The Mathematics Department

PS 3

The Foundations of Algebra

Name:

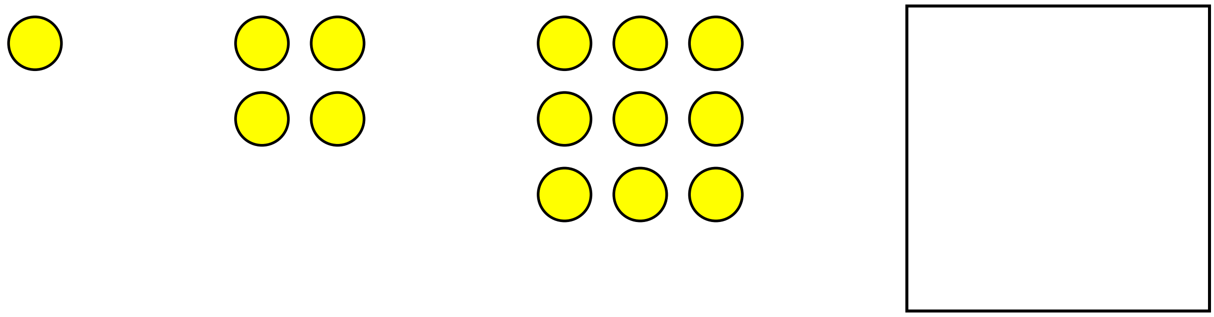
Contents

Chapter	Activities	Page Number
Patterns	Picture patterns. Number patterns. Triangle numbers and square numbers. Patterns with prime numbers. Calon Lân. The Fibonacci sequence. Frayer model.	3
Commutativity	Things that are commutative and noncommutative. The commutativity of addition. Using the additive inverse. The noncommutativity of division. Using the multiplicative inverse. A pair of brackets.	14
Distributivity	What is distributivity? Using distributivity.	34
Associativity	Associative operations. Using associativity. Commutativity and associativity.	38

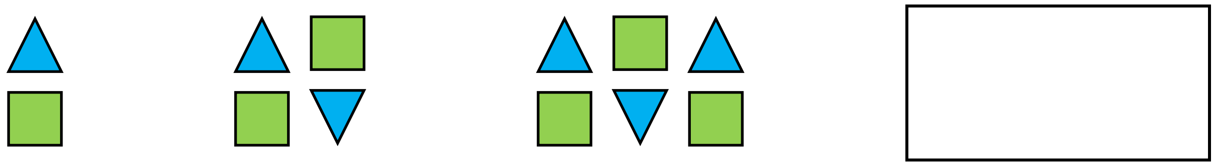
Patterns

In the boxes, draw what comes next in the following patterns.

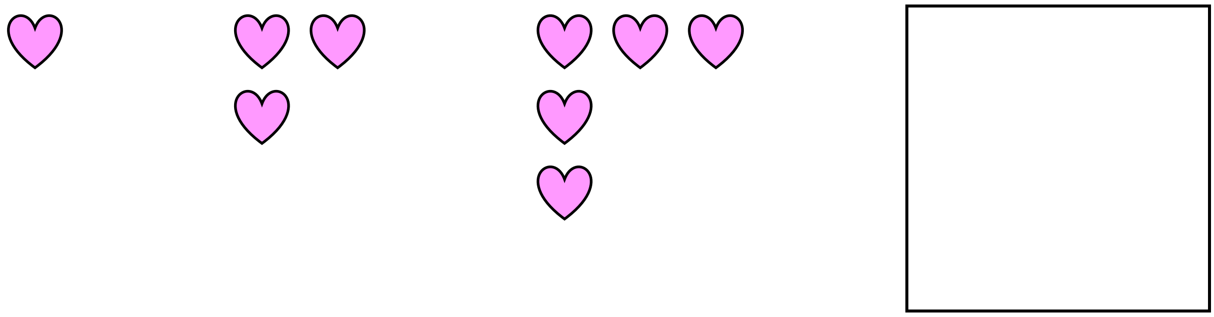
(a)



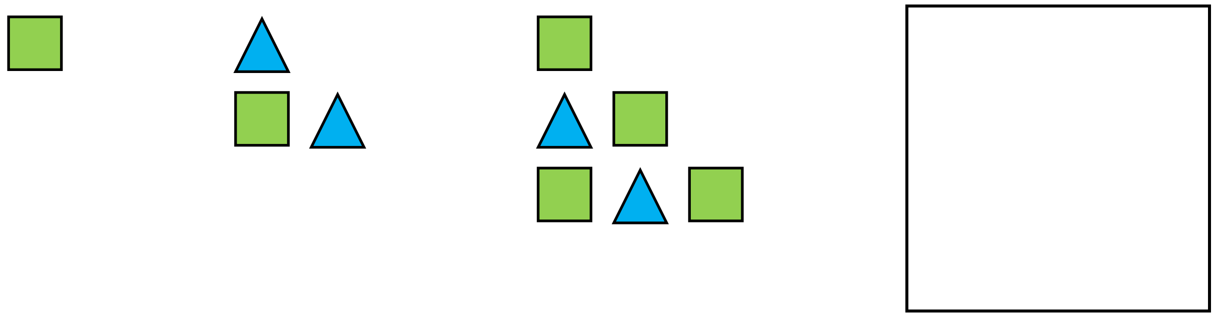
(b)



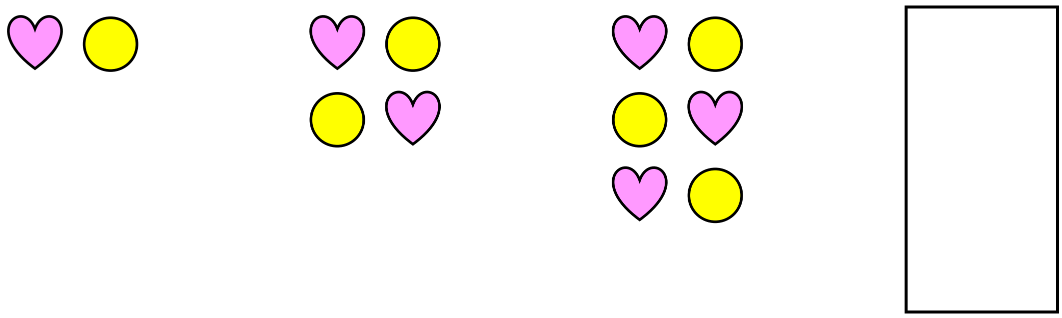
(c)



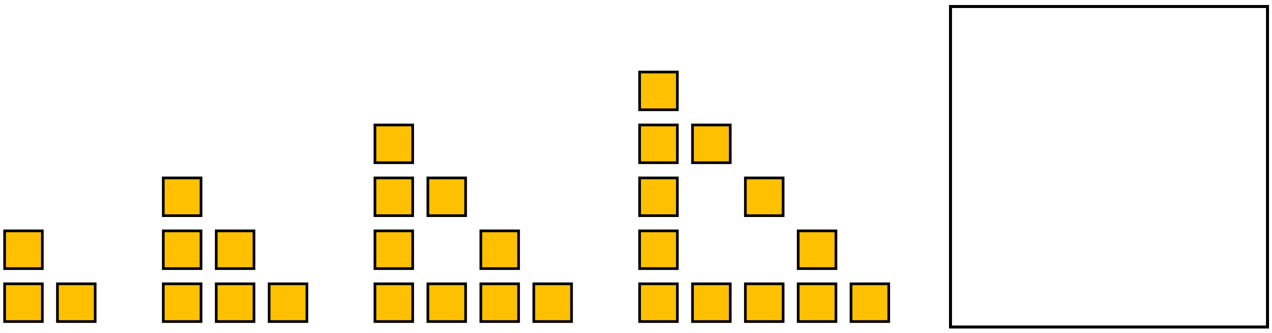
(d)



(e)

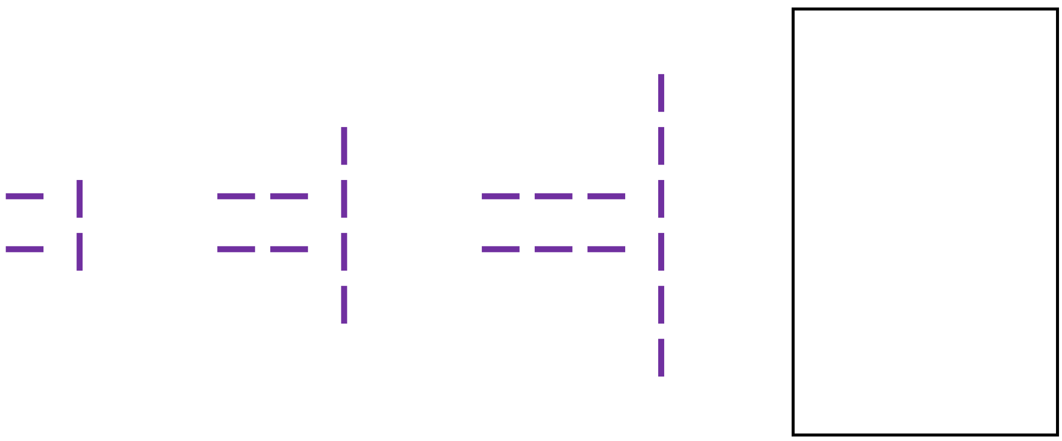


(f)



(g)

(h)



Discuss the following questions with a partner.

- In each question, what did you have to do to draw the next picture in the sequence?
- What would be required to create the 10th picture in each pattern?
- In each question, what was the number pattern involving the individual symbols appearing in each picture?

Number Patterns

Look carefully at each of the following number patterns, considering how the patterns continue. Fill in the blanks with the missing numbers to extend each pattern both ways. Underneath each pattern, explain in words how to continue the pattern.

