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WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

CYD-BWYLLGOR ADDYSG CYMRU

Tystysgrif Addysg Gyffredinol

Safon Uwch

MATHEMATICS A 1

A.M. MONDAY, 14 June 1982

(3 Hours)

Answer **seven** *questions.*

1. (*a*) Prove by mathematical induction, or otherwise, that $3^{2n}-1$ is divisible by 8 for all positive integers n. [5]

(*b*) Given that $\{r(r+1)\}^2-\{(r-1)r\}^2 \equiv 4r^3$ find the sum of the series

$$1^3+2^3+3^3+\ldots+n^3. \quad [3]$$

Deduce the sum of the series

$$1^3-2^3+3^3-4^3+\ldots+(2n+1)^3. \quad [3]$$

(*c*) Solve

$$x^2+xy+y^2 = 28$$
$$x^2-xy+y^2 = 12. \quad [4]$$

2. (*a*) Show, without using tables, that

$$\tan 22\tfrac{1}{2}^\circ = \sqrt{2}-1. \quad [3]$$

(*b*) Show that $\cos 3\theta = 4\cos^3\theta-3\cos\theta$ and hence, or otherwise, find all values of θ between 0° and 360° satisfying

$$4\cos 3\theta = \cos^2\theta. \quad [2, 5]$$

(*c*) Write down the first three terms of the binomial expansion for $(1+2y)^n$. Show that if y is greater than 2 then $(1+2y)^n+(1-2y)^n$ is greater than $16n^2-16n+2$. [5]

Turn over

3. (*a*) The matrices A and B are given by

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & -2 & 2 \\ 4 & 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{pmatrix}.$$

Find the inverse of A. Hence find the matrix C such that

$$CA = B. \qquad [3, 3]$$

Show that $BB^t = 9I$, where I is the 3×3 identity matrix. Given two matrices D, E such that $E = B^tDB$, express D in terms of B, B^t and E. [2, 3]

(*b*) Given that the equation

$$ax^2 + bx + c = 0$$

has roots α and β, find an expression for

$$\left(\alpha^2 + \frac{1}{\alpha}\right) + \left(\beta^2 + \frac{1}{\beta}\right). \qquad [4]$$

4. (*a*) Write down and simplify the 2×2 matrix representing a reflection in the line $y = \sqrt{3}x$. [3]

(*b*) Find the 2×2 matrix representing the transformation that transforms the point $(-k, 0)$ to the point $(0, k)$ and the point $(-k, k)$ to $(k, 3k)$, where $k \neq 0$. Find all the points that remain fixed under the transformation. What type of transformation is represented by the matrix? [3, 2, 2]

(*c*) Show by means of a sketch, or otherwise, that $x - 1 - \sin x = 0$ has only one root and that it lies between 0 and π.

Given an approximate value for the root, explain how an iterative method could be used to obtain a better approximation. (No calculation required.) [2, 3]

5. (*a*) The points $(0, 1)$, $(2, 6)$, $(6, 0)$ are denoted by A, B, and C respectively. Show that the equation of AC is

$$6y = -x + 6$$

and that the line through B perpendicular to AC is given by

$$y = 6x - 6.$$

This second line cuts AC in E. Given that F is the point $(\frac{36}{37}, \frac{15}{37})$ show that FE is parallel to AB. BE produced meets the x-axis at D. Show that the area of triangle ADC is $2\frac{1}{2}$ square units. [8]

(*b*) Show that the equation to the tangent to the curve $xy = a^2$ at the point $T(at, a/t)$ is given by

$$t^2y = -x + 2at.$$

Tangents at point T and the point $S(as, a/s)$ intersect at the point R. Given that R lies on the line $y = x$ and $s + t \neq 0$, show that $st = 1$. [3, 4]

6. (*a*) Show that the circle having the line joining the points $(1, 1)$ and $(2, 4)$ as diameter is given by the equation

$$x^2 + y^2 - 3x - 5y + 6 = 0. \qquad [3]$$

Find the equation satisfied by m if the line $y = mx$ is a tangent to this circle. If $y = m_1 x$, and $y = m_2 x$ are the tangents, find values for $(m_1 + m_2)$ and $m_1 m_2$ and hence find the angle between these tangents. [3, 3]

