Numbers: Diving Deeper

A workbook for Curriculum for Wales

Mathematics and Numeracy Area of Learning and Experience

Progression Step 3

Teacher's Guide

Version 1 (August 4th, 2021)

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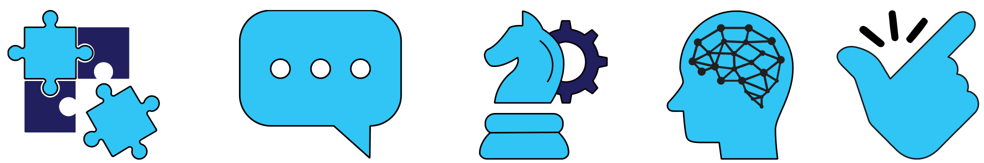
Mathematical proficiencies icons:



Conceptual understanding	*Communication using symbols*	*Strategic competence*	*Logical reasoning*	*Fluency*

Contents

This workbook deals with the following descriptions of learning.

The number system is used to represent and compare relationships between numbers and quantities.

THE NUMBER SYSTEM

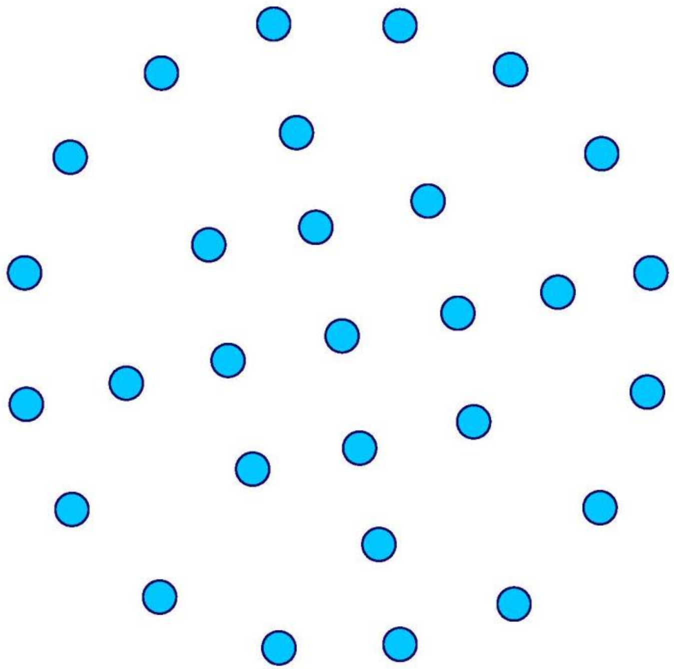
- I can use a range of representations to develop and secure my understanding that the value of a digit is related to its position.
- I can use a range of representations to extend my understanding of the number system to include negative values and fractions.

RELATIONSHIPS WITHIN THE NUMBER SYSTEM

- I can demonstrate my understanding that non-integer quantities can be represented using fractions (including fractions greater than 1). I can use my knowledge of equivalence to compare the size of simple fractions.
- I can demonstrate my understanding that a fraction can be used as an operator or to represent division. I can understand the inverse relation between the denominator of a fraction and its value.

Place Value

How many counters are shown below?

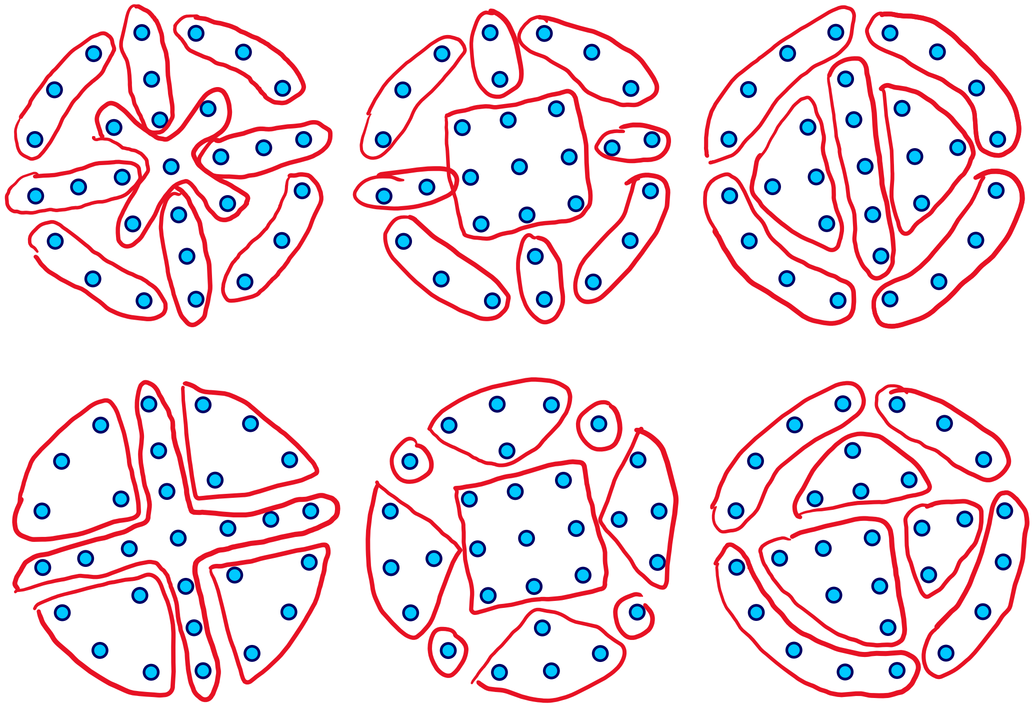


Circle the counters to show how you counted them.

Did other people count the counters in different ways? Discuss.

The purpose of this exercise is to discuss different strategies of counting the counters in the picture.

Do you start with the '9 square' in the middle, or with the circle on the outside? If you start with the circle on the outside, how do you split it up?



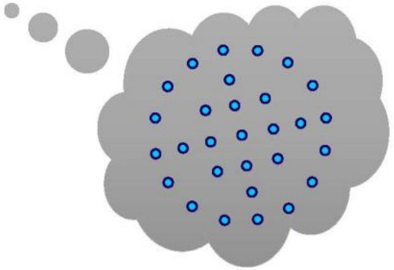
It is likely that nobody grouped the counters in sets of ten, even though our usual number system is based on grouping in sets of ten. This workbook discusses different ways of counting, based on grouping not by ten, but by a different *base* number.

Encourage the learners to use appropriate mathematical language whilst discussing how to count the counters.

Number, numeral and digit

Hopefully you agree that twenty-nine counters were shown on the previous page. Usually, we write this **number** as the **numeral** 29, with the **digit** 2 showing two tens, and the **digit** 9 showing nine units.

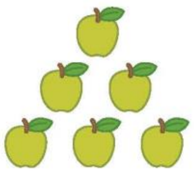
- **Digits** are used to form **numerals**.
- Numerals represent the idea of a **number**.



Exercise

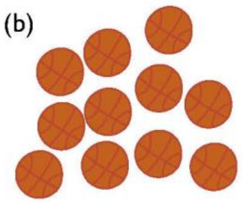
Which numbers are shown in the pictures below? Circle the correct numeral.

(a)



(i) 8 (ii) 7 (iii) 6

(b)

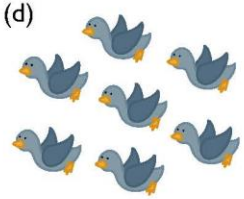


(i) 15 (ii) 13 (iii) 11

(c)

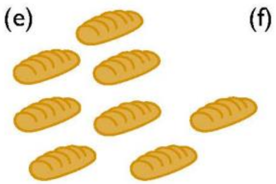
(i) 14 (ii) 16 (iii) 18

(d)



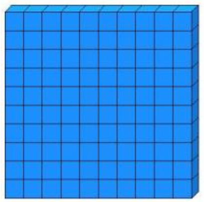
(i) 8 (ii) 7 (iii) 6

(e)



(i) 5 (ii) 7 (iii) 8

(f)



(i) 100 (ii) 10 (iii) 50

The following YouTube video discusses the terms number, numeral and digit:
https://www.youtube.com/watch?v=jkWMJVBmYL0

In our usual system of writing numbers, we have ten different digits available:

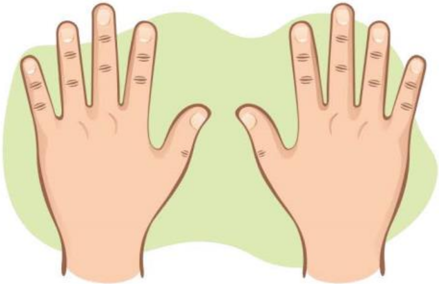
0 1 2 3 4 5 6 7 8 9

It is possible to use these digits, either on their own or in combination with other digits, to form numerals, e.g.

3 7 16 72 109 167 267 926 1029 3928

Each numeral represents the idea of a particular number.
For example, the numeral 3 represents the idea of 'three', which could (for example) represent three hot air balloons flying together.

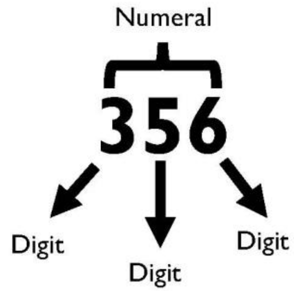
Base ten



Usually, a person has ten fingers and thumbs on their hands, so humans have learnt to write numerals using a **base ten** system.

In this system, we place the **ten digits** 0 to 9 in columns to write different numerals. For example, it is possible to write the number three hundred and fifty-six as follows:

hundreds	tens	units
3	**5**	**6**
$10^2 = 100$	10	1



Exercise

A 3-digit code is used to open the lock shown on the right. The 3-digit code must use all of the digits

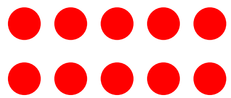
2, 5 and 9.

Write down all the different numerals that could represent the code.

We are used to writing numerals in base ten, but is this the best base to use?

The factors of 10 are 1, 2, 5 and 10, which means that there are only four ways of arranging ten counters in the shape of a rectangle:

1 × 10 One row of ten
2 × 5 Two rows of five



5 × 2 Five rows of two
10 × 1 Ten rows of one

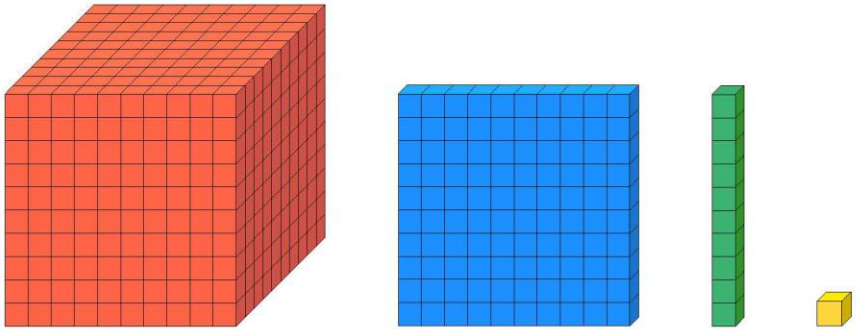
It is not straightforward to divide ten counters into three parts, or into four parts.

Base twelve would have been better, because 12 has more factors (1, 2, 3, 4, 6 and 12). Base twelve was not adopted of course because we do not have twelve fingers and thumbs on our hands. BUT! If you look at your fingers, you have 3 knuckles on each finger. So, it is possible to count in twelves using the four fingers on one of your hands. You can even use your thumb to count, touching each of the knuckles in turn.

Even though we do not write numbers in base twelve, counting in sets of twelve is used in different places. Twelve hours are shown on a standard clock; there are twelve inches in a foot; an old shilling was made up of twelve pennies; and there are twelve months in a year.

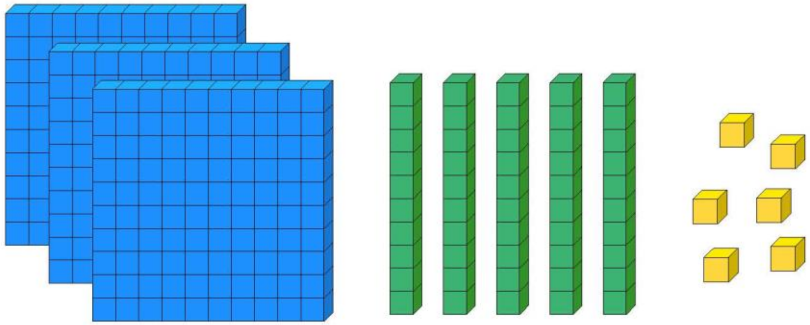
Dienes blocks (base ten)

It is possible to use the following Dienes blocks to show numbers that have been written in base ten.



$10^3 = 1000$, Thousand $10^2 = 100$, Hundred 10, Ten 1, Unit

For example, here is one way of using the blocks to show the numeral 356:



Exercise

Use the Dienes blocks to show the following numerals.

(a) 3	(b) 30	(c) 300	(d) 3000
(e) 303	(f) 3030	(g) 33	(h) 3330
(i) 2	(j) 24	(k) 243	(l) 2431
(m) 2016	(n) 4007	(o) 280	(p) 5309

Dienes blocks provide a concrete way of showing that the value of a digit depends upon its location in a numeral.

Physical sets of these manipulatives can be purchased, or electronic versions are available on the following websites:

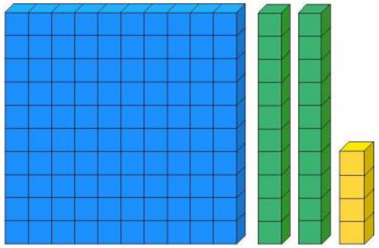
https://mathsbot.com/manipulatives/blocks

https://mathigon.org/polypad

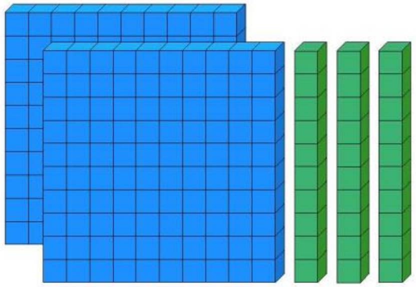
Exercise

Which numbers are shown below?

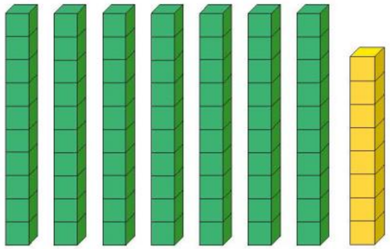
(a)



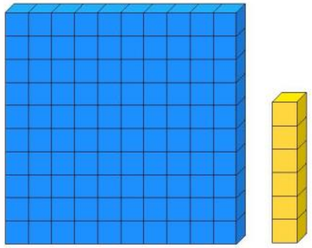
(b)



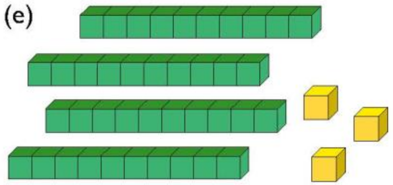
(c)



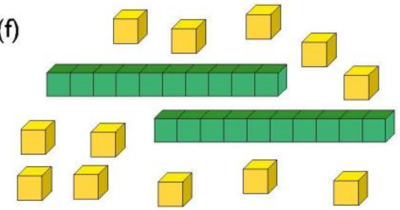
(d)



(e)



(f)



For the final question, (f), ask whether there is a better way of presenting the number under consideration.

