YSGOL CREUDDYN

DAWN DYSG DAIONI

Mathematics

Years 10–11

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Using the Workbooks

When you see a QR code (like the one on the left), scan it using your mobile device in order to reach a Welsh YouTube video hosted on the following channel.

www.youtube.com/adolygumathemateg

The letters in circles, for example I, show the tier of the work in the GCSE specification.

Tier	Foundation	Intermediate	Higher
GCSE Grades	U, G, F, E, D	U, E, D, C, B	U, C, B, A, A*

All the workbooks contain a variety of exercises, labelled as follows.

Skill
Exercises on new topics.

Applying
Answering a question in context, or solving a problem.

Extension
A more difficult question.

Revision
Revision of material from previous workbooks.

Evaluation
There are evaluation boxes at the end of each chapter to revise the completed work.

Supporting Materials:

- Diagnostic Questions
  - A quiz for each workbook on the website www.diagnosticquestions.com.
- Reflection Sheet
  - An opportunity to assess your understanding of a workbook.
- Old WJEC examination questions; worksheets; investigations; Tarsia puzzles
  - Available for some topics.

The website www.mathemateg.com contains an electronic copy of each workbook, alongside all the supporting materials.

At the end of each workbook, there is an intermediate tier reflection sheet and a higher tier reflection sheet.

Tracking Sheet
Year 10

End of KS3 Level: _____

Target Grade: _____

Tier: _________

Attainment for Term 1: _____

Data Handling and Statistics 4

%

Powers and Roots

%

Measuring Shapes 3

%

Fractions, Percentages and Decimals

%

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The Mathematics Department

10

Data Handling and Statistics 4

Name:

Contents

Chapter	Mathematics	Page Number
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Questionnaires

A **questionnaire** is a good way to collect data, but we must be careful when designing the questions.

(1) We must avoid asking **leading** questions which favour one answer over another. For example, the question "Do you agree that eating ice cream is bad for you?" leads people to agree with the statement. (Why?)

(2) We must avoid using answer boxes where the **options overlap**. For example, in the following question, people who are 20 years old can choose two different answer boxes.
How old are you?
Under 10 *10–20* *20–30* *Over 30*

(3) We must be careful **where, when and how** a questionnaire is conducted. For example, the following questionnaires would not be appropriate.
  a. Conducting a sports questionnaire outside of a football stadium.
  b. Conducting a questionnaire about night shift workers at mid-day.
  c. Conducting a questionnaire on mobile phone usage through a phone survey.

(4) We must use **clear** and **concise** questions. For example, the question "How often do you go to the gym?" is not appropriate without explaining the meaning of the word "often", i.e. daily, weekly, monthly…?

(5) We must use **appropriate** and **relevant** questions. For example,
  a. Some people would refuse to answer "How old are you?", however they might answer if options showing different age ranges were included.
  b. The question "What is your eye colour?" would not be appropriate in a questionnaire asking opinions about recycling.

Exercise 1

Skill

F

Write a criticism of the following questions.

Writing a criticism means finding something incorrect or out of place.

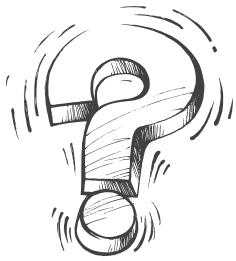
(a) Do you read books? Circle your answer.

Yes No Sometimes

(b) Do you agree that the cruel sport of fox hunting should be made illegal?

(c) How often do you use the gym in a typical month? Circle your answer.

Never Once or Twice 2–5 Times More than 5 times



(d) How old are you? Circle your answer.

10–15 16–20 21–25 26–30 31–35

(e) In your current job, how much money do you earn?

(f) How often do you shop in a supermarket? Circle your answer.

Three times a week Twice a week Once a week Once a month

(g) Do you agree that the amazing Liverpool are the best football team in the world? Circle your answer.

Yes Of course Absolutely

Exercise 2

Applying F

A survey is conducted to see how often teenagers buy trainers.

The following questions are included in the questionnaire.

Question 1: Where do you live?

Question 2: How often do you buy trainers?

Never	1–10 times	10–15 times	More than 15 times
☐	☐	☐	☐

(a) For each question state **one** reason why it is **not** appropriate.

(b) The survey is conducted by leaving copies of the questionnaire on the seats in a sports clothing shop. Give one criticism of how the survey was conducted.

Exercise 3

A survey was carried out to discover whether people would rather watch sports or detective programs on the television.

The following three questions were included.

Question 1: What is your address?

Question 2: Which type of TV program do you prefer? Tick one box.

Sports	Detective
☐	☐

Question 3: How many hours do you spend watching TV?

Less than 1 hour	1–5 hours	More than 5 hours
☐	☐	☐

(a) Give **one** reason why question 1 is not appropriate.

(b) Give **one** reason why question 3 is not appropriate.

(c) The survey was conducted by asking people who were leaving a football stadium one Saturday afternoon. Give **one** criticism of how the survey was carried out.

Exercise 4

Elen is conducting a survey in her school about the standard of food in the canteen. She asks every 20th person that goes to get hot food. Explain what is wrong with Elen's plan.

Hypotheses

A **hypothesis** is a statement like "boys spend more time on their homework than girls".
It is possible to create a questionnaire and collect data to test a hypothesis.

Applying

F

Exercise 5

Steffan wants to prove the following hypothesis.

'Most people spend more than two hours on the internet every night.'

He intends to

- give a short questionnaire to people in the local fitness centre,
- ask the following questions:
  - In your opinion, do people spend too much time on the internet?
  - How much time do you spend on the internet?
- ask each participant to post their completed questionnaire in an envelope with a stamp on it.

Write **three** unfavourable comments about this plan.

Exercise 6

Mari wants to prove the following hypothesis.

'Older pupils in a secondary school are better at remembering their times tables than younger pupils.'

She intends to

- give a short questionnaire to 50 randomly chosen pupils in each year,
- ask the same 5 multiplication questions to everyone, to complete in a maths lesson,
- ask the pupils to mark each other's work and return the questionnaires to her through the maths teacher.

Write **three** unfavourable comments about this plan.

Exercise 7

Iwan wants to prove the following hypothesis.

'The boys in year 10 spend more time on their homework than the girls'.

Write a questionnaire that Iwan could use to prove or disprove this hypothesis.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Sampling

In the previous chapter, we discussed how to design an appropriate questionnaire in order to prove a specific hypothesis. Often, it is not possible to ask the opinion of **all** members of a **population**. For example, it would be laborious to ask all pupils in a secondary school their opinion on a matter. Instead, we often use a **sample** of the population, which is a smaller group, and attempt to come to a conclusion about the opinion of the whole population based on the information from the sample.

We must be careful when selecting a sample. It must be **large enough,** and **representative** of the population. Asking only 5 pupils would not represent the opinion of a whole school, neither would asking only the pupils in year 8.

At GCSE level, we must be familiar with the following methods of selecting a sample.

Simple random sampling	Systematic sampling	Stratified sampling
Intermediate and Higher Tier	Intermediate and Higher Tier	Higher Tier only

Simple Random Sampling

In a **simple random sample**, every member of a population has **the same chance** of being chosen. There are two main methods of choosing a simple random sample:

- Using a table of random digits;
- Using the random number generator function on a calculator.

Example

A school hopes to change the start time for the school day and is eager to ask the opinion of the 600 pupils. The head teacher decides to choose a simple random sample of 10 pupils to question.

- The head teacher numbers all the pupils from 001 to 600.
- Starting from a random starting point in a table of random digits, the head teacher reads the numbers in groups of three.
- The head teacher accepts any number between 001 and 600, and rejects the others.
  The head teacher also ignores any repeated numbers.

Here is part of a table of random digits.

7087	0858	0164	1769	3218	1467	1938	8093	7918	2814
7796	7080	7227	3140	0933	0181	2013	7918	1177	4715
3830	9523	3653	8514	6061	0674	6025	9834	0499	3668
1347	1225	1910	3621	9722	8482	6298	1957	3507	7209

By starting at the digit in red (chosen at random), the head teacher chooses the following pupils: 218, 146, 388, 093, 147, 072, 273, 140, 301, 013.

Exercise 8

Skill I

Use the following table of random digits to choose a sample of 5 people out of 500, by
(a) starting at the first digit;
(b) starting at the red digit;
(c) starting at the blue digit.

0572	8836	4865	9430	8461	9978	1392	1166	7262	4438
8065	4455	5432	7323	9142	8933	4356	1767	0291	2037
9297	6827	1225	2158	8791	7847	6420	3726	1650	6365
3457	0248	5823	9512	1725	6247	0994	4066	8207	8813

Exercise 9

Use the following table of random digits to choose a sample of 8 people out of 75, by
(a) starting at the first digit;
(b) starting at the red digit;
(c) starting at the blue digit.

0003	3857	6162	2670	0883	5411	7163	3140	4505	6239
2415	1096	4182	7652	6254	5054	8743	2175	9256	8364
9570	0276	0303	6250	8236	3012	2980	7517	6803	1580
8478	6061	7948	2014	5047	0797	9177	3878	6272	5734

Exercise 10

Use the following table of random digits to choose a sample of 6 people out of 1600, by starting at the first digit.

3618	5991	8471	1714	0315	3185	2048	9874	5016	4707
5685	2304	2731	0092	7065	2428	0164	2798	1511	7259
9027	6444	9761	1197	5305	2910	3860	3490	7629	1963
2587	4167	6515	4516	0708	3449	5001	0437	6137	9031

Random Numbers on a Calculator

Instead of using a table of random digits to generate random numbers, it is possible to use a scientific calculator to generate random numbers. For example, to choose a random number between 001 and 600 (like the school example from the previous page), it is possible to press the following buttons on a Casio calculator:

ALPHA • 1 SHIFT) 6 0 0) =

We can generate further random numbers by pressing = again.

Exercise 11

Repeat Exercises 8 to 10, using the random number generator on your calculator to select the required samples.

Systematic Sampling

In a **systematic sample**, the sample is selected from the population in a **regular pattern**.

Example

A school hopes to change the start time for the school day and is eager to ask the opinion of the 600 pupils. The head teacher decides to choose a systematic sample of 10 pupils to question.

- The head teacher numbers all the pupils from 001 to 600.
- $600 \div 10 = 60$, therefore we work through the list of pupils in intervals of 60. (60 is the **sampling interval**.)
- To choose the starting number, we must use a table of random digits or the random number generator on a calculator to choose a number between 01 and 60.

If we use the table of random digits from page 6, and read the digits in groups of two, the first number between 01 and 60 we see is 08. Therefore, the systematic sample is

008, 068, 128, 188, 248, 308, 368, 428, 488, 548.

Example

A young farmers' club is considering holding a fair and wishes to collect the opinion of the members on the content of the fair. The chairperson decides to choose a systematic sample of 10 members to question. The club has a total of 87 members.

- The chairperson numbers all the members of the club from 01 to 87.
- $87 \div 10 = 8.7$, therefore we need to work through the list of members in intervals of 8. (Why doesn't working through the list in intervals of 9 work?)
- To choose the starting number, we must use a table of random digits or the random number generator on a calculator to choose a number between 01 and 08.

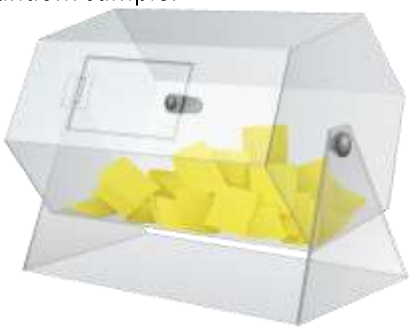
We must omit or ignore the digits after the decimal point.

The chairperson uses the random number generator function on a calculator to choose 03 as the starting number. Therefore, the systematic sample is

03, 11, 19, 27, 35, 43, 51, 59, 67, 75.

Notice, in the above example, that every member of the club doesn't have the same chance of being selected. The members between 01 and 80 have the same chance of being selected, $\frac{1}{8}$, but the members between 81 and 87 have **no** chance of being selected. Therefore, a systematic sample is not necessarily a random sample.

Would using a machine like this be a valid method of selecting a starting number?



Exercise 12

Find the sampling interval in the following systematic samples.

(a) Choosing 10 people out of 80.
(b) Choosing 5 people out of 45.
(c) Choosing 10 people out of 74.
(d) Choosing 4 people out of 18.
(e) Choosing 7 people out of 40.
(f) Choosing 9 people out of 63.
(g) Choosing 12 people out of 140.
(h) Choosing 20 people out of 1,500.

Exercise 13

Choose a systematic sample of 10 people out of 70, by starting with
(a) the second person (02);
(b) the fourth person (04);
(c) the seventh person (07).

Exercise 14

Choose a systematic sample of 10 people out of 140, by starting with
(a) the first person;
(b) the fifth person;
(c) the twelfth person.

Exercise 15

Choose a systematic sample of 8 people out of 100, by starting with
(a) the third person;
(b) the eighth person;
(c) the tenth person.

Exercise 16

Choose a systematic sample of 12 people out of 1,400, by starting with
(a) the sixth person;
(b) the twentieth person;
(c) the 89th person.

Exercise 17

Choose a systematic sample of 9 people out of 50. Use the following table of random digits to decide where to start.

6841	4804	3748	9980	4225	5215	8258	3707	2575	8524
6966	5346	1628	1375	8214	8630	5766	5942	1463	2818
4049	7245	5872	1469	0956	9848	1042	0684	4823	1716
2041	3672	9958	9099	5660	9092	4286	7496	8092	1236

Exercise 18

Choose a systematic sample of 12 people out of 80. Use the random number generator function on your calculator to choose a starting point.

Challenge!

Write a formula for calculating the sampling interval, using the size of the population and the size of the sample in your formula. Clue: look for “quotient and remainder” or the “floor function” on the internet.

Stratified Sampling

Higher Tier

Sometimes it is possible to split a population into strata or subgroups that reflect the composition of the population. For example, here are the details of the people working for a company creating computer games.

Role	Manager	Office staff	Programmers
Number of workers	3	7	27

An agency wants to sample the opinion of 10 people from the company. By looking at the table above, it would make sense to choose more programmers than managers to take part in the survey, since more programmers work for the company. Using a simple random sample or a systematic sample does not ensure that this will be the case, as it would be possible (for example) to choose all 3 managers – or none of them.

To reflect the strata (the subgroups) in the population, we use **stratified sampling** to ensure that every stratum of the population gets a fair representation in the sample.

Method:

Multiply by 10 since we want a sample of 10.

3 + 7 + 27 = 37 people work for the company.

We choose $\frac{3}{37} \times 10 = 0.8108$ … managers, which is 1 manager to the nearest whole number.
We choose $\frac{7}{37} \times 10 = 1.8918$ … office staff, which is 2 office staff to the nearest whole number.
We choose $\frac{27}{37} \times 10 = 7.2972$ … programmers, which is 7 programmers to the nearest whole number.

Check: 1 + 2 + 7 = 10, therefore we have selected the correct number of people to form the sample.

It would be possible to use a simple random sample or a systematic sample to choose which managers, which office staff and which programmers are questioned.

Example

An internet holiday club has members from 4 countries across the world. The number of members per country are given in the following table.

Country	Australia	China	Thailand	Mexico
Number of members	2,840	1,382	4,086	940

The company organises a meeting for 25 of the members to represent the opinions of all the members.
Use a stratified sample to calculate how many members from each country should be invited to the meeting.

Answer:

There are 2,840 + 1,382 + 4,086 + 940 = 9,248 members in total.

We choose $\frac{2840}{9248} \times 25 = 7.6773$ … people from Australia, which is 8 people to the nearest whole number.
We choose $\frac{1382}{9248} \times 25 = 3.7359$ … people from China, which is 4 people to the nearest whole number.
We choose $\frac{4086}{9248} \times 25 = 11.0456$ … people from Thailand, which is 11 people to the nearest whole number.
We choose $\frac{940}{9248} \times 25 = 2.5410$ … people from Mexico, which is 3 people to the nearest whole number.

Check: 8 + 4 + 11 + 3 = 26, therefore we have chosen one person too many. We adjust the country with the **most** members, Thailand, from 11 to 10 to ensure that the total is 25.

Exercise 19

Applying H

In a particular school, there are 359 girls and 467 boys. The school council includes 30 pupil members. Use the stratified sampling method to calculate how many girls and how many boys should be chosen for the school council.

Exercise 20

A sports company employs people from a number of different countries.
The following table shows the number of people employed by the company from each country.

Country	Canada	New Zealand	Turkey	China
Number of employees	2,785	804	1,207	8,763

The company is organising a promotional event and decides to invite a total of 45 employees to represent the opinion of all the employees. Use the stratified sampling method to calculate how many people from each country should be invited to the promotional event.

Exercise 21

A school in Wales has international links with schools in four countries across the world. The following table shows the number of pupils in each of the schools in the four countries.

Country	France	Australia	Canada	Brazil
Number of pupils	1,230	1,123	934	720

The school in Wales is organising a celebration and would like to invite a total of 35 pupils to represent the pupils from the four countries. Use the stratified sampling method to calculate the number of pupils who should be invited from each country.

Exercise 22

A movie society on the internet has members from four countries across the world.
In the following table the number of members from each country is shown.

Country	USA	UK	France	The Netherlands
Number of members	12,637	8,382	4,010	720

The movie society organises a meeting for 30 members to represent the opinion of the whole society. Use the stratified sampling method to calculate how many members from each country should be invited to the meeting.

Exercise 23

In the following table the populations of 5 villages are given.

Village	Aberford	Bronglas	Carmel	Dunwern	Eiderfalls
Population	1,550	3,700	600	980	5,500

A committee of 20 people from the 5 villages needs to be chosen. Use the stratified sampling method to calculate how many people from each village should be invited to join the committee.

Exercise 24 (Revision) I

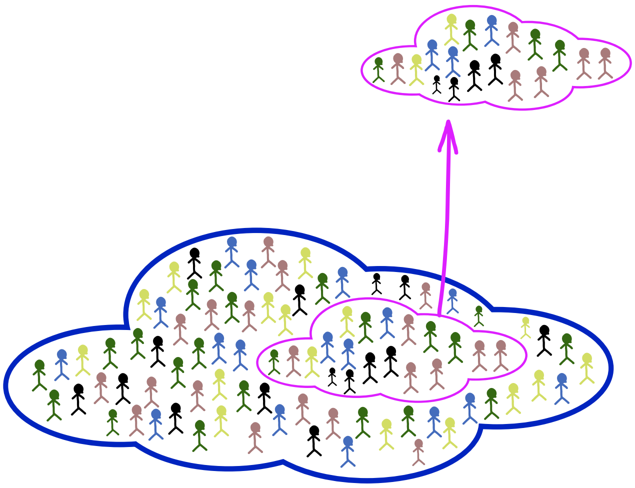
(a) By starting at the first digit in the following part of a table of random digits, choose a random sample of 4 people from a list of 45 people.

06 56 06 14 27 93 24

(b) The opinion of people in a queue is found by asking a number of people in the queue to answer a questionnaire. Explain why asking every tenth person in the queue is not a method of choosing a random sample to answer the questionnaire.

Exercise 25 (Revision) H

(a) Are the following statements TRUE or FALSE?



1) Choosing the first name on the register of every class will give a random sample.
2) The ratio of boys to girls in a school is 2 : 3. The school council of 30 pupils is chosen by using a gender stratified sample. There are 10 boys and 20 girls on the school council.
3) A phone survey is conducted to discover which political party people support.
   The sample of people in the survey is **not** a random sample of the whole population.
4) Stratified sampling always considers the proportions according to specific criteria.
5) A random sample means that everybody has an equal chance of being selected.

(b) An international organisation employs people from Australia, Belgium, Canada, Denmark and Ecuador. The following table shows the number of people employed by the organisation in each country.

Country	**Australia**	**Belgium**	**Canada**	**Denmark**	**Ecuador**
Number of employees	5,243	1,004	8,745	545	762

The organisation is organising a charity event and chooses to invite 25 employees to represent the employees from all 5 countries. Use the stratified sampling method to calculate how many people from each country should be invited to the charity event.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Frequency Polygons

Drawing Frequency Polygons

We draw **frequency polygons** for the following types of data.

- Grouped discrete quantitative data.
- Continuous quantitative data.

A frequency polygon is a **line graph** where we plot the **midpoint** for every class against the **frequency.**

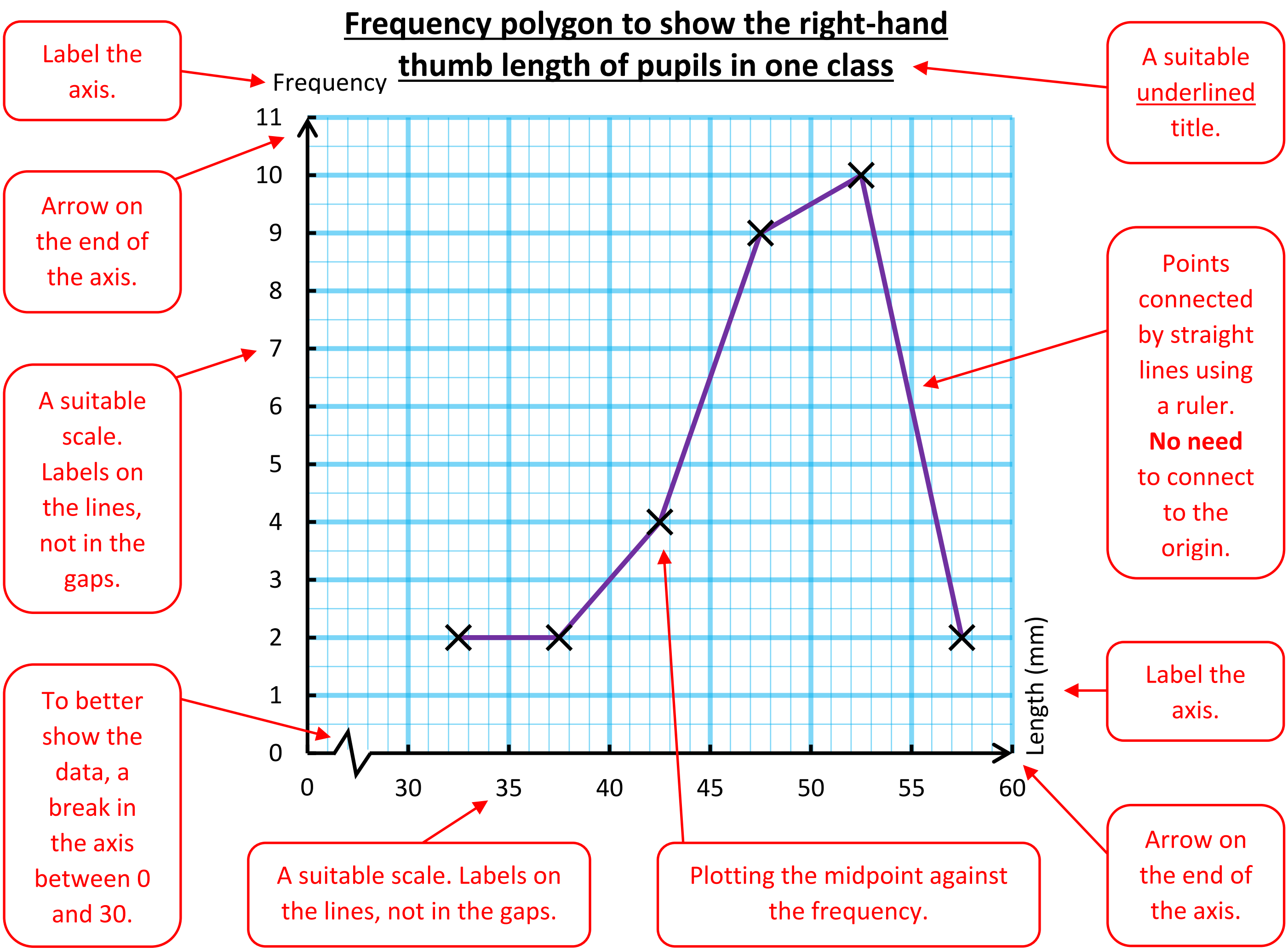
Example

The following frequency table shows the right-hand thumb length (*l*) for pupils in one class.

Class Interval (*l* mm)	Frequency
$30 \leqslant l < 35$	2
$35 \leqslant l < 40$	2
$40 \leqslant l < 45$	4
$45 \leqslant l < 50$	9
$50 \leqslant l < 55$	10
$55 \leqslant l < 60$	2

The midpoint for the class $30 \leqslant l < 35$ is 32.5.
Method 1: 32.5 is half way between 30 and 35.
Method 2: 30 + 35 = 65. 65 ÷ 2 = 32.5.

The frequency polygon below represents the data.



Exercise 26

Draw a **frequency polygon** for the following data on squared paper.

(a) Number of books bought by pupils in 7E during the last year.

None to four books: 12 Five to nine books: 5
Ten to fourteen books: 6 Fifteen to nineteen books: 1

(b) Number of minutes a dentist spends with each patient.

1–5 minutes: 2 6–10 minutes: 4 11–15 minutes: 9 16–20 minutes: 5
21–25 minutes: 3 26–30 minutes: 3 31–35 minutes: 0 36–40 minutes: 1

(c) Number of absent days from school for pupils in 7C last term.

0–4 days: 11 5–9 days: 8 10–14 days: 6 15–19 days: 0 20–24 days: 5

(d) Number of words in every sentence for the first 50 sentences of a book.

1–10 words: 2 11–20 words: 9 21–30 words: 14 31–40 words: 7
41–50 words: 4 51–60 words: 8 61–70 words: 6

Exercise 27

Draw a **frequency polygon** for the data in each of the following frequency tables.

(a) Weight was lost by people in a weight loss group over 6 months.

Weight (w kg)	Frequency
$0 \leqslant w < 6$	4
$6 \leqslant w < 12$	11
$12 \leqslant w < 18$	12
$18 \leqslant w < 24$	7
$24 \leqslant w < 30$	3

(b) Height of 60 pupils.

Height (h cm)	Frequency
$168 \leqslant h < 172$	2
$172 \leqslant h < 176$	6
$176 \leqslant h < 180$	17
$180 \leqslant h < 184$	22
$184 \leqslant h < 188$	10
$188 \leqslant h < 192$	3

(c) Sound of 60 electrical items.

Sound (s db)	Frequency
$15 \leqslant s < 20$	4
$20 \leqslant s < 25$	12
$25 \leqslant s < 30$	15
$30 \leqslant s < 35$	6
$35 \leqslant s < 40$	8
$40 \leqslant s < 45$	3
$45 \leqslant s < 50$	12

Exercise 28

The following data gives the time taken by 50 runners to complete a cross-country race, to the nearest minute.

30	37	43	55	52	47	49	36	44	40
41	49	52	53	39	41	46	42	50	49
39	53	54	57	43	59	34	38	40	42
48	53	50	52	37	36	45	53	48	42
52	39	41	46	50	52	38	58	57	46

Time (t minutes)	Tally Marks	Frequency
$30 \leqslant t < 35$		
$35 \leqslant t < 40$		
$40 \leqslant t < 45$		
$45 \leqslant t < 50$		
$50 \leqslant t < 55$		
$55 \leqslant t < 60$		

(a) Complete the **frequency table** for the data. (Remember that the data item 35 minutes would go into $35 \leqslant t < 40$, not $30 \leqslant t < 35$.)

(b) Draw a **frequency polygon** for the data.

Interpreting Frequency Polygons Applying

Exercise 29

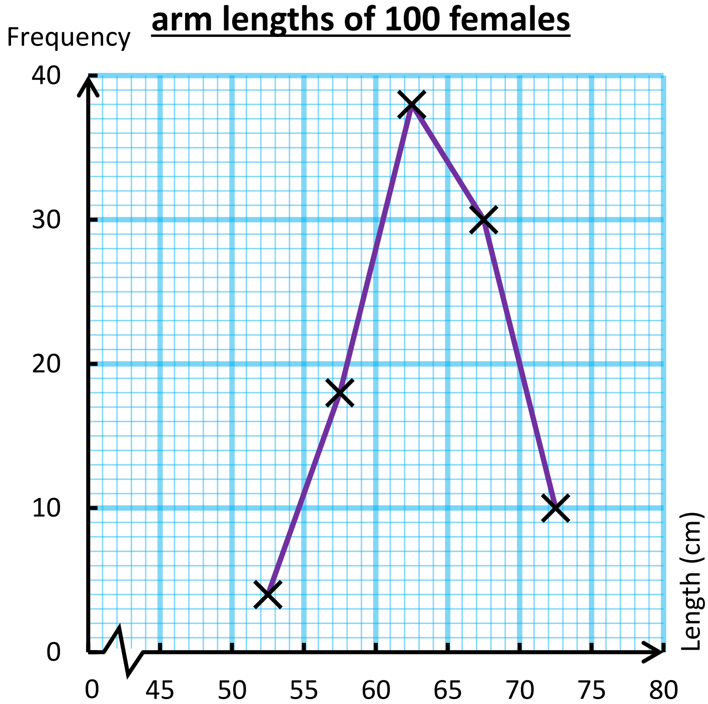
The frequency polygon on the right shows the arm lengths of 100 females.

(a) How many females have an arm length between 55 cm and 60 cm?

(b) How many more females have an arm length between 65 cm and 70 cm, compared to females with arm length between 70 cm and 75 cm?

(c) Complete the frequency table below, using the information from the frequency polygon.

<u>Frequency polygon to show the arm lengths of 100 females</u>



Arm length, l cm	Frequency
$50 < l \leqslant 55$	
$55 < l \leqslant 60$	
$60 < l \leqslant 65$	
$65 < l \leqslant 70$	
$70 < l \leqslant 75$	

(d) What is the modal class of the data?

Exercise 30

The daily rainfall for ten days was measured in Aberwen and Aberisel. The frequency polygon on the right shows the results. The purple line represents Aberwen, and the red line represents Aberisel.

(a) For how many days was the rainfall between 0 mm and 1 mm of rain in Aberwen?

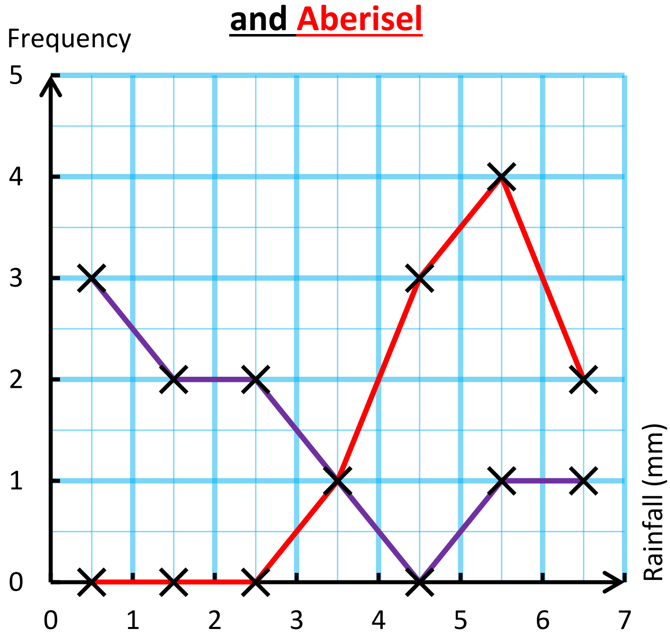
(b) For how many days was the rainfall between 6 mm and 7 mm of rain in Aberisel?

(c) Complete this sentence: Considering the days where the rainfall was between 5 mm and 6 mm of rain, Aberisel had this rainfall on _______ more days than Aberwen.

(d) Over these 10 days, in your opinion where was the wettest place? Explain your answer.

(e) Deiniol says "The frequency polygon shows that the rainfall in Aberwen and Aberisel is the same on the fourth day". Is Deiniol telling the truth? Explain your answer.

<u>Frequency polygon to show the rainfall over 10 days in Aberwen and Aberisel</u>



Challenge! ⚠ Extension

Use the frequency table in Exercise 29 to calculate an estimated mean arm length for the 100 females.

Challenge! ⚠

Use the frequency polygon from Exercise 30 to calculate the estimated mean rainfall in Aberwen and in Aberisel. Does your answer agree with your conclusion to part (d) of Exercise 30?

Exercise 31

I

The table below shows the marks for 10R in their English test. (The test was out of 30.)

Marks (m)	Frequency
$0 \leqslant m < 6$	3
$6 \leqslant m < 12$	8
$12 \leqslant m < 18$	7
$18 \leqslant m < 24$	6
$24 \leqslant m < 30$	2

(a) Eric looks at the table and says: *"Three people got 0 out of 30 in this test!"*. Is Eric telling the truth?

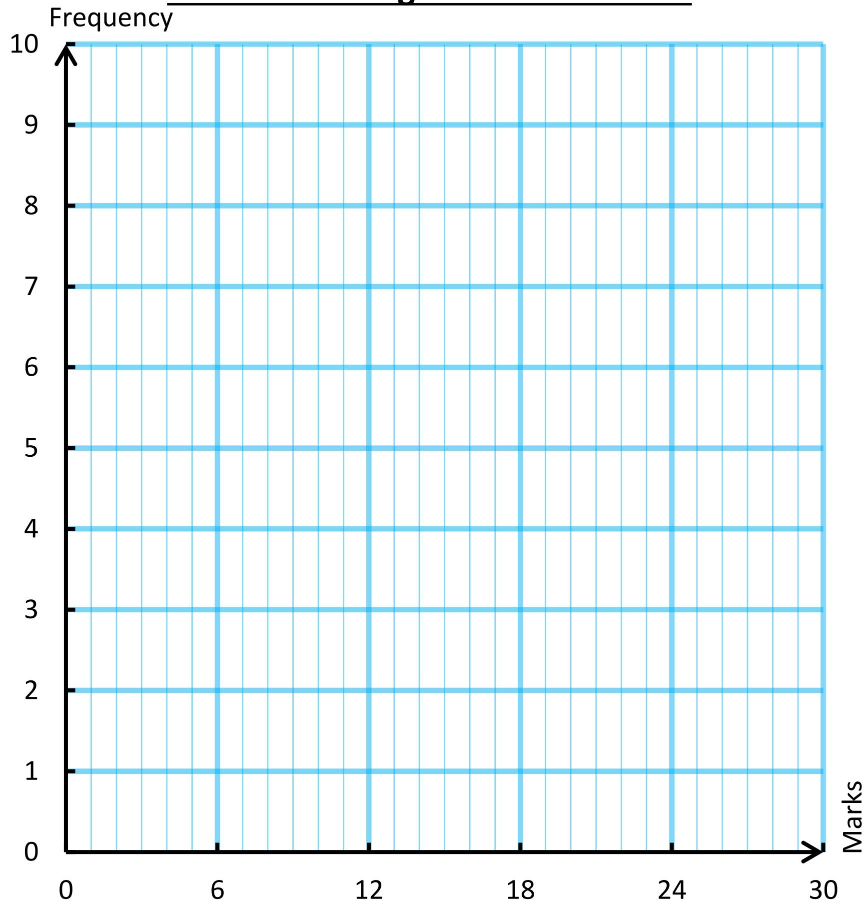
(b) Susan looks at the table and says: *"Nobody got full marks in this test!"*. Is Susan telling the truth?

(c) On the graph paper on the right, draw a frequency polygon for the data.

(d) On the same graph paper, draw a frequency diagram for the data.

(e) What is the connection between any frequency polygon and frequency diagram drawn for the same data?

Frequency polygon and frequency diagram to show the English marks for 10R



(f) Calculate the following for the data from the English test.

(i) The modal class.
(ii) The median class.
(iii) The estimated mean.
(iv) The estimated range.

Evaluation

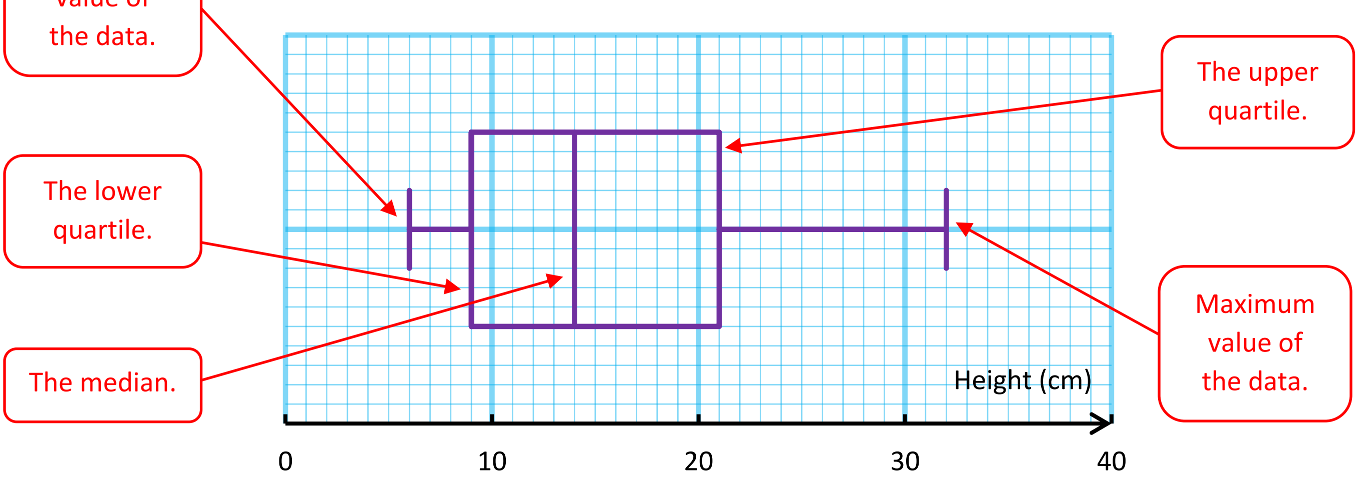
Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Box and Whisker Diagrams

(Or Box and Whisker Plots.)

A **box and whisker diagram** shows a number of statistics on one diagram.

Box and whisker diagram to show the height of 80 plants



Exercise 32

Skill I

For the box and whisker diagram shown above, write

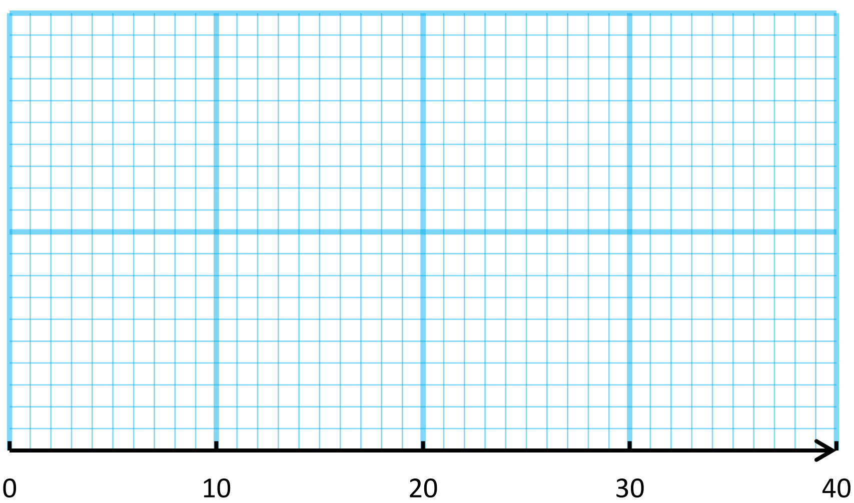
(a) The smallest plant height.
(b) The lower quartile.
(c) The median plant height.
(d) The upper quartile.
(e) The largest plant height.
(f) The range of plant heights.
(g) The interquartile range of plant heights.

Exercise 33

Moli calculated the following statistics for 50 numbers.
Use the statistics to draw a box and whisker diagram on the graph paper below.

Smallest number = 5 Lower quartile = 12 Median = 19 Upper quartile = 24 Largest number = 34

Box and whisker diagram for Moli's 50 numbers

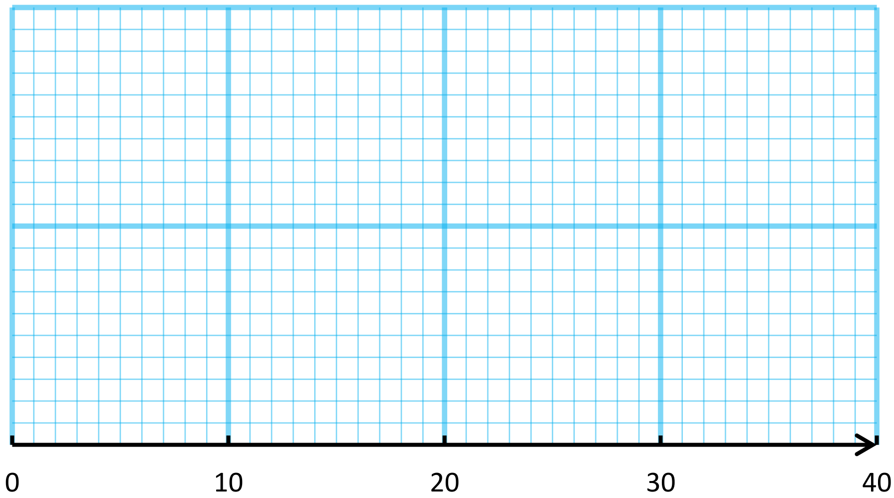


Exercise 34 I

Dafydd calculates the following statistics for 70 numbers.
Use the statistics to draw a box and whisker diagram on the graph paper below.

Smallest number = 8 Lower quartile = 15 Median = 20 Interquartile range = 12 Range = 27

<u>Box and whisker diagram for Dafydd's 70 numbers</u>



Exercise 35

Draw box and whisker diagrams for the following sets of data.

(a) 4, 11, 14, 15, 17, 19, 20, 22, 22, 26, 29, 34, 35, 35, 38.

(b) 24, 13, 9, 35, 3, 17, 21, 30, 12, 28.

(c) 2, 6, 14, 18, 26, 27, 27, 30, 31.

The Connection between Box and Whisker diagrams and Cumulative Frequency Diagrams

There is a useful connection between cumulative frequency diagrams and box and whisker diagrams.

Example

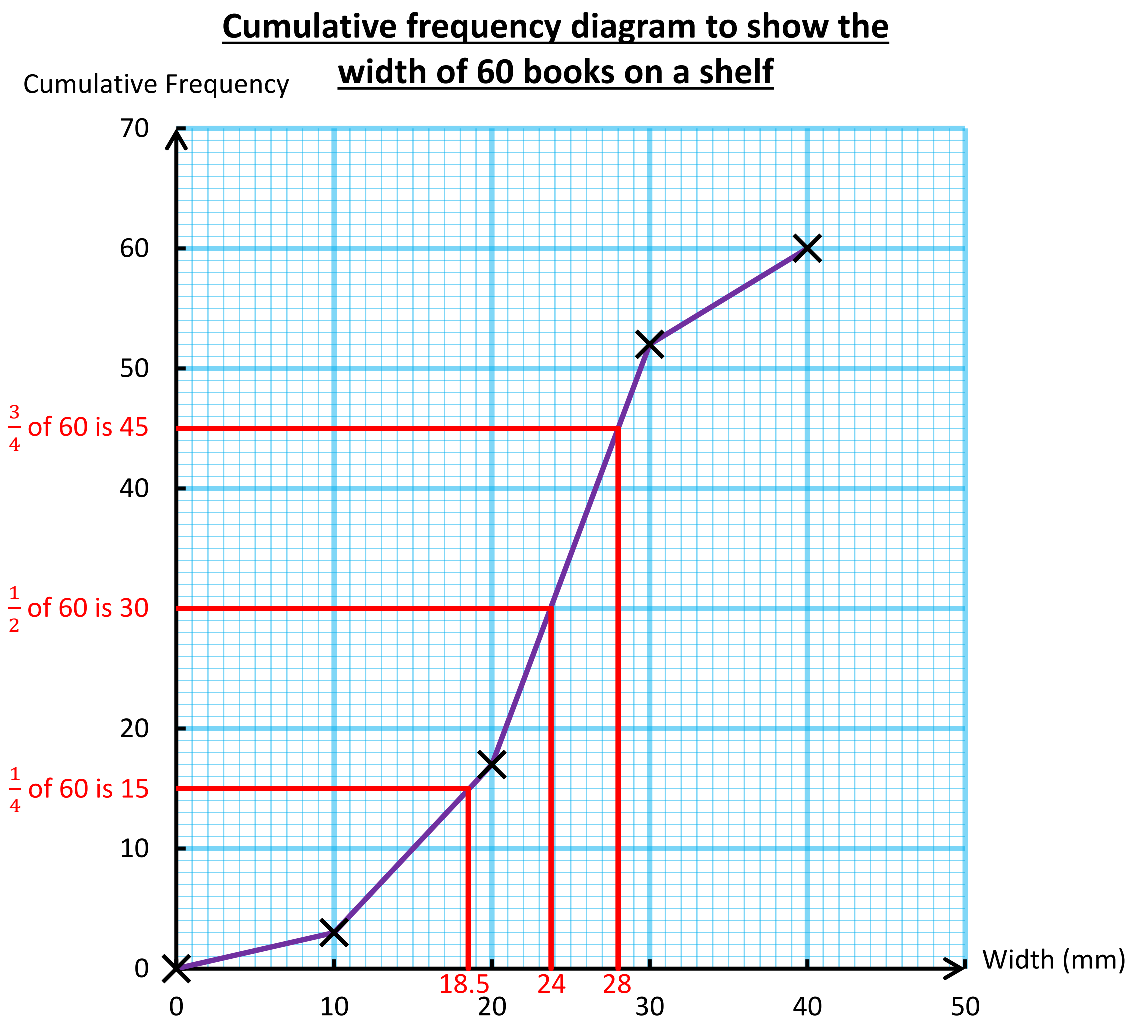
Consider the following data which shows the width of books sitting on a shelf in a library.

Book width (w mm)	Frequency
$0 < w \leqslant 10$	3
$10 < w \leqslant 20$	14
$20 < w \leqslant 30$	35
$30 < w \leqslant 40$	8

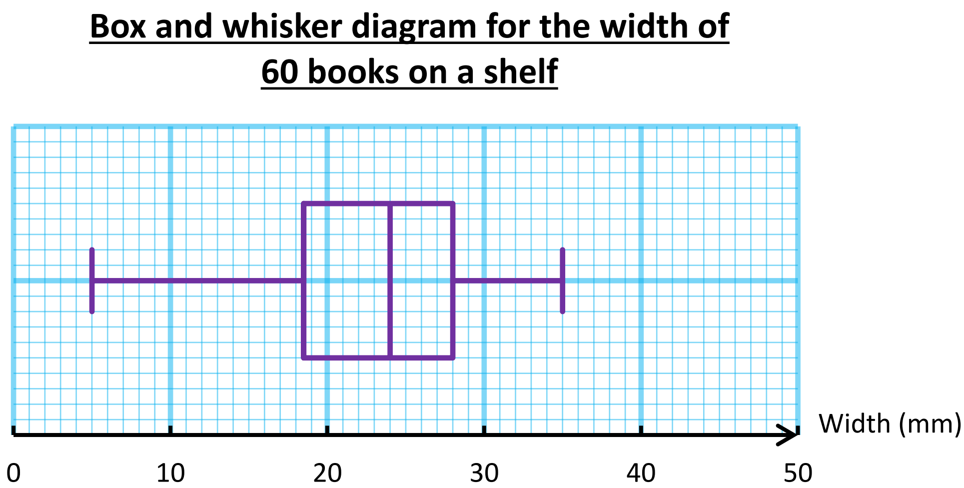
To draw a cumulative frequency diagram for the data we must first draw a cumulative frequency table.

Book width (w mm)	Frequency	Cumulative Frequency	
$0 < w \leqslant 10$	3	3	
$10 < w \leqslant 20$	14	17	(3 + 14 = 17)
$20 < w \leqslant 30$	35	52	(17 + 35 = 52)
$30 < w \leqslant 40$	8	60	(52 + 8 = 60)

We can now draw a cumulative frequency diagram for the data.



To be able to draw the box and whisker diagram for the data, we need to find the quartiles. We can use the cumulative frequency diagram to estimate these. (These are the red lines on the above diagram). We also need to estimate the minimum value and the maximum value for the data. For the minimum value, we use the midpoint of the first class ($0 < w \leqslant 10$) which gives 5 mm. For the maximum value, we use the midpoint of the last class ($30 < w \leqslant 40$) which gives 35 mm. We can now draw the box and whisker diagram for the data.



Exercise 36

Applying

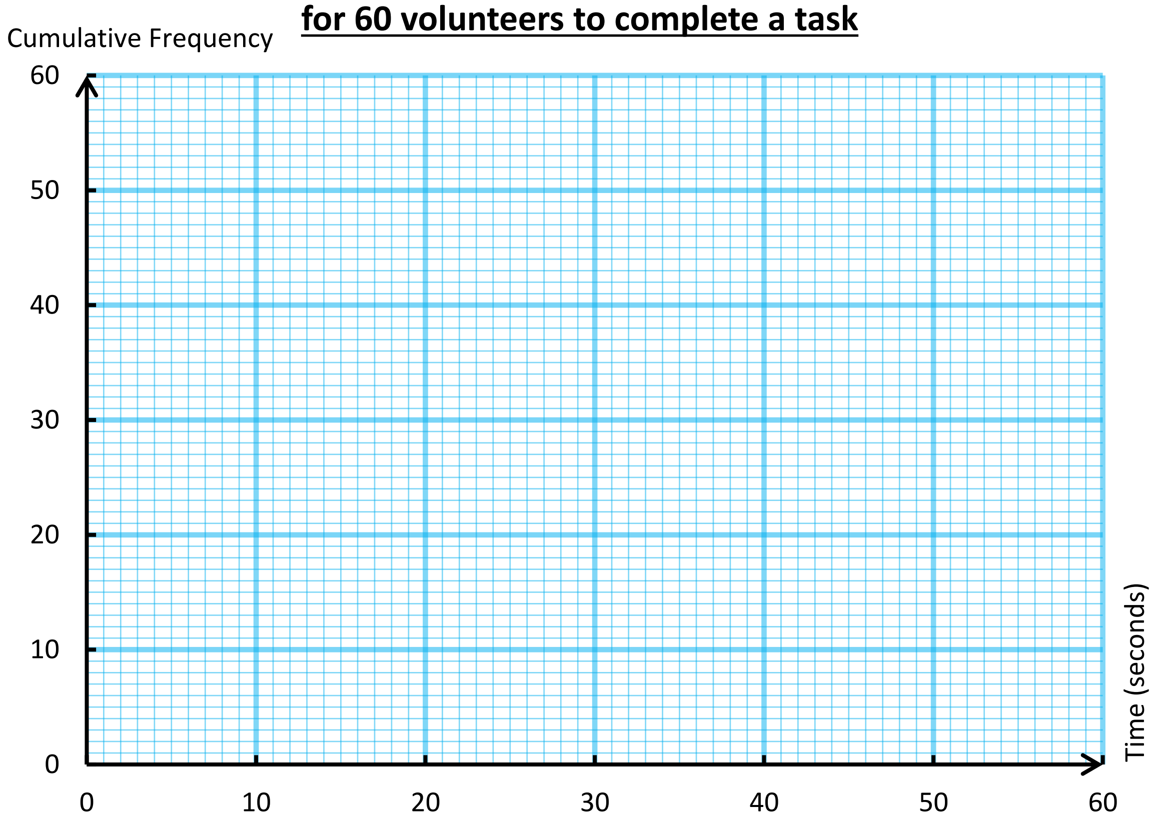
I

The times taken by 60 volunteers to complete a task were recorded, in seconds. The results are shown in the frequency table below. Complete the cumulative frequency table, the cumulative frequency diagram and the box and whisker diagram for the data.

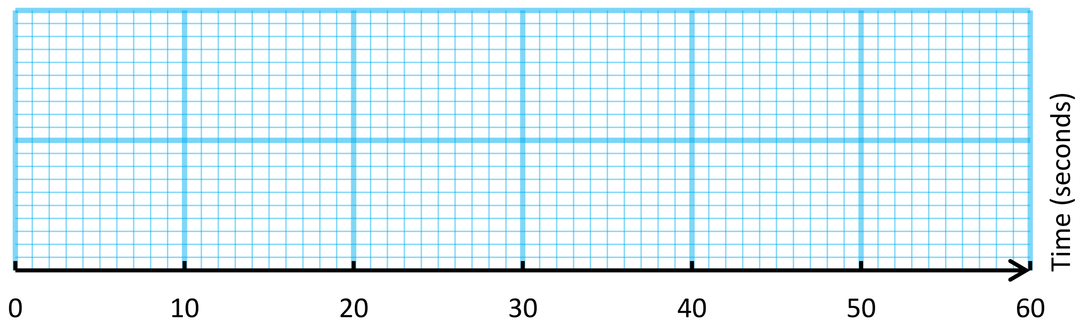
Time, t, to finish task (seconds)	Frequency
$15 < t \leqslant 20$	3
$20 < t \leqslant 25$	6
$25 < t \leqslant 30$	9
$30 < t \leqslant 35$	19
$35 < t \leqslant 40$	15
$40 < t \leqslant 45$	5
$45 < t \leqslant 50$	3

Time, t, to finish task (seconds)	Cumulative Frequency
$t \leqslant 15$	
$t \leqslant 20$	
$t \leqslant 25$	
$t \leqslant 30$	
$t \leqslant 35$	
$t \leqslant 40$	
$t \leqslant 45$	
$t \leqslant 50$	

<u>Cumulative frequency diagram to show the times taken for 60 volunteers to complete a task</u>



<u>Box and whisker diagram to show the times taken for 60 volunteers to complete a task</u>



Exercise 37 I

The frequency table below shows the time taken, in minutes, for company workers to travel to work every morning.

Time, t, in minutes	Frequency
$0 < t \leqslant 10$	3
$10 < t \leqslant 20$	8
$20 < t \leqslant 30$	14
$30 < t \leqslant 40$	6
$40 < t \leqslant 50$	7
$50 < t \leqslant 60$	2

(a) Draw a cumulative frequency table for the data.

(b) Draw a cumulative frequency diagram for the data.

(c) Draw a box and whisker diagram for the data.

(d) 5 years ago the median time was 22 minutes. How has the journey changed? Give a possible explanation.

(e) One day the median time was 24 minutes, the upper quartile was 50 minutes and the maximum time was 75 minutes. Explain what could have happened.

Exercise 38

Mr. Hughes and Mrs. Jones have bought the same type of seeds for planting a special plant. The two plant the seeds at the same time and measure the heights of the plants 6 months later. The box and whisker diagrams on the right show the results.

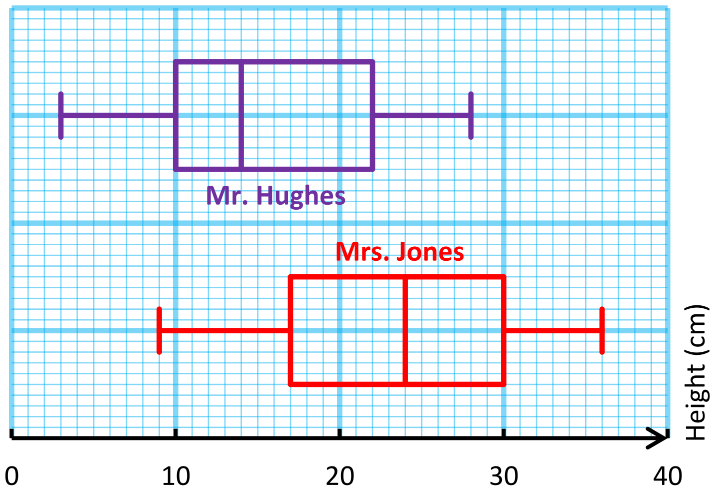
(a) What is the median height of Mr. Hughes' plants?

(b) What is the height of Mrs. Jones' tallest plant?

(c) Calculate the interquartile range of Mr. Hughes' plants.

(d) One of the people used fertiliser over the last 6 months. Who did this, in your opinion? Explain your answer.

Box and whisker diagrams for the heights of Mr. Hughes and Mrs. Jones' plants 6 months after planting



Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ❑ Target ❑

Comparing Averages

Revising Averages

We have studied three different types of averages over the last few years.

Year 7	Year 8	Year 9	Year 10
Mean	Mode	Median	**Comparing Averages**

Exercise 39

Revision

Fill in the boxes below to explain how to calculate the mean, the mode and the median.

F

The Mean	The Mode	The Median

Exercise 40

Calculate the mean, the mode and the median of the following data.

12, 14, 14, 15, 16, 17, 17, 17, 19, 20.

Choosing the Most Appropriate Average

I

	The Mean	The Mode	The Median
Advantages	• Uses all the data values.	• Not affected by outliers. • Can be used for qualitative data.	• Not affected by outliers.
Disadvantages	• Can be affected by outliers. • Needs to be calculated.	• Doesn't use all the data values. • There is no mode for some data sets.	• Doesn't use all the data values. • Need to rearrange the data to find it.
Used for	• Data which doesn't include outliers.	• Qualitative data. • Data which includes outliers.	• Data which includes outliers.

Challenge!

Extension

Use the internet to investigate the term *skewness*.
Which average is the best to use when the data has a skewed distribution?

Exercise 41

Applying I

Which average is most appropriate for the following data sets?

(a) Favourite football team:
Liverpool, Chelsea, Man City, Everton, Liverpool, Man Utd.

(b) Times in a 100 m race (in seconds):
9.81, 9.89, 9.91, 9.93, 9.94, 9.96, 10.04, 10.06.

(c) Price of Heinz Baked Beans in different shops:
75p, 60p, 74p, 80p, 70p, 95p, 85p.

(d) Age of players who start a football game:
28, 31, 19, 24, 25, 28, 30, 23, 20, 29, 26.

(e) Year 10 pupil heights:
162 cm, 160 cm, 161 cm, 148 cm, 163 cm, 161 cm.

(f) Favourite subject in school:
Science, Music, Drama, Mathematics, Music.

(g) Spelling test scores (out of 10):
5, 7, 8, 4, 5, 3, 6, 4, 5, 4, 7.

(h) Number of brothers:
0, 1, 2, 1, 0, 6, 1, 0, 1, 2.

Exercise 42

F

Are the following statements TRUE or FALSE?

(a) The mode is the most popular data item in a set of data.
(b) Half the values in a data set are greater than the mean.
(c) Half the values in a data set are greater than the median.
(d) When finding the median, it doesn't matter if you order the data from least to greatest or from greatest to least.
(e) It is always possible to find the mean of a data set.

Exercise 43

The following table shows the percentages of 10 pupils in Welsh and Mathematics tests.

Welsh	57	63	91	58	56	75	59	76	91	54
Mathematics	67	68	66	68	68	66	70	69	68	70

(a) Complete the following table.

	Welsh	**Mathematics**
The mean		
The median		
The mode		
The range		

(b) Which statistics from the table support the following newspaper headlines?

(i) **The Welsh results are very high this year.**

(ii) **Pupils do not perfom better in Mathematics than in Welsh.**

(iii) **There was lots of copying during the Mathematics test.**

(iv) **The Mathematics test was easier than the Welsh test.**

(v) **The Welsh results show that some people tried harder than others.**

Exercise 44 F

During a skiing trip, the PE department recorded the daily snowfall for 5 consecutive days. Here is some information about the daily snowfall.

Mean	Mode	Median	Range
5.8 cm	3 cm	5.6 cm	6.6 cm

(a) Use these statistics to calculate what was the daily snowfall for these 5 consecutive days.

(b) If it had snowed exactly 2 cm more each day, what would the new statistics be?

Mean	Mode	Median	Range

Exercise 45

In a game, it is possible for every player to score between 1 and 10 points.
Lois and Beca play the game 5 times.

The table below shows the points scored by Lois in every game.

	Game 1	Game 2	Game 3	Game 4	Game 5
Lois	5	2	8	5	1
Beca					

Beca had a higher mean score than Lois.
Beca had a lower median score than Lois.
Beca had a lower range of scores than Lois.

Complete the table above with a set of possible scores for Beca.

Exercise 46

Jim and Andy play for their local cricket team.
They scored the following runs in their six most recent games.

Jim	42	71	39	62	70	40
Andy	115	6	84	36	10	85

(a) Calculate Jim's mean and Andy's mean.

(b) Calculate Jim's median and Andy's median.

(c) There is no room for both Jim and Andy on the team for the next game. The management team need to choose either Jim or Andy to play for the team. Use your answers to parts (a) and (b) to give advice to the management team.

Exercise 47

I

The table below shows the number of ice hockey season tickets sold by a team last season, together with the prices for the tickets.

Ticket Price (£)	Number of tickets sold
250	180
300	230
350	230
500	150

For the cost of the tickets sold last season, calculate

(a) The mode; (b) The median; (c) The range.

(d) The ice hockey team owner says that more than half of the tickets sold were greater in value than £300. Explain why the team owner is incorrect.

Exercise 48

50 people took part in a charity walk.
The table below shows the grouped frequency distribution of the sums of money raised, to the nearest £.

Sum, s, in £	Number of people
$10 \leqslant s \leqslant 19$	2
$20 \leqslant s \leqslant 29$	18
$30 \leqslant s \leqslant 39$	29
$40 \leqslant s \leqslant 49$	1

(a) Find the modal class of the data.

(b) Find the median class of the data.

(c) Calculate an estimate for the mean sum of money raised per person.

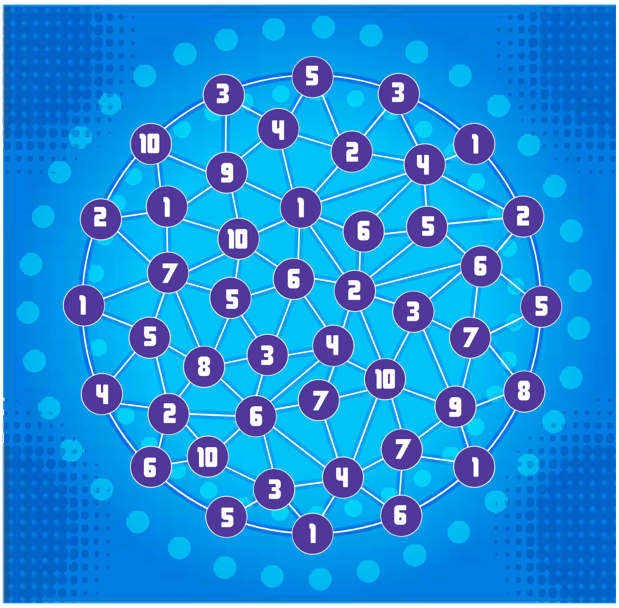
(d) Another 50 people took part in the same charity walk. The total of the money collected by these 50 extra people was £1,600. Is it possible to say that these 50 people raised more money than the original 50 people?

Evaluation

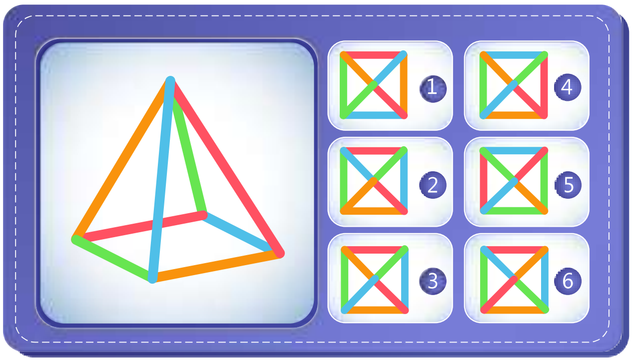
Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Puzzles

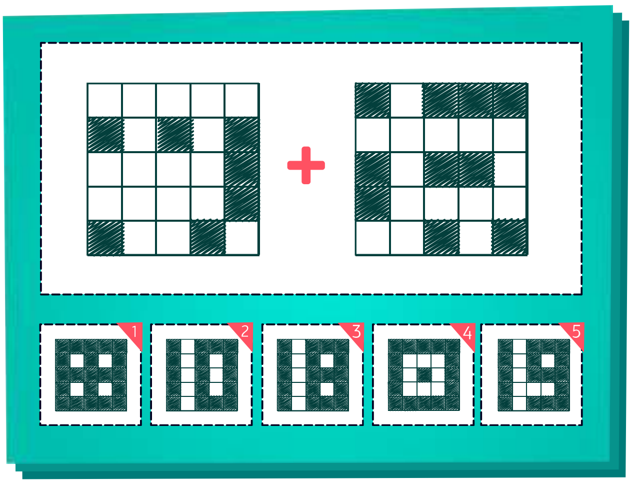
(a) Connect the numbers from 1 to 10.



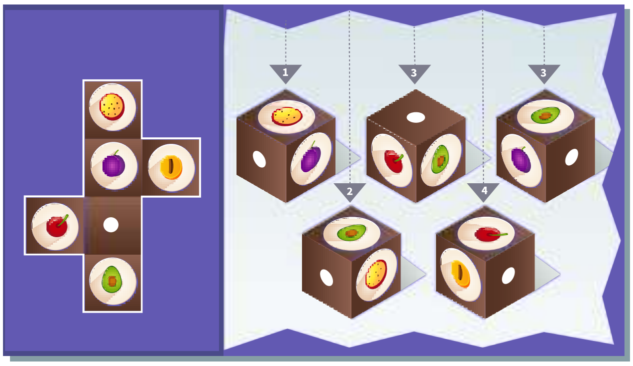
(b) Which is the correct plan view?



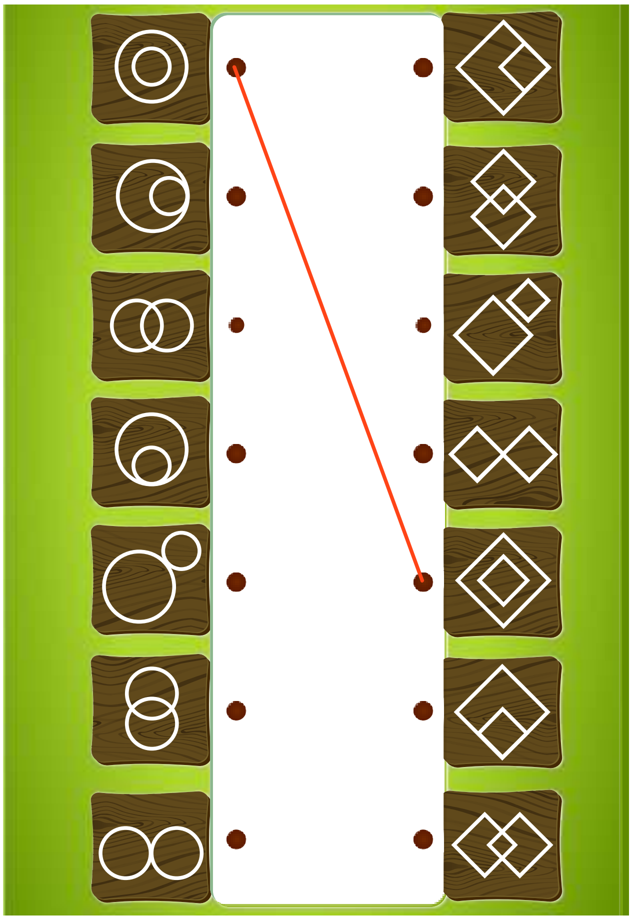
(c) Join the pictures to deduce which image is formed.



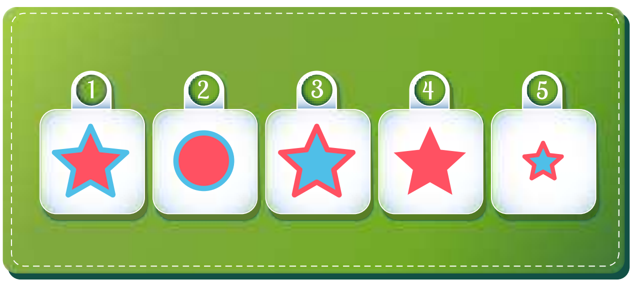
(d) Which cube is formed by folding the net?



(e) Pair the pictures.



(f) Which picture is the odd one out?



Data Handling and Statistics 4

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I know how to **criticise questions** in a **questionnaire**.			1	
I know how to write **unfavourable comments** about a plan to prove a specific **hypothesis**.			1	
I know how to choose a **simple random sample** using a **table of random digits** or using the **random number generator function on a calculator**.			2	
I know how to calculate the **sampling interval** for a systematic sample.			3	
I know how to choose a **systematic sample**.			3	
I know how to draw a **frequency polygon**.			4	
I know how to **interpret** a frequency polygon.				
I know how to draw a **box and whisker diagram**.			5	
I can use a **cumulative frequency diagram** in order to draw a box and whisker diagram.			7	
I can calculate the **mode**, the **median**, the **mean** and the **range** for **discrete data**.			10	
I can calculate the **modal class**, the **median class**, an **estimate of the mean** and an **estimate of the range** for **grouped data**.			4, 8	
I know how to decide **which average is most suitable** for a set of data.			9	
I know how to use averages and measures of spread to **compare** two data sets.			9	
I can **find the original data set** given information about averages and the range.			6	

Am I ready for the test?
Tick the boxes below...

- [ ] I have revised the work in my mathematics book.
- [ ] I have revised the workbook.
- [ ] I have watched the relevant videos on YouTube.
- [ ] I have completed the Diagnostic Questions quiz.
- [ ] I have completed at least 4 pages in my revision book.