(*b*) Show that the equation to the tangent to the curve $y = x^3$ at the point (c, c^3) is

$$y = 3c^2x - 2c^3. \qquad [2]$$

A tangent to the curve intercepts the Ox and Oy axes at P and Q respectively. Given that the triangle OPQ has area $\frac{32}{3}$, find the possible forms of the equations to the tangent. [4]

7. (*a*) Show that if $f(x) = \dfrac{(x-4)^2}{x-3}$ then $f'(x) = \dfrac{(x-4)(x-2)}{(x-3)^2}$. [2]

Find the local maxima or minima of the function f and show that it has no points of inflection. Sketch the graph of f. [6, 3]

(*b*) Given that the function g is defined by

$$g(x) = +\sqrt{\left(\frac{1-4x}{1-x}\right)}$$

find the maximum possible domain of g. [4]

8. (*a*) Differentiate $\log_e(\tan(3x+5))$, $\quad 0 < 3x+5 < \pi/2$. [3]

(*b*) Given $y = x\tan^{-1}x$, show that $\dfrac{d^2y}{dx^2} = \dfrac{2}{(1+x^2)^2}$. [4]

(*c*) The circular cross-section of a wooden log is of radius r. A beam of the same length as the log, with rectangular cross-section $ABCD$ is cut from the log, where $AB = x$ and $BC = y$. Find the maximum value of xy^3. [8]

9. (*a*) Given a function $f(x) = \dfrac{x^2+1}{2x^2+1}$ $(x \geqslant 2)$ find an expression for $f^{-1}(x)$ and verify that $(f^{-1} \circ f)(x) = x$. [3,4]

(*b*) Sketch the curve $y(x) = -x^2+4x$, between the origin O and the point A, where $x = 4$. The line $y = 3$ cuts the curve in points B and C. Find the area of $OBCA$. [2, 6]

10. (*a*) Use integration by parts to find

$$\int x\log_e x\,dx.$$ [4]

(*b*) Using the substitution $x = \sin\theta$, or otherwise, show that

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{(1-x^2)}}\,dx = \frac{\pi}{12} - \frac{\sqrt{3}}{8}.$$ [5]

(*c*) Find $\displaystyle\int \frac{3x^2+1}{x^2(x^2+1)}\,dx$. [6]

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General Certificate of Education — Tystysgrif Addysg Gyffredinol

Advanced Level — Safon Uwch

MATHEMATICS A 2

A.M. TUESDAY, 22 June 1982

(3 Hours)

Answer **seven** questions.

[$g = 9{\cdot}8$ m s^{-2}.]

1. (*a*) Verify that $\alpha_1 = -1-3i$ is a root of the equation

$$z^2+iz+5(1-i) = 0.$$

By considering the coefficient of z in the equation, or otherwise, find the second root α_2. [2, 2]

Find the modulus and argument of β where

$$\frac{1}{\beta} = \frac{1}{\alpha_1}+\frac{1}{\alpha_2}.$$ [5]

(*b*) (i) Show that the locus of points in the Argand plane satisfying the equation

$$z\bar{z}+(1+i)z+(1-i)\bar{z} = 1$$

is a circle. [2]

(ii) Find the complex numbers corresponding to the points where the locus

$$z^2+\bar{z}^2-14z\bar{z}+48 = 0$$

crosses the imaginary axis. [2]

(iii) Show that the locus

$$2z^2+2\bar{z}^2-z\bar{z}+15 = 0$$

does not cross the real axis. [2]

2. (*a*) Find the general solution of the differential equation

$$\frac{dy}{dx}+ky = a\sin mx,$$

where a, k, and m are constants.

If $a = 1$, $k = 2$ and $m = 1$, find the particular solution which has value 1 when $x = 0$. [4, 2]

$$\left[\int e^{kx}\sin mx\, dx = \frac{e^{kx}}{k^2+m^2}(k\sin mx - m\cos mx).\right]$$

(*b*) Find the general solution of the differential equation

$$2x\frac{dy}{dx}+y^2 = 1.$$

Sketch the particular solution passing through the point $(\frac{1}{2}, \frac{3}{5})$. [5, 4]

Turn over.