- It would be easier to recognise the number if the yellow blocks were arranged better.
- It would be possible to exchange ten of the yellow blocks for one green rod.

Look out for not including/mis-placing the zero digit in the answers to (b) and (d). That is, correct the misconception that the answer to (b) is 23 or 203, and the answer to (d) is 16 or 160.

Do not move on to work with other bases until the learners are comfortable with the idea of place value in base ten.

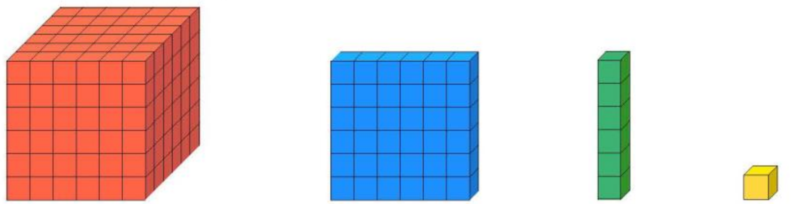
Other Bases

Base ten is not the only way of writing numbers. For example, base six uses the six digits 0, 1, 2, 3, 4, 5 to write numbers that have been grouped into sixes.

Counting in base six

1	One	20	Two sixes
2	Two	21	Two sixes and one
3	Three	22	Two sixes and two
4	Four	23	Two sixes and three
5	Five	24	Two sixes and four
10	Six	25	Two sixes and five
11	One six and one	30	Three sixes
12	One six and two	31	Three sixes and one
13	One six and three	32	Three sixes and two
14	One six and four	33	Three sixes and three
15	One six and five	34	Three sixes and four ....

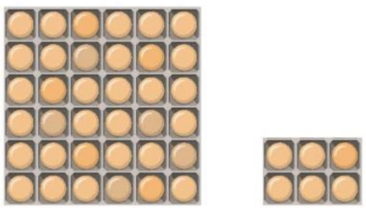
Dienes Blocks (base six)



$10^3 = 1000$, Six cubed $10^2 = 100$, Six squared   10, Six   1, Unit

Egg boxes

Base six is useful when packing eggs, as boxes of the following shapes are used.



From this page onwards, the workbook starts to develop the idea of writing numbers in a different base to base ten.

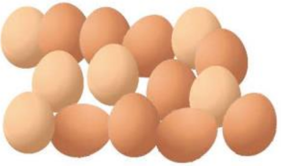
To the teacher, this will look strange to begin with, and perhaps difficult to understand, but remember that this is the experience of learners when writing numbers in base ten for the first time. By studying numbers in different bases, learners will deepen their understanding of place value, leading to a better appreciation of numerals in base ten.

For base six, notice that the digits 6, 7, 8 or 9 do not appear. There is also not a special word for 'six squared' (like a 'hundred' is a special word for 'ten squared'), so continuing to count in base six would look as follows.

35	Three sixes and five	52	Five sixes and two
40	Four sixes	53	Five sixes and three
41	Four sixes and one	54	Five sixes and four
42	Four sixes and two	55	Five sixes and five
43	Four sixes and three	100	Six squared
44	Four sixes and four	101	Six squared and one
45	Four sixes and five	102	Six squared and two
50	Five sixes	103	Six squared and three
51	Five sixes and one	104	Six squared and four

Be careful with your language when discussing numerals in base six. For example, you should not refer to 10 as 'ten'.

Egg packing



How many eggs are shown above?

To begin, write down your answer using base ten: ___________

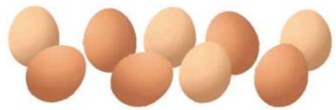
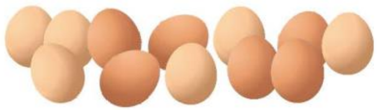
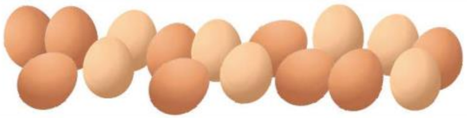
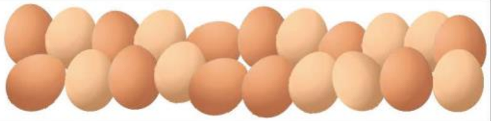
How many **full** boxes of 6 eggs would the above eggs fill? _______

How many eggs would be left over after filling the boxes? _______

How many eggs are shown above? This time, write down your answer using base six: _________

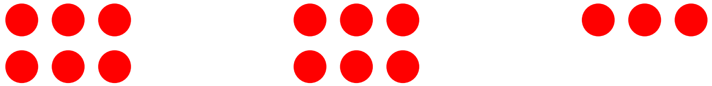
What is the connection between your answer in base six and the previous two answers? ____

Complete the following table.

Eggs	**Number of eggs in base ten**	**Number of eggs in base six**
		
		
		
		

This exercise emphasises the connection between writing a number in base six and grouping a set of objects in sixes.

If there are physical manipulatives available, e.g., counters, use them to model grouping fifteen eggs in sixes.

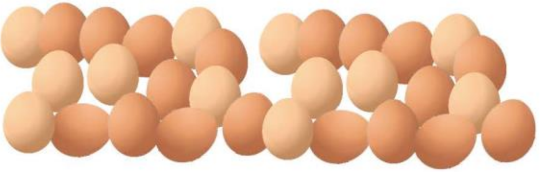


There are two full boxes and three eggs left over, so 15 in base six is written as 23, ‘two sixes and three’.

Digital counters are available here: https://mathsbot.com/manipulatives/counters

To change a number from base ten into base six, it must be reasoned how many sixes fit into the base ten number, and how many are left over. There is therefore a natural connection with division by six.

Egg packing in fours

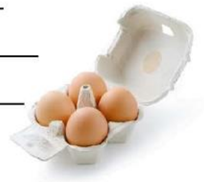


How many eggs are shown above?

To begin, write down your answer using base ten: ____________

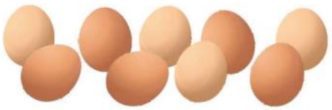
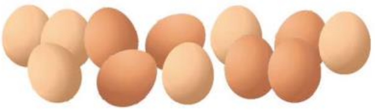
How many **full** boxes of 4 eggs would the above eggs fill? ________

How many eggs would be left over after filling the boxes? _______



How many eggs are shown above? This time, write down your answer using base four: ____________

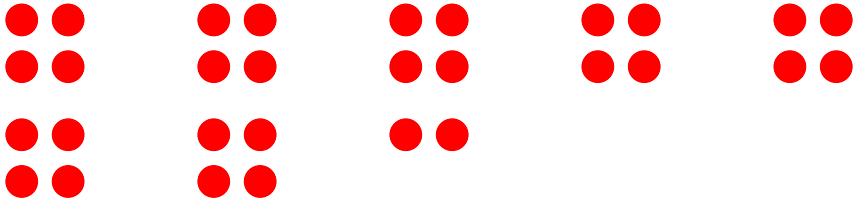
Complete the following table.

Eggs	Number of eggs in base ten	Number of eggs in base four
		
		

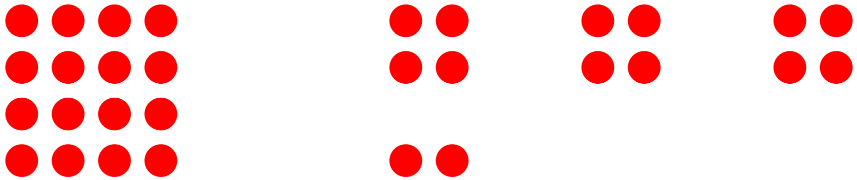
3 eggs are required to make one cheese and tomato omelette. Rhodri cooks 6 omelettes, one for each member of the family. How many boxes of 4 eggs does Rhodri need to buy?

This page introduces base four for the first time, where the digits 0, 1, 2 and 3 are used.

Notice that there are 30 eggs in the picture at the top of the page. Grouping them in fours, we arrive at the following arrangement.



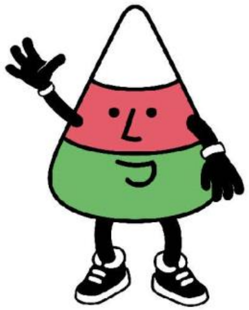
It is a temptation to write 30 in base four as 72, because we have 7 full boxes of eggs, and two eggs left over. Remember however that the digit 7 cannot appear in base four, so we must consider bigger groups of 16 eggs, or 4^2 eggs.



There is one group of 16 eggs, three groups of 4 eggs, and 2 left over, so the number 30 is written as 132 in base four. (This numeral would be read as 'four squared, three fours and two'.)

It would be possible to show the above process using the website https://mathsbot.com/manipulatives/blocks, changing the *Base* to be 4, and exchanging four green rods to form one blue square.

Base eight

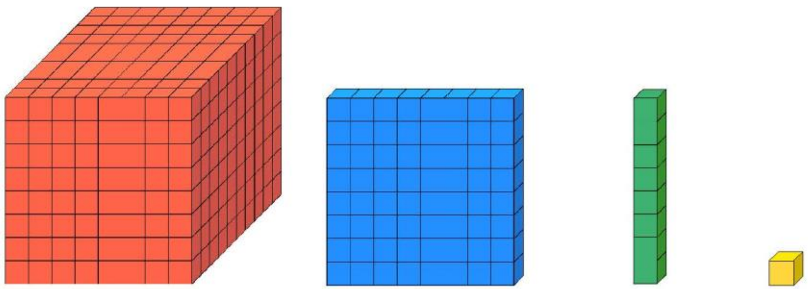


When watching cartoons, have you ever noticed that the characters very often have four fingers and thumbs on each hand? The reason for this is that it is quicker to animate a character with less fingers, but it does mean that Mickey Mouse, Spongebob Squarepants and Mr Urdd need to count using **base eight**.

Counting in base eight

1	One	15	One eight and five
2	Two	16	One eight and six
3	Three	17	One eight and seven
4	Four	20	Two eights
5	Five	21	Two eights and one
6	Six	22	Two eights and two
7	Seven	23	Two eights and three
10	Eight	24	Two eights and four
11	One eight and one	25	Two eights and five
12	One eight and two	26	Two eights and six
13	One eight and three	27	Two eights and seven
14	One eight and four	30	Three eights ....

Dienes blocks (base eight)



$10^3 = 1000$, Eight cubed $10^2 = 100$, Eight squared 10, Eight 1, Unit

Did you know?

The old American language Yuki used base eight to count as the spaces between the fingers were used to count, not the fingers themselves.

Here is a YouTube video showing the history behind using four fingers and thumbs in animation: https://www.youtube.com/watch?v=0QZFQ3gbd6I

More information about the Yuki language can be found here: https://en.wikipedia.org/wiki/Yuki_language

An alternative method of counting in eights would be to use the knuckles on a closed fist.

Biscuit packing

The company 'Delicious Biscuits' packs biscuits in packs of 8 biscuits.

On Monday, the baker bakes 30 delicious biscuits.

How many full packets of biscuits does this give? _______

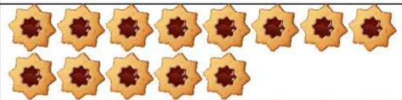
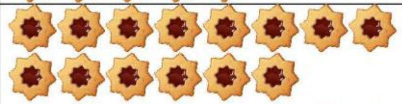
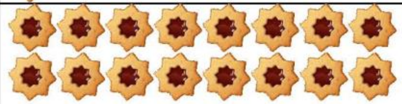
How many biscuits are left over on Monday after filling the boxes? _______

Use the two previous answers to write 30 in base eight. _______

The baker bakes 35 fresh delicious biscuits on Tuesday. He adds the left-over biscuits from Monday to the pile of biscuits. How many full packets of biscuits will be ready to sell on Tuesday? _______

How many biscuits are left over on Tuesday? _______

For the rest of the week, here are the number of fresh biscuits baked each day. Complete the table to show how many fresh biscuits were baked each day.

Day	Fresh biscuits baked today	Number of biscuits in base ten	Number of biscuits in base eight
Wednesday			
Thursday			
Friday			
Saturday			

The name of the system of writing numbers in base eight is the octal system. This system was used by early computers, before the hexadecimal system took over, where the sixteen digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F are used.

An extension task here would be to repeat the exercise in the table, but allowing biscuits left over from the previous day to be added to the fresh biscuits baked each day.

Notice that the shape of the biscuit here has been carefully chosen to have a rotational symmetry of order 8. What order of rotational symmetry do other biscuits have?

Addition in base eight

Example

	3	5			2	6	3			5	6			4	3
+		6		+	3	4	7		+	3	7		+	1	5
	4	3			6	3	2		1	1	5			6	0
	1				1	1				1				1	

Exercise

	2			2			2			2				3	2
+	4		+	5		+	6		+	7		+			7
	4	3			4	3			4	3			4	3	
+	2	1		+	2	6		+	3	6		+	7	0	
	3	4			2	6	4			2	7	3	4	2	4
+	7	6		+	6	2	3		+		3	6	5	3	1
	3	0	4	5				3	2	5				1	2
+	2	4	7	4		+	1	7	3	4		+	2	7	0

The aim of this page is to deepen learners' understanding of what happens in column addition when carrying over to the next column.

Example 1

- We start with 5 + 6. Usually in base ten the answer would be eleven, but in base eight we need to think of the eleven as one eight and three units. We carry over the one eight to the eights column on the left, and write the 3 units in the units column.
- To finish, we add 3 to 1 to obtain 4.

Example 2

- We start with 3 + 7. Usually in base ten the answer would be ten, but in base eight we need to think of the ten as one eight and two units. We carry over the one eight to the eights column on the left, and write the 2 units in the units column.
- Next, we add 6 + 4 + 1. Usually in base ten the answer would be eleven, but in base eight we have one eight and three units (and, technically, this would be one eight squared and three eights). We carry over the 1 to the eight squared column on the left, and write the 3 in the eights column.
- To finish, we add 2, 3 and 1 to obtain 6.

More exercises of this type are available on the website https://mathsbot.com/doNows/addition by changing the *Base* to be 8. (The website allows you to attempt addition sums in any base from 2 to 16.)

Subtraction in base eight

Example

					3	1			4	1			4	12	1
	6			7	~~4~~	3			~~5~~	2			~~5~~	~~3~~	6
–	2		–		1	5		–	2	6		–	1	4	7
	4			7	2	6			2	4			3	6	7

Exercise

	4	5			4	5			4	5			4	5	
–		4		–		5		–		6		–		7	
	3	4			3	4			3	4			3	4	
–	1	3		–	1	6		–	2	7		–	1	5	
	3	4	2			3	4	2			3	4	2		
–	2	1	6		–	2	5	1		–	1	6	4		
	3	1	5			4	1	3	2	5			4	0	3
–	1	2	6		–	1	3	4	0	6		–	2	3	5

The aim of this page is to deepen learners' understanding of what happens in column subtraction when 'borrowing' from the column to the left.

