The Foundations of Number

A workbook for Curriculum for Wales

Mathematics and Numeracy Area of Learning and Experience

Progression Step 3

Teacher's Guide

Version 1 (August 20th, 2021)

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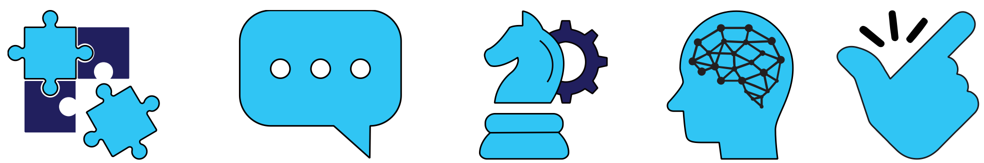
Mathematical proficiencies icons:



Conceptual understanding	*Communication using symbols*	*Strategic competence*	*Logical reasoning*	*Fluency*

Contents

This workbook deals with the following descriptions of learning.

Algebra uses symbol systems to express the structure of mathematical relationships.

PATTERNS

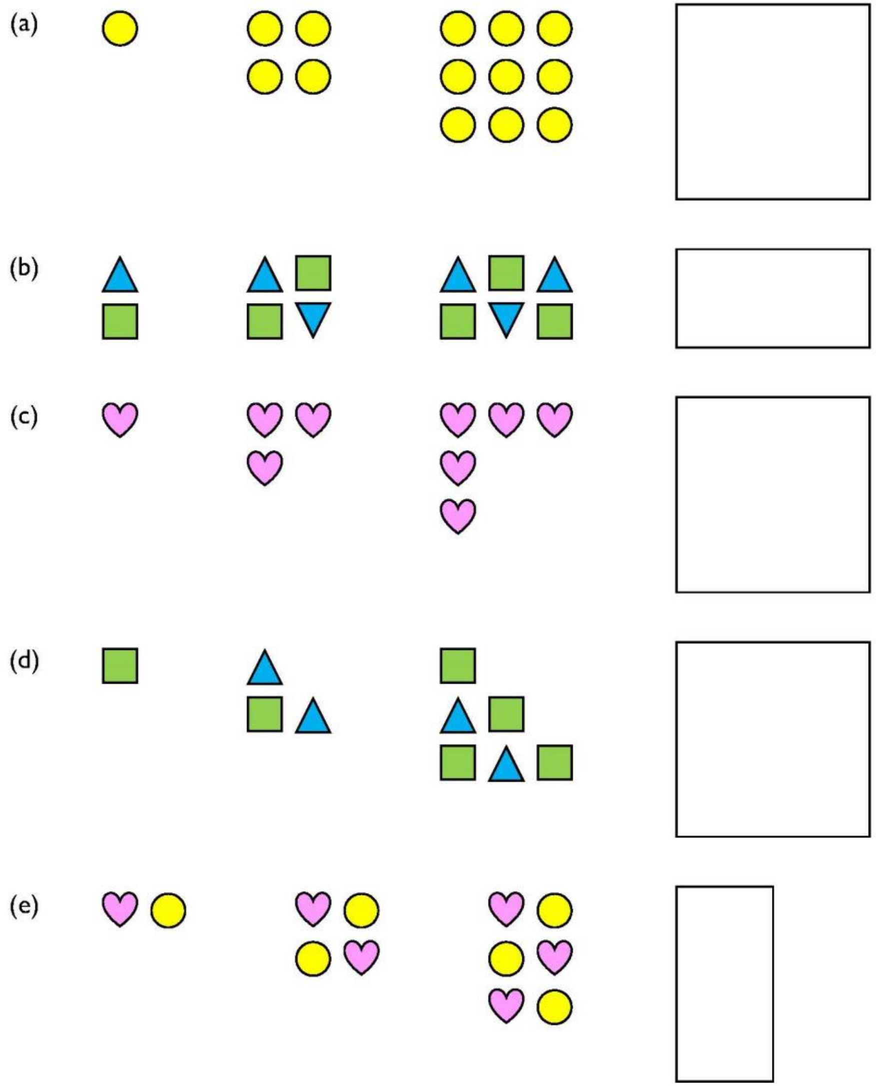
- I can explore and create patterns of numbers and shapes. I can explain numerical sequences and spatial patterns in words and by generalising them.

FORMING

- I can use commutativity, distributivity and associativity to explore equality and inequality of expressions.

Patterns

In the boxes, draw what comes next in the following patterns.



The learners reason how to complete the patterns.

(a) A pattern that forms the square numbers 1, 4, 9, 16, 25, … For the 10th picture, 100 yellow circles will be required, arranged into a 10 by 10 square.

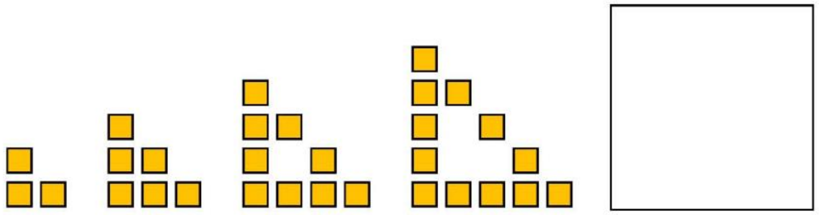
(b) A pattern that forms the even numbers 2, 4, 6, 8, 10, … For the 10th picture, two rows and ten columns will be required, giving 20 symbols in total.

(c) A pattern that forms the odd numbers 1, 3, 5, 7, 9, … For the 10th picture, a total of 19 hearts will be required, with 10 in the horizontal row and 10 in the vertical column.

(d) A pattern that forms the triangle numbers 1, 3, 6, 10, 15, … For the 10th picture, we will require a total of 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 55 symbols.

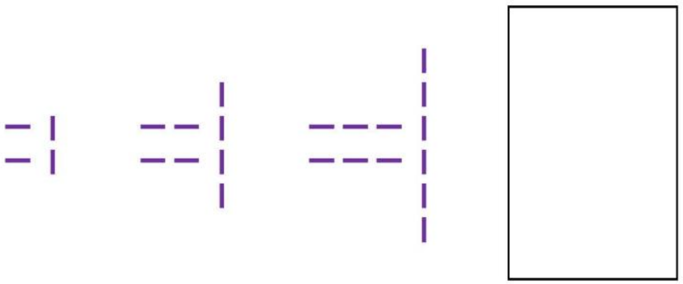
(e) The same pattern as (b), but continuing downwards instead of to the right.

(f)



(g)

(h)



Discuss the following questions with a partner.

- In each question, what did you have to do to draw the next picture in the sequence?
- What would be required to create the 10th picture in each pattern?
- In each question, what was the number pattern involving the individual symbols appearing in each picture?

The learners reason how to complete the patterns.

(f) A pattern that forms the perimeter of a triangle, and that shows the multiples of 3 when considering the number of symbols: 3, 6, 9, 12, 15, … For the 10th picture, 30 little squares will be required, forming the perimeter of a triangle with a base of 11 little squares. We can also think of this pattern as the difference between two triangle numbers.

(g) A cross-shaped pattern that adds four new stars each time: 1, 5, 9, 13, 17, … For the 10th picture, we will require 37 stars in total, forming a row of 19 stars and a column of 19 stars.

(h) Another pattern that adds four each time: 4, 8, 12, 16, 20, … For the 10th picture, 40 lines will be required in total, with 11 symbols in each of the two rows, and 20 vertical lines in the column on the right.

During the discussions, listen out for (and encourage) connections between the spatial pattern and concepts such as even numbers, square numbers, multiples of 3, vertical lines, etc.

Number Patterns

Look carefully at each of the following number patterns, considering how the patterns continue. Fill in the blanks with the missing numbers to extend each pattern both ways. Underneath each pattern, explain in words how to continue the pattern.

(a) ______ ______ 14, 17, 20, 23, ______ ______

____

(b) ______ ______ 65, 63, 61, 59, ______ ______

____

(c) ______ ______ 55, 64, 73, 82, ______ ______

____

(d) ______ ______ 54, 42, 30, 18, ______ ______

____

(e) ______ ______ 12, 22, 32, 42, ______ ______

____

(f) ______ ______ 75, 60, 45, 30, ______ ______

____

(g) ______ ______ –24, –18, –12, –6, ______ ______

____

(h) ______ ______ –110, –85, –60, –35, ______ ______

____

(i) ______ ______ 2.7, 3.4, 4.1, 4.8, ______ ______

____

(j) ______ ______ 0.5, 0.75, 1, 1.25, ______ ______

____

(k) ______ ______ $2\frac{5}{8}$, $3\frac{1}{8}$, $3\frac{5}{8}$, $4\frac{1}{8}$, ______ ______

____

Reasoning must be used here to recognise the pattern to begin with, and then to continue the pattern.

Aspects of this exercise also touch on the following description of learning for number (progression step 2):

THE NUMBER SYSTEM

- I can order and sequence numbers, including odd and even numbers, and I can count on and back in step sizes of any whole number and simple unit fractions.

(l) ______ ______ 4, 8, 16, 32, ______ ______

(m) ______ ______ 400, 200, 100, 50, ______ ______

(n) ______ ______ 12, 24, 48, 96, ______ ______

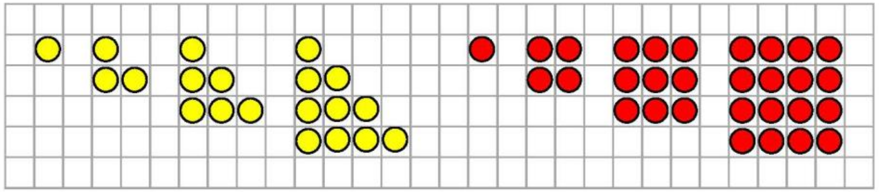
(o) ______ ______ 8, 4, 2, 1, ______ ______

(p) ______ ______ 4, 16, 64, 256, ______ ______

(q) ______ ______ 243, 81, 27, 9, ______ ______

(r) ______ ______ 3.6, 7.2, 14.4, 28.8, ______ ______

Triangle Numbers and Square Numbers



Write down the first ten triangle numbers.

______ ______ ______ ______ ______ ______ ______ ______ ______ ______

Write down the first ten square numbers.

______ ______ ______ ______ ______ ______ ______ ______ ______ ______

What is the connection between two consecutive triangle numbers and square numbers?

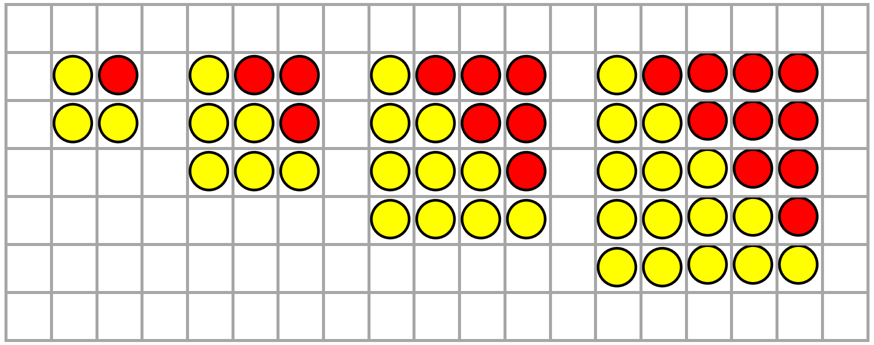
For your information: the exercises on the previous page are **arithmetic sequences** (where we need to add or subtract to obtain the next number), whilst the exercises on this page are **geometric sequences** (where we need to multiply or divide to obtain the next number).

The first ten triangle numbers are
1, 3, 6, 10, 15, 21, 28, 36, 45, 55. (Add one more each time.)

The first ten square numbers are
1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Adding two consecutive triangle numbers gives a square number.
For example,
1 + 3 = 4
3 + 6 = 9
6 + 10 = 16
10 + 15 = 25

If you have double sided counters, it would be possible to arrange them to show how two triangle numbers combine to give a square number.



Digital counters: https://mathsbot.com/manipulatives/counters

Odd one out

Circle the number that doesn't belong to each of the following patterns.