3. An aeroplane, which flies at speed u in still air, travels between landing sites A and B, B being a distance d from A in a direction N $\alpha°$ W. In a wind blowing with speed w in the direction N $\theta°$ W ($\theta < \alpha$), the journey from A to B takes time t_1 (ignoring take off and landing times) and the return journey, t_2.
Show that

$$d^2 = (u^2-w^2)t_1^2+2dwt_1\cos(\alpha-\theta)$$

and find the corresponding expression for d^2 in terms of t_2. [7]

By eliminating the cosine terms, deduce that

$$u^2-w^2 = v_1 v_2$$

where v_1 and v_2 are the ground speeds over the initial and return journeys respectively. [4]

If $u = 200$ km h^{-1}, $w = 20$ km h^{-1}, $\alpha = 60°$ and $\theta = 30°$, find the direction along which the plane should head on the journey from A to B. [4]

4. A projectile is fired from a point O with speed v at an angle α to the horizontal. Show that x and y, its horizontal and vertical displacements from O, are related by

$$y = x\tan\alpha-\frac{gx^2\sec^2\alpha.}{2v^2}$$ [4]

Hence or otherwise show that β, the angle between the tangent at point (x, y) on the trajectory and the horizontal, is given by

$$\tan\beta = \tan\alpha-\frac{gx}{v^2}\sec^2\alpha.$$ [3]

Find the two possible values of α when the projectile's horizontal range is $3v^2/5g$, and the ratio of the times of flight for the corresponding trajectories. [5, 3]

5. (*a*) Points A, B, and P have position vectors **a**, **b** and λ**a**$+(1-\lambda)$**b** respectively with respect to some origin. Prove that P lies on the line AB and that, if $0 < \lambda < 1$, it divides it internally in the ratio $(1-\lambda):\lambda$.

Points A, B and C have position vectors **a**, **b** and **c** respectively with respect to some origin. Point D divides BC in the ratio 2:3, point E divides CA in the ratio 3:1 and point F divides AB in the ratio 1:2. Find the position vectors of points F and G where G divides CF in the ratio 9:2.

Show that AD and BE pass through G and find the ratios in which G divides them. [3, 2, 5]

(*b*) A particle moves so that its position vector at time t is $\mathbf{r} = t\mathbf{i}+(1+t^2)\mathbf{j}$. Find **v** and **a**, its velocity and acceleration vectors respectively. Find also the speed of the particle at time $t = 1$ and a unit vector normal to its trajectory at time $t = 2$. [2, 1, 2]

6. Using standard notation, write down the differential equation for simple harmonic motion and derive the relation

$$v^2 = \omega^2(a^2-x^2).$$ [3]

A particle is moving in a straight line with simple harmonic motion, O being its centre of oscillation. When the particle is 12 cm from O its speed is 10 cm s^{-1}, and when 5 cm from O its speed is 24 cm s^{-1}. Find its amplitude and the period of oscillation. [6]

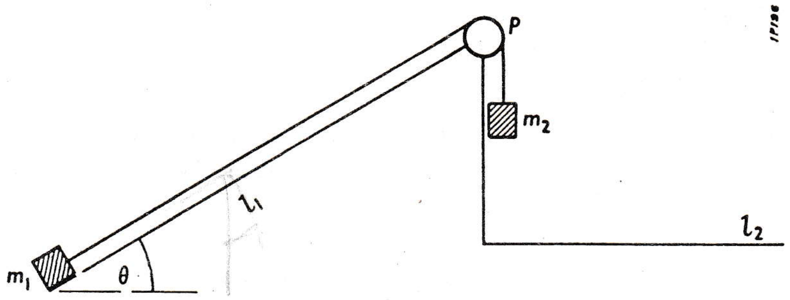
If A and B are points on opposite sides of O such that $OA = 12$ cm and $OB = 5$ cm, find the time taken by the particle to travel directly from A to B. [6]

7. A banked corner of a racing track can be regarded as an arc of a circle of radius r and the lateral gradient α of the bank is such that a car travelling at speed u has no tendency to side-slip. Find the relation between α, r and u. [5]

Show that the coefficient of friction necessary to prevent side-slip at a speed $v > u$, must be at least

$$\{(v^2-u^2)\sin\alpha\cos\alpha\}/(v^2\sin^2\alpha+u^2\cos^2\alpha).$$ [10]