Example 2

- We start with 3 – 5. This is not possible without using negative numbers, so we borrow eight from the eights column. The original four eights changes to be three eights, and the original 3 units changes to be one eight three. We can then do one eight three subtract five, to leave six. (In base ten, this would be 8 + 3 – 5 = 6.)
- Second, we do 3 – 1 in the eights column to leave 2.
- To finish, we do 7 – 0 = 7.

Example 3

- We start with 2 – 6. This is not possible without using negative numbers, so we borrow eight from the eights column. The original five eights changes to be four eights, and the original 2 units changes to be one eight two. We can then do one eight two subtract six, to leave four. (In base ten, this would be 8 + 2 – 6 = 4.)
- To finish, we do 4 – 2 = 2 in the eights column.

More exercises of this type are available on the website https://mathsbot.com/doNows/subtraction by changing the *Base* to be 8. (The website allows you to attempt subtraction sums in any base from 2 to 16.)

Base two: Binary Numbers

Binary numbers are the numerals that computers understand.
Only two digits are used in this system: 0 and 1. Simply put:

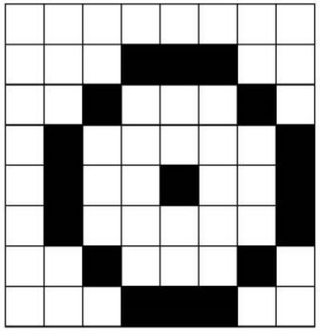
0 (zero) = off (no electrical current);

1 (one) = on (electrical current flowing).

Bitmaps

A bitmap uses binary numbers to represent simple black and white pictures.

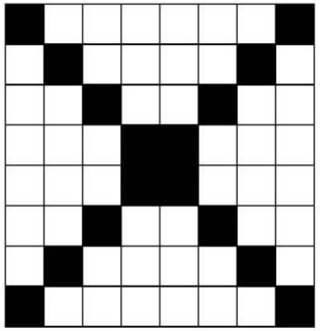
A '0' represents a white square and '1' represents a black square.



0	0	0	0	0	0	0	0
0	0	0	1	1	1	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	0	0	1
0	1	0	0	1	0	0	1
0	1	0	0	0	0	0	1
0	0	1	0	0	0	1	0
0	0	0	1	1	1	0	0

Exercise

Write down the bitmap for the following picture.



Because bitmaps provide one method of storing images on a computer, there is an opportunity here to link to the following Progression Step 3 description of learning for "Computation is the foundation of our digital world":

- I can explain how data is stored and processed.

The examples on this page are examples of black and white bitmaps, but the bitmap format can also be used to store colour images on a computer. The disadvantage of this format is that file sizes are usually large, with no compression used. The advantage of the format is that a perfect (lossless) copy of the image is stored, which is not true of a format like JPEG, where some information about the image is lost.

Exercise

On the following grids, design your own pictures. Then, write down the bitmap for the picture. Remember to use '0' to represent a white square and '1' to represent a shaded square.

Exercise

Complete the picture for the following bitmap.

0	0	1	0	0	1	0	0
0	1	0	1	1	0	1	0
1	0	0	0	0	0	0	1
0	1	0	0	0	0	1	0
0	0	1	0	0	1	0	0
0	0	0	1	1	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	0	0	1	0

For your information…

- One digit (0 or 1) on a computer is called a bit.
- There are eight bits in a byte. This means that a computer requires one byte of storage to store one row of the bitmaps on the left.
- There are 1,024 bytes in a kilobyte. (Note that $2^{10} = 1{,}024$.)
- There are 1,024 kilobytes in a megabyte.
- There are 1,024 megabytes in a gigabyte.
- There are 1,024 gigabytes in a terabyte.

Changing between base ten and base two

In our usual base ten system, the value of each new column to the left is **ten times** greater than the previous column.

Ten thousand	Thousand	Hundred	Ten	Unit
$10^4 = 10000$	$10^3 = 1000$	$10^2 = 100$	10	1

× 10 × 10 × 10 × 10

In the base two system however, the value of each new column to the left is **two times** greater than the previous column.

Sixteen	Eight	Four	Two	Unit
$10^4 = 10000$ ($2^4 = 16$)	$10^3 = 1000$ ($2^3 = 8$)	$10^2 = 100$ ($2^2 = 4$)	10 (2)	1

× 2 × 2 × 2 × 2

This means that the base two numeral 10110 represents the base ten numeral 16 + 4 + 2 = 22.

Exercise

Change the following base two numerals to be in base ten.

Base two						Base ten	Base two						Base ten
	16	8	4	2	1			16	8	4	2	1	
(a)	0	1	0	1	0		(g)	0	1	1	1	0	
(b)	1	0	1	0	1		(h)	1	0	0	0	1	
(c)	1	1	0	0	1		(i)	1	1	1	0	1	
(d)	0	0	1	0	0		(j)	0	0	1	1	1	
(e)	0	0	1	0	1		(k)	0	0	0	1	1	
(f)	1	0	0	1	0		(l)	1	1	1	1	0	

In general, in a base n number system, the value of each column to the left would be n times more than the previous column.

n^4	n^3	n^2	n	Unit
$10^4 = 10000$	$10^3 = 1000$	$10^2 = 100$	10	1

$\times n$ $\times n$ $\times n$ $\times n$

In order to write 9 in base two, we need to consider which columns can be used to give a total of 9.

16	**8**	**4**	**2**	**1**
0	1	0	0	1

Because 9 = 8 + 1, the base ten numeral 9 is written as 1001 in base two.

Exercise

Complete the following table.

Base ten	Base two					Base ten	Base two				
	16	8	4	2	1		16	8	4	2	1
1						17					
2						18					
3						19					
4						20					
5						21					
6						22					
7						23					
8						24					
9	0	1	0	0	1	25					
10						26					
11						27					
12						28					
13						29					
14						30					
15						31					
16											

Take the opportunity to discuss which strategies could be used to complete the table.

- If the table was completed in order (1, 2, 3, …), were any patterns seen? For example, notice that the contents of the final two columns (2 and 1) repeat every 4 rows (01, 10, 11, 00, ...).
- Which method did the learners use to change a base ten numeral to be in base two? For example, did they attempt to subtract (in order) 16, 8, 4, 2, 1 from the base ten numeral?

Extension 1: Consider how to write numbers greater than 31 in base two. To complete this, more columns to the left will be required, representing 32, 64, 128, etc. How far can the learners do the doubling in their heads?

Extension 2: Consider how to write decimals in different bases.

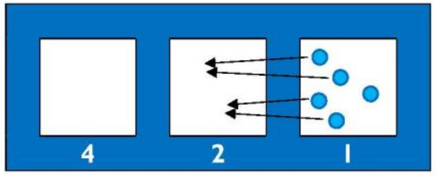
- Before starting, a visit to the website https://www.mathspad.co.uk/i2/teach.php?id=decimalBlocksTool would be useful, to revise how base ten decimals work.
- The 'decimal point' needs to be renamed, for example to the 'binary point'.
- How can the fraction $\frac{1}{2}$ be expressed in different bases?

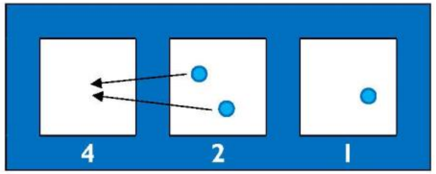
Learning more about bases: Exploding dots

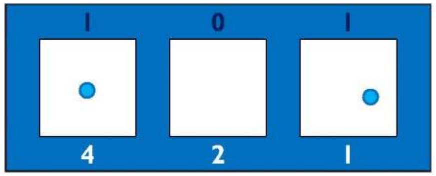
Go to the website https://www.explodingdots.org/ and work your way through ISLAND 1 (MECHANIA).

Example

This is what happens in a 2 → 1 machine if 5 dots are placed in the box on the right.







Exercise

Use squared paper to show what happens if the following number of dots are placed in the box on the right.

(a) 1 (b) 2 (c) 3

(d) 4 (e) 6 (f) 7

Extension

Repeat the above exercise, but this time use a 3 → 1 machine.

The website Exploding Dots (https://www.explodingdots.org/) introduces an alternative way of thinking about binary numbers (and numbers in other bases) by using dots to represent the numbers.

If you have access to a set of computers, it would be well worth spending a lesson looking at the website. It would be possible to complete the exercise on this page after completing the first part of the website, the '*Mechania*' island.

Remember also that the website https://mathigon.org/polypad has an interactive version of the Exploding Dots system, under the "Numbers" section.

Alien Invasion

5 aliens are shown below. Each one counts in a different way.

1st: Purple Peter counts in **twos.**

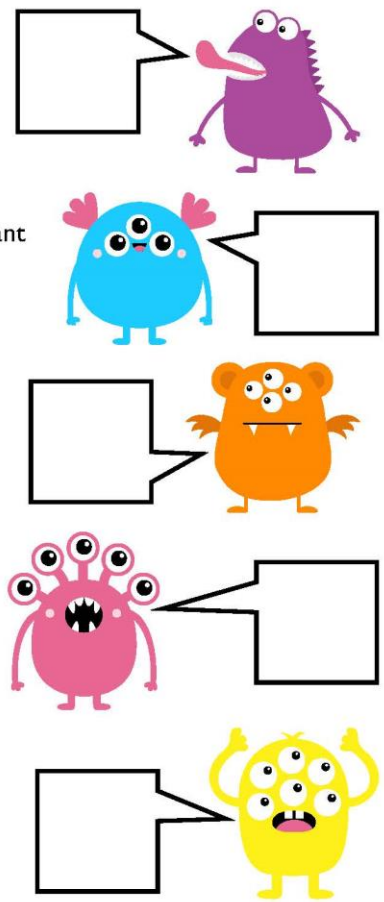
2nd: Blue Billie counts in **threes.**

3rd: Orange Ollie counts in **fours.**

4th: Pink Penny counts in **fives.**

5th: Yellow Yasmin counts in **sixes.**

The aliens have all just landed on Earth and want to count the number of fingers and thumbs shown below.



Write down their answers in the speech bubbles.

The aliens here count in different bases, based on the number of their eyes!

Purple Peter (Base 2)

16	8	4	2	1
1	1	0	0	1

Blue Billie (Base 3)

9	3	1
2	2	1

Orange Ollie (Base 4)

16	4	1
1	2	1

Pink Penny (Base 5)

25	5	1
1	0	0

Yellow Yasmin (Base 6)

6	1
4	1

Bases, bases, bases!

Complete the following table.

Base ten	Base nine	Base eight	Base seven	Base six	Base five	Base four	Base three	Base two
1								
2								
3								
4								
5								
6								
7								
8								
9								
10								
11								
12								
13								
14								
15								
16								
17								
18								
19								
20								
21								
22								
23								
24								
25								
26								
27								
28								
29								
30								
31								

Answers:

Base ten	Base nine	Base eight	Base seven	Base six	Base five	Base four	Base three	Base two
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	10
3	3	3	3	3	3	3	10	11
4	4	4	4	4	4	10	11	100
5	5	5	5	5	10	11	12	101
6	6	6	6	10	11	12	20	110
7	7	7	10	11	12	13	21	111
8	8	10	11	12	13	20	22	1000
9	10	11	12	13	14	21	100	1001
10	11	12	13	14	20	22	101	1010
11	12	13	14	15	21	23	102	1011
12	13	14	15	20	22	30	110	1100
13	14	15	16	21	23	31	111	1101
14	15	16	20	22	24	32	112	1110
15	16	17	21	23	30	33	120	1111
16	17	20	22	24	31	100	121	10000
17	18	21	23	25	32	101	122	10001
18	20	22	24	30	33	102	200	10010
19	21	23	25	31	34	103	201	10011
20	22	24	26	32	40	110	202	10100
21	23	25	30	33	41	111	210	10101
22	24	26	31	34	42	112	211	10110
23	25	27	32	35	43	113	212	10111
24	26	30	33	40	44	120	220	11000
25	27	31	34	41	100	121	221	11001
26	28	32	35	42	101	122	222	11010
27	30	33	36	43	102	123	300	11011
28	31	34	40	44	103	130	301	11100
29	32	35	41	45	104	131	302	11101
30	33	36	42	50	110	132	310	11110
31	34	37	43	51	111	133	311	11111

Further reading on different bases:

https://sketchcpd.com/viewBlog?id=25

The 1089 Problem

Here is a 3-digit number with decreasing digits:	742
Write the digits in reverse order:	247
Subtract:	495
Write the digits in reverse order:	594
Add:	1,089

What happens if you start with 832?

Here is a 3-digit number with decreasing digits:	832
Write the digits in reverse order:	_____
Subtract:	
Write the digits in reverse order:	_____
Add:	_____

Now start with a 3-digit number of your choice:

Here is a 3-digit number with decreasing digits:	
Write the digits in reverse order:	_____
Subtract:	
Write the digits in reverse order:	_____
Add:	_____

Why is the answer always 1089?

Challenge!

Repeat the above process, but work in base eight, not base ten.

Hint: Be extra careful when 'borrowing' during the subtraction sum.

If the final total is 1089 for base ten, what is the final total for base eight?

____

In base ten, why is the answer always 1089?

Let the initial number be abc, with $a > b > c$. We can write this number as $100a + 10b + c$ so that writing the digits in reverse order gives the number $100c + 10b + a$.

On subtraction, we obtain the number $(100a + 10b + c) - (100c + 10b + a) = 99a - 99c = 99(a - c)$. Because the original number had decreasing digits, $a - c$ must be between 2 and 9. So, $99(a - c)$ must be one of 198, 297, 396, 495, 594, 693, 792, 891. The following table shows that the final total must always be 1089.

$99(a-c)$	Reverse the digits	Add
198	891	1089
297	792	1089
396	693	1089
495	594	1089
594	495	1089
693	396	1089
792	297	1089
891	198	1089

So, what about base eight? For example, let us start with 542. Writing the digits in reverse order gives 245, and subtracting 542 – 245 in base eight gives 275. Reversing the digits gives 572, and adding 275 to 572 in base eight gives 1067. This is the total for any initial 3-digit number with decreasing digits; it would be possible to prove this by adapting the above proof for base eight.

So, 10**89** for base **10**; 10**67** for base **8**; is there a pattern here that continues in other bases?

Other Numerals

We are familiar with numerals such as 238 and 873, but other types of numerals exist.

Western Arabic Numerals

Chinese Numerals

Eastern Arabic Numerals

Roman Numerals

The Romans had their own way of writing numbers: they used the following digits.

1	5	10	50	100	500	1,000
I	V	X	L	C	D	M

The above digits can be combined to form the following numerals.

