set at the five values 0, 1, 2, 3, 4, respectively, and the corresponding values of y were observed. The chemist noted that the values of y increased fairly steadily with the increasing values of x, and, on applying the method of least squares to the results, the chemist produced the equation

$$y = 5{\cdot}8 + 2{\cdot}3x.$$

(i) Find the value of $\bar{y}$, the mean of the five observed values of y. [3]

Suppose that the experimentally observed values of y are subject to independent random errors that are normally distributed with mean zero and standard deviation 1·1. Assuming that the true relationship connecting y and x is linear, calculate

(ii) a 95% confidence interval for the true value of y when $x = 4$, [6]

(iii) a 90% confidence interval for the difference between the true values of y corresponding to $x = 1$ and $x = 4$, respectively. [6]

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WELSH JOINT EDUCATION COMMITTEE — CYD-BWYLLGOR ADDYSG CYMRU

General Certificate of Education — Tystysgrif Addysg Gyffredinol

Advanced Level — Safon Uwch

Special Paper — Papur Arbennig

MATHEMATICS S

A.M. MONDAY, 27 June 1983

(3 Hours)

Answer **seven** *questions.*

Not more than **four** *questions may be attempted from any section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

1. (*a*) Prove by mathematical induction that

$$1.2+2.2^2+3.2^3+\ldots+n.2^n = (n-1)2^{n+1}+2$$

for all positive integers n. [4]

(*b*) If $\log_b a = 4$, $\log_c b = 3$ and $a = 32c^2$, find a, b and c. [5]

(*c*) By first writing $u = \left(\dfrac{x+y}{x-y}\right)^{1/3}$ in the first equation, or otherwise, solve the simultaneous equations

$$(x+y)^{2/3}+2(x-y)^{2/3} = 3(x^2-y^2)^{1/3}$$
$$3x-2y = 13.$$ [6]

2. (*a*) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Starting with the equation $\sin 2\theta = \cos 3\theta$, show that

$$\sin 18° = \frac{\sqrt{5}-1}{4}.$$ [4]

(*b*) Express $5\sin^2 x - 3\sin x\cos x + \cos^2 x$ in the form $a + b\cos(2x-\alpha)$ where a, b, α are independent of x, and α is acute. Hence, or otherwise, find all solutions of

$$5\sin^2 x - 3\sin x\cos x + \cos^2 x = 2$$

in the range $0° \leqslant x \leqslant 360°$. [7]

(*c*) Given $16x^2+8xy+9y^2 = 64y$, find the range of values of x for which y can assume real values. [4]

Turn over.

3. (*a*) The matrices A, B, C are such that $ABC = I$, where I is the identity matrix. Find an expression for C, assuming that A^{-1} and B^{-1} exist.

Write down:

(i) the 3×3 matrix A representing a translation of the plane in which the origin moves from $(0, 0)$ to $(1, 2)$;

(ii) the 3×3 matrix B representing an anticlockwise rotation of the plane about the origin through an angle α where $\tan \alpha = \frac{3}{4}$.

The transformation represented by the matrix B is applied followed by the transformation represented by the matrix A. As a consequence of the transformations a point P moves to the point $P'(-1, 1)$. Find the coordinates of the point P. [8]

(*b*) The 2nd, 3rd, 4th terms in the expansion of $(a+b)^n$ are 240, 720, 1080 respectively. Find a, b and n. [7]

4. Prove that the circles

$$x^2+y^2+2x-8y+8 = 0, \qquad x^2+y^2+10x-2y+22 = 0$$

touch. [4]

Find the coordinates of P, the point of contact of these circles, and the equation of their common tangent at this point. The common tangent meets the x-axis at Q. Find the angle C_1QC_2, where C_1 and C_2 are the centres of the two circles. [4, 3, 4]

5. (*a*) Differentiate $\log_e(1+\sin 2x)+2\log_e \sec(\frac{1}{4}\pi - x)$ with respect to x, reducing the answer to its simplest form. [6]

(*b*) Show that

$$\frac{d}{dx}[e^{-x}(A\sin x + B\cos x)] = e^{-x}[(A-B)\cos x - (A+B)\sin x]. \qquad [2]$$

Hence or otherwise

(i) examine $e^{-x}[(\sqrt{3}+1)\sin x + (\sqrt{3}-1)\cos x]$ for local maximum and minimum values, [4]

(ii) find $\int_0^{\pi/2} e^{(\frac{1}{2}\pi - x)}[(\sqrt{5}+5)\sin x + (\sqrt{5}-5)\cos x]\,dx$. [3]