8. In the accompanying diagram, l_1 represents a smooth, fixed plane inclined at an angle θ to the horizontal and l_2 a fixed horizontal plane, P is a smooth pulley and the masses m_1, m_2 are connected by an inextensible string passing over the pulley. The system is released from rest with the string taut, m_1 in contact with l_1 and m_2 hanging freely. If m_1 starts to move up the plane, state the relation between m_1 and m_2 for this to happen and find the acceleration of the system and the tension in the string. [5]



Suppose that $m_1 = 6$ kg, $m_2 = 5$ kg, $\theta = 30°$ and that when m_2 has descended 3 m it strikes l_2. Find

(i) how far m_1 travels along l_1 after m_2 strikes l_2,

(ii) the time that m_2 remains in contact with l_2,

(iii) the speed with which the system starts to move after the string becomes taut again,

(iv) the impulsive tension in the string. [10]

9. A ladder of length $2l$ rests with one end against a rough vertical wall and the other end on rough horizontal ground. The centre of gravity of the ladder is above its centre and a distance x away from it, and the coefficient of friction at each point of contact is μ. Show that the ladder is on the point of slipping when it makes an angle α with the horizontal, where

$$\tan \alpha = \frac{1-\mu^2}{2\mu} + \frac{x}{l}\left(\frac{1+\mu^2}{2\mu}\right).$$

[10]

A man is climbing a *uniform* ladder 8 m long which has one end resting against a rough vertical wall and the other on equally rough ground. The ladder is inclined at 60° to the horizontal, the man weighs three times as much as the ladder and $\mu = 0{\cdot}4$. How far can the man climb before the ladder is on the point of slipping? [5]

10. A *non-uniform* straight rod is such that the mass of an element of length δx distant x from one end is proportional to $x\,\delta x$. Show that its centre of mass is at a point of trisection. Use this result to find the centre of mass of a uniform plane triangular lamina by dividing it into thin strips parallel to one side and replacing each strip by a suitable point mass. [3, 4]

Show that the centre of mass of such a triangular lamina is the same as that of three equal point masses situated one at each vertex. [3]

The lengths of the parallel sides of a uniform plane trapezium are in the ratio 5:6. Deduce that its centre of mass divides the line joining the mid-points of the parallel sides in the ratio 17:16. [5]

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WELSH JOINT EDUCATION COMMITTEE

General Certificate of Education

Advanced Level

CYD-BWYLLGOR ADDYSG CYMRU

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Safon Uwch

MATHEMATICS A 3

A.M. TUESDAY, 22 June 1982

(3 Hours)

Answer **seven** *questions.*

Mathematical and Statistical tables are provided.

A calculator may be used except where it is expressly forbidden.

1. A box contains twelve balls numbered from 1 to 12. The balls numbered 1 to 5 are red, those numbered 6 to 9 are white, and the remaining three balls are blue. Three balls are to be drawn at random without replacement from the box. Let A denote the event that each number drawn will be even, B the event that no blue ball will be drawn, and C the event that one ball of each colour will be drawn. Calculate (i) $P(A)$, (ii) $P(B)$, (iii) $P(C)$, (iv) $P(A \cap C)$, (v) $P(B \cup C)$, (vi) $P(A \cup B)$. [15]

2. Ann and Bethan play a series of two-stage games. In the first stage of each game two fair dice are thrown together; in the second stage one of the girls draws a card at random from a pack of ten cards numbered 1 to 10. Ann draws the card at the second stage when the sum of the scores on the two dice is 6 or less; otherwise Bethan draws the card. Whoever draws the card wins the game if the number on the drawn card is 1 or 2; otherwise the other girl wins the game. The two girls play a series of 10 games.

(i) Show that Ann's probability of winning the first game is 0·55. [5]

(ii) Given that Ann wins the first game find the conditional probability that she drew the card. [3]

Use tables to find the probabilities that the series will end with

(iii) each girl winning five games, [4]

(iv) Ann winning more games than Bethan. [3]

Turn over.