(a)	7	14	21	28	34	42	49
(b)	63	52	45	36	27	18	9
(c)	1	4	9	15	25	36	49
(d)	12	24	36	48	60	72	86
(e)	512	256	122	64	32	16	8
(f)	1.6	2	2.4	2.8	3.4	3.6	4
(g)	0.056	0.56	5.6	56	560	5600	560000
(h)	0.1	$\frac{1}{5}$	0.2	0.4	$\frac{1}{2}$	$\frac{3}{5}$	0.7

Investigation

Here is a set of instructions for generating a number pattern.

(a) For the first number of the pattern, think of any 4-digit number where the digits are not all identical. For example, 4,283, 6,816 and 3,025 would be fine, but not 452 (only 3 digits) or 7,777 (all digits are identical).

(b) Re-arrange the digits of your number to make the largest possible 4-digit number, and the smallest possible 4-digit number. For example, considering 4,283, the largest possible 4-digit number is 8,432, and the smallest 2,348.

(c) Subtract the smallest possible 4-digit number from the largest possible 4-digit number. For example, 8,432 – 2,348 = 6,084.

(d) Repeat the above process with the new number. (If the new number has less than 4 digits, add zeroes at the start to create a 4-digit number. For example, if we start with 3,233, the first subtraction sum would be 3,332 – 2,333 = 999. Then, we would need to treat the 999 as 0,999, and the second subtraction sum would be 9,990 – 0,999 = 8,991.)

Investigate the patterns produced by the above set of instructions.

Reasoning must be used here to recognise the pattern to begin with, and then to recognise the number that doesn't belong to the pattern.

More exercises on number sequences can be found on the page http://www.primaryresources.co.uk/maths/mathsB3.htm

In the investigation, learners must work systematically through the instructions in order to create the correct pattern. Starting with a 4-digit number (whose digits are not all identical), the pattern always reaches the number 6,174, and then stays at this number.

The number 6,174 is recognised as the **Kaprekar constant**, as the Indian mathematician Dattatreya Ramchandra Kaprekar discovered this pattern first. Every 4-digit number (whose digits are not all identical) reaches the number 6,174 in 8 steps or less. (What happens when you start with a 4-digit number whose digits are all identical?)

More information: https://math.hmc.edu/funfacts/kaprekars-constant/

Patterns with Prime Numbers

Here are the prime numbers between 1 and 100.

1	**2**	**3**	4	**5**	6	**7**	8	9	10
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48	49	50
51	52	**53**	54	55	56	57	58	**59**	60
61	62	63	64	65	66	**67**	68	69	70
71	72	**73**	74	75	76	77	78	**79**	80
81	82	**83**	84	85	86	87	88	**89**	90
91	92	93	94	95	96	**97**	98	99	100

What do you notice? What do you wonder?

Mathematicians believe that there is **no pattern** in how prime numbers appear on the number line.

There is a prize of $1,000,000 to prove that prime numbers have no pattern when placed on the number line. (This problem is one of the seven millennium problems.) The lack of pattern with prime numbers, and the fact that it is difficult to decide whether a large number is prime or not, is the reason why the internet is secure.

This page assumes that the learners have met prime numbers before (if not, you will need to discuss the concept).

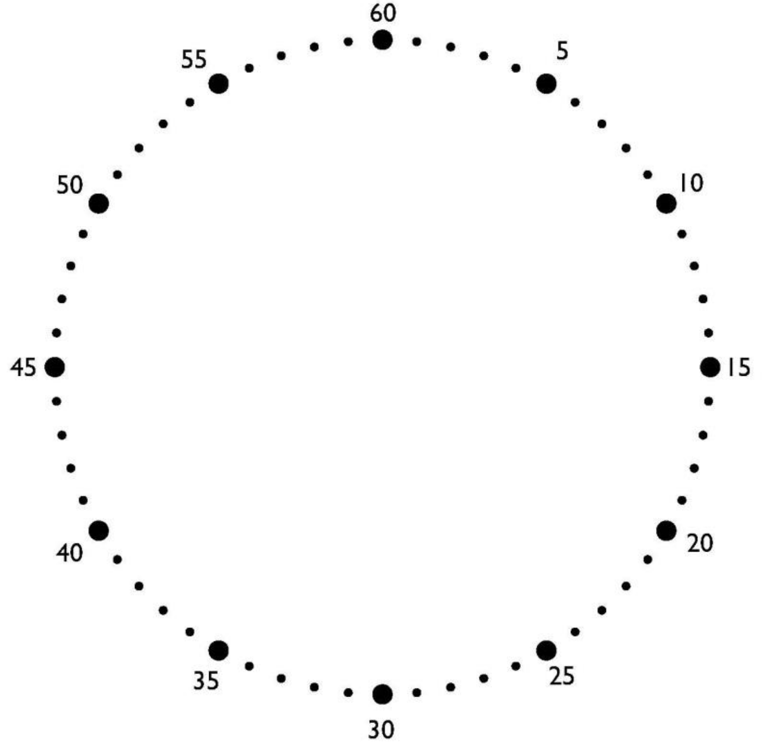
What type of things will the learners perhaps notice?

- Only one even number is here, namely 2, All other prime numbers are odd numbers.
- Exactly a quarter of the numbers in the grid are prime numbers, as 25 numbers out of 100 have been coloured yellow. This means that there are 75 composite numbers (numbers that are not prime numbers) in the grid.
- Apart from the first row, every prime number ends in a 1, 3, 7 or 9. Challenge! There are 4 prime numbers between 11 and 20. In which row will this happen next (supposing that the grid can be continued down vertically)?

One of the seven millennium problems has been solved by a mathematician named Grigori Perelman – but he declined the prize of $1 million! https://en.wikipedia.org/wiki/Grigori_Perelman

Calon Lân

The following diagram shows a blank clock face with the minutes labelled up to 60.



Use a ruler to connect each minute to its double. For example, you will need to connect 4 to 8, and 45 to 90 (which means connecting to 30 after travelling around the clock once). Perhaps the following table will help?

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	4	6	8																

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40

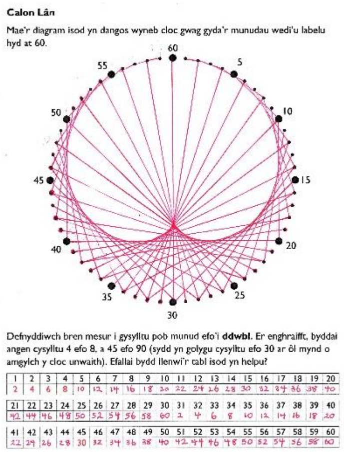
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
				30															

What happens when you multiply each minute by 3? By 4? By 5?

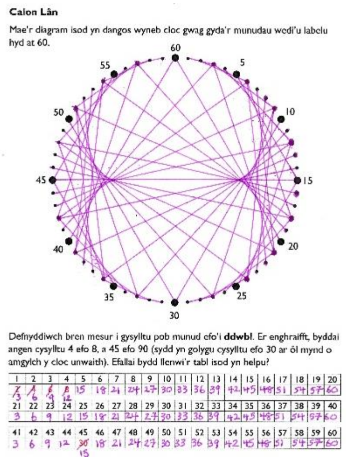
A shape known as a **cardioid** is formed on the clock face by completing the exercise correctly. Because it is similar to a heart shape, this explains the title of the page (heart = calon in Welsh).

Acknowledgement: This exercise is based upon one from the website https://www.artfulmaths.com/mathematical-art-lessons.html, where other exercises linking art to mathematics can also be found.

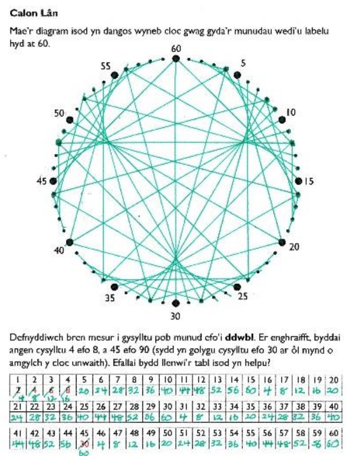
Answers: Multiplying by 2



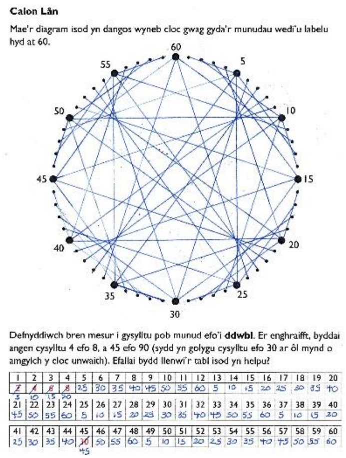
Multiplying by 3



Multiplying by 4



Multiplying by 5



The Fibonacci Sequence

For the Fibonacci sequence, we need to add the previous two numbers to give the next number in the sequence.

To begin, complete the first column in the following table to show how the sequence continues.

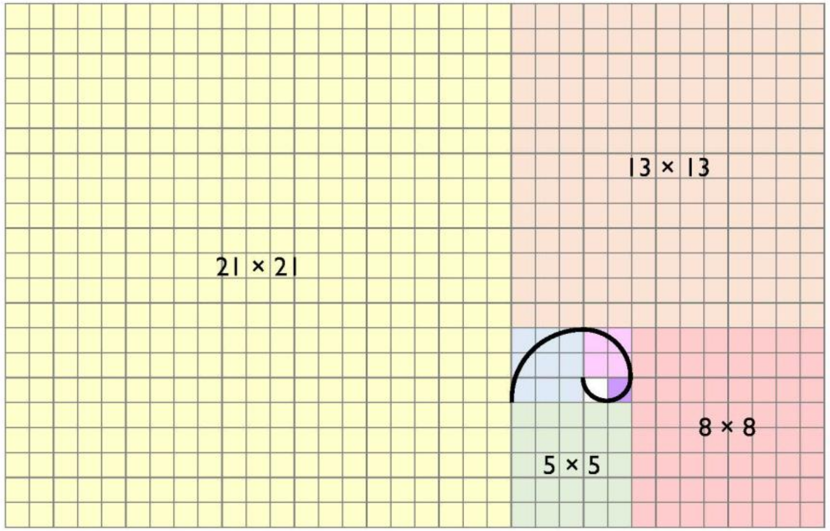
Fibonacci Sequence	Dividing two consecutive numbers
1	
1	1 ÷ 1 = 1
1 + 1 = 2	2 ÷ 1 = 2
1 + 2 = 3	3 ÷ 2 = 1.5
2 + 3 = 5	5 ÷ 3 = 1.6666666....

Next, use a calculator to divide two consecutive numbers in the sequence. You should see that your numbers settle down to become close to a special number known as **the golden ratio**. Use the internet to investigate this number, writing down what you find below.

Learners must work systematically to complete the table, taking care to add or divide the correct two numbers each time.

More information about the Fibonacci sequence:
https://www.mathsisfun.com/numbers/fibonacci-sequence.html

The Fibonacci Spiral



The above diagram shows part of the Fibonacci spiral. Use a compass or a piece of string to extend the spiral, connecting opposite corners in different coloured squares.

Colour of the square	Compass point is placed	Pencil starts
Green	Top right	Top left
Red	Top left	Bottom left
Orange	Bottom left	Bottom right
Yellow	Bottom right	Top right

Investigation

- What is the connection between the Fibonacci sequence and flower petals?
- Look at the scales on the surface of a pineapple (as shown in blue on the right). Count the number of scales whilst moving from top to bottom along one of the pineapple's diagonals. What do you notice?
- Who was Fibonacci?

Completing the spiral enables learners to practice their fine motor skills, using either a compass or a piece of string tied to a pencil.

More information about the connection between Fibonacci numbers and nature can be found here: http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html

Exercise

The following number sequences are related to the Fibonacci sequence.
Find the next two numbers in each sequence.

(a) 1, 2, 3, 5, 8, 13, _____, _____

(b) 2, 2, 4, 6, 10, 16, _____, _____

(c) 1, 3, 4, 7, 11, 18, _____, _____

(d) 5, 2, 7, 9, 16, 25, _____, _____

(e) 0.5, 0.3, 0.8, 1.1, 1.9, 3, _____, _____

(f) $\frac{1}{2}, \frac{1}{2}, 1, 1\frac{1}{2}, 2\frac{1}{2}, 4,$ _____, _____

(g) 1, 1, 1, 3, 5, 9, 17, _____, _____

(h) 1, 0, 1, 2, 3, 6, 11, _____, _____

Did you know?