Data Handling and Statistics 4

Reflection Sheet

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I know how to calculate the **sampling interval** for a systematic sample.			3	
I know how to choose a **systematic sample**.			3	
I know how to choose a **stratified sample**.			10	
I know how to draw a **frequency polygon**.			4	
I know how to **interpret** a frequency polygon.				
I know how to draw a **box and whisker diagram**.			5	
I can use a **cumulative frequency diagram** in order to draw a box and whisker diagram.			7	
I can calculate the **mode**, the **median**, the **mean** and the **range** for **discrete data**.				
I can calculate the **modal class**, the **median class**, an **estimate of the mean** and an **estimate of the range** for **grouped data**.			4, 8	
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The Mathematics Department

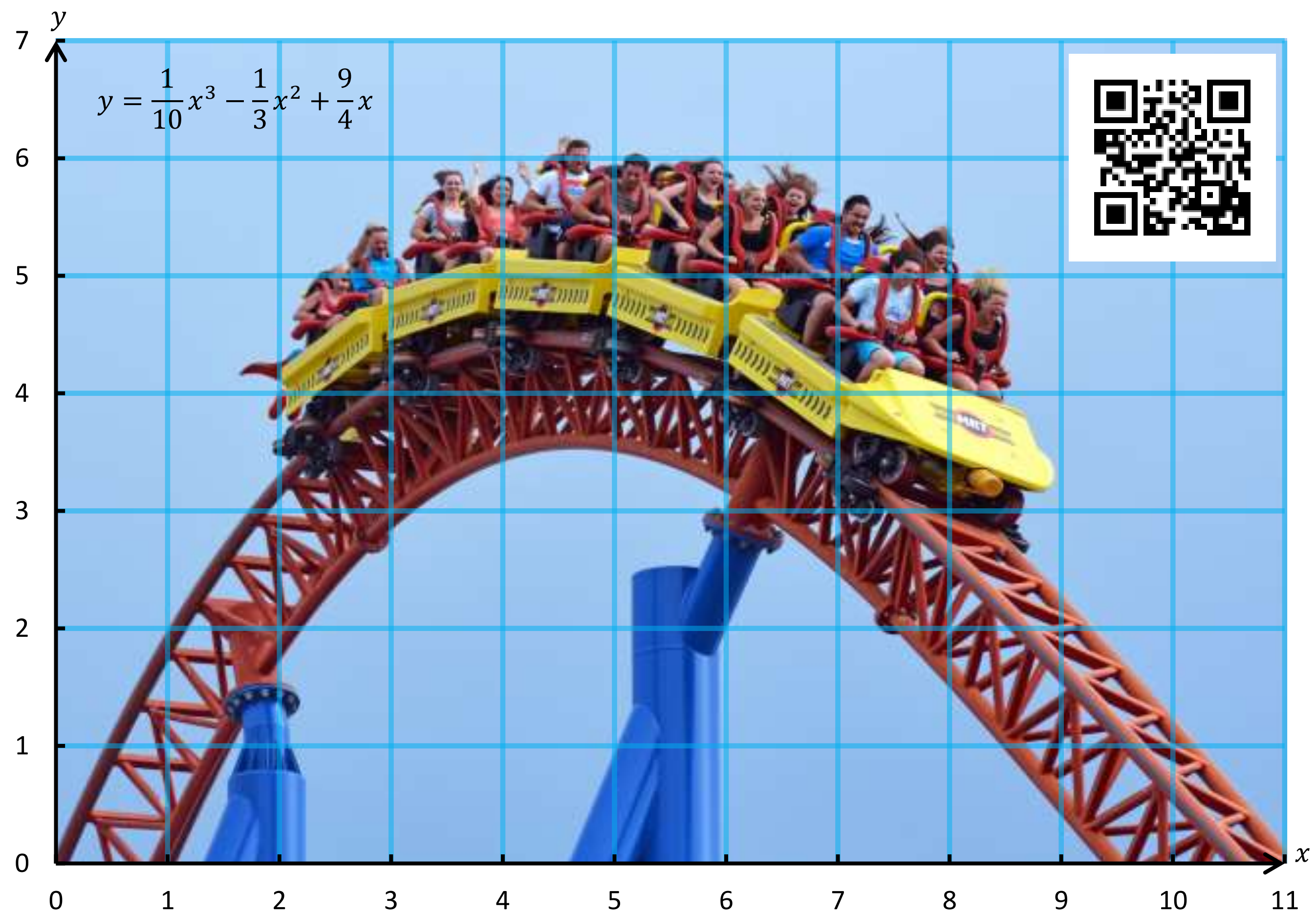
10

Powers and Roots

Name:

Contents

Chapter	Mathematics	Page Number
Rules of Indices	The index form. Evaluating the index form. The multiplication rule. The division rule. The zeroth index rule. Raising a power to another index. The negative index rule. The reciprocal as a negative index. Unitary fraction index rule. Algebra and rules of indices. **The general fraction index rule.**	3
Standard Form	Writing numbers $x \geq 1$ in standard form. Writing numbers $0 < x < 1$ in standard form. Changing from standard form to an ordinary number. Adding and subtracting in standard form. Almost in standard form. Multiplying and dividing in standard form.	11
Efficient Percentage Changes	Repeated percentage changes. Calculating compound interest efficiently. Fractional changes. Reverse percentages.	17
Graph Plotting	Quadratic graphs. Recognising and sketching graphs of the form $y = ax^2 + b$. Graphical method of solving equations of the form $x^2 + ax + b = 0$. **Other graphs.**	23



Rules of Indices

The Index Form

In year 8, we considered how to write a number as a product of its prime factors, in **index form.**
For example, we can write 72 as a product of its prime factors, in index form, like this.

$$72 = 2^3 \times 3^2.$$

The index form is a product of terms of the form n^a. Each of these terms include a **base** and an **index**.

The base. This shows which number is getting multiplied in the term. $\rightarrow n^a \leftarrow$ **The index**. This shows how many times the number n appears in the multiplication sum.

For example, we can write $7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7$ as 7^{10}.
The **base** is 7, since 7 is the number which is being multiplied. The **index** is 10, since 7 appears 10 times.

Other examples

$5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^7$

The number **5** is being multiplied. It appears **7** times.

$4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6$

The number **4** is being multiplied. It appears **6** times.

$2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$

The numbers **2** and **3** are being multiplied. **2** appears **3** times, and **3** appears **2** times.

Exercise 1

Skill I

Write the following in index form.

(a) $3 \times 3 \times 3 \times 3 \times 3$
(b) $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$
(c) $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$
(d) $3 \times 3 \times 3 \times 5 \times 5 \times 5 \times 5 \times 5$
(e) $3 \times 3 \times 3 \times 3 \times 3 \times 5 \times 5 \times 5$
(f) $3 \times 3 \times 3 \times 3 \times 3 \times 5$
(g) $2 \times 2 \times 7 \times 7 \times 7 \times 7 \times 9 \times 9 \times 9$
(h) $8 \times 8 \times 8 \times 33 \times 33 \times 33 \times 33$
(i) $3 \times 8 \times 8 \times 3 \times 8 \times 3 \times 3 \times 8$
(j) $3 \times 5 \times 5 \times 5 \times 3 \times 3 \times 7 \times 7$
(k) $2 \times 5 \times 7 \times 7 \times 2 \times 2 \times 2 \times 5$
(l) $13 \times 11 \times 7 \times 7 \times 11 \times 7$

Exercise 2

Write the following as multiplication sums without indices.

(a) 2^5
(b) 2^3
(c) 2^1
(d) 4^6
(e) 17^8
(f) 256^5
(g) $\left(\frac{1}{3}\right)^4$
(h) $2^4 \times 5^3$
(i) $4^4 \times 5^5$
(j) $24^3 \times 45^4$
(k) $\left(\frac{1}{5}\right)^3 \times \left(\frac{3}{4}\right)^3$
(l) $5^3 \times 13^2 \times 27^4$
(m) $3^2 \times 5^4 \times 10^2 \times 14^3$
(n) $2^3 \times \left(\frac{1}{2}\right)^3 \times 4^3$
(o) $\left(\frac{3}{7}\right)^2 \times \left(\frac{3}{4}\right)^4 \times \left(\frac{7}{9}\right)^3$

Investigation

Who was **Pierre de Fermat**? What was his contribution to mathematics?
What was his last theorem? When was this theorem proved?
Are there other theorems connected to index form?

Evaluating the Index Form

Evaluating index form is the process of writing a number written in index form as an ordinary number. For example, we can write 3^4 as $3 \times 3 \times 3 \times 3$, which is equal to 81.

Other examples

$$\begin{aligned} 4^3 &= 4 \times 4 \times 4 \\ &= 64 \end{aligned} \qquad \begin{aligned} 5^4 &= 5 \times 5 \times 5 \times 5 \\ &= 625 \end{aligned} \qquad \begin{aligned} 2^4 + 7^3 &= (2 \times 2 \times 2 \times 2) + (7 \times 7 \times 7) \\ &= 16 + 343 \\ &= 359 \end{aligned}$$

Exercise 3

I

Evaluate the following, without using a calculator.

(a) 3^4 (b) 6^3 (c) 10^5 (d) 2^9 (e) 20^4
(f) $3^2 + 2^5$ (g) $6^3 - 3^4$ (h) $6^3 \times 2^2$ (i) $10^4 \div 2^2$ (j) $5^4 + 4^4$

Exercise 4

Use your calculator to evaluate the following. If appropriate, write your answer correct to 2 decimal places.

(a) 125^2 (b) 17^4 (c) $29^3 + 5$ (d) $9^3 + 5$ (e) $12^4 - 5^6$
(f) $12^3 + 3^7$ (g) $3^4 \times 4^5$ (h) $2^3 \times 4^2 + 3^2$ (i) $(4^3)^4$ (j) $4^6 \div 2^6 + 10^3$
(k) $11^3 - 4^4$ (l) $4^5 - 5^6$ (m) $3^8 + 4^{10} - 5^6$ (n) $4^6 - 3^2 \times 8^3$ (o) $3^4 + 8^8 \div 4^{10}$
(p) $\frac{5^6}{3^7}$ (q) $\frac{4^4+3^6}{2^4}$ (r) $\frac{11^3}{2^5\times3^5}$ (s) $\frac{4^3+6^4\times2^3}{10^3-5^3}$ (t) $\left(\frac{4^3+6^4\times2^3}{10^3-5^3}\right)^3$

Exercise 5

Applying

(a) Without using a calculator, calculate the numbers which fill the following spaces.

(i) $2^{12} = 4096$, $2^{11} =$ ____ (ii) $4^6 = 4096$, $4^7 =$ ____ (iii) $3^{11} = 177147$, $3^{12} =$ ____ (iv) $5^5 = 3125$, $5^4 =$ ____ (v) $8^3 = 512$, $8^4 =$ ____

(vi) $2^8 = 256$, $2^{10} =$ ____ (vii) $6^4 = 1296$, $6^6 =$ ____ (viii) $3^7 = 2187$, $3^9 =$ ____ (ix) $5^9 = 1953125$, $5^7 =$ ____ (x) $7^4 = 2401$, $7^6 =$ ____

(b) Put the numbers 1 to 6 into the following boxes to make:

$$\square^{\square} + \square^{\square} + \square^{\square}$$

(i) The largest possible number;
(ii) The smallest possible number;
(iii) A total of 147.

(c) Put the numbers 1 to 6 into the following boxes to make the calculation correct.

$$\square^{\square} + \square^{\square} = \square\square$$

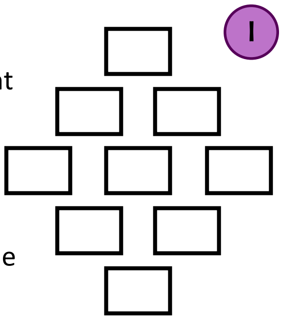
(d) Write any whole numbers in the following boxes to make the calculation correct.
How many possible solutions are there?

$$\square^{\square} = 64$$

Exercise 6

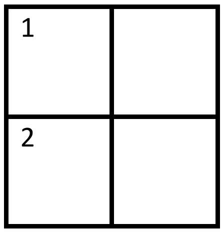
Write the digits 1 to 9 in the grid on the right so that each row (reading across) is a square number. You may use each digit once only.

Can you prove that there is only one possible solution?



Exercise 7

Complete the following crossnumber.



Clues:

Across
1. Cube number
2. Cube number

Down
1. One less than a cube number

Rules of Indices

When performing calculations in index form, we notice several different patterns.
The **rules of indices** write these patterns in a convenient way.

The Multiplication Rule

$$n^a \times n^b = n^{a+b}$$

When multiplying one variable or letter to an index, by the **same** number or variable to another index, we must **add** the indices. We can see below why this is true.

$$\begin{aligned} 8^4 \times 8^3 &= 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \\ &= 8^7. \end{aligned}$$

Other examples

$$\begin{aligned} 4^3 \times 4^6 &= 4^{3+6} \\ &= 4^9 \end{aligned}$$

$$\begin{aligned} 8^2 \times 8^9 &= 8^{2+9} \\ &= 8^{11} \end{aligned}$$

$$\begin{aligned} a^5 \times a^6 \times a^4 &= a^{5+6+4} \\ &= a^{15} \end{aligned}$$

$$\begin{aligned} 7^9 \times 7^{-3} &= 7^{9+-3} \\ &= 7^6 \end{aligned}$$

Exercise 8

Skill

Simplify each of the following expressions.

(a) $7^5 \times 7^3$ (b) $7^3 \times 7^5$ (c) $8^5 \times 8$ (d) $x^5 \times x^3$ (e) $7^4 \times 7^8$
(f) $a^5 \times a^3$ (g) $a^5 \times a^7 \times a^{10}$ (h) $3^2 \times 3^9 \times 3^4$ (i) $5^5 \times 5^2 \times 5^{12}$ (j) $y^3 \times y^{13} \times y^{16}$
(k) $7^{15} \times 7^{-4}$ (l) $14^9 \times 14^{-6}$ (m) $8^{-10} \times 8^3$ (n) $d^5 \times d^{-8}$ (o) $f^{-4} \times f^{-3}$
(p) $i^{-5} \times i^{11} \times i^{-3}$ (q) $p^{-9} \times p^{-2} \times p^5$ (r) $4^{-17} \times 4^{-7} \times 4^{31}$ (s) $(-5)^5 \times (-5)^3$ (t) $(-5)^5 \times (-5)^{-3}$
(u) $(-5)^{-5} \times (-5)^{-3}$ (v) $a^3 \times a^{\frac{1}{2}}$ (w) $a^{\frac{3}{5}} \times a^{\frac{1}{5}}$ (x) $a^{\frac{2}{3}} \times a^{\frac{4}{7}}$ (y) $a^{\frac{8}{3}} \times a^{\frac{5}{4}}$

Exercise 9

Find the missing numbers that go into the boxes in all of the following questions.

(a) $7^5 \times 7^{\square} = 7^8$ (b) $7^{\square} \times 7^4 = 7^6$ (c) $7^{13} \times 7^{\square} = 7^{11}$ (d) $7^8 \times 7^4 = 7^{\square}$ (e) $x^2 \times x^{\square} = x^{14}$
(f) $5^5 \times 5^{\square} = 5^6$ (g) $4^9 \times 4^{\square} = 4^8$ (h) $11^{\square} \times 11^{10} = 11^7$ (i) $2^{\square} \times 2^{-5} = 2^9$ (j) $8^{-2} \times 8^{\square} = 8^{-9}$

Challenge!

Extension

An **Armstrong Number** is a whole number where the sum of the digits, each raised to the number of digits, is equal to the original number.
For example, the number 371 is an Armstrong number since $3^3 + 7^3 + 1^3 = 371$.
1,634 is also an Armstrong number since $1^4 + 6^4 + 3^4 + 4^4 = 1{,}634$.
How many Armstrong numbers are there between 1 and 10,000?

The Division Rule

$$n^a \div n^b = n^{a-b} \quad \text{or} \quad \frac{n^a}{n^b} = n^{a-b}$$

When dividing one number or variable to an index, by the **same** number or variable to another index, we must **subtract** the indices. We can see below why this is true.

$$\begin{aligned}\frac{7^8}{7^5} &= \frac{7\times7\times7\times7\times7\times7\times7\times7}{7\times7\times7\times7\times7} \\ &= \frac{7\times7\times7\times\cancel{7\times7\times7\times7\times7}}{\cancel{7\times7\times7\times7\times7}} \\ &= 7\times7\times7 \\ &= 7^3.\end{aligned}$$

Other Examples

$4^6 \div 4^3 = 4^{6-3}$
$= 4^3$

$8^9 \div 8^2 = 8^{9-2}$
$= 8^7$

$\frac{a^7}{a^4} = a^{7-4}$
$= a^3$

$6^5 \div 6^{-2} = 6^{5--2}$
$= 6^7$

Exercise 10

Skill I

Simplify each of the following expressions.

(a) $7^5 \div 7^3$ (b) $7^3 \div 7^5$ (c) $7^{11} \div 7$ (d) $7^8 \div 7^4$ (e) $x^5 \div x^3$
(f) $a^5 \div a^3$ (g) $a^5 \div a^7$ (h) $\frac{3^9}{3^2}$ (i) $\frac{5^{15}}{5^{12}}$ (j) $\frac{y^3}{y^{16}}$
(k) $7^{15} \div 7^{-4}$ (l) $14^9 \div 14^{-6}$ (m) $8^{-10} \div 8^3$ (n) $d \div d^{-8}$ (o) $f^{-4} \div f^{-3}$
(p) $i^{-5} \div i^{-3}$ (q) $p^{-9} \div p^{-2} \div p^5$ (r) $4^{-17} \div 4^{-7} \div 4^{31}$ (s) $(-5)^5 \div (-5)^3$ (t) $(-5)^5 \div (-5)^{-3}$
(u) $(-5)^{-5} \div (-5)^{-3}$ (v) $a^3 \div a^{\frac{1}{2}}$ (w) $a^{\frac{3}{5}} \div a^{\frac{1}{5}}$ (x) $a^{\frac{2}{3}} \div a^{\frac{4}{7}}$ (y) $a^{\frac{8}{3}} \div a^{\frac{5}{4}}$

Exercise 11

Find the missing numbers that go into the boxes in all of the following questions.

(a) $7^5 \div 7^{\square} = 7^2$ (b) $7^{\square} \div 7^4 = 7^6$ (c) $7^{13} \div 7^{\square} = 7^{11}$ (d) $7^6 \div 7^4 = 7^{\square}$ (e) $x^{15} \div x^{\square} = x^7$
(f) $5^5 \div 5^{\square} = 5^7$ (g) $\frac{4^9}{4^{\square}} = 4^5$ (h) $\frac{11^{30}}{11^{\square}} = 11^{14}$ (i) $2^{\square} \div 2^{-5} = 2^{-2}$ (j) $8^{-2} \div 8^{\square} = 8^{-9}$

Example

$$\begin{aligned}3^7\times3^5\div3^2 &= 3^{7+5}\div3^2 \\ &= 3^{12}\div3^2 \\ &= 3^{12-2} \\ &= 3^{10}\end{aligned}$$

$$\begin{aligned}3^9\div3^2\times3^5 &= 3^{9-2}\times3^5 \\ &= 3^7\times3^5 \\ &= 3^{7+5} \\ &= 3^{12}\end{aligned}$$

$$\begin{aligned}\frac{y^7}{y^4}\times y^{10} &= y^{7-4}\times y^{10} \\ &= y^3\times y^{10} \\ &= y^{3+10} \\ &= y^{13}\end{aligned}$$

Exercise 12

Simplify each of the following expressions.

(a) $3^5\times3^6\div3^2$ (b) $6^8\times6^6\div6^7$ (c) $7^{11}\times7^5\div7^8$ (d) $5^8\times5^3\div5^4$ (e) $x^9\times x^3\div x^4$
(f) $a^5\times a^4\div a^8$ (g) $a^{10}\times a^{-5}\div a^7$ (h) $2^{13}\div2^3\times2^4$ (i) $5^7\div5^2\times5^4$ (j) $8^9\div8\times8^3$
(k) $5^8\times5^3\times5^7\div5^4$ (l) $4^9\div4^2\times4^3\div4^5$ (m) $8^{-4}\times8^3\times8^6$ (n) $d^5\div d^{-8}\times d^4$ (o) $u^{-3}\div u^{-3}\times u^{-3}$
(p) $\frac{10^6}{10^2}\times10^8$ (q) $\frac{7^9}{7^3}\times7^2$ (r) $19^4\times\frac{19^{12}}{19^3}$ (s) $4^9\div\frac{4^6}{4^2}$ (t) $e^8\times\frac{e^6}{e^3}$
(u) $\frac{4^5\times4^7}{4^3}$ (v) $\frac{15^7\times15}{15^2}$ (w) $\frac{6^{10}}{6^2\times6^5}$ (x) $\frac{r^6\times r^{-1}}{r^{-2}}$ (y) $\frac{q^{-3}}{q^4\times q^{-8}}\times q^3$

The Zeroth Index Rule

$$\boldsymbol{n^0 = 1}$$

Any number or variable raised to a zero index gives the answer **1**. Why? Let us consider the following sequences.

$7^3 = 7 \times 7 \times 7$
$\div 7$
$7^2 = 7 \times 7$
$\div 7$
$7^1 = 7$

$15^3 = 15 \times 15 \times 15$
$\div 15$
$15^2 = 15 \times 15$
$\div 15$
$15^1 = 15$

What will appear next in these sequences?

$7^1 = 7$
$\div 7$
$7^0 = 1$

$15^1 = 15$
$\div 15$
$15^0 = 1$

In both cases, the number raised to a zero index is equal to 1. This would work for any number, not just 7 or 15.

Exercise 13

I

Evaluate the following.

(a) 3^0 (b) 28^0 (c) 37648^0 (d) $19^0 \times 27^0$ (e) $19^0 + 27^0$
(f) x^0 (g) π^0 (h) $2^3 \times 2^0$ (i) $3^4 \div 3^0$ (j) $\frac{7^5}{7^5}$

Raising a Power to Another Index

$$\boldsymbol{(n^a)^b = n^{a \times b}}$$

If a power of the form n^a is raised to another index, then we need to **multiply** the indices.
We can see below why this is true.

$$\begin{aligned}(5^3)^4 &= \overbrace{5^3 \times 5^3 \times 5^3 \times 5^3}^{\text{4 times}} \\ &= 5^{3+3+3+3} \\ &= 5^{12} \\ &= 5^{3\times 4}.\end{aligned}$$

Exercise 14

Simplify the following, giving your answers in index form.

(a) $(5^2)^4$ (b) $(5^4)^2$ (c) $(7^2)^4$ (d) $(x^2)^4$ (e) $((-5)^2)^4$
(f) $(5^{-2})^4$ (g) $(5^2)^{-4}$ (h) $(5^{-2})^{-4}$ (i) $\left(\left(\frac{1}{5}\right)^2\right)^4$ (j) $(5^2)^0$
(k) $(6^3)^6$ (l) $(11^{25})^3$ (m) $(2^7)^8$ (n) $(43^5)^{-7}$ (o) $(10^{-3})^{-9}$
(p) $(9^0)^6$ (q) $(0.3^6)^9$ (r) $(11^{-9})^7$ (s) $((-3)^{-4})^{-3}$ (t) $(y^{14})^3$
(u) $(5^2)^4 \times 5^6$ (v) $(3^6)^2 \div 3^4$ (w) $(8^5)^2 \div 8^{10}$ (x) $6 \times (6^5)^3$ (y) $(4^{12})^5 \times 4^0$

Challenge!

In the computer world, 1 kilobyte is not equal to 1,000 bytes, but to 2^{10} bytes.
1 megabyte is 2^{10} kilobytes.
How many bytes are in one megabyte?
Write your answer in index form first, then use your calculator to calculate the answer.

The Negative Index Rule

$$n^{-a} = \frac{1}{n^a}$$

Any number or variable raised to a **negative** index can be written as a fraction whose numerator is **1**.
Why? Let us consider the following sequences.

$7^3 = 7 \times 7 \times 7$
$\div 7$
$7^2 = 7 \times 7$
$\div 7$
$7^1 = 7$
$\div 7$
$7^0 = 1$

$15^3 = 15 \times 15 \times 15$
$\div 15$
$15^2 = 15 \times 15$
$\div 15$
$15^1 = 15$
$\div 15$
$15^0 = 1$

What will appear next in these sequences?

$7^0 = 1$
$\div 7$
$7^{-1} = \frac{1}{7}$
$\div 7$
$7^{-2} = \frac{1}{7 \times 7}$

$15^0 = 1$
$\div 15$
$15^{-1} = \frac{1}{15}$
$\div 15$
$15^{-2} = \frac{1}{15 \times 15}$

Exercise 15

I

Complete the following table.

n	5	4	3	2	1	0	–1	–2	–3	–4	–5
2^n	32										

Exercise 16

Write the following as ordinary fractions, without using indices.

(a) 3^{-2} (b) 4^{-2} (c) 5^{-2} (d) 3^{-3} (e) 4^{-3}
(f) 3^{-4} (g) 8^{-2} (h) 7^{-1} (i) 10^{-4} (j) 11^{-2}
(k) $2^{-2} \times 3$ (l) $6^{-2} \div 2$ (m) $8^{-1} \times 4$ (n) $9^{-2} \times 2^{-1}$ (o) $2^{-1} + 2^{-3}$

The Reciprocal as a Negative Index

In year 9, we defined the **reciprocal** of a number as follows:

The reciprocal of a number is $\frac{1}{\text{the number}}$.

For example, the reciprocal of 4 is $\frac{1}{4}$ and the reciprocal of 15 is $\frac{1}{15}$. Because $n^{-1} = \frac{1}{n^1} = \frac{1}{n}$, we can now define the reciprocal of a number in an alternative way:

The reciprocal of a number n is n^{-1}.

Exercise 17

Prove that multiplying a number n by its reciprocal n^{-1} always gives an answer of 1.
(Clue: You will need to use the multiplication rule from page 5.)

Unitary Fraction Index Rule

$$n^{\frac{1}{a}} = \sqrt[a]{n}$$

Any number or variable raised to an index that is a **unitary fraction** of the form $\frac{1}{a}$ can be written as the ***a*–th root** of n. For example, if $a = 4$ then n raised to a quarter $\left(n^{\frac{1}{4}}\right)$ can be written as the fourth root of n $\left(\sqrt[4]{n}\right)$.
Why is this true? Consider the following application of the rule $(n^a)^b = n^{a\times b}$ from page 7.

$$\left(n^{\frac{1}{2}}\right)^2 = n^{\frac{1}{2}\times 2}$$
$$\left(n^{\frac{1}{2}}\right)^2 = n^1$$
$$\left(n^{\frac{1}{2}}\right)^2 = n$$

Because the square root of a number is a number that squares to give the original number, we have $n^{\frac{1}{2}} = \sqrt{n}$.

$$\left(n^{\frac{1}{3}}\right)^3 = n^{\frac{1}{3}\times 3}$$
$$\left(n^{\frac{1}{3}}\right)^3 = n^1$$
$$\left(n^{\frac{1}{3}}\right)^3 = n$$

Because the cube root of a number is a number that cubes to give the original number, we have $n^{\frac{1}{3}} = \sqrt[3]{n}$.

Example

$9^{\frac{1}{2}} = \sqrt{9}$
$= 3$

$216^{\frac{1}{3}} = \sqrt[3]{216}$
$= 6$

$625^{\frac{1}{4}} = \sqrt[4]{625}$
$= 5$

$5^4 = 5 \times 5 \times 5 \times 5$
$= 625$ therefore $\sqrt[4]{625} = 5$.

Exercise 18

I

Evaluate the following.

(a) $16^{\frac{1}{2}}$ (b) $25^{\frac{1}{2}}$ (c) $49^{\frac{1}{2}}$ (d) $8^{\frac{1}{3}}$ (e) $27^{\frac{1}{3}}$
(f) $16^{\frac{1}{4}}$ (g) $81^{\frac{1}{4}}$ (h) $64^{\frac{1}{2}}$ (i) $64^{\frac{1}{3}}$ (j) $64^{\frac{1}{6}}$
(k) $36^{\frac{1}{2}} \times 9^{\frac{1}{2}}$ (l) $125^{\frac{1}{3}} + 81^{\frac{1}{2}}$ (m) $100^{\frac{1}{2}} \div 4^{\frac{1}{2}}$ (n) $32^{\frac{1}{5}} - 1^{\frac{1}{3}}$ (o) $121^{\frac{1}{2}} \times 0^{\frac{1}{10}}$

Exercise 19 (Revision)

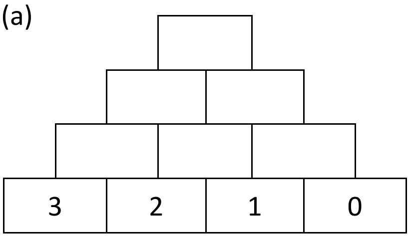
Simplify each of the following expressions.

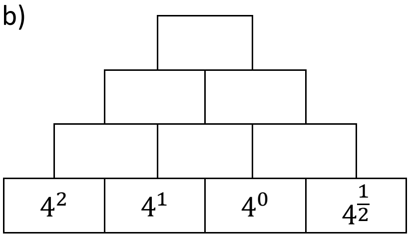
(a) $2^{10} \times 2^5$ (b) $3^5 \times 3$ (c) $15^6 \times 15^{-2}$ (d) $x^{-4} \times x^9$ (e) $4^{-3} \times 4^{-2}$
(f) $2^{10} \div 2^5$ (g) $3^5 \div 3$ (h) $15^6 \div 15^{-2}$ (i) $x^{-4} \div x^9$ (j) $4^{-3} \div 4^{-2}$
(k) 2^0 (l) 45^0 (m) $(2^{10})^5$ (n) $(15^6)^{-2}$ (o) $(4^{-3})^{-2}$
(p) 7^{-2} (q) 5^{-3} (r) $81^{\frac{1}{2}}$ (s) $343^{\frac{1}{3}}$ (t) $\frac{16^{\frac{1}{2}}}{2^{-2}}$

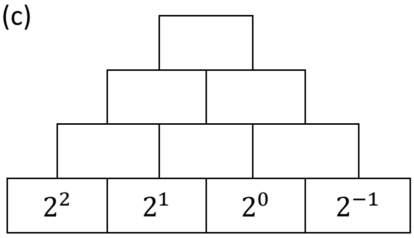
Exercise 20

Applying

Complete the following pyramids, where each number is the product of the two numbers in the boxes underneath.

(a)


(b)


(c)


Algebra and Rules of Indices

Multiply the numbers, add the indices.

Divide the numbers, subtract the indices.

Example

$4x^2y^3 \times 3x^4y^5 = 12x^6y^8$ $\qquad$ $20a^8b^6 \div 4a^2b^3 = 5a^6b^3$ $\qquad$ $\frac{42p^{12}q^{15}}{6p^3q^5} = 7p^9q^{10}$

Exercise 21

Skill

I

Simplify the following algebraic expressions.

(a) $2x^3y^4 \times 3x^4y^2$ (b) $8a^5b^3 \times 4a^3b^6$ (c) $9p^5q^3 \times 3p^4q$

(d) $16x^{10}y^{12} \div 2x^2y^4$ (e) $24a^6b^{15} \div 4a^2b^3$ (f) $80p^{32}q^{20} \div 10p^4q^{10}$

(g) $\frac{8x^{14}y^{10}}{2x^2y^2}$ (h) $\frac{28a^{16}b^4}{7a^4b}$ (i) $\frac{100p^4q^8}{25p^2q^4}$

(j) $4g^5h^3 \times -2g^5h^3$ (k) $6s^4t^6 \times 5s^{-2}t^3$ (l) $-3u^{-5}v^7 \times -9u^3v^{-2}$

(m) $25c^8d^{-12} \div 5c^2d^3$ (n) $\frac{-32e^{-4}f^{10}}{2ef^2}$ (o) $\frac{84x^5y^{-14}z}{2x^{-2}y^2z^{-2}}$

The General Fraction Index Rule

Higher Tier

$$\left(\sqrt[b]{n}\right)^a = n^{\frac{a}{b}} = \sqrt[b]{n^a}$$

There are two ways or writing any number or variable raised to a **general fraction** of the form $\frac{a}{b}$.

(1) Take the b–th root of n to begin with, and then raise everything to the index a. $\quad \left(\sqrt[b]{n}\right)^a$

(2) Raise n to the index a to begin with, and then take the b–th root of everything. $\quad \sqrt[b]{n^a}$

Example

Evaluate $27^{\frac{2}{3}}$.

Method (1)
$\sqrt[3]{27} = 3$
$3^2 = 9$

Method (2)
$27^2 = 729$
$\sqrt[3]{729} = 9$

Evaluate $32^{\frac{3}{5}}$.

Method (1)
$\sqrt[5]{32} = 2$
$2^3 = 8$

Method (2)
$32^3 = 32{,}768$
$\sqrt[5]{32{,}768} = 8$

Usually, method (1) is easier to calculate without a calculator.

Exercise 22

H

Evaluate the following.

(a) $8^{\frac{2}{3}}$ (b) $8^{\frac{4}{3}}$ (c) $125^{\frac{2}{3}}$ (d) $125^{\frac{4}{3}}$ (e) $81^{\frac{3}{4}}$

(f) $16^{\frac{3}{2}}$ (g) $32^{\frac{2}{5}}$ (h) $32^{\frac{4}{5}}$ (i) $49^{\frac{3}{2}}$ (j) $64^{\frac{2}{3}}$

(k) $16^{\frac{3}{4}}$ (l) $1024^{\frac{2}{5}}$ (m) $144^{\frac{3}{2}}$ (n) $3125^{\frac{2}{5}}$ (o) $1296^{\frac{3}{4}}$

(p) $625^{\frac{3}{4}}$ (q) $243^{\frac{2}{5}}$ (r) $36^{\frac{3}{2}}$ (s) $4^{\frac{5}{2}}$ (t) $729^{\frac{2}{3}}$

Challenge!

Extension

(a) Given that x is a number such that $x > 1$, put x, x^2, x^{-1} in ascending order.

(b) Given that x is a number such that $0 < x < 1$, put x, x^2, x^{-1} in ascending order.

(c) Given that x is a number such that $0 < x < 1$, put $x, x^0, x^{\frac{1}{2}}$ in ascending order.

(d) Given that x is a number such that $-1 < x < 0$, put $x, x^2, x^0, x^{-1}, x^{-3}$ in ascending order.

Combining the Rules

Example

$$4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\left(\sqrt[3]{8}\right)^2} = \frac{1}{2^2} = \frac{1}{4}$$

The negative index gives a unitary fraction

The fractional index gives a root

Evaluate the answer

Exercise 23

Skill H

Evaluate the following.

(a) $25^{-\frac{1}{2}}$ (b) $36^{-\frac{1}{2}}$ (c) $64^{-\frac{1}{2}}$ (d) $64^{-\frac{1}{3}}$ (e) $27^{-\frac{1}{3}}$

(f) $16^{-\frac{1}{4}}$ (g) $1024^{-\frac{1}{5}}$ (h) $144^{-\frac{1}{2}}$ (i) $125^{-\frac{1}{3}}$ (j) $81^{-\frac{1}{4}}$

(k) $27^{-\frac{2}{3}}$ (l) $4^{-\frac{3}{2}}$ (m) $216^{-\frac{2}{3}}$ (n) $81^{-\frac{3}{4}}$ (o) $32^{-\frac{3}{5}}$

(p) $16^{-\frac{3}{2}}$ (q) $256^{-\frac{3}{4}}$ (r) $25^{-\frac{3}{2}}$ (s) $243^{-\frac{3}{5}}$ (t) $8^{-\frac{4}{3}}$

Exercise 24

Evaluate the following.

(a) $\left(\frac{9}{16}\right)^{\frac{1}{2}}$ (b) $\left(\frac{27}{125}\right)^{\frac{1}{3}}$ (c) $\left(\frac{1}{36}\right)^{\frac{1}{2}}$ (d) $\left(\frac{1}{16}\right)^{\frac{1}{4}}$ (e) $\left(\frac{36}{49}\right)^{\frac{1}{2}}$

(f) $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$ (g) $\left(\frac{36}{121}\right)^{-\frac{1}{2}}$ (h) $\left(\frac{8}{27}\right)^{-\frac{1}{3}}$ (i) $\left(\frac{125}{343}\right)^{-\frac{1}{3}}$ (j) $\left(\frac{1}{81}\right)^{-\frac{1}{4}}$

(k) $\left(3\frac{1}{16}\right)^{\frac{1}{2}}$ (l) $\left(3\frac{3}{8}\right)^{\frac{1}{3}}$ (m) $\left(2\frac{1}{4}\right)^{-\frac{1}{2}}$ (n) $\left(2\frac{10}{27}\right)^{-\frac{1}{3}}$ (o) $\left(37\frac{1}{27}\right)^{-\frac{1}{3}}$

Challenge!

What is the answer to $\sqrt{9}$? One answer is 3, as $3^2 = 9$. But –3 is also an answer, as $(-3)^2 = 9$.
How many different answers do the following calculations have?

(a) $\sqrt{16}$ (b) $\sqrt[3]{27}$ (c) $\sqrt[4]{16}$ (d) $9^{\frac{1}{2}}$ (e) $3125^{\frac{1}{5}}$

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Standard Form

The standard form is a special way to write numbers, usually very large or very small numbers.

A number is written in standard form if it has the form

$$a \times 10^n,$$

where a is a number between 1 and 10 $(1 \leq a < 10)$ and n is an integer.

Example

Circle the numbers below that are written in standard form.

0.2 is not an integer.

3.2×10^5 $\quad$ 14.2×10^9 $\quad$ 7.2×10^{-6} $\quad$ $6.21 \times 10^{0.2}$

Not a number between 1 and 10.

Division, not multiplication.

0.34×10^{23} $\quad$ -3.2×10^{-5} $\quad$ 1×10^7 $\quad$ $8.7 \div 10^8$

Exercise 25

Skill I

Circle the numbers below that are written in standard form.

(a) 6.7×10^5 (b) $5.4 \div 10^9$ (c) 9×10^{-3} (d) 14.3×10^{12}

(e) 0.38×10^6 (f) 9.3×10^0 (g) -3.2×10^6 (h) 10×10^4

(i) 4.5×5^5 (j) 3×10^{189} (k) $6 \times 10^{1.4}$ (l) 5.6721×10^{-9}

Writing Numbers $x \geq 1$ in Standard Form

Given a number that is greater than or equal to one, this is how we write it in standard form.

a) Add a **decimal point** to the number, if there isn't already one present.

For example, the number 320 would change to be 320.0, and the number 73,000 would change to be 73,000.0.

b) Consider how many times we must **divide the number by 10** in order to reach a number a that is between 1 and 10 $(1 \leq a < 10)$. We do this by counting how many times we "jump" the decimal point to the left.

4 3 2 1

73,000.0

c) Use the number between 1 and 10 and the number of times we divided by 10 in order to write the original number in standard form.

73,000

$= 7.3 \times 10^4$

Exercise 26

Write the following numbers in standard form.

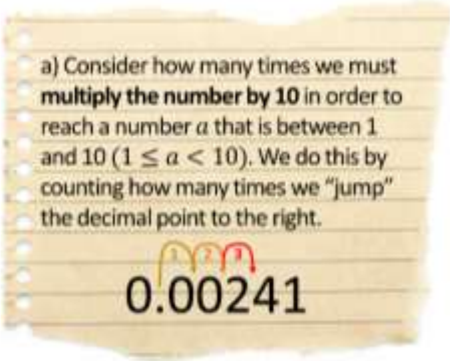
(a) 54,000 (b) 234,000 (c) 8,000 (d) 3,000,000

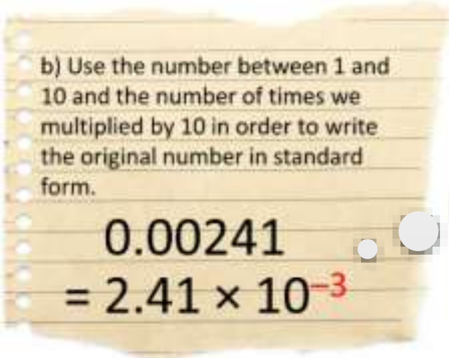
(e) 340 (f) 43,000,000 (g) 4,328,000,000 (h) 7

(i) 98,000,000,000 (j) 823,240,000,000 (k) 10 (l) 1

Writing Numbers $0 < x < 1$ in Standard Form

Given a number between 0 and 1, this is how we write it in standard form.





Remember that $10^{-3} = \frac{1}{10^3}$, therefore multiplying by 10^{-3} is the same as dividing by 10^3.

Exercise 27

I

Write the following numbers in standard form.

(a) 0.00428	(b) 0.000027	(c) 0.021	(d) 0.87
(e) 0.00000689	(f) 0.4	(g) 0.0009873	(h) 0.0901
(i) 0.00000000728	(j) 0.000000429	(k) 0.0000502	(l) 0.999999999

Exercise 28

Write the following numbers in standard form.

(a) 84,200	(b) 0.000647	(c) 5,000,000	(d) 0.005183
(e) 502,050	(f) 0.0000004	(g) 0.98	(h) 852,000,000,000
(i) 0.000201	(j) 2,384,900,000	(k) 1.03	(l) 0.03

Changing from Standard Form to an Ordinary Number

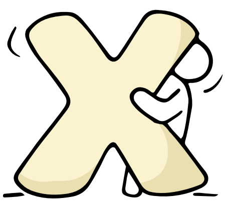
Example

(a) Write 6.962×10^6 as an ordinary number.

We must multiply 6.962 by 10 six times.

69.62	1 time
696.2	2 times
6962	3 times
69620	4 times
696200	5 times
6962000	6 times

The answer is 6,962,000.

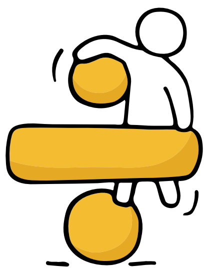


(b) Write 9.62×10^{-5} as an ordinary number.

We must divide 9.62 by 10 five times.

0.962	1 time
0.0962	2 times
0.00962	3 times
0.000962	4 times
0.0000962	5 times

The answer is 0.0000962.



Exercise 29

Write the following numbers, which are in standard form, as ordinary numbers.

(a) 8.243×10^6	(b) 4.2×10^4	(c) 8×10^5	(d) 3.704×10^8
(e) 6.25×10^{-5}	(f) 1.75×10^{-2}	(g) 8.02×10^{-3}	(h) 6.2829×10^{-7}
(i) 7×10^{-2}	(j) 9.2×10^1	(k) 3.504×10^{-1}	(l) 8.6284×10^{-6}
(m) 4×10^0	(n) 5.289×10^8	(o) 8.2×10^{-9}	(p) 8.28465×10^{10}

Adding and Subtracting Numbers in Standard Form

Change to ordinary numbers; calculate; change back to standard form.

Example

Calculate the following, giving your answer in standard form.

(a) $(3.4 \times 10^5) + (7.18 \times 10^4)$

	3	4	0	0	0	0
+		7	1	8	0	0
	4	1	1	8	0	0
	1					

Answer: 4.118×10^5

(b) $(7.36 \times 10^{-3}) - (1.9 \times 10^{-4})$

	0.	0	0	7	${}^{2}\cancel{3}$	${}^{1}6$
−	0.	0	0	0	1	9
	0.	0	0	7	1	7

Answer: 7.17×10^{-3}

Exercise 30

I

Calculate the following, giving your answers in standard form.

(a) $(2.7 \times 10^3) + (5.26 \times 10^2)$
(b) $(6.152 \times 10^5) + (7.64 \times 10^4)$
(c) $(2.09 \times 10^4) + (4 \times 10^3)$
(d) $(6.29 \times 10^6) + (3.283 \times 10^5)$
(e) $(5 \times 10^4) + (8.024 \times 10^6)$
(f) $(4.2 \times 10^7) + (1.59 \times 10^8)$

(g) $(2.7 \times 10^3) - (5.26 \times 10^2)$
(h) $(6.152 \times 10^5) - (7.64 \times 10^4)$
(i) $(2.09 \times 10^4) - (4 \times 10^3)$
(j) $(8 \times 10^6) - (4.6 \times 10^3)$
(k) $(2.07 \times 10^4) - (9.442 \times 10^3)$
(l) $(1.4 \times 10^2) - (4.6 \times 10^1)$

(m) $(2.7 \times 10^{-3}) + (5.26 \times 10^{-2})$
(n) $(6.152 \times 10^{-5}) + (7.64 \times 10^{-4})$
(o) $(2.09 \times 10^{-4}) + (4 \times 10^{-3})$
(p) $(6.4 \times 10^{-1}) + (7.28 \times 10^{-2})$
(q) $(8 \times 10^{-4}) + (7.4 \times 10^{-3})$
(r) $(1.02 \times 10^{-7}) + (7.32 \times 10^{-6})$

(s) $(5.26 \times 10^{-2}) - (2.7 \times 10^{-3})$
(t) $(7.64 \times 10^{-4}) - (6.152 \times 10^{-5})$
(u) $(4 \times 10^{-3}) - (2.09 \times 10^{-4})$
(v) $(6.43 \times 10^{-4}) - (3.82 \times 10^{-5})$
(w) $(4.6 \times 10^{-7}) - (6 \times 10^{-10})$
(x) $(3.814 \times 10^2) - (4.76 \times 10^{-2})$

Almost in Standard Form

In order to multiply and divide numbers given in standard form, we must first learn how to change numbers that are almost in standard form to be in standard form.

Example

Change the following numbers, that are almost in standard form, to be in standard form.

(a) 45×10^7 $\xrightarrow{\div 10}$ / $\xrightarrow{+1}$ $= 4.5 \times 10^8$

(b) 0.4×10^3 $\xrightarrow{\times 10}$ / $\xrightarrow{-1}$ $= 4 \times 10^2$

(c) 68×10^{-4} $\xrightarrow{\div 10}$ / $\xrightarrow{+1}$ $= 6.8 \times 10^{-3}$

(d) 0.064×10^{-7} $\xrightarrow{\times 100}$ / $\xrightarrow{-2}$ $= 6.4 \times 10^{-9}$

Divide the number / move decimal point to the left: increase the index.
Multiply the number / move decimal point to the right: decrease the index.

Exercise 31

Change the following numbers, that are almost in standard form, to be in standard form.

(a) 61×10^7
(b) 532×10^7
(c) 0.61×10^7
(d) 0.54×10^7
(e) 61×10^{-7}
(f) 532×10^{-7}
(g) 0.61×10^{-7}
(h) 0.54×10^{-7}
(i) 83×10^9
(j) 0.325×10^{14}
(k) 7324×10^{-5}
(l) 53×10^{-14}
(m) 0.025×10^8
(n) 0.0024×10^{-16}
(o) 10×10^5
(p) 0.63×10^{-43}

Multiplying Numbers in Standard Form

Example

Calculate $(2.5 \times 10^5) \times (6 \times 10^3)$, giving your answer in standard form.

Answer: $(2.5 \times 10^5) \times (6 \times 10^3)$

$= (2.5 \times 6) \times (10^5 \times 10^3)$	Rearrange (the order in multiplication sums doesn't matter).
$= 15 \times 10^{5+3}$	Multiply the numbers; use rules of indices to add the indices.
$= 15 \times 10^8$	This is almost in standard form; we must divide the 15 by 10 to correct...
$= 1.5 \times 10^9$	Final answer (divide by 10 so add 1 to the index).

Exercise 32

I

Calculate the following, giving your answer in standard form.

(a) $(2 \times 10^5) \times (4 \times 10^3)$
(b) $(2 \times 10^5) \times (8 \times 10^3)$
(c) $(2 \times 10^5) \times (4 \times 10^{-3})$
(d) $(1.3 \times 10^6) \times (2 \times 10^8)$
(e) $(4 \times 10^9) \times (3 \times 10^4)$
(f) $(7 \times 10^{14}) \times (6 \times 10^2)$
(g) $(4.6 \times 10^7) \times (3 \times 10^4)$
(h) $(7.5 \times 10^{14}) \times (8 \times 10^{23})$
(i) $(7 \times 10^7) \times (3.8 \times 10^9)$
(j) $(6 \times 10^{-4}) \times (6 \times 10^{14})$
(k) $(3 \times 10^6) \times (2 \times 10^{-2})$
(l) $(1 \times 10^{-4}) \times (8 \times 10^{-3})$
(m) $(2.4 \times 10^4) \times (1.5 \times 10^7)$
(n) $(5.3 \times 10^{14}) \times (6.2 \times 10^3)$
(o) $(5.13 \times 10^{-6}) \times (7.4 \times 10^2)$

Dividing Numbers in Standard Form

Example

Calculate $(4 \times 10^8) \div (5 \times 10^2)$, giving your answer in standard form.

Answer: $(4 \times 10^8) \div (5 \times 10^2)$

$= (4 \div 5) \times (10^8 \div 10^2)$	Rearrange
$= 0.8 \times 10^{8-2}$	Divide the numbers; use rules of indices to subtract the indices.
$= 0.8 \times 10^6$	This is almost in standard form; we must multiply the 0.8 by 10 to correct...
$= 8 \times 10^5$	Final answer (multiply by 10 so subtract 1 from the index).

Exercise 33

Calculate the following, giving your answer in standard form.

(a) $(8 \times 10^8) \div (4 \times 10^2)$
(b) $(4 \times 10^8) \div (8 \times 10^2)$
(c) $(8 \times 10^8) \div (4 \times 10^{-2})$
(d) $(3.6 \times 10^6) \div (3 \times 10^3)$
(e) $(6.4 \times 10^{12}) \div (4 \times 10^3)$
(f) $(9.3 \times 10^5) \div (3 \times 10^5)$
(g) $(8.6 \times 10^7) \div (2 \times 10^2)$
(h) $(7.5 \times 10^{14}) \div (5 \times 10^{20})$
(i) $(1 \times 10^8) \div (3 \times 10^4)$
(j) $(4.2 \times 10^{-3}) \div (3 \times 10^4)$
(k) $(2 \times 10^5) \div (5 \times 10^{-2})$
(l) $(1 \times 10^{-4}) \div (8 \times 10^{-2})$
(m) $(2.4 \times 10^8) \div (4 \times 10^3)$
(n) $(5.25 \times 10^{50}) \div (1.5 \times 10^{10})$
(o) $(2 \times 10^{-5}) \div (8 \times 10^{-15})$

Exercise 34

Calculate the following, giving your answer in standard form.

(a) $(6 \times 10^4) + (4 \times 10^3)$
(b) $(6 \times 10^4) - (4 \times 10^3)$
(c) $(6 \times 10^4) \times (4 \times 10^3)$
(d) $(6 \times 10^4) \div (4 \times 10^3)$
(e) $\frac{6 \times 10^4}{4 \times 10^3}$
(f) $\frac{(6 \times 10^4) + (4 \times 10^3)}{4 \times 10^3}$
(g) $(8.4 \times 10^6) + (2 \times 10^2)$
(h) $(8.4 \times 10^6) - (2 \times 10^2)$
(i) $(8.4 \times 10^6) \times (2 \times 10^2)$
(j) $(8.4 \times 10^6) \div (2 \times 10^2)$
(k) $(8.4 \times 10^6) \times 5$
(l) $(8.4 \times 10^6) + (2 \times 10^{-2})$

Challenge!

Use your calculator to check your answers to Exercise 34, making sure that your calculator shows the answer in standard form.

Applying

I

Exercise 35

The Earth is more or less spherical.

(a) The radius of the Earth is 6,378.1 km. Calculate the circumference of the Earth, writing your answer in standard form correct to 3 significant figures.

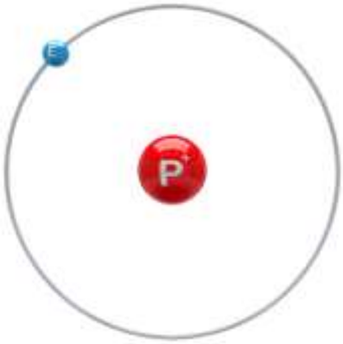
(b) The surface area of the whole Earth is approximately 5.112×10^{8} km². The oceans cover approximately 3.618×10^{8} km² of surface area and the rest of the surface area is covered by land. Calculate the amount of surface area of the Earth covered by land, giving your answer in standard form.

Exercise 36

The Millennium Stadium in Cardiff has enough seats for 74,500 people. The population of Wales would fill the Millennium Stadium 41 times.

Use this information to calculate the approximate population of Wales. Give your answer in standard form correct to 3 significant figures.

Exercise 37



The mass of one hydrogen atom is about

$$1.66 \times 10^{-24} \text{ kg.}$$

One litre of air contains approximately 2.51×10^{22} hydrogen atoms.

(a) What is the mass of hydrogen in one litre of air?
Give your answer in standard form.

(b) Express your answer to (a) without using standard form.

Challenge!

Investigate using the internet the meaning of the word googol.
Use your findings to write the number 50 googol in standard form.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ **Target** ☐

Efficient Percentage Changes

Consider how you would answer the following question.

Chris' weekly wage is £480. If Chris is awarded a pay rise of 10%, what will his wage be next week?

One way to answer this question would be to calculate 10% of £480, which is £480 ÷ 10 = £48, and then add the £48 onto the original wage: £480 + £48 = £528. On a calculator, it would be possible to type the following two calculations to find the correct answer.

$$480 \times 10\% = 48$$
$$480 + 48 = 528$$

It is possible however to reach the correct answer in **one** calculation, and therefore in a **more efficient** way. Since Chris' wage is increasing 10%, it is growing from 100% of the original wage to 110% of the original wage. It would therefore be possible to find the new wage by finding 110% of the original wage. On a calculator, we can do this by typing the following calculation.

$$480 \times 110\% = 528$$

Exercise 38

Skill I

Complete the following table.

Original Price	Original Percentage	Percentage Change	New Percentage	Sum to Find the New Price	New Price
£480	100%	Increase 10%	110%	£480 × 110%	£528
£54	100%	Decrease 20%	80%	£54 × 80%	£43.20
£250		Increase 25%			
£1,700		Increase 5%			
£20		Decrease 10%			
$800		Decrease 30%			
€460		Increase 2%			
£7.50		Decrease 30%			
£2,500		Increase 80%			
£100,000		Decrease 64%			
$650		Increase 16%			
£82.50		Decrease 26%			
£1.87		Increase 100%			
£74.50		Decrease 100%			
€2,925		Increase 250%			
£74,000		Decrease 99%			
£5.68		Increase 2.5%			
£100		Decrease 12.5%			
£276.40		Increase 0.4%			
€9.99		Decrease 60%			
£2,000,000		Decrease 1.4%			

Repeated Percentage Changes

Example

Depreciation means the value **decreases**.

Geraint buys a second hand car for **£6,500**.
Every year, the value of the car **depreciates 15%.**
What value will the car have after **three years?**

Inefficient Method

Year 1
£6,500 × 15% = £975
£6,500 – £975 = £5,525
Year 2
£5,525 × 15% = £828.75
£5,525 – £828.75 = £4,696.25
Year 3
£4,696.25 × 15% = £704.44, to the nearest penny.
£4,696.25 – £704.44 = £3,991.81

Better Method

Every year, the value of the car decreases from 100% to 85%.
Year 1
£6,500 × 85% = £5,525
Year 2
£5,525 × 85% = £4,696.25
Year 3
£4,696.25 × 85% = £3,991.81, to the nearest penny.

Efficient Method

Every year, the value of the car decreases from 100% to 85%.
This happens three times.
Answer
£6,500 × 85%^{3} = £3,991.81, to the nearest penny.

Exercise 39

Applying

I

Use an efficient method to answer the following questions.

(a) Megan buys a second-hand car for £8,000. Every year, the value of the car depreciates 12%. What value will the car have after three years?

(b) Aled buys a second-hand car for £14,500. Every year, the value of the car depreciates 20%. What value will the car have after five years?

(c) Ffion buys a pair of earrings for £400. Every year, the value of the earrings increases 15%. What value will the earrings have after four years?

(d) Steffan buys antique furniture for £1,500. Every year, the value of the furniture increases 3%. What value will the furniture have after 6 years?

(e) A football team buys a player for £75,000,000. The value of the player increases 24% every year. The club sells the player after three years. What was the sale price of the player after three years?

(f) Steven buys a field for £7,500. For every one of the next 5 years the price increases 10%.
(i) What value will the field have after two years?
(ii) How much does the price increase between the end of the third year and the end of the fifth year?

(g) A tropical rainforest loses 7% of its trees every year.
(i) What percentage of the rainforest will be left after a year?
(ii) What percentage of the rainforest will be left after two years?
(iii) After how many years will there be less than 50% of the current rainforest left?
(iv) How much of the rainforest will be left after 50 years?

(h) Five years ago, Mark bought a house for £180,000. For the first three years, the value of the house increased by 12% each year. For the next two years, the value of the house decreased by 7% each year. What is the value of the house today?

Calculating Compound Interest Efficiently

Example

Nia borrows **£8,500** from NatWest Bank at a compound interest rate of **4% per year**. Nia wants to pay back all the money after **three years.** How much money will Nia have to pay back after three years?

Inefficient method (Year 9)

Year 1
£8,500 × 4% = £340
£8,500 + £340 = £8,840
Year 2
£8,840 × 4% = £353.60
£8,840 + £353.60 = £9,193.60
Year 3
£9,193.60 × 4% = £367.74,
to the nearest penny.
£9,193.60 + £367.74 = £9,561.34

Better Method

Every year, the value of the money increases from 100% to 104%.
Year 1
£8,500 × 104% = £8,840
Year 2
£8,840 × 104% = £9,193.60
Year 3
£9,193.60 × 104% = £9,561.34,
to the nearest penny.

Efficient Method

Every year, the value of the money increases from 100% to 104%.
This happens three times.
Answer
£8,500 × 104%^{3} = £9,561.34,
to the nearest penny.

Exercise 40

I

Use an efficient method to calculate the answer to the following questions.

(a) Sophie borrows £11,000 from Barclays Bank at a compound interest rate of 3% per year. Sophie wants to pay back all the money after three years. How much money will Sophie have to pay back after three years?

(b) Bryn borrows £6,700 from HSBC Bank at a compound interest rate of 6% per year. Bryn wants to pay back all the money after five years. How much money will Bryn have to pay back after five years?

(c) Owen wants to invest £4,000 into Lloyds Bank at a compound interest rate of 5% per year. Owen wants to withdraw all the money from the bank after three years. How much money will Owen be able to take out after three years?

(d) Lorraine wants to invest £25,000 into Barclays Bank at a compound interest rate of 3.4% per year. Lorraine wants to withdraw all the money from the bank after eight years. How much money will Lorraine be able to take out after eight years?

Exercise 41

The following table shows the compound interest rates for different sums of money invested into a bank over a period of time. Complete the table.

	Sum to invest	Compound interest rate per year	Time period for the investment	Calculation to calculate the sum of money at the end	Sum of money at the end	Compound interest earnt
	£5,400	4%	3 years	£5,400 × 104%^{3}	£6,074.27	£674.27
(a)	£2,800	3%	5 years			
(b)	£19,000	2%	4 years			
(c)	£150,000	5%	2 years			
(d)	£8,500	7%	3 years			
(e)	£24,300	2.5%	4 years			
(f)	£100,000	4.8%	5 years			
(g)	£10,000	7.3%	15 years			

Fractional Changes

Revision F

Exercise 42

Calculate the following. (It is not necessary to convert any improper fractions to mixed numbers.)

(a) $1 - \frac{1}{3}$ (b) $1 - \frac{1}{7}$ (c) $1 - \frac{2}{5}$ (d) $1 - \frac{3}{10}$

(e) $1 + \frac{1}{3}$ (f) $1 + \frac{1}{7}$ (g) $1 + \frac{2}{5}$ (h) $1 + \frac{3}{10}$

(i) $1 - \frac{1}{45}$ (j) $1 + \frac{3}{45}$ (k) $1 - \frac{14}{25}$ (l) $1 + \frac{3}{50}$

Example

A radio station measures how many listeners are listening at least once per week to their morning show between 7am and 10am. Over the past two months, the number of listeners has decreased $\frac{1}{8}$ per month. If the number of listeners was 32,000 two months ago, what is the number of listeners today?

Inefficient method

Month 1

$32{,}000 \times \frac{1}{8} = 4{,}000$

$32{,}000 - 4{,}000 = 28{,}000$

Month 2

$28{,}000 \times \frac{1}{8} = 3{,}500$

$28{,}000 - 3{,}500 = 24{,}500$

Better Method

Instead of calculating $\frac{1}{8}$ of the number and then taking away, we can multiply by $1 - \frac{1}{8} = \frac{7}{8}$.

Month 1

$32{,}000 \times \frac{7}{8} = 28{,}000$

Month 2

$28{,}000 \times \frac{7}{8} = 24{,}500$

Efficient Method

We can multiply by $\frac{7}{8}$ twice.

$32{,}000 \times \left(\frac{7}{8}\right)^2 = 24{,}500$

Exercise 43

Applying I

Use an efficient method to answer the following questions.

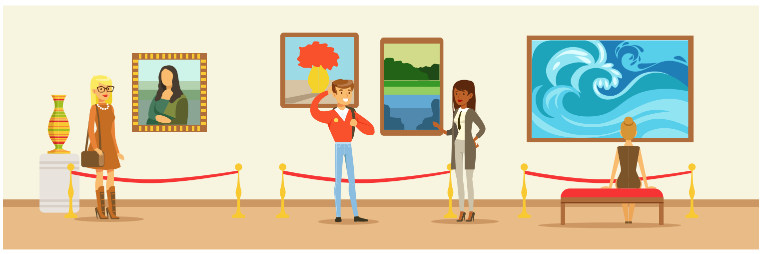
(a) Monthly sales for a magazine have dropped $\frac{1}{10}$ every month over the past three months. If the monthly sales were 60,000 three months ago, what are the monthly sales today?

(b) Boris decides to join a weight loss club. Boris accepts a target to lose $\frac{1}{80}$ of his weight every month, over a period of six months. If Boris weighs 90 kg today, how much weight does Boris need to lose over the next 6 months to reach his target? Write your answer to two decimal places.

(c) At the beginning of last year, 150,000 people visited a website every day. Over the year, the website saw a monthly increase of $\frac{1}{20}$ in their daily visitor numbers. How many people visited the website daily at the end of last year?

(d) A business saw a yearly increase of $\frac{2}{5}$ in product sales over a period of four years. If the yearly sales were £3,400,000 at the beginning of the period, what were the yearly sales at the end of the period?

(e) The number of people who visit a specific gallery in a month has reduced by $\frac{1}{200}$ every month over a period of 8 months. If 40,000 people visited the gallery eight months ago, how many people visited the gallery during the past month?



Reverse Percentages

There was a “10% off” sale in a clothes shop. William went into the shop and bought a t-shirt for £18.
If the sale *wasn’t* on, how much would William have paid for the t-shirt?

Exercise 44 I

Clumsy Clive attempts to answer the question above on the note paper on the right. Explain why Clive is wrong.

10% of £18 is
£18 ÷ 10 = £1.80.
£18 + £1.80 = £19.80.
William paid £19.80 for the t-shirt. ✗

The correct answer

We must consider what happens to the **percentage** to find the correct answer. The price in the sale is 90% of the original price, since the sale subtracts 10% from the original price.

Method 1 (with a calculator)

Original Price	× 90%	New Price
100%	→	90%
?	←	£18
	÷ 90%	

Answer: £18 ÷ 90% = £20. ✓

Method 2 (no calculator)

90% of the original price is £18.
↓ ÷ 9 ↓ ÷ 9
10% of the original price is £2.
↓ × 10 ↓ × 10
100% of the original price is £20. ✓

Exercise 45

Answer the following questions using a calculator.

(a) There was a “10% off” sale in a book shop. Dafydd went into the shop and bought a book for £14.40. How much would Dafydd have paid for the book if the sale was not on?

(b) In a sale there was a discount of 5% off every item. The price of a laptop in the sale was £570. What was the price of the laptop before the sale?

(c) There was a “20% off” sale in a furniture shop. Heledd went into the shop and bought a set of chairs for £320. How much would Heledd have paid if the sale was not on?

Exercise 46

Answer the following questions without using a calculator.

(a) There was a “10% off” sale in a clothes shop. Siwan went into the shop and bought a skirt for £27. How much would Siwan have paid for the skirt if the sale was not on?

(b) In a sale there was a discount of 20% off all items. A set of dishes were on sale for £56. What was the price of the dishes before the sale?

(c) There was a “50% off” closing down sale in a clothes shop. Simon bought a shirt in the sale for £30. What was the price of the shirt before the sale?

Exercise 47

Complete the following table.

	Original Price	Percentage Increase	New Price
(a)	£70	15%	
(b)		12%	£67.20
(c)	£250	3%	
(d)		2.5%	$615
(e)	£900	150%	
(f)		0.4%	€251

Exercise 48

Complete the following table.

	Original Price	Percentage Decrease	New Price
(a)	£30	35%	
(b)		18%	£69.70
(c)	$200	9%	
(d)		98%	£40
(e)	€985	3.5%	
(f)		0.25%	£47.88

Exercise 49

(a) Abigail invested in a bond which was increasing at a rate of 8% per year. After one year the value of Abigail's investment was £972. How much was Abigail's original investment?

(b) The cost of a holiday was £600 including VAT at a rate of 20%. What was the price before VAT?

(c) Emily had a pay rise of 6%. Her salary after the pay rise was £25,970. What was her salary before the pay rise?

Exercise 50

(a) Over the last four years, the value of Mr. Davies' car has dropped 10% per year. If Mr. Davies' car is worth £8,000 today, what was it worth four years ago?

(b) Over the last nine years, the value of Mrs. Jones' house has increased 4% per year. If Mrs. Jones' house is worth £140,000 today, how much was her house worth 9 years ago?

Challenge!

There are 36 more girls than boys in a school. 54% of the pupils in the school are girls. How many girls are in the school?

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ❑ Target ❑

Plotting Graphs

In year 9, you learnt how to plot linear equations of the form $y = mx + c$. In this chapter, we will learn how to plot equations that have different forms.

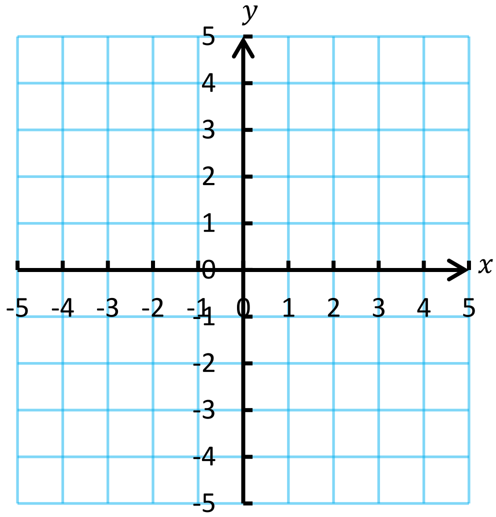
Exercise 51

Revision

F

On the graph paper on the right, plot the equation $y = 2x - 3$.

x	0	1	2	3
y				



Quadratic Graphs

Equations of the form $y = ax^2 + bx + c$ are **quadratic** equations.

The word quadratic refers to the x^2 term in the equation, a **squared** term. Quadratic graphs have a U or ∩ shape.

Exercise 52

Skill

I

Complete the following table to note whether each equation is linear or quadratic.

Equation	Type	Equation	Type
$y = 4x - 2$	Linear	$y = 3x^2 - 4x + 2$	Quadratic
$y = 2x^2 + 4x + 7$		$y = 5x^2 - 4x + 6$	
$y = 2x + 5$		$y = -4x + 2$	
$y = 4x + 7x^2 - 3$		$y = 2 + 6x + 3x^2$	
$y = 3 + 4x$		$y = 6x$	
$y = 5$		$y = 4 + 3x^2$	
$y = -3x^2 + 3$		$y = x(x + 2)$	

Substitution

To plot a quadratic graph, we can **substitute** values into the associated equation.

Exercise 53

Complete the following table by substituting in the whole numbers between –5 and 5. (No calculator allowed.)

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
x^2											
$x^2 - 5$											
x^2											
$2x$											
$x^2 + 2x - 8$											
x^2											
$2x^2$											
$-x$											
$2x^2 - x - 28$											

Exercise 54

Check your answers to Exercise 53 by using the *Table Mode* on your calculator.

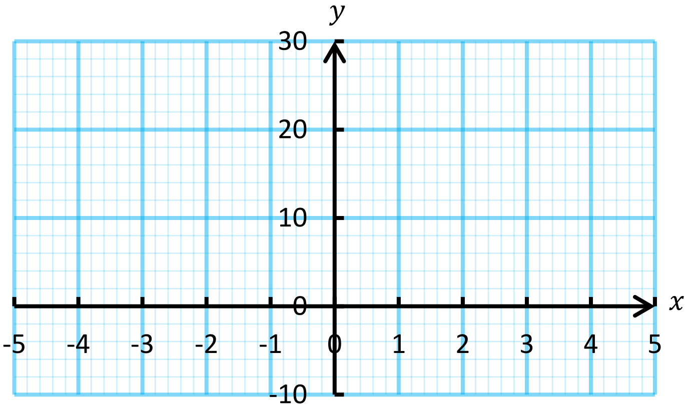
Exercise 55

I

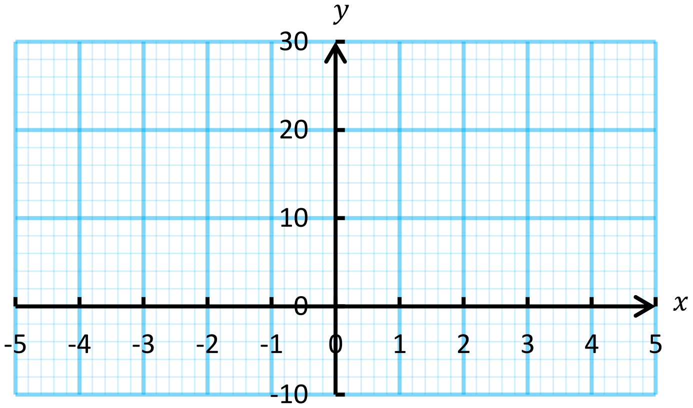
Use the table from Exercise 53 to plot the following equations on the graph paper below.