3. (*a*) The random variable X has a Poisson distribution and is such that $P(X = 2) = 3P(X = 4)$. Find, correct to three decimal places, the values of (i) $P(X = 0)$, (ii) $P(X \leqslant 4)$. [6]

(*b*) The number of characters that are mistyped by a copytypist in any assignment has a Poisson distribution, the average number of mistyped characters per page being 0·8. In an assignment of 80 pages calculate, to three decimal places,

(i) the probability that the first page will contain exactly two mistyped characters,

(ii) the probability that the first mistyped character will appear on the third page,

(iii) an approximate value for the probability that the total number of mistyped characters in the 80 pages will be at most 50. [9]

4. The maximum length to which a string of natural length a metres can be stretched before it snaps is $a(1+X^2)$ metres, where X is a continuous random variable whose probability density function is

$$f(x) = 4x, \quad 0{\cdot}25 \leqslant x \leqslant 0{\cdot}75,$$
$$f(x) = 0, \quad \text{otherwise.}$$

(i) Calculate the probability that a string can be stretched to $1\frac{1}{2}$ times its natural length without snapping. [4]

(ii) Find the value of $E(X^2)$ and hence find μ, the mean maximum stretched length of strings of natural length 1 metre. [4]

(iii) Find the probability density function of $Y = 1+X^2$, the maximum stretched length of a string of natural length 1 metre, and use it to verify the value of μ you obtained in (ii). [7]

5. The lengths, in centimetres, of manufactured rods have a continuous distribution with probability density function

$$f(x) = k(7-x), \quad 1{\cdot}9 \leqslant x \leqslant 2{\cdot}1,$$
$$f(x) = 0, \quad \text{otherwise,}$$

where k is a constant. A medial rod is defined to be one whose length lies in the range from 1·96 cm to 2·04 cm.

(i) Find the value of k. [2]

(ii) Show that there is a probability of 0·4 that a randomly chosen rod will be a medial rod. [2]

(iii) If four rods are chosen at random find the probability that exactly three of them will be medial rods. [2]

(iv) If 96 rods are chosen at random find an approximate value for the probability that at least 50 of them will be medial rods. [5]

(v) A person requires 20 medial rods for a specific purpose. Find the smallest number of rods the person should order for the expected number of medial rods among those ordered to be at least 20. [4]

6. A boy is to have three attempts at performing a certain task. The probability that he will succeed on his first attempt is $\frac{1}{2}$, while on each subsequent attempt the probability that he will succeed is $\frac{3}{4}$ if his preceding attempt was successful, and $\frac{1}{2}$ if his preceding attempt was not successful. Let X denote the number of successes by the boy in his three attempts, and let Y denote the number of the attempt on which he is first successful; define Y to be zero if the boy fails on all three attempts.

(i) Display the joint probability distribution of X and Y in a two-way table. [8]

(ii) Having completed his three attempts, the boy is awarded a score of 2 if $X > Y$, a score of 1 if $X = Y$, and a score of 0 if $X < Y$. Find the mean and the standard deviation of the boy's score. [7]

7. The lifetimes of electric light bulbs of brands A and B are independent and normally distributed. For brand A bulbs the mean and the standard deviation of the lifetimes are 1010 hours and 5 hours, respectively, while for brand B bulbs the corresponding values are 1020 hours and 10 hours, respectively.

(i) Calculate the probability that a brand A bulb will have a lifetime in excess of 1020 hours. [1]

(ii) Calculate the probability that a brand A bulb will have a longer lifetime than a brand B bulb. [4]

A box contains 8 bulbs of brand A and 2 bulbs of brand B.

(iii) If one bulb is drawn at random from the box, calculate the probability that its lifetime will exceed 1020 hours. [5]

(iv) Calculate the probability that the mean of the lifetimes of the 10 bulbs in the box will exceed 1010 hours. [5]

8. A random variable X can take only the values 1, 2 and 3, the respective probabilities of these values being θ, θ, and $1-2\theta$, where $0 < \theta < \frac{1}{2}$. Determine the mean and the variance of X in terms of θ. [3]

In a random sample of n values of X, let $\bar{X}$ denote the mean of the sample values, and let R denote the number of 3 s among the sample values. Show that $p_1 = 1-(\bar{X}/3)$ and $p_2 = \frac{1}{2}\{1-(R/n)\}$ are both unbiased estimators of θ. Determine which of p_1 and p_2 has the smaller standard error. [12]

9. (a) When an object is weighed on a chemical balance the readings obtained are subject to random errors which are known to be independent and normally distributed with mean zero and standard deviation 1 mg. A certain object is to be weighed 9 times on such a balance and the mean of the 9 readings is to be calculated. Find the probability that the mean of the 9 readings will be within 0·5 mg of the true weight of the object. [5]

(b) Another weighing device is undergoing tests to determine its accuracy. A certain object of known true weight 50 mg was weighed 10 times on this device and the readings in mg were

49, 51, 49, 52, 49, 50, 52, 51, 49, 48.