1	2	3	4	5	6	7	8	9	10
I	II	III	IV	V	VI	VII	VIII	IX	X

Rule 1

If a digit appears **after** a greater (or equal) digit, then the digit is **added**.

For example:

VI = V + I or LXX = L + X + X
= 5 + 1 = 50 + 10 + 10
= 6 = 70

Rule 2

If a digit appears **before** a greater digit, then the digit is subtracted from that digit.

For example:

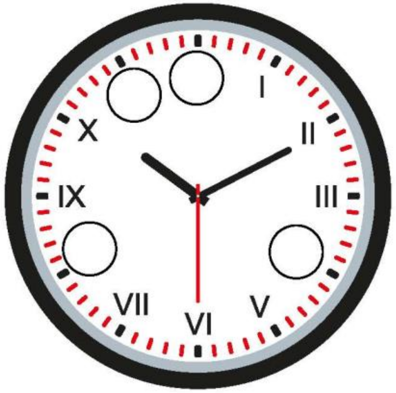
IV = V – I or IX = X – I
= 5 – 1 = 10 – 1
= 4 = 9

Further information about different numerals is available on the website https://en.wikipedia.org/wiki/Numeral_system

With Roman numerals, usually IV is used for 4, but on clocks IIII is often seen.

Exercise

Write the missing Roman numerals in the clock on the right.



Exercise

Change the following base ten numerals to be Roman numerals.

(a) 31 (b) 160

____________ ____________

(c) 1,530 (d) 2,025

____________ ____________

Exercise

Change the following Roman numerals to be base ten numerals.

(a) XXVI (b) CXVI (c) CDLII

(d) MMMCLXVII (e) MMDCCLXXIV (f) MCLXXVIII

Exercise

Here are two sets of digits from 0 to 9.

0 1 2 3 4 5 6 7 8 9

0 1 2 3 4 5 6 7 8 9

Use the above digits to create the following 4-digit numerals.
Use each digit at most once.

Greatest odd number __________ Greatest even number __________

Smallest multiple of 5 __________ Greatest multiple of 3 __________

Closest number to 5,000 ________ Closest multiple of 9 to 2,000 ________

For the exercise on the bottom of the page, the answers must contain four digits, so that (e.g.) ‘5’ is not acceptable as the smallest multiple of 5. Similar problems to this one can be found on the website of the late Don Steward: https://donsteward.blogspot.com/

Negative Numbers

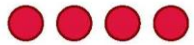
Double sided counters

A yellow counter is worth +1 and a red counter is worth –1.
We say "one" for +1 and "negative one" for –1.

Example

(a) The diagram below shows 6.

(b) The diagram below shows –4.



Exercise

Which numbers are shown in the following diagrams?

(a)

(b)

(c)

(d)

Exercise

Use double sided counters (physical or on-line) to show the following numbers.

(a) 5 (b) –2 (c) 9 (d) –6

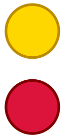
Zero Pairs

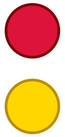
One yellow counter and one red counter, together, represents zero.
(Discuss why this is true.)

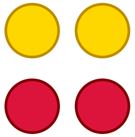
We call a pair of yellow and red counters a "zero pair". It is possible to add a zero pair to a set of counters, or (if possible) remove a zero pair from a set of counters, without affecting the value of the counters.

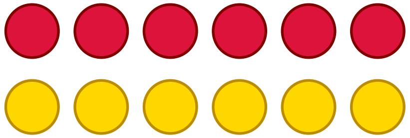
We use double sided counters here to represent directed numbers. Physical counters can be purchased, or the website https://mathsbot.com/manipulatives/doubleSidedCounters provides an electronic version of the counters.

The concept of a **zero pair** is an important one in this chapter. Show the learners different ways of arranging zero pairs, agreeing each time that zero is shown. For example:



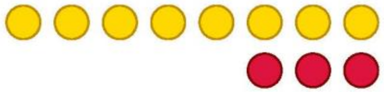




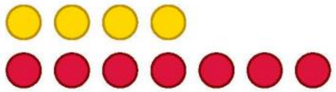


Example

(a) The diagram below shows 5.



(b) The diagram below shows –3.

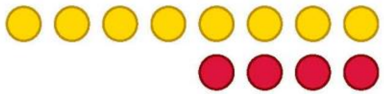


Exercise

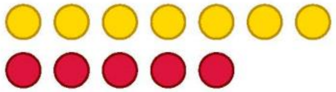
Which numbers are shown in the following diagrams?

(a)

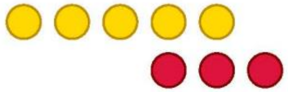
(b)



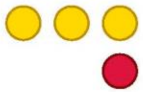
(c)



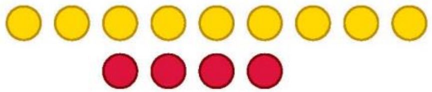
(d)



(e)

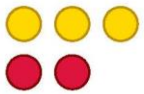


(f)

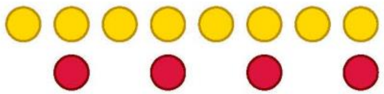


(g)

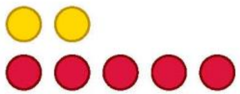
(h)



(i)



(j)



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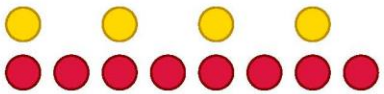

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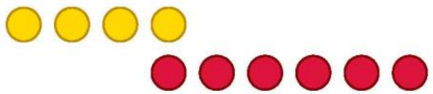


This page introduces the idea of recognising the value of a set of counters by identifying zero pairs, removing them, and counting what is left.

Notice that the counters in questions (d) and (l) must be re-arranged in order to identify the zero pairs.

On the website https://mathsbot.com/manipulatives/doubleSidedCounters, it is possible to drag a red counter on top of a yellow counter in order to remove a zero pair.

Further reading: 20 things to do with double sided counters: https://booleanmathshub.org.uk/files/4315/5845/2568/Abby_Cotton_-_20_Things_to_do_with_Double_Sided_Counters.pdf

What's the sum?

Below we show three ways of representing the same addition sum.
Fill in the blanks.

Example

4 + –2

Four add negative two

–6 + 3

Negative six add three

Exercise

(a)

Five add negative two

(b)

Negative five add four

(c)

6 + –1

(d)

–3 + 4

(e)

(f)

–2 + 5

(g)

Three add negative four

(h)

Negative one add seven

We represent the same addition sum in three different ways on this page:

- Using manipulatives, namely the double sided counters;
- Using symbols;
- Using words.

If red and yellow colours are not available to do the colouring, it is possible to label the counters with either "Y" and "R" or "+1" and "–1".

Emphasise the connection between the addition operation, +, and the process of adding more counters to a row.

Ensure that e.g., "negative three" is used for –3, and not "minus three". This is because the word "minus" has two meanings:

- The sign of a number, e.g., "minus two" for –2;
- A subtraction sum, e.g., "nine minus three" for 9 – 3.

To avoid the confusion of having two meanings for the word "minus", we avoid using it completely here and use

- "Negative" for the sign of a number, e.g., "negative two" for –2;
- "Subtract" for a subtraction sum, e.g., "nine subtract three" for 9 – 3.

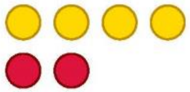
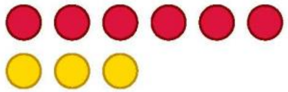
Addition with double sided counters

Example

4 + –2 –6 + 3

Step 1: Arrange the counters in a row to show the sum.


Step 2: Re-arrange the counters to identify any zero pairs.

Step 3: Remove any zero pairs and count what is left.


Answer: 2 Answer: –3

Exercise

Use double sided counters to find the answer to the following sums.

(a) 5 + –3	(b) 5 + –4	(c) 5 + –5
(d) 5 + –6	(e) 5 + –7	(f) 5 + –10
(g) –5 + 2	(h) –5 + 3	(i) –5 + 4
(j) –5 + 5	(k) –5 + 6	(l) –5 + 7
(m) 4 + –7	(n) –4 + –7	(o) –4 + 7
(p) 4 + 7	(q) 4 + –4	(r) –4 + 4
(s) –4 + –4	(t) –4 + 6	(u) –6 + 4
(v) –2 + –3	(w) 9 + –4	(x) –8 + –1
(y) 7 + –8	(z) –7 + 8	(α) –7 + –8

Challenge!

(a) 6 + –2 + 3	(b) –4 + –5 + 7	(c) –2 + 8 + –6
(d) –1 + –5 + 6	(e) 8 + –2 + –4	(f) –9 + 5 + –2 + 4

This page discusses how to add directed numbers using double sided counters.

Step 1 concentrates on the meaning of the addition sum, namely starting with a number (a set of counters) and adding another number (adding another set of counters at the end of the row).

Step 2 identifies any zero pairs by re-arranging the counters.

Step 3 removes any zero pairs and recognises the value of the remaining counters.

Confident learners may wish to start with step 2; this is acceptable! But insist that less confident learners start with step 1 in order to avoid doing two things at once (recognising the meaning of the sum and identifying the zero pairs).

Notice the patterns in the questions:

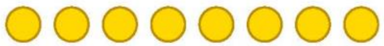
- The first 6 questions all start with 5 and change from having a positive answer to having a negative answer.
- The next 6 questions all start with –5 and change from having a negative answer to having a positive answer.
- The next 6 questions involve different combinations of the digits 4 and 7.

By completing more of this type of sum, learners will make the journey from the concrete (working with the physical counters) to the abstract (doing the sum mentally). In the middle, perhaps some learners will use a pictorial step (drawing pictures of counters on paper).

Subtraction with double sided counters

Example

To find the answer to the sum 8 – 2, we start with eight yellow counters...



...and subtract (remove) two of them to leave six yellow counters.

Therefore, 8 – 2 = 6.

Exercise

Use double sided counters to find the answers to the following sums.

(a) 7 – 2 (b) 7 – 4 (c) 7 – 6

(d) 6 – 1 (e) 5 – 5 (f) 12 – 7

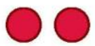
Removing what isn't there

How do you find the answer to the sum 5 – 7 using double sided counters?

Let us start with five yellow counters.

At the moment, we cannot subtract seven yellow counters (as we only have five of them), so we need to introduce two zero pairs.

Now it is possible to subtract seven yellow counters, and this leaves two red counters. Therefore, 5 – 7 = –2.



Exercise

Use double sided counters to find the answer to the following sums.

(a) 5 – 8 (b) 3 – 6 (c) 2 – 7

(d) 4 – 8 (e) 5 – 9 (f) 9 – 13

Regarding the double sided counters, the concept of subtraction in mathematics means removing a set of counters. This is simple if there are counters present to remove (the first half of this page), but if there aren't enough counters physically present to remove, zero pairs must be introduced. It should be reinforced here that introducing a zero pair does not alter the value of a set of counters, and this then allows us to complete the subtraction sum with the counters.

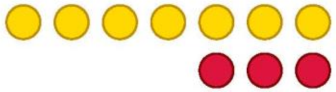
Regarding how many zero pairs must be introduced to complete a subtraction sum, the order of the sum can be reversed to determine this. For example, with the sum 5 – 7, we obtain the sum 7 – 5 by reversing the order, which gives an answer of 2. So, we need to introduce two zero pairs in order to complete the sum 5 – 7 with double sided counters.

Subtracting a negative number

What is the meaning of the sum 4 – –3, 'four subtract negative three'?

Let us start with four yellow counters.

We can add three zero pairs without affecting the value of the counters (four).



We can now subtract negative three by removing three red counters.

We see that seven yellow counters remain, so that 4 – –3 = 7.

Exercise

Use double sided counters to find the answer to the following sums.

(a) 4 – –2	(b) 4 – –1	(c) 4 – –4
(d) 5 – –2	(e) 5 – –1	(f) 5 – –4
(g) 6 – –3	(h) 6 – –2	(i) 6 – –5
(j) 3 – –2	(k) 3 – –4	(l) 3 – –6
(m) 8 – –3	(n) 8 – –1	(o) 8 – –5
(p) 7 – –4	(q) 7 – –2	(r) 7 – – 3

Starting with a negative number

Discuss how you would use double sided counters to find the answer to the sum –4 – –2.

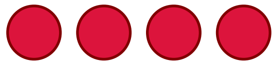
Exercise

(a) –4 – –1	(b) –4 – –3	(c) –4 – –4
(d) –5 – –1	(e) –5 – –3	(f) –5 – –4
(g) –7 – –1	(h) –7 – –3	(i) –7 – –4

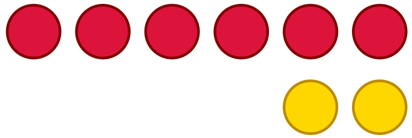
For the sum 4 – –3, three zero pairs must be introduced to be able to remove three red counters. So, when subtracting a negative number from a positive number, the negative number tells us how many zero pairs to introduce.

Extension work: Sums like –4 – –6, where we need to cross past zero.

We start with 4 red counters.



At the moment, we cannot remove six red counters (as only four red counters are present), so we need to introduce two zero pairs.



Now, it is possible to remove six red counters, and this leaves two yellow counters. Therefore, –4 – –6 = 2.

Further reading: some more ideas for adding and subtracting with directed numbers: http://ticktockmaths.co.uk/adding-and-subtracting-directed-numbers/

Directed Numbers Grids

How does the grid below work? What is the missing number?

4	5	9	14	23
6	7	13	20	33
10	12	22	34	56
16	19	35	54	89
26	31	57	88	?

Exercise

Fill in the numbers in the following grids.

1	1			
1	2			

1	−1			
0	1			

	3	3		
	3	1		

			5	8
			6	13

		1	5	
		3	5	

2	−5			
−4	7			

−4	−3			
−2	−1			

		1	2	
		3	4	

3	3			
−7	1			

These exercises provide **purposeful practice** for addition and subtraction of directed numbers.

- In each row, two consecutive numbers are added to give the next number on the right.
- In each column, two consecutive numbers are added to give the next number below.

In order to complete the grids where the four yellow numbers are not given, reasoning must be used to find the numbers above and to the left (an assortment of subtraction sums must be completed).

Acknowledgement: This exercise is based upon one found on the late Don Steward's web site: https://donsteward.blogspot.com/2020/03/directed-number-grid.html

Negative numbers in context

Golf

In a round of golf, players hit a ball into each of 18 different holes. Players need to do this using the least number of strokes possible; the person with the least score wins.

Here are the details of the famous golf course at the Celtic Manor hotel, located near Newport. This course was used during the Ryder Cup competition in 2010.

Hole	1	2	3	4	5	6	7	8	9
Length (yards)	461	605	189	458	447	436	212	437	614
Par	4	5	3	4	4	4	3	4	5

Hole	10	11	12	13	14	15	16	17	18
Length (yards)	211	560	454	187	480	365	502	209	608
Par	3	5	4	3	4	4	4	3	5

What do you notice? What do you wonder?