(a) $y = x^2$ (b) $y = x^2 - 5$ (c) $y = x^2 + 2x - 8$ (d) $y = 2x^2 - x - 28$

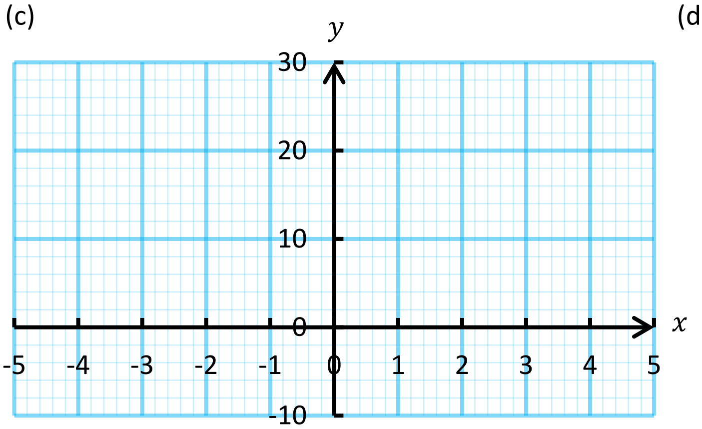
(a)



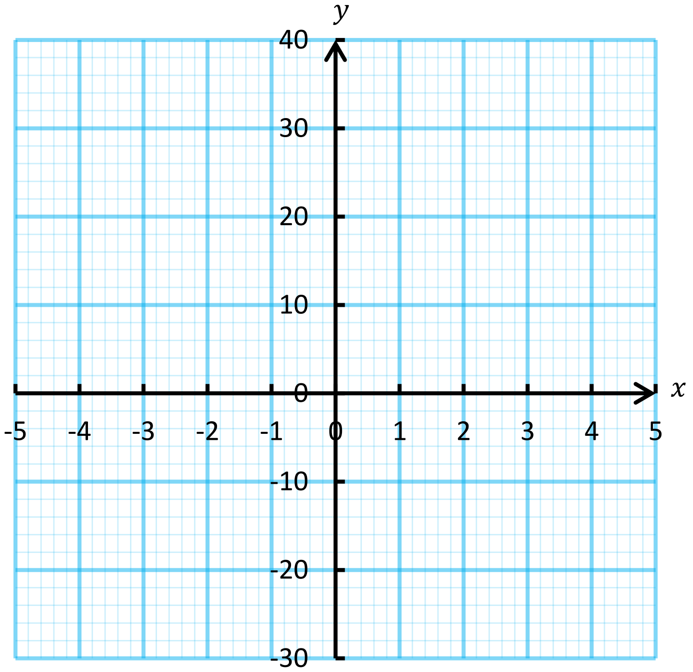
(b)



(c)



(d)



Exercise 56

Fill the blanks in the tables below. Then, in your books, plot suitable graphs for the equations.

(a)

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y = x^2 + 4$	29	20		8	5	4	5	8	13		29

(b)

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y = 3x^2 - 10$	65	38	17		-7	-10	-7		17	38	65

(c)

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y = 4x^2 + x - 7$	88		26	7	-4	-7	-2	11		61	98

Exercise 57

I

Plot appropriate graphs for the following equations.

(a) $y = x^2 - 4x$ (b) $y = x^2 - 3x + 4$ (c) $y = 3x - x^2$

(d) $y = x^2 - x - 5$ (e) $y = 5 - 2x^2$ (f) $y = 3x^2 + 4x + 2$

(g) $y = 15 - x^2 + 3x$ (h) $y = 4x^2 - x + 7$ (i) $y = -2x^2 + 5x - 6$

Recognising and sketching graphs of the form $y = ax^2 + b$

Exercise 58

You will need to use the website www.desmos.com/calculator to complete this exercise.

Type $y = ax^2 + b$ into the box. When the "add slider" option appears click "all".

(a) Which values for a and b does the computer set?

(b) What happens as you change the value of a?

(c) What stays the same as you change the value of a?

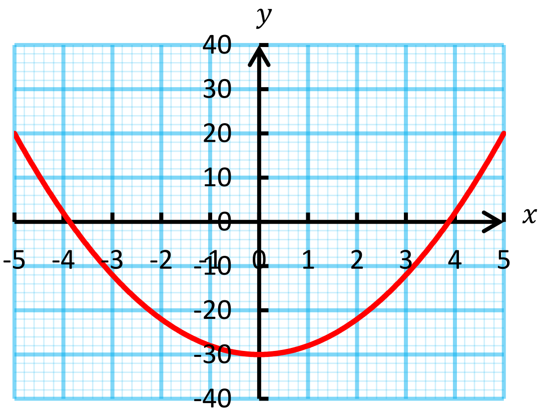
(d) What happens as you change the value of b?

(e) What stays the same as you change the value of b?

(f) Complete the following sentences. If a is positive, then the shape of the graph is similar to the letter ______.
If a is negative, then the shape of the graph is similar to the letter ______.

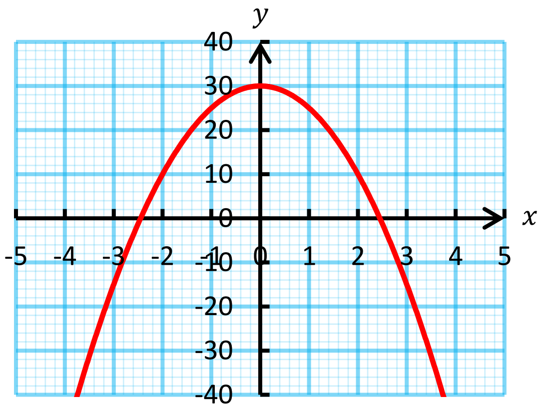
Exercise 59

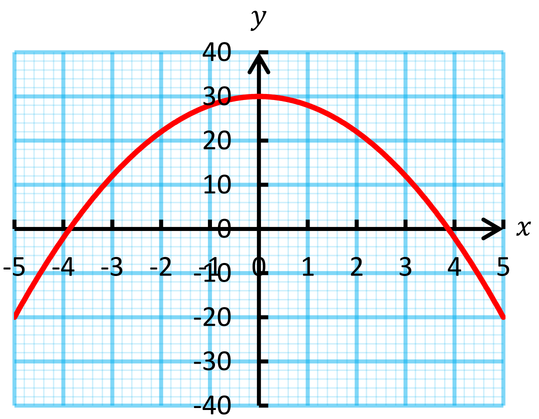
Pair the graphs and the equations.



$y = -5x^2 + 30$

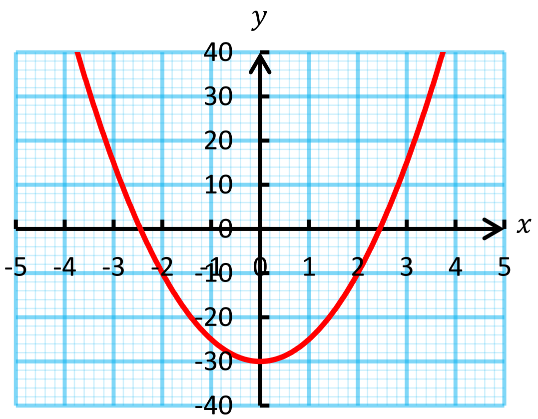
$y = 5x^2 - 30$





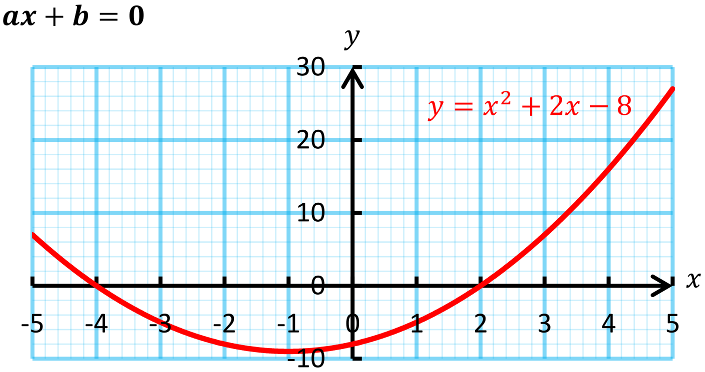
$y = -2x^2 + 30$

$y = 2x^2 - 30$



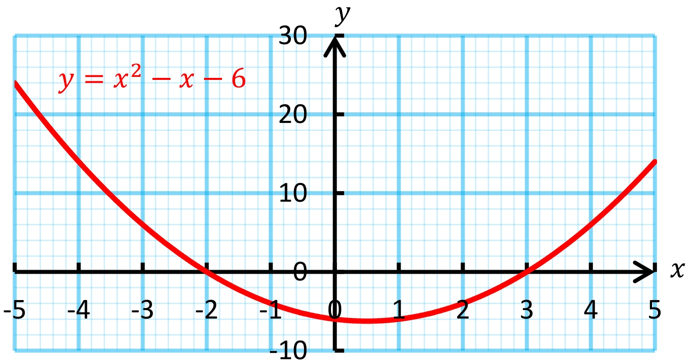
Graphical method of solving equations of the form $x^2 + ax + b = 0$

Consider the graph shown on the right for the equation $y = x^2 + 2x - 8$. To solve the equation $x^2 + 2x - 8 = 0$, we can use the graph to see where the graph crosses the x–axis (this is where the function $x^2 + 2x - 8$ is zero). We see that the graph crosses the x–axis at the points where $x = -4$ and $x = 2$, therefore the solutions to the equation $x^2 + 2x - 8 = 0$ are $x = -4$ and $x = 2$.

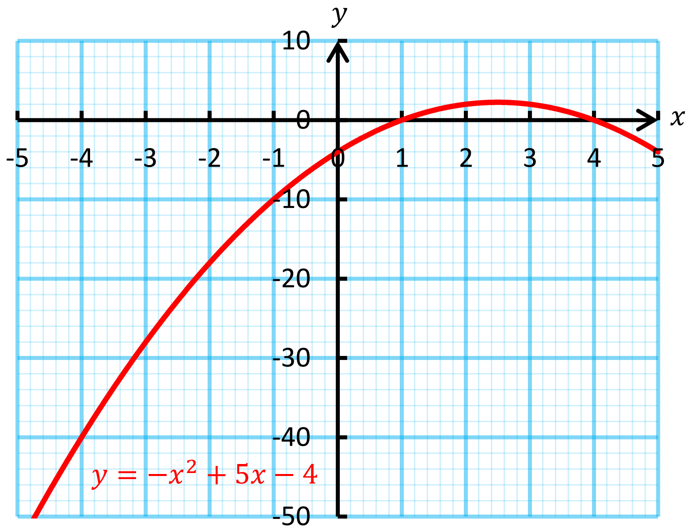


Exercise 60

(a) Use the graph below to solve the equation $x^2 - x - 6 = 0$.

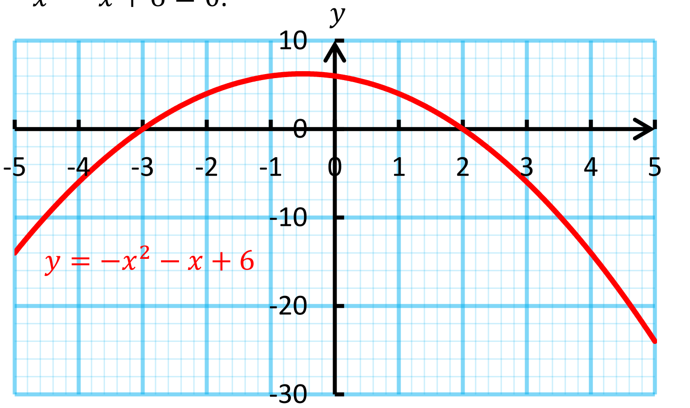


(b) Use the graph below to solve the equation $-x^2 + 5x - 4 = 0$.

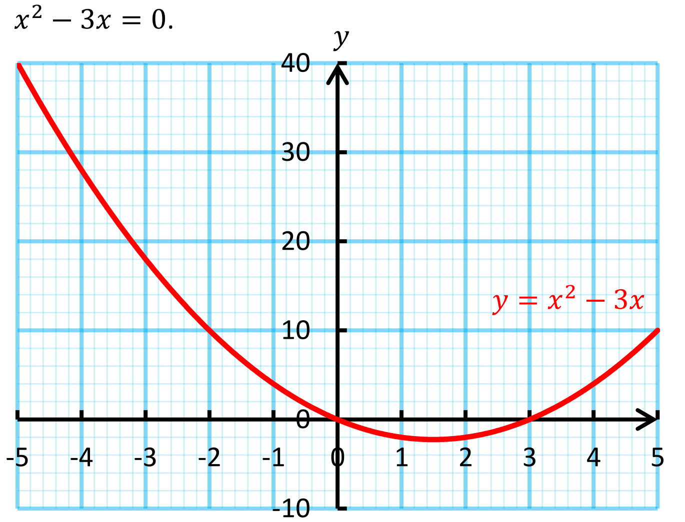


(c) Use the graph above to solve the equation $x^2 - x - 6 = 10$.
Give your answers correct to one decimal place.

(d) Use the graph below to solve the equation $-x^2 - x + 6 = 0$.



(e) Use the graph below to solve the equation $x^2 - 3x = 0$.



(f) Use the graph above to solve the equation $-x^2 - x + 6 = -10$.
Give your answers correct to one decimal place.

(g) Use the graph above to solve the equation $x^2 - 3x = 4$.

(h) (i) By drawing a suitable graph, solve the equation $x^2 + 3x - 4 = 0$.
(ii) Use your graph from part (i) to solve the equation $x^2 + 3x - 4 = 5$.
Give your answers correct to one decimal place.

Other Graphs

For the higher tier, you must be able to…

- recognise and sketch **reciprocal** graphs of the form $y = \frac{a}{x}$;
- recognise and sketch **cubic** graphs of the form $y = ax^3$;
- draw and interpret **reciprocal** graphs of the form $y = ax + b + \frac{c}{x}$;
- draw and interpret **cubic** graphs of the form $y = ax^3 + bx^2 + cx + d$;
- draw and interpret **exponential** graphs of the form $y = k^x$.

Exercise 61

Fill in the blanks in the following tables. Then, on the graph paper at the bottom of the page, plot appropriate graphs for the equations.

(a)

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
$y = \frac{1}{x}$											

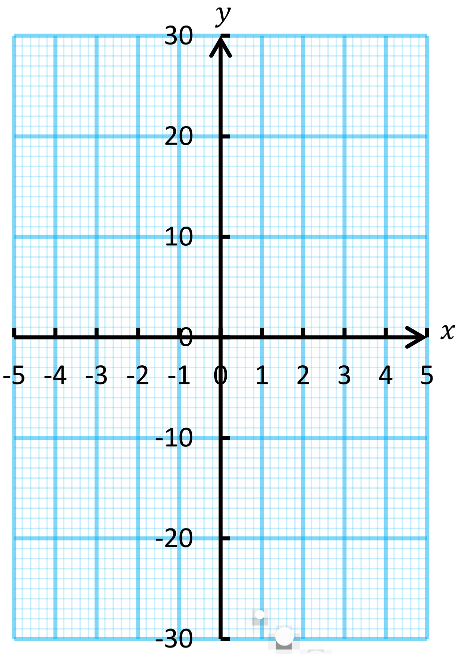
(b)

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
$y = x^3$											

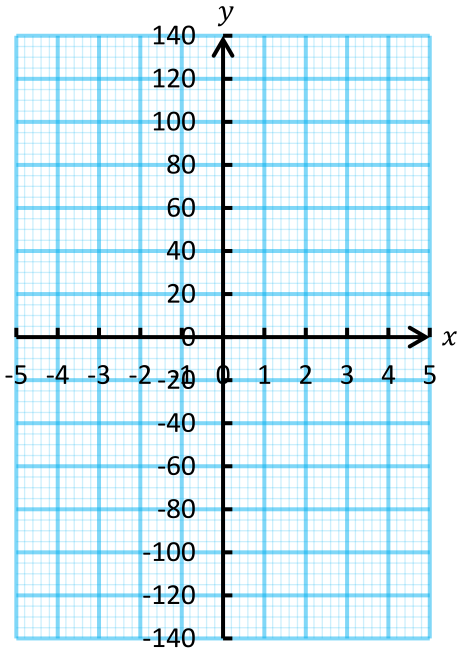
(c) (Use the first chapter on rules of indices to help fill in this table.)

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
$y = 2^x$											

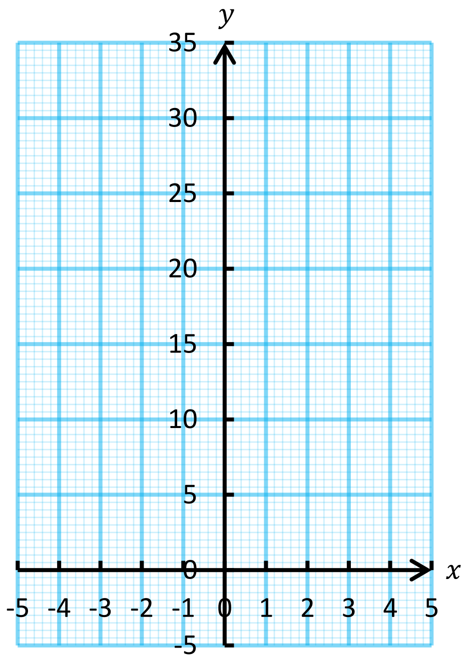
(a)



(b)



(c)



Plot additional points between −1 and 1
for a more accurate graph.

Exercise 62

H

Plot suitable graphs for the following equations.

(a) $y = 3^x$ (b) $y = \frac{2}{x} + 3$ (c) $y = x^3 - 4x$

(d) $y = \frac{1}{x} + x - 1$ (e) $y = x^3 + 5x^2 - 2x + 5$ (f) $y = -\frac{3}{x} + 10x$

(g) $y = 5^x$ (h) $y = -x^3$ (i) $y = \frac{1}{2}x^3 - 15$

Exercise 63

Use your graphs from Exercise 62 to solve the following equations.
Where necessary, write your answers correct to one decimal place.

(a) $3^x = 4$ (b) $\frac{2}{x} + 3 = 2$ (c) $x^3 - 4x = 0$

(d) $\frac{1}{x} + x - 1 = 2$ (e) $x^3 + 5x^2 - 2x + 5 = 20$ (f) $-\frac{3}{x} + 10x = 0$

(g) $5^x = 3$ (h) $-x^3 = 0$ (i) $\frac{1}{2}x^3 - 15 = -30$

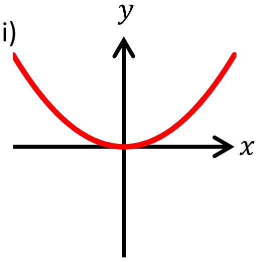
Exercise 64

Use the Desmos website ([www.desmos.com/calculator)](www.desmos.com/calculator) to investigate the graphs for the equations on the top of the previous page. Write a paragraph summarising your findings. Remember to discuss the general shape of each graph, and describe what happens as you change the parameters a, b, c, d and k.

Exercise 65

Pair each equation with its sketch.

(a) $y = 4^x$ (b) $y = x^2$ (c) $y = \frac{3}{x} - 4$ (d) $y = 2x^3$ (e) $y = 2x + 3$

(i)


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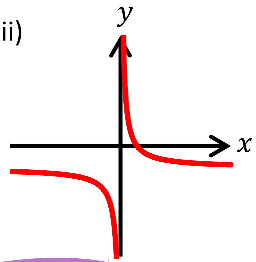

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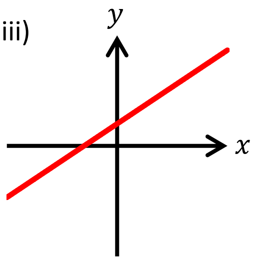

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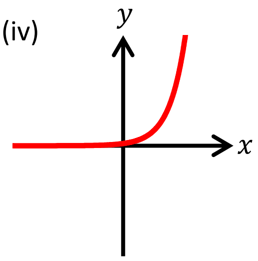

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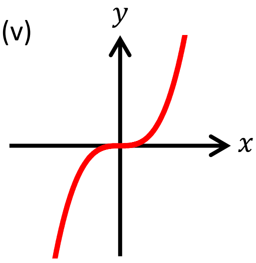


Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Powers and Roots

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test Without a calculator With a calculator	Correct in the test?
I know how to change between **ordinary numbers** and numbers written in **index form**.			1	
I know how to use the **multiplication rule** $\boldsymbol{n^a \times n^b = n^{a+b}}$.			2	
I know how to use the **division rule** $\boldsymbol{n^a \div n^b = n^{a-b}}$.			2	
I know how to use the **zeroth index rule** $\boldsymbol{n^0 = 1}$.			2	
I know how to use the rule where **a power is raised to another index**, $\boldsymbol{(n^a)^b = n^{a\times b}}$.			2	
I know how to use the **negative index rule** $\boldsymbol{n^{-a} = \frac{1}{n^a}}$.			2	
I know how to use the **unitary fraction index rule** $\boldsymbol{n^{\frac{1}{a}} = \sqrt[a]{n}}$.			2	
I can **simplify algebraic expressions** using the rules of indices.			3	
I can write numbers in **standard form**.			4	
I can **add and subtract** numbers written in standard form.			5	
I can **multiply and divide** numbers written in standard form.			5	
I can **solve problems** using standard form.			1	
I can calculate percentage changes **efficiently**.			2	
I can calculate **repeated percentage changes efficiently**.			2	
I can calculate **compound interest efficiently**.			3	
I can calculate **fractional changes efficiently**.				
I know how to answer questions involving **reverse percentages**.			6, 4	
I can **plot quadratic graphs**.			7	
I know how to **recognise and sketch quadratic graphs**.			5	
I know how to use a **graphical method** to solve quadratic equations.			7	

Am I ready for the test?
Tick the boxes below...

I have revised the work in my mathematics book.	☐
I have revised the workbook.	☐
I have watched the relevant videos on YouTube.	☐
I have completed the Diagnostic Questions quiz.	☐
I have completed at least 4 pages in my revision book.	☐

Powers and Roots

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test Without a calculator With a calculator	Correct in the test?
I know how to change between **ordinary numbers** and numbers written in **index form**.			1	
I know how to use the **multiplication rule** $\boldsymbol{n^a \times n^b = n^{a+b}}$.			2	
I know how to use the **division rule** $\boldsymbol{n^a \div n^b = n^{a-b}}$.			2	
I know how to use the **zeroth index rule** $\boldsymbol{n^0 = 1}$.			2	
I know how to use the rule where **a power is raised to another index**, $\boldsymbol{(n^a)^b = n^{a \times b}}$.			2	
I know how to use the **negative index rule** $\boldsymbol{n^{-a} = \frac{1}{n^a}}$.			2	
I know how to use the **unitary fraction index rule** $\boldsymbol{n^{\frac{1}{a}} = \sqrt[a]{n}}$.			2	
I can **simplify algebraic expressions** using the rules of indices.			3	
I know how to use the **general fraction index rule** $\boldsymbol{\left(\sqrt[b]{n}\right)^a = n^{\frac{a}{b}} = \sqrt[b]{n^a}}$.			2	
I can **combine** the rules of indices.			2	
I can write numbers in **standard form**.			4	
I can **add and subtract** numbers written in standard form.			5	
I can **multiply and divide** numbers written in standard form.			5	
I can **solve problems** using standard form.			1	
I can calculate percentage changes **efficiently**.			2	
I can calculate **repeated percentage changes efficiently**.			2	
I can calculate **compound interest efficiently**.			3	
I can calculate **fractional changes efficiently**.				
I know how to answer questions involving **reverse percentages**.			6, 4	
I can **plot quadratic graphs**.			7	
I know how to **recognise and sketch quadratic graphs**.			5	
I know how to use a **graphical method** to solve quadratic equations.			7	
I know how to **recognise and sketch** other graphs, e.g. **reciprocal graphs**; **cubic graphs**; **exponential graphs**.			5	

Am I ready for the test?
Tick the boxes below...

I have revised the work in my mathematics book.	☐
I have revised the workbook.	☐
I have watched the relevant videos on YouTube.	☐
I have completed the Diagnostic Questions quiz.	☐
I have completed at least 4 pages in my revision book.	☐

The Mathematics Department

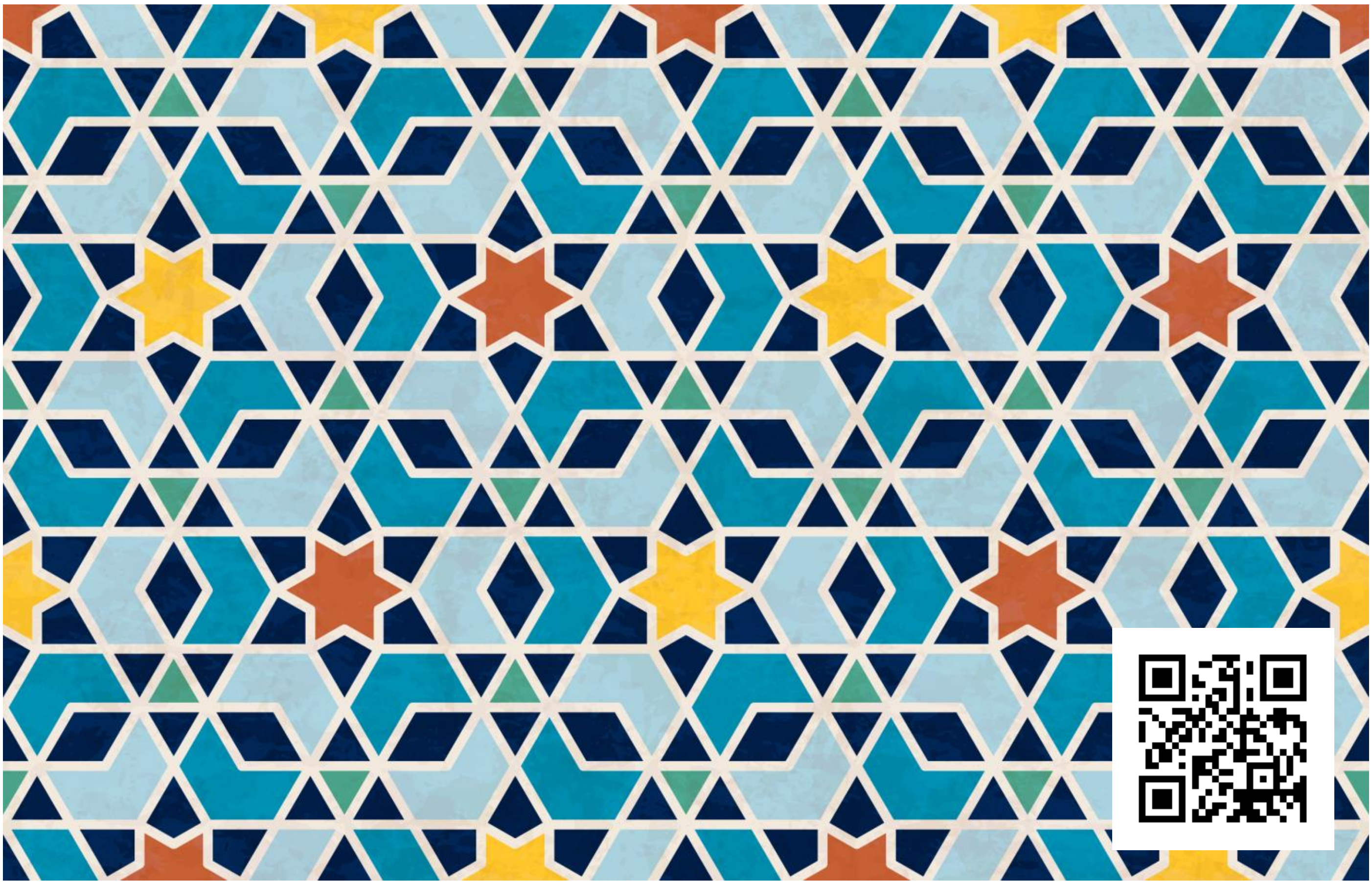
10

Measuring Shapes 3

Name:

Contents

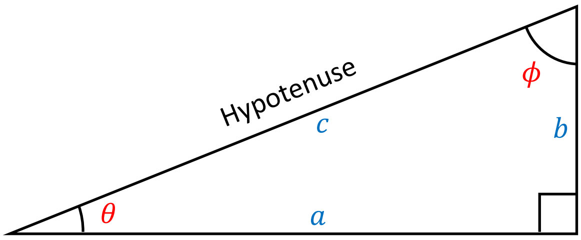
Chapter	Mathematics	Page Number
Trigonometry of Right-Angled Triangles	Labelling triangles. Calculating the length of a side. Calculating the size of an angle.	3
Transformations: Enlargement	Positive scale factor. Fractional scale factor. Centre of enlargement. Finding the centre of enlargement. **Negative scale factor**.	10
Tessellations	Tessellating with triangles. Tessellating with quadrilaterals.	15
Perimeter and Area of Composite Shapes	Composite shapes. **Length of an arc and the area of a sector**.	17



Trigonometry

Right-Angled Triangles

Any right-angled triangle has:



- One angle that is a right angle, or 90°;
- Two acute angles θ and ϕ;
- A hypotenuse c, which is always opposite the right angle;
- Two sides a and b which are shorter than the hypotenuse.

In year 9, we introduced Pythagoras' Theorem, which connects the lengths a, b and c:

$$c^2 = a^2 + b^2$$

Given the length of any two sides in a triangle, we can use Pythagoras' Theorem to calculate the length of the third side.

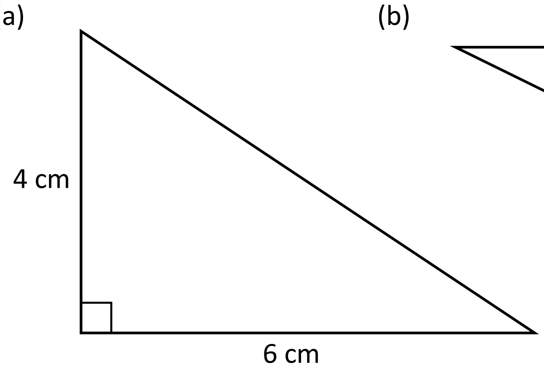
Exercise 1

Revision

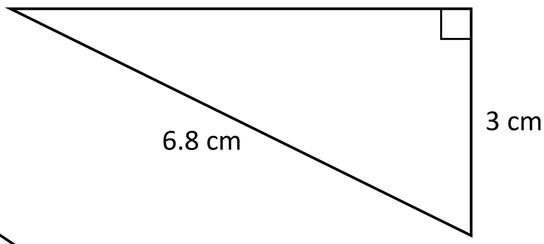
I

Use Pythagoras' Theorem to calculate the length of the third side in these right-angled triangles. Round off your answers to two decimal places.

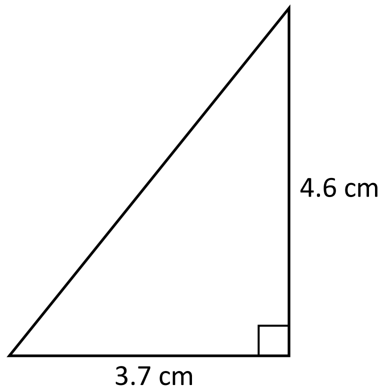
(a)



(b)



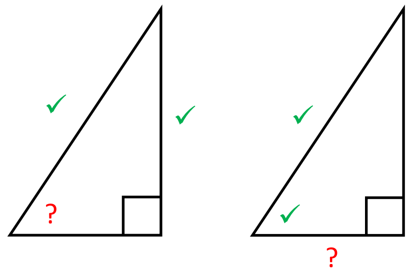
(c)



What is trigonometry?

Trigonometry is used for:

- Calculating the size of one of the acute angles in a right-angled triangle, given the length of any two sides;
- Calculating the length of one of the sides in a right-angled triangle given the length of one other side and the size of one acute angle.



How?

Trigonometry uses the relationship between the size of the angles and the lengths of the sides in any right-angled triangle.

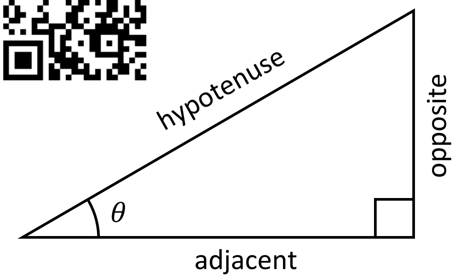
Exercise 2

Applying

Draw any three right-angled triangles where one of the angles measures 30°.
Measure the length of the hypotenuse and the length of the side opposite the 30° angle. What do you notice?

Labelling the sides of a right-angled triangle

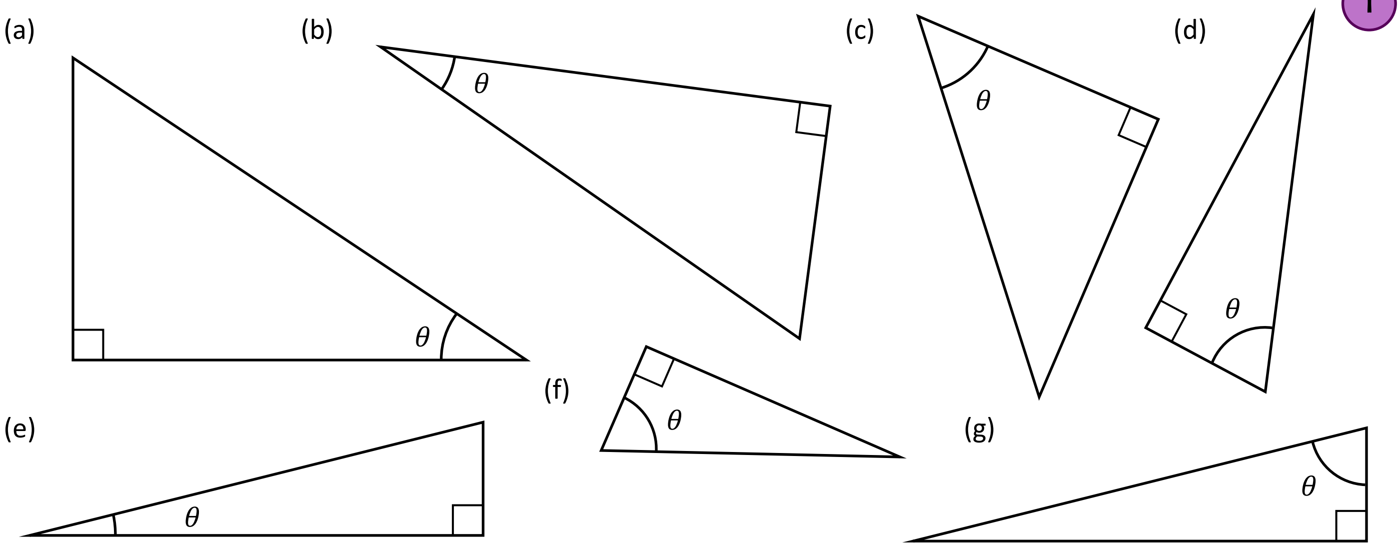
Let θ represent the size of one of the acute angles in a right-angled triangle. We follow these conventions when labelling the sides of the triangle.



- The **opposite** is the side opposite the angle θ.
- The **hypotenuse** is the side opposite the right angle.
- The **adjacent** is the side left over (it's close to the angle θ).

Exercise 3

Label the sides of these triangles using the words "opposite", "hypotenuse" and "adjacent".



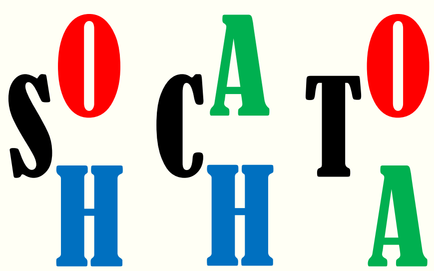
Sin, Cos, Tan

For a specific angle θ, we define the **sin**, **cos** and **tan** of the angle as follows.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

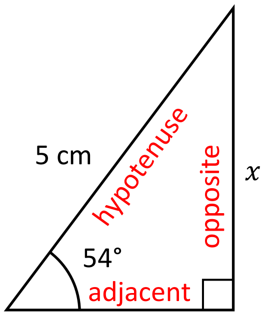
How to remember the formula...

In mathematics, we worship the King SOHCAHTOA.



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Finding lengths using trigonometry



Consider the right-angled triangle shown on the right.
Let us use trigonometry to calculate the length of the side x.

To start with, we label the sides of the triangle using the words "opposite", "adjacent" and "hypotenuse".

We see that we want to calculate the length of the **opposite** (x) side, and we know the length of the **hypotenuse** (5 cm). The trigonometric ratio that uses the words **opposite** and **hypotenuse** is sin, therefore we must use the formula

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

in this question. By substituting values into the formula, we obtain

$$\sin 54° = \frac{x}{5}.$$

By multiplying both sides of the equation by 5, we obtain

$$x = 5 \times \sin 54°.$$

x on **top** of the fraction leads to a **multiplication** sum in the answer.

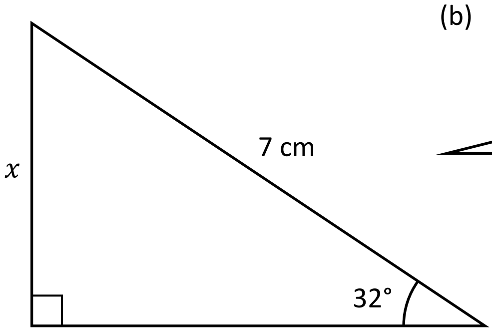
By typing this sum on a calculator, we find that $x = 4.05$ cm, correct to two decimal places.

Exercise 4

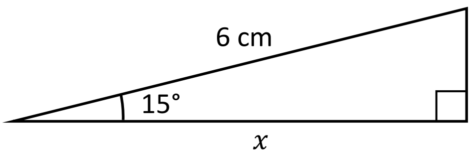
I

For the following right-angled triangles, calculate the length of the side that is labelled with the variable x.

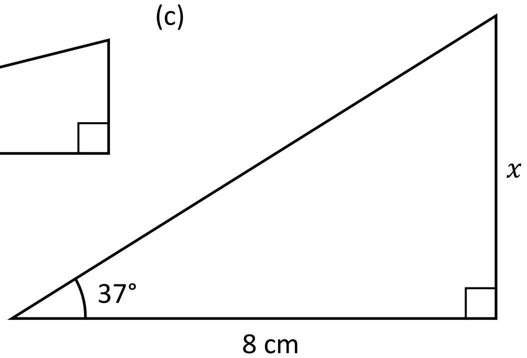
(a)



(b)



(c)



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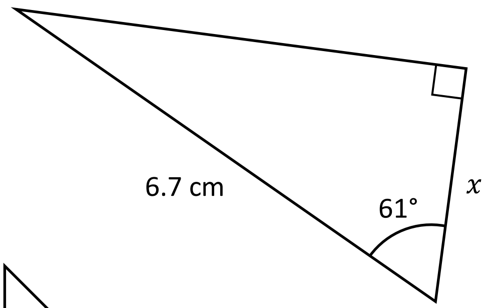
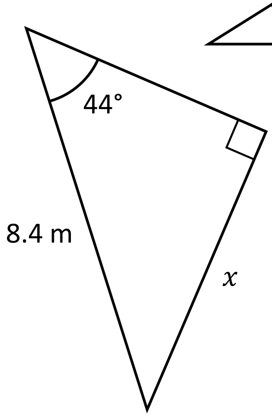


(e)



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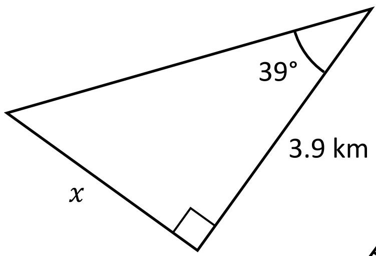

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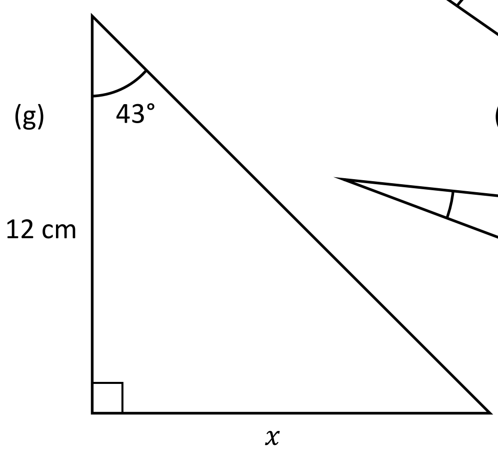

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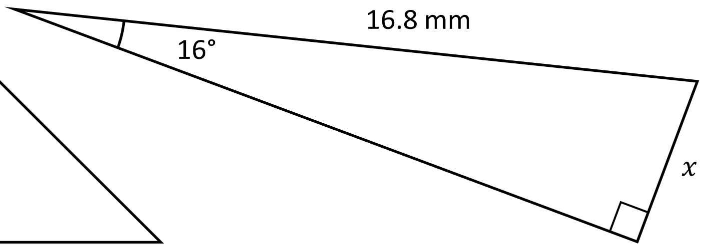
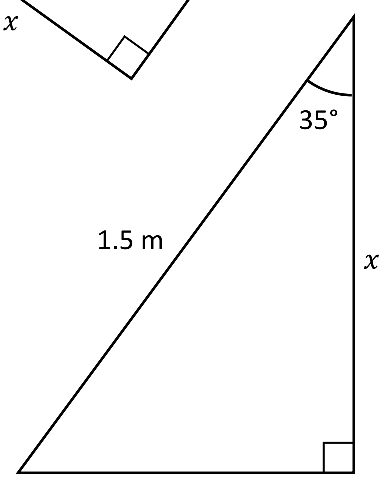
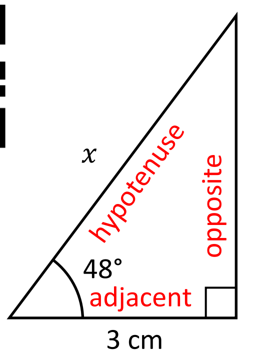


(i)



Example



Consider the right-angled triangle shown on the right.
Let us use trigonometry to calculate the length of the side x.

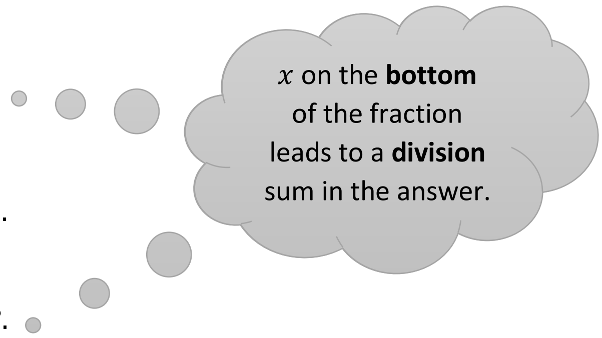
To start with, we label the sides of the triangle using the words "opposite", "adjacent" and "hypotenuse".

We see that we know the length of the **adjacent** (3 cm), and we want to calculate the length of the **hypotenuse** (x). The trigonometric ratio that uses the words **adjacent** and **hypotenuse** is cos, therefore we must use the formula

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

in this question. By substituting values into the formula, we obtain

$$\cos 48° = \frac{3}{x}.$$



By multiplying both sides of the equation by x, we obtain

$$x \times \cos 48° = 3.$$

By dividing both sides of the equation by $\cos 48°$, we obtain

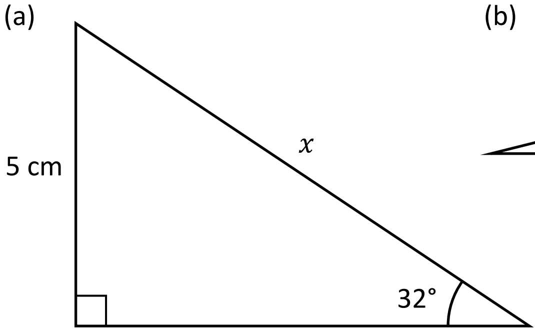
$$x = 3 \div \cos 48°.$$

By typing this sum on a calculator, we find that $x = 4.48$ cm, correct to two decimal places.

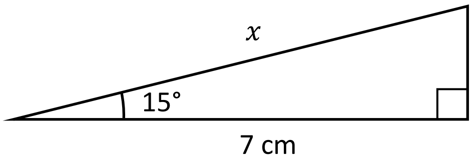
Exercise 5 I

For the following right-angled triangles, calculate the length of the side that is labelled with the variable x.

(a)



(b)



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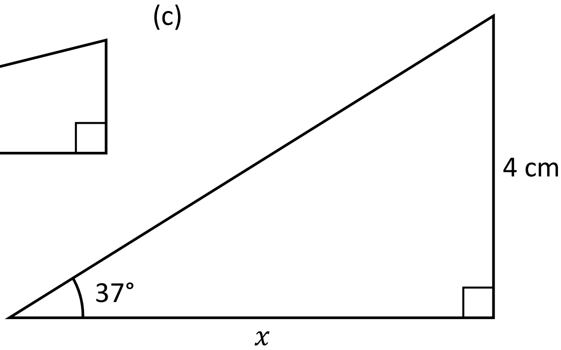

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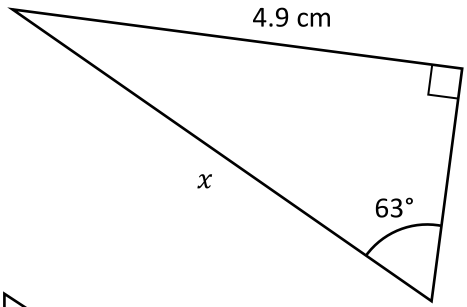

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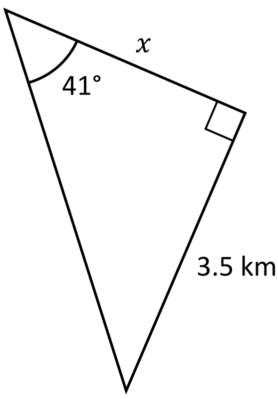

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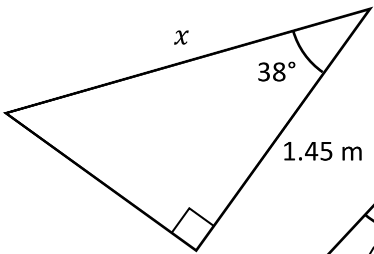

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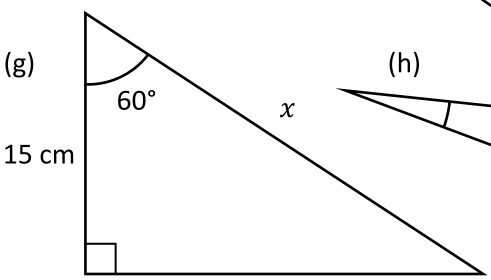

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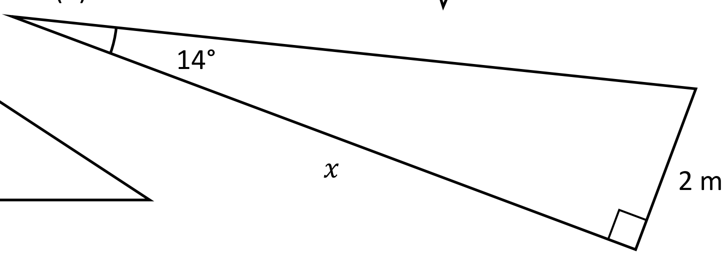

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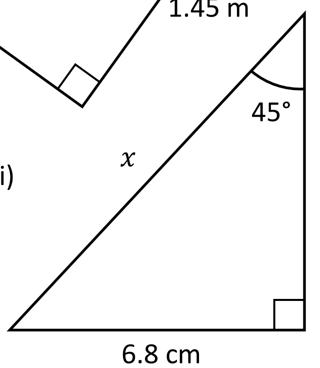
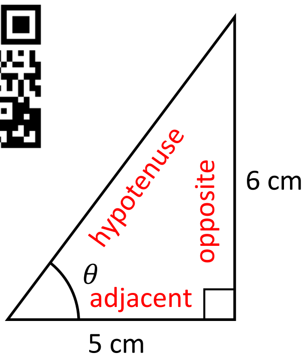


Finding angles using trigonometry



Consider the right-angled triangle shown on the right.
Let us use trigonometry to calculate the size of the angle θ.

To start with, we label the sides of the triangle using the words "opposite", "adjacent" and "hypotenuse".

We see that we know the length of the **opposite** (6 cm) and the length of the **adjacent** (5 cm). The trigonometric ratio that uses the words **opposite** and **adjacent** is tan, therefore we must use the formula

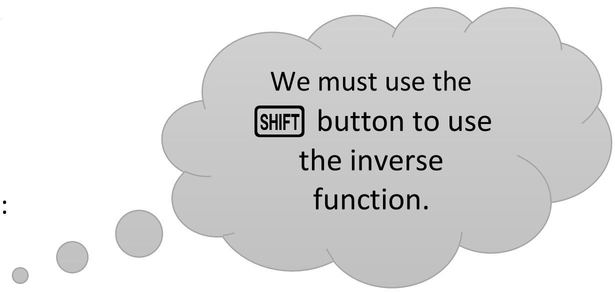
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

in this question. By substituting values into the equation, we obtain

$$\tan\theta = \frac{6}{5}.$$

To find the size of the angle θ we must use the inverse tan function:

$$\theta = \tan^{-1}\left(\frac{6}{5}\right).$$



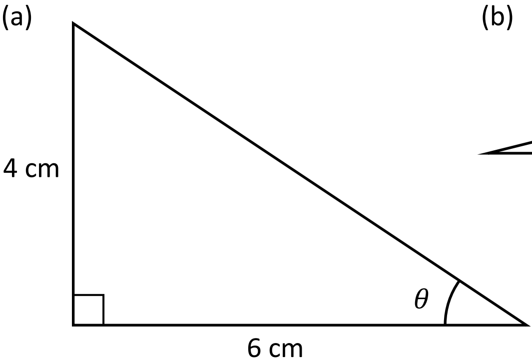
By typing this sum on a calculator, we find that $\theta = 50.19°$, correct to two decimal places.

Exercise 6

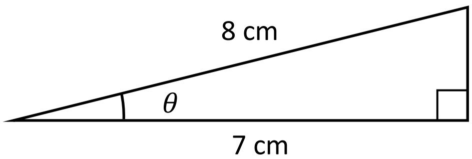
I

For the following right-angled triangles, calculate the size of the angle θ.

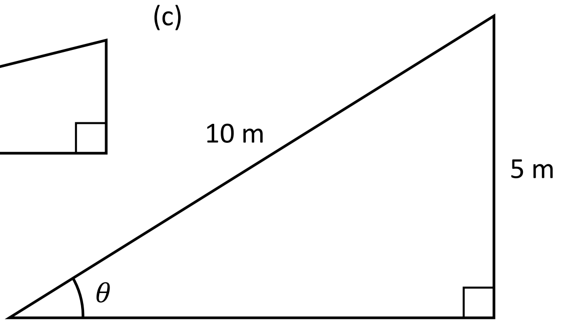
(a)



(b)



(c)



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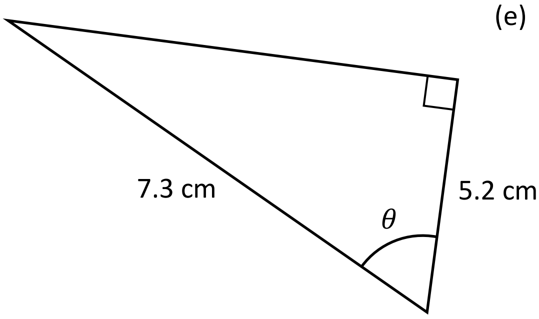

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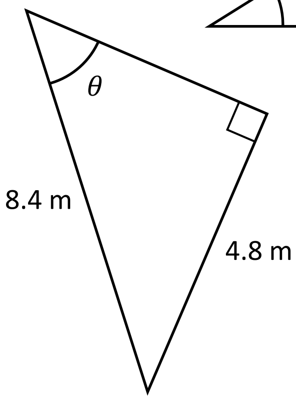

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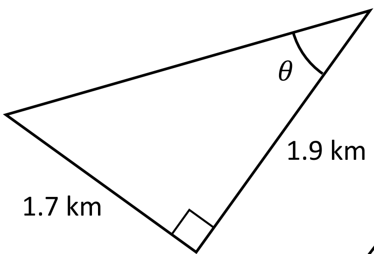

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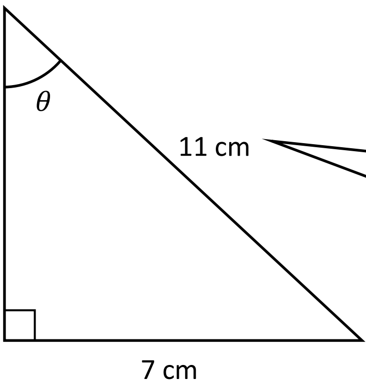

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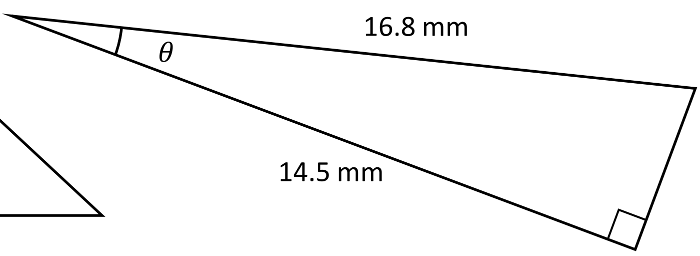

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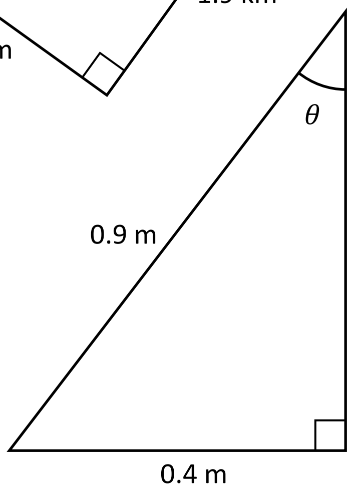
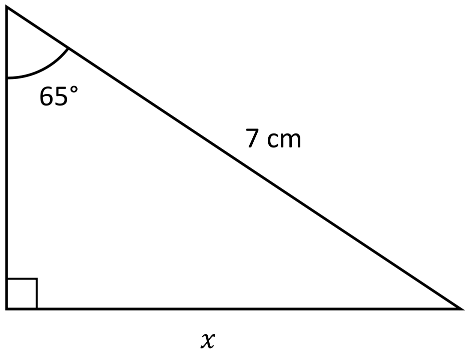


Exercise 7

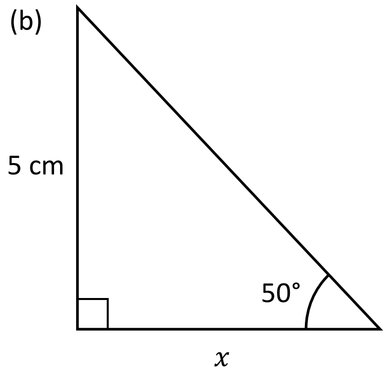
I

For the following right-angled triangles, calculate the length of the side x, or the size of the angle θ. Round off your answers correct to two decimal places.

(a)



(b)



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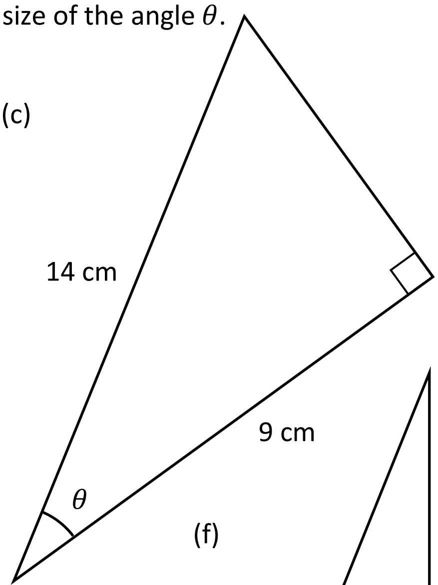

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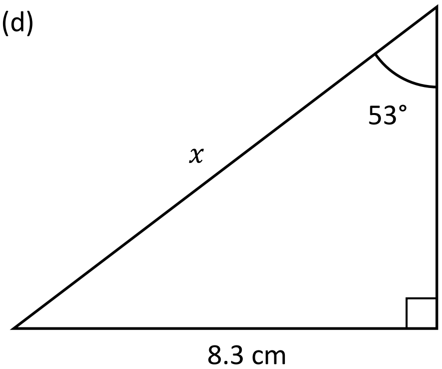
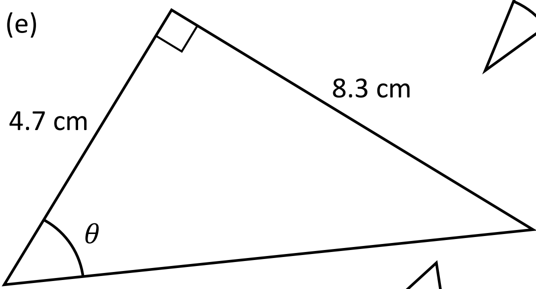
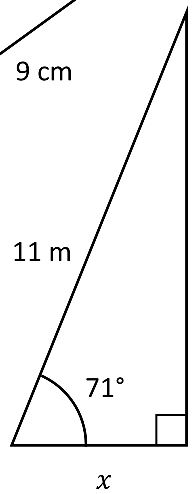


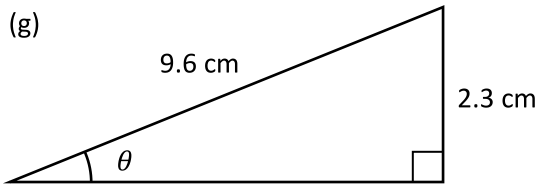
(e)



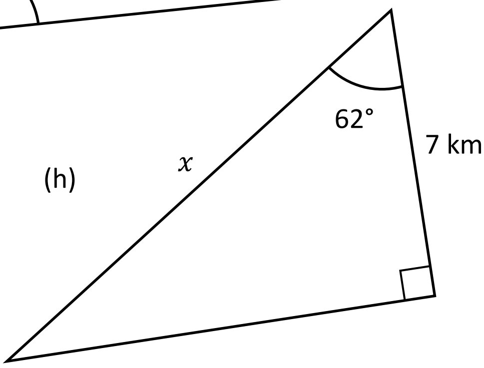
(f)



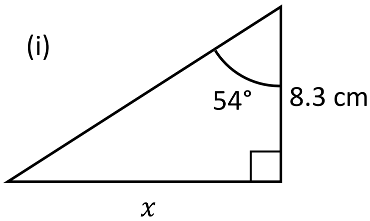
(g)



(h)



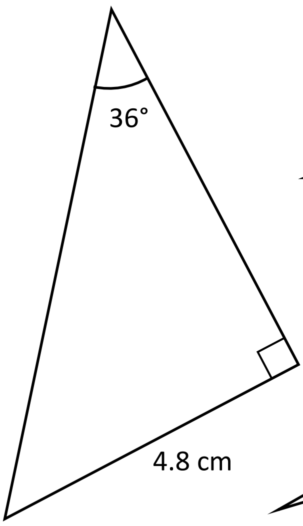
(i)



Exercise 8

For the following right-angled triangles, find the size of **every** missing angle and the length of **every** missing side.

(a)

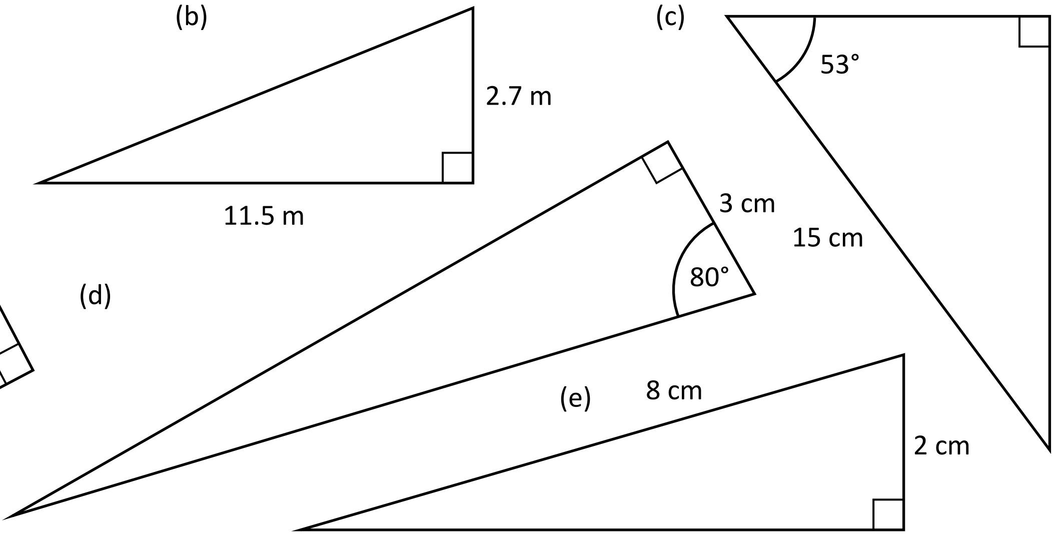


(b)

(c)

(d)

(e)



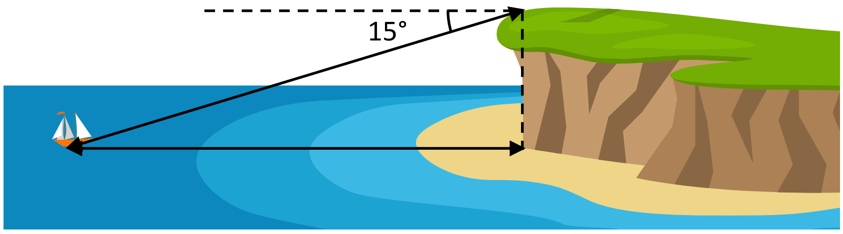
Exercise 9

Applying I

(a)

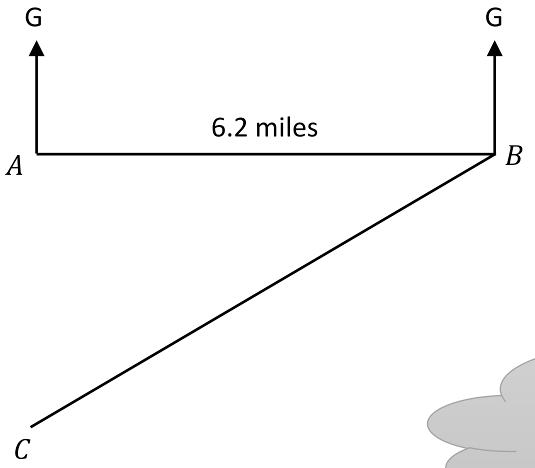
The vertical height of a tree is 3.2 metres. The horizontal distance from point A to the bottom of the tree is 7 metres. Calculate the angle of elevation of the top of the tree from the point A.

(b)



From the top of a vertical cliff, the angle of depression of a sailing boat is 15°. If the sailing boat is 200 m from the bottom of the cliff, calculate the height of the cliff above sea level.

(c)



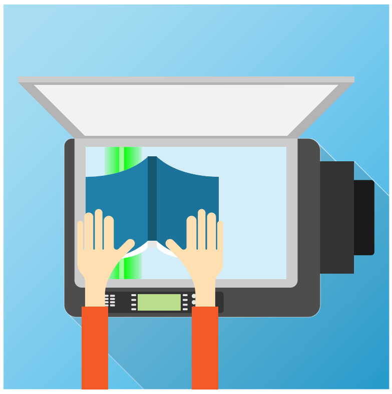
A ship leaves a port and sails 6.2 miles at a bearing of 090° to reach B. Then it turns and sails at a bearing of 224° until it reaches a point C, which is south of the port A. Calculate the distance between the point C and the port A.

The word trigonometry comes from the Greek language: "*trigon*" means triangle and "*metry*" means measure.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Enlargement



Enlargement is one of the four transformations.

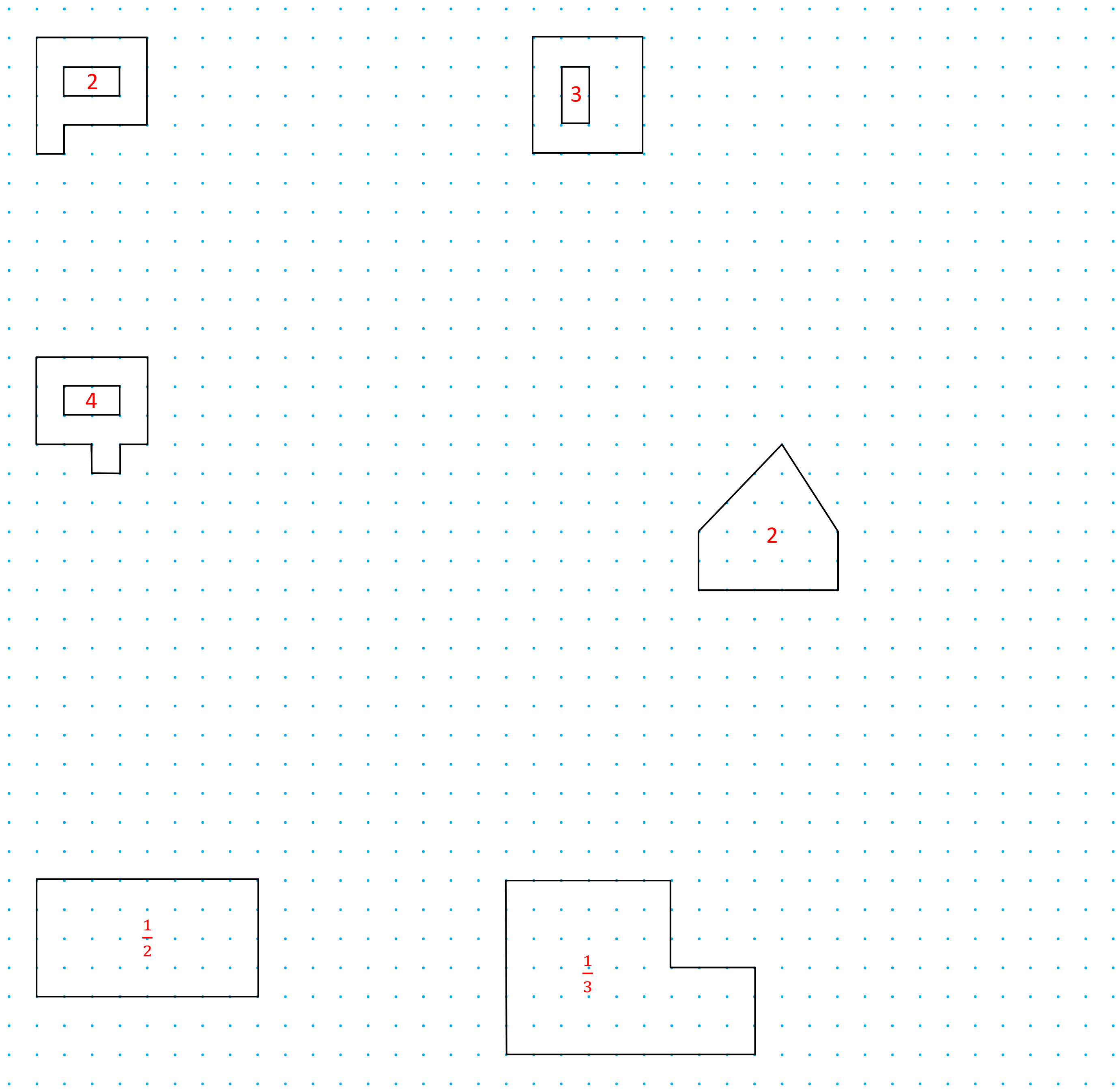
Year 7	Year 8	Year 9	Year 10
Translation	Rotation	Reflection	**Enlargement**

When a shape is enlarged, the **size** of the shape changes. The **scale factor** decides how the shape changes. For example, if the scale factor is 2, then the size of the shape doubles. If the scale factor is $\frac{1}{2}$, then the size of the shape halves.