(i) Calculate an unbiased estimate of the variance of the errors in readings using this device. [4]

(ii) Calculate 95 per cent confidence limits for the mean error in readings using this device. [6]

10. An investigation of a possible linear relationship between two variables x and y was conducted with x having five prespecified values, the corresponding values of y being observed. The results obtained are given in the following table:

$x =$	5	10	15	20	25
$y =$	55	52	50	48	45

You may assume that $\sum x^2 = 1375$ and that $\sum xy = 3630$.

Given that the true relationship between x and y is $y = \alpha + \beta x$, determine the least-squares estimates of α and β. [4]

Suppose that the observed values of y are subject to independent random errors that are normally distributed with a mean of zero and a standard deviation of 0·5. Calculate 90 per cent confidence limits for

(i) the true value of y when $x = 20$, [5]

(ii) the difference between the true values of y when $x = 5$ and $x = 25$, respectively. [6]

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WELSH JOINT EDUCATION COMMITTEE	**CYD-BWYLLGOR ADDYSG CYMRU**
General Certificate of Education	**Tystysgrif Addysg Gyffredinol**
Advanced Level	**Safon Uwch**
Special Paper	**Papur Arbennig**

MATHEMATICS S

A.M. MONDAY, 28 June 1982

(3 Hours)

Answer **seven** *questions.*

Not more than **four** *questions may be attempted from any section.*

A calculator may be used except where it is expressly forbidden.

SECTION A

1\. (*a*) A sequence such as $a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots$
is said to be a geometric progression (G.P.) of common ratio r.

A sequence such as $\alpha, \alpha+d, \alpha+2d, \alpha+3d, \ldots, \alpha+(n-1)d, \ldots$
is said to be an arithmetic progression (A.P.) of common difference d.

The 5th, 8th and 9th terms of a particular geometric progression are in arithmetic progression. Given that the common ratio is neither 0 nor 1, show that the sum of the first three terms of the G.P. is equal to the 4th term. [5]

(*b*) The roots of the equation $x^2+bx+c=0$ are α and β, and the roots of the equation $x^2+2bx+4c=0$ are γ and δ. Find the equation whose roots are $\frac{\alpha}{\gamma}+\frac{\beta}{\delta}, \frac{\alpha}{\delta}+\frac{\beta}{\gamma}$. [7]

(*c*) Given that $\log_e(x^2y^3)=7$ and $\log_e(x^4/y^2)=6$, find x and y. [3]

2\. (*a*) Express $2\cos(\alpha+r\beta)\sin\frac{\beta}{2}$ as the difference of two sines and hence, or otherwise, show that

$$\cos\alpha+\cos(\alpha+\beta)+\ldots+\cos(\alpha+n\beta)=\frac{\cos\left(\alpha+\frac{n\beta}{2}\right)\sin\left(\frac{n+1}{2}\right)\beta}{\sin\frac{\beta}{2}}.$$ [2, 5]

(*b*) Write down the matrix M representing M, the anticlockwise rotation of the plane through an angle α, and the matrix N representing N, the reflection of the plane in the line $y=x\tan\frac{\beta}{2}$.

The action of the transformation N followed by M results in a reflection about the line $y=x$; whilst the transformation M followed by N results in a reflection of the plane in the y axis. Find α and β. [2, 6]

3. (a) Show that the equation of the circle which touches the x-axis at the point $(4, 0)$ and cuts the positive y-axis in two points A and B, where $AB = 6$, is

$$x^2+y^2-8x-10y+16 = 0.$$

A second circle is given by $4x^2-64x+4y^2-72y+579 = 0$. Show that the two circles lie entirely outside each other. [4, 4]

(b) Show that no real values of x and θ can be found to satisfy the simultaneous equations

$$3\cos\theta = x+\frac{2}{x},$$

$$2\sin\theta = 2x-\frac{1}{x}.$$ [7]