There is a connection between the Fibonacci sequence and measurements in miles and kilometres. 5 miles is approximately 8 km; 8 miles is approximately 13 km; 13 miles is approximately 21 km, and so on. (Why?)

Challenge! ⚠

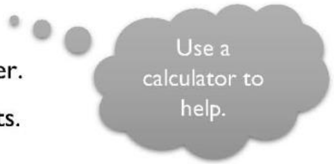
Choose any 3 consecutive numbers from the Fibonacci sequence.

Multiply the middle number by itself.

Now multiply the first number by the third number.

Try this several times using different starting points.

What do you notice about your answers?



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Puzzle

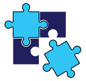
What would be the best prize in a competition: winning £1,000,000 today, or winning 1p today, 2p tomorrow, 4p the day after, and so on, for 30 days?

....

....

....

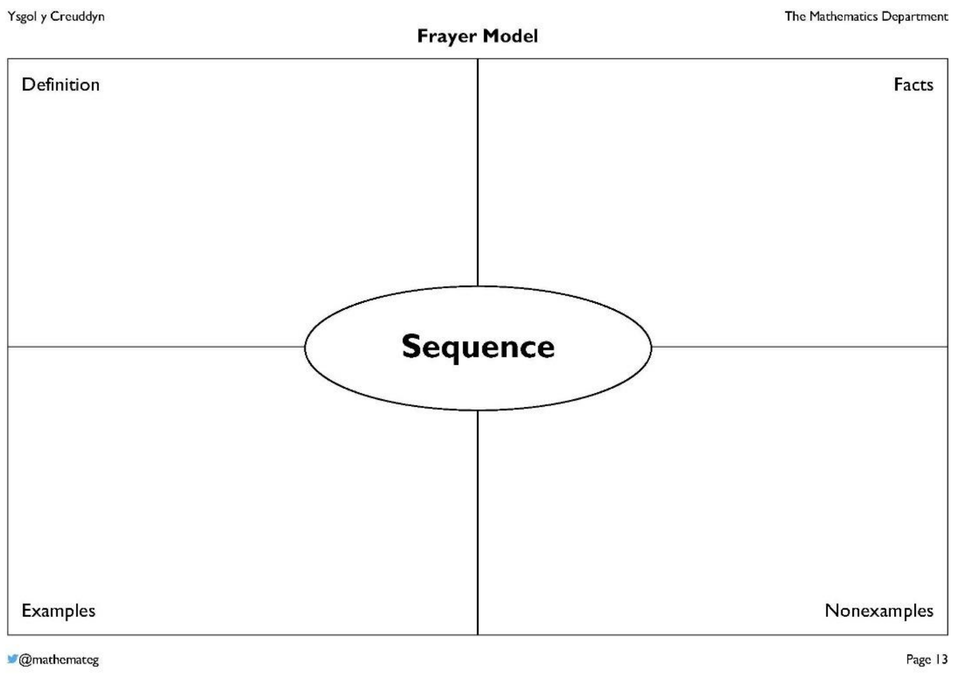
For the exercise on the top of the page, questions (a) to (f) are ones in which the previous two numbers must be added together to obtain the next number in the sequence, whereas questions (g) and (h) are ones where the previous three numbers must be added together to obtain the next number in the sequence.



1 mile is approximately 1.60934 kilometres, whereas the golden ratio is approximately 1.61803. Because these two numbers are close to each other, it is possible to use the Fibonacci sequence to estimate distances in miles that are Fibonacci numbers to be in kilometres.

For the challenge, there is always a difference of one between the two products. Sometimes the square of the middle number is one more, and sometimes one less; this alternates as we work through the Fibonacci numbers. It would be possible to model what is going on using a spreadsheet.

In the puzzle, the best option is to wait until the end of the 30 days: the prize on day 30 is £5,368,709.12, and the total amount of prizes from day 1 until day 30 is £10,737,418.23. Again, this situation could be modelled using a spreadsheet, or by using a calculator.


Ysgol y Creuddyn

The Mathematics Department

Frayer Model

Definition

Facts

Sequence

Examples

Nonexamples

@mathemateg

Page 13

A Frayer model presents information about a particular term, here the term 'Sequence'. Several further examples can be found on the website https://www.frayer-model.co.uk/.

Example responses:

Definition

'A list of numbers or objects in a special order'.
'A sequence is a collection of numbers or pictures with a rule to decide what is the next number or picture in the collection.'

Facts

'In a sequence of numbers, the numbers are recognised as the terms of the sequence. For example, in the sequence 5, 8, 11, 14, ..., the first term is 5, and the third term is 11'.

Examples

'9, 11, 13, 15, 17, 19, ...'
'1000, 500, 250, 125, 62.5, ...'

Nonexamples

'2, 4, 6, 8, 11, 12, 14, 16, ...'
'A, B, C, D, E, F, H, I, J, ...'

If you are not familiar with the idea of a nonexample, please visit the website https://nonexamples.com/compare for more information.

Commutativity

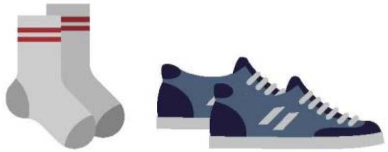
Is the order in which things are done important? If the order is *not* important, then we say that the things are **commutative**. If the order *is* important, then we say that the things are **noncommutative**.

Example

Siwan intends to mix yellow and blue paint to make green paint.

Putting the yellow paint in the empty bucket first and mixing in the blue paint would give the same result as putting the blue paint in the empty bucket first and mixing in the yellow paint. So, this process is **commutative**.

Arwyn gets up in the morning and finishes dressing by putting on a pair of socks and shoes.



Wearing the socks first and then putting on the shoes would give a different result to wearing the shoes first and then putting on the socks. So, this process is **noncommutative**.

Exercise

Are the following situations commutative or noncommutative?

Adding salt and vinegar to a plate of freshly made chips.

____

Studying for a spelling test and taking the spelling test.

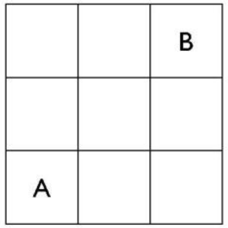
____

This page introduces the idea of commutativity, where we discuss whether the order in which things are done is important. In formal algebra, commutativity is written as $a + b = b + a$, but you do not need to refer to this equation in this work. Discuss with the learners the situations that are described, debating whether the results each time are the same (commutative) or different (noncommutative).

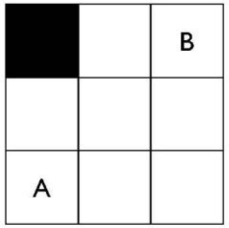
Shaking a bottle of water and opening it.

Shaking a bottle of pop and opening it.

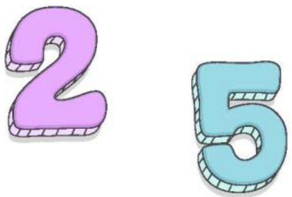
Moving 2 steps right and moving 2 steps up when moving from A to B.



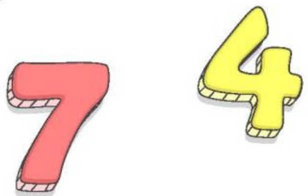
Moving 2 steps right and moving 2 steps up when moving from A to B.



Adding two numbers together.



Subtracting two numbers from each other.



Can you think of more examples of commutativity or noncommutativity?

The purpose of this exercise is to help learners understand that the addition of numbers is commutative (the order in which two numbers are added together is unimportant), whereas the subtraction of numbers is noncommutative (it does matter in which order two numbers are subtracted from each other). Spend some time investigating this, if required, considering examples such as 5 + 3 and 3 + 5, or 5 – 3 and 3 – 5. (Use double sided counters to help with the modelling.) Once the learners are happy that addition is commutative, and that subtraction is noncommutative, move on.

Further examples of commutativity and noncommutativity:

- Making orange squash by mixing water with orange cordial.
- Brushing your teeth in the morning and eating toast with strawberry jam.
- Lighting the candles on a birthday cake and blowing on the candles.
- Walking three steps forwards and walking three steps backwards.
- Washing your hair and drying your hair.
- Blowing air into a balloon and putting a pin into the balloon.
- Measuring heart rate and running 100 m.

Addition is commutative, whilst **subtraction is noncommutative**.

Example

$6 + 4 \quad = \quad 4 + 6$	$6 - 4 \quad \neq \quad 4 - 6$

Exercise

Write = or ≠ in the middle.

$8 + 3$	$3 + 8$	$8 - 3$	$3 - 8$
$3 + 8$	$8 + 3$	$3 - 8$	$8 - 3$
$7 - 5$	$5 - 7$	$7 + 5$	$5 + 7$
$9 + 4$	$4 + 9$	$9 - 4$	$4 - 9$
$2.5 + 3.5$	$3.5 + 2.5$	$3.4 - 0.6$	$0.6 - 3.4$
$7 + 0$	$0 + 7$	$7 - 0$	$0 - 7$
$\frac{3}{4} - \frac{2}{3}$	$\frac{2}{3} - \frac{3}{4}$	$\frac{7}{12} + \frac{5}{12}$	$\frac{5}{12} + \frac{7}{12}$

Take care with these next ones!

$4 + 5$	$5 + 6$	$9 - 3$	$3 + 3$
$3 + 4 + 5$	$5 + 4 + 3$	$10 - 3$	$3 - 4$
$-2 + 4$	$4 + -2$	$-2 - 4$	$4 - -2$
$5 + 2 - 3$	$2 + 5 - 3$	$5 + 2 - 3$	$5 + 3 - 2$

This exercise develops fluency in using commutativity.

- The first set of questions concentrates exclusively on reversing the order of the two numbers in the sum.
- The equality of the expressions needs to be considered in the second set of questions. Notice that the questions involving three numbers have been carefully chosen so that it does not matter in which order the sums are completed.

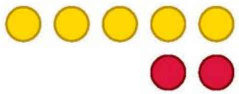
Subtraction as addition: using the additive inverse

Let us consider the subtraction sum 5 – 2. We can think of this sum as starting with five yellow counters

and then subtracting two counters to leave three counters.

Here is a different way of thinking about the sum. Instead of subtracting two, we can add the **additive inverse** of two, namely –2. So, instead of writing 5 – 2, we write 5 + –2. We can think of this sum as starting with five yellow counters

and then adding two red counters (to represent –2).



There are two zero pairs here, leaving us with three yellow counters as before.

Why do we want to use this new way of writing down the sum? Well, the first sum is noncommutative, as 5 – 2 is different to 2 – 5. (What are the two different answers here?) The second sum however **is** commutative, as 5 + –2 is the same as –2 + 5. This will help us answer sums like –2 + 5 on the next page.

Exercise

Write the following subtraction sums as addition sums, using the additive inverse.

(a) 6 – 2	(b) 7 – 2	(c) 10 – 2
(d) 7 – 5	(e) 10 – 3	(f) 14 – 8
(g) 5 – 3	(h) 11 – 4	(i) 18 – 6
(j) 2.7 – 1.2	(k) $\frac{7}{8} - \frac{3}{8}$	(l) 27% – 4%

This page introduces the idea of the **additive inverse**, which allows any subtraction sum to be written as an addition sum.

The additive inverse of 1 is –1.
The additive inverse of 2 is –2.
The additive inverse of 3 is –3.
And so on...

Adding a number to its additive inverse always gives an answer of zero. For example,

1 + –1 = 0
2 + –2 = 0
3 + –3 = 0

The additive inverse will be used further on in this chapter to decide on the order of operations for addition and subtraction sums.

Take care when pronouncing the sum 5 + –2. The correct pronunciation is 'five add negative two', not 'five add minus two'. (See the workbook Numbers: Diving Deeper for more information, or the document https://www.cambridgemaths.org/Images/espresso_15_introducing_negative_numbers.pdf.)

Exercise

Write the following addition sums as subtraction sums.