Skill

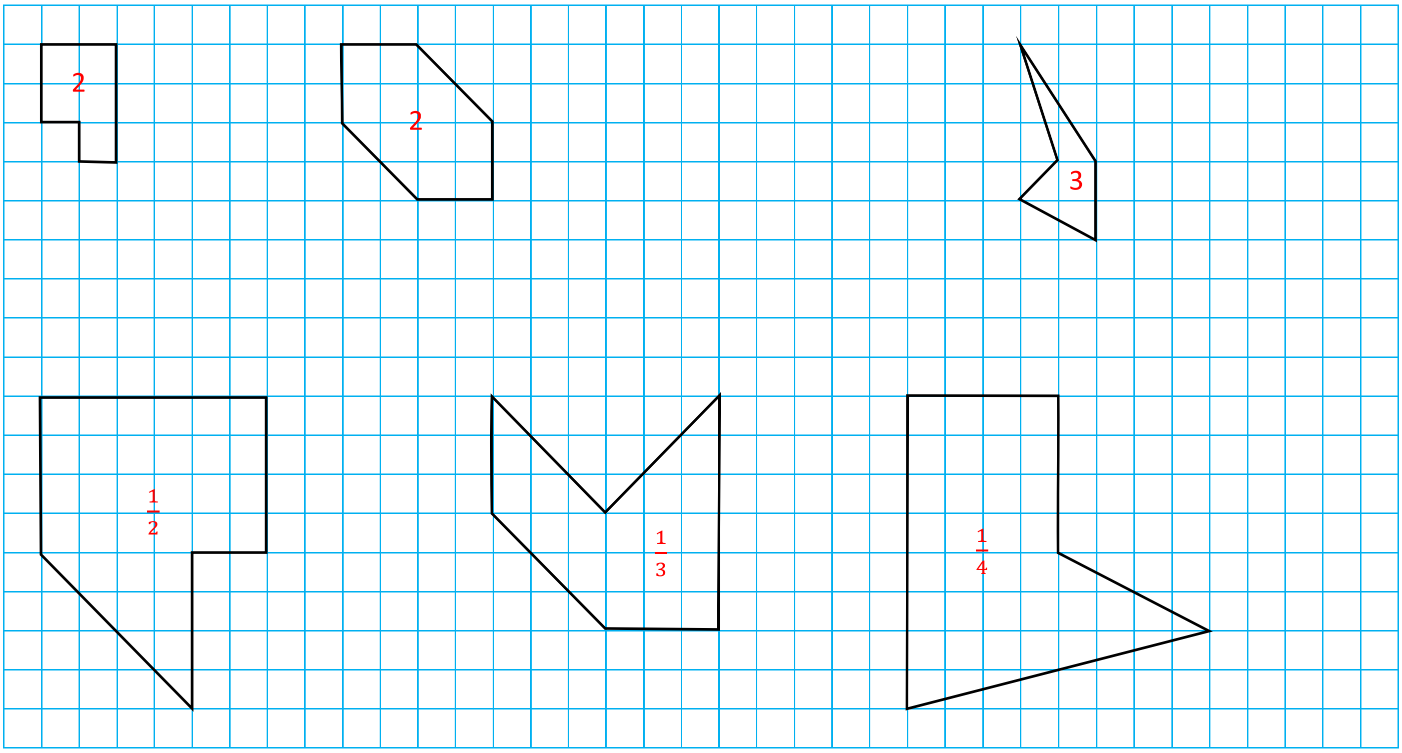
Exercise 10

Enlarge the following shapes using the scale factor that is given in the centre of each shape. I



Exercise 11 I

Enlarge the following shapes using the scale factor that is given in the centre of each shape.

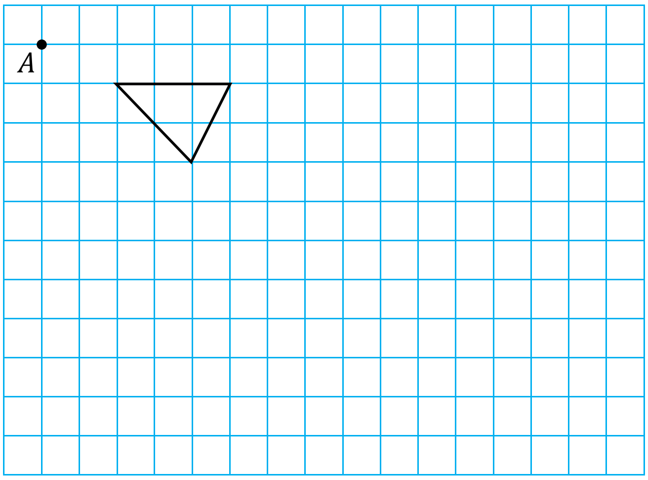


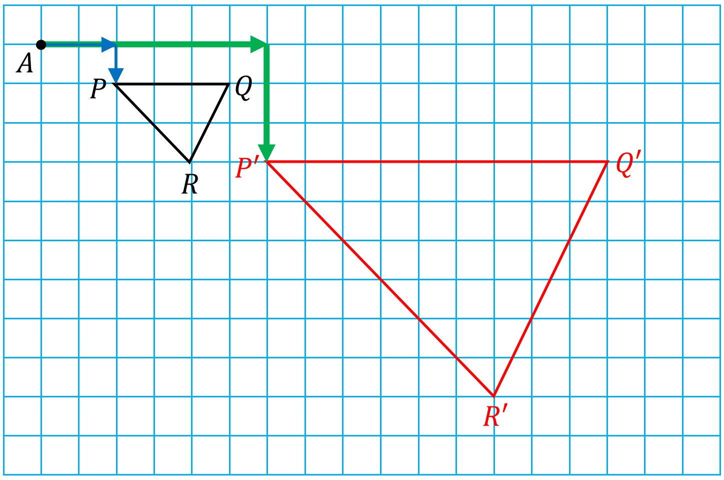
Centre of Enlargement

If a question states a point as the **centre of enlargement,** then the enlargement must appear in a certain location (it cannot appear *anywhere* like in Exercises 10 and 11).

Example

Enlarge the following triangle using a scale factor of 3 and the point A as the centre of enlargement.



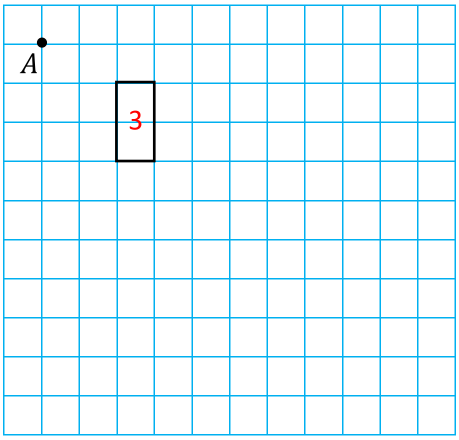


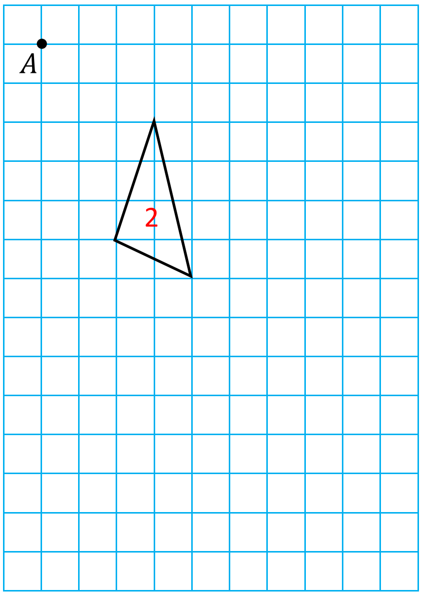
To go from the point A to the vertex P in the original triangle, we must go 2 units to the right and 1 unit down. Since the scale factor is 3, to go from the point A to the vertex P' in the new triangle, we must go $2 \times 3 = 6$ units to the right, and $1 \times 3 = 3$ units down. We can repeat this with the other vertices (Q and R), or you can start at the vertex P' and draw a triangle that is three times larger.

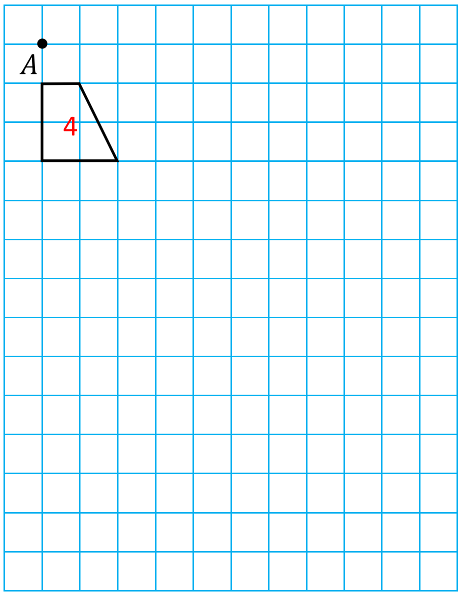
Exercise 12

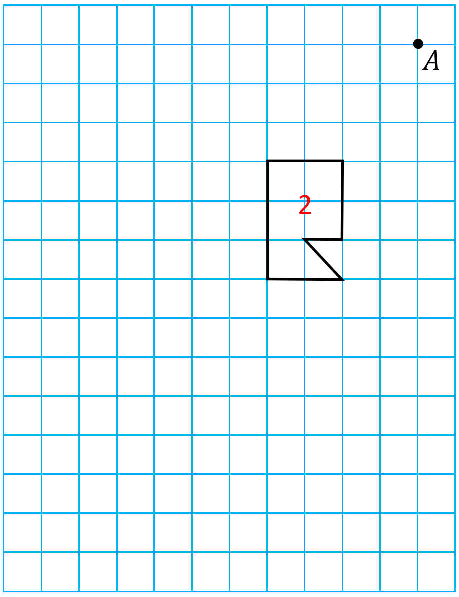
I

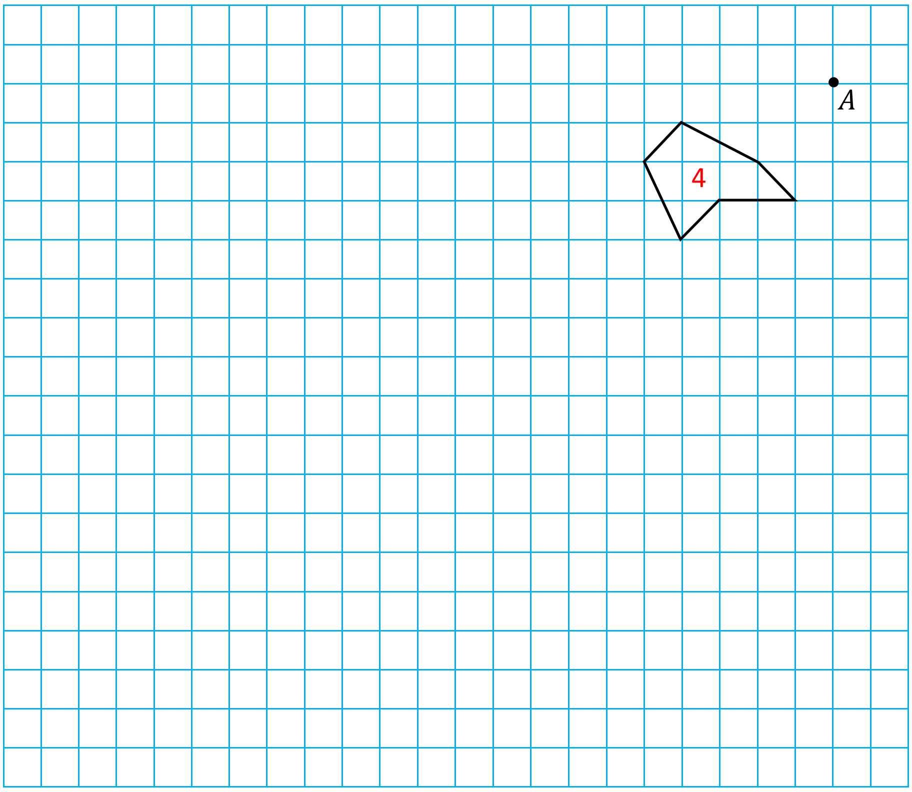
Enlarge the following shapes using the point A as the centre of enlargement and the number in the centre of each shape as the scale factor.

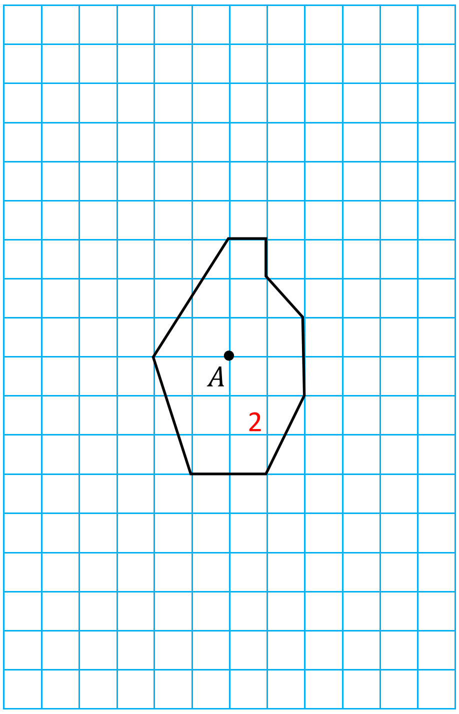


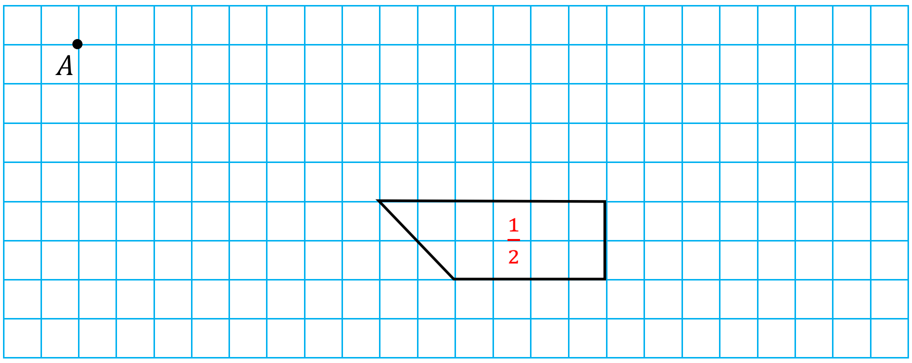




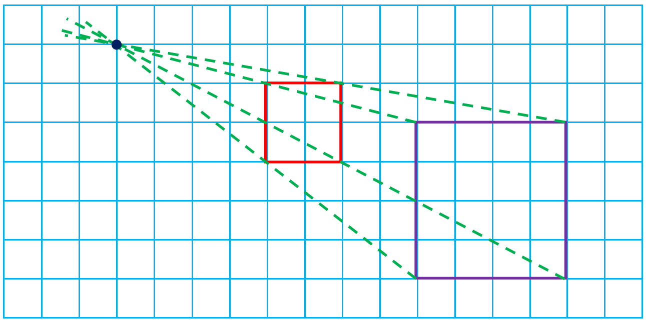








Finding the centre of enlargement

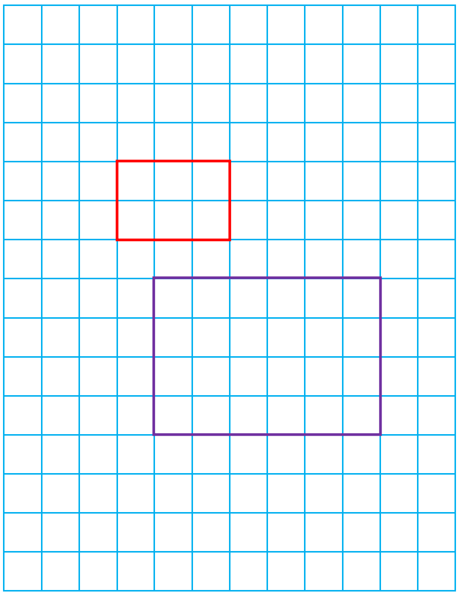


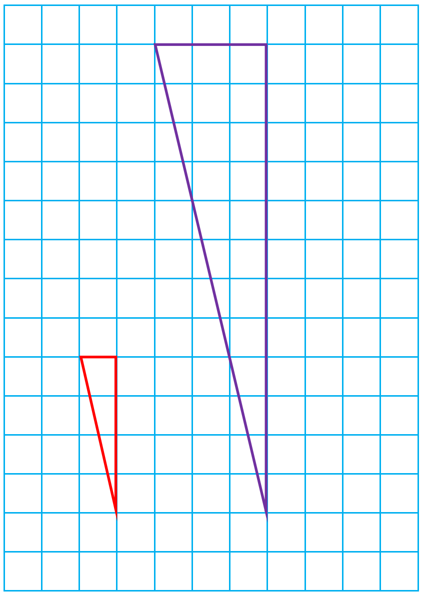
Given a shape and its enlargement, connect the corresponding vertices and extend the lines to find the location of the centre of enlargement.

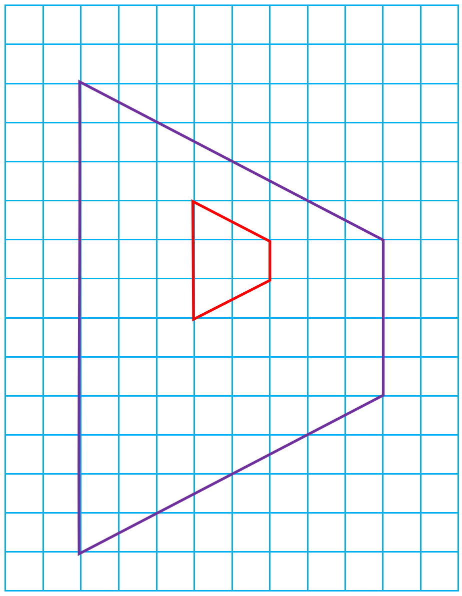
You can find the scale factor by comparing the sizes of the shapes.

Exercise 13

Find the scale factor and the centre of enlargement for the following enlargements.



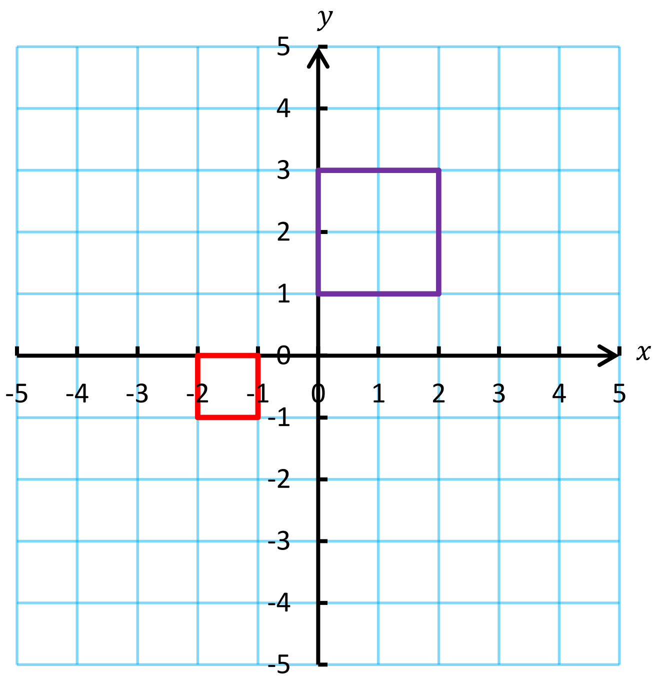




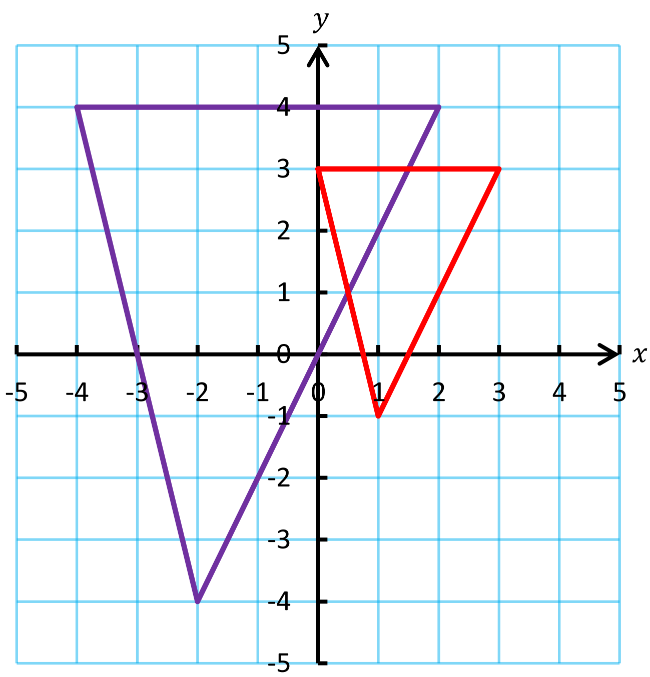
Exercise 14

Find the scale factor and the centre of enlargement for the following enlargements.

(a)



(b)



Negative Scale Factor

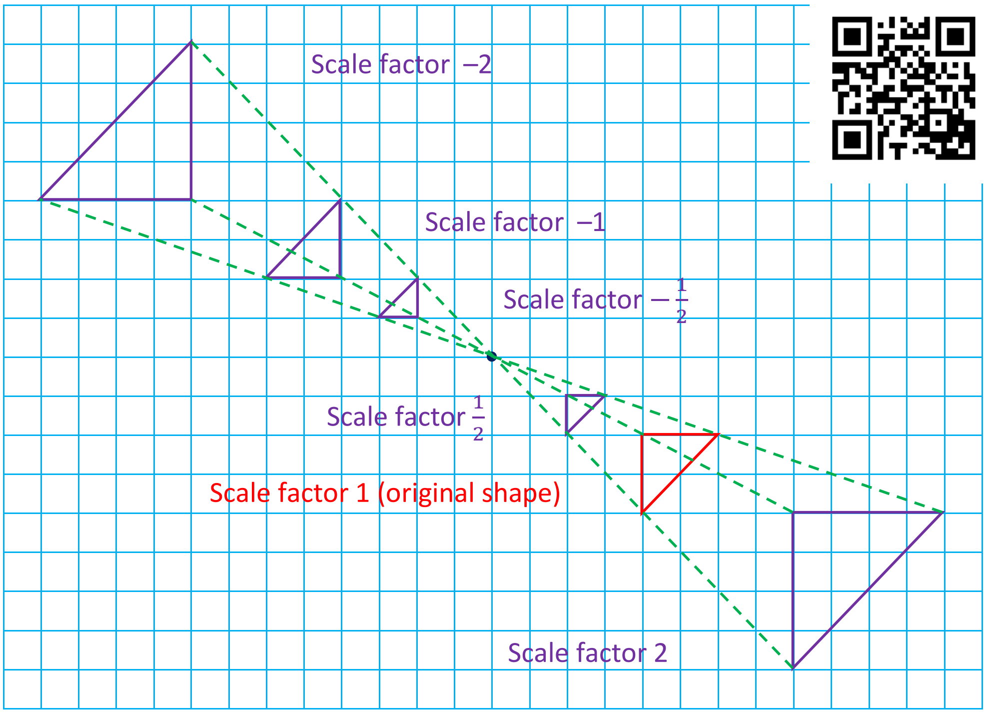
Higher Tier

A **negative** scale factor means working from the centre of enlargement to the opposite direction.

For example, consider the diagram on the right. To go from the centre of enlargement to the top left vertex of the original triangle, we must go 4 units right and 2 units down.

With a scale factor of 2, we must go 4 × 2 = 8 units right and 2 × 2 = 4 units down.

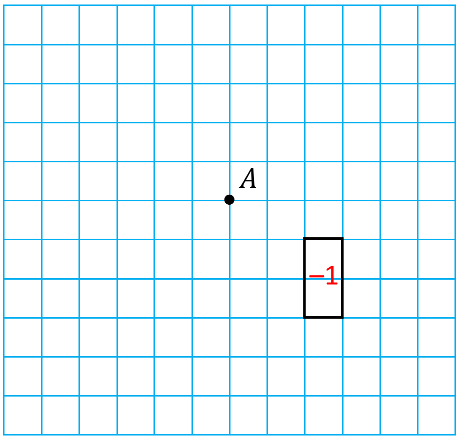
With a scale factor of −2, we must go 4 × 2 = 8 units **left** and 2 × 2 = 4 units **up**.

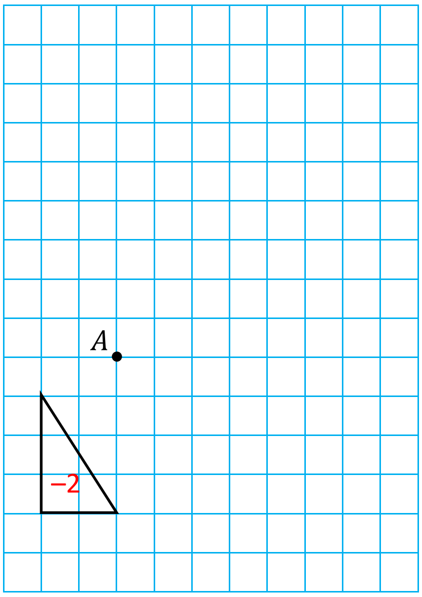


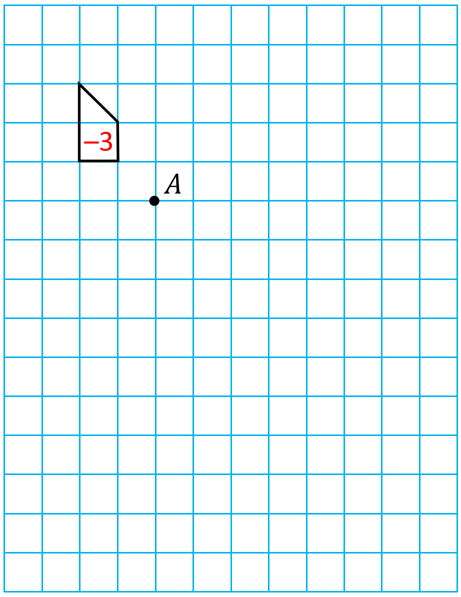
Exercise 15

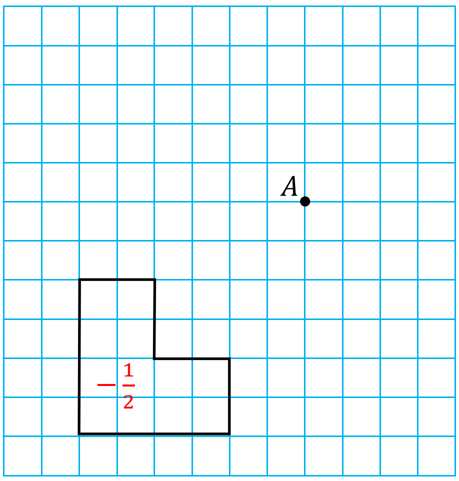
H

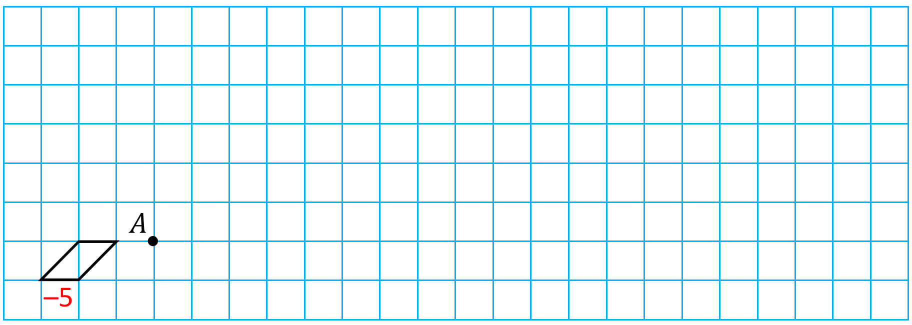
Enlarge the following shapes using the point A as the centre of enlargement and the number in the centre of each shape as the scale factor.









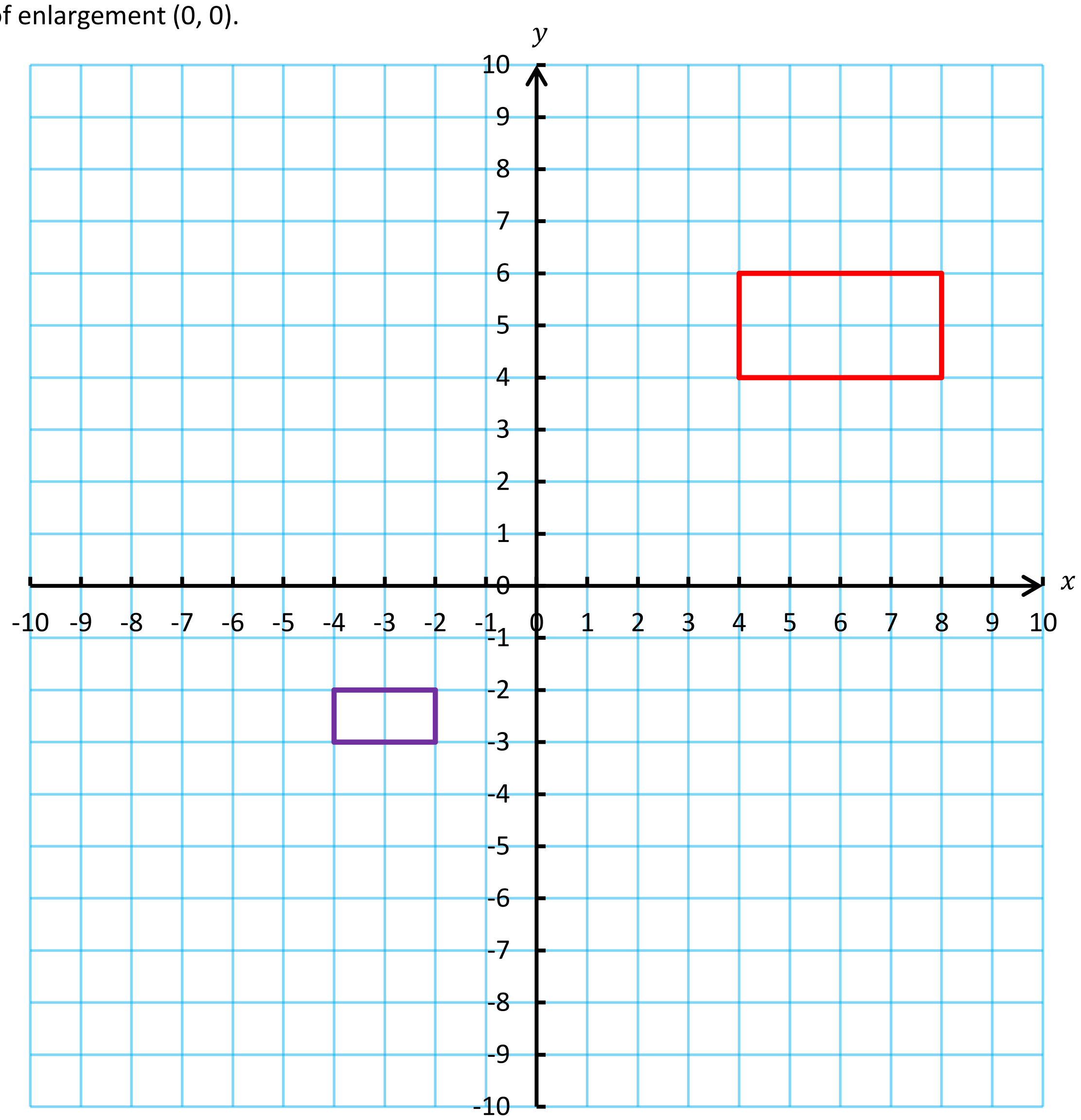


Exercise 16

H

The larger rectangle is transformed to the smaller rectangle. The co-ordinates of the centre of enlargement are (0, 0). Complete the following sentence to fully describe this transformation.

The transformation from the larger rectangle to the smaller rectangle is an enlargement using scale factor _______ and centre of enlargement (0, 0).



Evaluation

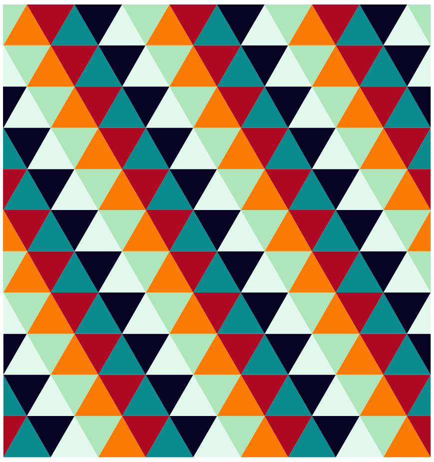
Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

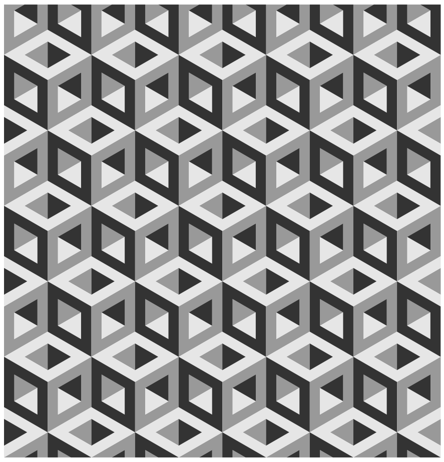
Tessellations

A **tessellation** involves repeating a shape (or a number of shapes) so that they fill the space entirely, without leaving any gaps. You can translate, rotate or reflect shapes to create a tessellation.

Example

The following pictures show examples of tessellations.

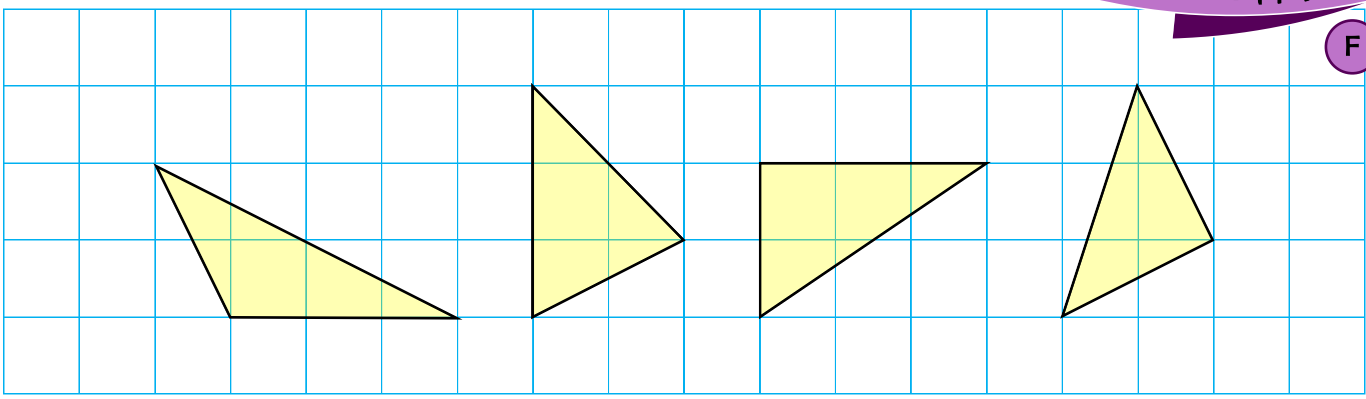




Exercise 17

Applying

F



Choose one of the triangles above. Using the squared paper below, tessellate the triangle to create a tiled pattern. Colour your design, using no more than three colours.

Did you know? The artist M.C. Escher used tessellations in his art.

Challenge!

Extension

Does every triangle tessellate? If not, give an example of a triangle that does not tessellate. If so, try to show how any triangle will tessellate.

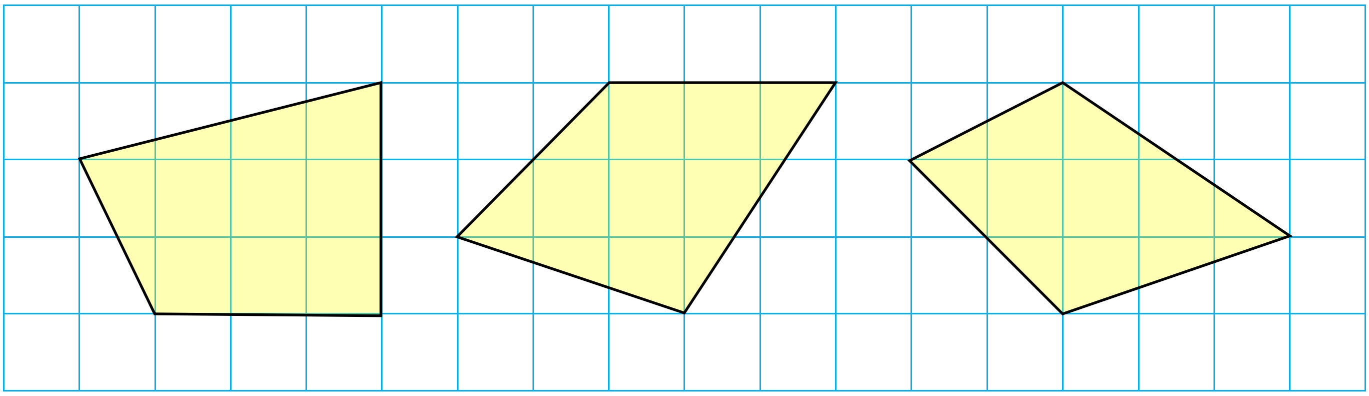
Exercise 18

Applying

F

Investigate different tessellations using the MAT tiles.

Exercise 19



Choose one of the above quadrilaterals. Using the squared paper below, tessellate your quadrilateral to form a tiled pattern. Colour your design, using no more than four colours.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

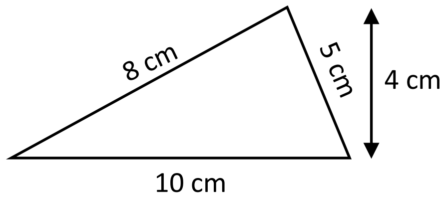
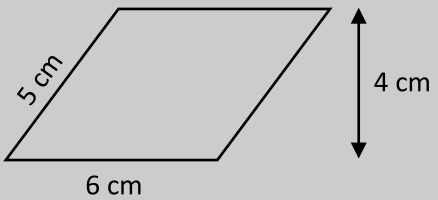
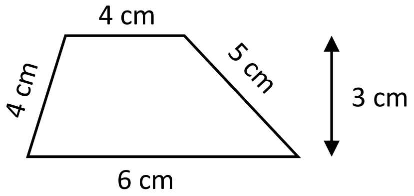
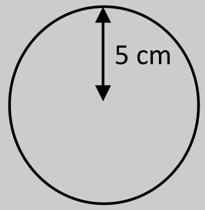
Composite Shapes

Revision

F

Exercise 20

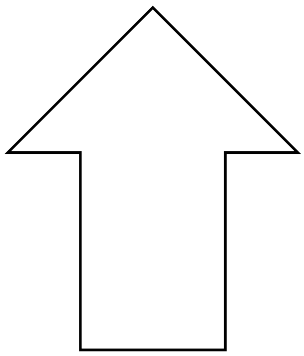
Complete the following table.

Shape	Name of the shape	Formula to find the area of the shape	Calculate the area of the shape
			
			
			
			
			

Exercise 21

Calculate the perimeter of each shape in the above table.

Composite Shapes

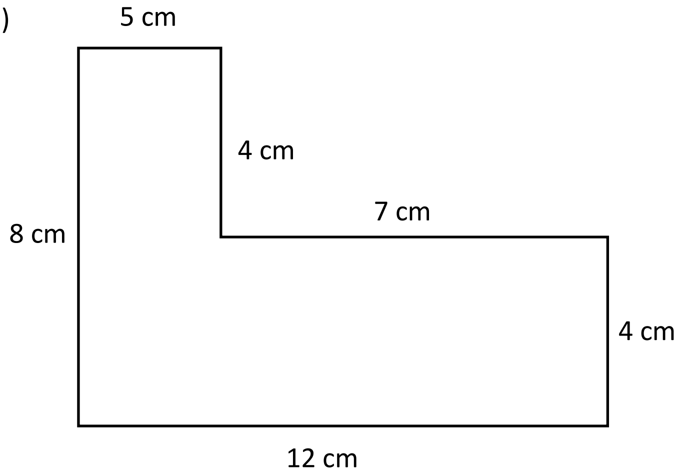


Composite shapes are shapes that you can split into simpler shapes, like the shapes in the above table. For example, the shape on the right is a composite shape – it is possible to split the shape into a rectangle (at the bottom) and a triangle (at the top). We can calculate the area of the composite shape by adding the area of the rectangle to the area of the triangle.

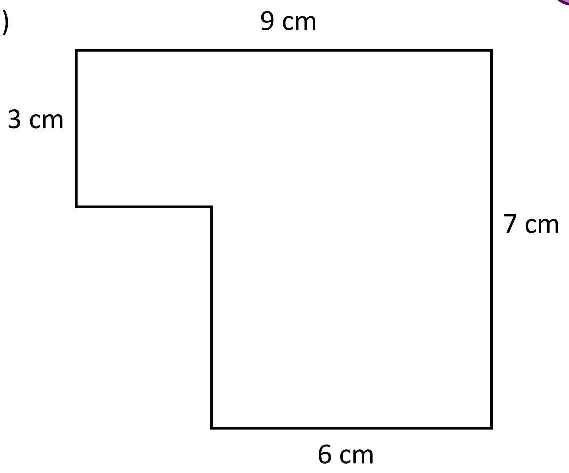
Exercise 22

Calculate the area of each of the following composite shapes.

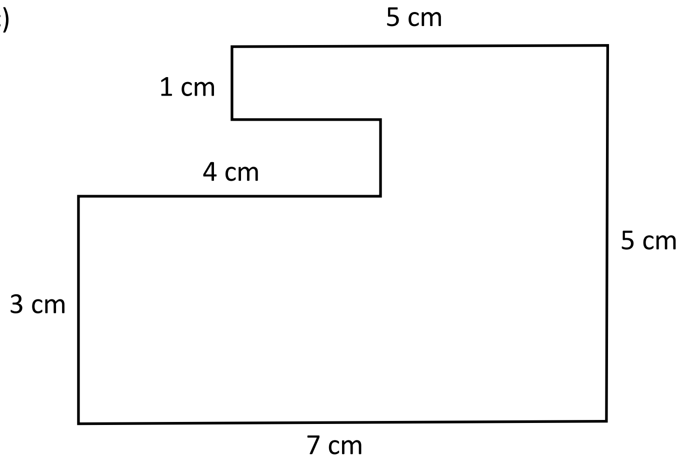
(a)



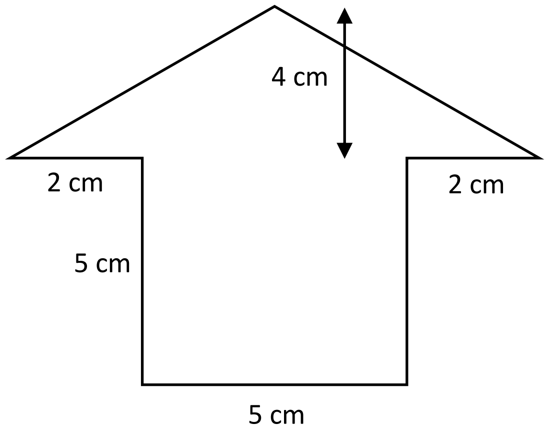
(b)



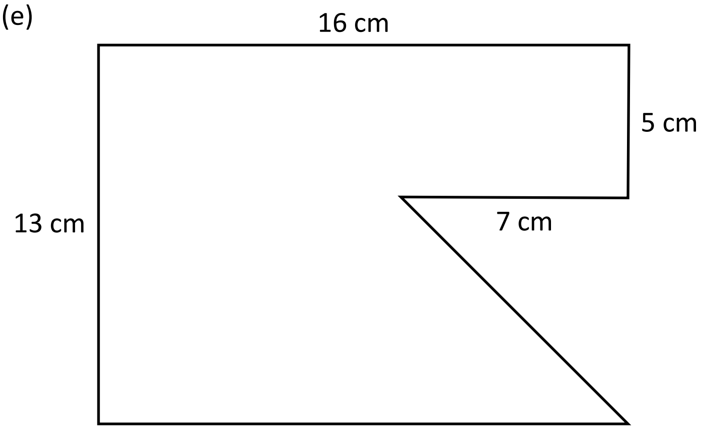
(c)



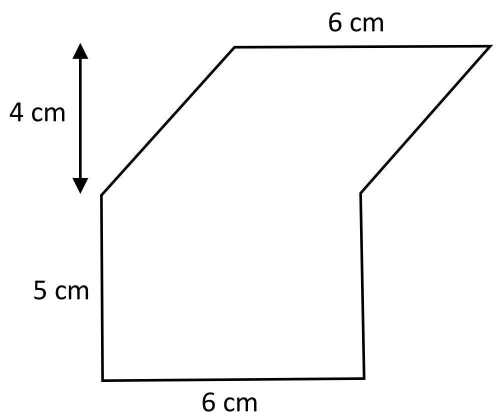
(d)



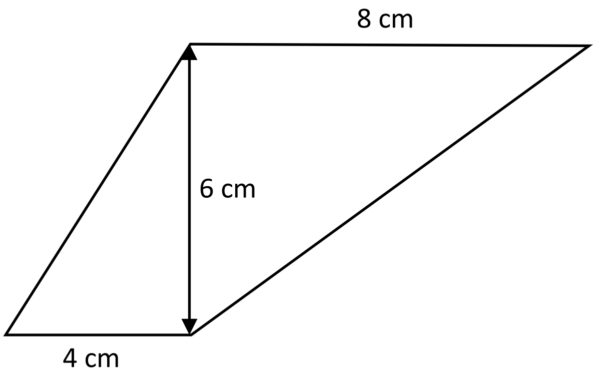
(e)



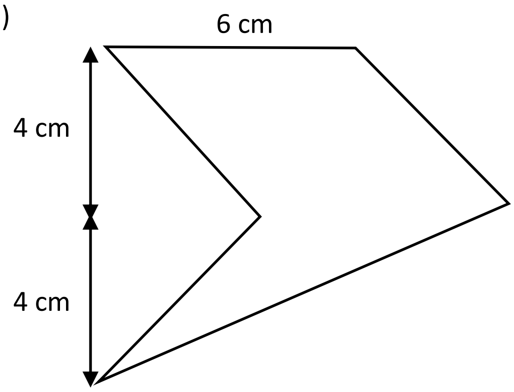
(f)



(g)



(h)

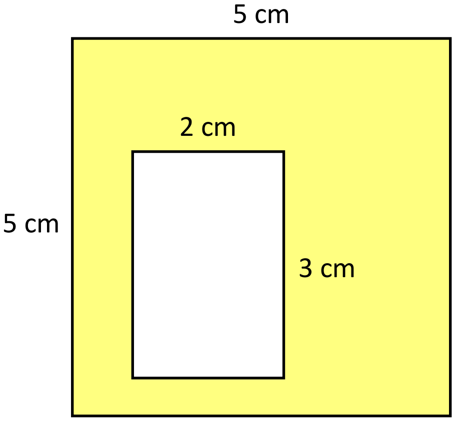


Exercise 23

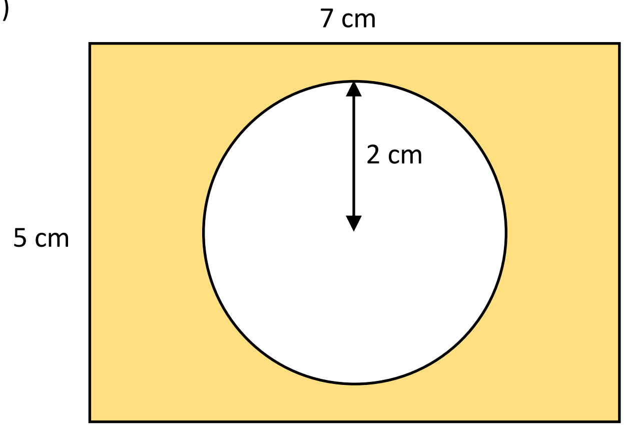
F

Calculate the area of the coloured region.

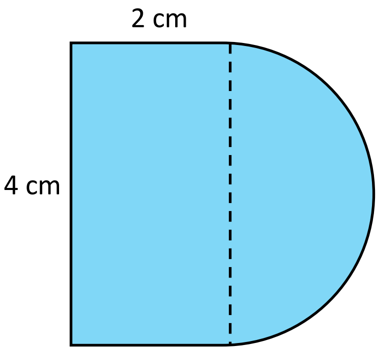
(a)



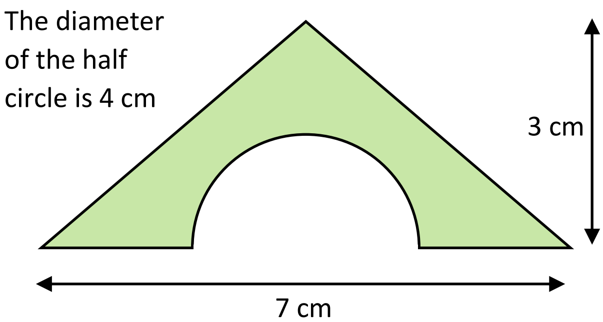
(b)



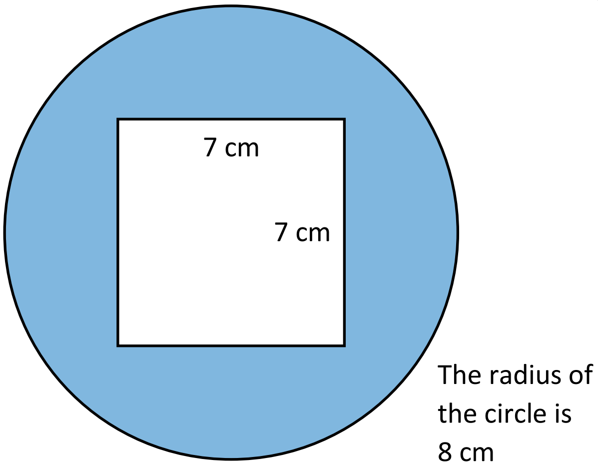
(c)



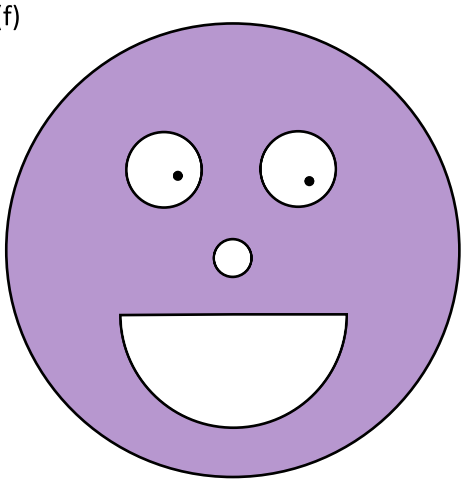
(d)



(e)

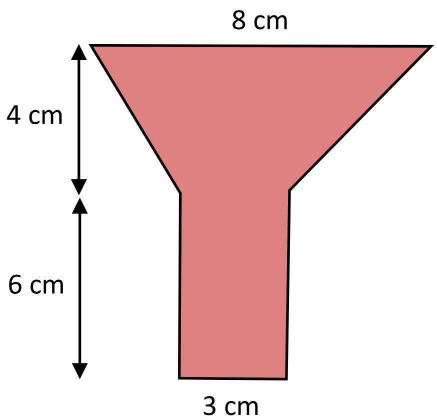


(f)



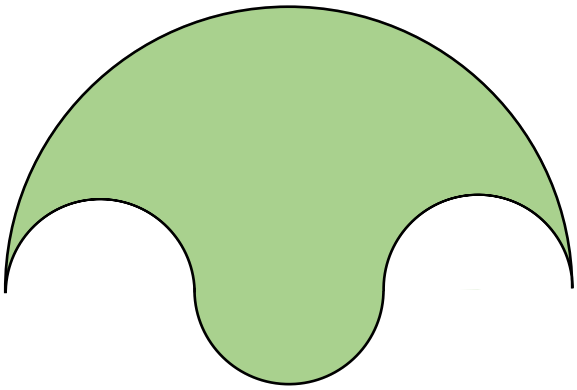
Circle	**Radius**
Face	6 cm
Eye	1 cm
Nose	0.5 cm
Mouth	3 cm

(g)



(h)

Circle	**Radius**
Large (1)	7.5 cm
Small (3)	2.5 cm



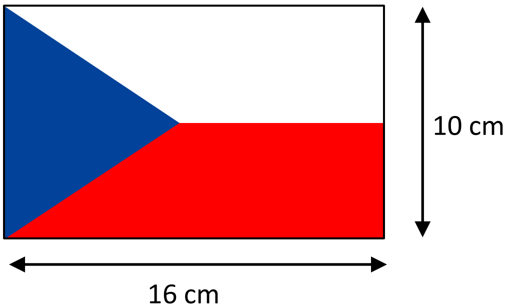
Exercise 24

F

Calculate the area of each colour in the following flags.

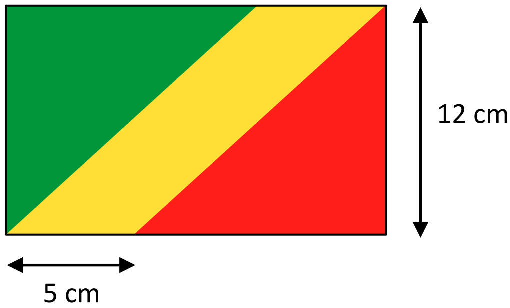
(a) *Czech Republic*

The vertex of the blue triangle is in the centre of the flag.



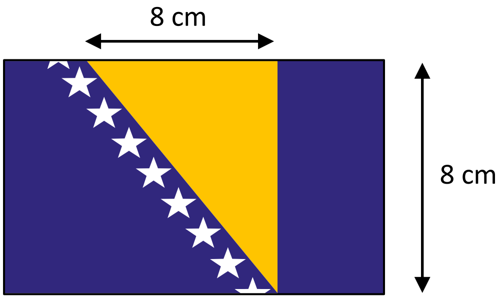
(b) *Republic of the Congo*

The green and red triangles have the same area. The base of the triangle is double the width of the parallelogram.



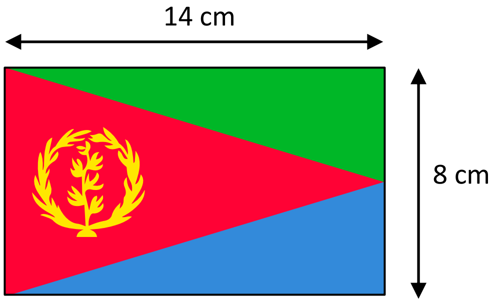
(c) *Bosnia Herzegovina*

The area of each white star is 2 cm². The area of the yellow triangle is $\frac{1}{4}$ of the area of the whole flag.



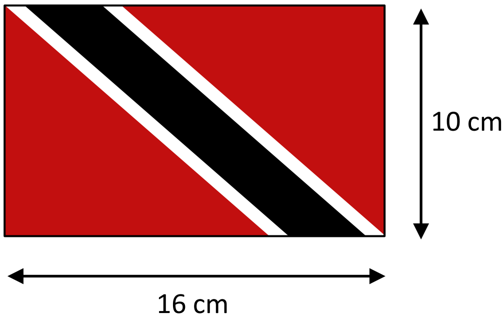
(d) *State of Eritrea*

The green and blue triangles have the same area.
The yellow picture has an area of 12 cm².



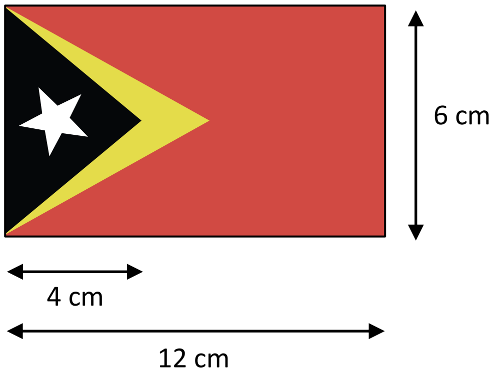
(e) *Republic of Trinidad and Tobago*

The width of the white stripe is 1 cm.
The width of the black stripe is 4 cm.



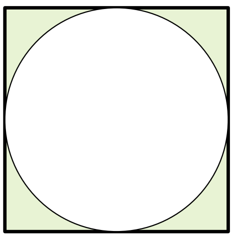
(f) *Democratic Republic of Timor-Leste*

The area of the white star is 4 cm².
The vertex for the yellow triangle is in the centre of the flag.



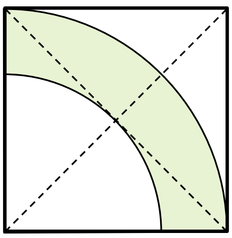
Challenge!

Which fraction of the square is shaded?



Challenge 2!

Which fraction of the square is shaded?

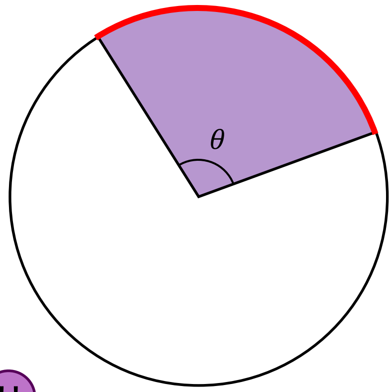


Length of an Arc and the Area of a Sector

Higher Tier

The length of an arc is a fraction of the circumference of a circle, whilst the area of a sector is a fraction of the area of the circle.

Arc Length = $\frac{\theta}{360°} \times$ circle circumference   Area of a Sector = $\frac{\theta}{360°} \times$ circle area



Arc Length = $\frac{\theta}{360°} \times \pi \times$ diameter

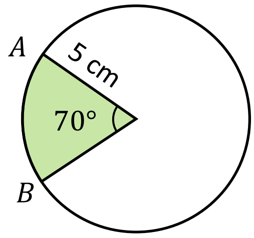
Area of a Sector = $\frac{\theta}{360°} \times \pi \times$ radius²

Exercise 25

H

For the circle shown on the right,

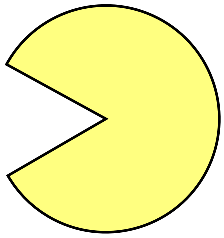
(a) Calculate the length of the minor (*smaller*) arc AB.
(b) Calculate the area of the minor sector AB.
(c) Calculate the length of the major (*larger*) arc AB.
(d) Calculate the area of the major sector AB.
(e) What fraction of the circle is shaded in green? Give your answer in its simplest form.



Exercise 26

Pacman, the computer game character, is in the shape of a sector.
The angle in the centre of the shape is 300°. If the radius of Pacman is 6 cm,

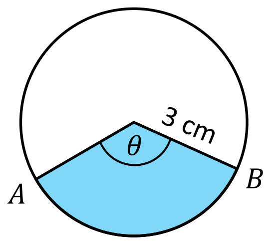
(a) What is the area of Pacman?
(b) What is the perimeter of Pacman?



Exercise 27

The length of the minor arc AB in the diagram on the right is 7 cm.

(a) What is the size of the angle θ?
(b) Calculate the area of the blue sector.



Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Measuring Shapes 3

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	**I need to revise this.**	**Question in the test**	**Correct in the test?**
I can recognise the **opposite**, the **adjacent** and the **hypotenuse** in a right-angled triangle.			1, 2, 11	
I can **calculate the lengths of edges** in a right-angled triangle using **trigonometry**.			1	
I can **calculate the angles** of a right-angled triangle using **trigonometry**.			2, 11	
I can **calculate the lengths of edges** in a right-angled triangle using **Pythagoras' Theorem**.			11	
I can **enlarge** shapes using a scale factor that is a **positive whole number**.			3	
I can **enlarge** shapes using a scale factor that is **fractional**.			5	
I know how to use the **centre of enlargement** whilst enlarging shapes.			4, 5	
Given a shape and its enlargement, I can **find the scale factor** and the **centre of enlargement**.			6	
I can form a **tessellation** by repeating given shapes.			7	
I can calculate the **perimeter of a composite shape**.			8	
I can calculate the **area of a composite shape**.			9, 10	

Am I ready for the test?
Tick the boxes below...

- [ ] I have revised the work in my mathematics book.
- [ ] I have revised the workbook.
- [ ] I have watched the relevant videos on YouTube.
- [ ] I have completed the Diagnostic Questions quiz.
- [ ] I have completed at least 4 pages in my revision book.

Measuring Shapes 3

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test	Correct in the test?
I can recognise the **opposite**, the **adjacent** and the **hypotenuse** in a right-angled triangle.			1, 2	
I can **calculate the lengths of edges** in a right-angled triangle using **trigonometry**.			1	
I can **calculate the angles** of a right-angled triangle using **trigonometry**.			2	
I can **calculate the lengths of edges** in a right-angled triangle using **Pythagoras' Theorem**.				
I can **enlarge** shapes using a scale factor that is a **positive whole number**.			3	
I can **enlarge** shapes using a scale factor that is **fractional**.			5	
I know how to use the **centre of enlargement** whilst enlarging shapes.			4, 5	
Given a shape and its enlargement, I can **find the scale factor** and the **centre of enlargement**.			6	
I can **enlarge** shapes using a scale factor that is **negative**.			11	
I can form a **tessellation** by repeating given shapes.			7	
I can calculate the **perimeter of a composite shape**.			8	
I can calculate the **area of a composite shape**.			9, 10	
I can calculate the **length of an arc** in a circle.			12	
I can calculate the **area of a sector** in a circle.			12	

Am I ready for the test?
Tick the boxes below...

- I have revised the work in my mathematics book. ☐
- I have revised the workbook. ☐
- I have watched the relevant videos on YouTube. ☐
- I have completed the Diagnostic Questions quiz. ☐
- I have completed at least 4 pages in my revision book. ☐

The Mathematics Department

10

Fractions, Percentages and Decimals

Name:

Contents

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Calculating

Mastering the Techniques

When working with fractions, decimals and percentages, we must develop fluency with many techniques.
Try the following questions, before going back and revising the techniques if you got the answer wrong.

Exercise 1

Revision F

Question	Revision	Revision Video	My answer	The correct answer	Fluent in the technique?
(a) 398 + 4829	Year 7 Numeracy Workbook	[001 Rh/S]			
(b) 693 – 246		[002 Rh/S]			
(c) 372 × 68		[004 Rh/S]			
(d) 925 ÷ 37		[006 Rh/S]			
(e) 2.6 × 10	Introducing Percentages Year 7	[059 Rh/S]			
(f) 63 ÷ 100					
(g) 0.0247 × 1000					
(h) 46.27 × 8		[060 Rh/S]			
(i) 29.3 + 2.43		[061 Rh/S]			
(j) 52.6 – 7.84		[062 Rh/S]			
(k) 14.7 ÷ 6	The End of Year 8	[126 Rh/S]			
(l) 0.3 × 0.2	Year 9 Numeracy Workbook	[132 Rh/S]			
(m) 4 ÷ 0.5		[133 Rh/S]			
(n) 50% o 80	Introducing Percentages Year 7	[064 Rh/S]			
(o) 10% o 72					
(p) 36% o £84		[065 Rh/S]			
(q) $\frac{3}{5}$ o £35	Parts of a Number Year 9	[158 Rh/S]			
(r) $1 - \frac{3}{7}$		[159 Rh/S]			
(s) $\frac{5}{9} + \frac{2}{9}$		[161 Rh/S]			
(t) $\frac{5}{9} - \frac{2}{9}$					
(u) $\frac{2}{3} + \frac{3}{5}$		[164 Rh/S] or [165 Rh/S]			
(v) $\frac{2}{3} - \frac{3}{5}$					
(w) $\frac{2}{3} \times \frac{3}{5}$		[140 Rh/S]			
(x) $\frac{2}{3} \div \frac{3}{5}$		[166 Rh/S]			

Exercise 2

Check that you are able to use your calculator to find the correct answers to the questions in Exercise 1.

Exercise 3

Extension F

Use the clues on the cards to solve the problem.

(1) The wage of Number 6 is half the wage of Number 3.

(2) Number 9 has the highest wage in the team.

(3) The wage of Number 10 is $\frac{6}{7}$ of the wage of Number 7.

(4) There's a 15% bonus for every player who scores a goal in a game.

(5) The manager's wage is the mean wage of Number 8 and Number 11.

(6) Number 7 earns £8,000 per day.

(7) There are 11 players in a football team.

(8) If the goalie doesn't conceed a goal in a game, he gets a bonus of £5,000.

(9) The wage of Number 3 is 48% of the wage of Number 11.

(10) There's a bonus of 40% for any player who scores a hat-trick in a game.

(11) This week the team won 2–0.

(12) Your task is to calculate this week's wage for the manager and every player who started the game this week.

(13) Number 5 earns £200 per hour.

(14) Number 11 has the third highest wage in the team.

(15) Number 9's usual wage is double the goalie's usual wage.

(16) Number 4's usual wage is $\frac{5}{6}$ of Number 3's usual wage.

(17) One of the players in the team has the same wage as the goalie's usual wage.

(18) There's a difference of £2,000 between the wages of Number 10 and Number 11.

(19) The team only played one game this week.

(20) Number 2's wage is £6,000 higher than the lowest wage.

(21) The crowd for the recent game was 34,500.

(22) Number 1's wage is $\frac{5}{8}$ of Number 10's usual wage.

(23) The word "wage" means "weekly wage" in this task.

(24) During this week's game, Number 4 and Number 9 scored.

You will need to use the following timetable to answer Exercises 4 and 5…

Summary of North Wales to South Wales train services — Saturdays
Crynodeb o wasanaethau rhwng Gogledd Cymru a De Cymru — Dydd Sadwrn

		◊ ⌶	◊ ⌶	◊ ⌶	R ⌶	◊ ⌶ L	◊ ⌶	◊ ⌶
Holyhead / Caergybi	d	0425	0635	0820	1033	1238	1423	1650
Valley / Y Fali	d	0432x	0641x	0826x	1039x			
Rhosneigr	d			0832x				
Ty Croes	d			0835x				
Bodorgan	d			0840x				
Llanfairpwll	d	0449x	0658x	0849x	1056x			
Bangor	d	0457	0707	0902	1105	1307	1453	1718
Llanfairfechan	d			0909x			1500x	
Penmaenmawr	d			0913x			1504x	
Conwy	d			0919x			1510x	
Llandudno Junction / Cyffordd Llandudno	a	0513	0723	0923	1121	1323	1515	1734
Llandudno Junction / Cyffordd Llandudno	d	0515	0725	0925	1125	1325	1516	1736
Colwyn Bay / Bae Colwyn	d	0521	0731	0931	1131	1331	1522	1742
Abergele & Pensarn	d							
Rhyl	d	0531	0741	0941	1141	1341	1533	1752
Prestatyn	d	0537	0747	0947	1147	1347	1538	1758
Flint / Y Fflint	d	0550	0800	1000	1200	1400	1552	1811
Shotton	d							
Chester / Caer	a	0604	0816	1016	1216	1414	1605	1825
Chester / Caer	d	0612	0819	1019	1219	1419	1619	1829
Wrexham General / Wrecsam Cyffredinol	d	0638	0834	1035	1234	1434	1635	1845
Ruabon / Rhiwabon	d	0645	0841	1042	1241	1441	1642	1852
Chirk / Y Waun	d	0651	0847	1048	1247	1447	1648	1858
Gobowen	d	0657	0853	1054	1253	1453	1654	1904
Shrewsbury / Yr Amwythig	a	0717	0913	1114	1313	1513	1714	1924
Shrewsbury / Yr Amwythig	d	0719	0915	1115	1315	1515	1716	1926
Church Stretton	d				1330			
Craven Arms	d				1338			
Ludlow	d	0745	0942	1141	1344	1541	1742	1951
Leominster	d	0755						
Hereford / Henffordd 7	d	0812	1007	1206	1410	1606	1807	2017
Abergavenny / Y Fenni	d	0835	1030	1229	1432	1629	1830	2040
Pontypool & New Inn / Pontypwl	d	0845	1040	1239	1443	1639	1840	
Cwmbran	d	0850	1045	1244	1448	1644	1845	2052
Newport / Casnewydd	d	0902	1057	1256	1506	1656	1857	2117
Cardiff Central / Caerdydd Canolog 7	a	0924	1115	1317	1526	1708	1915	2142

Notes

a	Arrival time
d	Departure time
x	Service stops on request
7	Recommended connecting time
L	To Maesteg
◊	Seat reservations available
R	Seat reservations recommended
⌶	At seat service

Nodiadau

a	Amser cyrraedd
d	Amser gadael
x	Yn aros ar gais
7	Amser cysylltu argymelledig
L	I Faesteg
◊	Seddi cadw ar gael
R	Argymhellir seddi cadw
⌶	Gwasanaeth troli

Exercise 4

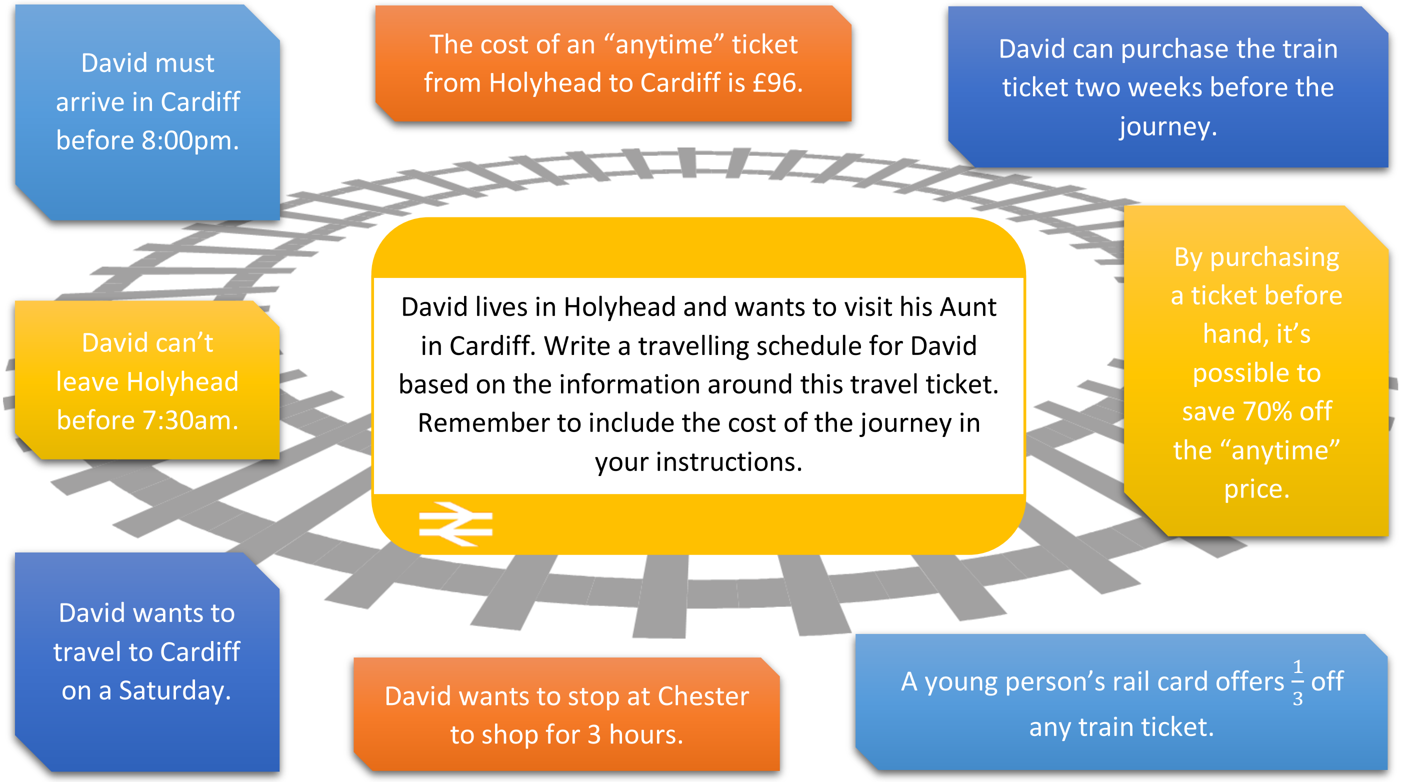
F

Use the timetable on the previous page to answer the following questions.