(a) ______ ______ 14, 17, 20, 23, ______ ______

____

(b) ______ ______ 65, 63, 61, 59, ______ ______

____

(c) ______ ______ 55, 64, 73, 82, ______ ______

____

(d) ______ ______ 54, 42, 30, 18, ______ ______

____

(e) ______ ______ 12, 22, 32, 42, ______ ______

____

(f) ______ ______ 75, 60, 45, 30, ______ ______

____

(g) ______ ______ –24, –18, –12, –6, ______ ______

____

(h) ______ ______ –110, –85, –60, –35, ______ ______

____

(i) ______ ______ 2.7, 3.4, 4.1, 4.8, ______ ______

____

(j) ______ ______ 0.5, 0.75, 1, 1.25, ______ ______

____

(k) ______ ______ $2\frac{5}{8}$, $3\frac{1}{8}$, $3\frac{5}{8}$, $4\frac{1}{8}$, ______ ______

____

(l) ______ ______ 4, 8, 16, 32, ______ ______

(m) ______ ______ 400, 200, 100, 50, ______ ______

(n) ______ ______ 12, 24, 48, 96, ______ ______

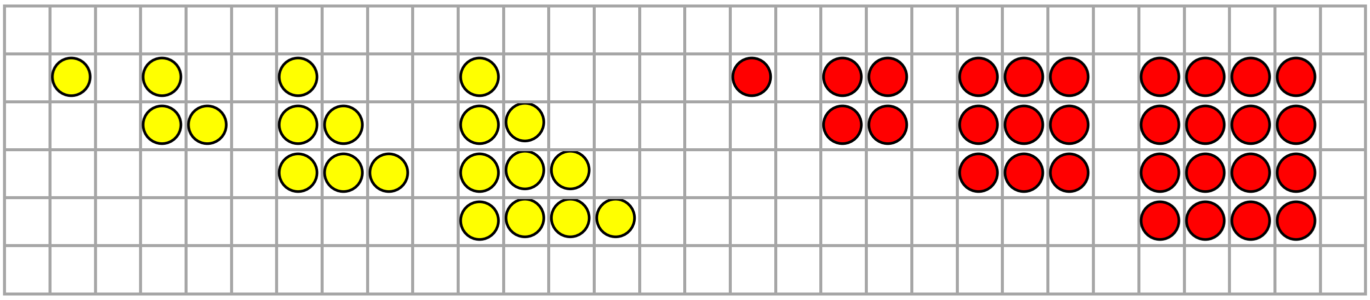
(o) ______ ______ 8, 4, 2, 1, ______ ______

(p) ______ ______ 4, 16, 64, 256, ______ ______

(q) ______ ______ 243, 81, 27, 9, ______ ______

(r) ______ ______ 3.6, 7.2, 14.4, 28.8, ______ ______

Triangle Numbers and Square Numbers



Write down the first ten triangle numbers.

______ ______ ______ ______ ______ ______ ______ ______ ______ ______

Write down the first ten square numbers.

______ ______ ______ ______ ______ ______ ______ ______ ______ ______

What is the connection between two consecutive triangle numbers and square numbers?

Odd one out

Circle the number that doesn't belong to each of the following patterns.

(a)	7	14	21	28	34	42	49
(b)	63	52	45	36	27	18	9
(c)	1	4	9	15	25	36	49
(d)	12	24	36	48	60	72	86
(e)	512	256	122	64	32	16	8
(f)	1.6	2	2.4	2.8	3.4	3.6	4
(g)	0.056	0.56	5.6	56	560	5600	560000
(h)	0.1	$\frac{1}{5}$	0.2	0.4	$\frac{1}{2}$	$\frac{3}{5}$	0.7

Investigation

Here is a set of instructions for generating a number pattern.

(a) For the first number of the pattern, think of any 4-digit number where the digits are not all identical. For example, 4,283, 6,816 and 3,025 would be fine, but not 452 (only 3 digits) or 7,777 (all digits are identical).

(b) Re-arrange the digits of your number to make the largest possible 4-digit number, and the smallest possible 4-digit number. For example, considering 4,283, the largest possible 4-digit number is 8,432, and the smallest 2,348.

(c) Subtract the smallest possible 4-digit number from the largest possible 4-digit number. For example, 8,432 – 2,348 = 6,084.

(d) Repeat the above process with the new number. (If the new number has less than 4 digits, add zeroes at the start to create a 4-digit number. For example, if we start with 3,233, the first subtraction sum would be 3,332 – 2,333 = 999. Then, we would need to treat the 999 as 0,999, and the second subtraction sum would be 9,990 – 0,999 = 8,991.)

Investigate the patterns produced by the above set of instructions.

Patterns with Prime Numbers

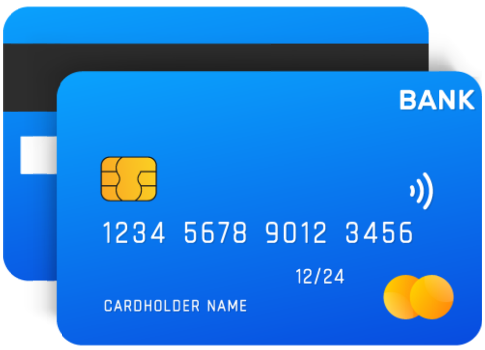
Here are the prime numbers between 1 and 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

What do you notice? What do you wonder?

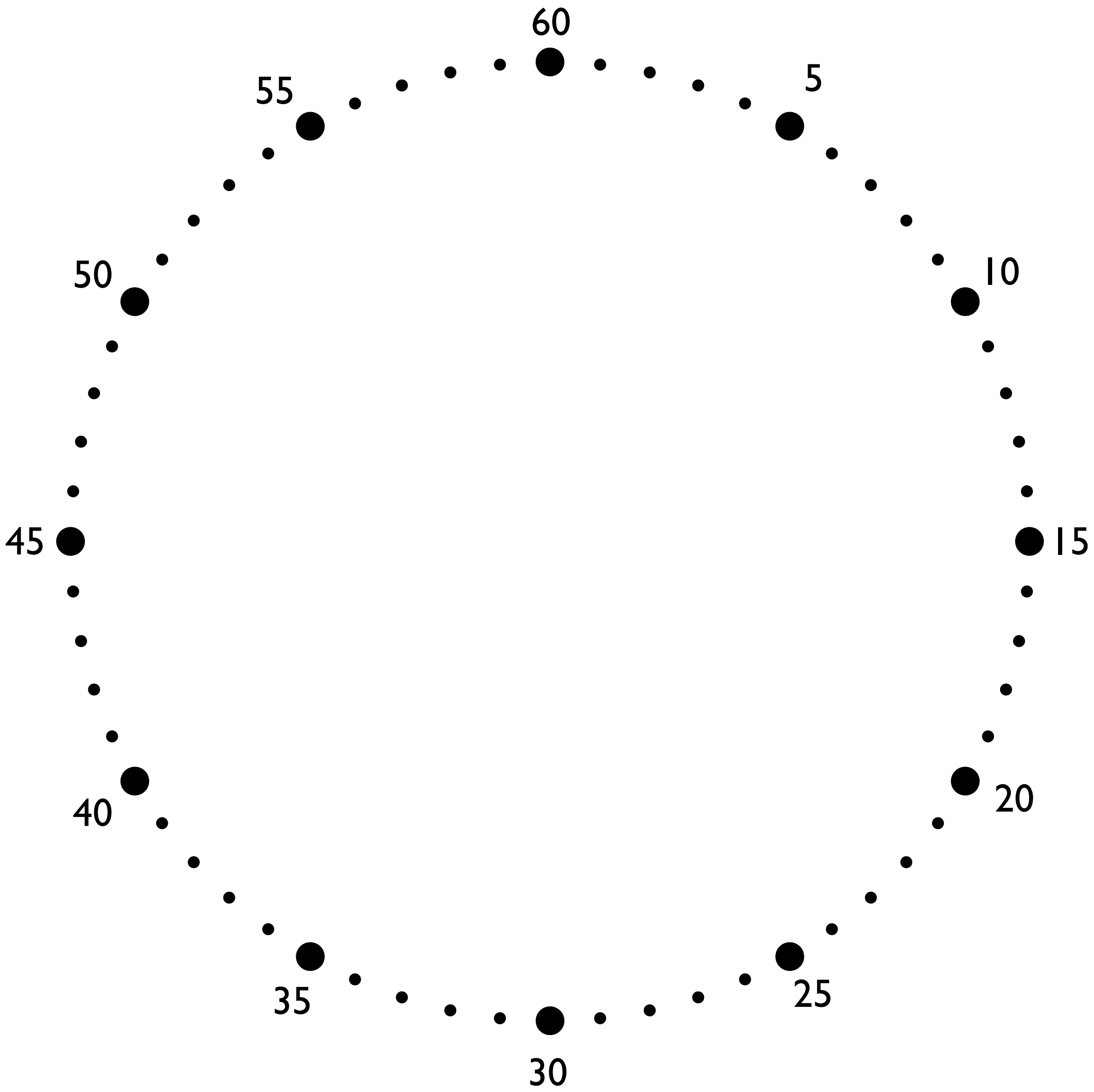
Mathematicians believe that there is **no pattern** in how prime numbers appear on the number line.

There is a prize of $1,000,000 to prove that prime numbers have no pattern when placed on the number line. (This problem is one of the [seven millennium problems].) The lack of pattern with prime numbers, and the fact that it is difficult to decide whether a large number is prime or not, is the reason why the internet is secure.



Calon Lân

The following diagram shows a blank clock face with the minutes labelled up to 60.



Use a ruler to connect each minute to its double. For example, you will need to connect 4 to 8, and 45 to 90 (which means connecting to 30 after travelling around the clock once). Perhaps the following table will help?

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	4	6	8																

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40

41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
				30															

What happens when you multiply each minute by 3? By 4? By 5?

The Fibonacci Sequence

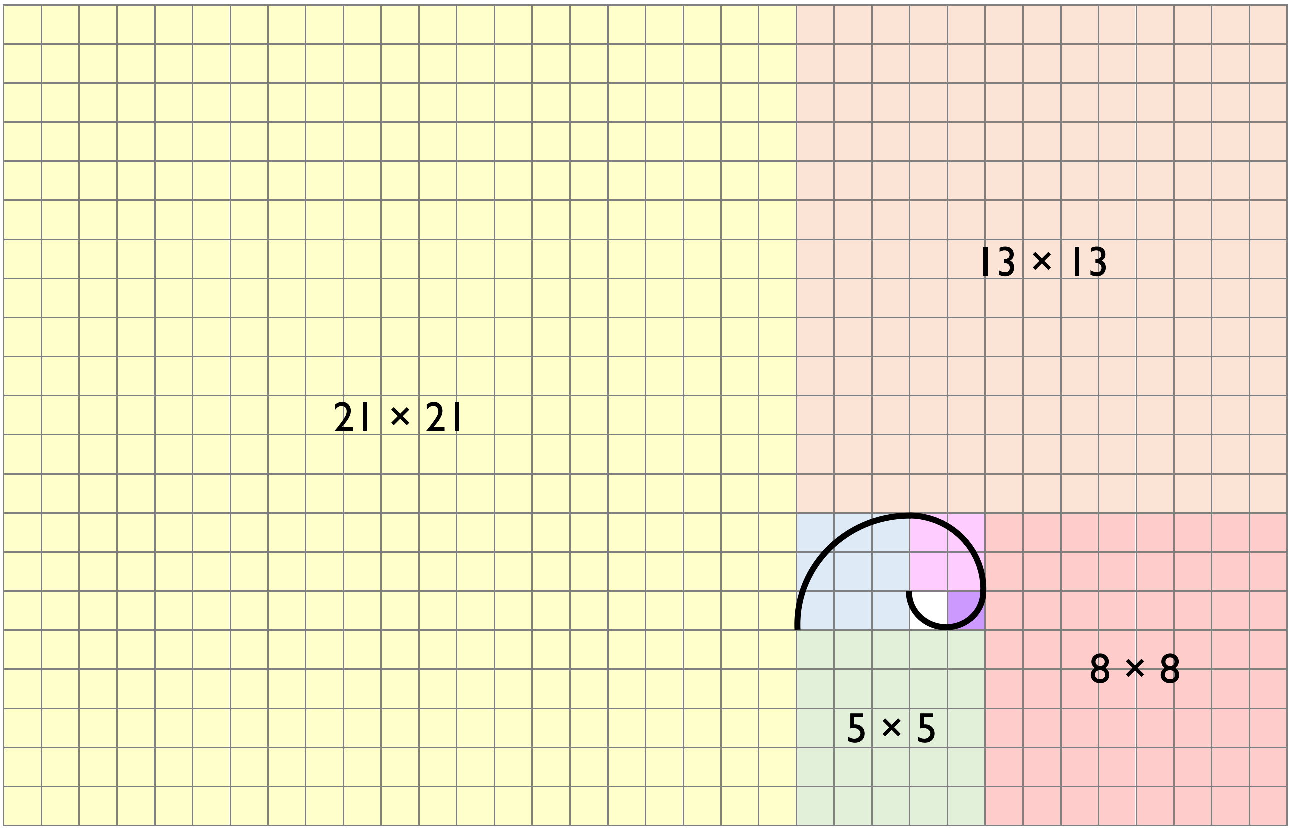
For the Fibonacci sequence, we need to add the previous two numbers to give the next number in the sequence.