(a) 5 + –2 (b) 5 + –3 (c) 5 + –4

(d) 6 + –2 (e) 8 + –1 (f) 9 + –3

(g) 7 + –5 (h) 10 + –4 (i) 13 + –4

Example

How do we find the answer to the sum –2 + 5?

Method A

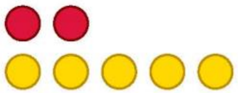
Step 1 We use the commutativity of addition to write the sum as 5 + –2.

Step 2 Instead of adding negative two (which is the additive inverse of 2), we can subtract 2, changing the sum to be 5 – 2. So, the answer is 3.

Method B

We use double sided counters to model: we start with two red counters (to represent –2)

and then add 5 yellow counters:



There are two zero pairs here, leaving us with three yellow counters.

Exercise

Use a method of your choice to answer the following sums.

(a) –2 + 6 (b) –2 + 7 (c) –2 + 4

(d) –3 + 7 (e) –5 + 8 (f) –6 + 9

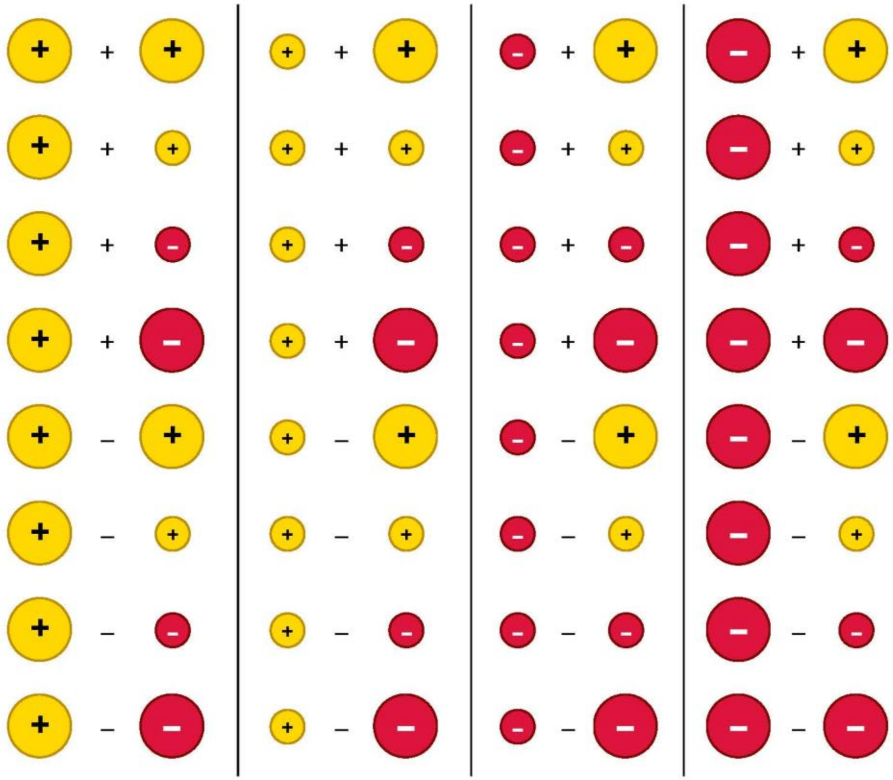
(g) –2 + 9 (h) –4 + 5 (i) –3 + 12

(j) –7 + 10 (k) –8 + 13 (l) –10 + 15

The exercise at the bottom of the page introduces a set of addition sums where we cross zero on the number line to go from the negative side to the positive side. Two methods for tacking these sums are given; "Method C" of using the number line to help would also be acceptable. (For –2 + 5, using the number line to help would mean locating –2 on the number line, and then moving 5 units in the positive direction to reach 3.)

Exercise

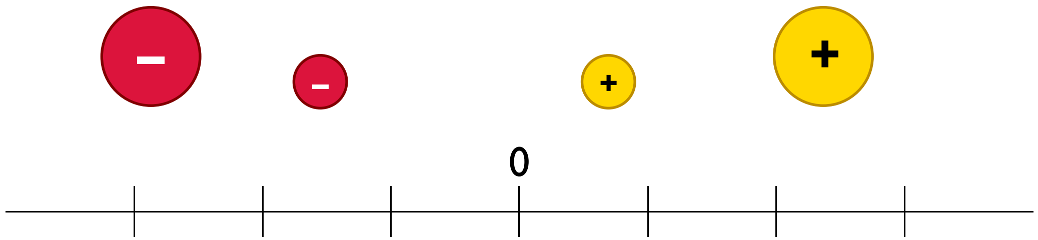
Discuss the meaning of the following sums.



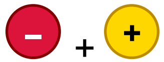
Space for calculations:

This task hopefully leads to some natural discussions amongst the learners.

Relative locations of the symbols on a number line:



The discussions should decide on the sign of each of the answers, in turn. Is the sign always positive, always negative, or is it impossible to conclude what the sign should be? Perhaps also some of the answers are zero, having decided that similar symbols have the same magnitude?

An example of possible sums for  :

–3 + 5 = 3	a positive answer
–3 + 3 = 0	zero
–3 + 1 = –2	a negative answer

So, it is impossible to conclude what the sign of the answer should be.

Acknowledgement: this task is based upon the work of Don Steward (https://donsteward.blogspot.com/2011/03/addingsubtracting-directed-numbers.html) and Chris McGrane (https://startingpointsmaths.com/2020/06/01/alternative-representation-of-integers/).

Exercise

(a) 4 + 2 = (b) 4 – 2 = (c) 4 + –2 =

(d) –4 + 2 = (e) –4 – 2 = (f) –4 + –2 =

(g) –2 + 4 = (h) –2 – 4 = (i) –2 + –4 =

(j) 4 – –2 = (k) 2 – –4 = (l) 2 – 4 =

Exercise

Fill in the blanks in the following tables. (Blue add green, then blue multiply by green.)

+	–5	–4	–3	–2	–1	0	1	2	3	4	5
–5											
–4											
–3											
–2											
–1											
0											
1											
2											
3											
4											
5											

–	–5	–4	–3	–2	–1	0	1	2	3	4	5
–5											
–4											
–3											
–2											
–1											
0											
1											
2											
3											
4											
5											

This exercise develops fluency in adding and subtracting directed numbers.

- The first exercise considers different combinations of the numbers 2 and 4. All the answers are either –6, –2, 2 or 6. (What about grouping the questions into ones that give these answers?)
- Patterns in the two tables should be recognised and used when filling in the blanks.

What is the missing number?

Example

$2-8=\boxed{-6}$	$8+\boxed{-2}=6$	$4-2=\boxed{5}-3$

Exercise

$8+3=\square$	$8-3=\square$	$8+-3=\square$
$3+8=\square$	$3-8=\square$	$-3+8=\square$
$-8+3=\square$	$-8-3=\square$	$-8+-3=\square$
$8+\square=11$	$8+\square=5$	$8+\square=-3$
$4+\square=6$	$5+\square=6$	$6+\square=6$
$7+\square=6$	$8+\square=6$	$15+\square=6$
$\square+6=11$	$\square+6=4$	$\square+6=-4$
$\square-4=6$	$\square-4=-1$	$\square-4=-6$
$\square+-2=2$	$\square+-2=-5$	$\square--2=7$
$3+4=2+\square$	$3+4=\square+5$	$3+\square=2+5$
$9-2=10-\square$	$9-3=\square-5$	$9-\square=12-7$
$2-9=1-\square$	$2-9=\square-7$	$2-9=\square+2$
$\square+3=-1+6$	$\square+2=-4+10$	$\square-1=-3+11$
$-4+5=7-\square$	$-4+\square=8-6$	$-4+7=\square+8$
$10-\square=1+5$	$-10-4=1-\square$	$-10+4=\square+2$
$8+-3=2+\square$	$8+-3=\square-4$	$8+-3=\square+7$
$1.2+1.5=2.4+\square$	$1.2+1.5=\square+1.1$	$1.2+\square=2+0.8$
$\square-0.5=7.3+0.6$	$0.4-0.6=0.7-\square$	$2-0.4=1+\square$
$2-\frac{3}{4}=1+\square$	$2-\frac{1}{4}=1+\square$	$2-\square=1+\frac{1}{2}$

To complete this exercise, conceptual understanding of the equal sign is needed. Watch out for learners adding (or subtracting) all the numbers in a question to reach their answer. Model by using double sided counters if learners prove difficulty with a particular question.

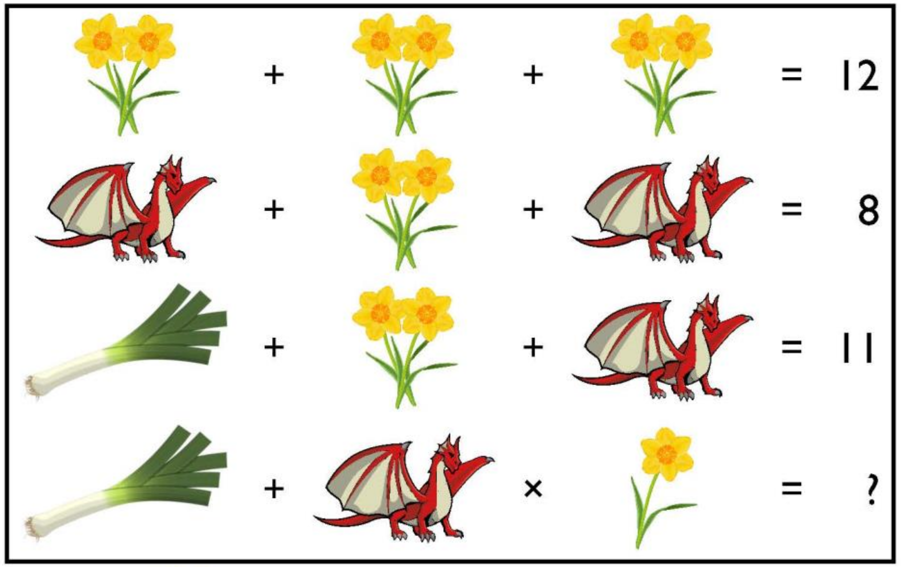
Aspects of this exercise also touch on (and develop fluency in) the following description of learning for number (progression step 2):

CALCULATION

- I have explored additive relationships, using a range of representations. **I can add and subtract whole numbers, using a variety of written and mental methods**.

Order of Operations

What is the answer to the following puzzle?



The flowers  are worth _____

The dragon  is worth _____

The leek  is worth _____

The sum in the final line is _______________

In the final line, if we do the addition sum first, the answer is _____.

In the final line, if we do the multiplication sum first, the answer is _____.

Which answer is correct?

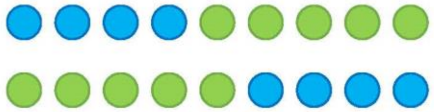
Reasoning is required here to decide on the value of each symbol in order to answer the puzzle. Watch out for the twist in the final row: one flower is shown, not two (so the sum for the final row is $5 + 2 \times 2$). I wonder if similar puzzles exist on the internet?...

We have previously discussed that addition is commutative. So, for example,

$$4 + 5 = 5 + 4.$$

4 + 5

5 + 4



By writing 2 × 2 instead of 4 on both sides of the equation, we arrive at

$$2 \times 2 + 5 = 5 + 2 \times 2.$$

By working from left to right, calculate the answer to the sum on the left side of the equation, and the answer to the sum on the right side of the equation.

Left side: $2 \times 2 + 5$ Right side: $5 + 2 \times 2$

Because the two answers above are different, it looks like we have broken the equation! To fix this, we must decide that **multiplication takes priority over addition** in a sum like $5 + 2 \times 2$. This way, we must calculate 2×2 to begin with, to obtain 4, and then calculate $5 + 4$ to obtain 9. This agrees with the answer on the left side, and with the number of counters shown in each row of the diagram above. Therefore, 9 is the correct answer to the puzzle!

RULE: Given a choice between an addition and a multiplication sum in mathematics, the **multiplication** sum must be done first.

Giving priority to multiplication over addition ensures that we **do not break the commutativity of addition**.