(a) How much time does the 0425 train from Holyhead take to reach Llandudno Junction?
(b) For how many minutes does the 1423 train from Holyhead wait in Chester?
(c) In minutes, what is the fastest journey between Chester and Shrewsbury?
(d) Out of the seven trains, how many trains stop in Pontypool?
(e) On the train that recommends seat reservations, how long does it take to travel from Prestatyn to Cwmbran?
(f) Which train waits the longest in Llandudno Junction?

Exercise 5

Use the timetable on the previous page to solve the following problem.



Exercise 6

(a) A man gave $\frac{1}{4}$ of the money in his will to his eldest son; $\frac{1}{4}$ of the rest to his second son; and so on until there was only £40,500 left for his only daughter. The value of his will was £128,000. How many sons did he have?

(b) How much is 30% of 40% of 50% of £60?

(c) Gwen's necklace broke. $\frac{1}{3}$ of the beads went on the floor. $\frac{1}{5}$ went down the side of the chair. Gwen found $\frac{1}{6}$ of the beads on the table and her Mum found the rest, which was $\frac{1}{10}$. 12 were still on the string. How many beads were on the string before it broke?

(d) In the sequence $\left(1-\frac{1}{2}\right), \left(\frac{1}{2}-\frac{1}{3}\right), \left(\frac{1}{3}-\frac{1}{4}\right)$, ... what is the sum of the first hundred terms?

(e) I bought a number of apples from a local market. On the way home, I saw William and gave him $\frac{2}{3}$ of the apples. Then I ate one apple before visiting Beth. She had $\frac{3}{4}$ of the remaining apples. I arrived home with only 4 apples. How many apples did I have originally?

F

Exercise 7: Upside-down calculator story: the golf tournament.

Use the yellow and green clues to complete the story.
Key: 1 = I, 2 = Z, 3 = E, 4 = H, 5 = S, 6 = G, 7 = L, 8 = B, 9 = b, 0 = O.
Ignore decimal points in your answers.

It was the final of the 8^3+106 golf tournament. $\frac{202\times4096}{16\times64}$ was in second place with only one $7^4+434\times3+1$ to go. $192\frac{3}{4}\times2\frac{2}{3}$ bitter rival $\sin(30)+0.201$ had already finished, and had 5^5+580 possession of first place. The final $13719616^{0.5}$ was a par 3. A $170683\frac{1}{5}\div5\frac{2}{5}$ would mean that $\sqrt{0.040401}+\frac{1}{2}$ was champion; a birdie would see $9^2-\frac{\frac{1}{10}}{\frac{1}{2}}$ win. On the tee $7!\times\frac{1}{8}+178$'s $7!+(9^3-132)$ were shaking like jelly and the club felt as slippery as an $\frac{38^2}{2}+11$. "Here $\sqrt{281536.36}$ nothing", $(10^2+1)\times8$ thought to himself, before striking the ball $\sqrt{\sqrt[3]{15625}}\times923-1$ and straight onto the green – a great shot!

This year had been one $8^2-3+\frac{\frac{2}{5}}{\frac{1}{2}}$ $\frac{\cos(45)\times12150}{\sqrt{2}}$ for $\frac{16\times101}{9^2}\times40\frac{1}{2}$. $2\times e^0\times17$ hated to $5555+(-2)^{11}$, but this year the feeling had become familiar; $405\frac{3}{4}+402\frac{1}{4}$ had more $2142028\times25\%$ to $\left(\frac{1000}{701}\right)^{-1}$ than he could remember. Today was $\frac{20200\times4}{10^2}$'s final chance to $\frac{7\times2267}{2^2\times5^3}$ the critics and feel the $31610\frac{4}{5}+23567\frac{1}{5}$ of victory. On reaching the green $8+\frac{1}{10}-\frac{2}{100}$ tried not to notice the $3^6\times4+(300-1)$ of the crowd watching on. He could $\tan(45)+2\times167$ that the ball was around 10 feet from the $463\times2\div\frac{1}{4}$, up a slight $6^5-5^4+4^3-3^2+2^1-1^0+507$. "$1032658225^{0.5}$ the day", $\sqrt{2^6\times101^2}$ thought, before putting the ball straight – and into the middle of the $7!-167\times2^3$! Victory at last – $\frac{4\times211}{5^2}$ for $404\times20\%$, and at last a dent to $\sqrt{\frac{1}{2}-0.008599}$'s $\frac{3^2\times7}{4\times25}$!

Challenge!

The name of the triangle on the right is the Leibniz Harmonic Triangle.

What patterns can you see?

Write down the next row in the pattern.

$$\frac{1}{1}$$
$$\frac{1}{2}\quad\frac{1}{2}$$
$$\frac{1}{3}\quad\frac{1}{6}\quad\frac{1}{3}$$
$$\frac{1}{4}\quad\frac{1}{12}\quad\frac{1}{12}\quad\frac{1}{4}$$
$$\frac{1}{5}\quad\frac{1}{20}\quad\frac{1}{30}\quad\frac{1}{20}\quad\frac{1}{5}$$
$$\frac{1}{6}\quad\frac{1}{30}\quad\frac{1}{60}\quad\frac{1}{60}\quad\frac{1}{30}\quad\frac{1}{6}$$

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Recurring Decimals

Changing Fractions to Decimals

If the denominator of a fraction is a power of 10, for example 10, 100 or 1,000, then it is easy to convert the fraction to a decimal.

Example

$\frac{3}{10} = 0.3$ $\frac{12}{100} = 0.12$ $\frac{6}{100} = 0.06$ $\frac{427}{1000} = 0.427$ $\frac{31}{1000} = 0.031$

Exercise 8

Skill F

Convert the following fractions to decimals.

(a) $\frac{7}{10}$ (b) $\frac{5}{10}$ (c) $\frac{17}{100}$ (d) $\frac{7}{100}$ (e) $\frac{95}{100}$ (f) $\frac{1}{10}$ (g) $\frac{93}{100}$

(h) $\frac{731}{1000}$ (i) $\frac{402}{1000}$ (j) $\frac{32}{1000}$ (k) $\frac{86}{1000}$ (l) $\frac{4}{1000}$ (m) $\frac{200}{1000}$ (n) $\frac{760}{1000}$

(o) $\frac{4237}{10000}$ (p) $\frac{29}{10000}$ (q) $\frac{3}{100000}$ (r) $\frac{17}{10}$ (s) $\frac{40}{100}$ (t) $\frac{329}{10}$ (u) $\frac{2053}{100}$

Using Equivalent Fractions

With some fractions, it is possible to find an equivalent fraction whose denominator is a power of 10, which then allows us to change the original fraction to a decimal.

Example

$\frac{3}{5} \xrightarrow{\times 2} \frac{6}{10} = 0.6$ $\frac{11}{25} \xrightarrow{\times 4} \frac{44}{100} = 0.44$ $\frac{143}{200} \xrightarrow{\times 5} \frac{715}{1000} = 0.715$

Exercise 9

Convert the following fractions to decimals.

(a) $\frac{2}{5}$ (b) $\frac{4}{5}$ (c) $\frac{1}{5}$ (d) $\frac{1}{2}$ (e) $\frac{6}{25}$ (f) $\frac{21}{25}$ (g) $\frac{1}{25}$

(h) $\frac{3}{20}$ (i) $\frac{19}{20}$ (j) $\frac{11}{20}$ (k) $\frac{43}{50}$ (l) $\frac{9}{50}$ (m) $\frac{55}{50}$ (n) $\frac{304}{500}$

(o) $\frac{1}{200}$ (p) $\frac{64}{200}$ (q) $\frac{136}{200}$ (r) $\frac{31}{250}$ (s) $\frac{147}{250}$ (t) $\frac{9}{250}$ (u) $\frac{53}{5000}$

Terminating and Recurring Decimals

The fractions we have seen so far convert to **terminating** decimals, which means that the decimals have a **specific number** of digits after the decimal point. For example, the fraction $\frac{11}{25}$ from the above example converted to the decimal 0.44, which is a decimal that has **two digits** appearing after the decimal point.

As we will see on the next page, some fractions convert to **recurring** decimals, which are decimals with an **infinite number** of digits appearing after the decimal point. The decimal $0.222\ldots$ is an example of a recurring decimal, where the dots ... show that the digit 2 repeats forever. In mathematics, we have a special notation to write recurring decimals, which is the **dot notation.** We write the decimal $0.222\ldots$ in dot notation as $0.\dot{2}$, where the dot on top of the digit 2 means the 2 repeats forever.

Example

$0.45222\ldots = 0.45\dot{2}$ $0.434343\ldots = 0.\dot{4}\dot{3}$ $0.243243243\ldots = 0.\dot{2}4\dot{3}$

Exercise 10

Write the following recurring decimals using dot notation.

(a) $0.777\ldots$ (b) $0.444\ldots$ (c) $0.5333\ldots$ (d) $5.222\ldots$ (e) $0.52888\ldots$

(f) $0.737373\ldots$ (g) $0.262626\ldots$ (h) $0.909090\ldots$ (i) $0.2454545\ldots$ (j) $0.0818181\ldots$

(k) $0.265265265\ldots$ (l) $0.405405405\ldots$ (m) $0.5274274274\ldots$ (n) $0.47812812812\ldots$ (o) $0.216737373\ldots$

Converting fractions to recurring decimals

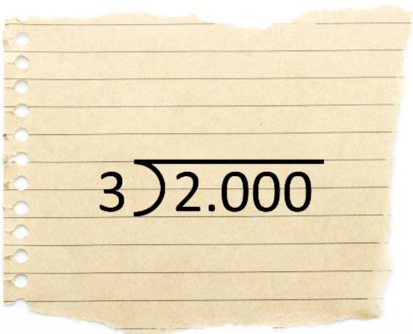
Not all fractions are equivalent to a fraction that has a denominator which is a power of 10. With these fractions we need to use a division frame to convert the fraction to a recurring decimal.

Example

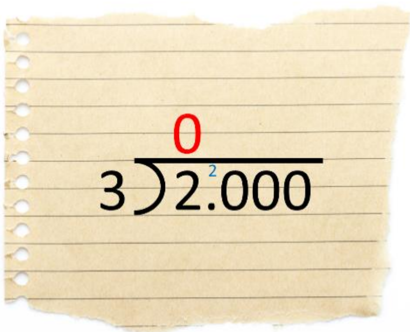
We can add more zeroes to the 2.000, if needed.

Convert the fraction $\frac{2}{3}$ to a recurring decimal.

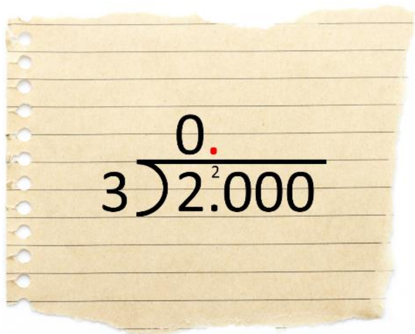
1. Put the sum 2 ÷ 3 into a division frame, writing 2 as the equivalent decimal 2.000.



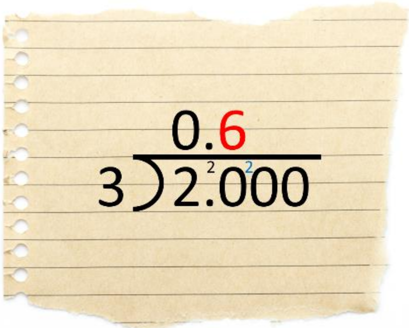
2. "How many times does 3 fit into 2?" It's too large, therefore it fits in 0 times, with 2 left over.



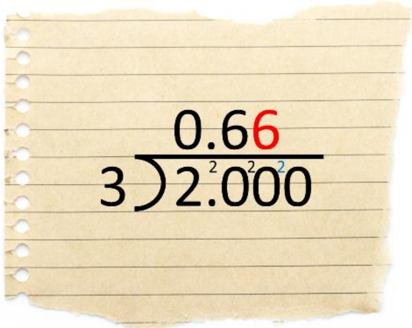
3. Remember to add the decimal point in the correct place.



4. "How many times does 3 fit into 20?" It fits in 6 times, with 2 left over.



5. "How many times does 3 fit into 20"? It fits in 6 times, with 2 left over.



6. We notice the pattern in the calculations, writing … to show that they repeat forever.

Answer: As a decimal, the fraction $\frac{2}{3}$ is equal to $0.666\ldots$ or, in dot notation, $0.\dot{6}$.

Exercise 11

Use a division frame to write the following fractions as recurring decimals, using dot notation.

(a) $\frac{1}{3}$ (b) $\frac{2}{9}$ (c) $\frac{5}{9}$ (d) $\frac{1}{9}$ (e) $\frac{3}{11}$ (f) $\frac{8}{11}$ (g) $\frac{10}{11}$

(h) $\frac{1}{6}$ (i) $\frac{2}{6}$ (j) $\frac{5}{6}$ (k) $\frac{1}{7}$ (l) $\frac{5}{7}$ (m) $\frac{6}{7}$ (n) $\frac{4}{13}$

Exercise 12

F

Use a calculator to check your answers to Exercise 11.

Challenge!

Use a division frame to write the following fractions as recurring decimals.

(a) $\frac{13}{17}$ (b) $\frac{11}{19}$ (c) $\frac{2}{23}$

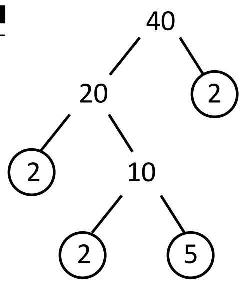
Terminating or recurring decimals?

It is possible to determine whether a fraction converts to a terminating or a recurring decimal by following these steps.

1. **Simplify** the fraction, if possible.
2. Write the denominator of the fraction as a **product of its prime factors.**
3. If the product of prime factors includes the numbers 2 and/or 5 only, then the fraction converts to a **terminating decimal.** If any other prime number appears, then the fraction converts to a **recurring decimal**.

Example

Determine whether the fraction $\frac{46}{80}$ converts to a terminating decimal or to a recurring decimal.



1. We can simplify the fraction by halving: $\frac{46}{80} \xrightarrow{\div 2} \frac{23}{40}$.
2. By using a factor tree (on the right), we see that 40, as a product of its prime factors, is 40 = 2 × 2 × 2 × 5.
3. Since the product of prime factors for 40 includes the prime numbers 2 and 5 only (and no other primes), we can say that the fraction $\frac{46}{80}$ converts to a **terminating** decimal.

Exercise 13

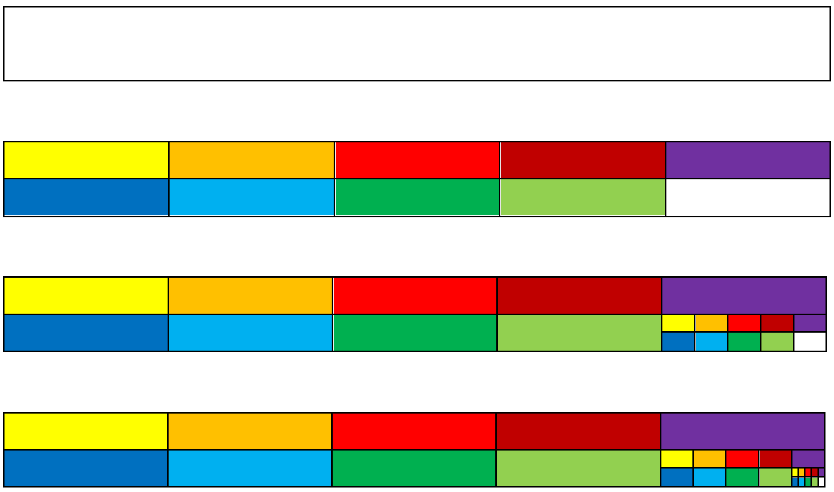
Determine whether the following fractions convert to terminating decimals or to recurring decimals.

(a) $\frac{1}{2}$ (b) $\frac{2}{5}$ (c) $\frac{5}{6}$ (d) $\frac{7}{8}$ (e) $\frac{4}{9}$ (f) $\frac{6}{12}$ (g) $\frac{8}{12}$

(h) $\frac{11}{15}$ (i) $\frac{12}{15}$ (j) $\frac{3}{25}$ (k) $\frac{12}{30}$ (l) $\frac{29}{30}$ (m) $\frac{1}{80}$ (n) $\frac{87}{125}$

Exercise 14

Convert the fractions from Exercise 13 to decimals.



Investigation

The series of images on the right attempts to show that $0.\dot{9} = 1$.

Can you think of a different way to show that $0.\dot{9} = 1$?

Suggestion: Use the decimal for $\frac{1}{3}$.

Converting recurring decimals to fractions

Higher Tier

We can use an algebraic method to convert recurring decimals to fractions.

Example

Convert the recurring decimal $0.5\dot{3}$ to a fraction.

Let the fraction for $0.5\dot{3}$ be represented by the letter a, so that $a = 0.5333\ldots$

In this recurring decimal, one digit is repeating (the digit 3), so we multiply a by 10. (If two digits repeat, we multiply a by 100. If three digits repeat, we multiply a by 1,000. And so on…)

$$\begin{array}{rrl} & 10a = & \overset{4}{\cancel{5}}.\overset{1}{3}333\ldots \\ - & a = & 0.5333\ldots \\ \hline & 9a = & 4.8 \\ \hline \end{array}$$

We subtract the original a from the $10a$. This leaves a terminating decimal since the digits that repeat cancel each other.

After subtracting a from $10a$, we see that $9a$ is equal to the terminating decimal 4.8.

This gives an equation that we can solve to give a value for our original number: $a = \frac{4.8}{9}$. But this is not quite a fraction, as a decimal appears as the numerator. We can deal with this by multiplying the top and bottom of the fraction by 10, to give $a = \frac{48}{90}$. To finish, we can simplify this fraction (by dividing by 6) which gives $a = \frac{8}{15}$. This is the simplest fraction that is equivalent to the recurring decimal $0.5\dot{3}$.

Exercise 15

H

Convert the following recurring decimals to fractions.

(a) $0.\dot{5}$	(b) $0.4\dot{7}$	(c) $0.7\dot{2}$	(d) $0.24\dot{6}$	(e) $0.0\dot{4}$
(f) $0.\dot{3}\dot{6}$	(g) $0.\dot{6}\dot{3}$	(h) $0.\dot{7}\dot{4}$	(i) $0.4\dot{6}\dot{7}$	(j) $0.41\dot{4}\dot{0}$
(k) $0.\dot{3}5\dot{7}$	(l) $0.\dot{7}1\dot{5}$	(m) $0.5\dot{2}4\dot{7}$	(n) $3.\dot{5}$	(o) $0.\dot{4}20\dot{7}$

Exercise 16

Check your answers to Exercise 15 by using the ($\dot{\blacksquare}$) button on your calculator.

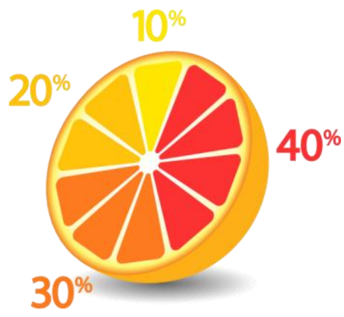
Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Converting

In this chapter, we will discuss how to convert between fractions, decimals and percentages.

Starting with a percentage



Percentage
e.g. 36%

Remove the % symbol.
Divide by 100.

Decimal
e.g. 0.36

Remove the % symbol.
Write as a fraction over 100.
Simplify the fraction.

Fraction
e.g. $\frac{36}{100} \xrightarrow{\div 4} \frac{9}{25}$

Skill F

Exercise 17

Complete the following table.

Fraction	Percentage	Decimal
	27%	
	68%	
	93%	
	4%	
	7%	
	100%	
	150%	
	400%	
	1%	
	0.5%	
	0.2%	
	0.05%	
	3.1%	
	10.2%	

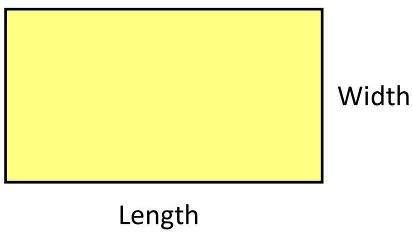
Exercise 18

Circle each number that has the same value as 40%.

0.04 $\frac{4}{10}$ 0.4 $\frac{10}{4}$ 0.40 $\frac{2}{5}$ 4%

Challenge!

If the length of a rectangle increases 10% but the area remains the same, by what fraction must the width be reduced by?



Starting with a fraction

Fraction
e.g. $\frac{2}{5}$

1) Try to find an equivalent fraction which has a denominator that is a power of 10. Not possible? 2) Use a division frame.

Decimal
e.g. 0.4

Multiply by 100. Add a % symbol.

Percentage
e.g. 40%

Exercise 19

F

Complete the following table.

Fraction	Percentage	Decimal
$\frac{87}{100}$		
$\frac{4}{100}$		
$\frac{7}{10}$		
$\frac{3}{5}$		
$\frac{13}{20}$		
$\frac{18}{25}$		
$\frac{184}{500}$		
$\frac{4}{9}$		
$\frac{3}{8}$		
$\frac{3}{2}$		
$\frac{4}{7}$		
$\frac{5}{11}$		
$\frac{12}{40}$		
$\frac{43}{200}$		

Exercise 20

(a) Write 13%, 0.2 and $\frac{3}{25}$ in ascending order.

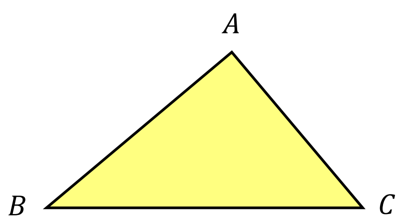
(b) Write $\frac{3}{4}$, 77% and 0.73 in ascending order.

(c) Write $\frac{32}{50}$, 0.63 and 67% in descending order.

(d) In a mathematics test, Rachel scored $\frac{3}{5}$ of the highest possible mark.
Jimmy scored 62% and Susie's mark was 0.58 of the highest possible mark.
State which student scored the most marks and which student scored the least marks.

Challenge!

In a triangle ABC, the angle B is $\frac{3}{4}$ of the angle C and $1\frac{1}{2}$ of the angle A.
What is the size of the angle B?



Starting with a terminating decimal

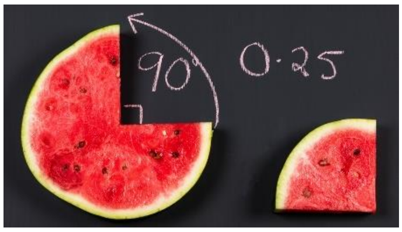
Decimal
e.g. 0.236

Multiply by 100.
Add a % symbol.

Percentage
e.g. 23.6%

Write as a fraction over 10^n, where n is the number of times you must multiply the decimal by 10 for it to become a whole number. Simplify the fraction.

Fraction
e.g. $\frac{236}{1000} \xrightarrow{\div 4} \frac{59}{250}$



F

Exercise 21

Complete the following table.

Fraction	Percentage	Decimal
		0.99
		0.9
		0.5
		0.2
		0.14
		0.08
		0.01
		1.6
		2.4
		12.5
		1.09
		0.452
		1.452
		10.2

Exercise 22

A band hires a concert hall for two nights. They intend to hire the concert hall for a third night, but only if at least 0.9 of the tickets are sold, either for the first night or for the second night.

On the first night, 82% of the tickets were sold.
On the second night, $\frac{3}{4}$ of the tickets were sold.

Did the band hire the hall for the third night? You must show your method and explain how you made your decision.

Challenge!

Find the value of x, where $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots - \frac{1}{1024} = \frac{x}{1024}$.

Starting with a recurring decimal

Higher Tier

Decimal
e.g. $0.2\dot{4}$

Multiply by 100.
Add a % symbol.

Percentage
e.g. $24.\dot{4}\%$

Let a represent the fraction for the decimal. Multiply the decimal by 10^n, where n is the number of digits that repeat. Subtract a from $(10^n)a$ to leave an equation to solve for a.

Fraction
e.g. $\frac{11}{45}$

This is the techinque from the end of the previous chapter.

Exercise 23

H

Complete the following table.

Fraction	Percentage	Decimal
		$0.\dot{2}$
		$0.4\dot{3}$
		$0.\dot{2}\dot{9}$
		$3.\dot{4}\dot{8}$
		$0.\dot{5}2\dot{5}$
		$0.2\dot{4}3\dot{7}$
		$6.\dot{3}2\dot{9}$

Challenge! ⚠

Here is an alternative method to convert the decimal $0.4\dot{7}$ to a fraction. Can you use this method to convert $0.81\dot{4}$ to a fraction?

Step 1: Split into terminating and recurring parts: $0.4\dot{7} = 0.4 + 0.0\dot{7}$

Step 2: Convert the decimals to fractions: $0.4 = \frac{4}{10},\ 0.0\dot{7} = \frac{7}{90}$

Step 3: Add the fractions:
$$\frac{4}{10} + \frac{7}{90} = \frac{36}{90} + \frac{7}{90}$$
$$= \frac{43}{90}$$

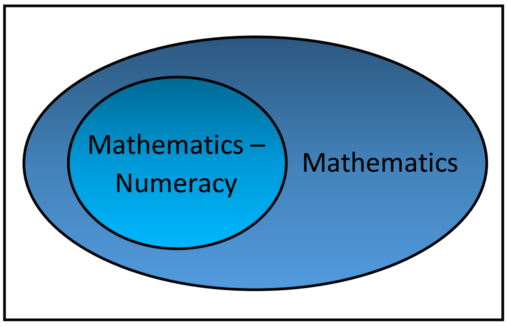
Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Exam Technique

There are two GCSE qualifications available in mathematics:

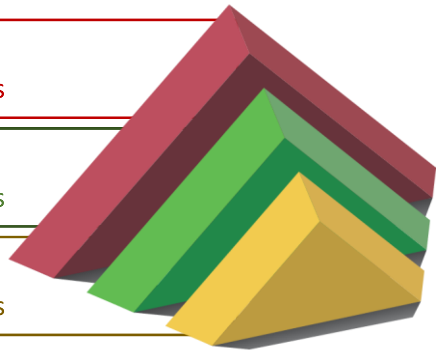
(1) GCSE Mathematics – Numeracy. (2) GCSE Mathematics.



Higher Tier: Grades A*–C
Exam: 1 hour 45 minutes / 80 marks

Intermediate Tier: Grades B–E
Exam: 1 hour 45 minutes / 80 marks

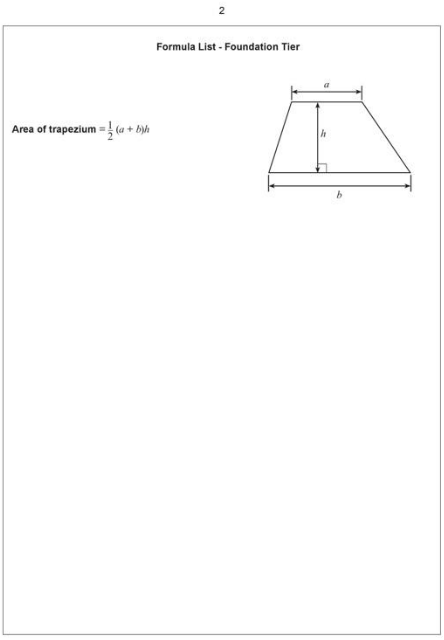
Foundation Tier: Grades D–G
Exam: 1 hour 30 minutes / 65 marks



As you can see in the blue diagram above, the Mathematics – Numeracy qualification is a subset of the Mathematics qualification, which means that the Mathematics exam contains all the topics from the Numeracy exam, and more.

Page 2 of the exam paper always includes the following formulae.

Foundation Tier

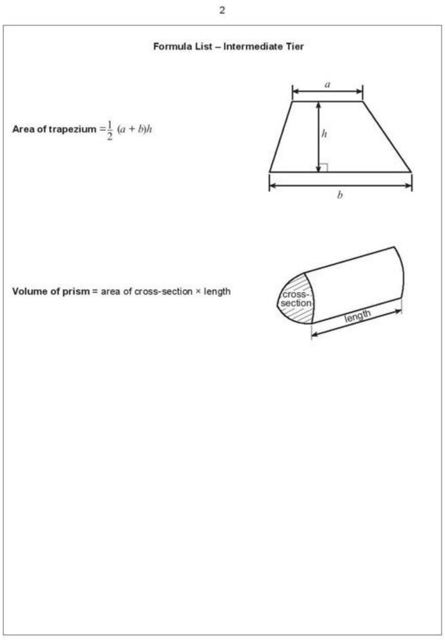


2

Formula List - Foundation Tier

Area of trapezium $= \frac{1}{2}(a+b)h$

Intermediate Tier



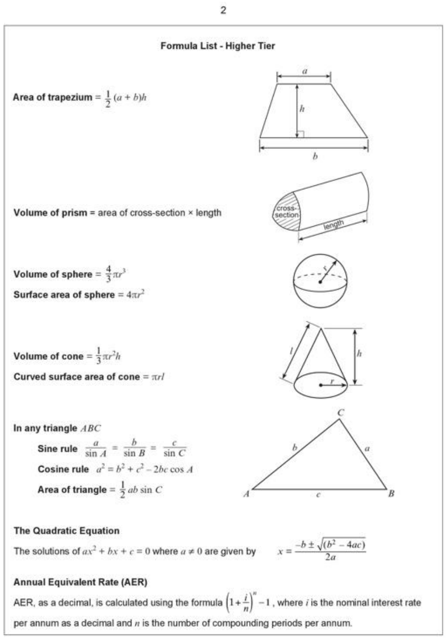
2

Formula List – Intermediate Tier

Area of trapezium $= \frac{1}{2}(a+b)h$

Volume of prism = area of cross-section × length

Higher Tier



2

Formula List - Higher Tier

Area of trapezium $= \frac{1}{2}(a+b)h$

Volume of prism = area of cross-section × length

Volume of sphere $= \frac{4}{3}\pi r^3$
Surface area of sphere $= 4\pi r^2$

Volume of cone $= \frac{1}{3}\pi r^2 h$
Curved surface area of cone $= \pi r l$

In any triangle ABC
Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$
Area of triangle $= \frac{1}{2}ab\sin C$

The Quadratic Equation
The solutions of $ax^2 + bx + c = 0$ where $a \neq 0$ are given by $x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$

Annual Equivalent Rate (AER)
AER, as a decimal, is calculated using the formula $\left(1+\frac{i}{n}\right)^n - 1$, where i is the nominal interest rate per annum as a decimal and n is the number of compounding periods per annum.

The exams are taken in the following order.

(1) Unit 1 GCSE Mathematics – Numeracy (non-calculator).
(2) Unit 2 GCSE Mathematics – Numeracy (with a calculator).
(3) Unit 1 GCSE Mathematics (non-calculator).
(4) Unit 2 GCSE Mathematics (with a calculator).

Summary of exam content.

Assessment Objective	Numeracy	Mathematics
Recall and use their knowledge of the specified content	15% – 25%	50% – 60%
Select and use mathematical methods	50% – 60%	10% – 20%
Interpret and analyse problems and produce strategies to solve them	20% – 30%	25% – 35%

Assessing Written Communication

There is one question in every exam paper where "the assessment will take into account the quality of your linguistic and mathematical organisation, communication and accuracy in writing". This means that you will get marks not just for the right answer, but for **how** you set out your answer.

For **organisation and communication** marks, you will be expected to:

- Present your answer in a structured way.
- Explain to the examiner what you are doing at each step of your answer.
- Set out your explanations and calculations in a way that is clear and logical.
- Write a conclusion that brings together the results and explains the meaning of the answer.

For **accuracy in writing** marks, you will be expected to:

- Show all of your calculation work.
- Make minimal spelling, punctuation and grammar mistakes.
- Use correct mathematical form in your calculations.
- Use appropriate terminology, appropriate units etc.

To make sure you obtain these marks (usually two marks in each examination paper), split the answer page in half, using the **Sum** and **Explanation** headings. On the left-hand side ("Sum"), write down any mathematical calculations you make. On the right-hand side ("Explanation"), explain why you have made that particular calculation. This needs to be done even if the sum is 1 + 1 = 2.

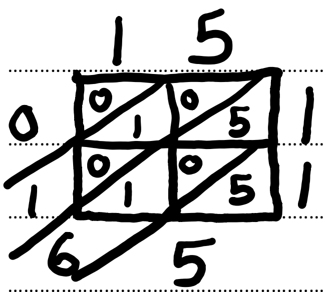
Example

Geraint has a Saturday job to save money for a holiday.
He earns £18 each week.

Geraint saves $\frac{5}{6}$ of the money that he earns each week and spends the rest.

How much money does he save in 11 weeks?
You must show all of your calculations.

Answer:

Sum	Explanation
£18 ÷ 6 = £3	Calculate $\frac{1}{6}$ of the wage
£3 × 5 = £15	Calculate $\frac{5}{6}$ of the wage
	Geraint saves £15 every week.
	11 × £15 = £165, therefore Geraint saves £165 in 11 weeks

Exercise 24 (no calculator allowed)

Applying F

In all of the following questions, **you will be assessed on the quality of your organisation, communication and accuracy in your writing.**

(a) For her birthday, Casey received two gift vouchers for her favourite shop.

She bought a pair of jeans for £26 and two tops at £15.99 each.

Casey used her two gift vouchers to buy the jeans and tops. How much more did she have to pay? You must show all of your calculations.

(b) A band was hired to play in the local hall.

The hall was hired for 4 hours at a rate of £20 per hour.
The band cost £150 to hire.
The price for a ticket to the event was £5 each. 128 tickets were sold.

Calculate how much money was spent, how much money was collected, and the profit or loss made on the event.

(c) A group of 14 Youth Club members want to go to a Theme Park.

Five of the members are 15 years old.
The rest of the members are younger than 15 years old.

How much money does the group save by going to the theme park on a weekend rather than on a weekday?

(d)

Small pot of daffodils
£1.20

Large pot of daffodils
£[redacted]

Marjorie spent £20 on daffodils. She received 50p in change.
She bought 10 small pots of daffodils.
The large pots of daffodils were 25% more expensive to buy than the small pots of daffodils.
How many large pots of daffodils did she buy?
You must show all of your calculations.

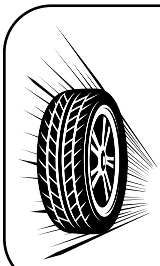
Exercise 25 (calculator allowed)

In all of the following questions, **you will be assessed on the quality of your organisation, communication and accuracy in your writing.** F

(a) Two car rental companies have the following deals available for renting the same model of car.

Just Go Cars

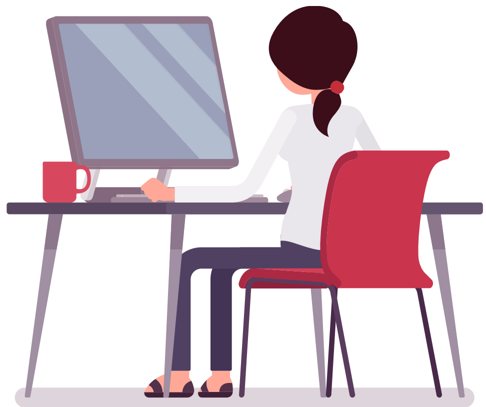
£32 per day
18p per mile
VAT of 20% to be added



Speedy Wheels

£68 per day
No charge for mileage
Includes VAT
Special offer: $\frac{1}{4}$ off

Dylan wants to rent a car for 7 days. He plans to drive 500 miles.
Which company should Dylan choose to get the cheapest bargain?



(b) Bethan's current annual salary is £30,000. After tax and other deductions, she receives 70% of this salary. Over one year, her job means that she travels 8,000 miles. Her car travels at 40 miles per gallon, and a gallon of petrol costs £6.25.

Bethan is considering a new job, working from home. Her new salary would be $\frac{2}{3}$ of her current salary, with the same percentage of deductions.

Calculate the difference, in monetary terms, that changing job would have. You must show all of your calculations.

(c) Here is the cost of buying electricity from *North Electricity.*

- A fixed charge of 28p per day.
- An energy charge of 14p for every kWh used.
- VAT of 5% to pay on the total cost.

Evan uses 850 kWh of electricity over a period of 90 days. Calculate the total bill for Evan to purchase electricity from *North Electricity.*

(d) A paint colour called ochre is made by using a recipe that includes white, red, blue and yellow paint. The percentages of each different colour used to create ochre paint is shown on the right.

Catrin has already bought 2.5 litres of blue paint.
She decides to buy white, red and yellow paint to use with **all** of her blue paint to make as much ochre paint as possible.

The sizes of paint tins available are 1 litre, 2.5 litres and 10 litres.

Only full tins of paint can be purchased.
Catrin only has a small shed to store the paint, therefore she wants minimal white, red and yellow paint left over.

Calculate how much of each paint colour Catrin needs to buy. You must show all of your calculations.

Mark Schemes

After you complete your exam, the paper is sent away to be marked. The markers follow a mark scheme for the exam paper, which uses the following codes for the marks.

Code	Explanation
M	'M' marks are awarded for any correct method applied to appropriate working, even though a numerical error may be involved. Once earned they cannot be lost.
m	'm' marks are dependent method marks. They are only given if the relevant previous 'M' mark has been earned.
A	'A' marks are given for a numerically correct stage, for a correct result or for an answer lying within a specified range. They are only given if the relevant M/m mark has been earned either explicitly or by inference from the correct answer.
B	'B' marks are independent of method and are usually awarded for an accurate result or statement.
S	'S' marks are awarded for strategy.
E	'E' marks are awarded for explanation.
U	'U' marks are awarded for units.
P	'P' marks are awarded for plotting points.
C	'C' marks are awarded for drawing curves.

Exercise 26 F

Here is a past paper question together with its mark scheme. Mark the four attempts using the mark scheme.

Question

Megan has some blocks. 10% of the blocks are white. $\frac{3}{5}$ of the blocks are red. The remaining blocks are green. 33 blocks are green. How many blocks are there in total? [5 marks]

Mark Scheme

White 10% Red 60% OR 0.6 OR $\frac{6}{10}$	B1	
Green 100% – 10% – 60% = 30% OR 0.3 OR $\frac{3}{10}$	M1 A1	Must use a common measure, i.e. percentages or fractions or decimals.
30% of the blocks is 33 (blocks). 100% of the blocks is 110 (blocks).	M1 A1	Follow through 'their 30%'.

Candidate 1

Red $\frac{3}{5} \xrightarrow{\times 2} \frac{6}{10} \xrightarrow{\times 10} \frac{60}{100} = 60\%$

Green 100% − 10% − 60%
= 30%

33 blocks is 30% $\div 3$
11 blocks is 10% $\times 10$
110 blocks is 100%

Candidate 2

$5 \overline{)3.{}^{3}0}$ = 0.6 ← Red

Green 30%

33 blocks is 30% $\div 3$
11 blocks is 10% $\times 10$
110 blocks is 100%

Candidate 3

White $\frac{1}{10}$ Red $\frac{3}{5} \xrightarrow{\times 2} \frac{6}{10}$

Green $1 - \frac{7}{10} = \frac{3}{10}$

$33 \times 3 = 99$

$99 \div 10 = 9.9$

10 blocks in total

Candidate 4

$10\% + \frac{3}{5} = \frac{4}{5}$

Green $1 - \frac{4}{5} = \frac{1}{5}$

$= 20\%$

33 blocks is 20%) ×5

165 blocks is 100%

$$\begin{array}{r} 33 \\ \times\ 5 \\ \hline 165 \\ \hline {}_{1} \end{array}$$

Timing

It's a good idea to complete any maths exam paper at a rate of **one mark per minute.** This way, you will have time left over at the end to check your answers.

Foundation Tier	Intermediate and Higher Tier
65 marks in 1 hour 30 minutes, therefore 65 marks in 90 minutes. Working at a rate of one mark per minute, there will be 25 minutes left over at the end to check your answers.	80 marks in 1 hour 45 minutes, therefore 80 marks in 105 minutes. Working at a rate of one mark per minute, there will be 25 minutes left over at the end to check your answers.

What should you do if you have time left over at the end of an exam?

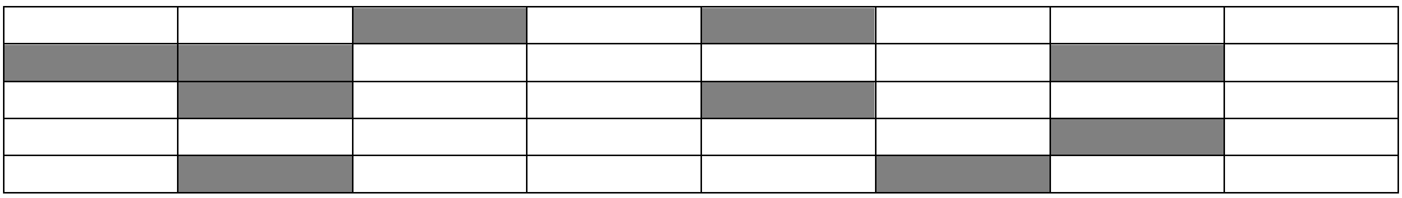
- Check there are no gaps in your exam paper.
- Make sure you have included the correct units, e.g. £, cm², ml.
- Make sure you have shown enough method.
- Make sure you have used the correct equipment, e.g. a ruler for drawing diagrams.
- Attempt to redo the question, perhaps using a different method, to check your answer.

Exercise 27

F

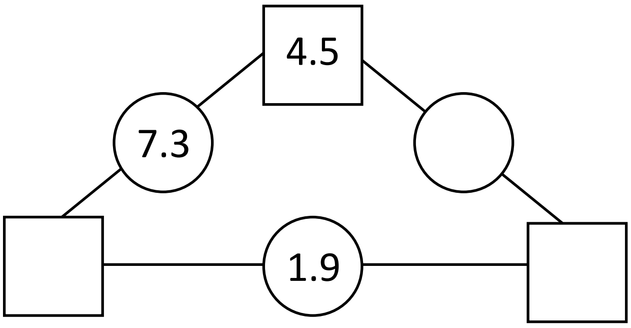
The following six questions are worth a total of 20 marks. Attempt to complete these is **only 25 minutes**. Mark the questions using the mark scheme provided by your teacher.

(1) What fraction of the following shape is shaded? Give your answer in its simplest form. [2]



(2) *You will be assessed on the quality of your organisation, communication and accuracy in writing in this question.* [5]
Gwilym bought a camera for £150.
When he sold the camera, he made a loss of 6%.
How much did Gwilym sell his camera for?

(3) The number in the circle is equal to the two numbers in the squares on either side of the circle. [3]
Fill in the missing numbers.



(4) Evaluate $\frac{3}{8} + \frac{1}{2}$. [2]

(5) New pylons are needed in an area in Wales. [3]

- The pylons are in a straight line.
- The distance between the first pylon and the last pylon is 9 km.
- The pylons must be 0.5 km apart from each other.

How many pylons, in total, are needed for a 9 km piece of land?

(6) Maria sells ribbon. [5]

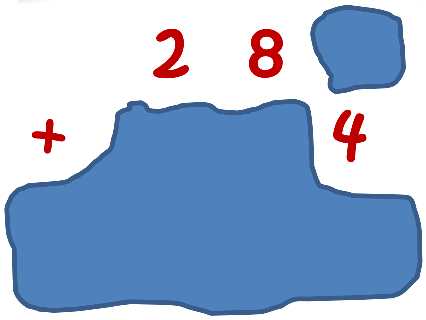
The length of the ribbon she has is 400 cm.
Maria cuts off 30% of the ribbon and sells this piece to a customer.
She uses $\frac{2}{5}$ of the **leftover** ribbon to decorate a card.
Then, Maria cuts the leftover ribbon into three equal pieces.
What is the length of each of the three pieces that are left over?

Challenge!

Ink is spilled onto a sum in a textbook.

All of the digits from 0 to 9 have been used in the sum.

Find the correct locations for the other 7 digits.



Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Fractions, Percentages and Decimals

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test	Correct in the test?
I can **calculate using fractions**, e.g. $\frac{3}{11}+\frac{5}{11}$, $\frac{7}{9}-\frac{1}{3}$, $\frac{2}{3}\times\frac{5}{7}$, $\frac{4}{5}\div\frac{2}{9}$.			1	
I can **calculate using percentages**, e.g. 10% of £75, 45% of 140.			5	
I can **calculate using decimals**, e.g. 6.5 + 2.79, 6 – 4.31, 0.4 × 0.3, 14.4 ÷ 6.			1	
I know how to use the **dot notation** for recurring decimals.			2	
I know how to decide whether a fraction is equivalent to a **terminating decimal** or to a **recurring decimal**.				
I can convert a **percentage** to a **decimal**.			3, 4	
I can convert a **percentage** to a **fraction**.			3, 4	
I can convert a **fraction** to a **decimal**.			2, 3, 4	
I can convert a **fraction** to a **percentage**.			3, 4	
I can convert a **decimal** to a **percentage**.			3, 4	
I can convert a **terminating decimal** to a **fraction**.			3, 4	
I know how to answer a question where you are "**assessed on the quality of your organisation, communication and accuracy in your writing**".			6, 7	

Am I ready for the test?

Tick the boxes below...

I have revised the work in my mathematics book.	☐
I have revised the workbook.	☐
I have watched the relevant videos on YouTube.	☐
I have completed the Diagnostic Questions quiz.	☐
I have completed at least 4 pages in my revision book.	☐

Fractions, Percentages and Decimals

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test	Correct in the test?
I can **calculate using fractions**, e.g. $\frac{3}{11}+\frac{5}{11}$, $\frac{7}{9}-\frac{1}{3}$, $\frac{2}{3}\times\frac{5}{7}$, $\frac{4}{5}\div\frac{2}{9}$.			1	
I can **calculate using percentages**, e.g. 10% of £75, 45% of 140.			5	
I can **calculate using decimals**, e.g. 6.5 + 2.79, 6 – 4.31, 0.4 × 0.3, 14.4 ÷ 6.			1	
I know how to use the **dot notation** for recurring decimals.			2	
I know how to decide whether a fraction is equivalent to a **terminating decimal** or to a **recurring decimal**.				
I can convert a **percentage** to a **decimal**.			3, 4	
I can convert a **percentage** to a **fraction**.			3, 4	
I can convert a **fraction** to a **decimal**.			2, 3, 4	
I can convert a **fraction** to a **percentage**.			3, 4	
I can convert a **decimal** to a **percentage**.			3, 4	
I can convert a **terminating decimal** to a **fraction**.			3, 4	
I can convert a **recurring decimal** to a **fraction**.			7	
I know how to answer a question where you are "**assessed on the quality of your organisation, communication and accuracy in your writing**".			6	

Am I ready for the test?
Tick the boxes below...

I have revised the work in my mathematics book.	☐
I have revised the workbook.	☐
I have watched the relevant videos on YouTube.	☐
I have completed the Diagnostic Questions quiz.	☐
I have completed at least 4 pages in my revision book.	☐

Grade after the examination: _____
Target Grade: _________
Tracking Sheet
Year 10
Tier: _________
Sheet 2
The Examination
%
Developing Algebra 2
%
Measuring Solids
%
Accuracy of Measurements
%

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The Mathematics Department

10

Developing Algebra 2

Name:

Contents

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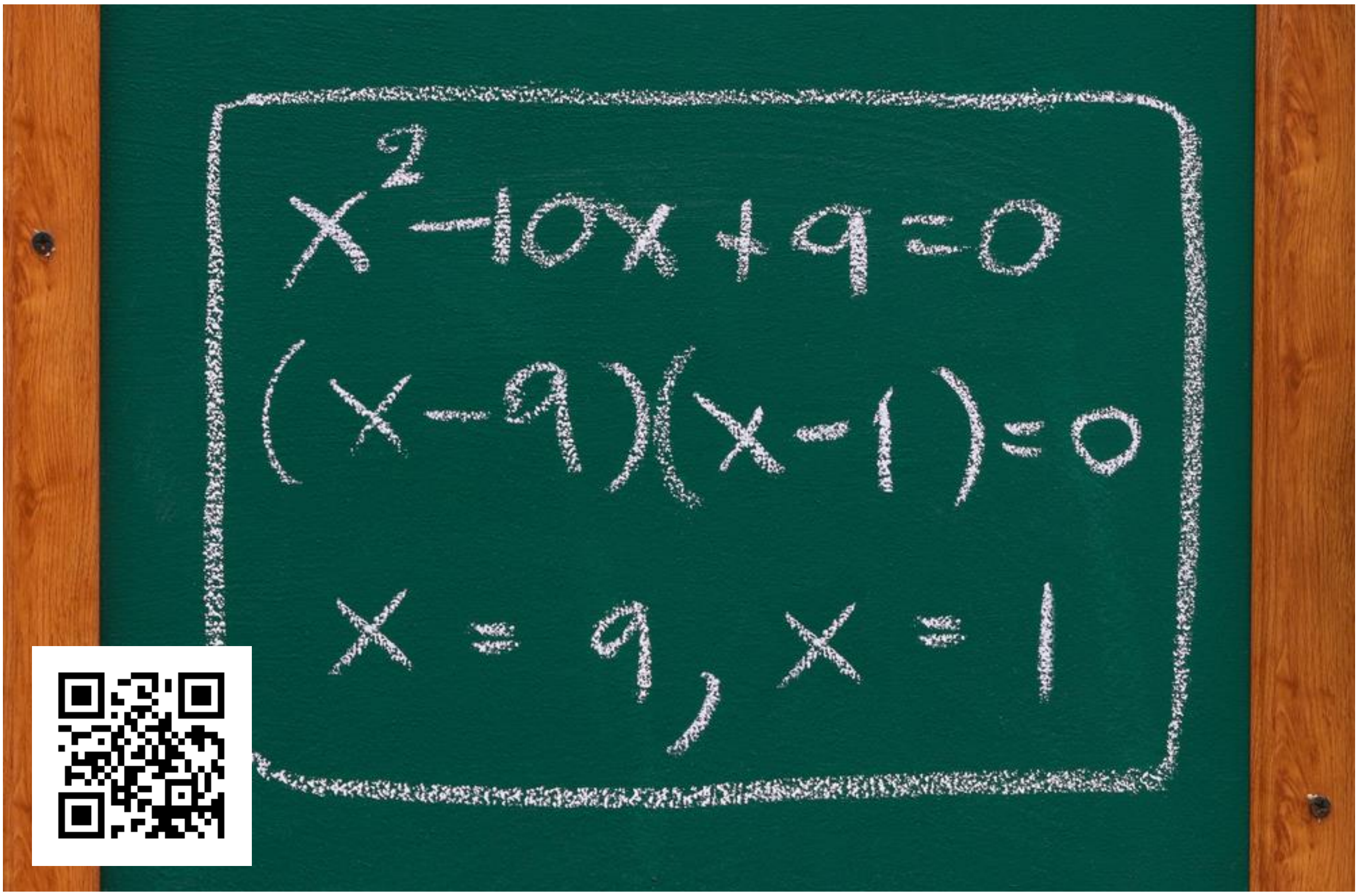


Simple Factorising

Reversing Expansion

Like subtraction reverses addition, and division reverses multiplication, **factorising** reverses **expanding brackets,** a technique seen previously in the **Developing Algebra** workbook.

Revision

F

Exercise 1

Expand the following algebraic expressions.

(a) $2(x+3)$	(b) $5(x+3)$	(c) $5(x-3)$
(d) $5(3-x)$	(e) $2(y+3)$	(f) $2(x+3+y)$
(g) $x(x+3)$	(h) $x(3+x)$	(i) $2x(x+3)$
(j) $5x(x+3)$	(k) $5x(2x+3)$	(l) $2x(5x-3)$
(m) $4(x-7)$	(n) $y(y+9)$	(o) $2z(z+4)$
(p) $7x(3x+4)$	(q) $-7x(3x+4)$	(r) $-7x(3x-4)$

Expanding gets rid of brackets, whereas factorising **reintroduces brackets.** Also, whilst expanding using multiplication sums, factorising uses **division** sums. To this intent, an important skill when factorising is to recognise the **highest common factor** of a set of numbers, which is the largest number that divides into the list of numbers.

Exercise 2

What is the highest common factor of the following numbers?

(a) 6 and 8	(b) 12 and 15	(c) 20 and 30	(d) 20 and 40	(e) 18 and 24
(f) 16 and 40	(g) 22 and 33	(h) 24 and 36	(i) 35 and 56	(j) 36 and 54
(k) 12, 14 and 16	(l) 12, 16 and 20	(m) 25, 35 and 45	(n) 30, 45 and 60	(o) 7, 11 and 13

In order to factorise an algebraic expression such as $12x+18$, we start by considering the highest common factor of the terms $12x$ and 18 in the expression.

Example

Factorise $12x+18$.

1. "What is the highest common factor of $12x$ and 18?" The answer is **6**, therefore we write 6 followed by a pair of brackets.

$$12x+18$$
$$=6(\qquad\quad)$$

2. "6 multiplied by what gives $12x$?" 6 multiplied by $\mathbf{2x}$ is $12x$, therefore we write $2x$ inside the brackets.

$$12x+18$$
$$=6(2x\qquad)$$

3. "6 multiplied by what gives 18?" 6 multiplied by **3** gives 18, therefore we write $+3$ inside the brackets.

$$12x+18$$
$$=6(2x+3)$$

Exercise 3

Factorise the following algebraic expressions.

(a) $4x+6$ (b) $6+4x$ (c) $4x-6$

(d) $6-4x$ (e) $4x+8$ (f) $4z+8$

(g) $6x+8$ (h) $6x+12$ (i) $12-6x$

(j) $9x+21$ (k) $25x+30$ (l) $18x+30$

(m) $14x+21$ (n) $28+14x$ (o) $30x-40$

(p) $24y+36$ (q) $60x+80$ (r) $36x+45$

(s) $36x+54$ (t) $33y-55$ (u) $33y-66$

(v) $45+30x$ (w) $300x+500$ (x) $80z-240$

(y) $2x+4y+6$ (z) $5x+10y+20$ (α) $12x-20y+24$

Not all algebraic expressions can be factorised. For example, $5x+7$ cannot be factorised since the highest common factor of $5x$ and 7 is 1. (We don't factorise $5x+7$ as $1(5x+7)$.)

Exercise 4

Factorise all of the algebraic expressions that do factorise, and note which expressions do not factorise.

(a) $5x+10$ (b) $5x+11$ (c) $5x+5$

(d) $16x$ (e) $16x+2$ (f) $16x+9$

(g) $8y-12$ (h) $8y-13$ (i) $8y-14$

When factorising, it's not just numbers that can appear before the first bracket – we can include variables like x too.

Example

Factorise $6x^2+14x$.

1. "What is the highest common factor of $6x^2$ and $14x$?" The answer is $\mathbf{2x}$, therefore we write $2x$ followed by a pair of brackets.

$$6x^2+14x = 2x(\qquad)$$

2. "$2x$ multiplied by what gives $6x^2$?" $2x$ multiplied by $\mathbf{3x}$ is $6x^2$, therefore we write $3x$ inside the brackets.

$$6x^2+14x = 2x(3x\qquad)$$

3. "$2x$ multiplied by what gives $14x$?" $2x$ multiplied by $\mathbf{7}$ gives $14x$, therefore we write $+\,7$ inside the brackets.

$$6x^2+14x = 2x(3x+7)$$

Exercise 5

What is the highest common factor of the following terms?

(a) $4x^2$ and $18x$ (b) $4x^2$ and $16x$ (c) $5x^2$ and $20x$ (d) $5x^2$ and $7x$ (e) $18x^2$ and $24x$

(f) x^2 and x (g) x^3 and x^2 (h) $4x^3$ and $18x^2$ (i) $9x^4$ and $3x^2$ (j) $12x^4$ and $15x^3$

Exercise 6

I

Factorise the following algebraic expressions.

(a) $3x^2 + 6x$ (b) $6x + 3x^2$ (c) $3x^2 - 6x$

(d) $x^2 + x$ (e) $x^3 + x$ (f) $x^3 + x^2$

(g) $4x^2 + 2x$ (h) $2x^3 - 6x$ (i) $2x^3 + 8x^2$

(j) $12x^2 + 4x$ (k) $4x - 12x^2$ (l) $10x^2 + 15x$

(m) $6x^4 + 9x^2$ (n) $24y^3 - 16$ (o) $21z^2 + 14z$

(p) $3x^2 + 5x$ (q) $7y - 11y^2$ (r) $4z^3 + 17z^2$

(s) $22x^2 + 33x^5$ (t) $12n - 4n^2$ (u) $2a + a^2$

(v) $x^6 + 6x$ (w) $x^6 + 6x^2$ (x) $6x^6 + 4x^4$

(y) $2x^3 + 4x^2 + 6x$ (z) $2x^2 + 4x + 6$ (α) $42x^4 - 30x^2 + 12x^3$

Exercise 7

What is the highest common factor of the following terms?

(a) x^2 and xy (b) $4x^2$ and $6xy$ (c) $12x^2y$ and $16xy$ (d) $5x^2y^2$ and $9y^2$ (e) $18x^3y$ and $12x^2y$

Example

$2x^2y + 4x = 2x(xy + 2)$ $30y^3z^2 - 24y^2z = 6y^2z(5yz - 4)$

Exercise 8

Factorise the following algebraic expressions.

(a) $xy + x$ (b) $4xy + 10y$ (c) $x^2y + 5xy$

(d) $2x^2y + 6xy$ (e) $10yz + 5yz^2$ (f) $12x^2y - 4x$

(g) $x^2 + xy^2$ (h) $16x^3z - 12z^2$ (i) $15x^4y + 25x^2y^2$

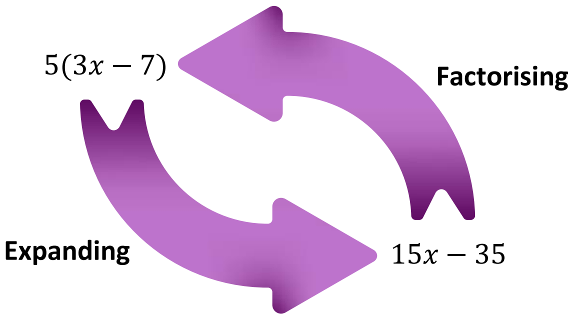
(j) $x^2yz + 4xyz$ (k) $8ab^2c^3 - 18abc^2$ (l) $26\pi x^2 + 65\pi x$

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Factorising Quadratic Expressions

As we saw in the previous chapter, factorising and expanding are two sides of the same coin.



Our aim in this chapter is to develop our understanding of factorising to be able to **factorise quadratic expressions.**

Preparation

To be able to factorise quadratic expressions, we must develop the following skill: given two numbers, for example 7 and 10, is it possible to find two numbers that **add to make 7** and **multiply to make 10**?

$$\square + \square = 7$$

$$\square \times \square = 10$$

In this case, the numbers we are looking for are **2 and 5**, since 2 + 5 = 7, and 2 × 5 = 10.

Exercise 9 I

Find a pair of numbers that satisfy the following calculations.

(a) $\square + \square = 10$, $\square \times \square = 24$

(b) $\square + \square = 11$, $\square \times \square = 24$

(c) $\square + \square = 14$, $\square \times \square = 24$

(d) $\square + \square = 25$, $\square \times \square = 24$

(e) $\square + \square = 9$, $\square \times \square = 18$

(f) $\square + \square = 11$, $\square \times \square = 18$

(g) $\square + \square = 8$, $\square \times \square = 16$

(h) $\square + \square = 17$, $\square \times \square = 16$

(i) $\square + \square = 13$, $\square \times \square = 30$

(j) $\square + \square = 17$, $\square \times \square = 30$

(k) $\square + \square = 8$, $\square \times \square = 12$

(l) $\square + \square = 7$, $\square \times \square = 12$

(m) $\square + \square = 15$, $\square \times \square = 14$

(n) $\square + \square = 11$, $\square \times \square = 28$

(o) $\square + \square = 5$, $\square \times \square = 4$

(p) $\square + \square = 2$, $\square \times \square = 1$

(q) $\square + \square = 13$, $\square \times \square = 42$

(r) $\square + \square = 23$, $\square \times \square = 42$

(s) $\square + \square = 17$, $\square \times \square = 42$

(t) $\square + \square = 43$, $\square \times \square = 42$

(u) $\square + \square = 32$, $\square \times \square = 60$

(v) $\square + \square = 16$, $\square \times \square = 60$

(w) $\square + \square = 19$, $\square \times \square = 60$

(x) $\square + \square = 17$, $\square \times \square = 60$

(y) $\square + \square = 16$, $\square \times \square = 55$

(z) $\square + \square = 24$, $\square \times \square = 80$

(α) $\square + \square = 29$, $\square \times \square = 100$

(β) $\square + \square = 36$, $\square \times \square = 99$

Challenge! ⚠

(a) $\square + \square = 27$, $\square \times \square = 72$

(b) $\square + \square = 19$, $\square \times \square = 84$

(c) $\square + \square = 22$, $\square \times \square = 96$

(d) $\square + \square = 39$, $\square \times \square = 360$

Factorising Quadratic Expressions of the form $x^2 + ax + b$

In the previous Developing Algebra workbook, we saw how to expand double brackets, for example $(x + 2)(x + 5)$, using the acronym **FOIL**.

$$(x + 2)(x + 5)$$

$$= x^2 + 5x + 2x + 10$$
$$= x^2 + 7x + 10$$

FIRST
OUTSIDE
INSIDE
LAST

Exercise 10

Revision
I

Expand the following algebraic expressions.

(a) $(x + 3)(x + 5)$ (b) $(x + 2)(x + 8)$ (c) $(x - 2)(x + 6)$

(d) $(x - 4)(x - 5)$ (e) $(y + 1)(y + 3)$ (f) $(x + 9)(x - 3)$

The acronym FOIL explains the process of expanding a double bracket like $(x + 2)(x + 5)$, and finishing with the quadratic expression $x^2 + 7x + 10$. Let us now look at the **reverse** process of factorising the quadratic expression $x^2 + 7x + 10$, and finishing with the double bracket $(x + 2)(x + 5)$.

Example

Factorise the quadratic expression $x^2 + 7x + 10$.

Answer: We need to consider the following question: 'Which two numbers add to make 7 (the coefficient of the x term) and multiply to make 10 (the constant)?'.

Add	Multiply
7	10

The answer is 2 and 5, since 2 + 5 = 7, and 2 × 5 = 10. **Therefore**, $x^2 + 7x + 10$ factorises to give $(x + 2)(x + 5)$.

Exercise 11

Skill

Factorise the following algebraic expressions.

(a) $x^2 + 6x + 8$ (b) $x^2 + 7x + 12$ (c) $x^2 + 8x + 12$

(d) $x^2 + 8x + 15$ (e) $x^2 + 16x + 15$ (f) $x^2 + 2x + 1$

(g) $x^2 + 11x + 18$ (h) $x^2 + 9x + 18$ (i) $x^2 + 19x + 18$

(j) $x^2 + 12x + 20$ (k) $x^2 + 9x + 20$ (l) $x^2 + 21x + 20$

(m) $x^2 + 18x + 32$ (n) $x^2 + 12x + 32$ (o) $x^2 + 33x + 32$

(p) $x^2 + 11x + 24$ (q) $x^2 + 25x + 24$ (r) $x^2 + 10x + 24$

(s) $x^2 + 14x + 33$ (t) $x^2 + 15x + 36$ (u) $x^2 + 16x + 39$

(v) $x^2 + 17x + 42$ (w) $x^2 + 26x + 48$ (x) $x^2 + 15x + 50$

(y) $x^2 + 16x + 60$ (z) $x^2 + 19x + 60$ (α) $x^2 + 23x + 60$

Exercise 12

I

Factorise the following algebraic expressions.

(a) $x^2 + 10x + 25$

(b) $x^2 + 25 + 10x$

(c) $25 + 10x + x^2$

(d) $x^2 + 40 + 14x$

(e) $40 + 13x + x^2$

(f) $22x + 40 + x^2$

Example

Factorise the following algebraic expressions.

(a) $x^2 + 2x - 15$

Add	Multiply
2	–15

$\boxed{-3} + \boxed{5} = 2$

$\boxed{-3} \times \boxed{5} = -15$

$x^2 + 2x - 15 = (x-3)(x+5)$

(b) $x^2 - 2x - 15$

Add	Multiply
–2	–15

$\boxed{-5} + \boxed{3} = -2$

$\boxed{-5} \times \boxed{3} = -15$

$x^2 - 2x - 15 = (x-5)(x+3)$

(c) $x^2 - 8x + 15$

Add	Multiply
–8	15

$\boxed{-3} + \boxed{-5} = -8$

$\boxed{-3} \times \boxed{-5} = 15$

$x^2 - 8x + 15 = (x-3)(x-5)$

Exercise 13

Factorise the following algebraic expressions.

(a) $x^2 + 4x - 12$

(b) $x^2 - 4x - 12$

(c) $x^2 + x - 12$

(d) $x^2 - x - 12$

(e) $x^2 + 23x - 24$

(f) $x^2 - 23x - 24$

(g) $x^2 + 10x - 24$

(h) $x^2 - 10x - 24$

(i) $x^2 + 5x - 24$

(j) $x^2 - 5x - 24$

(k) $x^2 + 2x - 24$

(l) $x^2 - 2x - 24$

(m) $x^2 + 39x - 40$

(n) $x^2 - 39x - 40$

(o) $x^2 + 18x - 40$

(p) $x^2 - 18x - 40$

(q) $x^2 + 6x - 40$

(r) $x^2 - 6x - 40$

(s) $x^2 + 3x - 40$

(t) $x^2 - 3x - 40$

(u) $x^2 + 4x - 32$

(v) $x^2 - 4x - 32$

(w) $x^2 - 14x - 32$

(x) $x^2 + 14x - 32$

(y) $x^2 - 31x - 32$

(z) $x^2 + 31x - 32$

(α) $x^2 - 2x - 8$

Exercise 14

Factorise the following algebraic expressions.

(a) $x^2 - 7x + 12$

(b) $x^2 - 8x + 12$

(c) $x^2 - 13x + 12$

(d) $x^2 - 10x + 24$

(e) $x^2 - 11x + 24$

(f) $x^2 - 25x + 24$

(g) $x^2 - 14x + 40$

(h) $x^2 - 13x + 40$

(i) $x^2 - 22x + 40$

(j) $x^2 - 33x + 32$

(k) $x^2 - 12x + 32$

(l) $x^2 - 18x + 32$

(m) $x^2 - 5x + 6$

(n) $x^2 - 6x + 9$

(o) $x^2 - 11x + 18$

Challenge!

Factorise the following algebraic expressions.

(a) $x^2 + 4x - 96$

(b) $x^2 - 5x - 84$

(c) $x^2 + x - 240$

Exercise 15

Factorise the following algebraic expressions.

(a) $x^2 + 8x + 16$ (b) $x^2 - 8x + 16$ (c) $x^2 + 10x + 16$

(d) $x^2 - 10x + 16$ (e) $x^2 + 6x - 16$ (f) $x^2 - 6x - 16$

(g) $x^2 + 17x + 16$ (h) $x^2 + 15x - 16$ (i) $x^2 - 17x + 16$

(j) $x^2 + 11x + 28$ (k) $x^2 + 16x + 28$ (l) $x^2 + 29x + 28$

(m) $x^2 - 11x + 28$ (n) $x^2 - 12x - 28$ (o) $x^2 + 27x - 28$

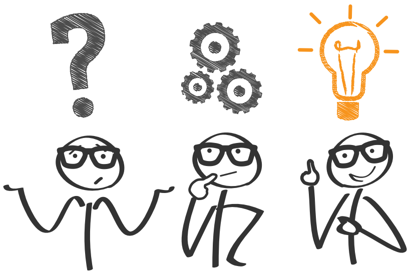
(p) $x^2 - 3x - 28$ (q) $x^2 - 16x + 28$ (r) $x^2 - 27x - 28$

(s) $x^2 + 7x + 10$ (t) $x^2 + 11x + 10$ (u) $x^2 - 7x + 10$

(v) $x^2 + 9x - 10$ (w) $x^2 - 9x - 10$ (x) $x^2 - 11x + 10$

(y) $x^2 + x - 20$ (z) $x^2 - 8x - 20$ (α) $x^2 - 21x + 20$

Solving Quadratic Equations through Factorising



Example

Solve the quadratic equation $x^2 + 6x + 8 = 0$.