Exercise

(a) Which hole is the longest? (b) Which hole is the shortest?

(c) How many 'par 3' holes are there? (d) How many 'par 4' holes are there?

(e) How many 'par 5' holes are there? (f) What is the total par of the course?

Challenge! ⚠ What is the total length of the course?

Encourage the learners to discuss the information that is given in the table. As well as the questions given in the exercise, what about discussing…

- The connection between the length of a hole and the par for the hole (the longer the hole, the greater the par).
- How much time would it take to walk along one of the holes? (Remember that 1 yard is similar to an adult's stride when walking.)
- What type of clubs should be used on each hole? (Search for "golf club distances" on the internet.)
- How does the golf course closest to the school compare to the course at the Celtic Manor?

Scoring

A player receives a score for each completed hole of the golf course.

- If Peter uses 6 strokes to get the ball into hole 1, then his score for hole 1 is +2, as 6 is two **more** than the par for the hole (4).
- If Elen uses 2 strokes to get the ball into hole 7, then her score for hole 7 is –1, as 2 is one **less** than the par for the hole (3).

Here is Peter's score card for a round of golf on the course.

Hole	1	2	3	4	5	6	7	8	9
Peter's strokes	6	4	3	5	4	3	2	4	3
Par	4	5	3	4	4	4	3	4	5
Score on the hole	+2	+1	0	–1	0	–1	–1	0	–2
Score	+2	+3	+3	+2	+2	+1	0	0	–2

Hole	10	11	12	13	14	15	16	17	18
Peter's strokes	3	6	4	5	4	3	5	2	4
Par	3	5	4	3	4	4	4	3	5
Score on the hole	0	+1	0	+2	0	–1	+1	–1	–1
Score	–2	–1	–1	+1	+1	0	1	0	–1

Exercise

Complete Elen's score card for a round of golf on the course.

Hole	1	2	3	4	5	6	7	8	9
Elen's strokes	3	5	3	5	3	3	2	4	6
Par	4	5	3	4	4	4	3	4	5
Score on the hole									
Score									

Hole	10	11	12	13	14	15	16	17	18
Elen's strokes	3	7	4	3	5	4	3	3	3
Par	3	5	4	3	4	4	4	3	5
Score on the hole									
Score									

Who had the better round of golf on the course: Peter or Elen?

Golf lends itself well to discuss directed numbers in context as the players' scores are often negative numbers. The least score is the best score in golf, which gives a good opportunity to compare the value of different negative numbers. For example, who would win in each row of the following table?

Score for player 1	*Score for player 2*
–8	–6
–12	–15
–4	2
8	–5
0	–3

If the learners are not familiar with golf, try to find a clip of a recent tournament on the internet, discussing the numbers that appear on screen. Alternatively, perhaps you have the resources to create a simple golf course on the school grounds?

Temperature

Here is the temperature in 10 cities at 18:00 and 06:00 during January.

Paris (France): 10°C and 7°C.

New York (USA): 5°C and –2°C.

Tokyo (Japan): 10°C and 3°C.

Cardiff (Wales): 8°C and –1°C.

Buenos Aires (Argentina): 29°C and 21°C.

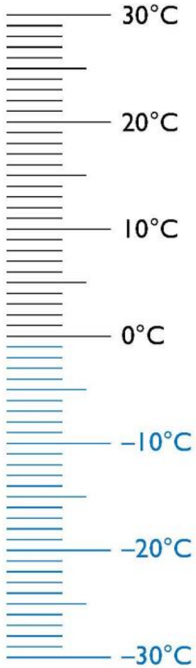
Oslo (Norway): –2°C and –7°C.

Sydney (Australia): 23°C and 18°C.

Delhi (India): 17°C and 14°C.

Moscow (Russia): –4°C and –10°C.

Berlin (Germany): 3°C and –3°C.



Exercise

The change in temperature for Paris was
10°C – 7°C = 3°C.
Write down the change in temperature for the other nine cities in a similar way.

Exercise

The temperature falls 7°C in the following cities. Fill in the blanks.

Los Angeles (USA): 18°C to ________.

Beijing (China): ________ to –3°C.

Edinburgh (Scotland): 7°C to ________.

Rome (Italy): ________ to 2°C.

Rio de Janeiro (Brazil): ________ to 24°C.

Madrid (Spain): 9°C to ________.

Reykjavik (Iceland): ________ to –4°C.

Seoul (South Korea): 2°C to _____.

Exercise

Explain what is happening to the temperature in the following sums.

(a) –5°C + 9°C

(b) –8°C – 4°C

(c) 3°C – 4°C

(d) –7°C + 3°C

Take the opportunity here to highlight the connection between subtracting two numbers and finding the difference between the numbers. Use the thermometer on the page (which is a type of number line) to show the connection between the change in temperature and the difference between the two temperatures.

For the final exercise on the page, an explanation of the following type would work:

(a) The temperature rises 9°C from –5°C to 4°C.

If you have a weather station, perhaps you will have an opportunity to discuss negative numbers during the cold months of winter? (A weather station would also provide several other opportunities to collect and discuss mathematical data.)

Money

Many people use a **current account** in a bank to receive and pay money. Often the account allows an **overdraft**, which means that the account **balance** can be negative. (The account is said to be "in the red" when this happens.)

Here is an example of a bank **statement**, showing the payments and deposits for a particular month.

Mrs Meinir Jones Current account statement Sort code: 12-34-56 Account number: 21436587		**Banc Cymru** **1 January – 31 January** Money in: £506.60 Money out: £476.22 Overdraft: £500		
Date	**Description**	**Money out**	**Money in**	**Balance**
01 Jan	Start balance			305.00
04 Jan	Direct Debit: Electricity Bill	103.00		202.00
07 Jan	Card Payment to Tesco	86.25		115.75
10 Jan	Card Payment to Amazon	49.99		65.76
14 Jan	Card Payment to Paypal	45.20		20.56
18 Jan	Card Payment to Lidl	74.35		–53.79
20 Jan	Direct Debit: Water Bill	55.00		–108.79
24 Jan	Received from your employer		506.60	397.81
30 Jan	Card Payment to Shell	62.43		335.38
31 Jan	End balance			335.38

What do you notice? What do you wonder?

There are a number of potentially unfamiliar terms on this page – take the time to discuss them fully.

- Historically, negative numbers on a bank statement were printed with red ink, whilst positive numbers were printed with black ink. So, a person with a positive balance is "in the black" and a person with a negative balance is "in the red".
- A bank account has a "balance" because if you add all the deposits and subtract all the payments, the number you end up with (positive or negative) is the number required *to balance* one side with the other side.
- A debit card for a bank account has 16 digits.
  - The first digit shows what type of card it is (for example, Visa cards start with a 4).
  - The final digit is the check digit, which is used to recognise errors when inputting the card number e.g., on a computer. (Challenge! Can you understand how the check digit is created? https://en.wikipedia.org/wiki/Luhn_algorithm)

Aspects of these two pages touch on the following description of learning for number:

FINANCIAL LITERACY

- I can demonstrate an understanding of income and expenditure, and I can apply calculations to explore profit and loss.

Exercise

Fill in the blanks in the following bank statement.

Mrs Meinir Jones
Current account statement

Sort code: 12-34-56
Account number: 21436587

Banc Cymru
1 February – 28 February

Money in: ________
Money out: ________
Overdraft: £500

Date	Description	Money out	Money in	Balance
01 Feb	Start balance			335.38
04 Feb	Direct Debit: Electricity Bill	103.00		______
05 Feb	Card Payment to Aldi	72.48		159.90
07 Feb	Card Payment to Posh Restaurant	48.52		______
10 Feb	Card Payment to Paypal	34.50		76.88
12 Feb	Card Payment to Amazon	______		52.48
17 Feb	Card Payment to Cinema	12.50		______
20 Feb	Direct Debit: Water Bill	55.00		______
21 Feb	Card Payment to McDonalds	5.40		−20.42
21 Feb	Card Payment to Spar	12.38		______
22 Feb	Card Payment to Primark	39.96		−72.76
24 Feb	Received from your employer		541.36	______
25 Feb	Card Payment to Asda	64.20		404.40
25 Feb	Card Payment to Boots	8.42		______
27 Feb	Card Payment to Garej Gethin	______		358.98
28 Feb	End balance			______

Perhaps it would be a good idea to allow the use of a calculator to complete this exercise? (And, therefore, a discussion on how to enter negative numbers on a calculator will be required?)

Notice that the start balance follows on from the end balance on the previous page. Extension work: Compare the statement shown on this page to the statement shown on the previous page. What is different? What is the same?

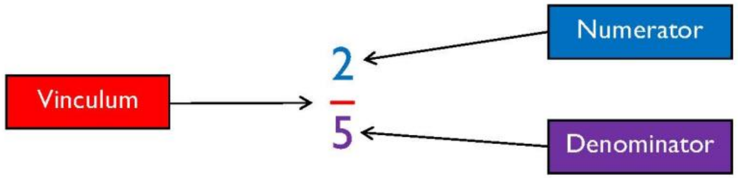
Further ideas for discussing directed numbers:

- Underground floors in a lift;
- Calculating the goal difference in sport;
- Distances above and below sea level.

Fractions

Terminology

Here are the names for the different parts of any fraction.



- The **vinculum** is the horizontal line in the middle of the fraction.
- The **numerator** is the integer at the top of the fraction.
- The **denominator** is the integer at the bottom of the fraction.

Exercise

(a) Circle the fractions below with a numerator of 3.

$\frac{3}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{3}{5}$ $\frac{3}{7}$ $\frac{2}{3}$ $\frac{33}{34}$ $\frac{3}{7}$

(b) Circle the fractions below with a denominator of 4.

$\frac{4}{5}$ $\frac{1}{4}$ $\frac{4}{7}$ $\frac{3}{4}$ $\frac{1}{44}$ $\frac{5}{4}$ $\frac{4}{9}$ $\frac{4}{11}$

(c) Write down four fractions with a denominator of 7.

(d) Write down three fractions with a numerator of 5.

(e) Write down five fractions where the numerator is 2 less than the denominator.

(f) Write down four fractions where the denominator is double the numerator.

(g) Write down three fractions where the numerator is 1.

(h) Write down four fractions where the sum of the numerator and the denominator is seven.

(i) Write down three fractions where the difference between the numerator and the denominator is four.

(j) Write down two fractions where the denominator is four times the numerator.

This chapter on fractions begins by considering the names for the different parts of a fraction.

- The word vinculum is used for any horizontal line in mathematical notation. For example, the vinculum forms part of the symbol for a square root, or forms part of a division frame.
- The name for a fraction where the numerator is 1 is a unit fraction.

The remainder of the chapter concentrates on different ways of using fractions, including

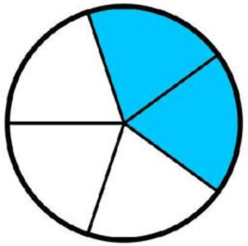
- Fraction of a shape, e.g., shading $\frac{3}{4}$ of a rectangle;
- Fraction of a number, e.g., calculating $\frac{3}{4}$ of £12;
- Fraction as a number, e.g., calculating $3 \div 4$ to locate $\frac{3}{4}$ on the number line.

Fraction of a Shape

To shade a fraction of a shape,

- Divide the shape into equal parts according to the number shown in the denominator;
- Shade the number of equal parts shown in the numerator.

For example, to shade $\frac{2}{5}$ of a circle,



- Divide the circle into 5 equal parts;
- Shade 2 of the equal parts.

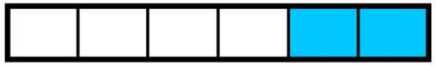
Notice that the parts must be **equal parts**.

Exercise

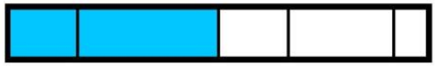
Which pictures below show $\frac{2}{5}$ of a shape being shaded?

(a)

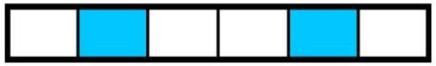
(b)



(c)



(d)



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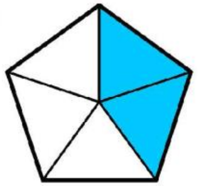
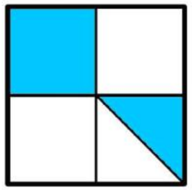
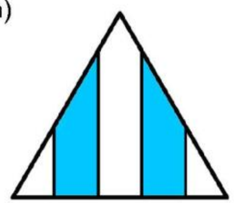


(f)

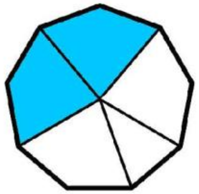


(g)

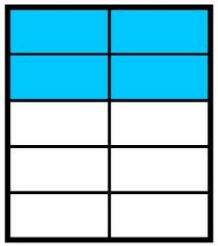
(h)



(i)



(j)



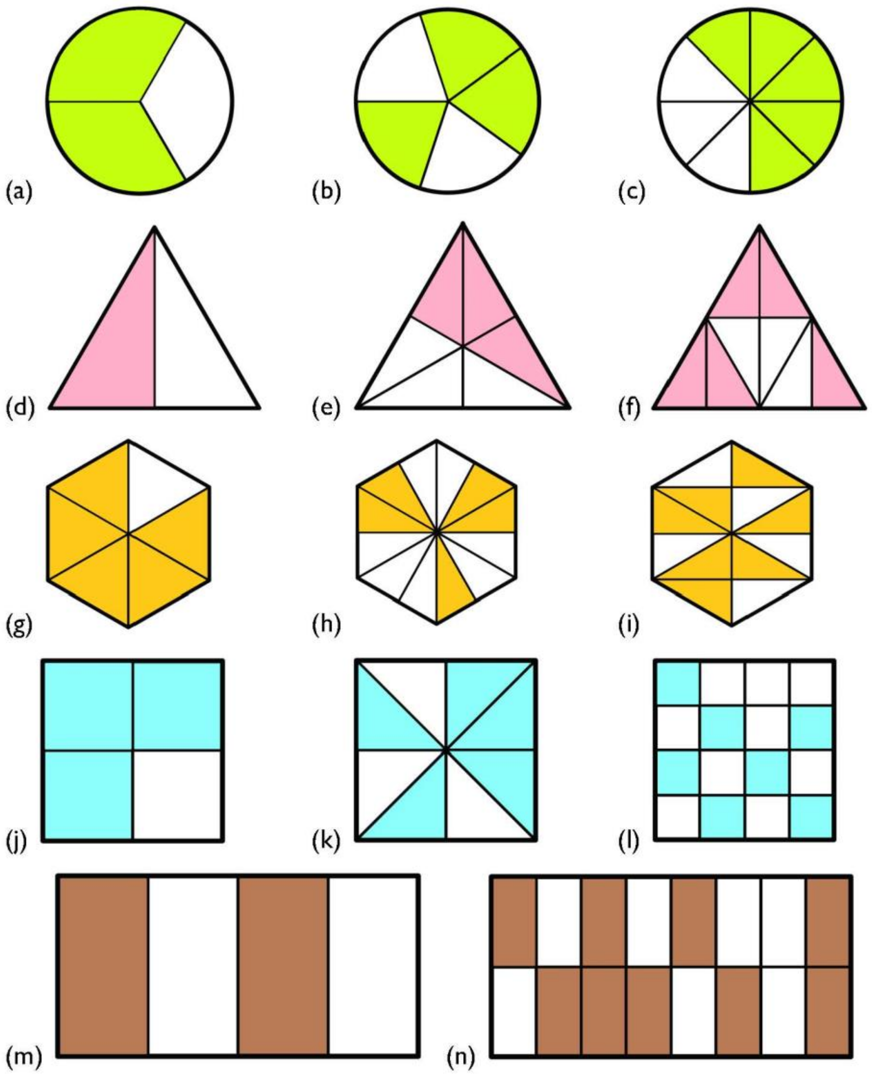
This page introduces a method for shading a particular fraction of a shape. The exercise provides examples and nonexamples of shading $\frac{2}{5}$ of a shape.