Exercise

(a) $5 + 3 \times 2$ (b) $5 \times 3 + 2$ (c) $5 + 2 \times 3$

(d) $9 + 2 \times 5$ (e) $2 + 9 \times 5$ (f) $2 + 5 \times 9$

(g) $12 + 3 \times 3$ (h) $3 + 12 \times 3$ (i) $3 + 3 \times 12$

(j) $1 + 2 \times 3 + 4$ (k) $1 \times 2 + 3 \times 4$ (l) $1 + 2 + 3 \times 4$

(m) $1 \times 2 + 3 + 4$ (n) $1 \times 2 \times 3 \times 4$ (o) $1 \times 2 \times 3 + 4$

This page explains why multiplication takes priority over addition in mathematics.

- We start by accepting that the addition of numbers is commutative.
- Second, we must accept that 2 × 2 is equal to 4.
- If the sum 5 + 2 × 2 is to have an answer of 9, then we must choose multiplication to take priority over addition. (Otherwise, the answer to the sum would be 14, which is different to the intention at the top of the page.)

We should note here that the order of operations in mathematics is just a *convention*. Addition could have been chosen to take priority over multiplication, but then brackets would have had to be introduced in the sum 5 + 2 × 2, changing it to be 5 + (2 × 2). Further information about this, and on the order of operations in general, can be found on the blog https://www.mathemateg.com/mod/page/view.php?id=4162.

Exercise

Choose three **different** numbers out of the five numbers given, to complete the sums in different ways.

(a) 2, 3, 4, 5, 6
□ × □ + □ = 14
6 different answers

(b) 2, 3, 4, 5, 6
□ × □ + □ = 18
2 different answers

(c) 2, 3, 4, 5, 6
□ × □ + □ = 22
4 different answers

(d) 2, 3, 4, 5, 6
□ + □ × □ = 23
4 different answers

(e) 2, 3, 4, 5, 6
□ + □ × □ = 27
2 different answers

(f) 2, 3, 4, 5, 6
□ + □ × □ = 34
2 different answers

(g) 2, 3, 4, 5, 7
□ × □ + □ = 13
6 different answers

(h) 2, 3, 4, 5, 7
□ + □ × □ = 19
6 different answers

(i) 2, 3, 4, 5, 7
□ × □ + □ = 33
2 different answers

(j) 3, 4, 6, 7, 9
□ + □ × □ = 25
4 different answers

(k) 3, 4, 6, 7, 9
□ × □ + □ = 27
6 different answers

(l) 3, 4, 6, 7, 9
□ + □ × □ = 43
2 different answers

(m) 4, 5, 7, 8, 9
□ × □ + □ = 27
2 different answers

(n) 4, 5, 7, 8, 9
□ + □ × □ = 41
4 different answers

(o) 4, 5, 7, 8, 9
□ × □ + □ = 49
4 different answers

This page provides **purposeful practice** on multiplying and adding numbers together.

- Remember to consider the order of operations in questions where addition appears first.
- The answers appear in pairs. For example, on finding the answer 3 × 4 + 2 in question (a), by reversing the order of the numbers in the multiplication sum (using commutativity), another answer 4 × 3 + 2 appears.

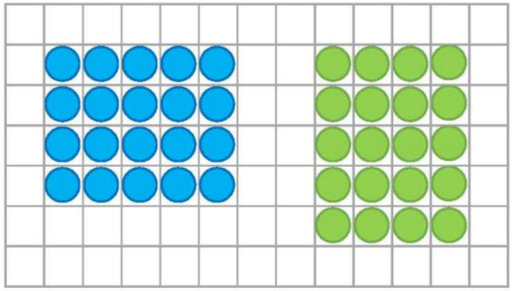
Acknowledgement: this exercise is based upon one from the late Don Steward's web site: https://donsteward.blogspot.com/2017/11/multiplication-problems.html

Answers:

(a) 2 × 4 + 6, 4 × 2 + 6, 2 × 5 + 4, 5 × 2 + 4, 3 × 4 + 2, 4 × 3 + 2.
(b) 3 × 4 + 6, 4 × 3 + 6.
(c) 3 × 6 + 4, 6 × 3 + 4, 4 × 5 + 2, 5 × 4 + 2.
(d) 3 + 4 × 5, 3 + 5 × 4, 5 + 3 × 6, 5 + 6 × 3.
(e) 3 + 4 × 6, 3 + 6 × 4
(f) 4 + 5 × 6, 4 + 6 × 5
(g) 2 × 3 + 7, 3 × 2 + 7, 2 × 4 + 5, 4 × 2 + 5, 2 × 5 + 3, 5 × 2 + 3.
(h) 4 + 3 × 5, 4 + 5 × 3, 5 + 2 × 7, 5 + 7 × 2, 7 + 3 × 4, 7 + 4 × 3.
(i) 4 × 7 + 5, 7 × 4 + 5.
(j) 4 + 3 × 7, 4 + 7 × 3, 7 + 3 × 6, 7 + 6 × 3.
(k) 3 × 6 + 9, 6 × 3 + 9, 3 × 7 + 6, 7 × 3 + 6, 4 × 6 + 3, 6 × 4 + 3.
(l) 7 + 4 × 9, 7 + 9 × 4.
(m) 4 × 5 + 7, 5 × 4 + 7.
(n) 5 + 4 × 9, 5 + 9 × 4, 9 + 4 × 8, 9 + 8 × 4.
(o) 5 × 8 + 9, 8 × 5 + 9, 5 × 9 + 4, 9 × 5 + 4.

Multiplication

Is multiplication commutative? For example, is the answer to 4 × 5 the same as, or different to, the answer to 5 × 4?



We see from the above diagrams that multiplication is commutative, because the answers to 4 × 5 and 5 × 4 are both 20. We can therefore write the equation

$$4 \times 5 = 5 \times 4.$$

Agreeing that 5 is the answer to the sum 2 + 3, it is reasonable to write 2 + 3 instead of 5 in the above equation:

$$4 \times 2 + 3 = 2 + 3 \times 4.$$

Remembering that multiplication takes priority over addition, what is the answer to the sums on both sides of this equation?

Left side: 4 × 2 + 3

Right side: 2 + 3 × 4

None of the answers above are equal to 20, so we have a problem! We fix this by introducing a pair of brackets into each side of the equation, writing

$$4 \times (2 + 3) = (2 + 3) \times 4.$$

Each side does now give 20, if we insist that any sum in brackets takes priority over any other addition or multiplication sums.

This page explains why brackets are used in mathematical sums to prioritise specific parts of the sum.

- We start by accepting that the multiplication of numbers is commutative.
- Second, we must accept that 2 + 3 is equal to 5.
- If the sums 4 × 2 + 3 and 2 + 3 × 4 are to have an answer of 20, then brackets must be introduced, to give 4 × (2 + 3) and (2 + 3) × 4, and sums in brackets need to be given priority. (Otherwise, the sums would have answers of 11 and 14, which is different to the intention at the top of the page.)

RULE: In any sum including brackets, the sum in the brackets must be completed before doing anything else.

Giving priority to sums in brackets ensures that we **do not break the commutativity of multiplication**.

Exercise

(a) 5 × (4 + 2) (b) (4 + 2) × 5 (c) 4 × (5 + 2)
(d) (5 + 2) × 4 (e) 5 + (4 + 2) (f) 2 × (4 + 5)
(g) (5 + 4) × 2 (h) 2 × (5 × 4) (i) 5 × (2 + 4)

Challenge!

Repeat the above exercise, but this time ignoring all the brackets (so that question (a), for example, changes to 5 × 4 + 2). Two of the answers stay the same – which ones?

Exercise

(a) (4 + 2) × (3 + 5) (b) 10 × (2 + 6) (c) (5 × 2) + (4 × 3)
(d) (4 × 2) + (3 × 5) (e) 10 + (2 × 6) (f) (5 + 2) × (4 + 3)
(g) 4 × (2 + 3) × 5 (h) (10 + 2) × 6 (i) 5 × (2 + 4) × 3
(j) 4 × 2 + 3 × 5 (k) 10 × 2 + 6 (l) 5 × 2 + 4 × 3
(m) 4 + (2 × 3 + 5) (n) 10 + 2 × 6 (o) (5 × 2 + 4) × 3

So far, we have decided on the following order of operations.

Where do subtraction and division sums fit into this puzzle?

In the challenge, the answers to questions (e) and (h) stay the same, because it does not matter in which order the two sums in each question are completed. This hints at work to come in the final chapter, where we consider the associativity of addition and multiplication.

The scroll introduces our first list of how to prioritise operations in mathematics. At this current time, only brackets, multiplication and addition appear, as these are the only operations considered thus far in this section on the order of operations.
A final version will appear further on in this chapter.

Addition and subtraction sums

Let us consider the following four sums.

8 + 4 + 2 8 + 4 – 2 8 – 4 + 2 8 – 4 – 2

What do you notice about the sums?

Complete the table below to answer the sums in two different ways: first by doing the red sum to begin with, and then by doing the blue sum to begin with. Leave the final column of the table empty, for now.

Sum	Answer when starting with the red sum	Answer when starting with the blue sum	Which sum needs to be done first?
8 + 4 + 2			Does not matter
8 + 4 – 2			Does not matter
8 – 4 + 2			
8 – 4 – 2			

The answers in the first two rows are the same, but the answers in the final two rows are different. Which answer is correct each time?

Because it is possible to write every subtraction sum as an addition sum, using the additive inverse, we can re-write the final two sums as addition sums like this:

8 + –4 + 2 8 + –4 + –2

Answer these sums in two ways:

Sum	Answer when starting with the red sum	Answer when starting with the blue sum
8 + –4 + 2		
8 + –4 + –2		

The answers in each row of the above table are the same, so we can use them to complete the final column of the first table, and state which sum needed to be done first in 8 – 4 + 2 and 8 – 4 – 2.

This page discusses why addition and subtraction sums must be completed from left to right in mathematics. For this purpose, we consider the four sums shown at the top of the page, where the numbers 8, 4 and 2 are used with every possible combination of addition and subtraction placed between them.

+, + +, – –, + –, –

With the first two sums in the first table, it does not matter in which order the sums are completed. The order is important however in the final two sums, so we re-write these sums using the additive inverse, to decide which answer is correct.

For 8 – 4 + 2, if we do the red sum first, the answer is 6, and if we do the blue sum first, the answer is 2. By re-writing the sum as 8 + –4 + 2, we see that an answer of 6 is calculated no matter which of the red or blue sums is completed first. This means that the red sum (the subtraction sum) is the one that must be completed first in the sum 8 – 4 + 2.

For 8 – 4 – 2, if we do the red sum first, the answer is 2, and if we do the blue sum first, the answer is 6. By re-writing the sum as 8 + –4 + –2, we see that an answer of 2 is calculated no matter which of the red or blue sums is completed first. This means that the red sum (the first subtraction sum) is the one that must be completed first in the sum 8 – 4 – 2.

All of the above means that working from left to right (completing the red sum first) ensures that *any* combination of addition or subtraction sums is completed correctly. This is the rule that is introduced on the next page.

In each of the four examples on the previous page, starting with the sum appearing **first on the left** (the red sum) leads to obtaining a correct answer. This in turn enables us to establish the following rule.

RULE: Given two or more addition or subtraction sums, we must complete them by working from left to right.

Exercise

(a) 9 – 6 + 3 (b) 9 + 6 – 3 (c) 9 – 6 – 3

(d) 9 + 6 + 3 (e) 6 – 3 + 9 (f) 6 + 3 – 9

(g) 6 + 9 – 3 (h) 6 – 9 + 3 (i) 3 – 6 + 9

(j) 10 – 3 + 5 (k) 10 + 3 – 5 (l) 10 – 5 + 3

(m) 12 – 7 + 1 (n) 7 – 12 + 1 (o) 1 + 7 – 12

Exercise

(a) 2 × 3 + 4 – 5 (b) 2 + 3 × 4 – 5 (c) 2 + 3 + 4 × 5

(d) 2 – 3 + 4 × 5 (e) 2 + 3 – 4 × 5 (f) 2 – 3 × 4 + 5

(g) 6 × 3 – 2 + 7 (h) 3 × 2 – 7 + 6 (i) 2 × 7 – 6 + 3

(j) 6 + 3 × 2 – 7 (k) 3 + 2 × 7 – 6 (l) 2 + 7 × 6 – 3

(m) 6 – 3 + 2 × 7 (n) 3 – 2 + 7 × 6 (o) 2 – 7 + 6 × 3

Exercise

Complete the following equations by writing +, – or × in the boxes.