6. (*a*) Find the constants A and B such that

$$\cos\theta \equiv A(\sin\theta+\cos\theta)+B(\cos\theta-\sin\theta).$$

Hence, or otherwise, evaluate

$$\int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta.$$

Deduce the value of $\displaystyle\int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta}\,d\theta.$ [1, 3, 1]

(*b*) Using integration by parts, or otherwise, find

$$\int x^3 \tan^{-1} x\,dx. \qquad [4]$$

(*c*) Sketch the curves $y^2 = 4(x+1)$ and $y^2 = 8(2-x)$ and show that the area included between them is $8\sqrt{2}$. [6]

SECTION B

7. (*a*) For any two complex numbers w, z, prove that

$$(w\bar{z}+\bar{w}z)^2+(iw\bar{z}-i\bar{w}z)^2 = 4w\bar{w}z\bar{z}.$$

Show also that $w\bar{z}+\bar{w}z$ and $iw\bar{z}-i\bar{w}z$ are both real.

Deduce that

$$|w\bar{z}+\bar{w}z| \leqslant 2|w||z|$$

and, by considering $|w+z|^2$, that

$$|w+z| \leqslant |w|+|z|.$$

Show that

$$\left|\frac{1}{z^2+2z}\right| \geqslant \tfrac{1}{3}$$

for all z such that $|z| = 1$. [3, 4, 2]

(*b*) Vectors $\mathbf{p}$ and $\mathbf{q}$ are such that

$$|\mathbf{p}+\mathbf{q}| = |\mathbf{p}-3\mathbf{q}| = |2\mathbf{p}-\mathbf{q}|.$$

Find the ratio of the length of $\mathbf{p}$ to the length of $\mathbf{q}$ and the angle between $\mathbf{p}$ and $\mathbf{q}$. [6]

8. Ship B is required to intercept ship A which left port O steering a straight course at 12 km h^{-1} in an unknown direction. B leaves O six hours after A and sails due East for nine hours at 20 km h^{-1} to a location C. Subsequently, still sailing at 20 km h^{-1}, it follows a course such that, relative to O as origin, its radial component of velocity is 12 km h^{-1} and its transverse component is counterclockwise. Explain why B is bound to intercept A. [2]

If ship B is at P at time t after leaving C, and $OP = r$ and angle $POC = \theta$, show that $\dot{r} = 12$, $r\dot{\theta} = 16$, and deduce that, relative to O, B's path from C is given by

$$r = 180e^{3\theta/4}.$$ [7]

For this path, find the differential equation satisfied by θ with time as independent variable. Show that the time required to intercept A is less than $15e^{3\pi/2}-6$ hours. [6]

9. A particle, suspended from a fixed point by a light elastic string of natural length l, makes vertical oscillations of amplitude a ($< l$). The modulus of elasticity of the string is equal to the weight of the particle. As the particle rises through its equilibrium position, a second stationary particle, of the same mass as the first, is attached to the first and the combined masses continue to oscillate. Show that the amplitude of that oscillation is $(l^2+a^2/2)^{1/2}$. [8]

Find the time from when the masses combine until when they are first instantaneously at rest. [7]

10. A uniform chain of total length l and mass m per unit length lies partly in a straight line along a rough horizontal table, perpendicular to the edge, with the remainder hanging over the edge. Initially, a length x_0 is hanging over the edge and this is just sufficient to cause the chain to begin to slip. If the coefficient of friction between the chain and the table is μ, show that when a length x ($> x_0$) is overhanging, the speed v of the chain is given by

$$v^2 = (x-x_0)[(1+\mu)g(x+x_0)/l-2\mu g].$$ [12]

Find the tension in the chain at that point of its length which is at the edge of the table. [3]

11. Inside a fixed hollow circular cylinder of radius b, whose axis is horizontal, are placed symmetrically and longitudinally two equal uniform smooth circular cylinders of radius a. A third uniform smooth circular cylinder, of the same length and radius as the other two but of twice their individual weights, is placed symmetrically on the first two so that all the axes are parallel and the centres of mass are in the same vertical plane. Prove that the third cylinder will force the other two apart if $b > a(1+\sqrt{13})$. [15]

SECTION C

12. (*a*) An electronic device, which consists of 10 components, will function only if at least 7 of the components are functioning. Each component, independently of the others, has probability 0·9 of functioning throughout a period of 12 hours.