- Questions (a), (d), (e), (g) and (j) correctly show $\frac{2}{5}$ of the shape being shaded.
- Question (b) shows the incorrect number of parts (6 instead of 5).
- Questions (c), (f), (h) and (i) show parts that are not equal in size.

Extension work: For questions (b), (f) and (i), having agreed that $\frac{2}{5}$ of the shape is not shaded, *what* fraction of the shape is shaded?

Recognising a Fraction of a Shape

Which fractions of the following shapes are shaded?

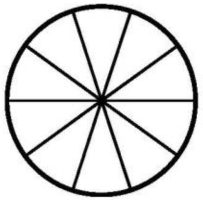


Each of these shapes have been split into equal parts, so an effective strategy for finding the answer would be to count the number of equal parts (the denominator of the fraction), and then count how many parts have been shaded (the numerator of the fraction).

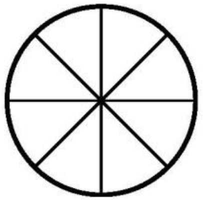
Exercise

Shade the fraction shown of each shape.

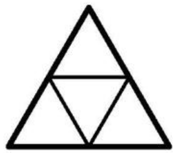
(a) $\frac{3}{10}$



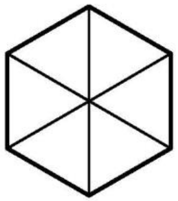
(b) $\frac{5}{8}$



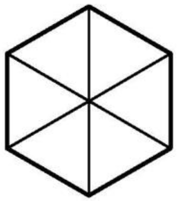
(c) $\frac{1}{4}$



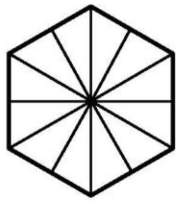
(d) $\frac{2}{6}$



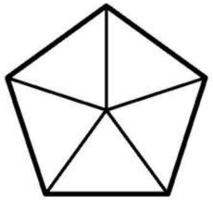
(e) $\frac{6}{6}$



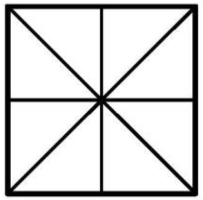
(f) $\frac{7}{12}$



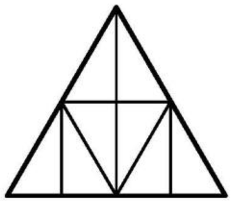
(g) $\frac{3}{5}$



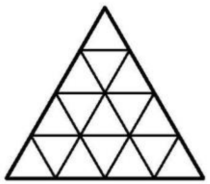
(h) $\frac{7}{8}$



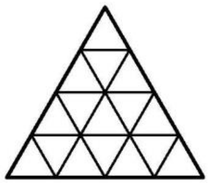
(i) $\frac{3}{8}$



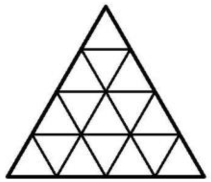
(j) $\frac{9}{16}$



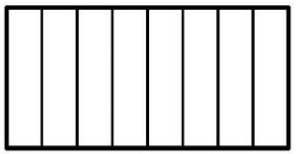
(k) $\frac{1}{2}$



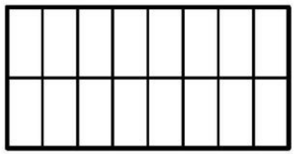
(l) $\frac{1}{4}$



(m) $\frac{5}{8}$



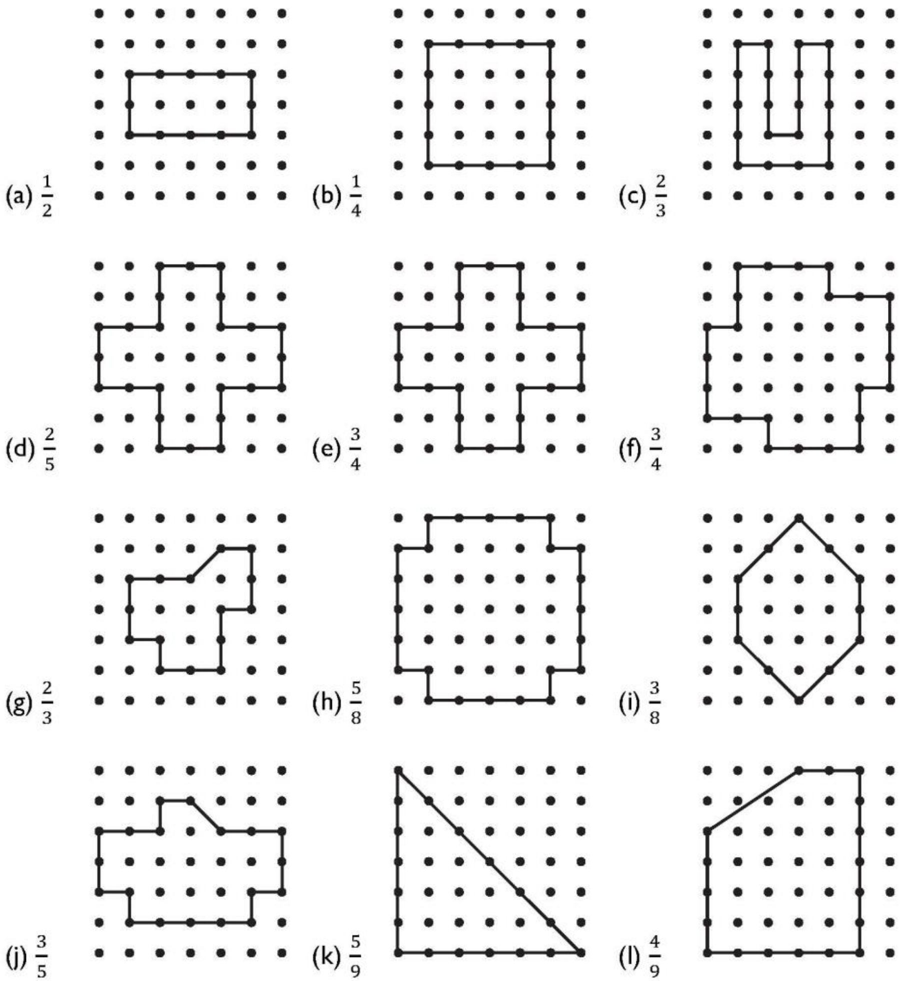
(n) $\frac{11}{16}$



Each of these shapes have been split into equal parts. An extension task would be to complete a similar exercise where the shapes have not been split into equal parts to begin with.

Exercise

By dividing each shape into parts of equal area, shade the fraction that is shown beside each shape.



Are there different ways of answering the above questions? Discuss.

Learners must reason here how to split the shapes into the required number of equal parts. For example, the area of shape (g) is $10\frac{1}{2}$ square units (found by counting whole squares and the half square). This needs to be divided into three equal parts, so the parts need to have an area of $3\frac{1}{2}$ square units each.

This activity is based upon the one found on the page https://donsteward.blogspot.com/2017/12/fraction-shading.html

Aspects of this page touch upon the following description of learning for geometry:

SHAPE AND SPACE

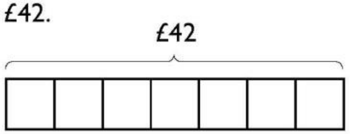
- I can use efficient methods for finding the perimeter and area of two-dimensional shapes.

Calculating a Fraction of a Number (fraction as an operator)

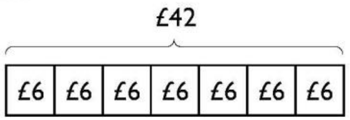
Example

Calculate $\frac{3}{7}$ of £42.

Because the denominator of the fraction is 7, we draw a bar model containing 7 blocks, to represent the £42.



Next, we calculate the value of one block, by dividing the £42 equally between the 7 blocks: £42 ÷ 7 = £6.



The question asks for $\frac{3}{7}$ of £42, so we need to find the value of three of the blocks: 3 × £6 = £18. This is the answer to the question.

Exercise

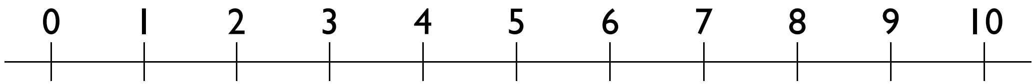
Draw a suitable bar model to answer the following questions.

(a) $\frac{2}{3}$ of £12 (b) $\frac{3}{5}$ of £15 (c) $\frac{1}{4}$ of £12

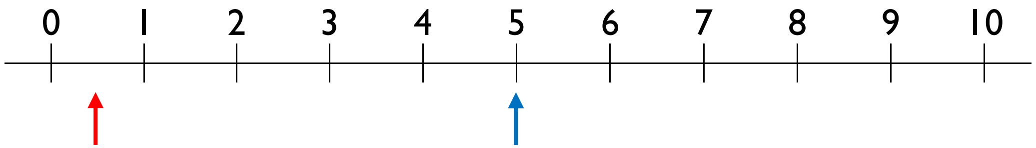
(d) $\frac{5}{6}$ of £36 (e) $\frac{2}{7}$ of £21 (f) $\frac{7}{8}$ of £40

This page introduces using a fraction as an operator. This means that the fraction operates on a value, as opposed to representing a value itself (which is discussed further on in this chapter).

To see the difference between these two concepts, consider a number line going from 0 to 10.



Given the question 'Add an arrow to show the location of $\frac{1}{2}$', where would you place the arrow: in the red location below, or in the blue location below?



The blue location considers the fraction $\frac{1}{2}$ as an operator (finding $\frac{1}{2}$ of 10), whilst the red location considers the fraction $\frac{1}{2}$ as a number (finding the location of $\frac{1}{2}$ on the number line). For the question that was asked, the red location is correct. For the blue location, the question's wording would need to change to 'Add an arrow to show the location of $\frac{1}{2}$ of 10'.

Further information about bar models (as used on this page) is available here: https://thirdspacelearning.com/blog/teach-bar-model-method-arithmetic-maths-word-problems-ks1-ks2/

Exercise

Connect the following cards in groups of three.

$\frac{1}{2}$ of 30	7 | 7 | 7 | 7 | 7 | 7 | 7	16
$\frac{2}{3}$ of 24	5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5	15
$\frac{7}{9}$ of 45	5 | 5 | 5 | 5 | 5 | 5	28
$\frac{4}{7}$ of 49	15 | 15	35
$\frac{5}{6}$ of 30	8 | 8 | 8	25

Exercise

(a) My dog has had 15 puppies, with $\frac{2}{3}$ of the puppies being sold. How many puppies have been sold, and how many puppies are left?

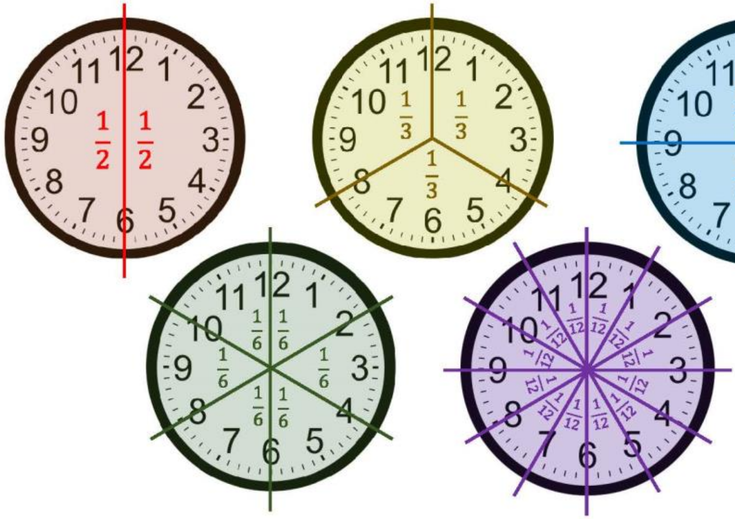
(b) John has 54 stickers. He gives $\frac{4}{9}$ of the stickers to Tom. How many stickers has he given to Tom, and how many stickers are left?

(c) Sam had £5 of pocket money. He spent $\frac{7}{10}$ of the money on a book. How much money did Sam spend, and how much money does he have left?

(d) In a game of cricket, Sali managed to hit $\frac{4}{5}$ of the balls being bowled towards her. If 40 balls in total were bowled towards Sali, how many balls did she manage to hit, and how many balls did she miss?

Encourage the learners to draw a bar model if they have difficulty with the second exercise on the page.

Fractions and Time



Exercise

(a) In _____ minutes, the minute hand of the clock goes $\frac{1}{4}$ of the way around the clock.

(b) In _____ minutes, the minute hand of the clock goes $\frac{1}{6}$ of the way around the clock.

(c) In 30 minutes, the minute hand of the clock goes ______ of the way around the clock.

(d) In 5 minutes, the minute hand of the clock goes ______ of the way around the clock.

(e) In _____ minutes, the minute hand of the clock goes $\frac{1}{3}$ of the way around the clock.

(f) In _____ minutes, the minute hand of the clock goes $\frac{2}{3}$ of the way around the clock.

(g) In _____ minutes, the minute hand of the clock goes $\frac{3}{4}$ of the way around the clock.

(h) In 50 minutes, the minute hand of the clock goes ______ of the way around the clock.

(i) In 35 minutes, the minute hand of the clock goes ______ of the way around the clock.

(j) In 1 minute, the minute hand of the clock goes ______ of the way around the clock.

(k) In 60 minutes, the **hour** hand of the clock goes ______ of the way around the clock.

(l) In 180 minutes, the **hour** hand of the clock goes ______ of the way around the clock.