(a) 3 □ 4 □ 5 = 12 (b) 3 □ 4 □ 5 = 17 (c) 3 □ 4 □ 5 = 23

(d) 3 □ 4 □ 5 = 60 (e) 3 □ 4 □ 5 = 7 (f) 3 □ 4 □ 5 = 2

(g) 3 □ 4 □ 5 = –6 (h) 3 □ 4 □ 5 = –17 (i) 3 □ 4 □ 5 = 4

Challenge!

Use the numbers 2, 3 and 4 and the operations +, – and × to create sums with as many different answers as possible.

Technical note:

Often in materials on the order of operations, addition and subtraction are considered as being "on the same level". This is not strictly true, as subtraction does take priority over addition. We saw this in the third sum on the previous page, 8 – 4 + 2. Here, if we complete the subtraction sum first, the correct answer is obtained, but if we complete the addition sum first, an incorrect answer appears. This means that addition and subtraction are not "on the same level" as there is not a free choice between completing the addition sum first or the subtraction sum first.

Why then does the rule not state "give priority to subtraction over addition"? The problem comes when we have two subtraction sums in the same sum, as in the sum 8 – 4 – 2. Here, we need to start by completing the first subtraction sum on the left (the red sum). It is therefore easier to talk about working from left to right, rather than prioritising subtraction over addition, and then having to deal with sums involving two subtraction sums. It is worth noting however that prioritising subtraction over addition would be a valid strategy. For example, consider the second sum on the previous page, 8 + 4 – 2. Here, we could start by completing the subtraction sum first (leading to 8 + 2 = 10), as well as following the usual strategy of working from left to right (12 – 2 = 10).

Division

Is dividing two numbers commutative or noncommutative? For example, is the answer to the sum 4 ÷ 2 the same as, or different to, the answer to 2 ÷ 4?

Exercise

Write = or ≠ in the middle.

8 × 4 4 × 8	8 ÷ 4 4 ÷ 8
4 × 8 8 × 4	4 ÷ 8 8 ÷ 4
12 ÷ 3 3 ÷ 12	12 × 3 3 × 12
5 × 2 2 × 5	5 ÷ 2 2 ÷ 5

Take care with these next ones!

5 × 4 4 + 5	9 ÷ 3 12 ÷ 4
10 ÷ 2 5 × 1	15 ÷ 3 30 ÷ 6
4 × 6 6 ÷ 4	2 × 3 × 4 4 × 3 × 2

Division as multiplication: using the multiplicative inverse

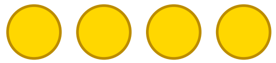
Let us consider the division sum 6 ÷ 2. We can think of this sum as starting with six yellow counters

and dividing them into two equal parts containing three counters each.

So, 6 ÷ 2 = 3.

The division of two numbers is noncommutative. We can show this by considering the difference between the answers to the sums 4 ÷ 2 and 2 ÷ 4.

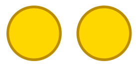
The first sum can be modelled by starting with four yellow counters



and dividing them into two equal parts containing two counters each.

So, 4 ÷ 2 = 2.

With the second sum, we start with two yellow counters



and divide them into four equal parts, shown by arranging four half counters separately in a row.

So, 2 ÷ 4 = $\frac{1}{2}$.

Because these two answers are different, it means that division is noncommutative.

Here is a different way of thinking about the sum. Instead of dividing by two, we can multiply by the **multiplicative inverse** of two, namely $\frac{1}{2}$. So, instead of writing $6 \div 2$, we write $6 \times \frac{1}{2}$. We can think of this sum as arranging six half counters in a row

which come together to form three whole counters.

So, as before, $6 \div 2 = 3$.

Exercise

Write the following division sums as multiplication sums, using the multiplicative inverse.

(a) $8 \div 2$	(b) $8 \div 4$	(c) $6 \div 3$
(d) $12 \div 6$	(e) $12 \div 3$	(f) $12 \div 12$
(g) $21 \div 7$	(h) $24 \div 8$	(i) $45 \div 9$

Exercise

Write the following multiplication sums as division sums.

(a) $10 \times \frac{1}{2}$	(b) $6 \times \frac{1}{3}$	(c) $20 \times \frac{1}{4}$
(d) $10 \times \frac{1}{5}$	(e) $6 \times \frac{1}{6}$	(f) $20 \times \frac{1}{10}$
(g) $42 \times \frac{1}{6}$	(h) $44 \times \frac{1}{11}$	(i) $28 \times \frac{1}{7}$

Exercise

Calculate the answers to all the sums in the above two exercises.

Challenge!

Huw weighs six times as much as his cat, Jess.
Holding his cat in his hands, Huw stands on a weighing scale.
The scales show 42 kg.
What is Huw's weight?

This page introduces the idea of the **multiplicative inverse**, which allows any division sum to be written as a multiplication sum.

The multiplicative inverse of 1 is $\frac{1}{1}$, or 1.

The multiplicative inverse of 2 is $\frac{1}{2}$.

The multiplicative inverse of 3 is $\frac{1}{3}$.

And so on...

Multiplying a number by its multiplicative inverse always gives an answer of one. For example,

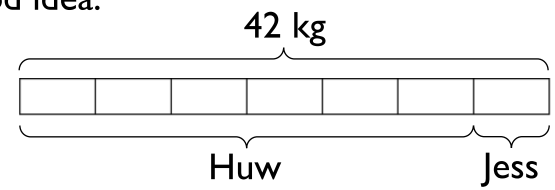
$1 \times 1 = 1$

$2 \times \frac{1}{2} = 1$

$3 \times \frac{1}{3} = 1$

The multiplicative inverse will be used further on in this chapter to decide on the order of operations for multiplication and division sums.

In the challenge, drawing a bar model to illustrate the situation would be a good idea.



Each part of the bar model is worth $42 \text{ kg} \div 7 = 6 \text{ kg}$, so that Huw's weight is $6 \text{ kg} \times 6 = 36 \text{ kg}$.

Multiplication and division sums

Let us consider the following four sums.

$8 \times 4 \times 2$ $8 \times 4 \div 2$ $8 \div 4 \times 2$ $8 \div 4 \div 2$

What do you notice about the sums?

Complete the table below to answer the sums in two different ways: first by doing the red sum to begin with, and then by doing the blue sum to begin with. Leave the final column of the table empty, for now.

Sum	Answer when starting with the red sum	Answer when starting with the blue sum	Which sum needs to be done first?
$8 \times 4 \times 2$			Does not matter
$8 \times 4 \div 2$			Does not matter
$8 \div 4 \times 2$			
$8 \div 4 \div 2$			

The answers in the first two rows are the same, but the answers in the final two rows are different. Which answer is correct each time?

Because it is possible to write every division sum as a multiplication sum, using the multiplicative inverse, we can re-write the final two sums as multiplication sums like this:

$8 \times \frac{1}{4} \times 2$ $8 \times \frac{1}{4} \times \frac{1}{2}$

Answer these sums in two ways: (Hint: $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$.)

Sum	Answer when starting with the red sum	Answer when starting with the blue sum
$8 \times \frac{1}{4} \times 2$		
$8 \times \frac{1}{4} \times \frac{1}{2}$		

The answers in each row of the above table are the same, so we can use them to complete the final column of the first table, and state which sum needed to be done first in $8 \div 4 \times 2$ and $8 \div 4 \div 2$.

This page discusses why multiplication and division sums must be completed from left to right in mathematics. For this purpose, we consider the four sums shown at the top of the page, where the numbers 8, 4 and 2 are used with every possible combination of multiplication and division placed between them.

$\times, \times$ $\times, \div$ $\div, \times$ $\div, \div$

With the first two sums in the first table, it does not matter in which order the sums are completed. The order is important however in the final two sums, so we re-write these sums using the multiplicative inverse, to decide which answer is correct.

For $8 \div 4 \times 2$, if we do the red sum first, the answer is 4, and if we do the blue sum first, the answer is 1. By re-writing the sum as $8 \times \frac{1}{4} \times 2$, we see that an answer of 4 is calculated no matter which of the red or blue sums is completed first. This means that the red sum (the division sum) is the one that must be completed first in the sum $8 \div 4 \times 2$.

For $8 \div 4 \div 2$, if we do the red sum first, the answer is 1, and if we do the blue sum first, the answer is 4. By re-writing the sum as $8 \times \frac{1}{4} \times \frac{1}{2}$, we see that an answer of 1 is calculated no matter which of the red or blue sums is completed first. This means that the red sum (the first division sum) is the one that must be completed first in the sum $8 \div 4 \div 2$.

All of the above means that working from left to right (completing the red sum first) ensures that *any* combination of multiplication or division sums is completed correctly. This is the rule that is introduced on the next page.

In each of the four examples on the previous page, starting with the sum appearing **first on the left** (the red sum) leads to obtaining a correct answer. This in turn enables us to establish the following rule.

RULE: Given two or more multiplication or division sums, we must complete them by working from left to right.

Exercise

(a) 6 × 2 ÷ 3	(b) 6 ÷ 2 × 3	(c) 6 ÷ 2 ÷ 3
(d) 6 × 2 × 3	(e) 2 × 6 ÷ 3	(f) 6 ÷ 3 × 2
(g) 12 ÷ 2 × 3	(h) 12 ÷ 3 × 2	(i) 12 ÷ 3 ÷ 2
(j) 5 × 6 ÷ 2	(k) 18 ÷ 3 × 6	(l) 1 × 2 × 3
(m) 8 × 3 ÷ 6	(n) 40 ÷ 5 ÷ 4	(o) 100 ÷ 10 × 10

Here is a summary of the order of operations in mathematics.

Sums in brackets

Multiplication and division, working from left to right

Addition and subtraction, working from left to right

Exercise

(a) 4 + 6 × 2	(b) (4 + 6) × 2	(c) 4 + 6 – 2
(d) 4 + (6 – 2)	(e) 4 + 6 ÷ 2	(f) (4 + 6) ÷ 2
(g) 12 + (3 × 2) – 9	(h) (12 + 3) × 2 – 9	(i) 12 ÷ (3 × 2) – 9
(j) 12 + 3 × 2 – 9	(k) (12 + 3 × 2) ÷ 9	(l) (12 – 3) × (2 + 9)
(m) 15 + 15 ÷ 3	(n) (15 + 15) ÷ 3	(o) 15 ÷ 15 + 3
(p) 15 – 15 ÷ 3	(q) (15 – 15) ÷ 3	(r) 15 + 15 – 3
(s) 8 + (2 × (2 + 3))	(t) (8 + 2) × (2 + 3)	(u) (8 + (2 × 2)) + 3

You have no doubt noticed the similarity between the previous page and the earlier material on addition and subtraction sums. This was intentional (with the numbers 8, 4 and 2 chosen carefully) to be able to see the connections between the two sections. The previous technical note could also be modified to note that multiplication and division are not "on the same level" (as is often stated), but that division takes priority over multiplication.

The scroll on the page presents our final list (in this workbook) for the order of operations in mathematics. Notice that we do not use an acronym (such as BODMAS) to describe the order, preferring instead to develop conceptual understanding of the order. See the following blog for a further discussion on using acronyms: https://jemmaths.wordpress.com/2018/12/06/thoughts-on-the-order-of-operations/

A pair of brackets

Here is a mathematical sum without brackets.

$$16 - 2 + 4 - 8$$

What is the answer to the sum?

Now add **one pair of brackets** to the sum in different ways.

How many different answers can be formed?

Exercise

(a) $5 \times 2 + 3$ (b) $3 + 5 \times 2$ (c) $4 + 5 \times 2$

(d) $4 + 5 \times 3$ (e) $4 + (5 \times 3)$ (f) $(4 + 5) \times 3$

(g) $3 \times (4 + 5)$ (h) $3 \times (4 + 6)$ (i) $3 \times 4 + 6$

(j) $3 \times 4 + 6 \times 2$ (k) $6 \times 2 + 3 \times 4$ (l) $6 \times 2 + (3 \times 4)$

(m) $6 \times (2 + 3) \times 4$ (n) $6 + (2 + 3) \times 4$ (o) $(6 + 2 \times 3) \times 4$

Exercise

Add one pair of brackets to the left side of each of the following equations in order to fix them.