4. (a) Find the stationary values of $\dfrac{\sin x}{(1+\sin^2 x)^{\frac{3}{2}}}$ for values of x between 0 and π, distinguishing between them. Deduce the stationary values for all values of x. [9]

(b) Show that if $a < 0$, $b < 0$ the equation

$$x^3-ax^2-b = 0$$

has only one real root. [6]

5. (a) Show that $\dfrac{d}{dx}\left(\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right) = \dfrac{1}{1-\sin x}$. [5]

(b) Show by induction that

$$\frac{d^n}{dx^n}(x^2 e^x) = [x^2+2nx+n(n-1)]\,e^x$$

for all positive integers n. [6]

(c) The function g is defined by

$$g(x) = \sqrt{(3-x)}, \qquad (x \leqslant 3).$$

The rule for the function h is

$$h(x) = 1+x^2.$$

Given that $g \circ h$ exists, find the maximum possible domain of h. Give the range of $g \circ h$. [4]

6. (a) Evaluate $\displaystyle\int_0^1 \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} dx$ by substituting $x = \cos 2\theta$, or otherwise. [7]

(b) Sketch the graphs of $y = \dfrac{4x}{\pi}$ and $y = \tan x$ between $x = 0$ and $x = \dfrac{\pi}{4}$. The area enclosed by these graphs is rotated about the Ox-axis through four right angles. Find the volume generated. [2, 6]

SECTION B

7. (a) Find the slopes of the straight lines through the origin whose equations satisfy the differential equation

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^2.$$ [3]

Find the general solution of the equation and show that, for each finite non-zero value of the arbitrary constant, it represents a pair of non-intersecting curves, one of which passes through the origin. What is the slope of the general solution at the origin?

Sketch the graph of the particular solution one of whose branches passes through the point $x = 1$, $y = 1$. [9]

(*b*) If the complex numbers z, w and v lie on the same line in the Argand plane, prove that

$$\operatorname{Im}\left(\frac{v-z}{w-z}\right) = 0. \qquad [3]$$

8. A bowl has the shape of the surface formed by rotating the parabola $y = x^2/2a$ about its axis. Show that the line of greatest slope of the tangent plane at a point of the bowl distance r from the axis makes an angle $\frac{1}{2}\pi - \tan^{-1}r/a$ with the axis. [2]

The bowl is rough, and rotates with constant angular velocity ω about its axis, which is vertical. A particle rests in relative equilibrium on the inside of the bowl at a distance r from the axis. Prove that the coefficient of friction between the particle and the bowl is at least

$$\frac{r}{ag+\omega^2 r^2}|g-a\omega^2|. \qquad [13]$$

9. One end of a light elastic string, of natural length a and modulus of elasticity $2mg$, is attached to a fixed point O and the other end to a particle of mass m. The particle, initially at rest at O, is allowed to fall. Find the greatest extension of the string during the motion and show that the particle will reach O again after a time

$$(\pi+2-\tan^{-1}2)(2a/g)^{\frac{1}{2}}. \qquad [3, 12]$$

10. The maximum speed of a car is V and the resistance to its motion varies as the square of its speed. If its engine works at a constant maximum rate, the car attains the speed $\frac{1}{2}V$ after travelling a distance a from rest. If its engine exerts a constant maximum tractive force, the car attains the speed $\frac{1}{2}V$ after travelling a distance b from rest. Prove that

$$a:b = 2\log_e(\tfrac{8}{7}):3\log_e(\tfrac{4}{3}). \qquad [15]$$

11. A thin beam AB, of length $2a$ and weight W, is hung from a fixed point O by equal light strings OA, OB. The centre of gravity of the beam is at a distance b from the mid point of AB. When a weight $w(> (b/a)W)$ is hung from A the beam is in equilibrium with A lowest and the beam inclined at an angle θ to the horizontal. When the same weight is hung from B the inclination is φ ($> \theta$) with B lowest. Show that

(*a*) $\dfrac{\tan\theta}{\tan\varphi} = \dfrac{aw-bW}{aw+bW}$,

(*b*) the beam will be in equilibrium in the horizontal position if a weight

$$w\frac{\sin(\varphi-\theta)}{\sin(\varphi+\theta)}$$

is hung from A. [9, 6]