Reading the time provides many opportunities to discuss fractions, as an analogue clock is divided into 12 equal parts. The Babylonians were mainly responsible for how we tell the time today; they used base 60 numerals to form the system. This is why an hour is split into 60 minutes, and a minute is split into 60 seconds. The number 60 has a large number of factors, which makes dividing (e.g.) an hour into equal parts relatively easy.

$$60 \div 2 = 30$$
$$60 \div 3 = 20$$
$$60 \div 4 = 15$$
$$60 \div 5 = 12$$
$$60 \div 6 = 10$$

If an hour had been chosen to be 10 minutes instead of 60 minutes, it would not be possible to divide an hour so easily into 3 equal parts or 4 equal parts.

Fraction as a Division Sum (fraction as a number)

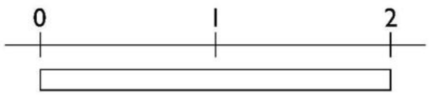
It is possible to write the fraction $\frac{2}{5}$ as the division sum $2 \div 5$.

The name of the symbol ÷ is the *obelus*, a symbol that looks like a fraction, with dots representing the numerator and the denominator.

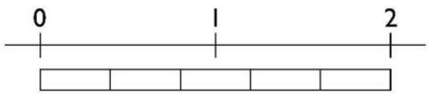
Each fraction has a specific location on the number line. Here are two methods for finding the location of $\frac{2}{5}$ on the number line.

Method 1
(the division method, $2 \div 5$)

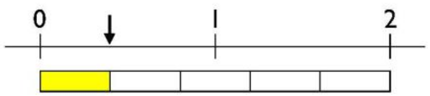
Start with a bar model showing 2 on a number line.



Divide the bar into 5 equal parts.

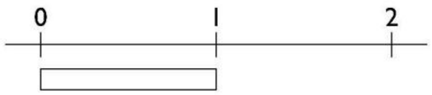


The location of $\frac{2}{5}$ on the number line is where the first part of the bar finishes.

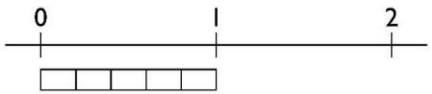


Method 2
(the unit method, finding $\frac{2}{5}$ of 1)

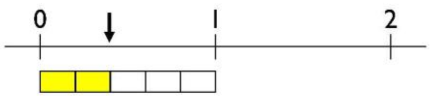
Start with a bar model showing 1 on a number line.



Divide the bar into 5 equal parts.

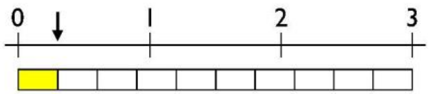


The location of $\frac{2}{5}$ on the number line is where the second part of the bar finishes.



Exercise

Discuss how the following two diagrams show the location of the fraction $\frac{3}{10}$ on the number line.



This page discusses two different methods of locating a fraction's value on the number line.

- The first method (the division method) starts with the numerator and divides by the denominator. The length of **one** of the parts corresponds to the location of the fraction on the number line. This method is advantageous for discussing improper fractions that have a value exceeding one.
- The second method (the unit method) always starts with a single unit, and calculates a fraction of this unit. This method is advantageous for comparing the size of different fractions.

Encourage the use of the terms "unit", "numerator" and "denominator" whilst completing the exercise at the bottom of the page.

In the first exercise, note that one of the boxes for $\frac{3}{2}$ has been drawn using a dotted line, to show that this box did not belong to the original unit.

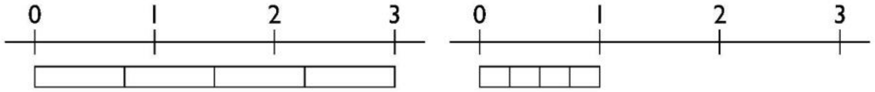
In the second exercise, reasoning must be used to determine how many squares of squared paper must be used to represent a single unit. The denominator influences this, of course, so that 5 squares for each unit is a wise choice for the first three questions.

In the challenge, visualising two of the white triangles coming together to form a rectangle would perhaps help with the solution?

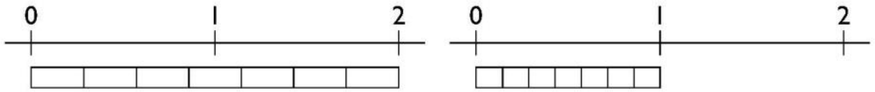
Exercise

Add arrows on the following number lines to show the location of the fractions.

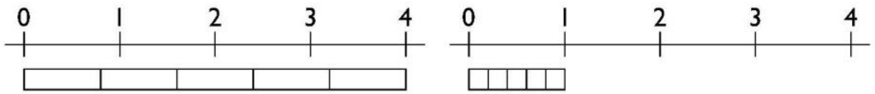
(a) $\frac{3}{4}$



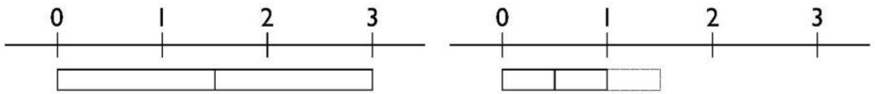
(b) $\frac{2}{7}$



(c) $\frac{4}{5}$



(d) $\frac{3}{2}$



Exercise

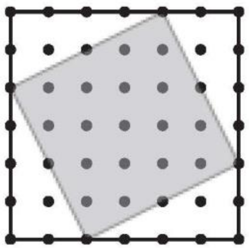
On a piece of squared paper, draw suitable diagrams to show the location of the following fractions on a number line.

(a) $\frac{3}{5}$ (b) $\frac{1}{5}$ (c) $\frac{4}{5}$

(d) $\frac{5}{2}$ (e) $\frac{3}{3}$ (f) $\frac{3}{7}$

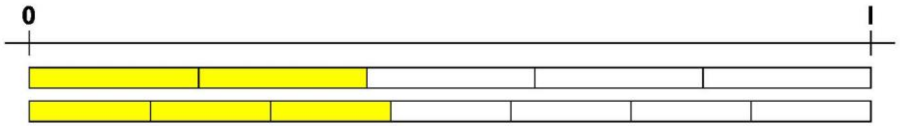
Challenge!

What fraction of the square on the right is shaded?



Comparing the Size of Fractions

Which fraction is the greatest: $\frac{2}{5}$ or $\frac{3}{7}$? Using the unit method, we can locate both fractions on a number line:



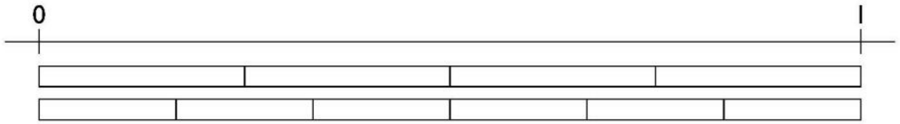
We can see from the diagram that $\frac{2}{5}$ is less than $\frac{3}{7}$, so we write $\frac{2}{5} < \frac{3}{7}$.

< 'less than' > 'greater than' = 'equal to'

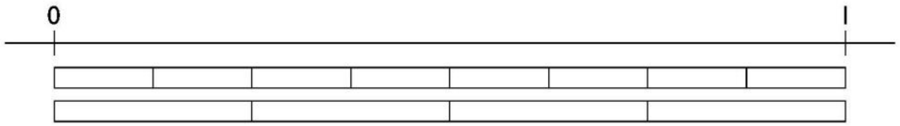
Exercise

Write the symbol <, > or = between the following pairs of fractions.

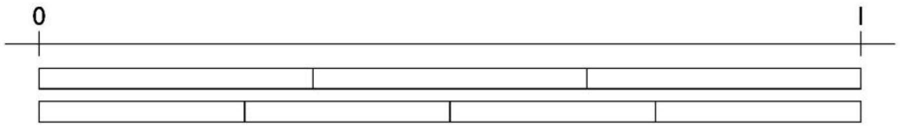
(a) $\frac{3}{4}$ $\frac{5}{6}$



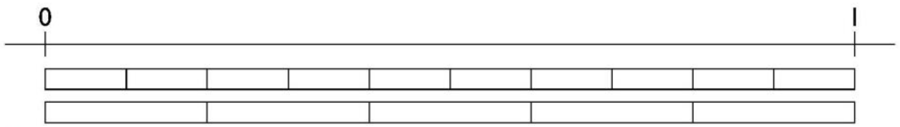
(b) $\frac{2}{8}$ $\frac{1}{4}$



(c) $\frac{2}{3}$ $\frac{1}{4}$



(d) $\frac{7}{10}$ $\frac{4}{5}$

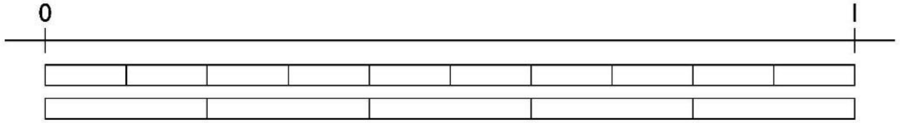


The unit method is convenient for comparing the size of two proper fractions (where the numerator is less than the denominator).

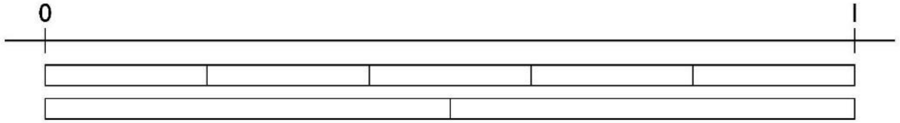
In follow-up work (not in this workbook), it would also be possible to compare the size of fractions using equivalent fractions. For example, considering the fractions $\frac{2}{5}$ and $\frac{3}{7}$, the equivalent fractions $\frac{14}{35}$ and $\frac{15}{35}$ would allow us to see that $\frac{3}{7}$ is the greatest fraction (because 15 is greater than 14).

Extension work: using the fraction wall on page 51 to compare the size of different fractions.

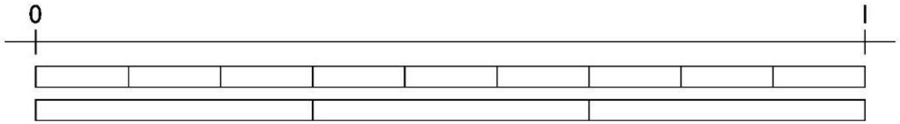
(e) $\frac{4}{10}$ $\frac{2}{5}$



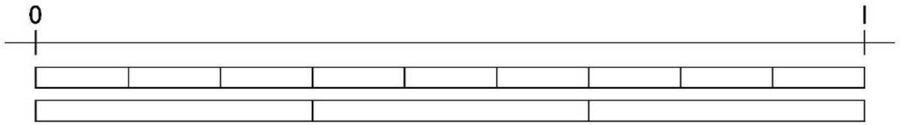
(f) $\frac{2}{5}$ $\frac{1}{2}$



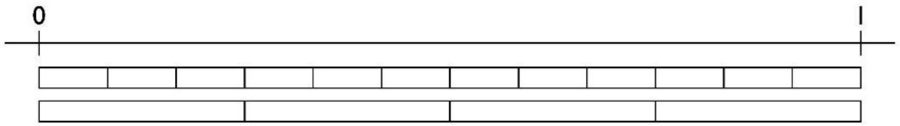
(g) $\frac{5}{9}$ $\frac{2}{3}$



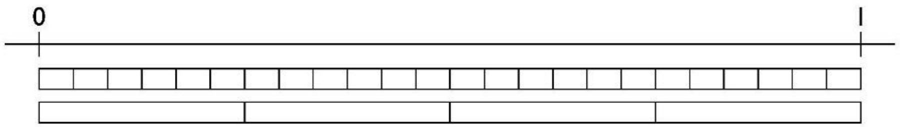
(h) $\frac{6}{9}$ $\frac{2}{3}$



(i) $\frac{7}{12}$ $\frac{3}{4}$



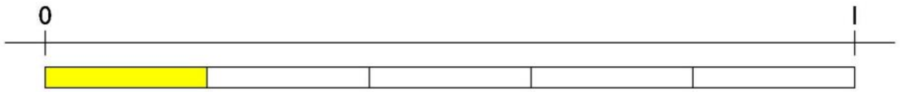
(j) $\frac{18}{24}$ $\frac{3}{4}$



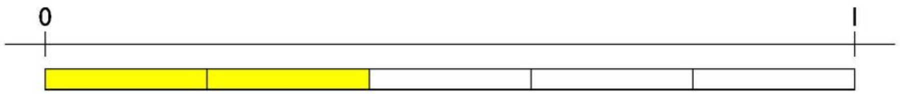
Changing the Numerator

What happens to the size of a fraction when the denominator is kept the same and the numerator is increased?

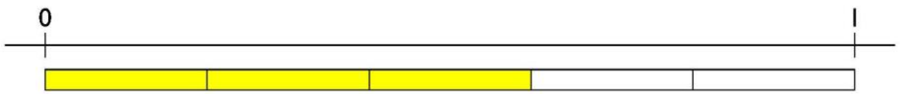
$\frac{1}{5}$



$\frac{2}{5}$



$\frac{3}{5}$



We see from the above diagrams that the size of a fraction **increases** when the denominator is kept the same and the numerator increases.

Exercise

Circle the **greatest** fraction in each of the following pairs of fractions.

(a) $\frac{2}{5}$ and $\frac{3}{5}$ (b) $\frac{3}{5}$ and $\frac{4}{5}$ (c) $\frac{2}{7}$ and $\frac{5}{7}$

(d) $\frac{6}{7}$ and $\frac{3}{7}$ (e) $\frac{3}{4}$ and $\frac{1}{4}$ (f) $\frac{7}{8}$ and $\frac{3}{8}$

(g) $\frac{4}{9}$ and $\frac{7}{9}$ (h) $\frac{4}{11}$ and $\frac{5}{11}$ (i) $\frac{10}{11}$ and $\frac{8}{11}$

(j) $\frac{3}{2}$ and $\frac{5}{2}$ (k) $\frac{5}{5}$ and $\frac{6}{5}$ (l) $\frac{76}{123}$ and $\frac{107}{123}$

Challenge!

Write a single digit between 1 and 9 in each of the boxes on the right to create the greatest fraction that is less than a half.

$$\frac{\square}{\square}$$

These two pages get to grip with the following description of learning from progression step 3:

RELATIONSHIPS WITHIN THE NUMBER SYSTEM

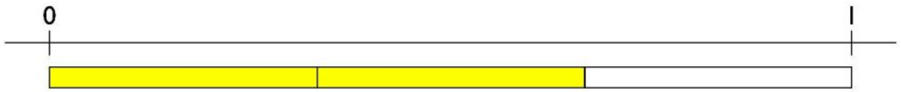
- I can understand the inverse relation between the denominator of a fraction and its value.

On this page, we investigate the relationship between the value of a fraction and increasing the fraction's numerator. There is a **direct proportion** here, which means that the value of a fraction increases as its numerator increases. Thus, in the exercise, the greatest fraction is the one with the greatest numerator.

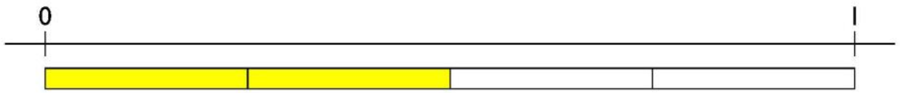
Changing the Denominator

What happens to the size of a fraction when the numerator is kept the same and the denominator is increased?