To begin, complete the first column in the following table to show how the sequence continues.

Fibonacci Sequence	Dividing two consecutive numbers
1	
1	1 ÷ 1 = 1
1 + 1 = 2	2 ÷ 1 = 2
1 + 2 = 3	3 ÷ 2 = 1.5
2 + 3 = 5	5 ÷ 3 = 1.6666666….

Next, use a calculator to divide two consecutive numbers in the sequence. You should see that your numbers settle down to become close to a special number known as **the golden ratio**. Use the internet to investigate this number, writing down what you find below.

The Fibonacci Spiral

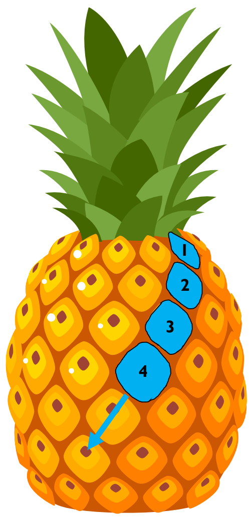


The above diagram shows part of the Fibonacci spiral. Use a compass or a piece of string to extend the spiral, connecting opposite corners in different coloured squares.

Colour of the square	Compass point is placed	Pencil starts
Green	Top right	Top left
Red	Top left	Bottom left
Orange	Bottom left	Bottom right
Yellow	Bottom right	Top right

Investigation

- What is the connection between the Fibonacci sequence and flower petals?
- Look at the scales on the surface of a pineapple (as shown in blue on the right). Count the number of scales whilst moving from top to bottom along one of the pineapple's diagonals. What do you notice?
- Who was Fibonacci?



Exercise

The following number sequences are related to the Fibonacci sequence.
Find the next two numbers in each sequence.

(a) 1, 2, 3, 5, 8, 13, _____, _____

(b) 2, 2, 4, 6, 10, 16, _____, _____

(c) 1, 3, 4, 7, 11, 18, _____, _____

(d) 5, 2, 7, 9, 16, 25, _____, _____

(e) 0.5, 0.3, 0.8, 1.1, 1.9, 3, _____, _____

(f) $\frac{1}{2}, \frac{1}{2}, 1, 1\frac{1}{2}, 2\frac{1}{2}, 4$, _____, _____

(g) 1, 1, 1, 3, 5, 9, 17, _____, _____

(h) 1, 0, 1, 2, 3, 6, 11, _____, _____

Did you know?

There is a connection between the Fibonacci sequence and measurements in miles and kilometres. 5 miles is approximately 8 km; 8 miles is approximately 13 km; 13 miles is approximately 21 km, and so on. (Why?)

Challenge!

Choose any 3 consecutive numbers from the Fibonacci sequence.

Multiply the middle number by itself.

Now multiply the first number by the third number.

Try this several times using different starting points.

What do you notice about your answers?

Use a calculator to help.

....

....

....

Puzzle

What would be the best prize in a competition: winning £1,000,000 today, or winning 1p today, 2p tomorrow, 4p the day after, and so on, for 30 days?

....

....

....

Frayer Model

Definition	Facts
Examples	Nonexamples

Sequence

Commutativity

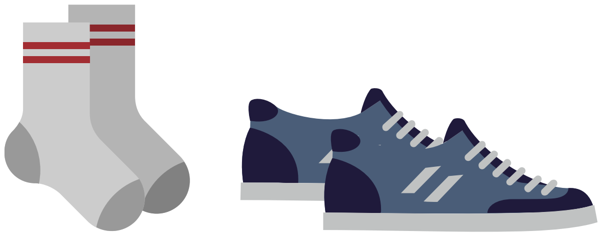
Is the order in which things are done important? If the order is *not* important, then we say that the things are **commutative**. If the order *is* important, then we say that the things are **noncommutative**.

Example

Siwan intends to mix yellow and blue paint to make green paint.

Putting the yellow paint in the empty bucket first and mixing in the blue paint would give the same result as putting the blue paint in the empty bucket first and mixing in the yellow paint. So, this process is **commutative**.

Arwyn gets up in the morning and finishes dressing by putting on a pair of socks and shoes.



Wearing the socks first and then putting on the shoes would give a different result to wearing the shoes first and then putting on the socks. So, this process is **noncommutative**.

Exercise

Are the following situations commutative or noncommutative?

Adding salt and vinegar to a plate of freshly made chips.

____

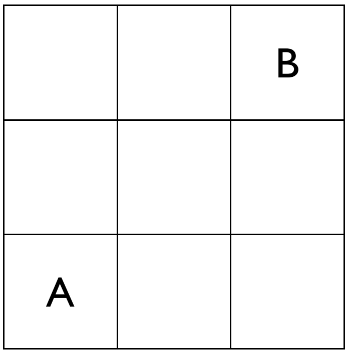
Studying for a spelling test and taking the spelling test.

____

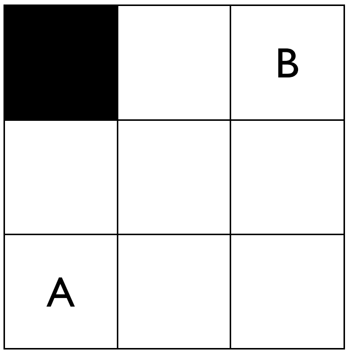
Shaking a bottle of water and opening it.

Shaking a bottle of pop and opening it.

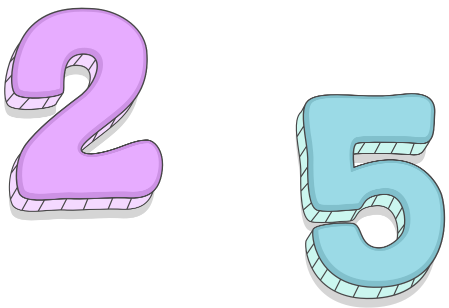
Moving 2 steps right and moving 2 steps up when moving from A to B.



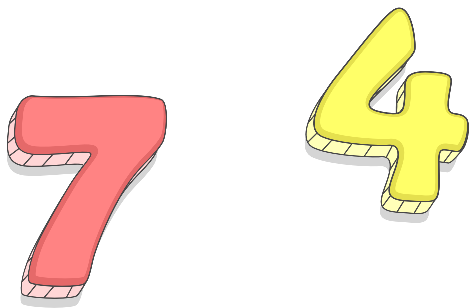
Moving 2 steps right and moving 2 steps up when moving from A to B.



Adding two numbers together.



Subtracting two numbers from each other.



Can you think of more examples of commutativity or noncommutativity?

Addition is commutative, whilst **subtraction is noncommutative**.

Example

$6 + 4 \quad = \quad 4 + 6$	$6 - 4 \quad \neq \quad 4 - 6$

Exercise

Write = or ≠ in the middle.

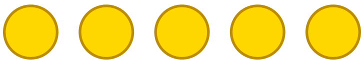
$8 + 3 \qquad 3 + 8$	$8 - 3 \qquad 3 - 8$
$3 + 8 \qquad 8 + 3$	$3 - 8 \qquad 8 - 3$
$7 - 5 \qquad 5 - 7$	$7 + 5 \qquad 5 + 7$
$9 + 4 \qquad 4 + 9$	$9 - 4 \qquad 4 - 9$
$2.5 + 3.5 \qquad 3.5 + 2.5$	$3.4 - 0.6 \qquad 0.6 - 3.4$
$7 + 0 \qquad 0 + 7$	$7 - 0 \qquad 0 - 7$
$\frac{3}{4} - \frac{2}{3} \qquad \frac{2}{3} - \frac{3}{4}$	$\frac{7}{12} + \frac{5}{12} \qquad \frac{5}{12} + \frac{7}{12}$

Take care with these next ones!

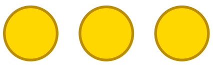
$4 + 5 \qquad 5 + 6$	$9 - 3 \qquad 3 + 3$
$3 + 4 + 5 \qquad 5 + 4 + 3$	$10 - 3 \qquad 3 - 4$
$-2 + 4 \qquad 4 + -2$	$-2 - 4 \qquad 4 - -2$
$5 + 2 - 3 \qquad 2 + 5 - 3$	$5 + 2 - 3 \qquad 5 + 3 - 2$

Subtraction as addition: using the additive inverse

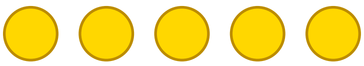
Let us consider the subtraction sum 5 – 2. We can think of this sum as starting with five yellow counters



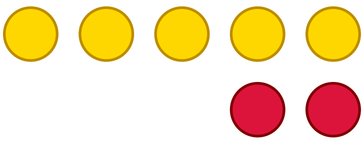
and then subtracting two counters to leave three counters.



Here is a different way of thinking about the sum. Instead of subtracting two, we can add the **additive inverse** of two, namely –2. So, instead of writing 5 – 2, we write 5 + –2. We can think of this sum as starting with five yellow counters



and then adding two red counters (to represent –2).



There are two zero pairs here, leaving us with three yellow counters as before.

Why do we want to use this new way of writing down the sum? Well, the first sum is noncommutative, as 5 – 2 is different to 2 – 5. (What are the two different answers here?) The second sum however **is** commutative, as 5 + –2 is the same as –2 + 5. This will help us answer sums like –2 + 5 on the next page.

Exercise

Write the following subtraction sums as addition sums, using the additive inverse.