SECTION C

12. Show that for any three events A, B, and C,

$$P(A\cap B') = P(A) - P(A\cap B),$$
$$P(A\cap B'\cap C') = P(A) - P(A\cap B) - P(A\cap C) + P(A\cap B\cap C).$$

Deduce that the probability that exactly one of the events A, B, C, will occur is

$$P(A)+P(B)+P(C)-2\{P(A\cap B)+P(A\cap C)+P(B\cap C)\}+3P(A\cap B\cap C). \qquad [6]$$

A fair die has two of its faces coloured red, two coloured blue and two coloured white. The die is to be tossed four times and the colour of the uppermost face on each toss is to be noted. Let A denote the event that no toss will result in a red uppermost face, B the event that no toss will result in a blue uppermost face, and C the event that no toss will result in a white uppermost face. Find the probability that exactly one of the events A, B, C, will occur in the four tosses. Deduce the probability that in repeated tosses of the die all three colours will have appeared uppermost for the first time on the fifth toss. [9]

13. Manufactured wire is cut into pieces whose lengths, in metres, are distributed with probability density function f defined by

$$f(x) = 2x, \quad 0 < x < 1.$$
$$f(x) = 0, \quad \text{otherwise.}$$

Two pieces of wire are chosen at random. Let Y denote the length in metres of the longer of the two pieces.

(i) Show that for any y between 0 and 1,

$$P(Y \leqslant y) = y^4,$$

and hence, or otherwise, find the probability density function of Y. [4]

(ii) Find the expected length of the longer of the two pieces of wire. [1]

(iii) Find the expected length of the shorter of the two pieces of wire. [3]

The longer piece of wire is bent to form a square.

(iv) Find the probability that the square has an area in excess of 0·01 m^2. [3]

(v) Find the probability density function of the area of the square. [4]

14. A factory uses two machines A and B for mass-producing a certain type of article. The lengths of the articles produced on A are normally distributed with mean 50 cm and standard deviation 1 cm, while the lengths of the articles produced on B are normally distributed with mean 50·5 cm and standard deviation 1·1 cm. For a specific purpose the articles have to be used in pairs; a pairing of two articles is satisfactory for the purpose only if their lengths do not differ by more than 2 cm. The present policy is to pair off articles from the same machine, two-thirds of the total number of pairings required consisting of pairs of articles chosen randomly from those produced on A, and the other one-third consisting of pairs of articles chosen randomly from those produced on B. Calculate, to three significant figures, the overall proportion of satisfactory pairings from this policy. [9]

One possible alternative policy is for each pairing to consist of one article chosen at random from those produced on A and one chosen at random from those produced on B. Determine whether or not this alternative policy will lead to a larger overall proportion of satisfactory pairings. [6]

15. A coin has probability p of showing head when tossed. In n independent tosses of the coin let X denote the number of heads that will be obtained.

(i) Given that $X = r$, find in terms of r and n, the conditional probability that both the first and the last of the n tosses showed head. [3]

(ii) Show that $Z = \{X(n-X)\}/\{n(n-1)\}$ is an unbiased estimator of pq, where $q = 1-p$. [5]

A second coin has probability $q = 1-p$ of showing head when it is tossed. In n independent tosses of this coin let Y denote the number of heads that will be obtained.

(iii) Find, in terms of n and p, the mean and the variance of $W = X+Y$, the combined number of heads in the $2n$ tosses. [3]

(iv) For the case when $p = q = \frac{1}{2}$, show that $P(X = Y) = P(X+Y = n)$. [4]

16. The random variable X has a Poisson distribution with mean μ. Show that, for any positive integer n,

(i) $\sum_{r=1}^{n} rP(X = r) = \mu P(X \leqslant n-1)$, [3]

(ii) $\sum_{r=1}^{n} r^2 P(X = r) = \mu^2 P(X \leqslant n-2) + \mu P(X \leqslant n-1)$. [4]

A newsagent receives 20 copies of a daily newspaper to sell each day. The daily demand for this newspaper from the newsagent may be assumed to have a Poisson distribution with mean 20. Let Y denote the number of copies sold on a particular day by this newsagent. Using tables and the results given above, or otherwise, find the mean and the variance of Y. [6]

The newsagent makes a profit of 5 pence on each copy sold and loses 1 penny on each unsold copy. Find, correct to the nearest penny, the newsagent's expected daily profit from the sale of this newspaper. [2]