(a) $2 + 3 \times 4 = 20$ (b) $10 - 3 + 2 = 5$ (c) $6 + 4 \div 2 = 5$

(d) $9 - 3 \times 2 = 12$ (e) $2 \times 3 + 3 \times 2 = 24$ (f) $2 \times 3 + 3 \times 2 = 18$

(g) $10 - 3 - 2 - 1 = 6$ (h) $10 - 3 - 2 - 1 = 8$ (i) $10 - 3 - 2 - 1 = 10$

Example

(a) $\frac{12}{3} = 4$ (b) $6 + \frac{12}{3} = 6 + 4 = 10$ (c) $\frac{6+12}{3} = \frac{18}{3} = 6$

Exercise

(a) $(4 + 10) \div 2$ (b) $4 + 10 \div 2$ (c) $4 + \frac{10}{2}$

(d) $\frac{10}{2} + 4$ (e) $\frac{10+4}{2}$ (f) $\frac{10}{2} + \frac{4}{2}$

(g) $\frac{10}{2} - 4$ (h) $4 - \frac{10}{2}$ (i) $\frac{10-4}{2}$

These exercises develop fluency in using the order of operations. Some of the questions are based upon exercises found on Craig Barton's website, https://variationtheory.com/category/number/order-of-operations-bidmas/, where the questions have been carefully designed to promote discussions based upon the connection between one question and the next.

(a) $5 \times 2 + 3$

(b) $3 + 5 \times 2$ Using the commutativity of addition

(c) $4 + 5 \times 2$ Changing the 3 to be a 4

(d) $4 + 5 \times 3$ Changing the 2 to be a 3

(e) $4 + (5 \times 3)$ Introducing brackets

(f) $(4 + 5) \times 3$ Changing the location of the brackets

(g) $3 \times (4 + 5)$ Using the commutativity of multiplication

and so on...

The final exercise introduces fraction notation to take the place of division sums. Brackets are implied by some of these questions, for example without using fraction notation $\frac{10+4}{2}$ would need to be written as $(10 + 4) \div 2$.

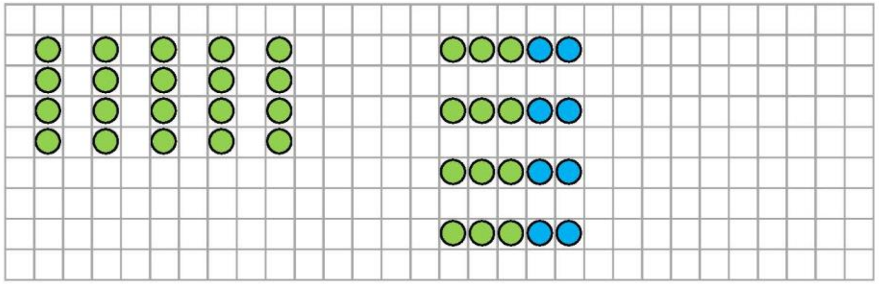
Distributivity

Siwan keeps 3 rabbits and 2 cats as pets.

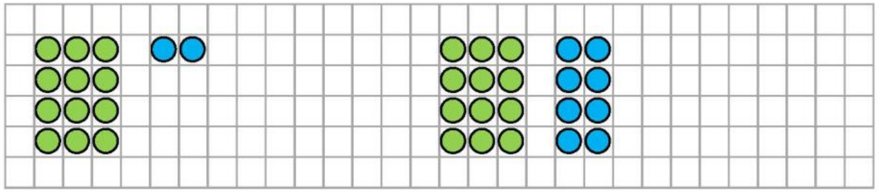
Carwyn asks Siwan: 'How many legs do all of your pets have'?

Siwan uses the following methods to count all the legs. Which method does not work?

[Method 1] 4 + 4 + 4 + 4 + 4 [Method 2] 4 × (3 + 2)



[Method 3] 4 × 3 + 2 [Method 4] 4 × 3 + 4 × 2



For each method that does work, try to explain *how* Siwan went about counting the number of legs using that method.

In formal algebra, the **distributivity** of multiplication over addition is written like this:

$$a \times (b + c) = a \times b + a \times c.$$

(There is no need to discuss this formal algebra with learners.)

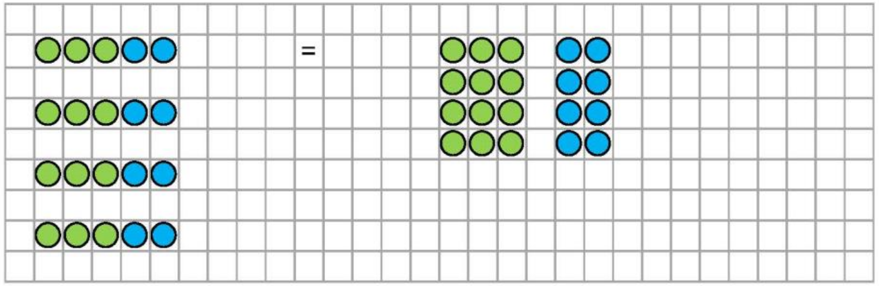
In the pets example, methods 1, 2 and 4 are valid.

- Method 1 treats each pet separately, counting the legs in fours.
- Method 2 counts how many pets there are in total, before multiplying by 4.
- Method 4 treats the rabbits and cats separately, counting the number of legs for the rabbits, the number of legs for the cats, and then adding them together.

Method 3 emphasises the need to include brackets in Method 2, as otherwise the order of operations means that an incorrect answer is obtained.

Let us consider two of the methods that did work from the previous page.

$4 \times (3 + 2)$ $\quad$ $4 \times 3 + 4 \times 2$



In the first method, we multiply 4 by a group of numbers added together in a bracket. In the second method, we multiply 4 by the individual numbers (the ones appearing in the previous bracket) and then add the answers. This is an example of **distributivity** in mathematics. When moving from the first method to the second method, we say that the 4 is **distributed** across the 3 + 2 to give 4 × 3 and 4 × 2. Here are some more examples.

$3 \times (2 + 4) = 3 \times 2 + 3 \times 4$ $\quad$ $5 \times (6 + 9) = 5 \times 6 + 5 \times 9$

$2 \times (7 + 11) = 2 \times 7 + 2 \times 11$ $\quad$ $12 \times (1 + 8) = 12 \times 1 + 12 \times 8$

Exercise

Fill in the blanks below.

(a) $4 \times (2 + 3) = 4 \times \square + 4 \times 3$ $\quad$ (b) $4 \times (2 + 3) = 4 \times 2 + \square \times 3$

(c) $4 \times (2 + \square) = 4 \times 2 + 4 \times 3$ $\quad$ (d) $\square \times (2 + 3) = 4 \times 2 + 4 \times 3$

(e) $5 \times (3 + 7) = 5 \times 3 + 5 \times \square$ $\quad$ (f) $5 \times (\square + 7) = 5 \times 4 + 5 \times 7$

(g) $\square \times (3 + 7) = 6 \times 3 + 6 \times 7$ $\quad$ (h) $6 \times (3 + 8) = \square \times 3 + 6 \times 8$

(i) $10 \times (4 + 9) = \square \times 4 + 10 \times 9$ $\quad$ (j) $8 \times (3 + 5) = 8 \times 3 + 8 \times \square$

(k) $7 \times 3 + 7 \times \square = 7 \times (3 + 4)$ $\quad$ (l) $7 \times 5 + 7 \times 6 = 7 \times (\square + 6)$

(m) $7 \times 7 + 7 \times 8 = 7 \times (\square + 8)$ $\quad$ (n) $17 \times 7 + \square \times 8 = 17 \times (7 + 8)$

(o) $2.3 \times 4 + 2.3 \times 6 = 2.3 \times (\square + 6)$ $\quad$ (p) $8 \times (9.2 + 0.8) = 8 \times \square + 8 \times 0.8$

This page continues the example from the previous page to explain the meaning of distributivity in mathematics.

It should be verified that the two sums 4 × (3 + 2) and 4 × 3 + 4 × 2 have the same answer, either by calculating the answer to the sums directly or by counting the counters.

The exercise encourages learners to see the patterns that are formed when using distributivity.

Using Distributivity

It is possible to use distributivity to split complex sums into simpler sums.

Example

$6 \times 17 = 6 \times (10 + 7)$
$= 6 \times 10 + 6 \times 7$
$= 60 + 42$
$= 102$

$4 \times 148 = 4 \times (100 + 40 + 8)$
$= 4 \times 100 + 4 \times 40 + 4 \times 8$
$= 400 + 160 + 32$
$= 592$

Exercise

Use distributivity to answer the following sums.

(a) 6 × 14 (b) 3 × 14 (c) 3 × 24

(d) 3 × 104 (e) 3 × 140 (f) 3 × 146

(g) 4 × 27 (h) 7 × 32 (i) 9 × 68

(j) 2 × 145 (k) 7 × 145 (l) 7 × 345

(m) 3 × 71 (n) 3 × 701 (o) 3 × 7001

(p) 9 × 18 (q) 9 × 38 (r) 9 × 98

(s) 5 × 64 (t) 5 × 136 (u) 5 × 2016

We can also use distributivity to combine sums in order to simplify them.

Example

$14 \times 6 + 14 \times 4 = 14 \times (6 + 4)$
$= 14 \times 10$
$= 140$

$6 \times 23 + 6 \times 7 = 6 \times (23 + 7)$
$= 6 \times 30$
$= 180$

Exercise

Use distributivity to answer the following sums.

(a) 18 × 7 + 18 × 3 (b) 18 × 8 + 18 × 2 (c) 18 × 98 + 18 × 2

(d) 27 × 6 + 27 × 4 (e) 5 × 13 + 5 × 7 (f) 4 × 34 + 4 × 6

(g) 24 × 9 + 24 × 1 (h) 132 × 7 + 132 × 3 (i) 2 × 178 + 2 × 22

(j) 63 × 2 + 63 × 8 (k) 3 × 41 + 3 × 59 (l) 2.7 × 4 + 2.7 × 6

A calculator should not be used to complete the exercises on this page. Perhaps it would be appropriate, before starting, to discuss the connection between the following two multiplication tables, and similar ones.

1 × 4 = 4	10 × 4 = 40
2 × 4 = 8	20 × 4 = 80
3 × 4 = 12	30 × 4 = 120
4 × 4 = 16	40 × 4 = 160
5 × 4 = 20	50 × 4 = 200
6 × 4 = 24	60 × 4 = 240
7 × 4 = 28	70 × 4 = 280
8 × 4 = 32	80 × 4 = 320
9 × 4 = 36	90 × 4 = 360
10 × 4 = 40	100 × 4 = 400

Distributivity also works with subtraction sums.

Example

3 × 29 – 3 × 23 = 3 × (29 – 23)
= 3 × 6
= 18

43 × 17 – 33 × 17 = (43 – 33) × 17
= 10 × 17
= 170

Exercise

Use distributivity to answer the following sums.

(a) 3 × 27 – 3 × 23 (b) 3 × 37 – 3 × 32 (c) 3 × 37 – 3 × 27

(d) 4 × 54 – 4 × 52 (e) 4 × 54 – 4 × 44 (f) 4 × 54 – 4 × 4

(g) 5 × 38 – 5 × 32 (h) 5 × 48 – 5 × 43 (i) 5 × 71 – 5 × 67

(j) 27 × 3 – 23 × 3 (k) 18 × 4 – 11 × 4 (l) 26 × 5 – 16 × 5

(m) 54 × 6 – 51 × 6 (n) 78 × 7 – 77 × 7 (o) 112 × 8 – 107 × 8

(p) 72 × 9 – 64 × 9 (q) 27 × 11 – 24 × 11 (r) 76 × 12 – 74 × 12

(s) 3 × 123 – 3 × 121 (t) 424 × 5 – 421 × 5 (u) 2 × 145 – 2 × 45

Exercise

Use distributivity to answer the following mixed sums.