(a) 6 – 2	(b) 7 – 2	(c) 10 – 2
(d) 7 – 5	(e) 10 – 3	(f) 14 – 8
(g) 5 – 3	(h) 11 – 4	(i) 18 – 6
(j) 2.7 – 1.2	(k) $\frac{7}{8} - \frac{3}{8}$	(l) 27% – 4%

Exercise

Write the following addition sums as subtraction sums.

(a) $5 + -2$ (b) $5 + -3$ (c) $5 + -4$

(d) $6 + -2$ (e) $8 + -1$ (f) $9 + -3$

(g) $7 + -5$ (h) $10 + -4$ (i) $13 + -4$

Example

How do we find the answer to the sum $-2 + 5$?

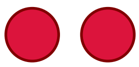
Method A

Step 1 We use the commutativity of addition to write the sum as $5 + -2$.

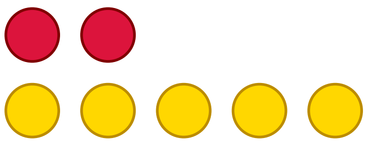
Step 2 Instead of adding negative two (which is the additive inverse of 2), we can subtract 2, changing the sum to be $5 - 2$. So, the answer is 3.

Method B

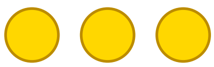
We use double sided counters to model: we start with two red counters (to represent -2)



and then add 5 yellow counters:



There are two zero pairs here, leaving us with three yellow counters.



Exercise

Use a method of your choice to answer the following sums.

(a) $-2 + 6$ (b) $-2 + 7$ (c) $-2 + 4$

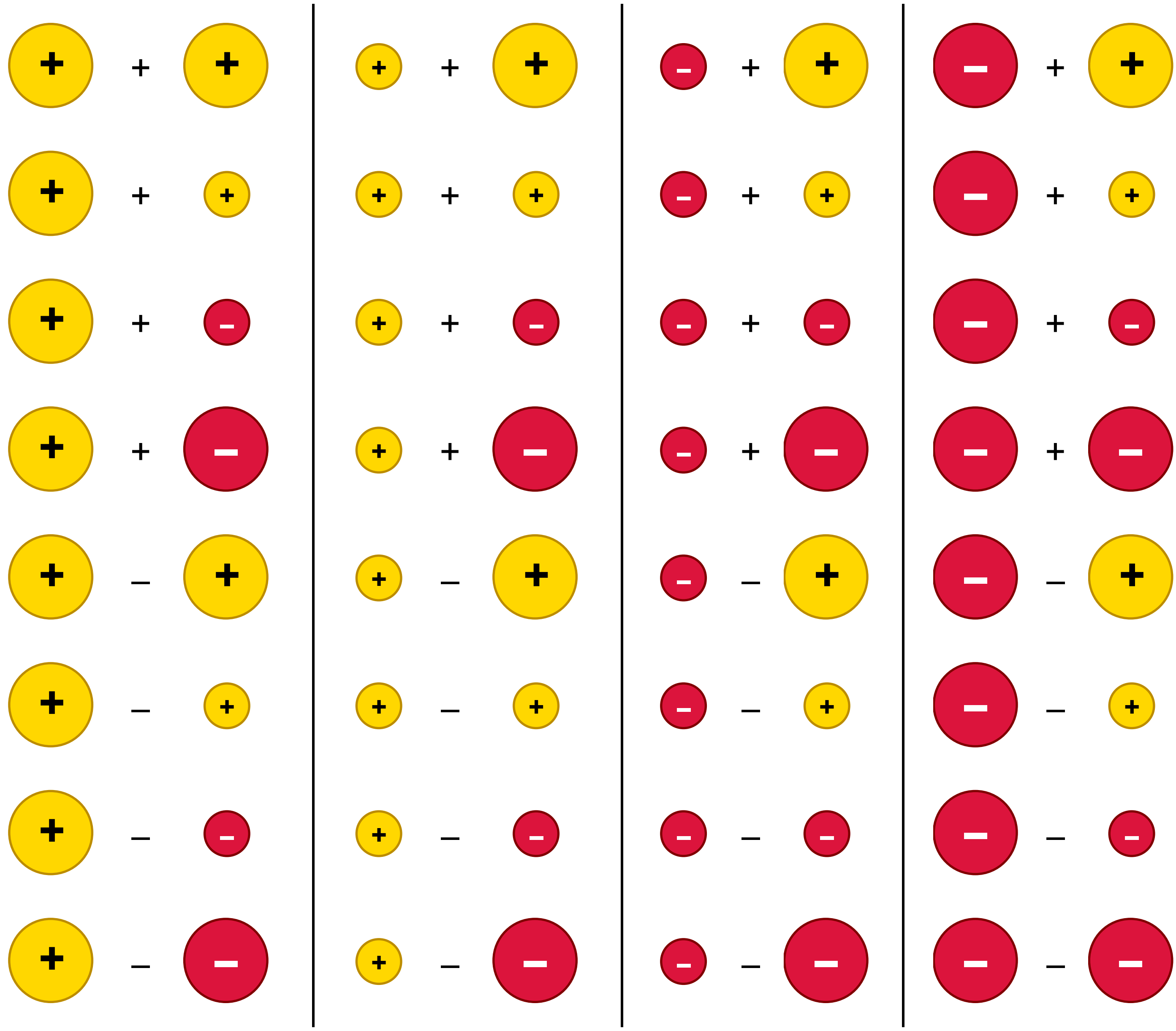
(d) $-3 + 7$ (e) $-5 + 8$ (f) $-6 + 9$

(g) $-2 + 9$ (h) $-4 + 5$ (i) $-3 + 12$

(j) $-7 + 10$ (k) $-8 + 13$ (l) $-10 + 15$

Exercise

Discuss the meaning of the following sums.



Space for calculations:

Exercise

(a) 4 + 2 =
(b) 4 – 2 =
(c) 4 + –2 =
(d) –4 + 2 =
(e) –4 – 2 =
(f) –4 + –2 =
(g) –2 + 4 =
(h) –2 – 4 =
(i) –2 + –4 =
(j) 4 – –2 =
(k) 2 – –4 =
(l) 2 – 4 =

Exercise

Fill in the blanks in the following tables. (Blue add green, then blue multiply by green.)

+	–5	–4	–3	–2	–1	0	1	2	3	4	5
–5											
–4											
–3											
–2											
–1											
0											
1											
2											
3											
4											
5											

–	–5	–4	–3	–2	–1	0	1	2	3	4	5
–5											
–4											
–3											
–2											
–1											
0											
1											
2											
3											
4											
5											

What is the missing number?

Example

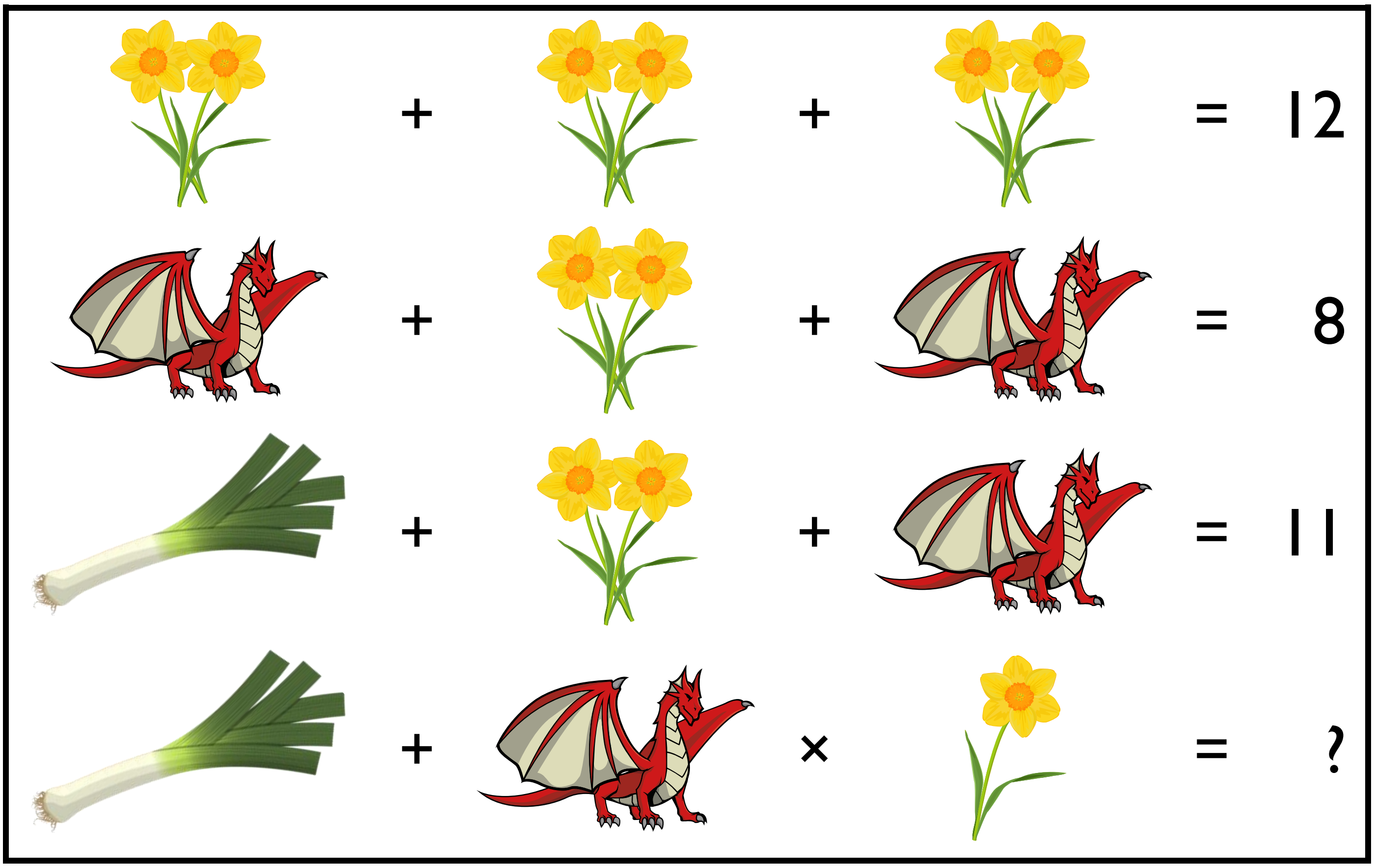
$2 - 8 = \boxed{-6}$	$8 + \boxed{-2} = 6$	$4 - 2 = \boxed{5} - 3$