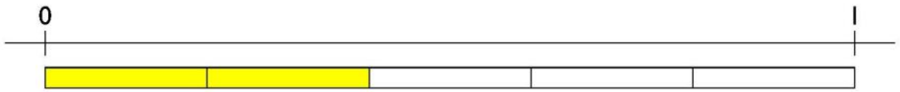
$\frac{2}{3}$



$\frac{2}{4}$



$\frac{2}{5}$



We can see from the above diagrams that the size of a fraction **decreases** when the numerator is kept the same and the denominator increases.

Exercise

Circle the **greatest** fraction in each of the following pairs of fractions.

(a) $\frac{2}{4}$ and $\frac{2}{5}$ (b) $\frac{3}{5}$ and $\frac{3}{4}$ (c) $\frac{2}{7}$ and $\frac{2}{5}$

(d) $\frac{6}{7}$ and $\frac{6}{11}$ (e) $\frac{3}{4}$ and $\frac{3}{2}$ (f) $\frac{7}{8}$ and $\frac{7}{9}$

(g) $\frac{2}{9}$ and $\frac{2}{7}$ (h) $\frac{4}{11}$ and $\frac{4}{13}$ (i) $\frac{10}{11}$ and $\frac{10}{7}$

(j) $\frac{4}{3}$ and $\frac{4}{5}$ (k) $\frac{5}{5}$ and $\frac{5}{8}$ (l) $\frac{76}{123}$ and $\frac{76}{122}$

Challenge!

Write a different single digit between 1 and 9 in each of the boxes on the right to form a true statement.

$$\frac{\square}{\square} < \frac{\square}{\square} < \frac{\square}{\square}$$

On this page, we investigate the relationship between the value of a fraction and increasing the fraction's denominator. There is an **inverse proportion** here, which means that the value of a fraction decreases as its denominator increases. Thus, in the exercise, the greatest fraction is the one with the least numerator.

It is important to develop the relationship between the value of a fraction and how the fraction's numerator or denominator changes. Here are the results from one experiment asking 14-year-old learners to choose the greatest fraction out of the following pairs of fractions.

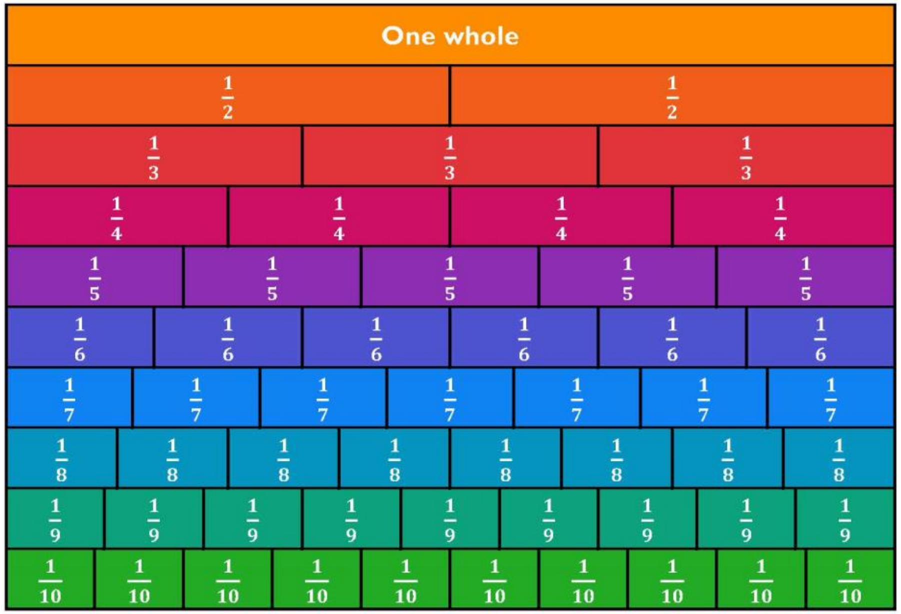
Fractions	Percentage choosing the greatest fraction correctly
$\frac{5}{8}$ and $\frac{7}{8}$	95%
$\frac{3}{4}$ and $\frac{4}{5}$	75%
$\frac{3}{8}$ and $\frac{3}{5}$	Less than 20%

Source:
https://www.dylanwiliam.org/Dylan_Wiliams_website/Papers_files/Devising%20learning%20progressions%20%28AERA%202011%29.doc

As you can see from the table, if the numerator stays the same and the denominator changes, a large percentage of 14-year-old learners choose the greatest fraction incorrectly.

The challenges (on the previous page and on this page) are based on tasks from the Open Middle website. Further similar exercises can be found here: https://www.openmiddle.com/category/grade-4/number-operations-fractions-grade-4/

The Fraction Wall



What do you notice? What do you wonder?

Exercise

(a) Which two fractions add together to give $\frac{1}{2}$?

(b) Find two different ways of writing $\frac{1}{3}$.

(c) Find a different way of writing $\frac{3}{4}$.

(d) Which fraction is the greatest: $\frac{3}{8}$ or $\frac{4}{10}$?

(e) Which fraction is the least: $\frac{3}{5}$ or $\frac{5}{8}$?

(f) How many tenths should be added to $\frac{2}{5}$ to make one whole?

(g) Look at the following pattern of fractions: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$

(i) What is the next fraction in the pattern?
(ii) What happens to the value of the fractions as the pattern continues?
(iii) Will a fraction with a value of more than one appear in the pattern?

The fraction wall is useful for comparing the size of fractions, and for discussing fractions in general.

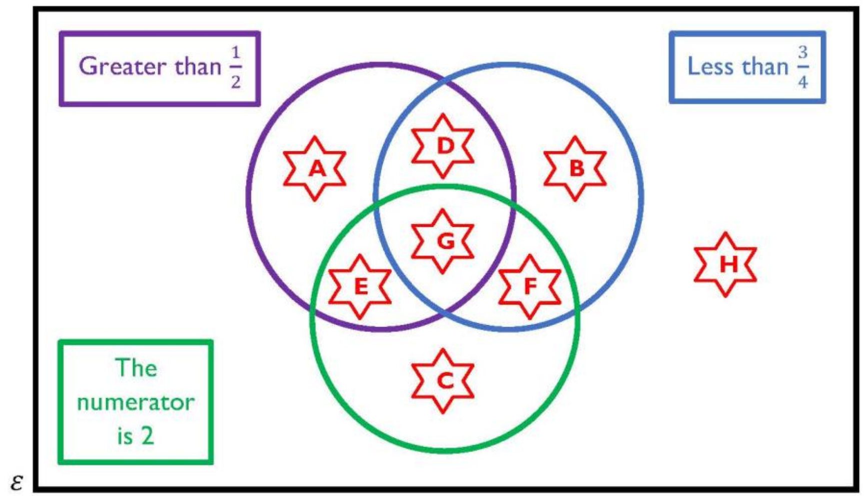
Here are some suggestions for further discussion:

- What happens to the unit fractions ($\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$) going down the wall?
- What would be the value of $1-\frac{1}{2}$, or $1-\frac{1}{3}$, or $1-\frac{1}{4}$, or …?
- How many different ways of showing $\frac{1}{2}$ can be seen in the wall? What about ones that are not shown in the wall?
- What is the connection between the $\frac{1}{3}$ row and the $\frac{1}{6}$ row? Is there a similar connection between other rows? Why?

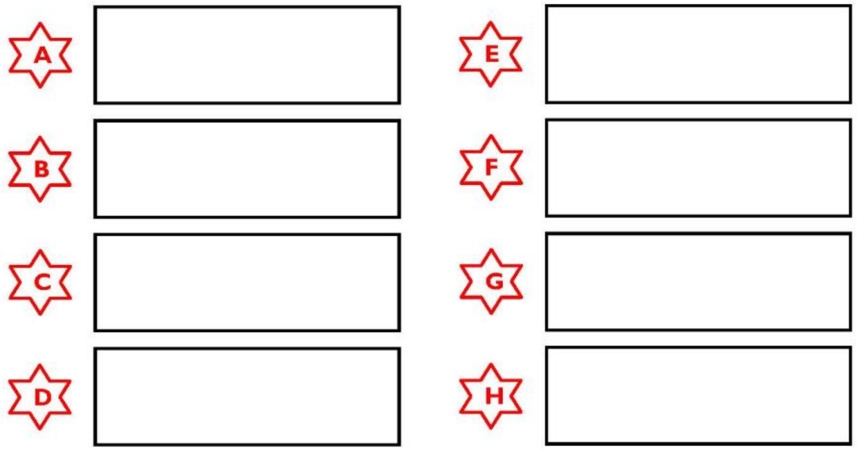
For an electronic version of the fraction wall, go to the website https://mathigon.org/polypad, where the colours on the left correspond to the colours on the website. Go to the "Fraction bars" section to see the pieces for the wall, and remember to experiment with the "Split Tiles" and "Rename" buttons.

Another version of the fraction wall is available here: https://mathsbot.com/manipulatives/fractionWall

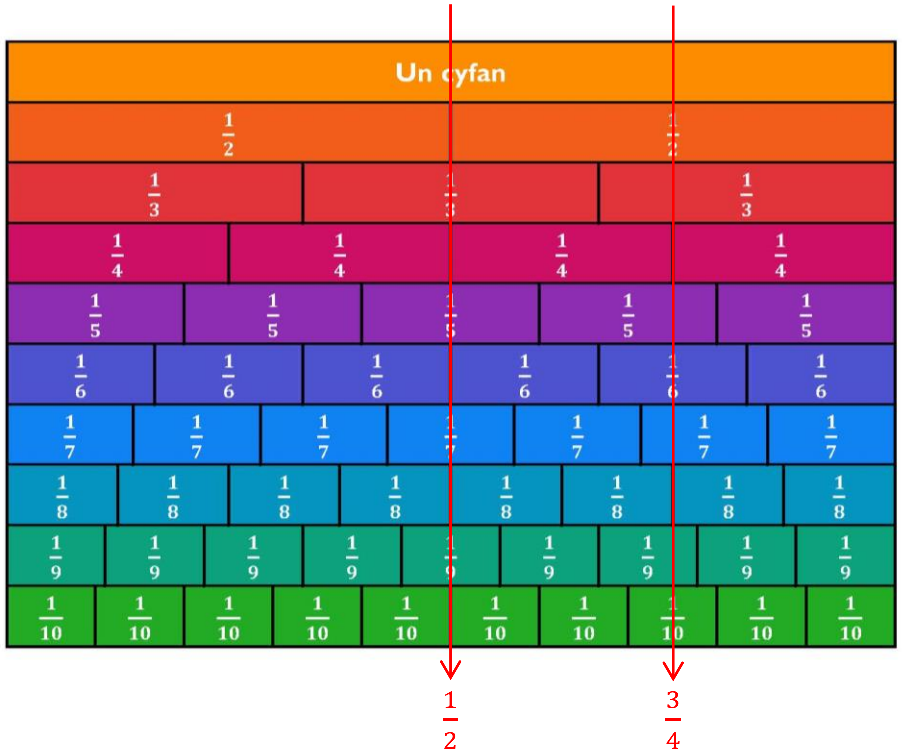
Venn Diagram Challenge



Think of a fraction that could fit into each region.
If you think a region is impossible to fill, explain why!

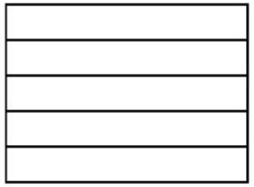


The first task here is to reason what is the requirement for each region. For example, considering region F, we require a fraction that is less than $\frac{3}{4}$ (in the blue region); where the numerator is 2 (in the green region); and where the fraction is not greater than $\frac{1}{2}$ (not in the purple region). Perhaps the following version of the fraction wall will be useful?

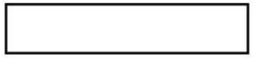


The Story of $\frac{2}{5}$

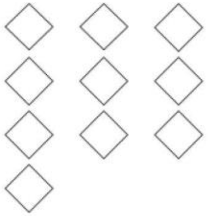
Shade $\frac{2}{5}$ of the shape.



Shade $\frac{2}{5}$ of the shape.



Shade $\frac{2}{5}$ of the shapes.



Write $\frac{2}{5}$ in words.

What is the numerator of $\frac{2}{5}$?

What is the denominator of $\frac{2}{5}$?

Locate $\frac{2}{5}$ on the number line.

Calculate $\frac{2}{5}$ of 35.

Calculate $\frac{2}{5} \times 15$.

Which division sum does $\frac{2}{5}$ represent?

Write down a fraction that is equivalent to (the same as) the fraction $\frac{2}{5}$.

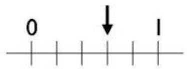
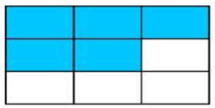
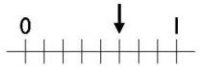
Write down two fractions that add to give $\frac{2}{5}$.

Catrin has 2 apples and 5 bananas. Fill in the missing fractions.

The number of apples is   of the total fruit.
The number of bananas is   of the total fruit.
The number of apples is   of the bananas.

This page brings together many of the ideas from this chapter. It is based upon a task by Chris McGrane, https://startingpointsmaths.com/2020/04/04/fraction-stories-multiple-representations/

True or False?

 $\frac{3}{8}$ of the shape is shaded.	$\frac{6}{12}$ The denominator is double the numerator.	48 children eat in the school's refectory. If $\frac{1}{4}$ of the children eat sandwiches, 12 children each sandwiches.	$\frac{3}{4} < \frac{3}{5}$
In 10 minutes, the minute hand of the clock goes $\frac{1}{6}$ of the way around the clock.	$\frac{2}{3}$ of £24 is £16.	$\frac{1}{3} < \frac{1}{10}$	$\frac{8}{15}$ The difference between the numerator and the denominator is seven.
There are 49 books in the library. If Nia has read $\frac{3}{7}$ of the books, she has read 14 of the books.	 The arrow points towards the fraction $\frac{3}{4}$.	 $\frac{5}{9}$ of the shape is shaded.	In 40 minutes, the minute hand of the clock goes $\frac{3}{4}$ of the way around the clock.
$\frac{4}{5}$ The numerator of the fraction is 5.	$\frac{4}{5} = \frac{12}{15}$	 The arrow points towards the fraction $\frac{5}{8}$.	$\frac{3}{8}$ of 32 is 7.
 $\frac{2}{5}$ of the stars have been shaded.	$\frac{4}{6}$, when written in words, is 'four sixths'.	$\frac{6}{7} > \frac{5}{7}$	$\frac{4}{7}$ The denominator of the fraction is 7.

This task is based upon the one from the website https://mhorley.wordpress.com/2015/03/18/uncovering-fraction-misconceptions-through-true-and-false-cards/

As on the previous page, it revises many of the topics introduced in this chapter.