Step 1: Factorise.

Add	Multiply
6	8

$\boxed{2} + \boxed{4} = 6$

$\boxed{2} \times \boxed{4} = 8$

$x^2 + 6x + 8 = (x + 2)(x + 4)$

Step 2: Solve.

$x^2 + 6x + 8 = 0$

$(x + 2)(x + 4) = 0$

Either $x + 2 = 0$ or $x + 4 = 0$

$x = -2 \qquad x = -4$

Exercise 16

Solve the following quadratic equations.

(a) $x^2 + 5x + 6 = 0$ (b) $x^2 + 9x + 14 = 0$ (c) $x^2 + 12x + 27 = 0$

(d) $x^2 - 5x + 6 = 0$ (e) $x^2 - 9x + 14 = 0$ (f) $x^2 - 12x + 27 = 0$

(g) $x^2 + x - 6 = 0$ (h) $x^2 - 5x - 14 = 0$ (i) $x^2 + 6x - 27 = 0$

(j) $x^2 - x - 6 = 0$ (k) $x^2 + 5x - 14 = 0$ (l) $x^2 - 6x - 27 = 0$

(m) $x^2 + 7x + 6 = 0$ (n) $x^2 - 15x + 14 = 0$ (o) $x^2 - 26x - 27 = 0$

(p) $x^2 - 5x - 6 = 0$ (q) $x^2 + 13x - 14 = 0$ (r) $x^2 + 26x - 27 = 0$

Exercise 17

Applying

Wilf thinks of a number, x.

His sister says that if Wilf subtracts 6 from his number and then multiplies this new number with the number he first thought of, he will get an answer of –5.

Use this information to do the following.
(a) Form a quadratic equation of the form $x^2 + ax + b = 0$.
(b) Solve the equation to find the possible values of x.

Factorising Quadratic Expressions of the form $ax^2 + bx + c$

Higher Tier

Example

Factorise the quadratic expression $2x^2 + 11x + 12$.

Method 1: The Splitting Method

1. Multiply 2 (the coefficient of the x^2 term) by 12 (the constant) to obtain 24. We must look for a pair of numbers that add to give 11 (the coefficient of the x term) and multiply to give 24.

$$2x^2 + 11x + 12$$
$$2 \times 12 = 24$$
$$\square + \square = 11$$
$$\square \times \square = 24$$

2. The numbers which work are 3 and 8. We rewrite the question by splitting the $11x$ to be $3x$ add $8x$.

$$2x^2 + 11x + 12$$
$$2 \times 12 = 24$$
$$\boxed{3} + \boxed{8} = 11$$
$$\boxed{3} \times \boxed{8} = 24$$
$$2x^2 + 3x \quad + 8x + 12$$

3. We split the four terms into two halves and factorise the first half.

$$2x^2 + 11x + 12$$
$$2 \times 12 = 24$$
$$\boxed{3} + \boxed{8} = 11$$
$$\boxed{3} \times \boxed{8} = 24$$
$$2x^2 + 3x \mid + 8x + 12$$
$$= x(2x + 3)$$

4. We copy the brackets, leaving a space between the old brackets and the new brackets.

$$2x^2 + 11x + 12$$
$$2 \times 12 = 24$$
$$\boxed{3} + \boxed{8} = 11$$
$$\boxed{3} \times \boxed{8} = 24$$
$$2x^2 + 3x \mid + 8x + 12$$
$$= x(2x + 3) \qquad (2x + 3)$$

5. $+\ 4$ must appear in the space, since 4 multiplied by $2x$ is $8x$, and 4 multiplied by 3 is 12.

$$2x^2 + 11x + 12$$
$$2 \times 12 = 24$$
$$\boxed{3} + \boxed{8} = 11$$
$$\boxed{3} \times \boxed{8} = 24$$
$$2x^2 + 3x \mid + 8x + 12$$
$$= x(2x + 3) + 4(2x + 3)$$

6. The expression has a common factor of $2x + 3$, therefore we factorise this out to leave the final answer.

$$2x^2 + 11x + 12$$
$$2 \times 12 = 24$$
$$\boxed{3} + \boxed{8} = 11$$
$$\boxed{3} \times \boxed{8} = 24$$
$$2x^2 + 3x \mid + 8x + 12$$
$$= x(2x + 3) + 4(2x + 3)$$
$$= (2x + 3)(x + 4)$$

Method 2: The Detective Method

1. Write a pair of brackets. At the start of the brackets, write a pair of terms that multiply to give $2x^2$.

$$2x^2 + 11x + 12$$
$$= (2x \qquad)(x \qquad)$$

2. At the end of the brackets, write a pair of terms that multiply to give 12. Use FOIL (in your head or on paper) to check your answer.

$$2x^2 + 11x + 12$$
$$= (2x + 2)(x + 6)$$

FOIL:

$$(2x + 2)(x + 6)$$
$$= 2x^2 + 12x + 2x + 12$$
$$= 2x^2 + 14x + 12 \quad ✗$$

3. If the answer is not correct, choose a different combination, repeating until you reach the correct answer.

$$2x^2 + 11x + 12$$
$$= (2x + 3)(x + 4)$$

FOIL:

$$(2x + 3)(x + 4)$$
$$= 2x^2 + 8x + 3x + 12$$
$$= 2x^2 + 11x + 12 \quad ✓$$

Exercise 18

Skill H

Factorise the following quadratic expressions.

(a) $2x^2 + 11x + 15$ (b) $2x^2 + 13x + 15$ (c) $2x^2 + 7x + 6$

(d) $3x^2 + 13x + 4$ (e) $3x^2 + 11x + 10$ (f) $3x^2 + 17x + 20$

(g) $4x^2 + 21x + 5$ (h) $4x^2 + 9x + 5$ (i) $4x^2 + 12x + 5$

(j) $5x^2 + 18x + 9$ (k) $5x^2 + 8x + 3$ (l) $6x^2 + 13x + 6$

(m) $2x^2 - x - 15$ (n) $3x^2 + x - 14$ (o) $5x^2 - 17x - 12$

(p) $3x^2 - 5x - 12$ (q) $4x^2 - 3x - 10$ (r) $2x^2 - 7x - 15$

(s) $4x^2 - 7x - 2$ (t) $3x^2 - 16x - 12$ (u) $4x^2 + 21x - 18$

(v) $3x^2 - 14x + 8$ (w) $5x^2 - 19x + 12$ (x) $3x^2 - 26x + 35$

(y) $2x^2 - 21x + 40$ (z) $2x^2 - 11x + 12$ (α) $4x^2 - 11x + 6$

Challenge! ⚠

Factorise the following quadratic expressions.

(a) $8x^2 - 2x - 15$ (b) $8x^2 - 19x + 6$ (c) $30x^2 - 42x + 12$

Example

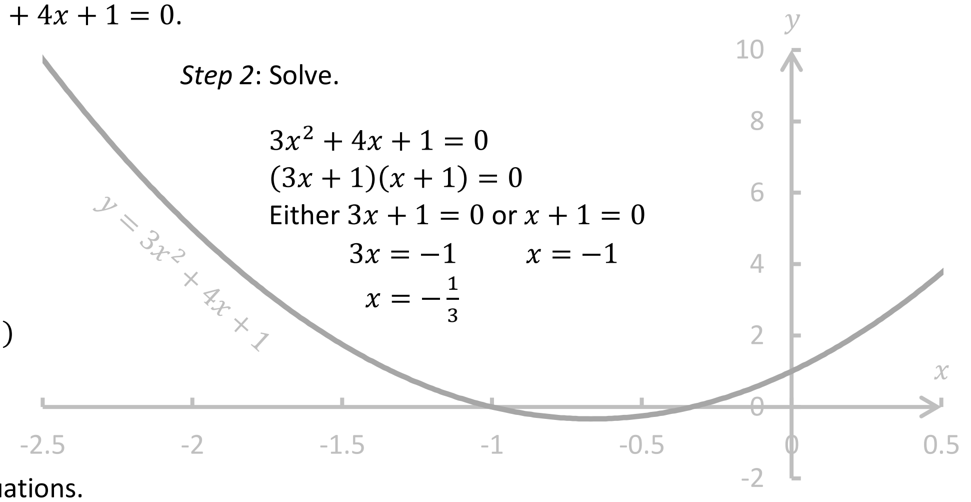
Solve the quadratic equation $3x^2 + 4x + 1 = 0$.

Step 1: Factorise.

$3 \times 1 = 3.$
$\boxed{1} + \boxed{3} = 4$
$\boxed{1} \times \boxed{3} = 3$
$3x^2 + 4x + 1$
$= 3x^2 + x + 3x + 1$
$= x(3x + 1) + 1(3x + 1)$
$= (3x + 1)(x + 1)$

Step 2: Solve.

$3x^2 + 4x + 1 = 0$
$(3x + 1)(x + 1) = 0$
Either $3x + 1 = 0$ or $x + 1 = 0$
$3x = -1$ $x = -1$
$x = -\frac{1}{3}$



Exercise 19

Solve the following quadratic equations.

(a) $2x^2 + 3x + 1 = 0$ (b) $2x^2 + 5x + 2 = 0$ (c) $2x^2 + 13x + 20 = 0$

(d) $2x^2 - 3x - 20 = 0$ (e) $2x^2 + 3x - 20 = 0$ (f) $2x^2 - 13x + 20 = 0$

(g) $3x^2 + 10x + 7 = 0$ (h) $3x^2 + 7x + 2 = 0$ (i) $3x^2 - 11x + 6 = 0$

(j) $2x^2 - 7x + 3 = 0$ (k) $2x^2 + 3x - 5 = 0$ (l) $2x^2 - 11x + 5 = 0$

(m) $4x^2 - 4x + 1 = 0$ (n) $4x^2 - 11x - 3 = 0$ (o) $5x^2 - 24x - 5 = 0$

(p) $6x^2 + x - 2 = 0$ (q) $6x^2 - 7x - 5 = 0$ (r) $15x^2 - 4x - 3 = 0$

Challenge! ⚠

Solve the following quadratic equations.

(a) $12x^2 + 28x - 5 = 0$ (b) $28x^2 + 15x + 2 = 0$ (c) $24x^2 - 2x - 15 = 0$

Example

Factorise the following quadratic expressions.

(a) $2x^2 + 20x + 42$

Answer: $2x^2 + 20x + 42$
$= 2(x^2 + 10x + 21)$
$= 2(x + 3)(x + 7)$

(b) $(x + 5)^2 + 8(x + 5)$

Answer: $(x + 5)^2 + 8(x + 5)$
$= (x + 5)((x + 5) + 8)$
$= (x + 5)(x + 13)$

(c) $2x^2 + 8x$

Answer: $2x^2 + 8x$
$= 2x(x + 4)$

Exercise 20 H

Factorise the following quadratic expressions.

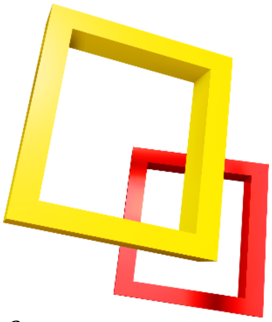
(a) $2x^2 + 22x + 56$ (b) $(x + 3)^2 + 7(x + 3)$ (c) $2x^2 + 20x$

(d) $3x^2 + 18x + 24$ (e) $(x - 5)^2 + 8(x - 5)$ (f) $3x^2 - 12x$

(g) $4x^2 + 12x - 40$ (h) $(x - 2)^2 - 4(x - 2)$ (i) $4x^2 - 18x$

(j) $4x^2 + 26x + 30$ (k) $7(x + 4)^2 + 3(x + 4)$ (l) $5x^2 + 45x$

The Difference of Two Squares



An expression of the form $a^2 - b^2$ factorises in a special way.

$$\boldsymbol{a^2 - b^2 = (a + b)(a - b)}$$

Example

Factorise the following expressions.

(a) $x^2 - 9$

Answer: $x^2 - 9$
$= (x + 3)(x - 3)$

(b) $4x^2 - 49$

Answer: $4x^2 - 49$
$= (2x + 7)(2x - 7)$

(c) $27x^2 - 75y^2$

Answer: $27x^2 - 75y^2$
$= 3(9x^2 - 25y^2)$
$= 3(3x + 5y)(3x - 5y)$

Exercise 21 H

Factorise the following expressions.

(a) $x^2 - 4$ (b) $x^2 - 16$ (c) $x^2 - 1$

(d) $x^2 - 144$ (e) $y^2 - 100$ (f) $z^2 - 36$

(g) $4x^2 - 25$ (h) $9x^2 - 4$ (i) $49x^2 - 81$

(j) $64x^2 - 1$ (k) $100y^2 - 9$ (l) $16z^2 - 121$

(m) $2x^2 - 18$ (n) $2x^2 - 50$ (o) $3x^2 - 48$

(p) $8x^2 - 18$ (q) $6x^2 - 24$ (r) $5x^2 - 125$

(s) $x^2 - y^2$ (t) $4x^2 - z^2$ (u) $x^2y^2 - 1$

(v) $16x^2 - \pi^2$ (w) $8x^2 - 72z^2$ (x) $4x^2z^2 - 36y^2$

(y) $x^4 - 4$ (z) $9y^4 - 16$ (α) $32z^6 - 128y^2$

Exercise 22

H

Solve the following quadratic equations.

(a) $x^2 - 25 = 0$ (b) $y^2 - 64 = 0$ (c) $z^2 - 169 = 0$

(d) $4x^2 - 49 = 0$ (e) $9x^2 - 1 = 0$ (f) $4x^2 - 16 = 0$

Exercise 23

The area of the rectangle on the right is 45 cm². Use the difference of two squares method to calculate the height and width of the rectangle.

$(x-2)$ cm

$(x+2)$ cm

Exercise 24 (Revision)

Solve the following quadratic equations.

(a) $x^2 + 15x + 44 = 0$ (b) $x^2 - 15x + 44 = 0$ (c) $x^2 + 7x - 44 = 0$

(d) $4x^2 + 14x = 0$ (e) $4x^2 - 14x = 0$ (f) $14x - 4x^2 = 0$

(g) $2x^2 + 13x + 21 = 0$ (h) $2x^2 - 13x + 21 = 0$ (i) $2x^2 + x - 21 = 0$

(j) $x^2 - 36 = 0$ (k) $9x^2 - 100 = 0$ (l) $4x^2 - 36 = 0$

Challenge!

The picture below shows Menai Bridge. We can model the cable between the two towers using the quadratic equation $y = \frac{43}{7744}x^2 - \frac{43}{44}x$. Given that the origin is at the highest point of one of the towers, solve the equation $\frac{43}{7744}x^2 - \frac{43}{44}x = 0$ to calculate the horizontal distance (in metres) between the top of the two towers.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Simultaneous Equations

Our aim in this chapter is to solve problems similar to the one below.

"Deiniol buys 2 *fish* and 3 *chips* in the local *fish & chips* shop, and he pays £8. Awel buys 4 *fish* and 2 *chips* in the same shop, and pays £12. What is the cost of 1 *fish* and 1 *chips* in the shop?"

By using the variable f to represent the cost of 1 *fish*, and c to represent the cost of 1 *chips*, we can write the following equations to represent the problem.

$$2f + 3c = 8$$
$$4f + 2c = 12$$

To solve the above equations, which are **simultaneous equations,** we must develop a number of **algebraic techniques** for our algebraic *toolbox* …

Multiplying Equations

Example

Multiply the equation $3x + 2y = 5$ by 4.

Answer: We multiply every term in the equation by four to obtain the equation $12x + 8y = 20$.

Exercise 25

Skill

I

Multiply the following equations by the numbers in the boxes.

(a) $2x + 6y = 4$ $\boxed{\times 2}$ (b) $3x + 4y = 5$ $\boxed{\times 2}$ (c) $7x + 2y = 6$ $\boxed{\times 2}$

(d) $4x + 3y = 7$ $\boxed{\times 3}$ (e) $8x + 11y = 3$ $\boxed{\times 4}$ (f) $6x + 3y = 11$ $\boxed{\times 5}$

(g) $2x - 5y = 3$ $\boxed{\times 2}$ (h) $-5x + 2y = 4$ $\boxed{\times 2}$ (i) $3x - 8y = -7$ $\boxed{\times 2}$

(j) $x + 7y = 3$ $\boxed{\times 6}$ (k) $8x + y = 9$ $\boxed{\times 7}$ (l) $3x - 4y = 1$ $\boxed{\times 8}$

(m) $3x + 6y = 7$ $\boxed{\times -2}$ (n) $5x - 2y = 10$ $\boxed{\times -3}$ (o) $-3x + 2y = -5$ $\boxed{\times -4}$

Subtracting Equations

Example

$$\begin{array}{rl} & 5x + 8y = 16 \\ - & \underline{2x + 3y = 7} \\ & \underline{3x + 5y = 9} \end{array} \qquad \begin{array}{rl} & 7x + 9y = 25 \\ - & \underline{3x + 9y = 4} \\ & \underline{4x \qquad = 21} \end{array} \qquad \begin{array}{rl} & 5x + 6y = 10 \\ - & \underline{5x - 2y = 3} \\ & \underline{\qquad 8y = 7} \end{array}$$

Exercise 26

Subtract the second equation from the first equation.

(a) $7x + 8y = 20$
$3x + 4y = 12$

(b) $9x + 5y = 13$
$3x + 2y = 5$

(c) $15x + 9y = 14$
$12x + 4y = 9$

(d) $6x + 2y = 31$
$2x + 7y = 8$

(e) $8x + 3y = 15$
$2x + y = 10$

(f) $19x - 8y = 8$
$4x + 4y = 3$

Exercise 26 (continued)

I

(g) $8x + 7y = 15$
$2x + 7y = 6$

(h) $11x + 4y = 27$
$11x + 2y = 3$

(i) $20x + 5y = -8$
$18x - 5y = 4$

(j) $-4x + 8y = 18$
$4x + 2y = 4$

(k) $6x - 4y = 8$
$2x - 4y = 10$

(l) $8x - 4y = 15$
$8x - 6y = 2$

(m) $x + 18y = 12$
$x + 17y = 2$

(n) $18x - 2y = -5$
$11x - 9y = 3$

(o) $-4x + 10y = 8$
$-4x - 2y = -3$

Multiplying Equations to Obtain Equal Coefficients

The **coefficient** of an algebraic term is the **number** which is part of the term. For example, the coefficient of $16x$ is 16.

Example

Consider the following simultaneous equations.

$3x + 10y = 16$
$4x + \;\; 5y = 13$

By multiplying the first equation by 4, and the second equation by 3, we can ensure that the $\boldsymbol{x}$ **coefficients** are equal.

$$3x + 10y = 16 \xrightarrow{\times 4} 12x + 40y = 64$$
$$4x + 5y = 13 \xrightarrow{\times 3} 12x + 15y = 39$$

On the other hand, by leaving the first equation as it is, and multiplying the second equation by 2, we can ensure that the $\boldsymbol{y}$ **coefficients** are equal.

$$3x + 10y = 16 \qquad\qquad 3x + 10y = 16$$
$$4x + 5y = 13 \xrightarrow{\times 2} 8x + 10y = 26$$

Try to use the **smallest possible** numbers that work.

Exercise 27

Which numbers do we need to multiply the following equations by to obtain equal x coefficients?

(a) $2x + 4y = 4$
$3x + 5y = 7$

(b) $2x + \;\; 4y = 4$
$4x + 12y = 7$

(c) $2x + 4y = 4$
$5x + 8y = 7$

(d) $5x + 7y = 6$
$2x + 2y = 8$

(e) $5x + 7y = 6$
$6x + 3y = 8$

(f) $5x + \;\; 7y = 6$
$15x + 14y = 8$

(g) $4x + 8y = 12$
$5x + 4y = 5$

(h) $4x + 8y = 12$
$6x + 3y = 5$

(i) $4x + \;\; 8y = 12$
$12x + 12y = 5$

(j) $24x + 32y = 20$
$8x + \;\; 8y = 20$

(k) $24x + 32y = 20$
$12x + 64y = 20$

(l) $24x + 32y = 20$
$16x + 16y = 20$

Exercise 28

Which numbers do we need to multiply the equations from Exercise 27 by to obtain equal y coefficients?

Solving Linear Equations

The final part of the jigsaw is to be able to solve linear equations like the following ones.

Exercise 29

Revision

F

Solve the following equations.

(a) $4x = 8$

(b) $4x = 32$

(c) $7x = 35$

(d) $4x = 2$

(e) $3x = 5$

(f) $8x = 7$

Solving Simultaneous Equations

We now have enough tools in our algebraic *toolbox* to return to the *fish & chips* problem from the beginning of this chapter.

"Deiniol buys 2 *fish* and 3 *chips* in the local *fish & chips* shop, and he pays £8. Awel buys 4 *fish* and 2 *chips* in the same shop, and pays £12. What is the cost of 1 *fish* and 1 *chips* in the shop?"

Step 1: Change the word problem into a pair of equations.

$$\begin{aligned} 2f + 3c &= 8 \\ 4f + 2c &= 12 \end{aligned}$$

Step 2: Multiply the first equation by 2 so that the f coefficients are equal.

$$\begin{array}{lcl} 2f + 3c = 8 & \xrightarrow{\boxed{\times 2}} & 4f + 6c = 16 \\ 4f + 2c = 12 & & 4f + 2c = 12 \end{array}$$

Step 3: Subtract the second equation from the first equation.

$$\begin{array}{lcr} 2f + 3c = 8 & \xrightarrow{\boxed{\times 2}} & 4f + 6c = 16 \\ 4f + 2c = 12 & & -\ 4f + 2c = 12 \\ \hline & & 4c = 4 \end{array}$$

Step 4: Solve the equation $4c = 4$ to obtain $c = 1$.

Conclusion: The cost of 1 *chips* in the shop is £1.

To find the value of f (and thus the cost of 1 *fish*), we can use any of the following methods.

Method A: Repeat steps 2–4 above, but this time making sure that the c coefficients are equal.

$$\begin{array}{lcr} 2f + 3c = 8 & \xrightarrow{\boxed{\times 2}} & 4f + 6c = 16 \\ 4f + 2c = 12 & \xrightarrow{\boxed{\times 3}} & -\ 12f + 6c = 36 \\ \hline & & -\ 8f \quad = -20 \\ & & f = \frac{-20}{-8} \\ & & f = \frac{5}{2} \end{array}$$

Method B: Substitute $c = 1$ into one of the original equations.

Substitute $c = 1$ into the equation $2f + 3c = 8$:

$$\begin{aligned} 2f + 3 \times 1 &= 8 \\ 2f + 3 &= 8 \\ 2f &= 5 \\ f &= \frac{5}{2} \end{aligned}$$

Conclusion: The cost of 1 *fish* is £2.50.

Note: You can check that the solutions are correct by substituting the values $c = 1$, $f = 2.5$ into the left-hand side of any of the original equations.

$$\begin{aligned} &2f + 3c \\ &= 2 \times 2.5 + 3 \times 1 \\ &= 5 + 3 \\ &= £8 \quad ✓ \end{aligned}$$

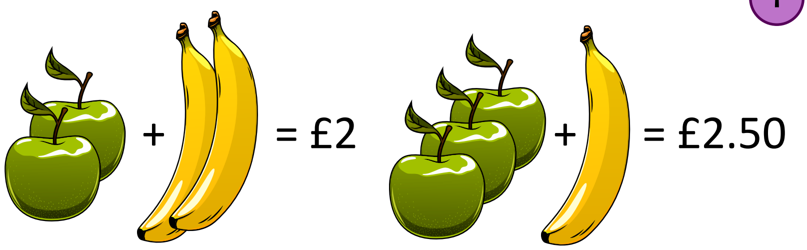
$$\begin{aligned} &4f + 2c \\ &= 4 \times 2.5 + 2 \times 1 \\ &= 10 + 2 \\ &= £12 \quad ✓ \end{aligned}$$

Applying

I

Exercise 30

Two apples and two bananas costs £2.
Three apples and one banana costs £2.50.
Find the cost of one apple and one banana.



Exercise 31

Skill I

Solve the following simultaneous equations.

(a) $3x + 4y = 18$
$2x + 2y = 10$

(b) $2x + 3y = 9$
$4x + y = 13$

(c) $2x + 4y = 16$
$2x + 3y = 14$

(d) $5x - 2y = 6$
$2x + 2y = 8$

(e) $6x + 3y = 18$
$-2x + 2y = 6$

(f) $-2x + y = -2$
$4x - 3y = 0$

(g) $2x + 3y = 3$
$2x - y = 7$

(h) $3x + 2y = 7$
$3x - y = -8$

(i) $2x + 3y = 14$
$3x + 2y = 16$

(j) $-3x + 2y = 0$
$3x - 4y = 6$

(k) $x + 5y = 9$
$2x + 3y = 11$

(l) $5x - 2y = 19$
$3x + y = 18$

(m) $4x + y = 8$
$7x + 3y = 9$

(n) $2x - 3y = 8$
$x + 2y = -10$

(o) $2x + 6y = 34$
$4x - 2y = 5$

(p) $2x + 3y = 10$
$5x - 6y = 16$

(q) $2x + 3y = 0$
$8x + 9y = -1$

(r) $7x + 8y = 19$
$3x - 2y = -19$

(s) $3x + 4y = 15$
$x - 6y = -6$

(t) $3x - 4y = 14$
$5x - 8y = 30$

(u) $3x + 5y = 21$
$4x + 3y = 17$

(v) $3x - 2y = 17$
$2x + 7y = 3$

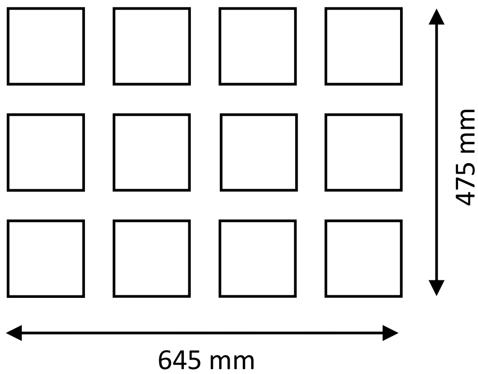
(w) $5x - 2y = 26$
$3x - 5y = 27$

(x) $2x + 4y = 5$
$5x + 7y = 8$

Exercise 32

(a) Aled buys 2 Cornish pasties and 3 sausage rolls in a shop, and pays £7. Ceinwen buys 4 Cornish pasties and 1 sausage roll in the same shop, and pays £9. What is the cost of 1 Cornish pasty and 1 sausage roll from the shop?

(b) A rectangle shape is made by using 12 square tiles with equal spaces between them. The total length of the rectangle is 645 mm and the total width of the rectangle is 475 mm. Find the dimensions of the tiles and the width of the space in mm.



(c) Glyn employs two people, Ben and Ceri. Ben and Ceri are paid at different hourly rates. Glyn has recorded how many hours both Ben and Ceri have worked on Monday and Tuesday. He has also noted the total he paid in wages.

Day	Number of hours worked		Total wages (£)
	Ben	Ceri	
Monday	6	5	116
Tuesday	4	8	138

Applying

Use an algebraic method to calculate how much Ben and Ceri are paid per hour.

(d) Ysgol Trefswm organised a concert to raise money for a charity.
All of the 120 tickets were sold for a total of £1,210.
The price of an adult ticket was £12.
The price of a child ticket was £7.
How many adult tickets and how many child tickets were sold?

Exercise 33: Old, Older, Oldest

Extension I

Use the clues on the cards to solve the problem.

The information for these people has been shuffled. To begin you must form simultaneous equations and solve them. Every answer is a whole number.

(1) 18 is the sum of Arwyn's age and Bedwyr's age.

Your job is to find everyone's age and list them from oldest to youngest. There are two sets of twins.

(2) 47 is the sum of Rhodri's age and Sali's age.

(4) Three times Wil's age subtract four times Tesni's age is 33.

(5) Three times Ynyr's age add Penri's age is 27.

(3) 38 is the sum of three times Gwen's age add two times Heledd's age.

(7) Two times Wil's age add three times Tesni's age is 107.

(8) Three times Iolo's age add four times John's age is 147.

(9) 20 is two times Cai's age subtract Dewi's age.

(6) Arwyn's age subtract Bedwyr's age is 8.

(10) 78 is two times Fred's age subtract Elfyn's age.

(11) Four times Kate's age add three times Lora's age is 40.

(12) The sum of Cai's age and Dewi's age is 73.

(13) 91 is two times Mali's age add Nia's age.

(14) 43 is five times Kate's age add two times Lora's age.

(15) Fred is older than Elfyn by thirteen years.

(16) By doubling Ynyr's age and adding Penri's age you get 25.

(17) Two times Rhodri's age add Sali's age is 76.

(18) Seven times Iolo's age subtract three times John's age is 10.

(19) Double of Gwen's age add three times Heledd's age gives 42.

(20) Mali is eight years older than Nia.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Changing the Subject

The purpose of **changing the subject** is to re-arrange formulae so that a particular **variable** appears on its own on the left-hand side of the formula. For example, consider the formula $p = 3w + d$, a formula to calculate the number of points (p) a football team has given how many games they have won (w) and how many drawn games (d) they have had. We can re-arrange the formula to give the number of games won by a football team.

$p = 3w + d$
$3w + d = p$ [Swap sides]
$3w = p - d$ [Subtract d from both sides]
$w = \frac{p-d}{3}$ [Divide both sides by 3]

After re-arranging the formula as shown above, we say that w is the **subject** of the formula.

There are a number of 'movements' we can perform to help re-arrange a formula to give a specific subject. Here are some of the most common movements.

Add a number to both sides of the formula

E.g. $y - 3 = x$
$y = x + 3$ [Add 3 to both sides]

Subtract a number from both sides of the formula

E.g. $y + 7 = x$
$y = x - 7$ [Subtract 7 from both sides]

Multiply both sides of the formula by a number

E.g. $\frac{y}{2} = 5x$
$y = 10x$ [Multiply both sides by 2]

Divide both sides of the formula by a number

E.g. $4y = x - 3$
$y = \frac{x-3}{4}$ [Divide both sides by 4]

Square both sides of the formula

E.g. $\sqrt{y} = 4x + 5$
$y = (4x + 5)^2$ [Square both sides]

Take the square root of both sides of the formula

E.g. $y^2 = 2x - 9$
$y = \sqrt{2x - 9}$ [Square root both sides]

Swap sides

E.g. $5x + 3 = y$
$y = 5x + 3$ [Swap sides]

Expand brackets

E.g. $4(y + 2) = 5x$
$4y + 8 = 5x$ [Expand brackets]

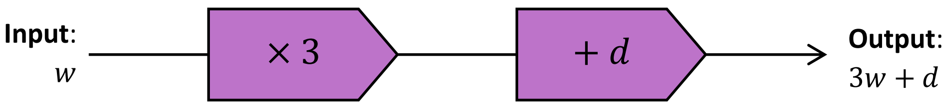
Let us reconsider the example at the top of this page. The question in the example can be set as follows.

Make w the subject of the formula $p = 3w + d$.

The aim in this question is to re-arrange the formula to leave only the variable w on the left-hand side of the formula. As the variable w initially appears on the right-hand side of the formula, it makes sense to start by swapping the sides of the formula, so that w appears on the left-hand side of the formula.

$p = 3w + d$
$3w + d = p$ [Swap sides]

There are several ways to proceed. You can think of the formula as an equation, and 'solve' to leave w on its own. Or you can think of how to calculate the left-hand side of the formula, if you start with the variable w.



By working backwards through the function machine, we can see the steps required to proceed, namely subtracting d from both sides of the formula, and then dividing both sides by 3.

Skill I

Exercise 34

(a) Make e the subject of the formula $p = 2e + c$.

(b) Make c the subject of the formula $p = 2e + c$.

(c) Make c the subject of the formula $y = mx + c$.

(d) Make m the subject of the formula $y = mx + c$.

(e) Make p the subject of the formula $c = p - 3t$.

(f) Make t the subject of the formula $c = p - 3t$.

(g) Make p the subject of the formula $A = p(q + r)$.

(h) Make q the subject of the formula $A = p(q + r)$.

(i) Make t the subject of the formula $F = \frac{m+4n}{t}$.

(j) Make m the subject of the formula $F = \frac{m+4n}{t}$.

(k) Make n the subject of the formula $F = \frac{m+4n}{t}$.

(l) Make r the subject of the formula $A = \pi r^2$.

(m) Make R the subject of the formula $I = \frac{PRT}{100}$.

(n) Make s the subject of the formula $A = \frac{su}{2}$.

(o) Make u the subject of the formula $C = \frac{1}{3}\pi r^2 u$.

(p) Make r the subject of the formula $C = \frac{1}{3}\pi r^2 u$.

(q) Make u the subject of the formula $A = \frac{1}{2}(a + b)u$.

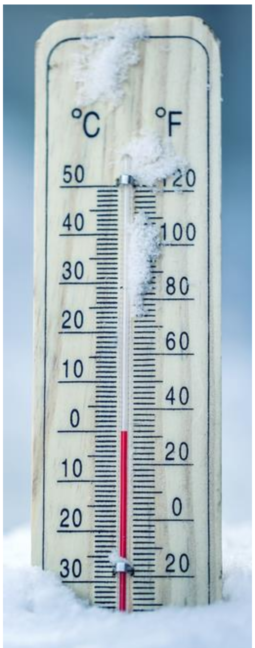
(r) Make b the subject of the formula $A = \frac{1}{2}(a + b)u$.

Exercise 35

The following formula was used by festival planners to calculate the parking fee for mini buses.

Parking Fee = Number of Passengers × 30p + £5

(a) What was the parking fee for a mini bus with 12 passengers?
(b) The parking fee for another mini bus was £7.40. How many passengers were on this mini bus?



Exercise 36

To change from degrees Celsius (°C) to degrees Fahrenheit (°F), you can use the following formula.

$$F = \frac{9}{5}(C + 40) - 40$$

(a) The temperature is 60°C. What is this in °F?
(b) Re-arrange the formula to find C in terms of F.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Expression, Equation, Formula, Identity

In algebra, what's the difference between expressions, equations, formulae and identities?

Expression

An expression is a collection of terms (e.g. $5x$ or 7) and operators (e.g. $+$ or $\times$).
$4x + 2$ and $\sqrt{6y - 4z}$ are examples of expressions.
There are no equals signs ($=$) in expressions.

Equation

An equation notes that two terms or expressions are equal. Two sides of an equation are separated by an equals sign ($=$). Sometimes, it is possible to solve an equation to find the value of a variable.

Formula

A formula is a special type of equation which shows the connection between different variables.
$P = 2a + 2b$ is an example of a formula, one which is used to calculate the perimeter of a rectangle with length a and width b.

Identity (Higher Tier Only)

An identity is an equation which is always true, no matter what the values of the variables are.
$2(x + 4) \equiv 2x + 8$ is an example of an identity. Two sides of an identity are separated by an equivalence sign ($\equiv$).

Example

$5x + 1 = 31$	Expression	$3x + 10$
$A = \frac{1}{2}(a + b)u$	Equation	$x(x + 2) \equiv x^2 + 2x$
$2x + 7z$	Formula	$\frac{y}{2} = 6$
$x^2 + 7x + 10 = 0$	Identity	$y = 2x + 5$
$(x + 2)(x + 4) \equiv x^2 + 6x + 8$		$C = \pi d$

Exercise 37

Skill

I H

Add arrows pointing to the correct descriptions.

$4x + 3 = 2x + 27$	Expression	$w = 8u + 17v$
$E = mc^2$	Equation	$z - 3 = 8$
$4x^2 + 6x \equiv 2x(2x + 3)$	Formula	$\sqrt{\frac{2x-3}{y}}$
$21x + 8$	Identity	$e^{i\pi} + 1 \equiv 0$
$f = e - v + 2$		$4x^2 + 2x - 6$
$\frac{x^3}{x^2} \equiv \frac{x^2}{x}$		$(2x - 4)(x + 3) = 0$

Higher Tier

Proving Identities

H

To prove an identity such as $(x+6)(x-2)-x(x+3) \equiv x-12$, we must use algebraic steps to change the left-hand side to be the right-hand side.

Left-hand side	$= (x+6)(x-2)-x(x+3)$	
	$= x^2-2x+6x-12-(x^2+3x)$	[Expand brackets]
	$= x^2+4x-12-x^2-3x$	[Collect like terms]
	$= x-12$	[Simplify]
	= Right-hand side ✓	

Exercise 38

Prove the following identities.

(a) $4(x+2) \equiv 4x+8$

(b) $2(x+4)+5(x+8) \equiv 7x+48$

(c) $(x+8)(x-3) \equiv x^2+5x-24$

(d) $6(x+8)-2(x-4) \equiv 4(x+14)$

(e) $(x+5)(x+2)+(x+8)(x+8) \equiv 2x^2+23x+74$

(f) $(x+6)(x-2)-(x+8)(x+2) \equiv -6x-28$

(g) $(y+4)(y-7)+3y(y-1) \equiv 4y^2-6y-28$

(h) $(2x+1)(x+2)-2x(x+4) \equiv -3x+2$

Exercise 39

Three of the following identities are incorrect. Which ones?

(a) $3(x-4) \equiv 3x-12$

(b) $(x+4)(x-2) \equiv x^2+2x-8$

(c) $(x+3)^2 \equiv x^2+9$

(d) $7(x+3)+2(x-2) \equiv 9x+17$

(e) $4(x+8)-2(x+8) \equiv 2(x+8)$

(f) $(x+2)(x-2) \equiv x^2+4$

(g) $5(y-2)-2(y-3) \equiv 3y-4$

(h) $\frac{x^2+6x+8}{x^2+5x+6} \equiv \frac{x+4}{x+3}$

(i) $(x+3)(x-3)-(x-4)(x+4) \equiv 7$

(j) $4(x+2)+(x-4)(x+7) \equiv x^2-20$

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Developing

Algebra 2

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test	Correct in the test?
I can **factorise** simple expressions such as $8x + 12$ or $15x^2 - 10$.			1	
I can **factorise quadratic expressions** of the form $x^2 + ax + b$.			1	
I can **solve quadratic equations** through factorisation.			2, 3	
I can **solve simultaneous equations**.			4, 5	
I can **re-arrange a formula** in order to make a specific variable (e.g. x) the **subject** of the formula.			6, 7	
I can recognise **expressions, equations, formulae** and **identities**.			8	

Am I ready for the test?
Tick the boxes below...

- [ ] I have revised the work in my mathematics book.
- [ ] I have revised the workbook.
- [ ] I have watched the relevant videos on YouTube.
- [ ] I have completed the Diagnostic Questions quiz.
- [ ] I have completed at least 4 pages in my revision book.

Developing Algebra 2

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test	Correct in the test?
I can **factorise** simple expressions such as $8x + 12$ or $15x^2 - 10$.			1	
I can **factorise quadratic expressions** of the form $x^2 + ax + b$.			1	
I can **factorise quadratic expressions** of the form $ax^2 + bx + c$.			2	
I can factorise quadratic expressions of the form $a^2 - b^2$ (a **difference of two squares**).			1	
I can **solve quadratic equations** through factorisation.			2, 3	
I can **solve simultaneous equations**.			4, 5	
I can **re-arrange a formula** in order to make a specific variable (e.g. x) the **subject** of the formula.			6, 7	
I can recognise **expressions, equations, formulae** and **identities**.			8	
I can **prove identities**.			9	

Am I ready for the test?
Tick the boxes below...

I have revised the work in my mathematics book.	☐
I have revised the workbook.	☐
I have watched the relevant videos on YouTube.	☐
I have completed the Diagnostic Questions quiz.	☐
I have completed at least 4 pages in my revision book.	☐

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The Mathematics Department

10

Measuring Solids

Name:

Contents

Chapter	Mathematics	Page Number
Solids	Volume and surface area of a cuboid. Volume of a prism. Volume and surface area of a cylinder. **Volume of a pyramid. Surface area of a cone. Volume and surface area of a sphere.**	3
Dimensions	Length, area and volume. Nonsensical formulae.	11
Composite Solids	Volume of composite solids. **Frustum of a cone. Hemisphere.**	15
Similar Shapes	Calculating the scale factor. Calculating missing lengths. Similar or not? **Similar triangles. Scale factor for length, area and volume. Using similar triangles to calculate the volume of a frustum of a cone.**	18
Pythagoras' Theorem (3-D)	**Calculating lengths in three-dimensional shapes.**	25

Solids

In this chapter, we will discuss how to calculate the volume and surface area of a variety of different solids.

The **volume** measures how much space a solid occupies or uses. It's measured in cube units, e.g. cm³.

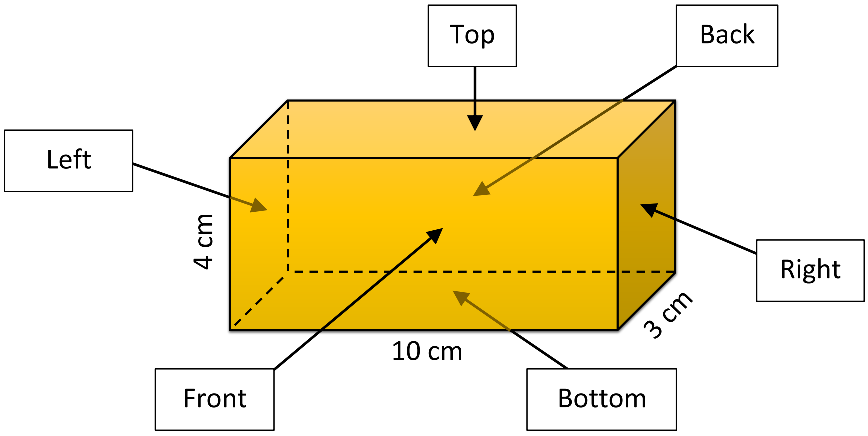
The **surface area** measures the area of the outside of the solid. We can think of the surface area as how much paper you would need to wrap the solid. Surface area is measured in square units, e.g. m².

Cuboid

We've previously seen the formula to **calculate** the volume of a cuboid in the "Measuring Shapes" workbook.

Volume of a Cuboid = Length × Width × Height

To calculate the **surface area** of a cuboid, we must add the areas of each of the six faces.



Example

The volume of the above cuboid is 10 × 3 × 4 = 120 cm³.

The surface area of the cuboid is the total area of the six faces.

Front	10 × 4 = 40
Back	40
Left	3 × 4 = 12
Right	12
Top	10 × 3 = 30
Bottom	30
Total	**164 cm²**

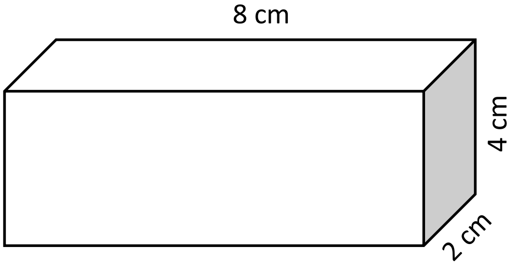
Notice that some of the areas are the same so we do not need to calculate them.

Exercise 1

Skill F

Calculate the volume and surface area of the following cuboids.

(a)



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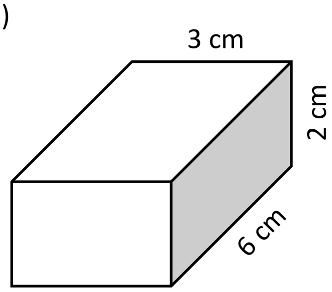

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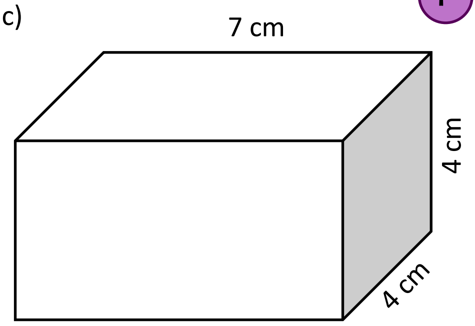
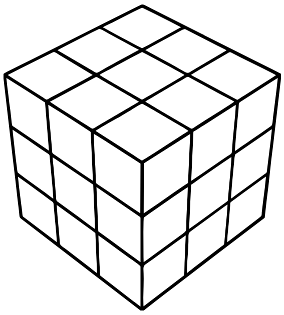


Applying

F

Exercise 2

The diagram shows a Rubik's cube before adding the coloured stickers.
The dimensions of one of the small cubes is 2 cm × 2 cm × 2 cm.



(a) What is the volume of one of the small cubes?

(b) How many small cubes form the Rubik's cube?

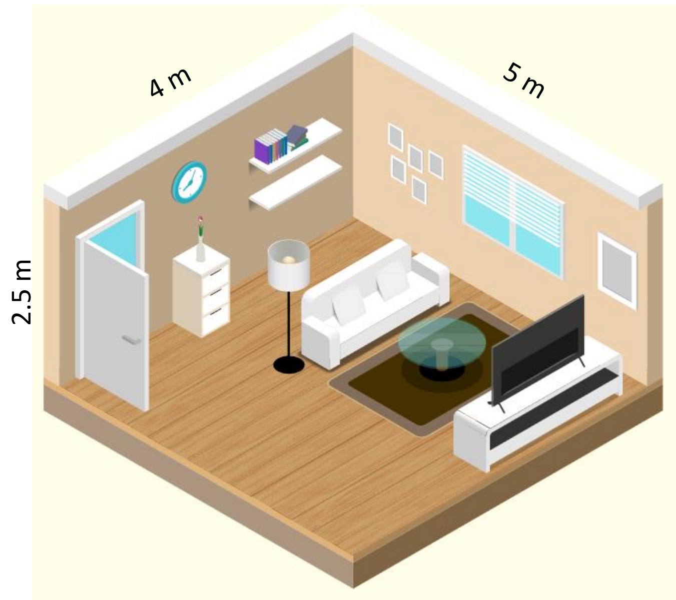
(c) What is the volume of the Rubik's cube?

(d) What is the surface area of the Rubik's cube?

(e) How many small 2 cm × 2 cm stickers are needed to be stuck on the Rubik's cube?

Exercise 3

The diagram on the right shows a floor plan for a living room.



(a) What is the volume of the living room?

(b) What is the area of the floor?

(c) The wood for the floor costs £14 per square metre. What was the cost of the wood for the whole floor?

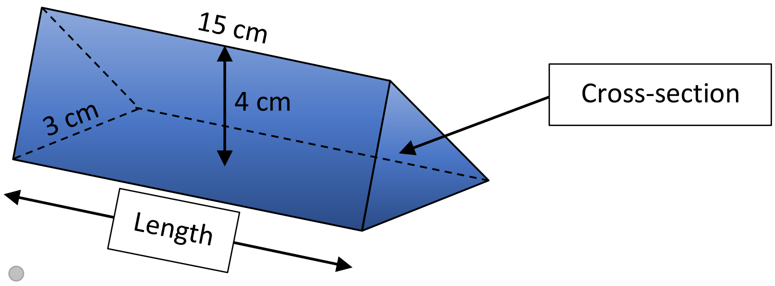
(d) Cerys wishes to repaint the wall on which the clock hangs. Given that the door measures 75 cm by 2 m, what area will need repainting?

(e) To repaint the wall, Cerys buys a 2.5 litre tin of paint. The tin states that the paint covers up to 10 m² of area for each litre of paint. Will Cerys have enough paint to give the wall two coats of paint?

Prism

A prism is a solid in which both ends are identical and the cross-section at any point is identical to the two ends. The shape of the two ends gives the prism its name. For example, the diagram below shows a triangular prism.

This formula is on page 2 of every GCSE exam paper.



Volume of a Prism = Area of Cross-section × Length

Example

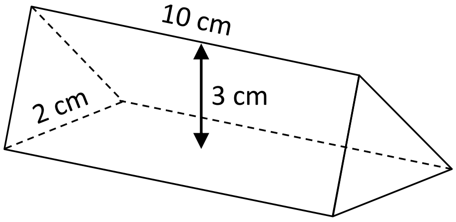
For the above triangular prism, the area of the cross-section is the area of the triangle, which is $\frac{3\times4}{2} = 6$ cm².
Therefore the volume of the prism is 6 × 15 = 90 cm³.

Exercise 4

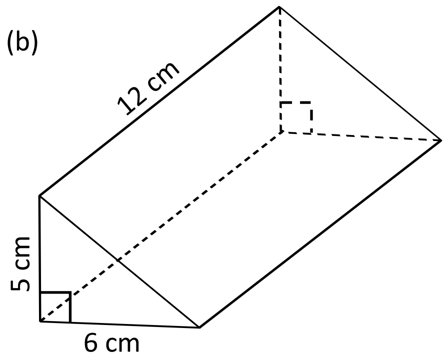
Skill I

Calculate the volume of the following prisms.

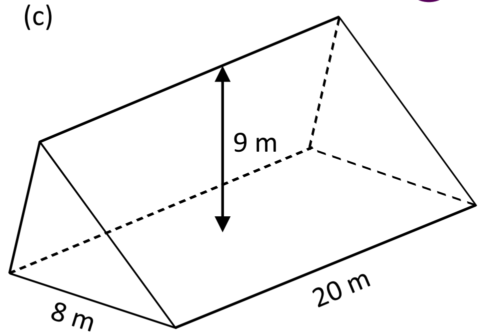
(a)



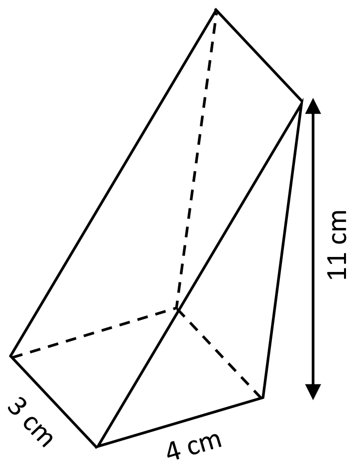
(b)



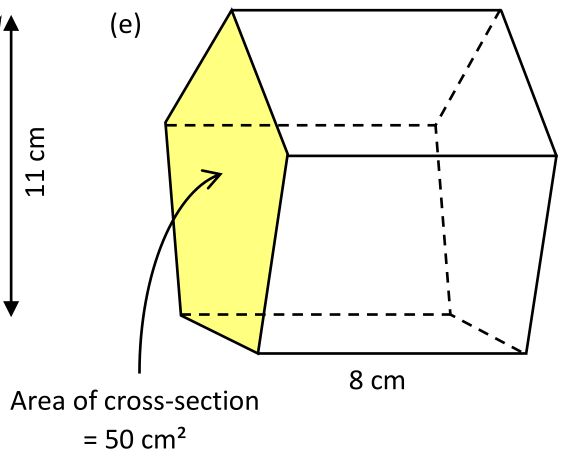
(c)



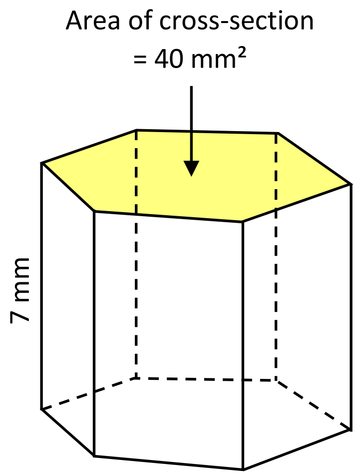
(d)



(e)



(f)



Challenge!

Calculate the surface area of the prism in question (b) above. Give your answer to 1 decimal place.

Exercise 5

Applying I

The picture on the right shows a water fountain.

The depth of the water in the lower part of the fountain is 40 cm.

The area of the cross-section of the water is 38,000 cm².

(a) What is the volume of the water in the fountain, in cm³?

(b) What is the volume of the water in the fountain, in ml?

(c) What is the volume of the water in the fountain, to the nearest litre?

(d) Buddug wants to empty the water fountain so that it can be cleaned. The water pump that Buddug uses to empty the fountain works at a rate of 100 litres per minute. To the nearest minute, how long will it take for the pump to empty the fountain?

Exercise 6

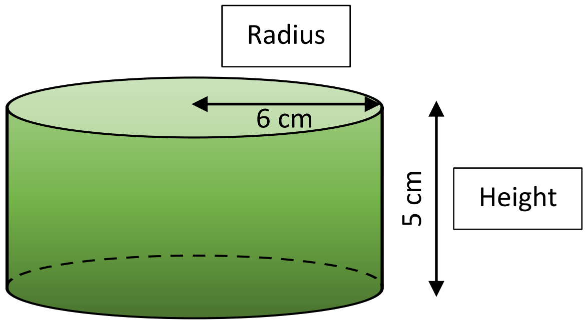
Skill I

What is the name of the prism that has the following shapes forming the cross-section?

(a) Rectangle (b) Square (c) Circle

Cylinder

A cylinder is a special type of prism where the cross-sectional shape is a circle.



Volume of a Cylinder = Area of the circle × Height or width of the cylinder

Volume of a Cylinder = π × Radius² × Height

Example

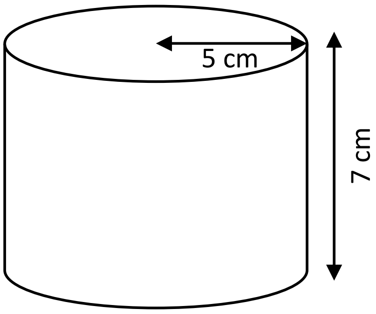
The volume of the above cylinder is $\pi \times 6^2 \times 5 = 565.49$ cm³, correct to two decimal places.

Exercise 7

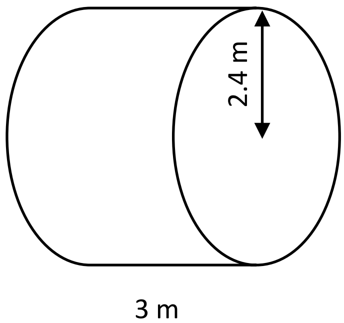
I

Calculate the volume of the following cylinders.

(a)



(b)



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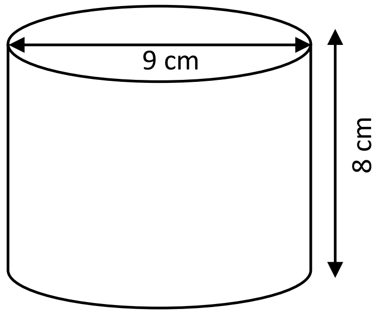


Applying

Exercise 8

I

The picture on the right shows an apple cake that has been baked in a baking tin.

(a) Given that the radius of the tin is 10 cm and its height is 6 cm, calculate the volume of the cake in the tin.

(b) The cake weighs 1.2 kg. Gwenda wants to cut the cake into equal pieces so that each piece weighs 150 g.
How many equal pieces will Gwenda need to cut?

(c) What is the volume of each of the pieces of cake from part (b)?

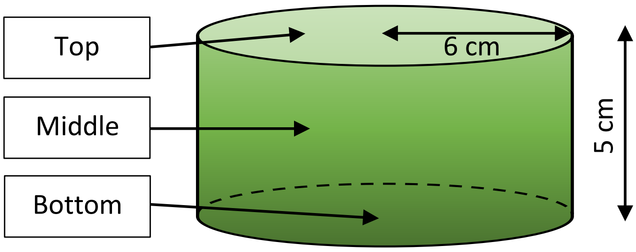
Exercise 9

I

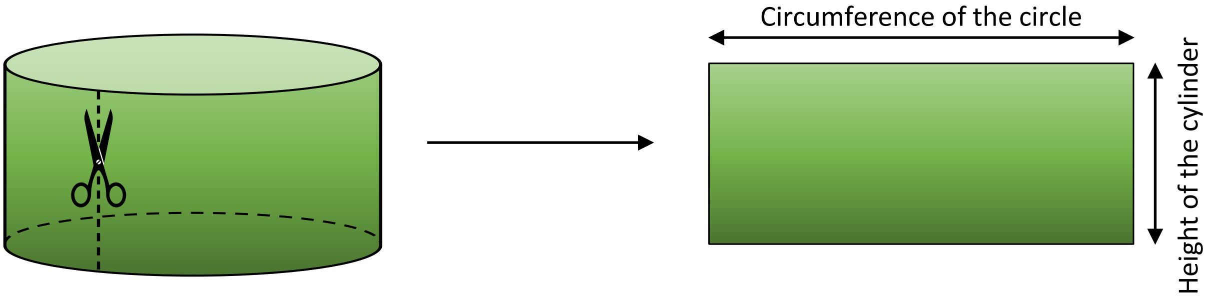
A hot water tank is in the shape of a cylinder. The height of the tank is 90 cm, and its diameter is 45 cm.
The manufacturer estimates that the tank holds 140 litres of water.
Has the manufacturer provided an overestimate or an underestimate?

Surface Area of a Cylinder

For a closed cylinder, we must add the areas of the top, middle and bottom faces in order to find the surface area of the cylinder.



The top and bottom are obviously circle shaped, but what is the shape of the middle face? Imagine cutting the middle face with scissors, vertically, and stretching the shape out. You would be left with a rectangle, with height the same as the cylinder's height and width equal to the circumference of the circle.



To find the area of this rectangle, we multiply the circumference of the circle (π × diameter of the circle) by the height of the cylinder.

Example

The surface area of the above cylinder is the total of the top, middle and bottom faces.

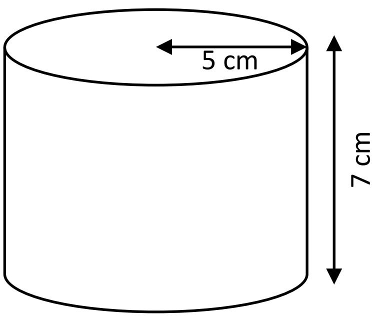
Top Middle Bottom	$\pi \times 6^2 = 113.10$ cm² $(\pi \times 12) \times 5 = 188.50$ cm² 113.10 cm²
Total	**414.7 cm², to one decimal place**

Exercise 10

Skill I

Calculate the surface area of the following closed cylinders.

(a)



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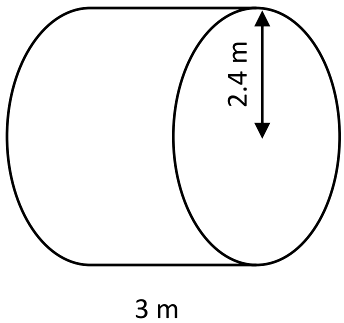

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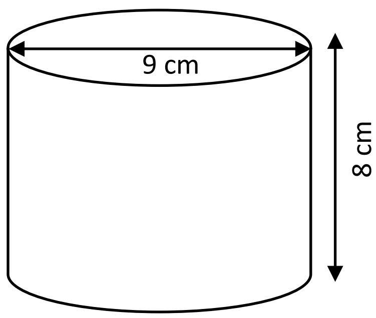


Exercise 11

If the cylinders in Exercise 10 had been open rather than closed cylinders (which means that they are empty cylinders without a top or a bottom), what would the surface area of the cylinders have been?

Exercise 12

Complete the following table. State your answers correct to two decimal places.

Type of cylinder	Radius	Diameter	Height	Volume	Surface Area
Open	14 cm		6 cm		
Closed		6.8 m	2.4 m		
Closed	9.3 mm		12 mm		
Open		0.7 km	0.3 km		
Closed	18 cm		1.2 cm		

Exercise 13

Applying

The cardboard tube of a toilet roll has the shape of a cylinder.
The diameter of the tube is 4.4 cm and the length of the tube is 11 cm.
Calculate the area of the cardboard used to create the tube.

Exercise 14

Pringles are sold in cylindrical packaging.

The height of the cylinder is 26 cm, and the diameter of the cylinder is 8 cm.

There is a layer of foil on the top along with a plastic lid.

The bottom is metal.

The centre is made from cardboard.

(a) What is the area of the metal bottom?

(b) What is the area of the top layer made of foil?

(c) Given that the plastic lid has a vertical edge of 0.8 cm, how much plastic is needed to create the lid?

(d) How much cardboard is needed to create one tube?

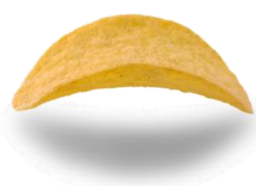
(e) How much cardboard is needed to create 10,000 tubes?

Exercise 15

The manufacturer of *Pringles* wants to save money by changing the height of the cylinder to be 25.9 cm and the diameter to be 7.9 cm.

Consider how much cardboard is required to make 10,000 of the original tubes (height 26 cm, diameter 8 cm). How many **additional** new tubes will it be possible to make using **the same amount** of cardboard?

Challenge!



Use the internet to find the mathematical name for the shape of a single *Pringle*.

What is the general equation for this type of shape?

What is the "*Pringles circle challenge*"?

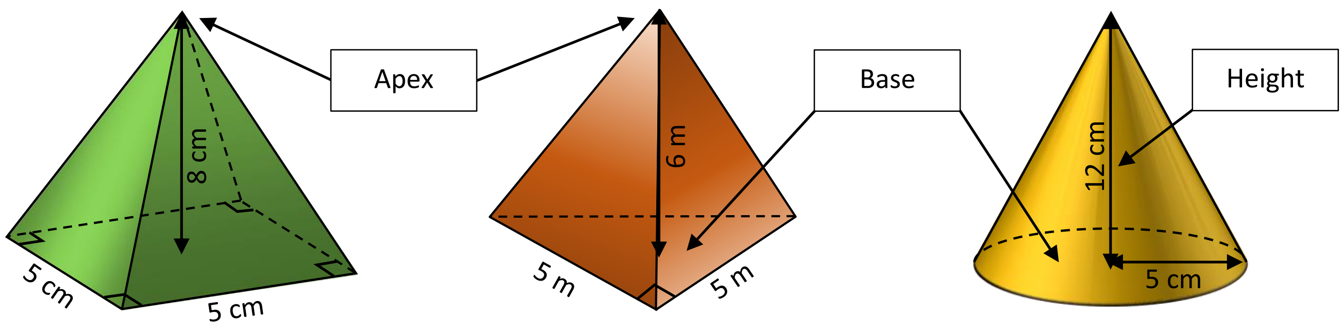
Pyramid

Higher Tier

A **pyramid** is any solid with a flat base where the whole perimeter of the base raises up to meet at one point above the base, the **apex** of the pyramid.

There are a number of different types of pyramids, for example:

Square based pyramid | Tetrahedron (triangle-based pyramid) | Cone (circle-based pyramid)



Volume of a pyramid = $\frac{1}{3}$ × Area of the base × Height

Example

The volume of the above square based pyramid is
$\frac{1}{3}$ × Area of the square × Height
$= \frac{1}{3} \times (5 \times 5) \times 8$
$= 66\frac{2}{3}$ cm³

The volume of the above tetrahedron is
$\frac{1}{3}$ × Area of the triangle × Height
$= \frac{1}{3} \times \left(\frac{5 \times 5}{2}\right) \times 6$
$= 25$ m³

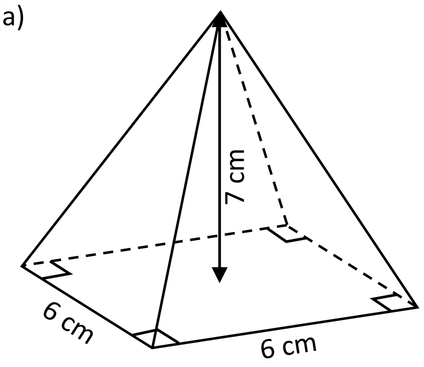
The volume of the above cone is
$\frac{1}{3}$ × Area of the circle × Height
$= \frac{1}{3} \times (\pi \times 5^2) \times 12$
$= 314.16$ cm³, correct to two decimal places.

Exercise 16

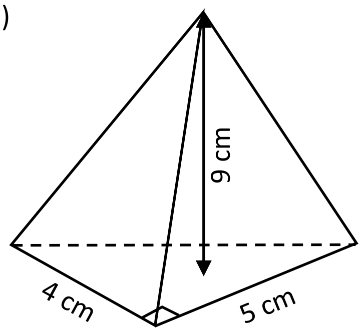
Skill H

Calculate the volume of the following pyramids.

(a)



(b)



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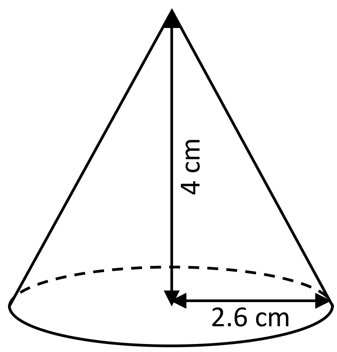
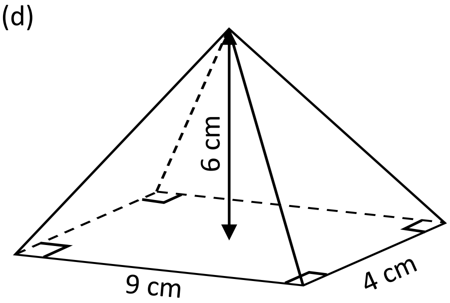
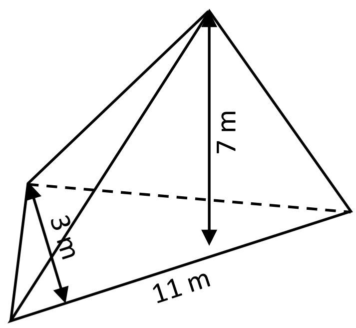


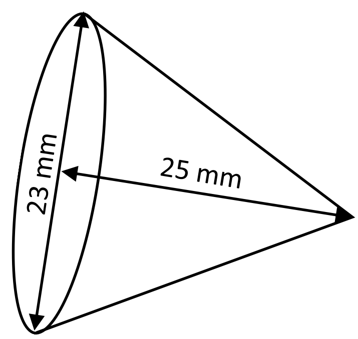
(d)



(e)

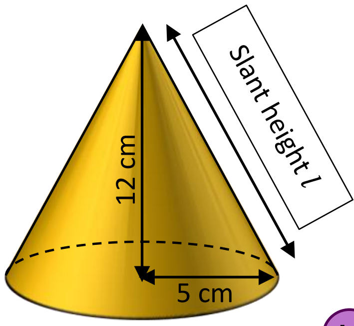


(f)



Surface Area of a Cone

A cone has two faces, the base (circle shaped) and the curved face (sector shaped).



Surface Area of a Cone = $\pi r^2 + \pi r l$

Example

By using Pythagoras' Theorem, the slant height, l, for the cone on the right is 13 cm.

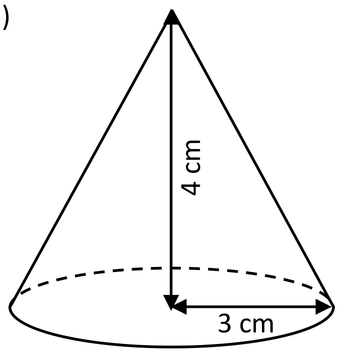
Therefore, the surface area of the cone is $\pi \times 5^2 + \pi \times 5 \times 13 = 282.74$ cm², correct to two decimal places.

H

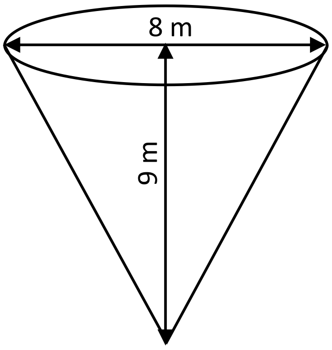
Exercise 17

Calculate the surface area of the following solid cones.

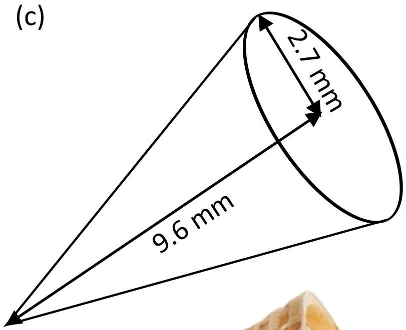
(a)



(b)



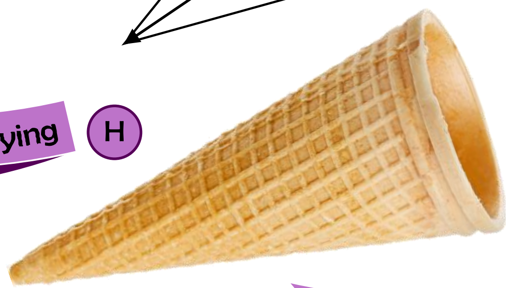
(c)



Exercise 18

Applying H

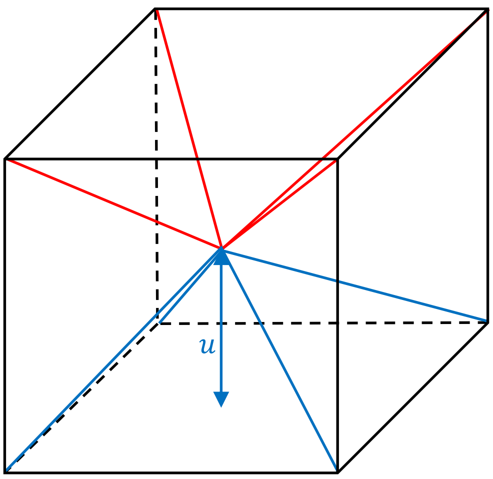
The picture on the right shows an empty ice-cream cone. The diameter of the top of the cone is 5 cm, and the height of the cone is 10 cm. What is the surface area of the wafer?



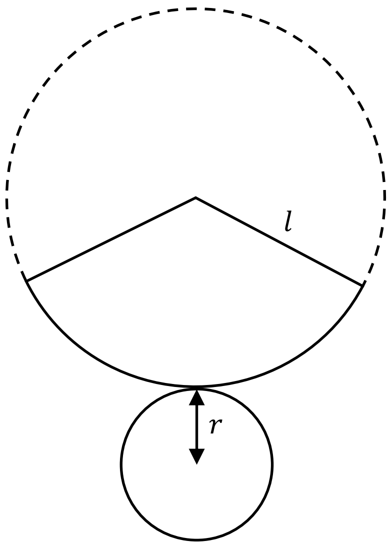
Exercise 19

Extension H

(a) The diagram below shows a cube with all vertices connected to the centre. How does the diagram explain the fraction $\frac{1}{3}$ in the formula for the volume of a pyramid?