Exercise

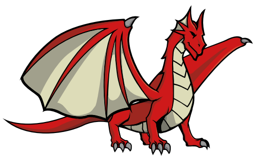
$8 + 3 = \square$	$8 - 3 = \square$	$8 + -3 = \square$
$3 + 8 = \square$	$3 - 8 = \square$	$-3 + 8 = \square$
$-8 + 3 = \square$	$-8 - 3 = \square$	$-8 + -3 = \square$
$8 + \square = 11$	$8 + \square = 5$	$8 + \square = -3$
$4 + \square = 6$	$5 + \square = 6$	$6 + \square = 6$
$7 + \square = 6$	$8 + \square = 6$	$15 + \square = 6$
$\square + 6 = 11$	$\square + 6 = 4$	$\square + 6 = -4$
$\square - 4 = 6$	$\square - 4 = -1$	$\square - 4 = -6$
$\square + -2 = 2$	$\square + -2 = -5$	$\square - -2 = 7$
$3 + 4 = 2 + \square$	$3 + 4 = \square + 5$	$3 + \square = 2 + 5$
$9 - 2 = 10 - \square$	$9 - 3 = \square - 5$	$9 - \square = 12 - 7$
$2 - 9 = 1 - \square$	$2 - 9 = \square - 7$	$2 - 9 = \square + 2$
$\square + 3 = -1 + 6$	$\square + 2 = -4 + 10$	$\square - 1 = -3 + 11$
$-4 + 5 = 7 - \square$	$-4 + \square = 8 - 6$	$-4 + 7 = \square + 8$
$10 - \square = 1 + 5$	$-10 - 4 = 1 - \square$	$-10 + 4 = \square + 2$
$8 + -3 = 2 + \square$	$8 + -3 = \square - 4$	$8 + -3 = \square + 7$
$1.2 + 1.5 = 2.4 + \square$	$1.2 + 1.5 = \square + 1.1$	$1.2 + \square = 2 + 0.8$
$\square - 0.5 = 7.3 + 0.6$	$0.4 - 0.6 = 0.7 - \square$	$2 - 0.4 = 1 + \square$
$2 - \frac{3}{4} = 1 + \square$	$2 - \frac{1}{4} = 1 + \square$	$2 - \square = 1 + \frac{1}{2}$

Order of Operations

What is the answer to the following puzzle?



The flowers  are worth ______

The dragon  is worth ______

The leek  is worth ______

The sum in the final line is ________________

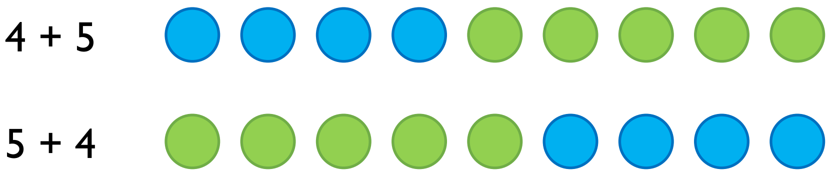
In the final line, if we do the addition sum first, the answer is ______.

In the final line, if we do the multiplication sum first, the answer is ______.

Which answer is correct?

We have previously discussed that addition is commutative. So, for example,

$$4 + 5 = 5 + 4.$$



By writing 2×2 instead of 4 on both sides of the equation, we arrive at

$$2 \times 2 + 5 = 5 + 2 \times 2.$$

By working from left to right, calculate the answer to the sum on the left side of the equation, and the answer to the sum on the right side of the equation.

Left side: $2 \times 2 + 5$

Right side: $5 + 2 \times 2$

Because the two answers above are different, it looks like we have broken the equation! To fix this, we must decide that **multiplication takes priority over addition** in a sum like $5 + 2 \times 2$. This way, we must calculate 2×2 to begin with, to obtain 4, and then calculate $5 + 4$ to obtain 9. This agrees with the answer on the left side, and with the number of counters shown in each row of the diagram above. Therefore, 9 is the correct answer to the puzzle!

RULE: Given a choice between an addition and a multiplication sum in mathematics, the **multiplication** sum must be done first.

Giving priority to multiplication over addition ensures that we **do not break the commutativity of addition**.

Exercise

(a) $5 + 3 \times 2$

(b) $5 \times 3 + 2$

(c) $5 + 2 \times 3$

(d) $9 + 2 \times 5$

(e) $2 + 9 \times 5$

(f) $2 + 5 \times 9$

(g) $12 + 3 \times 3$

(h) $3 + 12 \times 3$

(i) $3 + 3 \times 12$

(j) $1 + 2 \times 3 + 4$

(k) $1 \times 2 + 3 \times 4$

(l) $1 + 2 + 3 \times 4$

(m) $1 \times 2 + 3 + 4$

(n) $1 \times 2 \times 3 \times 4$

(o) $1 \times 2 \times 3 + 4$

Exercise

Choose three **different** numbers out of the five numbers given, to complete the sums in different ways.

(a) 2, 3, 4, 5, 6

□ × □ + □ = 14

6 different answers

(b) 2, 3, 4, 5, 6

□ × □ + □ = 18

2 different answers

(c) 2, 3, 4, 5, 6

□ × □ + □ = 22

4 different answers

(d) 2, 3, 4, 5, 6

□ + □ × □ = 23

4 different answers

(e) 2, 3, 4, 5, 6

□ + □ × □ = 27

2 different answers

(f) 2, 3, 4, 5, 6

□ + □ × □ = 34

2 different answers

(g) 2, 3, 4, 5, 7

□ × □ + □ = 13

6 different answers

(h) 2, 3, 4, 5, 7

□ + □ × □ = 19

6 different answers

(i) 2, 3, 4, 5, 7

□ × □ + □ = 33

2 different answers

(j) 3, 4, 6, 7, 9

□ + □ × □ = 25

4 different answers

(k) 3, 4, 6, 7, 9

□ × □ + □ = 27

6 different answers

(l) 3, 4, 6, 7, 9

□ + □ × □ = 43

2 different answers

(m) 4, 5, 7, 8, 9

□ × □ + □ = 27

2 different answers

(n) 4, 5, 7, 8, 9

□ + □ × □ = 41

4 different answers

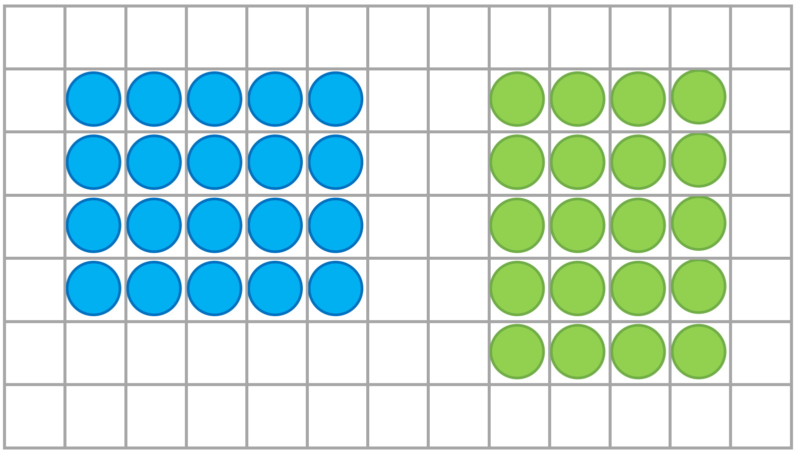
(o) 4, 5, 7, 8, 9

□ × □ + □ = 49

4 different answers

Multiplication

Is multiplication commutative? For example, is the answer to 4×5 the same as, or different to, the answer to 5×4?



We see from the above diagrams that multiplication is commutative, because the answers to 4×5 and 5×4 are both 20. We can therefore write the equation

$$4 \times 5 = 5 \times 4.$$

Agreeing that 5 is the answer to the sum $2 + 3$, it is reasonable to write $2 + 3$ instead of 5 in the above equation:

$$4 \times 2 + 3 = 2 + 3 \times 4.$$

Remembering that multiplication takes priority over addition, what is the answer to the sums on both sides of this equation?

Left side: $4 \times 2 + 3$

Right side: $2 + 3 \times 4$

None of the answers above are equal to 20, so we have a problem! We fix this by introducing a pair of brackets into each side of the equation, writing

$$4 \times (2 + 3) = (2 + 3) \times 4.$$

Each side does now give 20, if we insist that any sum in brackets takes priority over any other addition or multiplication sums.

RULE: In any sum including brackets, the sum in the brackets must be completed before doing anything else.

Giving priority to sums in brackets ensures that we **do not break the commutativity of multiplication**.

Exercise

(a) $5 \times (4 + 2)$ (b) $(4 + 2) \times 5$ (c) $4 \times (5 + 2)$

(d) $(5 + 2) \times 4$ (e) $5 + (4 + 2)$ (f) $2 \times (4 + 5)$

(g) $(5 + 4) \times 2$ (h) $2 \times (5 \times 4)$ (i) $5 \times (2 + 4)$

Challenge!

Repeat the above exercise, but this time ignoring all the brackets (so that question (a), for example, changes to $5 \times 4 + 2$). Two of the answers stay the same – which ones?

Exercise

(a) $(4 + 2) \times (3 + 5)$ (b) $10 \times (2 + 6)$ (c) $(5 \times 2) + (4 \times 3)$

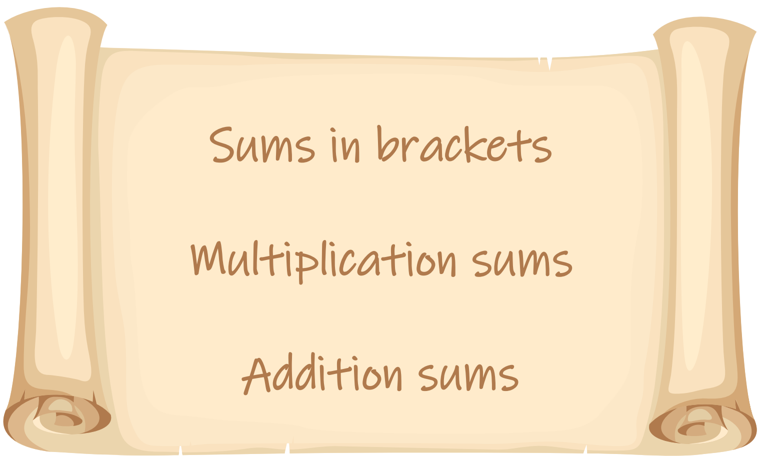
(d) $(4 \times 2) + (3 \times 5)$ (e) $10 + (2 \times 6)$ (f) $(5 + 2) \times (4 + 3)$

(g) $4 \times (2 + 3) \times 5$ (h) $(10 + 2) \times 6$ (i) $5 \times (2 + 4) \times 3$

(j) $4 \times 2 + 3 \times 5$ (k) $10 \times 2 + 6$ (l) $5 \times 2 + 4 \times 3$

(m) $4 + (2 \times 3 + 5)$ (n) $10 + 2 \times 6$ (o) $(5 \times 2 + 4) \times 3$

So far, we have decided on the following order of operations.



Where do subtraction and division sums fit into this puzzle?

Addition and subtraction sums

Let us consider the following four sums.

$8 + 4 + 2$ $8 + 4 - 2$ $8 - 4 + 2$ $8 - 4 - 2$

What do you notice about the sums?

Complete the table below to answer the sums in two different ways: first by doing the red sum to begin with, and then by doing the blue sum to begin with. Leave the final column of the table empty, for now.