(a) 14 × 2 + 14 × 8 (b) 14 × 2 + 14 × 3 + 14 × 5

(c) 14 × 2 + 14 × 3 + 14 × 4 + 14 × 1 (d) 14 × 13 – 14 × 3

(e) 14 × 7 + 14 × 5 – 14 × 2 (f) 14 × 21 + 14 × 5 – 14 × 16

(g) 17 × 3 + 17 × 7 (h) 3 × 17 + 7 × 17

(i) 17 × 7 – 17 × 5 (j) 18 × 17 – 5 × 17 – 3 × 17

(k) 6 × 18 (l) 4 × 29

(m) 16 × 7 (n) 36 × 8

(o) 16 × 7 + 4 × 7 (p) 40 × 8 – 4 × 8

(q) 154 × 3 (r) 416 × 2

(s) 154 × 3 – 54 × 3 (t) 416 × 2 – 2 × 2

Note the subtle difference between the two examples: the first example uses

$$a \times b - a \times c = a \times (b - c)$$

whilst the second example utilises

$$b \times a - c \times a = (b - c) \times a.$$

This is an example of using the commutativity of multiplication:

$$a \times (b - c) = (b - c) \times a.$$

Introducing subtraction sums leads to the possibility of considering different strategies of answering the questions. For example, in question (l) of the final exercise, what is the most efficient way of finding the answer: using

4 × (20 + 9)

or

4 × (30 – 1)?

Associativity

Associative operations

A mathematical **operation** (like addition or multiplication) is **associative** if re-arranging the brackets in a sum, as shown in the following exercise, does not have an effect on the answer to the sum.

Exercise

Answer the following sums.

(a) (8 + 4) + 2 (b) 8 + (4 + 2)

(c) (8 – 4) – 2 (d) 8 – (4 – 2)

(e) (8 × 4) × 2 (f) 8 × (4 × 2)

(g) (8 ÷ 4) ÷ 2 (h) 8 ÷ (4 ÷ 2)

Exercise

Write = or ≠ in the middle.

(8 + 4) + 2	8 + (4 + 2)
(8 – 4) – 2	8 – (4 – 2)
(8 × 4) × 2	8 × (4 × 2)
(8 ÷ 4) ÷ 2	8 ÷ (4 ÷ 2)

Fill in the blanks: Out of addition, subtraction, multiplication and division, the two associative operations are ________________ and ________________, and the two non-associative operations are ________________ and ________________.

With associative operations, it is possible to use their associativity to help us answer sums that contain these operations. For example, in 8 × 4 × 2, is it easier to answer the sum by doing (8 × 4) × 2, or by doing 8 × (4 × 2)?

In formal algebra, a general operation $*$ is **associative** if

$$(a * b) * c = a * (b * c)$$

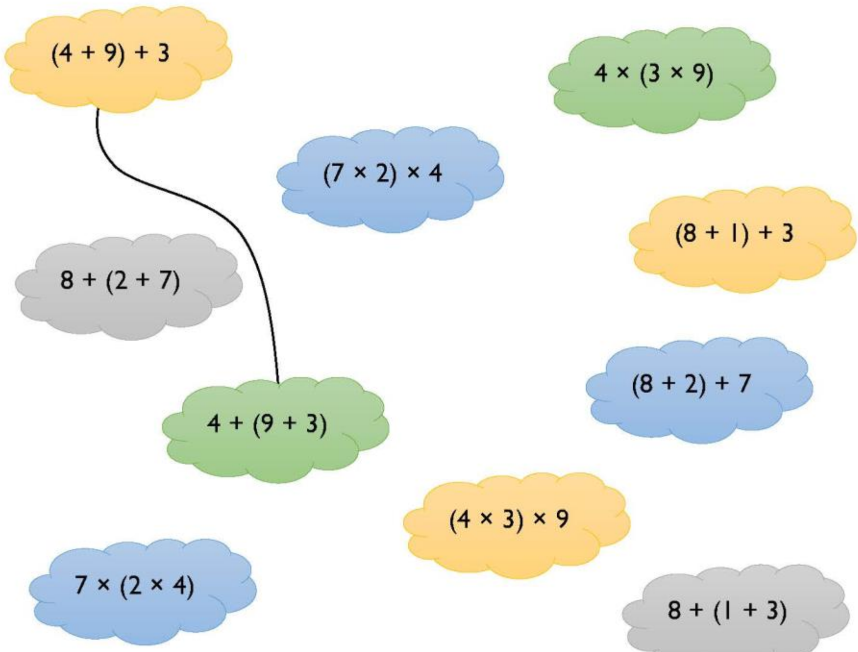
for any values a, b and c. (There is no need to discuss this formal algebra with learners.) The exercises lead learners to notice that addition and multiplication are associative, whilst subtraction and division are non-associative. Further information is available on the website https://www.mathsisfun.com/associative-commutative-distributive.html.

Addition, subtraction, multiplication and division are examples of **operations** in mathematics. The addition operation (for example) takes two numbers, the inputs of the operation, and adds them together to give the output of the operation.

For the discussion at the bottom of the page, perhaps some learners will argue that 32 × 2 is easier to calculate than 8 × 8, and others will argue the other way. The key point is that two different methods of calculating the same answer are available, and there is freedom to choose which one to complete.

Exercise

Pair the following bubbles. (The first one has been done for you.)



Exercise

Write = or ≠ in the middle.

10 × (5 × 2)	(10 × 5) × 2	12 ÷ (6 ÷ 2)	(12 ÷ 6) ÷ 2
(10 – 5) – 2	10 – (5 – 2)	12 + (6 + 2)	(12 + 6) + 2
10 × (5 × 2)	(10 × 5) × 3	12 – (6 – 2)	(12 – 2) – 2
(10 × 5) × 2	10 + (5 + 2)	(12 × 6) × 2	12 × (6 × 2)
10 × (5 × 2)	10 × (2 × 5)	0 ÷ (6 ÷ 2)	(0 ÷ 6) ÷ 2

The first exercise develops fluency in recognising expressions that are equal to each other due to associativity.

Reasoning is required in the second exercise to decide on the equality or inequality of the expressions.

- In the first two rows, learners need to differentiate between associative operations (addition and multiplication) and non-associative operations (subtraction and division).
- In the third row, the equality (or inequality) of the expressions needs to be examined. An inequality sign is required on the left (due to the change from the 2 to the 3), whereas an equality sign is needed on the right (these expressions are equal even though subtraction is non-associative).
- In the fourth row, an inequality sign is needed on the left because two different operations are present.
- In the fifth row, an equality sign is needed on the left (this is an example of commutativity), and an equality sign is also needed on the right (starting with zero is a special case of division: zero divided by any other number is zero).

Using Associativity

The following equations all show the associativity of addition or multiplication. In each equation, circle the side that is easiest to calculate.

Example

(a) $(17 + 3) + 18 = 17 + (3 + 18)$ (b) $(13 \times 5) \times 2 = 13 \times (5 \times 2)$

17 + 3 bonds to 20

13 × 10 is easier than 13 × 5

Exercise

(a) $(16 + 4) + 18 = 16 + (4 + 18)$ (b) $(17 \times 5) \times 2 = 17 \times (5 \times 2)$

(c) $(23 + 8) + 12 = 23 + (8 + 12)$ (d) $(2 \times 5) \times 27 = 2 \times (5 \times 27)$

(e) $(25 + 15) + 47 = 25 + (15 + 47)$ (f) $(23 \times 4) \times 5 = 23 \times (4 \times 5)$

(g) $(39 + 23) + 7 = 39 + (23 + 7)$ (h) $(25 \times 4) \times 7 = 25 \times (4 \times 7)$

Example

Calculate the answer to $(58 + 27) + 3$, using associativity if required.

$(58 + 27) + 3 = 58 + (27 + 3)$ Using associativity (recognising a bond to 30)
$= 58 + 30$ Calculating the sum in brackets
$= 88$ Adding the numbers

Exercise

Calculate the answer to the following sums, using associativity if required.

(a) $(42 + 8) + 31$ (b) $(5 \times 2) \times 18$

(c) $(23 + 8) + 12$ (d) $(17 \times 5) \times 2$

(e) $(34 + 17) + 3$ (f) $(7 \times 25) \times 4$

(g) $48 + (23 + 7)$ (h) $18 \times (5 \times 2)$

(i) $14 + (6 + 79)$ (j) $20 \times (5 \times 9)$

(k) $35 + (15 + 128)$ (l) $2 \times (8 \times 5)$

(m) $34 + 16 + 47$ (n) $19 \times 5 \times 2$

This page introduces the strategy of searching for a multiple of 10 (either as the answer to an addition sum, or as the answer to a multiplication sum) in order to make a calculation easier. In the first example, we circle the left side because the sum 20 + 18 is slightly easier than the sum 17 + 21. In the second example, we circle the right side because the sum 13 × 10 is easier than the sum 13 × 5 followed by doubling. The first exercise encourages learners to spot multiples of 10, whilst the second exercise encourages learners to use associativity to form a multiple of 10 and therefore make the sum easier.

Commutativity and Associativity

Example

Use commutativity and associativity to re-arrange the sum 2 × (17 × 5) to give the sum (2 × 5) × 17.

2 × (17 × 5)	
= 2 × (5 × 17)	Using the commutativity of multiplication
= (2 × 5) × 17	Using the associativity of multiplication

Exercise

Use commutativity and associativity to re-arrange the following sums.

	Original sum	Sum after re-arranging
(a)	2 × (18 × 5)	(2 × 5) × 18
(b)	2 × (16 × 5)	(5 × 2) × 16
(c)	4 × (24 × 5)	(4 × 5) × 24
(d)	5 × (18 × 6)	(6 × 5) × 18
(e)	(24 × 5) × 2	24 × (5 × 2)
(f)	(5 × 27) × 2	27 × (5 × 2)
(g)	(7 × 25) × 4	7 × (25 × 4)
(h)	(9 × 20) × 5	9 × (5 × 20)

Exercise

Calculate the answers to the sums in the above exercise.

Challenge!

Use commutativity and associativity to re-arrange the sum

8 × (2 × 23) × (19 × 5) × 5

to give the sum

(2 × 5) × 23 × 19 × (8 × 5)

This page combines the work on the previous page with earlier work on commutativity. Notice that the sum after re-arranging is easier to calculate than the original sum, every time.

Encourage learners to set out their work as shown in the example, working through each question step-by-step in a logical manner. Only two ‘moves’ are available each time: either reversing the order of the numbers in the multiplication sum (using commutativity) or changing the location of the bracket (using associativity).

Example

Use commutativity and associativity to re-arrange the sum 14 + (19 + 16) to give the sum (14 + 16) + 19.

14 + (19 + 16)

= 14 + (16 + 19) Using the commutativity of addition

= (14 + 16) + 19 Using the associativity of addition

Exercise

Use commutativity and associativity to re-arrange the following sums.

	Original sum	Sum after re-arranging
(a)	25 + (39 + 15)	(25 + 15) + 39
(b)	52 + (27 + 38)	(52 + 38) + 27
(c)	15 + (45 + 34)	(15 + 45) + 34
(d)	17 + (41 + 33)	(17 + 33) + 41
(e)	17 + (41 + 33)	41 + (17 + 33)
(f)	(56 + 21) + 24	21 + (56 + 24)
(g)	(31 + 11) + 49	31 + (11 + 49)
(h)	(32 + 126) + 118	126 + (32 + 118)
(i)	(32 + 126) + 118	(118 + 32) + 126
(j)	0.8 + (0.15 + 0.2)	(0.8 + 0.2) + 0.15
(k)	(0.45 + 0.12) + 0.15	0.12 + (0.45 + 0.15)

Exercise

Calculate the answers to the sums in the above exercise.

Challenge!

Use commutativity and associativity to re-arrange the sum

8 + (3 + 9) + (6 + 2) + 7

to give the sum

(2 + 3) + 6 + (7 + 8) + 9

This page repeats the work on the previous page, but this time with addition sums instead of multiplication sums. Notice again that the sum after re-arranging is easier to calculate than the original sum, every time.