(i) Find, to three decimal places, the probability that the device will function throughout a period of 12 hours.

A component which has functioned throughout a period of 12 hours has probability 0·5 of functioning throughout a second period of 12 hours. Find, to three decimal places, the probabilities that

(ii) a component will function throughout a period of 24 hours;

(iii) the device will function throughout a period of 24 hours. [6]

(*b*) When firing at a target, an archer's aim tends to improve with practice in such a way that, independently of the outcomes of all preceding attempts, his probability of scoring a 'bull' on his rth attempt is equal to $1-(\alpha/r)$, where α is a constant between 0 and 1.

(i) Show that the archer's probability of scoring exactly one 'bull' in his first three attempts is equal to $\frac{1}{2}\alpha^2(2-\alpha)$.

(ii) Find an expression for the probability that the archer's nth attempt will be his first to hit the 'bull'. Hence, or otherwise, show that the expected number of attempts for his first 'bull' is equal to e^{α}. [9]

13. At the end of its flowering season a hardy annual plant sheds its seeds and dies. Given the right conditions, each seed shed will independently develop into a flowering plant the following season, shed its seeds and then die. Independently for each plant, the probabilities that 0, 1, or 2 of the seeds it sheds will develop into flowering plants the following season are 0·2, 0·5, and 0·3, respectively. Starting with just one flowering plant, let X_1 denote the number of flowering plants one year later, X_2 the number two years later, and X_3 the number three years later. Find

(i) the probability distribution of X_2; [7]

(ii) $P(X_1 = 1 | X_2 = 1)$; [3]

(iii) $P(X_3 = 0)$, correct to three decimal places. [5]

14. A dealer has purchased two large batches of electric light bulbs, all of which are identical in appearance. The dealer knows that the bulbs in one of the two batches have lifetimes which are normally distributed with mean 1400 hours and standard deviation 60 hours, and that those in the other batch have lifetimes which are normally distributed with mean 1500 hours and standard deviation 80 hours.

(i) One of the two batches is chosen at random and a bulb is selected at random from it. Given that this bulb had a lifetime in excess of 1450 hours, calculate the probability, to three decimal places, that it came from the batch having the larger mean lifetime. [5]

(ii) If one bulb is drawn at random from each of the two batches, calculate the probability, to three decimal places, that their lifetimes will differ by at least 50 hours. [4]

(iii) The dealer wishes to identify which of the two batches has the larger mean lifetime. To do so, he is prepared to take a small random sample of n bulbs from each of the two batches, and to conclude that the batch having the larger mean lifetime is the one which had the larger sample mean. Find, in terms of n, an expression for the probability that the dealer's conclusion will be incorrect. Hence find the smallest value of n for which this probability will be less than 0·001. [6]

15. An alloy bar of weight 1 kg is produced from a mixture of X kg of metal A and $(1-X)$ kg of metal B, where $0{\cdot}4 \leqslant X \leqslant 0{\cdot}6$. The total cost £$Y$ of producing such a bar is given by

$$Y = 3(5X^2 - 4X + 1).$$

(*a*) Show that $0{\cdot}6 \leqslant Y \leqslant 1{\cdot}2$. [3]

(*b*) Given that X is uniformly distributed over the interval $(0{\cdot}4, 0{\cdot}6)$ find

(i) the mean value of Y, [4]

(ii) the probability density function of Y, and use it to check the answer you obtained in (i). [8]

16. The continuous random variable X has probability density function

$$f(x) = \frac{2}{\theta}\left(1 - \frac{x}{\theta}\right), \qquad 0 \leqslant x \leqslant \theta,$$

$$f(x) = 0, \qquad \text{otherwise},$$

where θ is an unknown positive constant.

(*a*) Find the mean and the variance of X in terms of θ. [3]

(*b*) Let $X_1, X_2, \ldots, X_n$ denote a random sample of n observations of X, and let

$$T = \sum_1^n X_i, \qquad S = \sum_1^n X_i^2.$$

(i) Show that $(3T)/n$ is an unbiased estimator of θ, and that $6S/n$ is an unbiased estimator of θ^2. [4]

(ii) Assuming that n is large enough to justify approximating the sampling distribution of $\bar{X} = T/n$ by a normal distribution, find a and b, in terms of θ and n, so that

$$P(\bar{X} < a) = P(\bar{X} > b) = 0{\cdot}025.$$

Given $n = 200$ and $\bar{X} = 2{\cdot}5$, deduce a 95% confidence interval for θ. [8]