Sum	Answer when starting with the red sum	Answer when starting with the blue sum	Which sum needs to be done first?
$8 + 4 + 2$			Does not matter
$8 + 4 - 2$			Does not matter
$8 - 4 + 2$			
$8 - 4 - 2$			

The answers in the first two rows are the same, but the answers in the final two rows are different. Which answer is correct each time?

Because it is possible to write every subtraction sum as an addition sum, using the additive inverse, we can re-write the final two sums as addition sums like this:

$8 + -4 + 2$ $8 + -4 + -2$

Answer these sums in two ways:

Sum	Answer when starting with the red sum	Answer when starting with the blue sum
$8 + -4 + 2$		
$8 + -4 + -2$		

The answers in each row of the above table are the same, so we can use them to complete the final column of the first table, and state which sum needed to be done first in $8 - 4 + 2$ and $8 - 4 - 2$.

In each of the four examples on the previous page, starting with the sum appearing **first on the left** (the red sum) leads to obtaining a correct answer. This in turn enables us to establish the following rule.

RULE: Given two or more addition or subtraction sums, we must complete them by working from left to right.

Exercise

(a) $9 - 6 + 3$ (b) $9 + 6 - 3$ (c) $9 - 6 - 3$

(d) $9 + 6 + 3$ (e) $6 - 3 + 9$ (f) $6 + 3 - 9$

(g) $6 + 9 - 3$ (h) $6 - 9 + 3$ (i) $3 - 6 + 9$

(j) $10 - 3 + 5$ (k) $10 + 3 - 5$ (l) $10 - 5 + 3$

(m) $12 - 7 + 1$ (n) $7 - 12 + 1$ (o) $1 + 7 - 12$

Exercise

(a) $2 \times 3 + 4 - 5$ (b) $2 + 3 \times 4 - 5$ (c) $2 + 3 + 4 \times 5$

(d) $2 - 3 + 4 \times 5$ (e) $2 + 3 - 4 \times 5$ (f) $2 - 3 \times 4 + 5$

(g) $6 \times 3 - 2 + 7$ (h) $3 \times 2 - 7 + 6$ (i) $2 \times 7 - 6 + 3$

(j) $6 + 3 \times 2 - 7$ (k) $3 + 2 \times 7 - 6$ (l) $2 + 7 \times 6 - 3$

(m) $6 - 3 + 2 \times 7$ (n) $3 - 2 + 7 \times 6$ (o) $2 - 7 + 6 \times 3$

Exercise

Complete the following equations by writing +, – or × in the boxes.

(a) $3 \square 4 \square 5 = 12$ (b) $3 \square 4 \square 5 = 17$ (c) $3 \square 4 \square 5 = 23$

(d) $3 \square 4 \square 5 = 60$ (e) $3 \square 4 \square 5 = 7$ (f) $3 \square 4 \square 5 = 2$

(g) $3 \square 4 \square 5 = -6$ (h) $3 \square 4 \square 5 = -17$ (i) $3 \square 4 \square 5 = 4$

Challenge!

Use the numbers 2, 3 and 4 and the operations +, – and × to create sums with as many different answers as possible.

Division

Is dividing two numbers commutative or noncommutative? For example, is the answer to the sum $4 \div 2$ the same as, or different to, the answer to $2 \div 4$?

Exercise

Write = or ≠ in the middle.

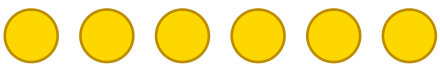
8×4		4×8	$8 \div 4$		$4 \div 8$
4×8		8×4	$4 \div 8$		$8 \div 4$
$12 \div 3$		$3 \div 12$	12×3		3×12
5×2		2×5	$5 \div 2$		$2 \div 5$

Take care with these next ones!

5×4		$4 + 5$	$9 \div 3$		$12 \div 4$
$10 \div 2$		5×1	$15 \div 3$		$30 \div 6$
4×6		$6 \div 4$	$2 \times 3 \times 4$		$4 \times 3 \times 2$

Division as multiplication: using the multiplicative inverse

Let us consider the division sum $6 \div 2$. We can think of this sum as starting with six yellow counters



and dividing them into two equal parts containing three counters each.

So, $6 \div 2 = 3$.

Here is a different way of thinking about the sum. Instead of dividing by two, we can multiply by the **multiplicative inverse** of two, namely $\frac{1}{2}$. So, instead of writing $6 \div 2$, we write $6 \times \frac{1}{2}$. We can think of this sum as arranging six half counters in a row

which come together to form three whole counters.

So, as before, $6 \div 2 = 3$.

Exercise

Write the following division sums as multiplication sums, using the multiplicative inverse.

(a) $8 \div 2$ (b) $8 \div 4$ (c) $6 \div 3$

(d) $12 \div 6$ (e) $12 \div 3$ (f) $12 \div 12$

(g) $21 \div 7$ (h) $24 \div 8$ (i) $45 \div 9$

Exercise

Write the following multiplication sums as division sums.

(a) $10 \times \frac{1}{2}$ (b) $6 \times \frac{1}{3}$ (c) $20 \times \frac{1}{4}$

(d) $10 \times \frac{1}{5}$ (e) $6 \times \frac{1}{6}$ (f) $20 \times \frac{1}{10}$

(g) $42 \times \frac{1}{6}$ (h) $44 \times \frac{1}{11}$ (i) $28 \times \frac{1}{7}$

Exercise

Calculate the answers to all the sums in the above two exercises.

Challenge! ⚠

Huw weighs six times as much as his cat, Jess.
Holding his cat in his hands, Huw stands on a weighing scale.
The scales show 42 kg.
What is Huw's weight?

Multiplication and division sums

Let us consider the following four sums.

$8 \times 4 \times 2$ $8 \times 4 \div 2$ $8 \div 4 \times 2$ $8 \div 4 \div 2$

What do you notice about the sums?

Complete the table below to answer the sums in two different ways: first by doing the red sum to begin with, and then by doing the blue sum to begin with. Leave the final column of the table empty, for now.

Sum	Answer when starting with the red sum	Answer when starting with the blue sum	Which sum needs to be done first?
$8 \times 4 \times 2$			Does not matter
$8 \times 4 \div 2$			Does not matter
$8 \div 4 \times 2$			
$8 \div 4 \div 2$			

The answers in the first two rows are the same, but the answers in the final two rows are different. Which answer is correct each time?

Because it is possible to write every division sum as a multiplication sum, using the multiplicative inverse, we can re-write the final two sums as multiplication sums like this:

$$8 \times \frac{1}{4} \times 2 \qquad 8 \times \frac{1}{4} \times \frac{1}{2}$$

Answer these sums in two ways: (Hint: $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$.)

Sum	Answer when starting with the red sum	Answer when starting with the blue sum
$8 \times \frac{1}{4} \times 2$		
$8 \times \frac{1}{4} \times \frac{1}{2}$		

The answers in each row of the above table are the same, so we can use them to complete the final column of the first table, and state which sum needed to be done first in $8 \div 4 \times 2$ and $8 \div 4 \div 2$.

In each of the four examples on the previous page, starting with the sum appearing **first on the left** (the red sum) leads to obtaining a correct answer. This in turn enables us to establish the following rule.

RULE: Given two or more multiplication or division sums, we must complete them by working from left to right.

Exercise

(a) $6 \times 2 \div 3$

(b) $6 \div 2 \times 3$

(c) $6 \div 2 \div 3$

(d) $6 \times 2 \times 3$

(e) $2 \times 6 \div 3$

(f) $6 \div 3 \times 2$

(g) $12 \div 2 \times 3$

(h) $12 \div 3 \times 2$

(i) $12 \div 3 \div 2$

(j) $5 \times 6 \div 2$

(k) $18 \div 3 \times 6$

(l) $1 \times 2 \times 3$

(m) $8 \times 3 \div 6$

(n) $40 \div 5 \div 4$

(o) $100 \div 10 \times 10$

Here is a summary of the order of operations in mathematics.

Sums in brackets

Multiplication and division, working from left to right

Addition and subtraction, working from left to right

Exercise

(a) $4 + 6 \times 2$

(b) $(4 + 6) \times 2$

(c) $4 + 6 - 2$

(d) $4 + (6 - 2)$

(e) $4 + 6 \div 2$

(f) $(4 + 6) \div 2$

(g) $12 + (3 \times 2) - 9$

(h) $(12 + 3) \times 2 - 9$

(i) $12 \div (3 \times 2) - 9$

(j) $12 + 3 \times 2 - 9$

(k) $(12 + 3 \times 2) \div 9$

(l) $(12 - 3) \times (2 + 9)$

(m) $15 + 15 \div 3$

(n) $(15 + 15) \div 3$

(o) $15 \div 15 + 3$

(p) $15 - 15 \div 3$

(q) $(15 - 15) \div 3$

(r) $15 + 15 - 3$

(s) $8 + (2 \times (2 + 3))$

(t) $(8 + 2) \times (2 + 3)$

(u) $(8 + (2 \times 2)) + 3$

A pair of brackets

Here is a mathematical sum without brackets.

$$16 - 2 + 4 - 8$$

What is the answer to the sum?

Now add **one pair of brackets** to the sum in different ways.

How many different answers can be formed?

Exercise

(a) $5 \times 2 + 3$ (b) $3 + 5 \times 2$ (c) $4 + 5 \times 2$

(d) $4 + 5 \times 3$ (e) $4 + (5 \times 3)$ (f) $(4 + 5) \times 3$

(g) $3 \times (4 + 5)$ (h) $3 \times (4 + 6)$ (i) $3 \times 4 + 6$

(j) $3 \times 4 + 6 \times 2$ (k) $6 \times 2 + 3 \times 4$ (l) $6 \times 2 + (3 \times 4)$

(m) $6 \times (2 + 3) \times 4$ (n) $6 + (2 + 3) \times 4$ (o) $(6 + 2 \times 3) \times 4$

Exercise

Add one pair of brackets to the left side of each of the following equations in order to fix them.