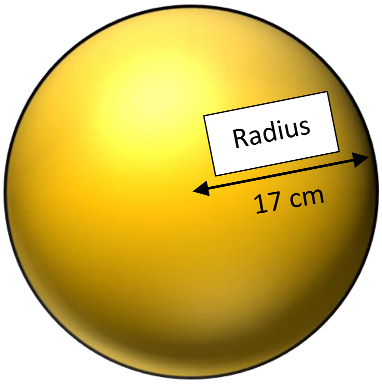
(b) The diagram below shows the net of a cone. How does the diagram help to explain the formula $\pi r l$ for the surface area of the curved face of a cone?



Sphere

Volume of a Sphere $= \frac{4}{3}\pi r^3$

Surface Area of a Sphere $= 4\pi r^2$



Example

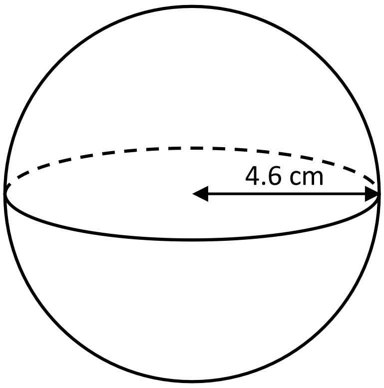
The volume of the sphere on the right is $\frac{4}{3} \times \pi \times 17^3 = 20{,}579.53$ cm³, correct to 2 decimal places. The surface area of the sphere is $4 \times \pi \times 17^2 = 3{,}631.68$ cm², correct to two decimal places.

Exercise 20

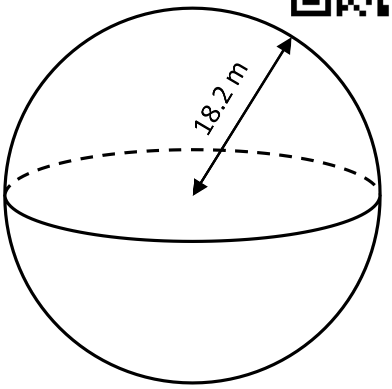
Skill

Calculate the volume and surface area of the following spheres.

(a)



(b)



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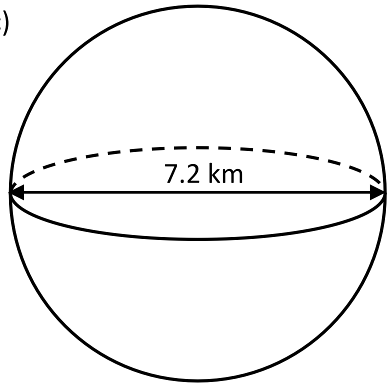


Exercise 21

Applying

H

It is possible to treat the Earth as a sphere of radius 6,371 km.

(a) What is the volume of the Earth? Give your answer to the nearest km³.

(b) What is the surface area of the Earth? Give your answer to the nearest km².

(c) About 71% of the surface of the Earth is covered by water. What is the surface area of the water that covers the Earth?

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ **Target** ☐

Dimensions

Length, Area and Volume

Given a particular formula, we need to be able to recognise, using **dimensions**, if the formula is for calculating a length, area, volume, or none of these.

Any formula for a **length** is a **one-dimensional** formula.
Any formula for an **area** is a **two-dimensional** formula.
Any formula for a **volume** is a **three-dimensional** formula.

Example

Formula	Purpose	Number of Dimensions
$\boldsymbol{P = 2a + 2b}$	Calculate the perimeter of a rectangle with length a and width b	1
$\boldsymbol{A = \pi r^2}$	Calculate the area of a circle with radius r	2
$\boldsymbol{C = \pi r^2 h}$	Calculate the volume of a cylinder of radius r and height h	3

It is possible to consider the following rules when deciding how many dimensions a formula has (and therefore to decide whether a formula is used to find length, area or volume).

Length + Length = Length
Area + Area = Area
Volume + Volume = Volume

Length – Length = Length
Area – Area = Area
Volume – Volume = Volume

Number × Length = Length
Number × Area = Area
Number × Volume = Volume

Length × Length = Area
Length × Area = Volume
Volume ÷ Length = Area
Area ÷ Length = Length

Length × Length × Length = Volume
Area × Length = Volume
Volume ÷ Area = Length

Exercise 22

Skill I

In the following formulae, a, b and c represent lengths.
Decide whether each formula represents a length, an area or a volume.

(a) $M = a + b$ (b) $M = a - b$ (c) $M = 3a$

(d) $M = ab$ (e) $M = 8bc$ (f) $M = ab + ac$

(g) $M = abc$ (h) $M = 4abc$ (i) $M = 4abc - cab$

(j) $M = a + 3c$ (k) $M = 3ab + 2ac$ (l) $M = 7a - 3c$

(m) $M = \frac{ab}{c}$ (n) $M = bc^2$ (o) $M = \frac{bc^2}{a}$

(p) $M = a(b + c)$ (q) $M = a^2(b + c)$ (r) $M = \frac{a^2}{b+c}$

(s) $M = \pi a$ (t) $M = 3\pi a$ (u) $M = 7\pi ab$

(v) $M = 4b^2 - 4ac$ (w) $M = ab - bc + ca$ (x) $M = \frac{abc}{4bc}$

(y) $M = 10\pi c^2$ (z) $M = \frac{ab^2}{a+b+c}$ (α) $M = ab(4 + \pi)$

Exercise 23

Each of the following quantities has a specific number of dimensions.
Write the number of dimensions for each quantity. The first has been completed for you.

Quantity	Number of dimensions
Capacity of a swimming pool	3
Perimeter of a hexagon	
Volume of a cylinder	
Distance between Llandudno and Liverpool	
Area of a trapezium	
Length of the shoelaces in a pair of shoes	

Nonsensical Formulae

It is possible to write a formula that doesn't make sense. For example, it doesn't make sense to add a volume to an area (there would be no meaning to the answer), therefore the formula $M = ab + abc$ doesn't make sense.
Here are some other combinations that lead to nonsensical formulae.

Length + Area	Length + Volume	Area + Volume
Length – Area	Length – Volume	Area – Volume
	Length × Volume	Area × Volume
Length ÷ Area	Length ÷ Volume	Area ÷ Volume

Exercise 24

In the following formulae, p, q and r represent lengths.
Decide which seven formulae do not make sense.

(a) $M = pq + rp$ (b) $M = 3p + qr$ (c) $M = p^2 - pq$

(d) $M = 2r + 3q$ (e) $M = pqr - 6pr$ (f) $M = q - r^3$

(g) $M = \frac{pq}{r}$ (h) $M = \frac{p}{qr}$ (i) $M = \frac{5q}{p^3}$

(j) $M = 2\pi r + q$ (k) $M = \pi r^2 + pqr$ (l) $M = \frac{3pq}{r} + 7\pi$

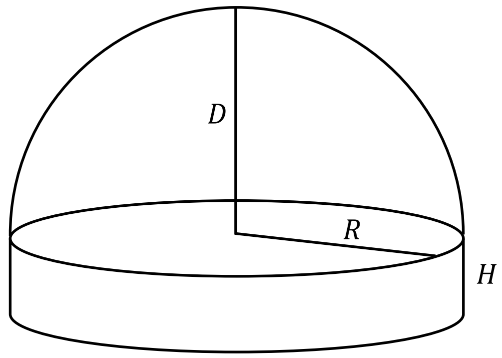
Exercise 25

In each of the following expressions, every letter represents a length. By considering the dimensions of the expressions write, for each one, what the expression may be describing: length, area, volume or none of these.
The first one has been completed for you.

	The formula could be for
(a) $e^2 + df$	Area
(b) $5d + 8e + 2f$	....
(c) $7de + 2d^2f$	....
(d) $(d + e)f$	....
(e) $5def - 2e^3$	....
(f) $\pi de + fde$	....

Exercise 26

The diagram on the right shows a solid.
The lengths D, R and H are noted on the diagram.



One of the following formulae can be used to estimate C, the volume of the solid.

$$C = 3H + 2R + 5D$$
$$C = 3R + 5DR$$
$$C = 3R^2H + 2R^2D$$
$$C = 3R(4D + 5H)$$

(a) Explain why we cannot use $C = 3H + 2R + 5D$ to estimate the volume of the solid.

(b) Note, stating your reasons, which of the above formulae can be used to estimate the volume of the solid.

Exercise 27

A factory uses wire to make the frame for a plant cover, as shown in the diagram on the right.
Every frame has a width L, depth D and upright height U.

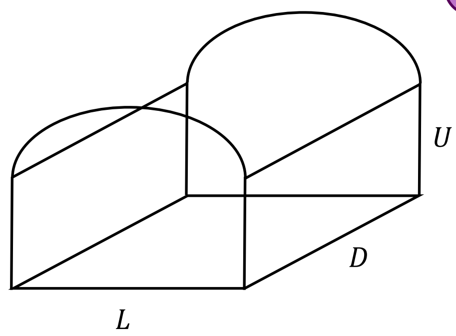


Diagram not drawn to scale.

One of the following formulae can be used to estimate C, the total length of wire needed to make the frame.

$$C = 5L + 4D + 4U$$
$$C = 5L + 4DU$$
$$C = 5L(4D + 4U)$$
$$C = 5LDU$$

(a) Explain why we cannot use the formula $C = 5LDU$ to estimate the total length of wire required.

(b) Note, stating your reasons, which of the above formulae can be used to estimate the total length of wire required.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Composite Solids

A **composite solid** is a solid we can split into simpler solids, like the solids considered in the first chapter of this workbook.

Example

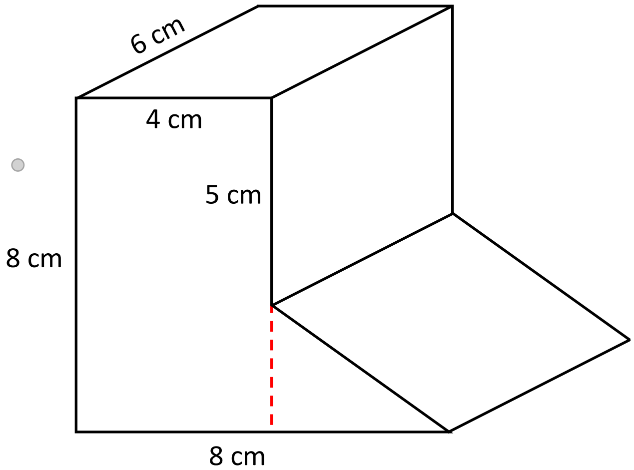
It is possible to split the composite solid on the right into a cuboid (on the left) and a triangular prism (on the right).

Cuboid
Volume = Length × Width × Height
= 4 × 6 × 8
= 192 cm³

Triangular Prism
Volume = Area of Cross-section × Length
$= \left(\frac{4\times3}{2}\right) \times 6$
= 36 cm³

Composite Solid
Volume = 192 + 36
= 228 cm³

It is also possible to treat this solid as one large prism.

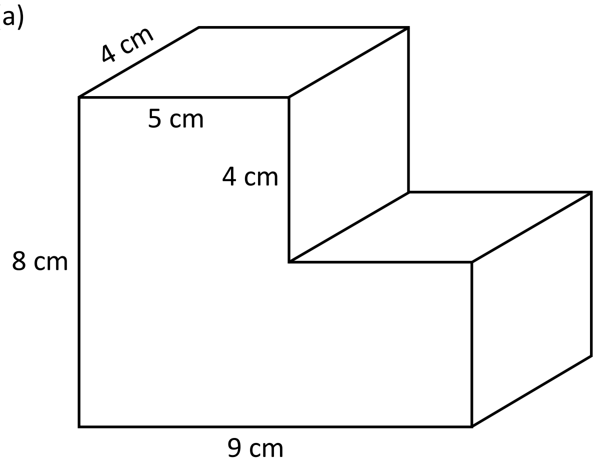


Exercise 28

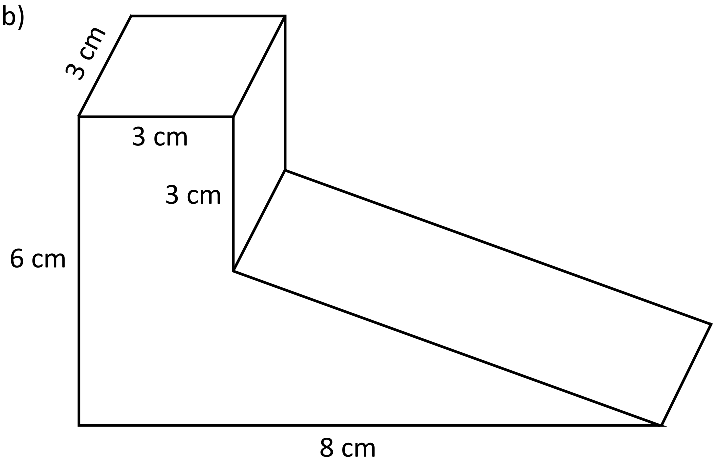
Skill

Calculate the volume of the following composite solids.

(a)



(b)



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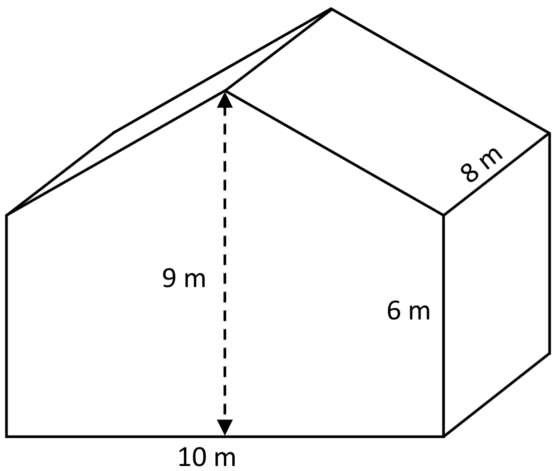

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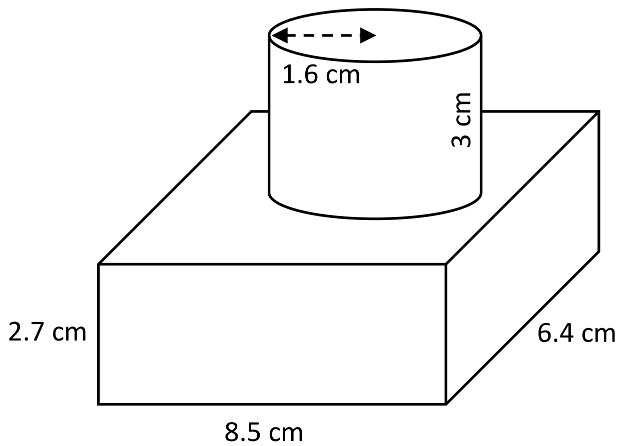
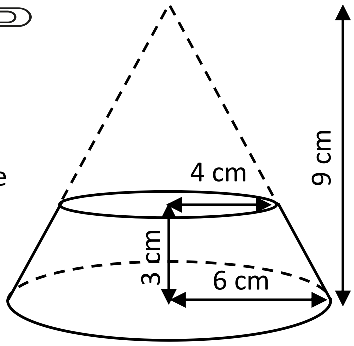


Frustum of a Cone

Higher Tier

A cone frustum is the shape left over after the top part of the cone is taken away.

Volume of the frustum = Volume of the whole cone – Volume of the missing cone



Example

Volume of the frustum (right) = Volume of the whole cone – Volume of the missing cone

$= \frac{1}{3} \times \pi \times 6^2 \times 9 - \frac{1}{3} \times \pi \times 4^2 \times 6$

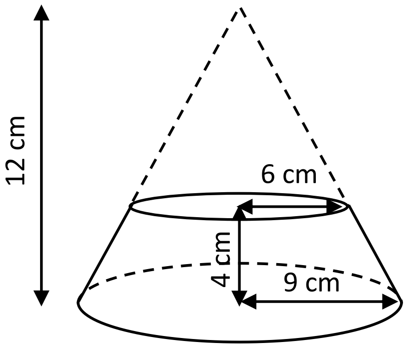
$= 238.76$ cm³, correct to two decimal places.

Exercise 29

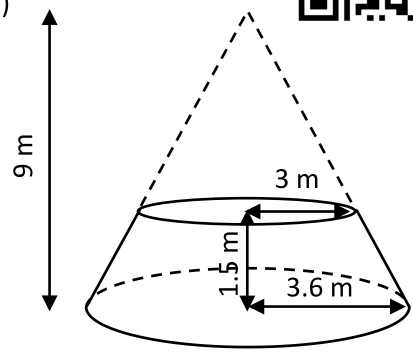
H

Calculate the volume of the following frustums.

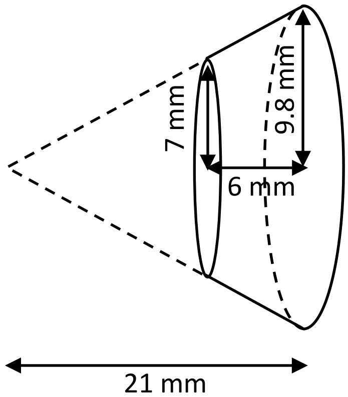
(a)



(b)



(c)



Challenge!

Calculate the surface area of the frustums in Exercise 29.

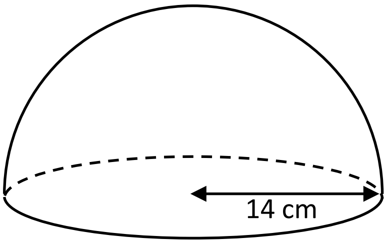
Half the surface of the sphere + area of the circle.

Hemisphere

A hemisphere is half a sphere.

Volume of a hemisphere $= \frac{2}{3}\pi r^3$

Surface area of a hemisphere $= 3\pi r^2$



Example

The volume of the hemisphere on the right is $\frac{2}{3} \times \pi \times 14^3 = 5{,}747.02$ cm³, correct to 2 decimal places.

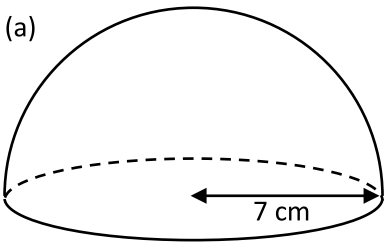
The surface area of the hemisphere is $3 \times \pi \times 14^2 = 1{,}847.26$ cm², correct to 2 decimal places.

Exercise 30

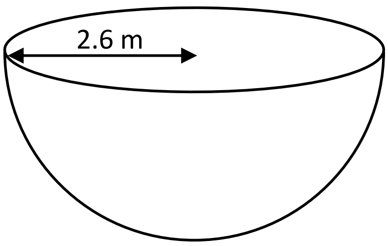
H

Calculate the volume and surface area of the following solid hemispheres.

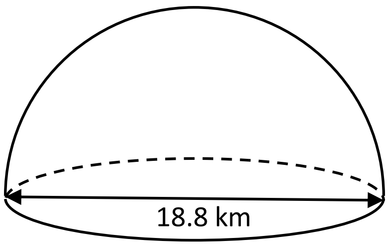
(a)



(b)



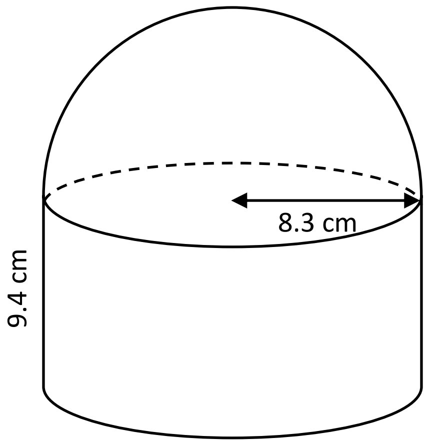
(c)



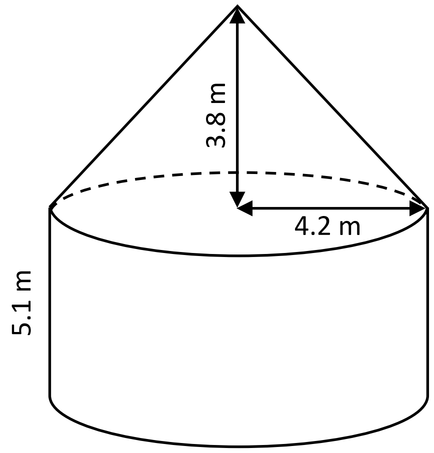
Exercise 31 H

Calculate the volume of the following composite solids.

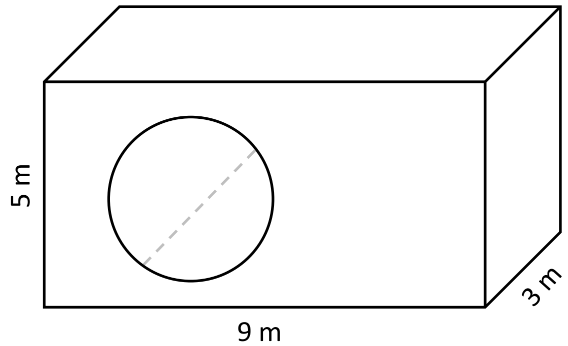
(a)



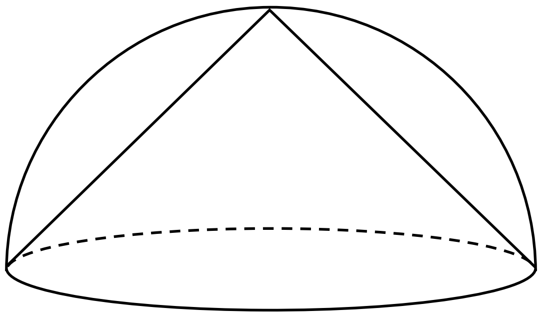
(b)



(c) A cuboid with a cylindrical hole in it.
The radius of the hole is 2.3 m.



(d) A hemisphere with a cone shaped hole in it.
The radius of the hemisphere is 3.5 cm.



Exercise 32

Applying H

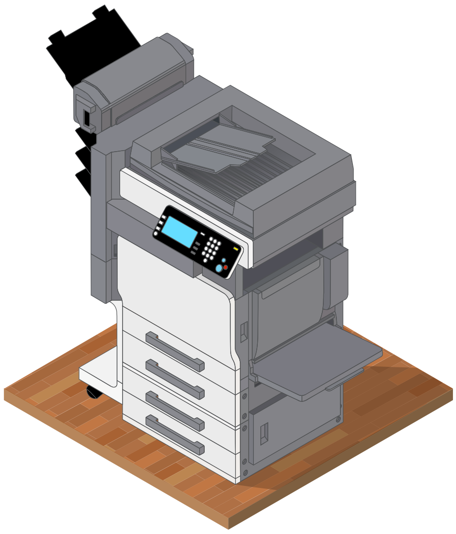
The inside of a plant pot is in the shape of a frustum.
The radius of the highest part of the frustum is 10 cm.
The radius of the lowest part of the frustum is 7.5 cm.
The height of the plant pot is 10 cm.

Calculate how many litres of soil the plant pot can hold.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Similar Shapes



Imagine using a photocopier to enlarge a diagram drawn on a piece of paper. If the original paper is A4 sized and the paper that comes out of the photocopier is A3 sized, then you have created a copy of the diagram twice the size.

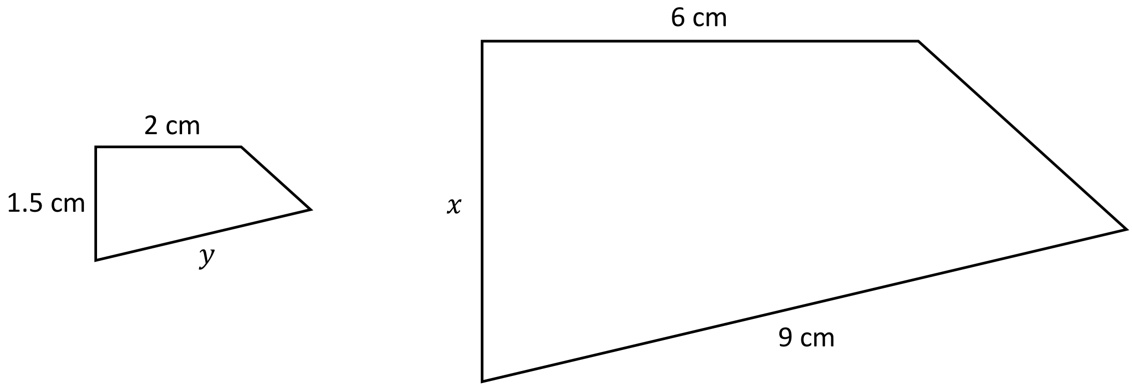
In mathematics, we say that the new diagram is **similar** to the original diagram. This means that the new diagram has the same **shape,** but the **size** has changed. Since the new diagram is twice as big, we can say that the **scale factor** is 2.

Similar shapes are the exact same shape, but of different sizes.

Given two shapes that are similar, we can use the measurements on the shapes either to find the scale factor or to find some missing lengths on the shapes.

Example

The two shapes shown below are similar shapes. Use the measurements on the shapes to find the lengths x and y.



Answer: The first step is to find the **scale factor.** To do this, we need to consider how much bigger the large shape is compared to the small shape. We see that the horizontal lengths at the top of both shapes are given. We can use these two lengths to find the scale factor, by calculating 6 ÷ 2 = 3. Therefore, the large shape is three times bigger than the small shape.

Having found the scale factor of 3, we can now use it to find the lengths x and y.

The edge that corresponds to the x edge in the small shape is 1.5 cm. We must **multiply** 1.5 cm by the scale factor to find the length of x, since we are going from the small shape to the large shape. Therefore

$$x = 1.5 \times 3$$
$$x = 4.5 \text{ cm}$$

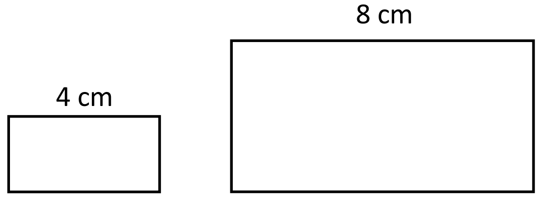
The edge that corresponds to the y edge in the large shape is 9 cm. We must **divide** 9 cm by the scale factor to find the length of y, since we are going from the large shape to the small shape. Therefore

$$y = 9 \div 3$$
$$y = 3 \text{ cm}$$

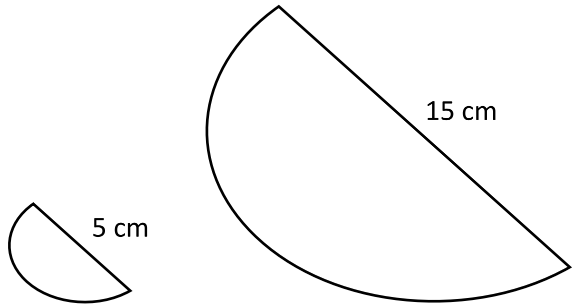
Exercise 33

The following pairs of shapes are similar shapes. Use the measurements on the shapes to find the scale factor.

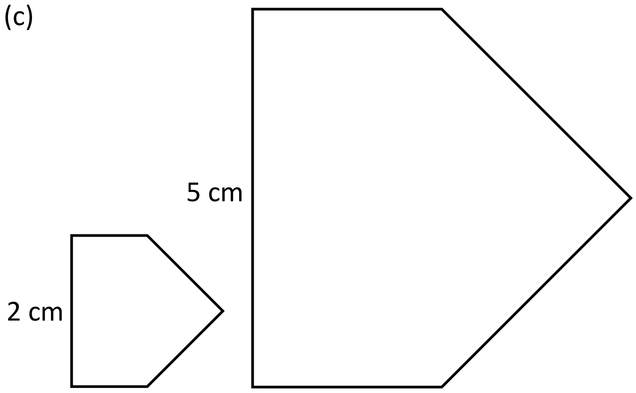
(a)



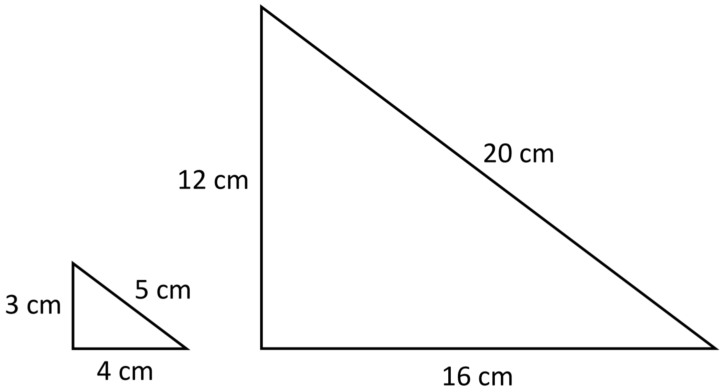
(b)



(c)



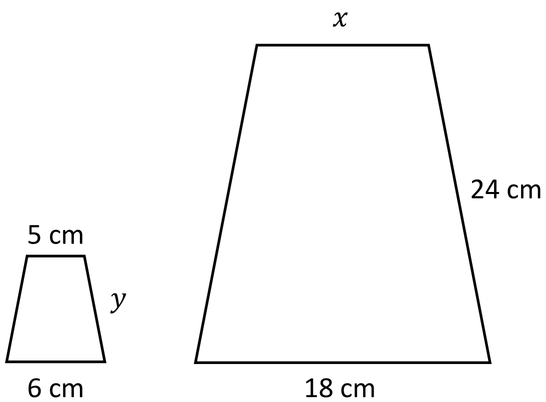
(d)



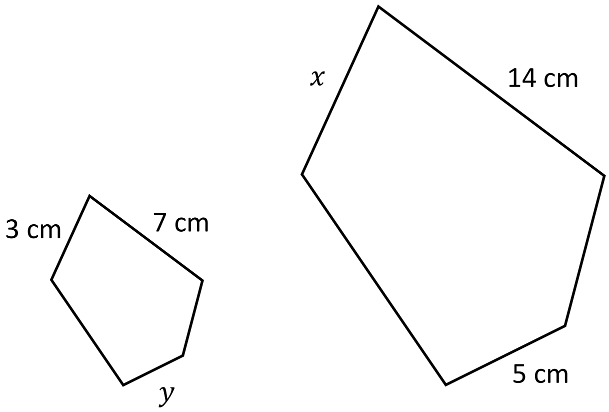
Exercise 34

The following pairs of shapes are similar shapes. Use the measurements on the shapes to find the lengths x and y.

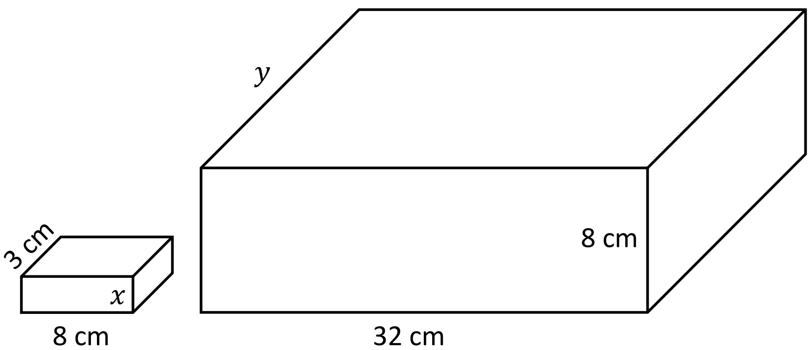
(a)



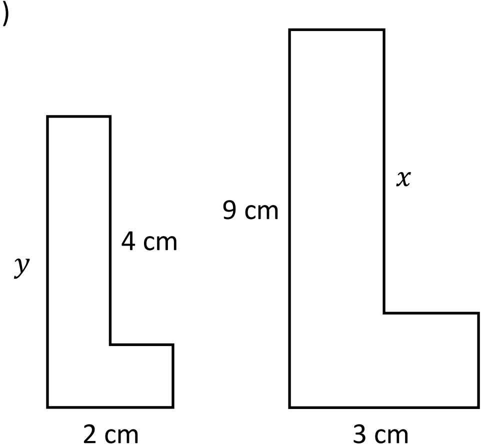
(b)



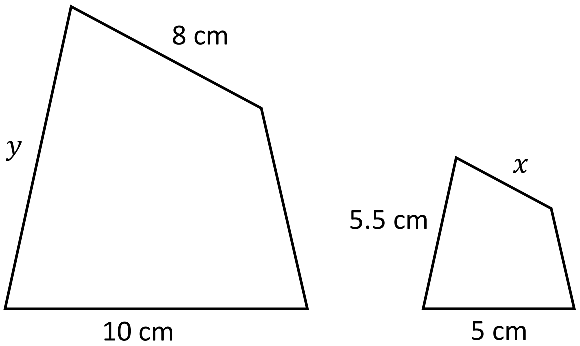
(c)



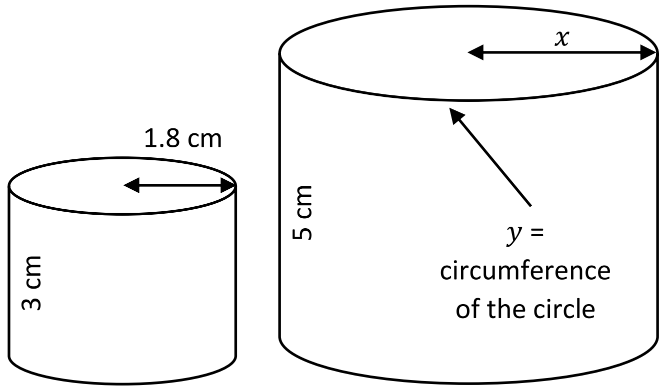
(d)



(e)



(f)

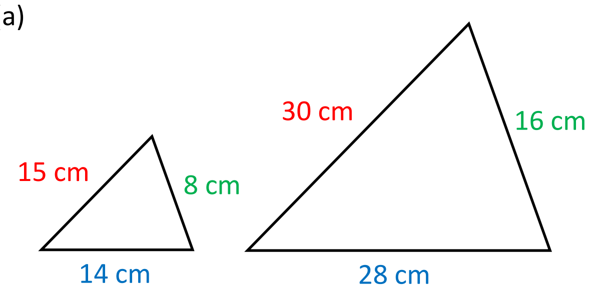


Similar or not?

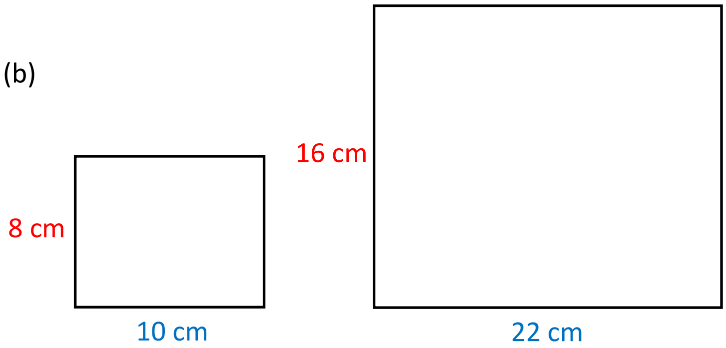
If two shapes are similar, then the **corresponding edges are in the same ratio.**
This means that if we divide a pair of corresponding edges, we will always obtain the same answer.

Example

(a)



(b)



For the two triangles above, the corresponding edges are in the same ratio.

30 ÷ 15 = 2
28 ÷ 14 = 2
16 ÷ 8 = 2

Therefore the two triangles are similar.

For the two rectangles above, the corresponding edges are not in the same ratio.

16 ÷ 8 = 2
22 ÷ 10 = 2.2

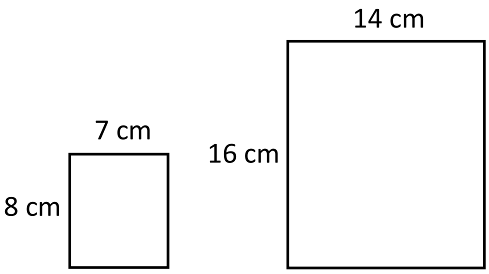
Therefore, the two rectangles are not similar.

Exercise 35

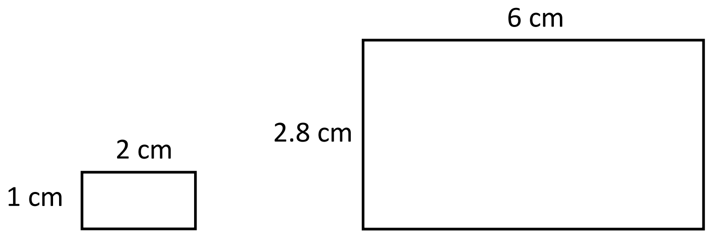
I

Decide whether the following pairs of shapes are similar or not.

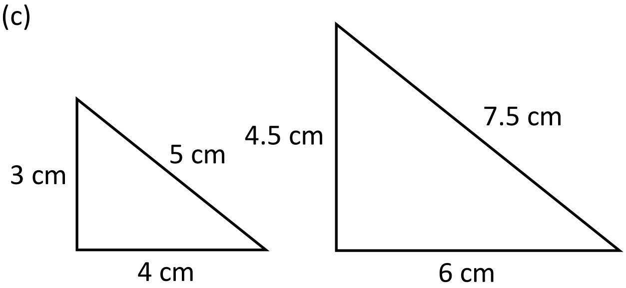
(a)



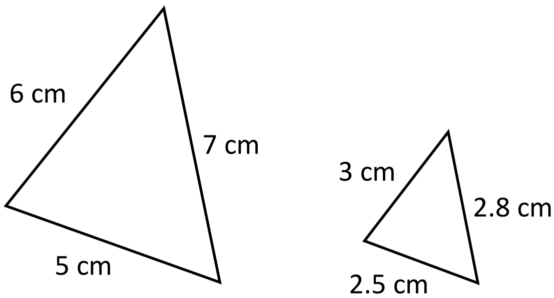
(b)



(c)



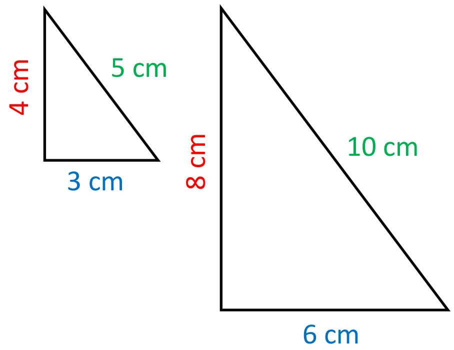
(d)



Similar Triangles

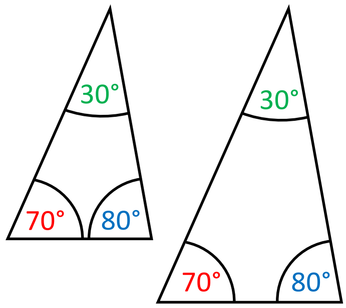
Two triangles are similar:

1) If the corresponding edges are in the same ratio;



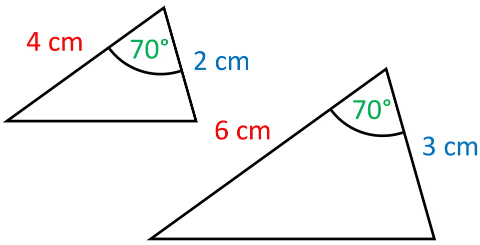
8 ÷ 4 = 2
6 ÷ 3 = 2
10 ÷ 5 = 2

2) if the corresponding angles are equal;



70° = 70°
80° = 80°
30° = 30°

3) if the ratio of two pairs of corresponding edges are the same and the angle between them is equal.

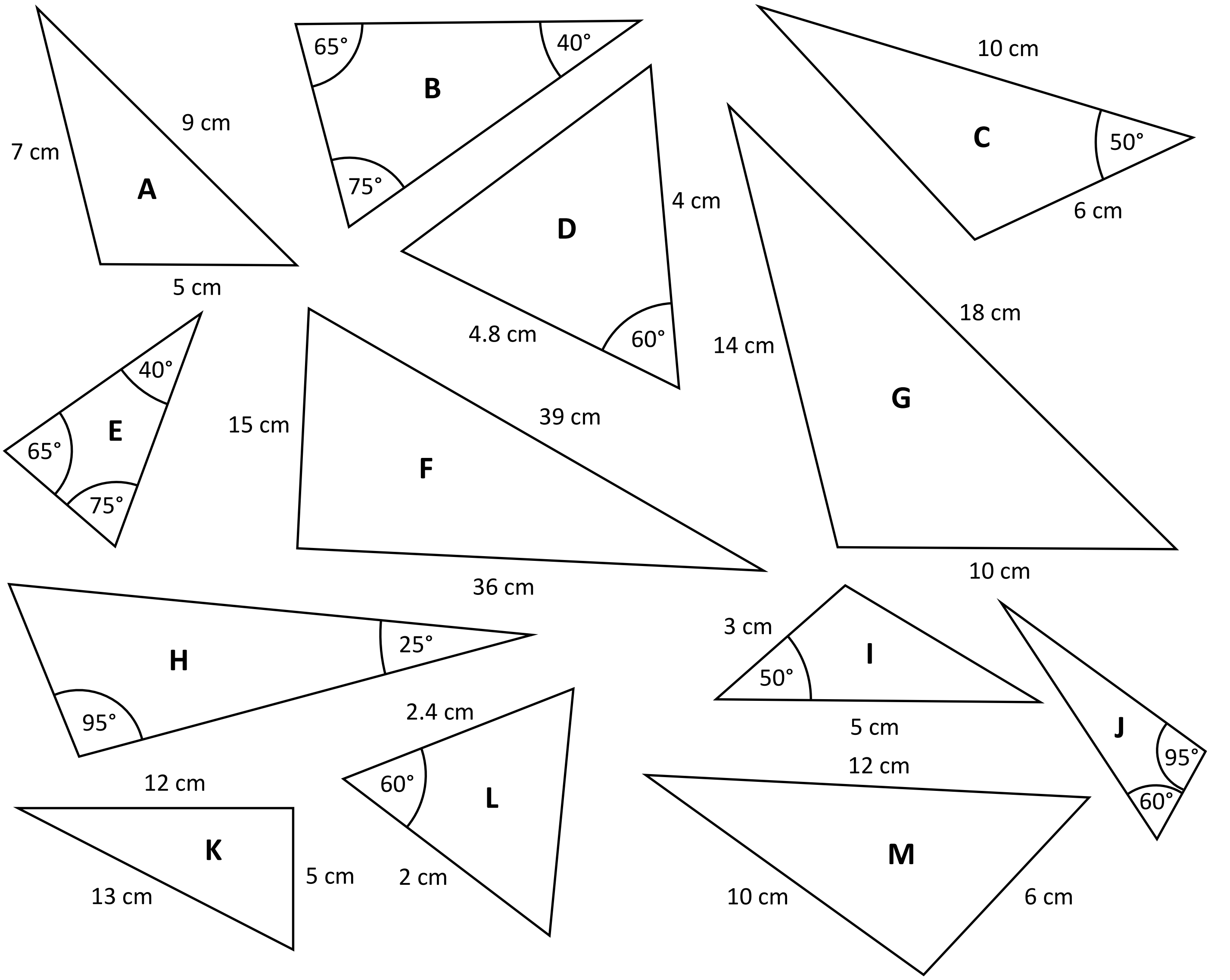


6 ÷ 4 = 1.5
3 ÷ 2 = 1.5
70° = 70°

Exercise 36

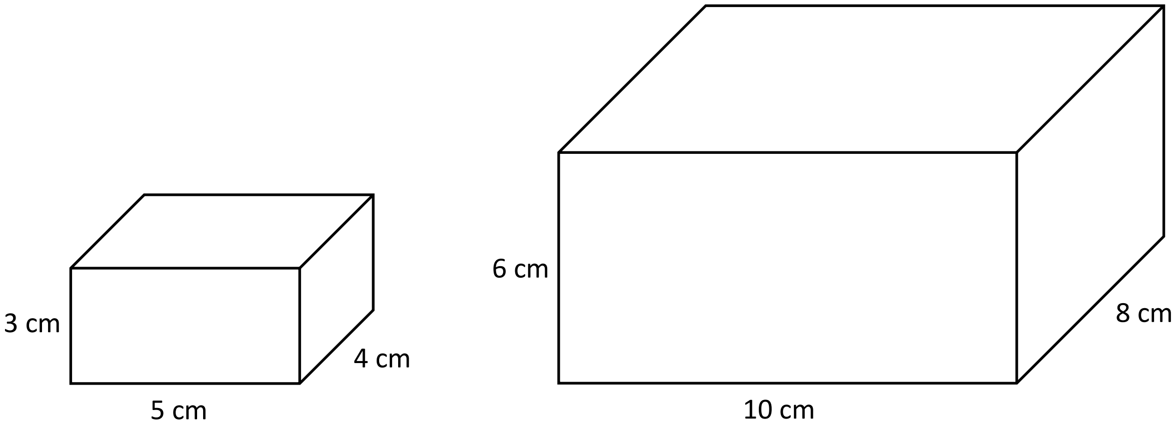
H

Here are 13 triangles. 6 pairs are similar and 1 is the odd one out. Find the similar pairs.



Scale factor for length, area and volume

Consider the two similar cuboids shown below.



By considering the corresponding edges, we can calculate that the scale factor is 2. This is the **length scale factor**, since the lengths (or the edges) were used in calculating the scale factor.

Next, we consider the top surface of the cuboids. For the small cuboid, the area of the top is $5 \times 4 = 20$ cm^2. For the large cuboid, the area of the top is $10 \times 8 = 80$ cm^2. It follows that the **area scale factor is** $80 \div 20 = 4$.
We can use the area scale factor to calculate corresponding areas. For example, the area of the front of the small cuboid is $3 \times 5 = 15$ cm^2. By multiplying by the area scale factor, the area of the front of the large cuboid is $15 \times 4 = 60$ cm^2. It would be possible to check this by calculating the area of the front of the large cuboid by using the dimensions of the cuboid: $6 \times 10 = 60$ cm^2. ✓

Lastly, consider the volume of the cuboids. For the small cuboid, the volume is $5 \times 4 \times 3 = 60$ cm^3. For the large cuboid, the volume is $10 \times 8 \times 6 = 480$ cm^3. It follows that the **volume scale factor** is $480 \div 60 = 8$.

For any three-dimensional shape, the following relationship exists between the length, area and volume scale factors.

If x is the length scale factor, then x^2 is the area scale factor and x^3 is the volume scale factor.

Exercise 37

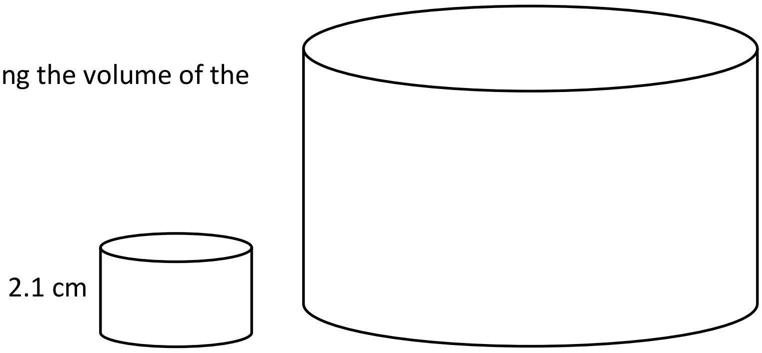
H

Complete the following table.

Length Scale Factor	Area Scale Factor	Volume Scale Factor
2	$2^2 = 4$	$2^3 = 8$
3		
	16	
		216
7		
	25	
		1,000
	81	
12		

Example

The diagram on the right shows two similar cylinders.
Given that the volume of the small cylinder is 40 cm³ and the volume of the large cylinder is 1,080 cm³, calculate the height of the large cylinder.



Answer: We can find the volume scale factor by dividing the volume of the large cylinder by the volume of the small cylinder.

1,080 ÷ 40 = 27

Next, we can find the length scale factor by taking the cube root of the volume scale factor.

$\sqrt[3]{27} = 3$

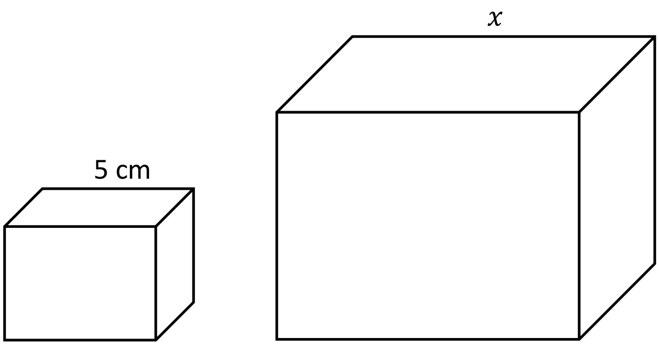
To calculate the height of the large cylinder, we must multiply the height of the small cylinder by the length scale factor.

2.1 × 3 = 6.3 cm.

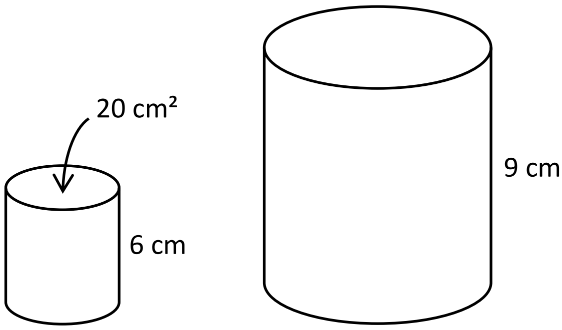
Exercise 38

H

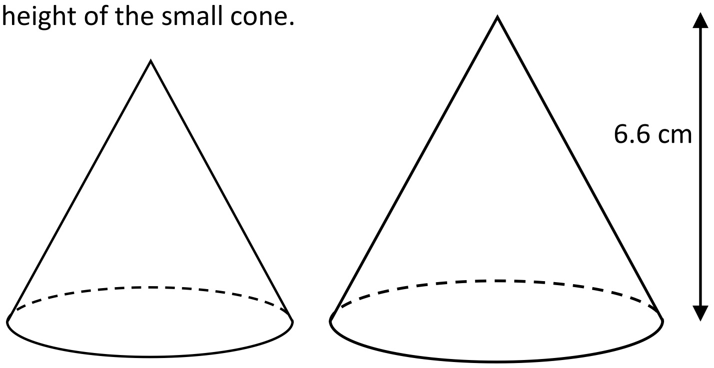
(a) The diagram below shows two similar cuboids. Given that the volume of the small cuboid is 30 cm³ and the volume of the large cuboid is 1,920 cm³, calculate the length x.



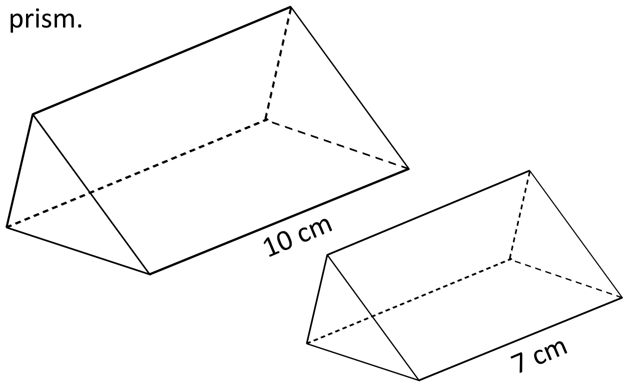
(b) The diagram below shows two similar cylinders. Calculate the area of the top of the large cylinder.



(c) The diagram below shows two similar cones. Given that the area of the base of the small cone is 40 cm² and the area of the base of the large cone is 48.4 cm², calculate the height of the small cone.



(d) The diagram below shows two similar prisms. Given the volume of the large triangular prism is 60 cm³, calculate the volume of the small triangular prism.



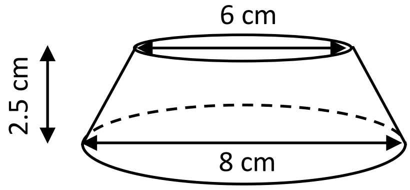
(e) Eleri has two similar spheres. The surface area of the small sphere is 60 cm² and the surface area of the large sphere is 194.4 cm². How much bigger is the volume of the large sphere compared to the volume of the small sphere?

(f) Dafydd has two similar pyramids. The volume of the large pyramid is 250 m³ and the volume of the small pyramid is 128 m³. How much taller is the large pyramid compared to the small pyramid?

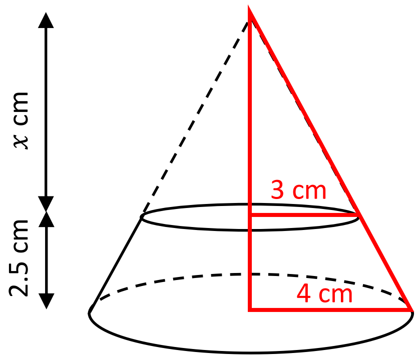
Using similar triangles to calculate the volume of a frustum of a cone

Example

Calculate the volume of the frustum shown on the right.



Answer: To start, let's add right-angled triangles to the diagram, as shown below.



To calculate the height of the large cone we can use the fact that the two red triangles are similar (the corresponding angles are equal).

$$\frac{\text{Base of the large triangle}}{\text{Base of the small triangle}} = \frac{\text{Height of the large triangle}}{\text{Height of the small triangle}}$$

$$\frac{4}{3} = \frac{x+2.5}{x}$$

$$4x = 3(x+2.5)$$

$$4x = 3x + 7.5$$

$x = 7.5$ cm

Therefore the height of the large cone is 10 cm and the volume of the frustum is

Volume of the whole cone – Volume of the missing cone

$= \frac{1}{3} \times \pi \times 4^2 \times 10 - \frac{1}{3} \times \pi \times 3^2 \times 7.5$

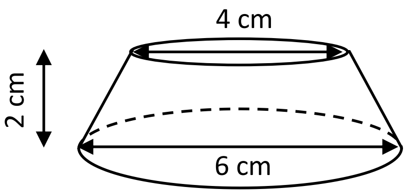
$= 96.87$ cm³, correct to two decimal places.

Exercise 39

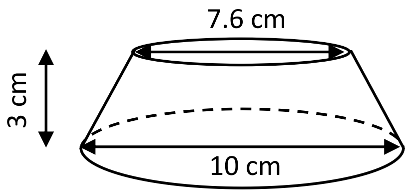
H

Calculate the volume of the following frustums of cones.

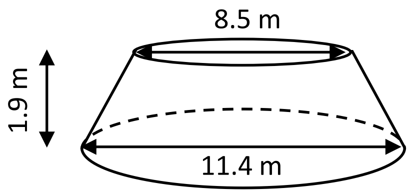
(a)



(b)



(c)



Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ❑ Target ❑

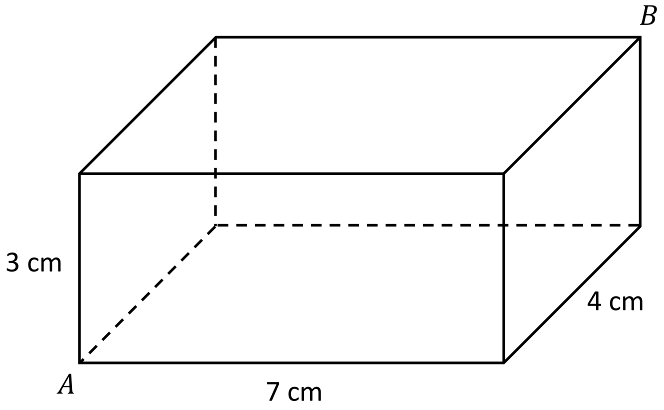
Pythagoras' Theorem (3-D)

Higher Tier

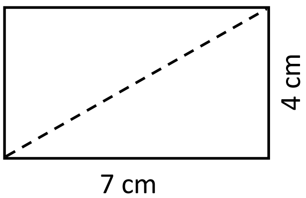
It is possible to use Pythagoras' Theorem to calculate lengths in three dimensional shapes.

Example

For the cuboid shown on the right, calculate the length of the longest diagonal, which is the distance between A and B.



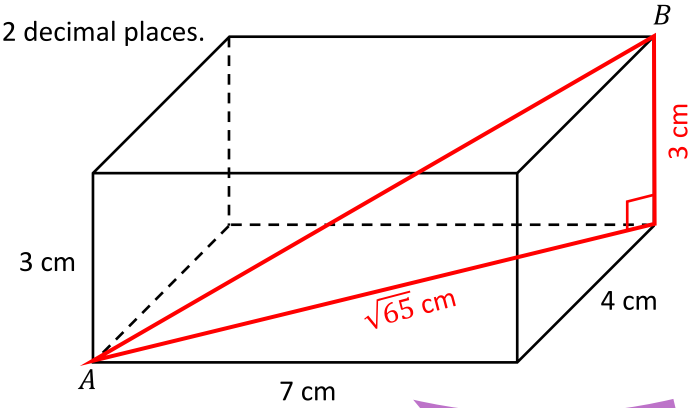
Answer: To start, let us use Pythagoras' Theorem to calculate the length of the diagonal on the base of the cuboid, which is the diagonal of this rectangle:



$$\begin{array}{l|rl} a^2 & 4^2 = & 16 \\ b^2 & 7^2 = & +\,49 \\ c^2 & & \overline{\;65\;} \end{array}$$

$\sqrt{65} = 8.06$ cm to 2 decimal places.

Next, we need to consider the red right-angled triangle shown on the right.



$$\begin{array}{l|rl} a^2 & 3^2 = & 9 \\ b^2 & \left(\sqrt{65}\right)^2 = & +\,65 \\ c^2 & & \overline{\;74\;} \end{array}$$

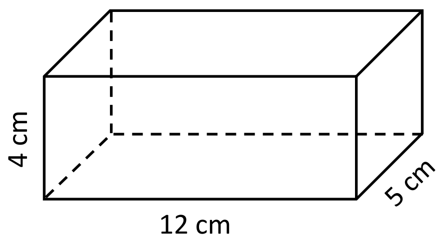
$\sqrt{74} = 8.60$ cm to 2 decimal places.

Exercise 40

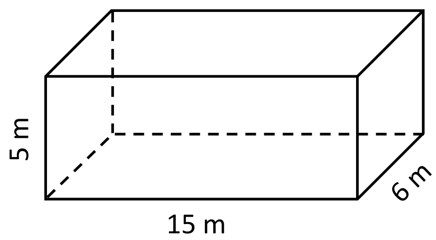
Skill H

Calculate the length of the largest diagonal in the following cuboids.

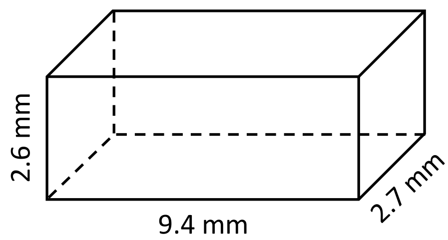
(a)



(b)



(c)

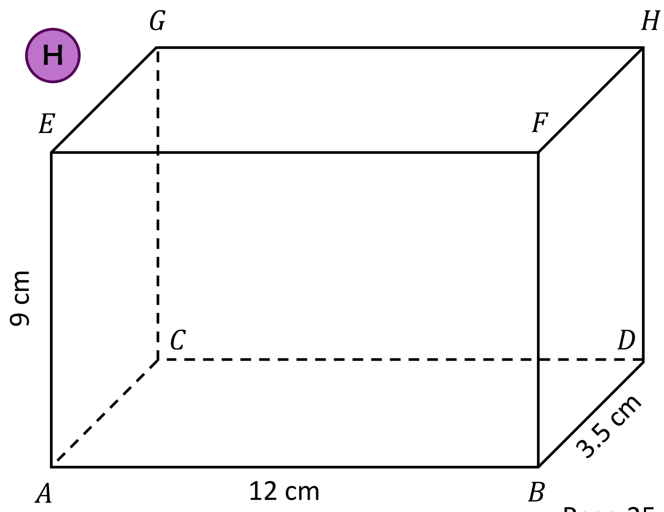


Exercise 41

H

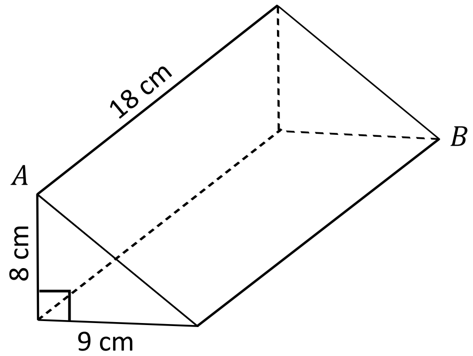
The diagram on the right shows a cuboid.
Calculate the shortest length between the following pairs of vertices.

(a) AD
(b) AG
(c) AF
(d) AH
(e) BG

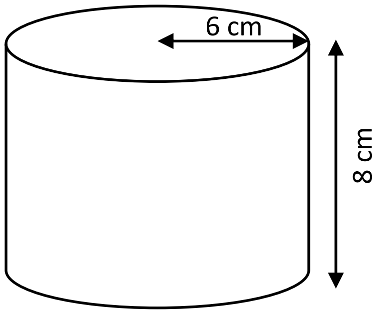


Exercise 42 H

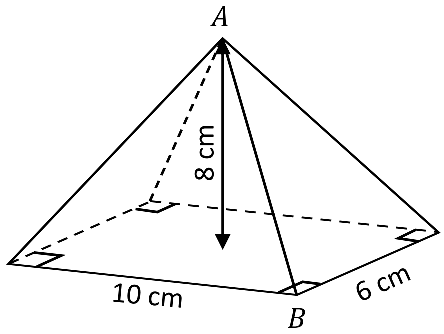
(a) Calculate the shortest distance between the vertices A and B.



(b) What is the length of the longest straw that can fit in this cylinder?



(c) Calculate the shortest distance between the vertices A and B.

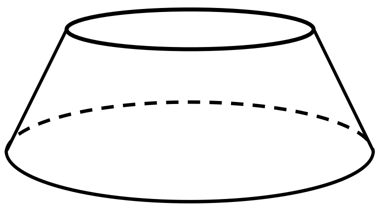


Exercise 43 H

The diagram on the right shows the frustum of a cone.

The radius of the base of the frustum is 9 cm. The radius of the top of the frustum is 6 cm. The **slant height** of the frustum is 5 cm.

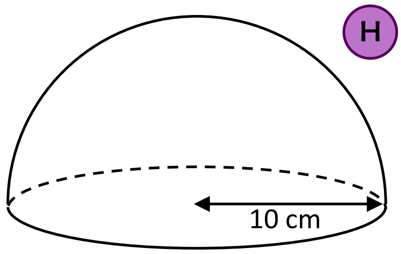
(a) Calculate the height of the frustum.
(b) Calculate the volume of the frustum.



Exercise 44 H

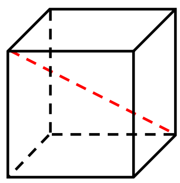
The diagram on the right shows a hemisphere.

(a) What is the height of the hemisphere?
(b) What is the shortest distance from the top of the hemisphere to a point on the circumference of the base of the hemisphere?



Challenge! ⚠

The length of the edge of a cube is x cm.
Find a general expression for the longest diagonal in the cube.



Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Measuring Solids

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test	Correct in the test?
I know how to calculate the **volume of a cuboid**.			1, 9	
I know how to calculate the **surface area of a cuboid**.			1	
I know how to calculate the **volume of a prism**.			2	
I know how to calculate the **volume of a cylinder**.			4, 9	
I know how to calculate the **surface area of a cylinder**.			4	
I know how to recognise whether a formula represents **a length, an area, a volume or none of these**.			5	
I can recognise the **number of dimensions** for specific quantities.			3	
I can calculate the **volume of composite solids**.			6	
Given two similar shapes, I can calculate the **scale factor**.			7	
Given two similar shapes, I can **calculate missing lengths**.			7	
I can decide **whether two shapes are similar**.			8	

Am I ready for the test?
Tick the boxes below...

- [ ] I have revised the work in my mathematics book.
- [ ] I have revised the workbook.
- [ ] I have watched the relevant videos on YouTube.
- [ ] I have completed the Diagnostic Questions quiz.
- [ ] I have completed at least 4 pages in my revision book.

Measuring Solids

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test	Correct in the test?
I know how to calculate the **volume and surface area of a cuboid**.				
I know how to calculate the **volume of a prism**.			3	
I know how to calculate the **volume and surface area of a cylinder**.			5	
I know how to calculate the **volume of a pyramid**.			1	
I know how to calculate the **surface area of a cone**.			2	
I know how to calculate the **volume and surface area of a sphere**.			1	
I know how to recognise whether a formula represents **a length, an area, a volume or none of these**.			6	
I can recognise the **number of dimensions** for specific quantities.			4	
I can calculate the **volume of composite solids**, including **hemispheres** and **frustums**.			7	
Given two similar shapes, I can calculate the **scale factor**.			8	
Given two similar shapes, I can **calculate missing lengths**.			8	
I can recognise whether **two triangles are similar**.			9	
I can work with **length, area and volume scale factors**.			10	
I can use **similar triangles to calculate the volume of a frustum**.			10	
I can use **Pythagoras' Theorem** to find lengths in **three dimensional** shapes.			7	

Am I ready for the test?
Tick the boxes below...

- [ ] I have revised the work in my mathematics book.
- [ ] I have revised the workbook.
- [ ] I have watched the relevant videos on YouTube.
- [ ] I have completed the Diagnostic Questions quiz.
- [ ] I have completed at least 4 pages in my revision book.

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The Mathematics Department

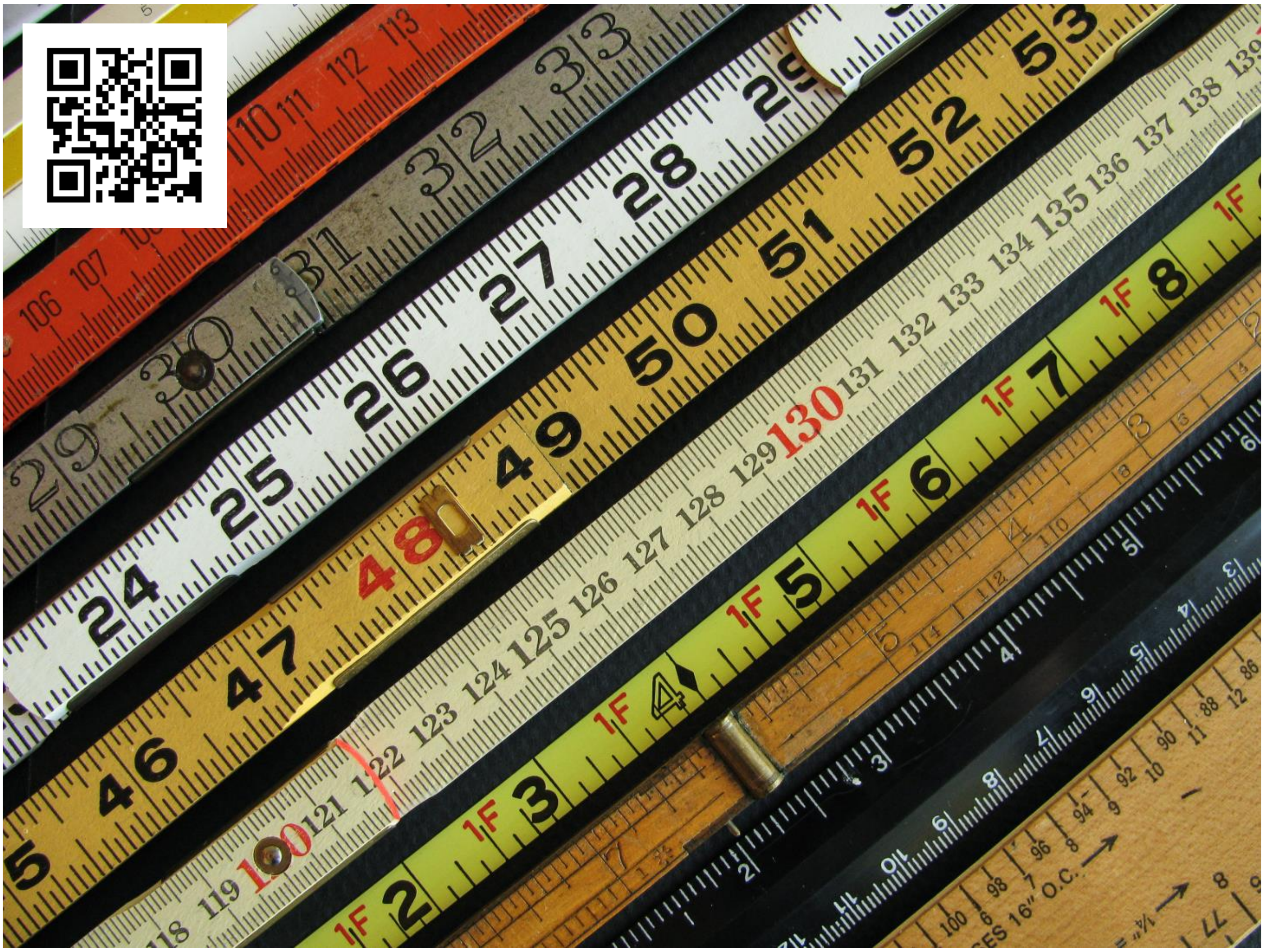
10

Accuracy of Measurements

Name:

Contents

Chapter	Mathematics	Page Number
Upper and Lower Bounds	Revision of how to round off. Upper and lower bounds. Problem solving (intermediate tier). **Problem solving (higher tier).**	3
Appropriate Degree of Accuracy	Calculating answers to an appropriate degree of accuracy.	8
Compound Measures	Distance, time and speed. Population, area and population density. Mass, volume and density. Other compound measures.	9



Upper and Lower Bounds

In order to understand how to find **upper and lower bounds**, it's a good idea to first revise different techniques of **rounding off** a number.