(a) $2 + 3 \times 4 = 20$ (b) $10 - 3 + 2 = 5$ (c) $6 + 4 \div 2 = 5$

(d) $9 - 3 \times 2 = 12$ (e) $2 \times 3 + 3 \times 2 = 24$ (f) $2 \times 3 + 3 \times 2 = 18$

(g) $10 - 3 - 2 - 1 = 6$ (h) $10 - 3 - 2 - 1 = 8$ (i) $10 - 3 - 2 - 1 = 10$

Example

(a) $\frac{12}{3} = 4$

(b) $6 + \frac{12}{3} = 6 + 4$
$= 10$

(c) $\frac{6 + 12}{3} = \frac{18}{3}$
$= 6$

Exercise

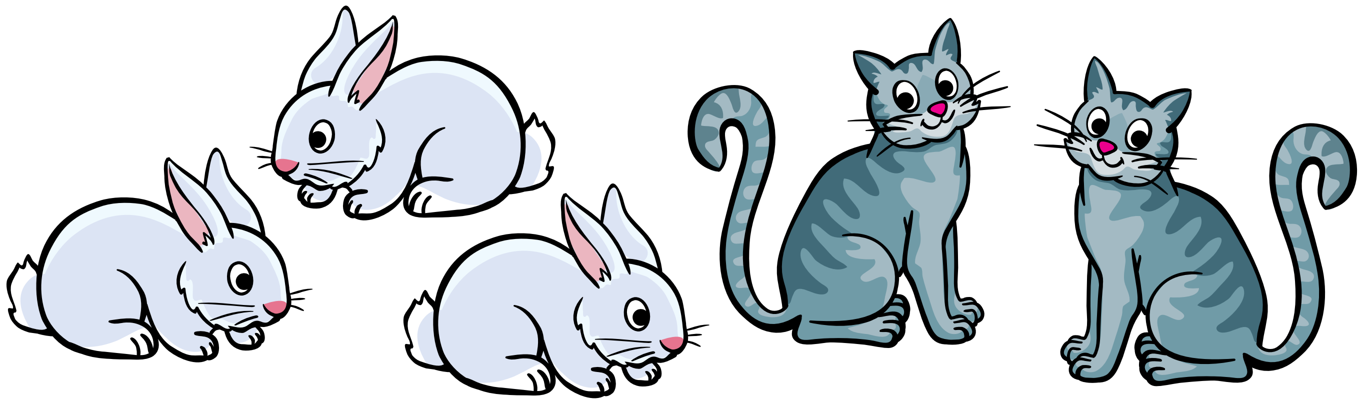
(a) $(4 + 10) \div 2$ (b) $4 + 10 \div 2$ (c) $4 + \frac{10}{2}$

(d) $\frac{10}{2} + 4$ (e) $\frac{10 + 4}{2}$ (f) $\frac{10}{2} + \frac{4}{2}$

(g) $\frac{10}{2} - 4$ (h) $4 - \frac{10}{2}$ (i) $\frac{10 - 4}{2}$

Distributivity

Siwan keeps 3 rabbits and 2 cats as pets.

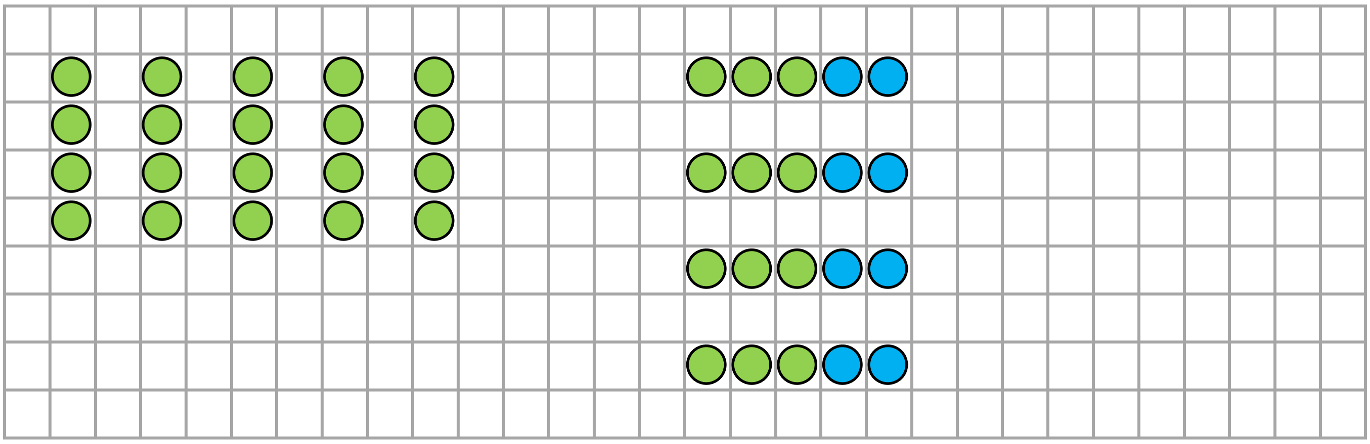


Carwyn asks Siwan: 'How many legs do all of your pets have'?

Siwan uses the following methods to count all the legs. Which method does not work?

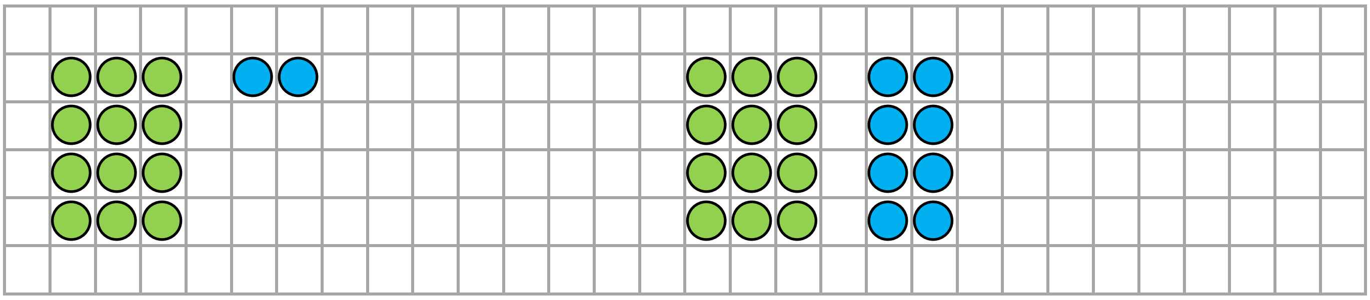
[Method 1] 4 + 4 + 4 + 4 + 4

[Method 2] 4 × (3 + 2)



[Method 3] 4 × 3 + 2

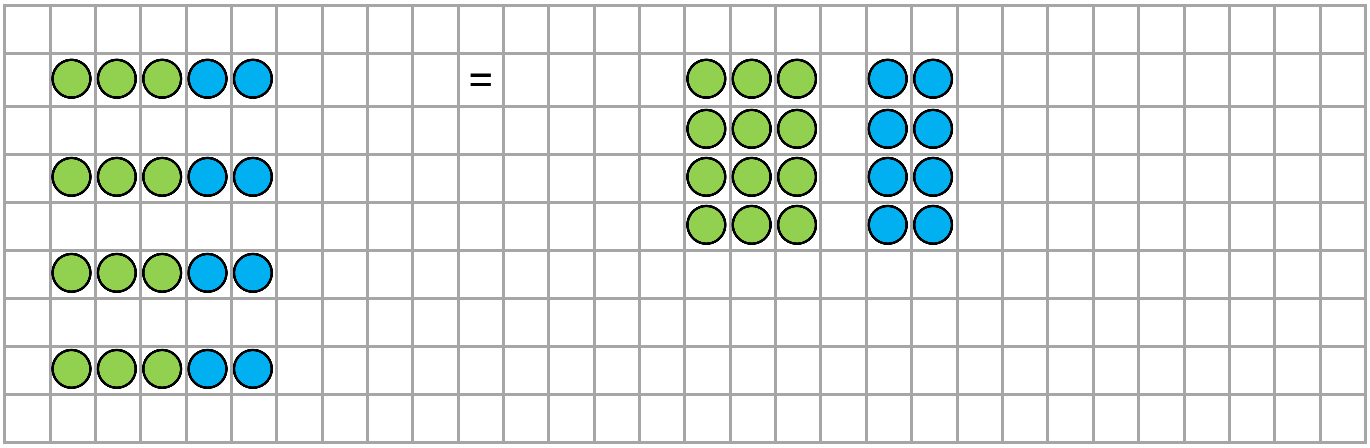
[Method 4] 4 × 3 + 4 × 2



For each method that does work, try to explain *how* Siwan went about counting the number of legs using that method.

Let us consider two of the methods that did work from the previous page.

$4 \times (3 + 2)$ $4 \times 3 + 4 \times 2$



In the first method, we multiply 4 by a group of numbers added together in a bracket. In the second method, we multiply 4 by the individual numbers (the ones appearing in the previous bracket) and then add the answers. This is an example of **distributivity** in mathematics. When moving from the first method to the second method, we say that the 4 is **distributed** across the 3 + 2 to give 4 × 3 and 4 × 2. Here are some more examples.

$3 \times (2 + 4) = 3 \times 2 + 3 \times 4$ $5 \times (6 + 9) = 5 \times 6 + 5 \times 9$

$2 \times (7 + 11) = 2 \times 7 + 2 \times 11$ $12 \times (1 + 8) = 12 \times 1 + 12 \times 8$

Exercise

Fill in the blanks below.

(a) $4 \times (2 + 3) = 4 \times \square + 4 \times 3$
(b) $4 \times (2 + 3) = 4 \times 2 + \square \times 3$
(c) $4 \times (2 + \square) = 4 \times 2 + 4 \times 3$
(d) $\square \times (2 + 3) = 4 \times 2 + 4 \times 3$
(e) $5 \times (3 + 7) = 5 \times 3 + 5 \times \square$
(f) $5 \times (\square + 7) = 5 \times 4 + 5 \times 7$
(g) $\square \times (3 + 7) = 6 \times 3 + 6 \times 7$
(h) $6 \times (3 + 8) = \square \times 3 + 6 \times 8$
(i) $10 \times (4 + 9) = \square \times 4 + 10 \times 9$
(j) $8 \times (3 + 5) = 8 \times 3 + 8 \times \square$
(k) $7 \times 3 + 7 \times \square = 7 \times (3 + 4)$
(l) $7 \times 5 + 7 \times 6 = 7 \times (\square + 6)$
(m) $7 \times 7 + 7 \times 8 = 7 \times (\square + 8)$
(n) $17 \times 7 + \square \times 8 = 17 \times (7 + 8)$
(o) $2.3 \times 4 + 2.3 \times 6 = 2.3 \times (\square + 6)$
(p) $8 \times (9.2 + 0.8) = 8 \times \square + 8 \times 0.8$

Using Distributivity

It is possible to use distributivity to split complex sums into simpler sums.

Example

$$6 \times 17 = 6 \times (10 + 7) = 6 \times 10 + 6 \times 7 = 60 + 42 = 102$$

$$4 \times 148 = 4 \times (100 + 40 + 8) = 4 \times 100 + 4 \times 40 + 4 \times 8 = 400 + 160 + 32 = 592$$

Exercise

Use distributivity to answer the following sums.

(a) 6×14 (b) 3×14 (c) 3×24

(d) 3×104 (e) 3×140 (f) 3×146

(g) 4×27 (h) 7×32 (i) 9×68

(j) 2×145 (k) 7×145 (l) 7×345

(m) 3×71 (n) 3×701 (o) 3×7001

(p) 9×18 (q) 9×38 (r) 9×98

(s) 5×64 (t) 5×136 (u) 5×2016

We can also use distributivity to combine sums in order to simplify them.

Example

$$14 \times 6 + 14 \times 4 = 14 \times (6 + 4) = 14 \times 10 = 140$$

$$6 \times 23 + 6 \times 7 = 6 \times (23 + 7) = 6 \times 30 = 180$$

Exercise

Use distributivity to answer the following sums.

(a) $18 \times 7 + 18 \times 3$ (b) $18 \times 8 + 18 \times 2$ (c) $18 \times 98 + 18 \times 2$

(d) $27 \times 6 + 27 \times 4$ (e) $5 \times 13 + 5 \times 7$ (f) $4 \times 34 + 4 \times 6$

(g) $24 \times 9 + 24 \times 1$ (h) $132 \times 7 + 132 \times 3$ (i) $2 \times 178 + 2 \times 22$

(j) $63 \times 2 + 63 \times 8$ (k) $3 \times 41 + 3 \times 59$ (l) $2.7 \times 4 + 2.7 \times 6$

Distributivity also works with subtraction sums.

Example

$3 \times 29 - 3 \times 23 = 3 \times (29 - 23)$
$= 3 \times 6$
$= 18$

$43 \times 17 - 33 \times 17 = (43 - 33) \times 17$
$= 10 \times 17$
$= 170$

Exercise

Use distributivity to answer the following sums.

(a) $3 \times 27 - 3 \times 23$ (b) $3 \times 37 - 3 \times 32$ (c) $3 \times 37 - 3 \times 27$

(d) $4 \times 54 - 4 \times 52$ (e) $4 \times 54 - 4 \times 44$ (f) $4 \times 54 - 4 \times 4$

(g) $5 \times 38 - 5 \times 32$ (h) $5 \times 48 - 5 \times 43$ (i) $5 \times 71 - 5 \times 67$

(j) $27 \times 3 - 23 \times 3$ (k) $18 \times 4 - 11 \times 4$ (l) $26 \times 5 - 16 \times 5$

(m) $54 \times 6 - 51 \times 6$ (n) $78 \times 7 - 77 \times 7$ (o) $112 \times 8 - 107 \times 8$

(p) $72 \times 9 - 64 \times 9$ (q) $27 \times 11 - 24 \times 11$ (r) $76 \times 12 - 74 \times 12$

(s) $3 \times 123 - 3 \times 121$ (t) $424 \times 5 - 421 \times 5$ (u) $2 \times 145 - 2 \times 45$

Exercise

Use distributivity to answer the following mixed sums.

(a) $14 \times 2 + 14 \times 8$ (b) $14 \times 2 + 14 \times 3 + 14 \times 5$

(c) $14 \times 2 + 14 \times 3 + 14 \times 4 + 14 \times 1$ (d) $14 \times 13 - 14 \times 3$

(e) $14 \times 7 + 14 \times 5 - 14 \times 2$ (f) $14 \times 21 + 14 \times 5 - 14 \times 16$

(g) $17 \times 3 + 17 \times 7$ (h) $3 \times 17 + 7 \times 17$

(i) $17 \times 7 - 17 \times 5$ (j) $18 \times 17 - 5 \times 17 - 3 \times 17$

(k) 6×18 (l) 4×29

(m) 16×7 (n) 36×8

(o) $16 \times 7 + 4 \times 7$ (p) $40 \times 8 - 4 \times 8$

(q) 154×3 (r) 416×2

(s) $154 \times 3 - 54 \times 3$ (t) $416 \times 2 - 2 \times 2$

Associativity

Associative operations

A mathematical **operation** (like addition or multiplication) is **associative** if re-arranging the brackets in a sum, as shown in the following exercise, does not have an effect on the answer to the sum.

Exercise

Answer the following sums.

(a) $(8 + 4) + 2$

(b) $8 + (4 + 2)$

(c) $(8 - 4) - 2$

(d) $8 - (4 - 2)$

(e) $(8 \times 4) \times 2$

(f) $8 \times (4 \times 2)$

(g) $(8 \div 4) \div 2$

(h) $8 \div (4 \div 2)$

Exercise

Write = or ≠ in the middle.

$(8 + 4) + 2$	$8 + (4 + 2)$
$(8 - 4) - 2$	$8 - (4 - 2)$
$(8 \times 4) \times 2$	$8 \times (4 \times 2)$
$(8 \div 4) \div 2$	$8 \div (4 \div 2)$

Fill in the blanks: Out of addition, subtraction, multiplication and division, the two associative operations are ________________ and ________________, and the two non-associative operations are ________________ and ________________.

With associative operations, it is possible to use their associativity to help us answer sums that contain these operations. For example, in $8 \times 4 \times 2$, is it easier to answer the sum by doing $(8 \times 4) \times 2$, or by doing $8 \times (4 \times 2)$?

Exercise

Pair the following bubbles. (The first one has been done for you.)

(4 + 9) + 3

4 × (3 × 9)

(7 × 2) × 4

(8 + 1) + 3

8 + (2 + 7)

(8 + 2) + 7

4 + (9 + 3)

(4 × 3) × 9

7 × (2 × 4)

8 + (1 + 3)

Exercise

Write = or ≠ in the middle.

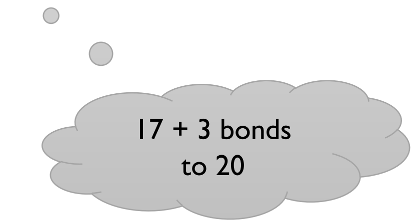
10 × (5 × 2)	(10 × 5) × 2	12 ÷ (6 ÷ 2)	(12 ÷ 6) ÷ 2
(10 – 5) – 2	10 – (5 – 2)	12 + (6 + 2)	(12 + 6) + 2
10 × (5 × 2)	(10 × 5) × 3	12 – (6 – 2)	(12 – 2) – 2
(10 × 5) × 2	10 + (5 + 2)	(12 × 6) × 2	12 × (6 × 2)
10 × (5 × 2)	10 × (2 × 5)	0 ÷ (6 ÷ 2)	(0 ÷ 6) ÷ 2

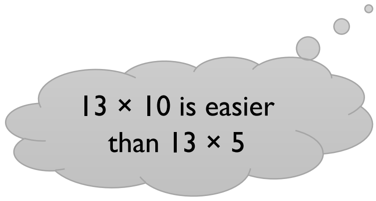
Using Associativity

The following equations all show the associativity of addition or multiplication. In each equation, circle the side that is easiest to calculate.

Example

(a) $(17 + 3) + 18 = 17 + (3 + 18)$ (b) $(13 \times 5) \times 2 = 13 \times (5 \times 2)$





Exercise

(a) $(16 + 4) + 18 = 16 + (4 + 18)$ (b) $(17 \times 5) \times 2 = 17 \times (5 \times 2)$

(c) $(23 + 8) + 12 = 23 + (8 + 12)$ (d) $(2 \times 5) \times 27 = 2 \times (5 \times 27)$

(e) $(25 + 15) + 47 = 25 + (15 + 47)$ (f) $(23 \times 4) \times 5 = 23 \times (4 \times 5)$

(g) $(39 + 23) + 7 = 39 + (23 + 7)$ (h) $(25 \times 4) \times 7 = 25 \times (4 \times 7)$

Example

Calculate the answer to $(58 + 27) + 3$, using associativity if required.

$(58 + 27) + 3 = 58 + (27 + 3)$ Using associativity (recognising a bond to 30)

$= 58 + 30$ Calculating the sum in brackets

$= 88$ Adding the numbers

Exercise

Calculate the answer to the following sums, using associativity if required.

(a) $(42 + 8) + 31$ (b) $(5 \times 2) \times 18$

(c) $(23 + 8) + 12$ (d) $(17 \times 5) \times 2$

(e) $(34 + 17) + 3$ (f) $(7 \times 25) \times 4$

(g) $48 + (23 + 7)$ (h) $18 \times (5 \times 2)$

(i) $14 + (6 + 79)$ (j) $20 \times (5 \times 9)$

(k) $35 + (15 + 128)$ (l) $2 \times (8 \times 5)$

(m) $34 + 16 + 47$ (n) $19 \times 5 \times 2$

Commutativity and Associativity

Example

Use commutativity and associativity to re-arrange the sum $2 \times (17 \times 5)$ to give the sum $(2 \times 5) \times 17$.

$2 \times (17 \times 5)$

$= 2 \times (5 \times 17)$	Using the commutativity of multiplication
$= (2 \times 5) \times 17$	Using the associativity of multiplication

Exercise

Use commutativity and associativity to re-arrange the following sums.

	Original sum	Sum after re-arranging
(a)	$2 \times (18 \times 5)$	$(2 \times 5) \times 18$
(b)	$2 \times (16 \times 5)$	$(5 \times 2) \times 16$
(c)	$4 \times (24 \times 5)$	$(4 \times 5) \times 24$
(d)	$5 \times (18 \times 6)$	$(6 \times 5) \times 18$
(e)	$(24 \times 5) \times 2$	$24 \times (5 \times 2)$
(f)	$(5 \times 27) \times 2$	$27 \times (5 \times 2)$
(g)	$(7 \times 25) \times 4$	$7 \times (25 \times 4)$
(h)	$(9 \times 20) \times 5$	$9 \times (5 \times 20)$

Exercise

Calculate the answers to the sums in the above exercise.

Challenge!

Use commutativity and associativity to re-arrange the sum

$$8 \times (2 \times 23) \times (19 \times 5) \times 5$$

to give the sum

$$(2 \times 5) \times 23 \times 19 \times (8 \times 5)$$

Example

Use commutativity and associativity to re-arrange the sum 14 + (19 + 16) to give the sum (14 + 16) + 19.

14 + (19 + 16)

= 14 + (16 + 19) Using the commutativity of addition

= (14 + 16) + 19 Using the associativity of addition

Exercise

Use commutativity and associativity to re-arrange the following sums.

	Original sum	Sum after re-arranging
(a)	25 + (39 + 15)	(25 + 15) + 39
(b)	52 + (27 + 38)	(52 + 38) + 27
(c)	15 + (45 + 34)	(15 + 45) + 34
(d)	17 + (41 + 33)	(17 + 33) + 41
(e)	17 + (41 + 33)	41 + (17 + 33)
(f)	(56 + 21) + 24	21 + (56 + 24)
(g)	(31 + 11) + 49	31 + (11 + 49)
(h)	(32 + 126) + 118	126 + (32 + 118)
(i)	(32 + 126) + 118	(118 + 32) + 126
(j)	0.8 + (0.15 + 0.2)	(0.8 + 0.2) + 0.15
(k)	(0.45 + 0.12) + 0.15	0.12 + (0.45 + 0.15)

Exercise

Calculate the answers to the sums in the above exercise.

Challenge!

Use commutativity and associativity to re-arrange the sum

$$8 + (3 + 9) + (6 + 2) + 7$$

to give the sum

$$(2 + 3) + 6 + (7 + 8) + 9$$