Revision I

Exercise 1

Complete the following table.

	Number	Round off to the nearest 10	Round off to 1 decimal place	Round off to 1 significant figure
E.g.	825.94	830	825.9	800
(a)	523.86			
(b)	49.15			
(c)	5,284.792			
(d)	3.67			
(e)	284.99			
(f)	43,704.75			
(g)	726			

Given a number rounded off in a particular way, we can consider what the original number was, before rounding occurred. For example, consider the number 470, which has been rounded off to the nearest 10. What could the original number have been? All the numbers between 465 and 474.9999⋯ round off (to the nearest 10) to be 470. If x represents the original number, then we can say that

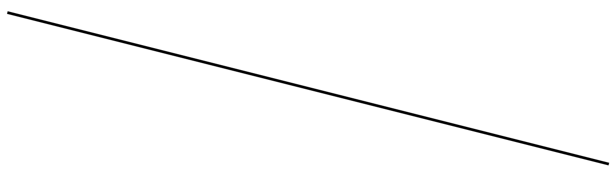
$$465 \leq x < 475.$$

We can also write

$$x = 470 \pm 5$$

but we must remember that x cannot be exactly 475, since this number would round off to give 480. We say that the **lower bound** is 465, and the **upper bound** is 475.

One place where upper and lower bounds are used are when taking measurements, since **every measurement is an approximation.** For example, measure the line below with a ruler.



You should measure the line to be 8.2 cm, but is the line exactly 8.2 cm? How do you know that the line isn't truly 8.19 cm, or 8.21 cm? We would need a more accurate measuring instrument than a ruler to check this, therefore the best we can say (by using a ruler and our eyes) is that the line is 8.2 cm, correct to the nearest 0.1 cm (or mm).

The accuracy limit of a measurement, approximation or calculation uses upper and lower bounds. To measure the above line with a ruler, the lower bound is 8.15 cm and the upper bound is 8.25 cm. The true measurement lies between these two limits. If l denotes the true length of the line in cm, then we can say that

$$8.15 \leq l < 8.25$$

or

$$l = 8.2 \pm 0.05.$$

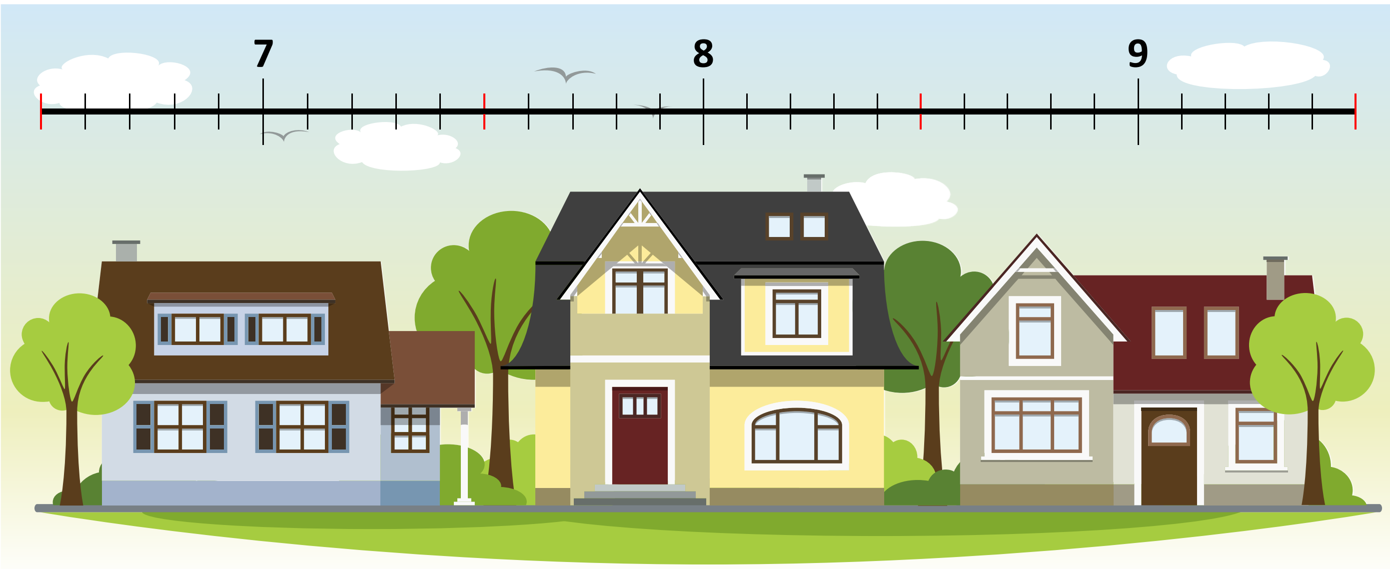
Example

Measurement	Lower Bound	Upper Bound
42 cm (to the nearest cm)	41.5 cm	42.5 cm
85 km (to the nearest 5 km)	82.5 km	87.5 km
60 m (to the nearest 10 m)	55 m	65 m

Exercise 2

Complete the following table.

	Measurement	Lower Bound	Upper Bound
(a)	34 cm (to the nearest cm)		
(b)	5 m (to the nearest m)		
(c)	148 kg (to the nearest kg)		
(d)	25 cm (to the nearest 5 cm)		
(e)	420 ml (to the nearest 5 ml)		
(f)	1,825 km (to the nearest 5 km)		
(g)	40 m (to the nearest 10 m)		
(h)	80 cm (to the nearest 10 cm)		
(i)	180 g (to the nearest 10 g)		
(j)	400 cm (to the nearest 100 cm)		
(k)	5,400 m (to the nearest 100 m)		
(l)	27,800 litres (to the nearest 100 litres)		
(m)	8,000 km (to the nearest 1,000 km)		
(n)	45,000 miles (to the nearest 1,000 miles)		
(o)	3,000 tons (to the nearest 1,000 tons)		
(p)	6 cm (to the nearest even number)		
(q)	154 cl (to the nearest even number)		
(r)	4,250 ml (to the nearest even number)		
(s)	80 cm (to the nearest 20 cm)		
(t)	250 ml (to the nearest 50 ml)		
(u)	320 kg (to the nearest 40 kg)		



Challenge!

How does the above picture explain how to find upper and lower bounds?

Example

Measurement	Lower Bound	Upper Bound
7.6 kg (to one decimal place)	7.55 kg	7.65 kg
50 cm (to one significant figure)	45 cm	55 cm
740 ml (to two significant figures)	735 ml	745 ml

Exercise 3

I

Complete the following table.

	Measurement	Lower Bound	Upper Bound
(a)	5.2 cm (to one decimal place)		
(b)	6.7 m (to one decimal place)		
(c)	92.0 inches (to one decimal place)		
(d)	8.24 m (to 2 decimal places)		
(e)	15.28 km (to 2 decimal places)		
(f)	104.09 m (to 2 decimal places)		
(g)	9.258 km (to 3 decimal places)		
(h)	435.205 miles (to 3 decimal places)		
(i)	9.984 seconds (to 3 decimal places)		
(j)	40 m (to one significant figure)		
(k)	400 cm (to one significant figure)		
(l)	4,000 cl (to one significant figure)		
(m)	430 cm (to two significant figures)		
(n)	6,500 ml (to two significant figures)		
(o)	5.2 cm (to two significant figures)		
(p)	500 cm (to two significant figures)		
(q)	7,450 kg (to three significant figures)		
(r)	7,300 kg (to three significant figures)		
(s)	7,000 kg (to three significant figures)		

Exercise 4

Applying

I

Usain Bolt's 100 m record is 9.58 seconds, correct to 2 decimal places.
How fast could Bolt have run the race, in reality?

Exercise 5

Square tiles are to be laid onto the floor of a room.
The side length of each of these tiles is 45 cm, measured to the nearest cm.

(a) Write down the least possible value and the greatest possible value for the length of a tile in cm.

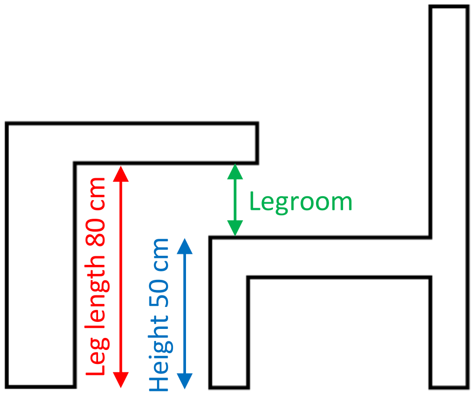
(b) Calculate the least possible value and the greatest possible value for the perimeter of a tile in cm.

Exercise 6

Two boxes are stacked on top of each other.
The height of one box is 57 cm, correct to the nearest cm.
The height of the other box is 28 cm, correct to the nearest cm.
Calculate the least possible height and the greatest possible height of the boxes stacked on top of each other.

Exercise 7

I



The legroom between a table and a chair is calculated by finding the difference between the leg length of the table and the height of the seat of the chair. In the diagram, the height of the seat of the chair and the leg length of the table are given to the nearest cm. Find, in centimetres, the least and greatest possible values of the legroom.

Exercise 8

A DIY shop sells lengths of kitchen worktops. There are two different suppliers for the worktops. One supplier, *Worktop Magic*, notes that the length of each worktop is 4,000 mm, measured to the nearest 5 mm. The other supplier, *Worktops 4 U*, notes that the length of each worktop is 4,000 mm ± 3 mm.

(a) Complete the following table.

	Shortest Possible Length	Longest Possible Length
Worktop Magic		
Worktops 4 U		

(b) A customer wants a worktop measuring at least 4.02 m. Would *Worktop Magic* or *Worktops 4 U* be able to supply a suitable worktop? Give a reason for your answer.

Exercise 9

Steffan measures the width of a table. He does this correct to the nearest cm.
Meinir measures the width of the same table. She does this correct to the nearest mm.

(a) What is the smallest possible difference between Steffan and Meinir's measurements?

(b) What is the largest possible difference between Steffan and Meinir's measurements?

Exercise 10

The diameter of the alloy wheel shown on the right is 15 inches ± 0.1 inches.

(a) Write down the smallest possible diameter of the alloy wheel.

(b) Write down the largest possible diameter of the alloy wheel.

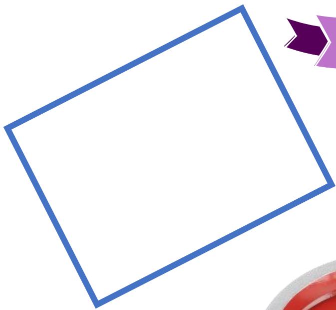
(c) Calculate the smallest possible circumference of the alloy wheel.

(d) Calculate the largest possible circumference of the alloy wheel.

(e) Copy and complete the following sentence:
The circumference of the alloy wheel is ☐ inches ± ☐ inches.

Exercise 11 H

Higher Tier



The length of a rectangle is 20 cm, correct to the nearest cm.
The width of the rectangle is 15 cm, correct to the nearest 5 cm.

(a) What is the longest possible length of the rectangle?
(b) What is the widest possible width of the rectangle?
(c) What is the smallest possible perimeter of the rectangle?
(d) What is the smallest possible area of the rectangle?

Exercise 12

The capacity of a paint pot is 250 ml, correct to the nearest 10 ml.

(a) What is the lowest possible capacity of the paint pot?
(b) What is the largest possible capacity of the paint pot?
(c) Dewi purchases three of these paint pots. The total volume of the paint is V ml.
Copy and complete this statement: $\square \leq V < \square$.

Exercise 13

A car travels 84 miles in 2.4 hours. The distance is measured correct to the nearest mile and the time is measured correct to the nearest 0.1 hours. Calculate the smallest possible average speed for the car over this distance. Give your answer in m.p.h. correct to one decimal place.

Exercise 14

Wall tiles for sale

Length 30 cm Width 15 cm

All measurements to the nearest cm

Is it always possible to tile an area up to 8,500 cm² using 20 of these tiles?
You must give a reason for your answer and show your method.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Appropriate Degree of Accuracy

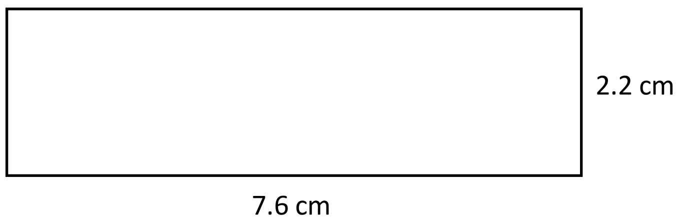
Measurements and calculations should not be too precise for their purpose. For example, advertising a television of size 39.97 inches would be too precise (it would be better to say that the television was 40 inches in size).

As a general rule, if a question asks you to give an answer to an **appropriate degree of accuracy**,

do not give an answer that is more precise than the values used in the calculation.

Example

Calculate the area of the following rectangle. Give your answer to an appropriate degree of accuracy.



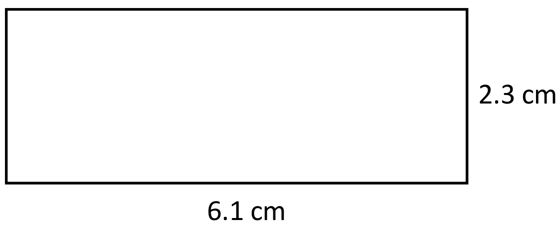
Answer: The area of the rectangle is $7.6 \times 2.2 = 16.72$ cm^2. Because the numbers in the question have been rounded off to one decimal place, then our answer should also be rounded off to one decimal place. Therefore, the area of the rectangle (to an appropriate degree of accuracy) is 16.7 cm^2.

Exercise 15

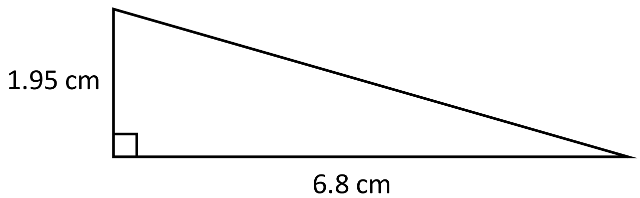
Skill I

Calculate the area of the following shapes. Give your answers to an appropriate degree of accuracy.

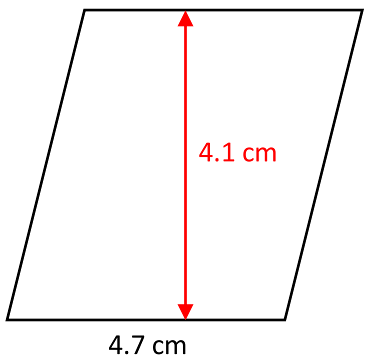
(a)



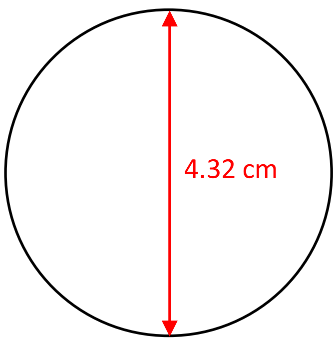
(b)



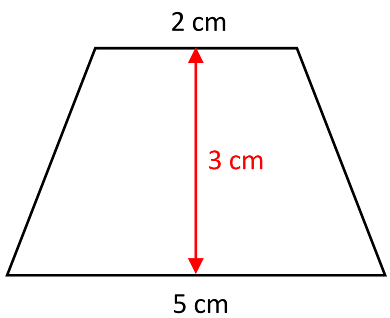
(c)



(d)



(e)



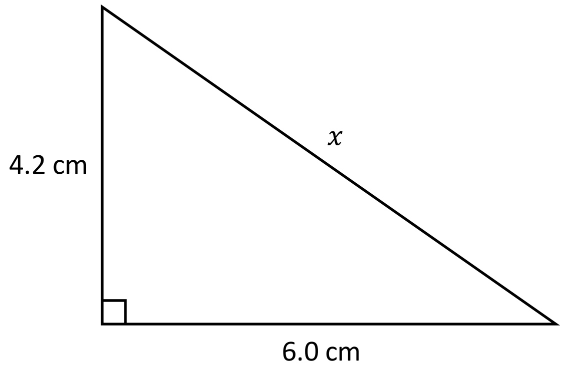
Exercise 16

Mabli has a cuboid that measures 5.2 cm by 8.9 cm by 12.8 cm. Calculate, to an appropriate degree of accuracy, the volume of the cuboid.

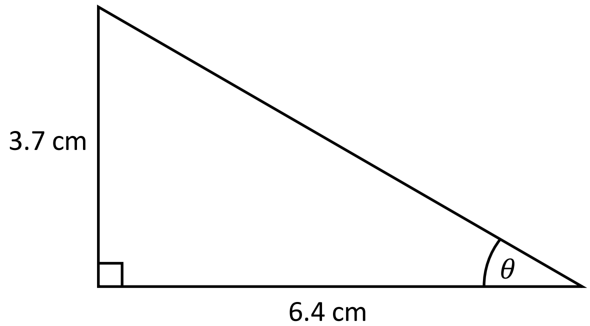
Exercise 17

I

(a) Calculate x to an appropriate degree of accuracy.



(b) Calculate θ to an appropriate degree of accuracy.



Exercise 18

Applying

Lisa wants to invest £6,000 into Barclays Bank at a compound interest rate of 3% per year. Lisa wants to withdraw all the money from the bank after four years. How much money will Lisa be able to withdraw after four years? Give your answer to an appropriate degree of accuracy.

Exercise 19

Rhys is going on holiday to China. The exchange rate for changing money in British pounds (£) to money in Chinese yuan (CYN) is shown below.

£1 = 9.28 CYN

Whilst in China, Rhys catches a taxi that costs 200 CYN. How much is this in pounds? Give your answer to an appropriate degree of accuracy.

Exercise 20

During an experiment, a scientist notices that the number of bacteria halves each second. There were 2.3×10^{30} bacteria at the start of the experiment. Calculate how many bacteria were present after 5 seconds. Give your answer to an appropriate degree of accuracy.

Exercise 21

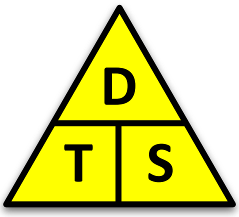
Cynan has a cylinder with diameter 5.3 cm and height 14 cm.
Calculate, to an appropriate degree of accuracy, the volume of the cylinder.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Compound Measures

A **compound measure** is a type of measure that combines two different measures.
In the "Movement with Sphero" workbook, we saw an example of a compound measure, namely **speed**.



Distance = Time × Speed
Time = Distance ÷ Speed
Speed = Distance ÷ Time

To use the triangle, *hide* the letter that you want to find.

Example

(a) Iwan cycles a distance of 32 km in $2\frac{1}{2}$ hours.
Calculate Iwan's average speed in km/hour.

Answer: Speed = Distance ÷ Time
= 32 ÷ 2.5
= 12.8 km/hour

(b) Moli runs an 800 m race in 2 minutes 48 seconds.
What is Moli's average speed in metres per second?

Answer: Speed = Distance ÷ Time
= 800 ÷ 168
= 4.8 metres per second (to one decimal place)

Exercise 22

Revision

F

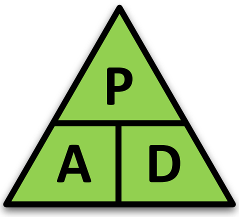
(a) Hannah cycles a distance of 49 km in $3\frac{1}{2}$ hours.
Calculate Hannah's average speed in km/hour.
(b) Elin runs a 400 m race in 1 minute 12 seconds.
What is Elin's average speed in metres per second?
(c) A bus travels the 12 miles from Llandudno to Abergele in 30 minutes.
Find the average speed of the bus in miles per hour.
(d) A train travels at an average speed of 90 km/hour for 2 hours 15 minutes.
How far has the train travelled during this time?
(e) In a 100 m race Dilwyn ran at an average speed of 7.1 metres per second.
How much time did Dilwyn take to finish the race?
(f) An aeroplane flies at an average speed of 885 km/hour. How far did the aeroplane travel between the times 0820 and 0900?
(g) The following chart shows the travelling distance for a car, in miles, between some places in Wales.

	Swansea			
Aberystwyth	70	Aberystwyth		
Bangor	157	86	Bangor	
Cardiff	41	98	180	Cardiff
Wrexham	130	79	70	139

(i) What is the travelling distance for a car between Cardiff and Bangor?
(ii) Dewi travels between Cardiff and Bangor, and then between Bangor and Wrexham. How far has Dewi travelled in total?
(iii) Elis travels between Bangor and Aberystwyth in 2 hours and a half. What is Elis' average speed, in mph?
(iv) Esyllt travels between Wrexham and Swansea at an average speed of 32 miles per hour.
How much time did Esyllt take to travel between Wrexham and Swansea?

Population Density

Population density compares the population of a place to its area.



Population = Area × Population Density
Area = Population ÷ Population Density
Population Density = Population ÷ Area

Population density is (usually) measured in people per square kilometre, and is used to compare how developed different areas are.

Example

Paris has a population of 2,265,886 and an area of 105.4 km². What is Paris' population density?

Answer: Population Density = Population ÷ Area
= 2,265,886 ÷ 105.4
= 21,498 people per km² (correct to the nearest whole number)

Exercise 23

Applying

I

(a) Mumbai has a population of 12,478,447 and an area of 603 km².
What is Mumbai's population density?

(b) San Francisco has a population of 805,816 and an area of 120.93 km². What is San Francisco's population density?

(c) Dublin has a population of 553,165 and an area of 114.99 km².
What is Dublin's population density?

(d) The population density of London is 5,584 people per km².
If London's population is 8,778,500, what is London's area?

(e) The population density of Chicago is 4,582 people per km².
Is Chicago's population is 2,695,598, what is Chicago's area?

(f) The population density of Cairo is 19,376 people per km².
If Cairo's area is 606 km², what is Cairo's population?

(g) The population density of Miami is 4,324 people per km².
If Miami's area is 92.38 km², what is Miami's population?

(h) The following table gives information about some Spanish cities.

City	*Madrid*	*Barcelona*	*Seville*
Population	3,141,991	1,621,537	703,021
Area	604.3 km²	101.9 km²	140 km²

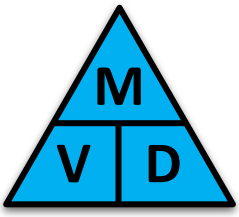
(i) Calculate the population density of the three Spanish cities.
(ii) Which city is the one where people live closest to each other?

Challenge!

Try to calculate the population density of where you live.

Density

Density compares an object's mass to its volume.



Mass = Volume × Density
Volume = Mass ÷ Density
Density = Mass ÷ Volume

To use the triangle, *hide* the letter that you want to find.

Density is (usually) measured in g/cm³ or kg/m³, and is used to compare how much mass an object has per unit of volume.

Example

(a) The mass of a 200 cm³ piece of metal is 1.2 kg. What is the metal's density, in g/cm³?

Answer: Density = Mass ÷ Volume
= 1,200 ÷ 200
= 6 g/cm³

(b) The density of a piece of aluminium is 2.7 g/cm³. Calculate the mass of a 20 cm³ piece of aluminium.

Answer: Mass = Volume × Density
= 20 × 2.7
= 54 g

Exercise 24

I

(a) The mass of a stone is 4.4 kg.
(i) What is the mass of the stone in grams?
(ii) Find the density of the stone (in g/cm³) if its volume is 2,000 cm³.

(b) A piece of cork weighs 10 kg. Its volume is 0.04 m³.
Calculate the density of the cork, in kg/m³.

(c) The mass of a 1,200 ml piece of ice is 1,104 grams. Find its density.
(Hint: 1 ml = 1 cm³.)

(d) Gold is expensive and desirable not just for how it looks but also because it does not rust easily. Lewis wishes to build a car made out of gold but is worried about its mass. The density of gold is 19,300 kg/m³ and the volume of Lewis' car is around 1.5 m³.
(i) Calculate the mass of Lewis' gold car.
(ii) Given that the mass of a normal car is around 1,500 kg, comment on the practicality of Lewis' plan.

(e) Megan is mixing and pouring concrete. She mixes and then pours a total of 2.4 m³ of concrete. Calculate the mass of this concrete if its density is 2,500 kg/m³.

(f) The density of water is 1 g/cm³. Calculate the mass of 1.5 litres of water.

(g) A clay statue has density 1.4 g/cm³. Another statue is carved from wood and has a density of 0.8 g/cm³. The two statues are weighed in order to find their mass. The wood statue weighs 4,500 g. The clay statue weighs 7,700 g. Which statue has the greatest volume?

(h) The dimensions of a metal cuboid are 8 cm, 6 cm and 5 cm.
The mass of the cuboid is 0.9 kg. Calculate the density of the metal, noting your units clearly.

Other Compound Measures

Exercise 25

(a) The fuel consumption of Bethan's car is 50 miles per gallon.
(i) If Bethan travels 200 miles in her car, how many gallons of petrol does she use?
(ii) Today, petrol costs 117.9 pence per litre. Given that a gallon of petrol is around 4.5 litres, calculate the cost of the petrol for Bethan's journey. Give your answer correct to the nearest penny.

(b) John travels 126 miles in his car, using 2.25 gallons of petrol. What is the fuel consumption of John's car, in miles per gallon?

(c) Janet types at a rate of 70 words per minute. How much time would Janet need to type a report containing 3,500 words?

(d) The average person can type at a rate of 40 words per minute. Bob types a 1,200-word report in 40 minutes. Does Bob type at a rate that is slower than the average person, or is quicker than the average person?

(e) A mill produces 20 metric tons/hour of flour.
How much flour is produced every 15 minutes?

(f) A paint tin notes that it can cover an area of 13 m^2/litre.
If the paint tin contains 2.5 litres of paint, what total area can the tin cover?

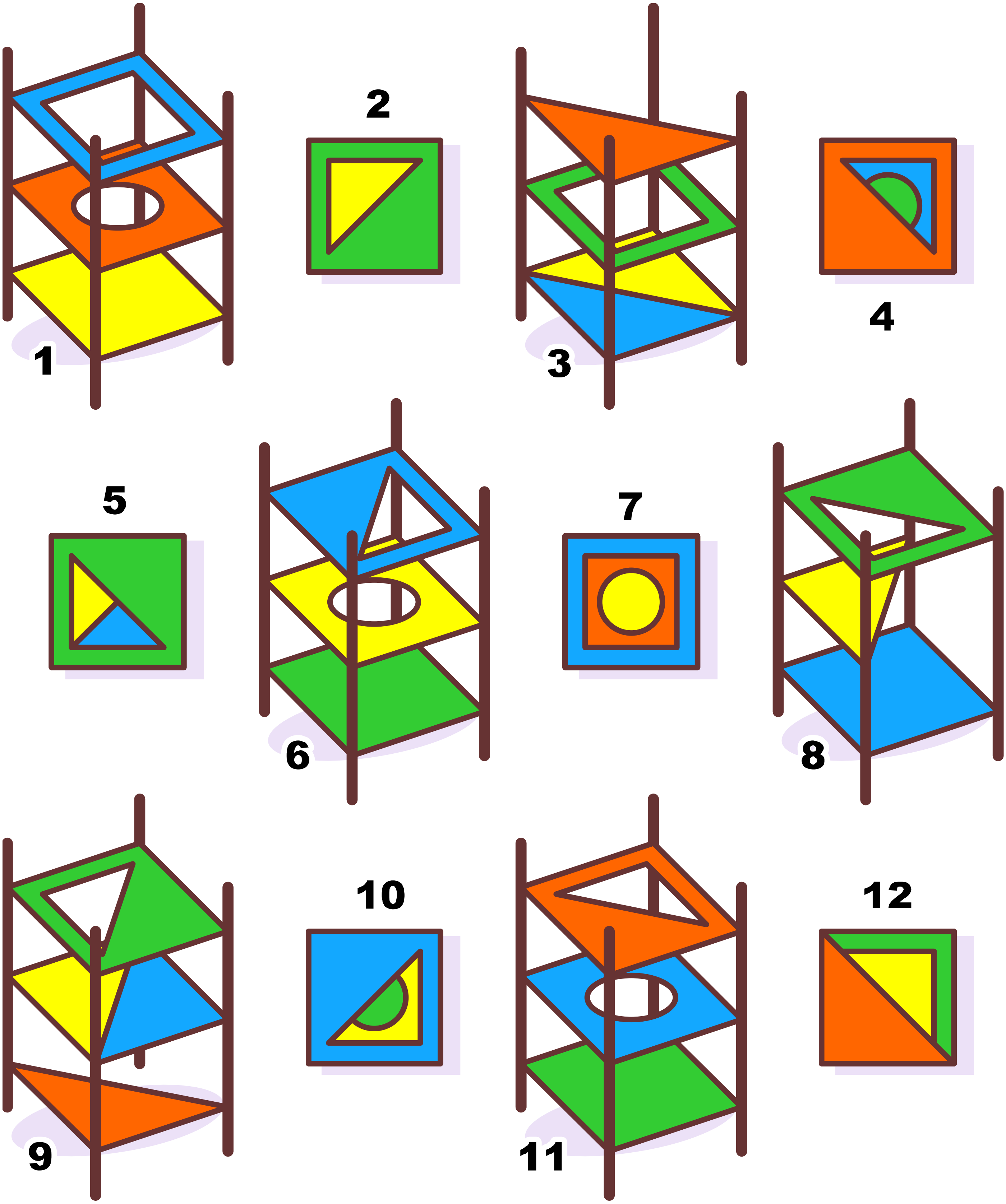
(g) A swimming pool is emptied in order to clean it. The pool (when full) can hold 375,000 litres of water. After it is cleaned, the pool is refilled by using a water pipe that can supply water at a rate of 200 litres per minute. How much time will it take to refill the pool? Give your answer in hours and minutes.

(h) During the first 20 games of a football season, Gareth Bale scored an average of 0.7 goals per game.
How many goals is it expected that Gareth Bale scores during the next 6 games?

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Puzzle: Can you find the top view for every stand?



Accuracy of Measurements

Reflection Sheet

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test	Correct in the test?
I know how to find the **upper and lower bounds** of a measurement.			1, 2, 3, 10	
I know how to interpret measurements that are given using the symbol ±, for example 32 cm ± 0.5 cm.			3	
I can use upper and lower bounds to **solve problems in context**.			2, 3, 10	
I know how to write answers to an **appropriate degree of accuracy**.			4, 10	
I can use the formulae that are associated with the compound measure **speed**.			6	
I can use the formulae that are associated with the compound measure **population density**.			5	
I can use the formulae that are associated with the compound measure **density**.			7	
I am aware of how to work with **other compound measures**.			8, 9	

Am I ready for the test?
Tick the boxes below...

I have revised the work in my mathematics book.	☐
I have revised the workbook.	☐
I have watched the relevant videos on YouTube.	☐
I have completed the Diagnostic Questions quiz.	☐
I have completed at least 4 pages in my revision book.	☐

Accuracy of Measurements

Reflection Sheet

Name: ....

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I can use upper and lower bounds to **solve problems in context**.			2, 3, 9, 10	
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I can use the formulae that are associated with the compound measure **population density**.			5	
I can use the formulae that are associated with the compound measure **density**.			7	
I am aware of how to work with **other compound measures**.			8, 9	

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Tracking Sheet
Year 11
End of Year 10
Grade: ____
Target Grade:
____
Tier: ________
Attainment for
Term 1: ____
%
Developing Algebra 3
%
Measuring Shapes 4
%
Developing Probability
%
Year 11
Mock Examination

YSGOL CREUDDYN
DAWN DYSG DAIONI

The Mathematics Department

11

Developing Algebra 3

Name:

Contents

Chapter	Mathematics	Page Number
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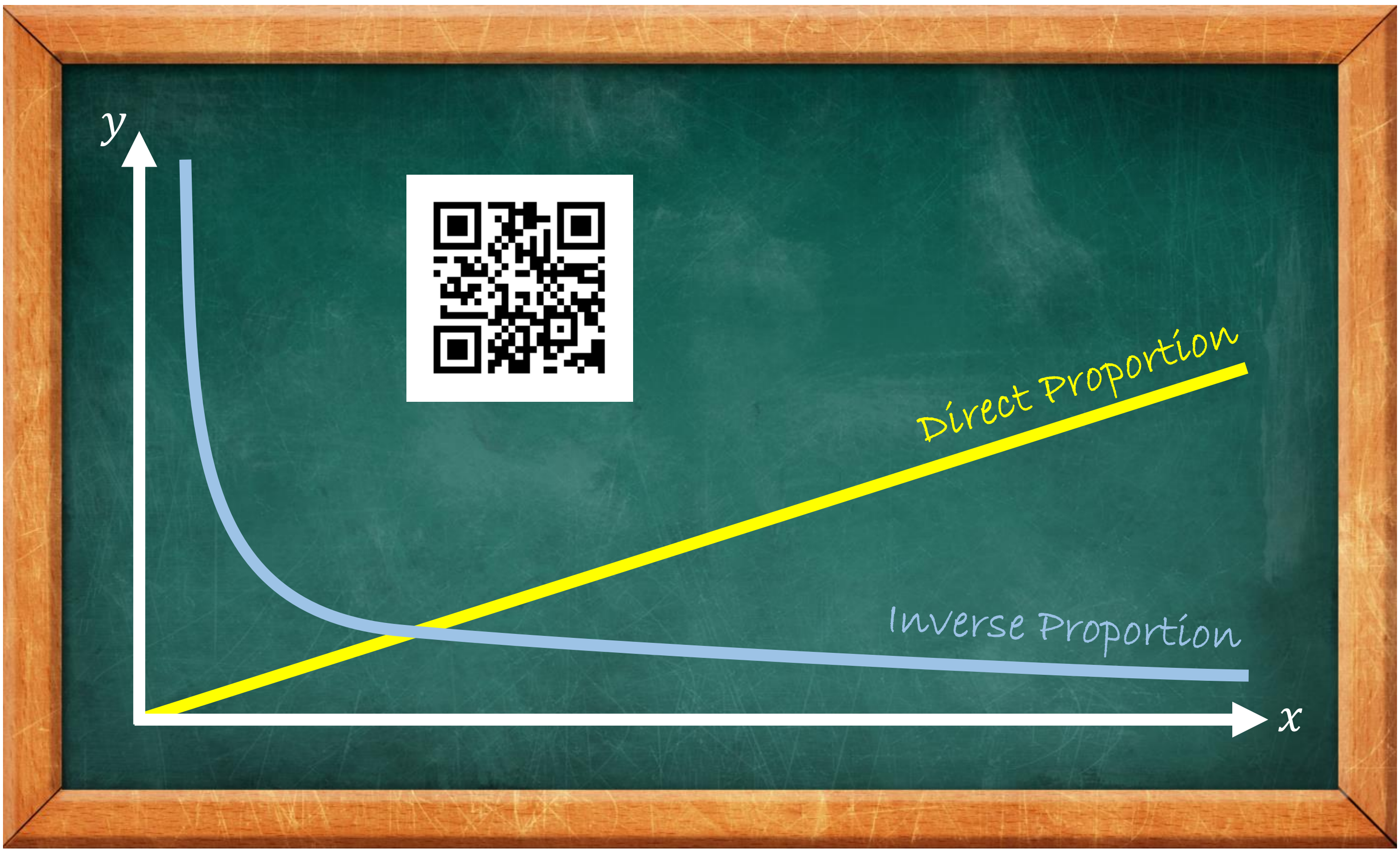
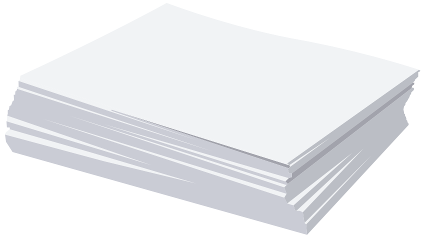
Proportion Equations	**Direct Proportion. Inverse Proportion. Finding Proportion Equations.**	10
Quadratic Nth Term	Linear Nth Term. The First Difference. Simple Quadratic Sequences. More Complex Quadratic Sequences.	13
Inequalities	Inequality Symbols. Inequalities on a Number Line. Solving Equations. Solving Inequalities.	18
Regions of Graphs	Revising plotting graphs of the form $x = a$ and $y = b$. Revising plotting graphs of the form $y = mx + c$. Plotting graphs of the form $ax + by + c = 0$. **Shading Regions.**	22



Proportion



Two measures are in **proportion to each other** if there is a **connection** between the measures. For example, the more pieces of paper there are in a pile of paper, the higher the pile will be. We say that the height of the pile of paper and the number of pages in the pile are in proportion to each other.

The **type** of proportion depends on the type of connection between the measures.

Direct Proportion	Inverse Proportion
As one measure increases, the other measure also increases.	As one measure increases, the other measure decreases.

Example

(a) The distance a car travels is in direct proportion to the amount of petrol it uses.

(b) The average speed of a car on a specific journey is in inverse proportion to the time the car takes to make the journey.

Exercise 1

Skill I

Note which type of proportion (direct proportion or inverse proportion) the following questions describe.

(a) The height of a pile of paper and the number of pages in the pile.

(b) The length of a piece of string and the mass of the string.

(c) The time taken to build a wall and the number of workers used to build the wall.

(d) The number of tins of soup bought and the total cost of the tins.

(e) The time taken to empty a water tank and the number of water pumps used to empty the tank.

(f) The number of pages in a book and the time taken to read the book.

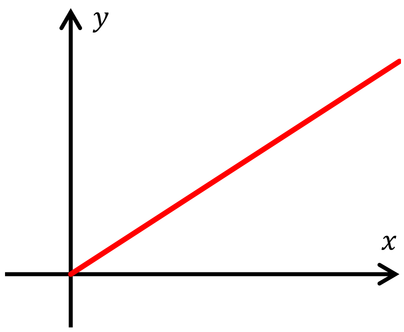
(g) The distance a car travels in half an hour and the average speed of the car.

(h) The age of a car and the monetary value of the car (during the first decade after the initial purchase).

(i) The mass of a bar of gold and the monetary value of the bar.

Direct Proportion

With direct proportion, when one measure increases (e.g. miles travelled, x), another measure must also increase (e.g. amount of petrol used, y). We can write this relationship as $y \propto x$. The symbol $\propto$ means "in proportion to". The graph on the right illustrates direct proportion. The gradient of the line (the multiplier of the proportion, k) can be any positive value.



Example

A digger can dig a 560 m long ditch in 21 days. How much time would it take to dig a 240 m long ditch?

Answer: To begin, we set out the information in a table, ensuring that the quantity we want to find appears in the bottom-right of the table.

Length of the ditch	Time
560 m	21 days
240 m	?

Method 1: Multiplier method

To change the number 560 to be the number 21, we must multiply by the fraction $\frac{21}{560}$. (Starting with 560, we divide by 560 to reach 1, and then multiply by 21 to reach 21.)

We multiply 240 m with the same fraction to obtain the answer: $240 \times \frac{21}{560} = 9$ days.

Method 2: DM method (Divide, Multiply)

We imagine an L shape formed using the numbers in the table.

Length of the ditch	Time
560 m	21 days
240 m	?

We follow the path of the L shape, **dividing** first and then **multiplying** by the numbers we encounter.
Either 240 ÷ 560 × 21 = 9 days
or 21 ÷ 560 × 240 = 9 days.

Exercise 2

Applying

I

(a) A train travels 165 metres in 3 seconds.
How far would it travel in 8 seconds?

(b) An aeroplane travels 216 miles in 27 minutes.
How far would it travel in 12 minutes?

(c) £50 is worth \$90. How much is £175 worth?

(d) A 7-metre ladder has 28 steps.
How many steps would a similar 5-metre ladder have?

(e) The mass of a 27 metre long piece of string is 351 grams.
What would be the mass of 15 metres of the same type of string?

(f) A rabbit can burrow a 4 metre long tunnel in 26 hours. How long would the rabbit take to burrow a 7 metre long tunnel?

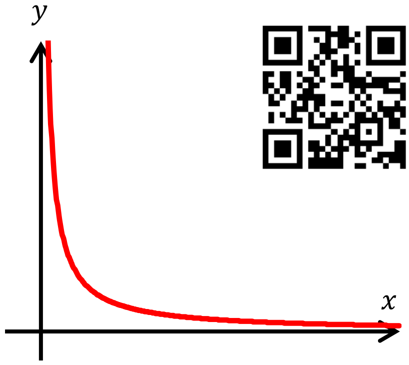
(g) A landscape gardener can paint 15 fence panels in 6 hours.
How long would it take to paint 40 fence panels?

(h) The cost of 12 printer cartridges is £90.
What is the cost of five of these printer cartridges?

(i) The height of 500 pieces of paper is 4.9 cm.
What would be the height of 800 pieces of the same type of paper?

Inverse Proportion

With inverse proportion, when one measure increases (e.g. average speed of a car, x), another measure decreases (e.g. the time taken to complete the journey, y). We can write this relationship as $y \propto \frac{1}{x}$. We read this as "y is inversely proportional to x". The graph on the right illustrates inverse proportion.



Example

If three diggers can dig a hole in 8 hours, how long would four diggers take to dig the same hole?

Method 1: Division method

To begin, we set out the information in a table, ensuring that the quantity we want to find appears in the bottom-right of the table.

Number of diggers	Time
3	8 hours
4	?

By multiplying together the numbers in the first row, we obtain 3 × 8 = 24. We can divide the 24 by the number of diggers to obtain the time taken. For 3 diggers, the time taken is 24 ÷ 3 = 8 hours (verifying the information given in the question). For 4 diggers, the time taken is 24 ÷ 4 = 6 hours.

Method 2: DM method (Divide, Multiply)

To begin, we set out the information in a table, ensuring that the quantity we want to find appears in the bottom-right of the table. (Note, because this is an inverse proportion question, the second column shows $\frac{1}{\text{Time}}$, not Time). We then imagine an L shape formed using the numbers in the table.

Number of diggers	$\frac{1}{\text{Time}}$
3	$\frac{1}{8}$
4	$\frac{1}{?}$

We follow the path of the L shape, **dividing** first and then **multiplying** by the numbers we encounter.

Either $4 \div 3 \times \frac{1}{8} = \frac{1}{6}$

or $\frac{1}{8} \div 3 \times 4 = \frac{1}{6}$

So, the answer is 6 hours, because 6 is the **reciprocal** of the fraction $\frac{1}{6}$.

Exercise 3

I

(a) Travelling at a constant speed of 32 kilometres per hour, a journey takes 18 minutes. How long would the same journey take travelling at a constant speed of 48 kilometres per hour?

(b) It takes a team of 8 people 6 weeks to paint a bridge. How long would the painting take if 12 people were employed?

(c) Usually, a swimming pool is filled using 4 water valves, over a period of 18 hours. Today however, one of the valves cannot be used. How long will it take to fill the pool using only 3 water valves?

(d) A journey can be completed in 44 minutes if we travel at an average speed of 50 miles per hour. How long would the same journey take if we travelled at an average speed of 40 miles per hour?

(e) A supply of hay is sufficient to feed 12 horses for 15 days. How long would the same supply of hay feed 20 horses?

(f) It takes 3 combine harvesters 6 hours to harvest a crop of wheat. How long would it take to harvest the same crop using only 2 combine harvesters?

(g) It takes a team of 18 people 21 weeks to dig a canal. How long would it take to dig the canal using only 14 people?

(h) A tank can be emptied using 6 pumps in 18 hours. How long would it take to empty the tank using 8 pumps?

(i) A gang of 9 bricklayers can build a wall in 20 days. How long would a gang of 15 bricklayers take to build the same wall?

Exercise 4

I

In this exercise, you will need to decide what type of proportion each question describes, before using an appropriate method to find the answer.

(a) The height of a stack of 150 pieces of paper is 9 mm.
What would be the heght of a stack of 350 pieces of similar paper?

(b) A small swimming pool can be filled in 9 minutes using 8 identical water pumps. How many pumps would be needed to fill the pool in 6 minutes?

(c) A car uses 24 litres of petrol to travel 216 km. How many litres of petrol are required to travel 162 km?

(d) A shop sells 8 apples for £1.40. What would be the price of 12 apples?

(e) A ship takes 12 days to complete a journey, travelling at a speed of 20 knots (1 knot = 1 nautical mile per hour). What speed is required to complete the journey in 10 days?

(f) For a Christmas party, a school arranges that 2 Christmas puddings are available for every 5 children. How many Christmas puddings must be purchased if there are 108 children?

(g) A car travels 180 km in 95 minutes. Find the time taken to travel 72 km at the same speed.

(h) Travelling at a speed of 84 km/hour, a train takes 2 hours to complete a journey. How long would the same journey take at a speed of 96 km/hour?

(i) If 12 pumps, all identical and working together, can empty a water tank in 60 minutes, how long would the tank take to empty if only 10 of the pumps were working?

(j) When a bike travels 145 m, each wheel rotates 58 times. How many times will each wheel rotate when completing a 1,000 m journey?

(k) It costs £1,450 to repair an 87 m long pavement.
Find the cost of repairing a 72 m long pavement at the same rate.

(l) A man owning 2,400 shares in a company receives a final dividend of £128. A final dividend of £164, from the same company, was received by a woman. How many shares does she own?

(m) An electric fire uses 8 units of electricity in 3 hours. How long would the electric fire work when using 20 units of electricity?

(n) A ship takes 45 days to complete a journey travelling at a speed of 16 knots. How long would the same journey take travelling at 18 knots?

(o) It costs £10.20 to feed a cat for 14 days. Find, to the nearent penny, the cost of feeding the same cat for 30 days.

(p) A machine can fill 580 bottles in 3 minutes.
How many bottles can the same machine fill in 1 hour?

(q) If 14 men can dig a ditch in 11 days, how many days would 22 men take to dig the same ditch?

(r) A bricklayer can lay 245 bricks in 3 hours. How many bricks could the bricklayer lay in 7 hours, working at the same rate?

John Napier

John Napier was born in Edinburgh, Scotland in 1550. He was a mathematician, a physicist and an astronomer. Napier was the first person to use logarithms (A Level work) and was responsible for the popularisation of the decimal point in mathematics. In 1570 he published a document that contained the following rhyme.

Multiplication is vexation,
Division is as bad;
The Rule of Three doth puzzle me,
And practice drives me mad.

Challenge!

Use the internet to investigate the "Rule of Three" in a mathematical context. What is the link to the "DM method" from this chapter?

More than one proportion

Example

A fruit grower knows that it will take 8 hours for 20 workers to pick 420 kg of strawberries. She needs to collect 360 kg of strawberries in 5 hours. What is the minimum number of workers she should employ?

Answer: In this question, there are three things that can vary, namely time, the number of workers, and the weight of the strawberries. We can, using the methods of proportion, change two of these at any one time, whilst keeping the third measure constant.

To start, let us keep the number of workers constant (20 workers), and consider how many strawberries they can pick in 5 hours. Because time and the weight of the strawberries are in direct proportion, we can form the following table.

Time	Weight of the strawberries
8 hours	420 kg
5 hours	?

Using the DM method, we can calculate that 20 workers can collect 5 ÷ 8 × 420 = 262.5 kg of strawberries in 5 hours.

Next, let us keep the time constant (5 hours), and consider how many workers are required to collect 360 kg of strawberries. Because the weight of the strawberries and the number of workers are in direct proportion, we can form the following table.

Weight of the strawberries	Number of workers
262.5 kg	20
360 kg	?

Using the DM method, we can calculate that $360 \div 262.5 \times 20 = 27.\dot{4}2857\dot{1}$ workers are required to pick 360 kg of strawberries in 5 hours. However a whole number of workers is required, so we round up to **28 workers** to ensure that 360 kg of strawberries can be collected in 5 hours.

Exercise 5

I

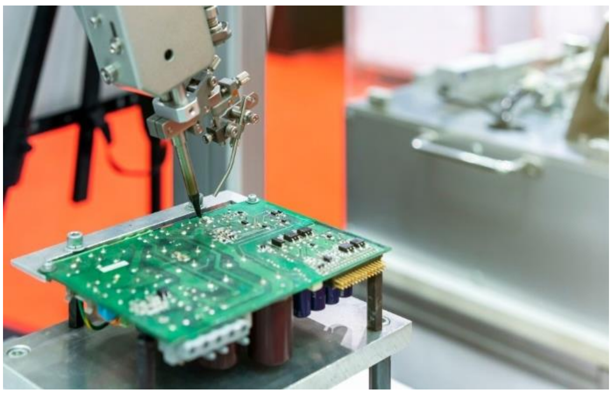
(a) 5 identical industrial water pumps can drain 600,000 litres of water in 8 hours. The local council wants to drain 450,000 litres of water from a flooded area. The work should not take more than 3 hours to complete. What is the minimum number of water pumps required for the task?

(b) Using their old printer, a printing company takes 12 hours to print 54,000 flyers. How long will it take to print another 72,000 flyers using a new printer that works twice as fast as the old one?

(c) A pump is used to fill empty tanks with oil. It takes 27 minutes to fill 6 identical tanks if the flow rate is 5 litres per second. Calculate how much time it would take to fill 8 of these tanks if the flow rate was 9 litres per second.

(d) A new school photocopier can photocopy 3 times as many pages as the old one in the same time. It used to take 20 minutes to copy 500 pages on the old photocopier. How much time would the new photocopier take to copy 600 pages?

(e) It takes 8 tractors 6 hours to plough 38 acres of land. What is the minimum number of tractors required to plough 76 acres of land in less than 9 hours? You may assume that each tractor works at the same rate and that all other conditions are similar.



(f) Machine A is three times as quick as Machine B at assembling identical circuit boards. Machine A is given two and a half times more circuit boards to assemble compared to Machine B. Machine B took 4 hours to complete all of its required assembly. How long did Machine A take to complete all of its required assembly? Give your answer in hours and minutes.

Proportion Graphs

You are required to recognise and interpret graphs that show direct proportion or inverse proportion.

Exercise 6

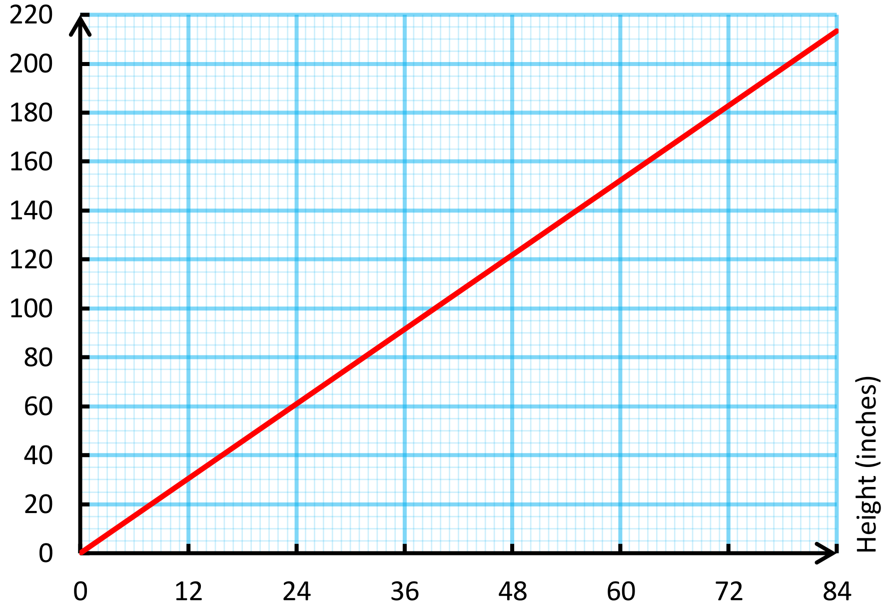
(a) What type of proportion (direct or inverse) is shown by the graph on the right?

(b) Siwan's height is 60 inches. What is Siwan's height in centimetres?

(c) Ben's height is 120 cm. What is Ben's height in inches?

(d) Huw's height is 170 cm. What is Huw's height in feet and inches?

Conversion graph for changing between height in inches and height in centimetres



Exercise 7

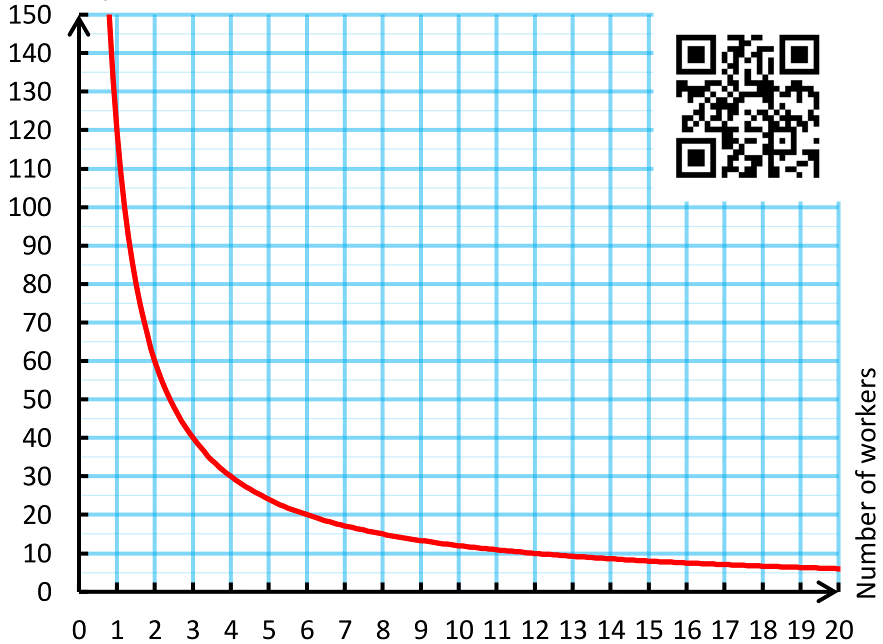
(a) What type of proportion (direct or inverse) is shown by the graph on the right?

(b) If 8 workers are available to build the wall, how many days will it take?

(c) Alan wishes to build the wall in less than 10 days. What is the minimum number of workers that Alan must employ?

(d) How long would it take for one person to build the wall?

A graph to show the time taken to build a wall using a specific number of workers



Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Proportion Equations

Higher Tier

Direct Proportion

If two measures x and y are in **direct proportion** to each other, then it is possible to write the relationship between x and y as $y \propto x$. As an **equation,** we can write the relationship as $y = kx$, where k represents the multiplier of the proportion. Given the value of y for a specific value of x, we can solve the equation to find k, and therefore write the equation that connects x to y.

Example

y is in direct proportion to x. Given that $y = 8$ when $x = 2$, find the equation that connects x to y.

Answer: If y is in direct proportion to x, then $y \propto x$, or $y = kx$ for some multiplier k.
Substituting the values of x and y from the question, we see that $8 = k \times 2$, so that $k = \frac{8}{2}$, which gives $k = 4$.
Therefore the equation that connects x to y is $y = 4x$.

Exercise 8

Skill

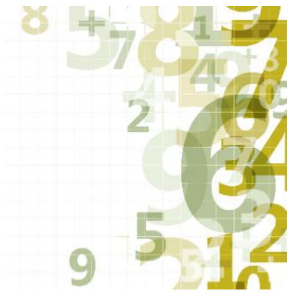
H

(a) y is in direct proportion to x. Given that $y = 12$ when $x = 3$, find the equation that connects x to y.

(b) y is in direct proportion to x. Given that $y = 35$ when $x = 5$, find the equation that connects x to y.

(c) y is in direct proportion to x. Given that $y = 2$ when $x = 8$, find the equation that connects x to y.

(d) y is in direct proportion to x. Given that $y = \frac{1}{3}$ when $x = 7$, find the equation that connects x to y.



Example

y is in direct proportion to x^2. Given that $y = 45$ when $x = 3$, find the equation that connects x to y.

Answer: If y is in direct proportion to x^2, then $y \propto x^2$, or $y = kx^2$ for some multiplier k.
Substituting the values of x and y from the question, we see that $45 = k \times 3^2$, so that $k = \frac{45}{3^2}$, which gives $k = 5$.
Therefore the equation that connects x to y is $y = 5x^2$.

Exercise 9

H

(a) y is in direct proportion to x^2. Given that $y = 80$ when $x = 4$, find the equation that connects x to y.

(b) y is in direct proportion to x^3. Given that $y = 500$ when $x = 5$, find the equation that connects x to y.

(c) y is in direct proportion to x^2. Given that $y = 16$ when $x = 8$, find the equation that connects x to y.

(d) y is in direct proportion to $\sqrt{x}$. Given that $y = 30$ when $x = 25$, find the equation that connects x to y.

Inverse Proportion

If two measures x and y are in **inverse proportion** to each other, then it is possible to write the relationship between x and y as $y \propto \frac{1}{x}$. As an **equation,** we can write the relationship as $y = \frac{k}{x}$, where k represents the multiplier of the proportion. Given the value of y for a specific value of x, we can solve the equation to find k, and therefore write the equation that connects x to y.

Example

y is in inverse proportion to x. Given that $y = 4$ when $x = 5$, find the equation that connects x to y.

Answer: If y is in inverse proportion to x, then $y \propto \frac{1}{x}$, or $y = \frac{k}{x}$ for some multiplier k.

Substituting the values of x and y from the question, we see that $4 = \frac{k}{5}$, so that $k = 4 \times 5$, which gives $k = 20$.

Therefore the equation that connects x to y is $y = \frac{20}{x}$.

Exercise 10

(a) y is in inverse proportion to x. Given that $y = 6$ when $x = 8$, find the equation that connects x to y.

(b) y is in inverse proportion to x. Given that $y = 2$ when $x = 14$, find the equation that connects x to y.

(c) y is in inverse proportion to x. Given that $y = \frac{2}{5}$ when $x = 8$, find the equation that connects x to y.

Example

y is in inverse proportion to x^2. Given that $y = 3$ when $x = 6$, find the equation that connects x to y.

Answer: If y is in inverse proportion to x^2, then $y \propto \frac{1}{x^2}$, or $y = \frac{k}{x^2}$ for some multiplier k.

Substituting the values of x and y from the question, we see that $3 = \frac{k}{6^2}$, so that $k = 3 \times 6^2$, which gives $k = 108$.

Therefore the equation that connects x to y is $y = \frac{108}{x^2}$.

Exercise 11

(a) y is in inverse proportion to x^2. Given that $y = 4$ when $x = 5$, find the equation that connects x to y.

(b) y is in inverse proportion to x^2. Given that $y = 15$ when $x = 10$, find the equation that connects x to y.

(c) y is in inverse proportion to x^3. Given that $y = \frac{3}{4}$ when $x = 2$, find the equation that connects x to y.

Exercise 12

Given that $y = 5$ when $x = 4$, write an equation to show each of the following relationships.

(a) $y \propto x$ (b) $y \propto x^2$ (c) $y \propto \sqrt{x}$ (d) $y \propto \frac{1}{x}$ (e) $y \propto \frac{1}{x^3}$ (f) $y \propto \frac{1}{\sqrt{x}}$

Exercise 13

H

(a) Given that y is in inverse proportion to x, and knowing that $y = 6$ when $x = 4$,
(i) find an expression for y in terms of x,
(ii) calculate y when $x = 2$,
(iii) calculate x when $y = 3$.

(b) Given that y is in proportion to x, and knowing that $y = 18$ when $x = 2$,
(i) find an expression for y in terms of x,
(ii) calculate y when $x = 7$,
(iii) calculate x when $y = 27$.

If the type of proportion is not stated, use direct proportion.

(c) Given that y is in direct proportion to x^2, and knowing that $y = 36$ when $x = 3$,
(i) find an expression for y in terms of x,
(ii) calculate y when $x = 4$,
(iii) calculate the two possible values for x when $y = 256$.

(d) Given that y is in inverse proportion to x^3, and knowing that $y = 5$ when $x = 4$,
(i) find an expression for y in terms of x,
(ii) calculate y when $x = 8$,
(iii) calculate x when $y = 40$.

Exercise 14

Applying

H

(a) In a science experiment, Susan takes measurements for t and m.
The following table shows her results.

t	2	6	8
m	4	108	256

Susan believes that either m is in proportion to t^2 or m is in proportion to t^3.
By considering both possibilities, find an expression for m in terms of t.
Show all of your calculations.

(b) In an experiment, a scientist saw that the force, F, between two particles was in inverse proportion to the square of the distance, d, between the particles. The unit of force was Netwons and the unit of distance was millimetres. When the particles were 5 mm apart, the force between them was 8 Newtons. How far apart were the particles when the force between them was 12.5 Newtons?

(c) Cerys takes people on hot air balloon trips. She knows that the pressure in the balloon, measured in atmospheres, is in inverse proportion to the square root of the height of the balloon above Earth. When the balloon is at a height of 36 metres above Earth, the pressure is 2 atmospheres.

(i) Write this information as an equation.

(ii) Cerys pilots her balloon up to a height of 400 m and then down to a height of 256 m. Calculate the change in pressure during the descent.

(d) Awel wants to paint the walls in her bedroom. The area of the walls is 75 m². The paint costs £6.80 per litre and 2 litres of paint covers 30 m². Write a formula that connects the area of the wall, A m², to the number of litres of paint required, L. Use the formula to calculate the cost of painting the walls in Awel's bedroom.

Finding Proportion Equations

Example

The following table shows two measures x and y.
Find the equation that shows the proportion that is between these measurements.

x	6	8
y	18	32

Answer: As x increases, so does y, so there is direct proportion between the measurements. Considering first whether the proportion is of the form $y \propto x$, or $y = kx$, let us substitute the data from the first column: $18 = k \times 6$, $k = \frac{18}{6}$, $k = 3$. To check whether we have a proportion of the form $y = 3x$, we must consider the data from the second column. The equation does not work for this data ($y = 3 \times 8 = 24$, not 32) so we must consider a different type of proportion. Considering whether we have a proportion of the form $y \propto x^2$, or $y = kx^2$, we must again substitute the data from the first column to find the multiplier of the proportion: $18 = k \times 6^2$, $k = \frac{18}{6^2}$, $k = \frac{1}{2}$. To check whether we have a proportion of the form $y = \frac{1}{2}x^2$, we must again consider the data from the second column. This time, the equation does work for the data ($y = \frac{1}{2} \times 8^2 = 32$), so the equation that shows the proportion that is between the measurements x and y is $y = \frac{1}{2}x^2$.

Exercise 15

Skill

H

The following tables show measures x and y.
Find the equation that describes the proportion that is between the measurements.

(a)

x	4	6
y	12	18

(b)

x	4	6
y	3	2

(c)

x	10	6
y	15	9

(d)

x	20	15
y	3	4

(e)

x	2	3
y	20	45

(f)

x	2	3
y	18	8

(g)

x	2	3
y	32	108

(h)

x	4	9
y	14	21

Evaluation

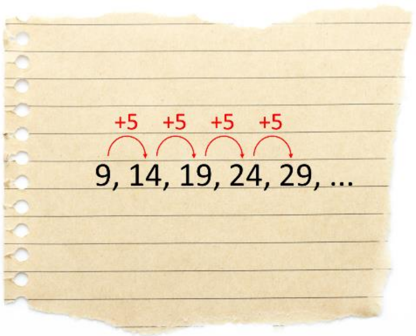
Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Quadratic Nth Term

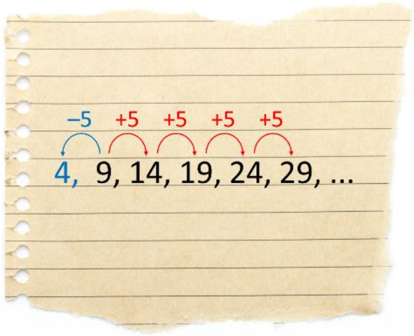
Linear Nth Term

In the Developing Algbera 1 workbook, we learnt how to find the formula for the nth term of a linear sequence such as 9, 14, 19, 24, 29, ...

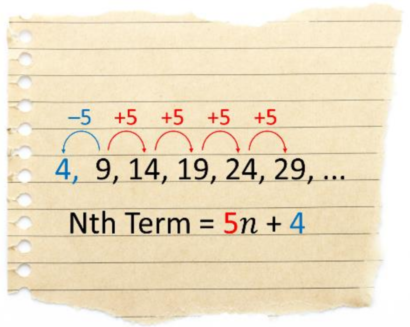
1. Consider what the rule is for finding the next number. Here, we must **add five** to find the next number.



2. If another number was added at the start of the sequence, what would this number have to be? Here, the number would have to be 9 – 5 = 4.



3. The ***n*th term** for this sequence is $5n + 4$. (The 5 and the 4 come from the previous steps.)



Exercise 16

Revision
F

Find the nth term for the following linear sequences.

(a) 4, 6, 8, 10, 12, ...
(b) 13, 15, 17, 19, 21, ...
(c) 14, 17, 20, 23, 26, ...
(d) 20, 18, 16, 14, 12, ...
(e) 34, 31, 28, 25, 22, ...
(f) 10, 14, 18, 22, 26, ...
(g) 5, 5.5, 6, 6.5, 7, ...
(h) 8, 9, 10, 11, 12, ...
(i) 3, 6, 9, 12, 15, ...
(j) –12, –10, –8, –6, –4, ...
(k) –7, –9, –11, –13, –15, ...
(l) –3, –1, 1, 3, 5, ...
(m) –26, –30, –34, –38, –42, ...
(n) 2, 7, 12, 17, 22, ...
(o) 10, 9.75, 9.5, 9.25, 9, ...

The First Difference

The sequences in Exercise 16 were linear sequences as the difference between any two consecutive numbers was constant. For example, the common difference in question (a) was 2.

$$4, \xrightarrow{+2} 6, \xrightarrow{+2} 8, \xrightarrow{+2} 10, \xrightarrow{+2} 12, \quad .....$$

In a quadratic sequence, the difference between any two consecutive numbers is not constant. For example, in the quadratic sequence 4, 15, 30, 49, 72, ... the difference between two consecutive numbers increases.

$$4, \xrightarrow{+11} 15, \xrightarrow{+15} 30, \xrightarrow{+19} 49, \xrightarrow{+23} 72, \quad .....$$

We can use this **first difference** to decide whether or not a specific sequence is linear.

Exercise 17

Are the following sequences linear or not?

(a) 9, 11, 13, 15, 17, ...
(b) 1, 4, 9, 16, 25, ...
(c) 16, 14, 12, 10, 8, ...
(d) 3, 6, 11, 18, 27, ...
(e) 5, 7, 5, −1, −11, ...
(f) −20, −10, 0, 10, 20, ...
(g) 9, 8, 7, 6, 5, ...
(h) 11, 23, 43, 71, 107, ...
(i) 8, 7.5, 7, 6.5, 6, ...

Simple Quadratic Sequences

The simplest quadratic sequence is the sequence of square numbers

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...

The nth term for this sequence is n^2. We can form another quadratic sequence by adding or subtracting the same number from each of the square numbers. For example:

Sequence	Nth Term
4, 7, 12, 19, 28, ...	$n^2 + 3$
−1, 2, 7, 14, 23, ...	$n^2 - 2$

Quadratic sequences of this form have the nth term $n^2 + a$, where a is a specific number.

Exercise 18

Find the nth term for each of the following simple quadratic sequences.

(a) 2, 5, 10, 17, 26, ...
(b) 11, 14, 19, 26, 35, ...
(c) 7, 10, 15, 22, 31, ...
(d) −4, −1, 4, 11, 20, ...
(e) −9, −6, −1, 6, 15, ...
(f) 0, 3, 8, 15, 24, ...
(g) 1.5, 4.5, 9.5, 16.5, 25.5, ...
(h) 1, 4, 9, 16, 25, ...
(i) −25, −22, −17, −10, −1, ...

Exercise 19

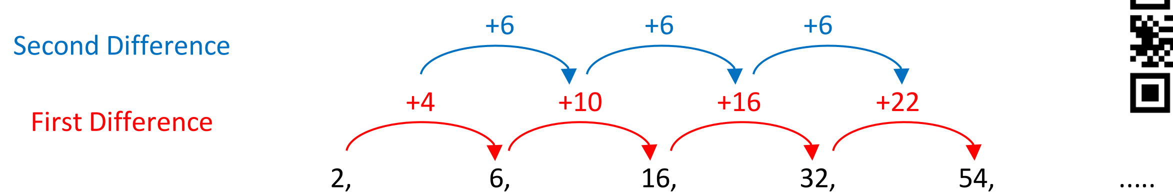
Write the first five terms of the following quadratic sequences.

(a) $n^2 + 4$
(b) $n^2 - 6$
(c) $n^2 + 13$
(d) $n^2 - \frac{1}{4}$
(e) $n^2 + 27$
(f) $n^2 - 50$

More Complex Quadratic Sequences

Consider the quadratic sequence 2, 6, 16, 32, 54, ... It is not possible to form this sequence by adding or subtracting the same number from the list of square numbers, so a different method is needed to find the nth term.

Step 1: Find the **second difference** for the sequence.



Step 2: Halve the second difference to find the coefficient[1] of n^2 in the formula for the nth term.

6 ÷ 2 = 3, so the nth term for the sequence contains the term $3n^2$.

[1] The coefficient of a term is the number that appears at the start of the term.

Step 3: Form a table to find the difference between $3n^2$ and the original sequence.

Original Sequence	2,	6,	16,	32,	54,	.....
n^2	1,	4,	9,	16,	25,	.....
$3n^2$	3,	12,	27,	48,	75,	.....
Original Sequence $-\ 3n^2$	−1,	−6,	−11,	−16,	−21,	.....

Step 4: Find the nth term of the linear sequence in the final row of the table.

$$-1, \xrightarrow{-5} -6, \xrightarrow{-5} -11, \xrightarrow{-5} -16, \xrightarrow{-5} -21, \quad .....$$

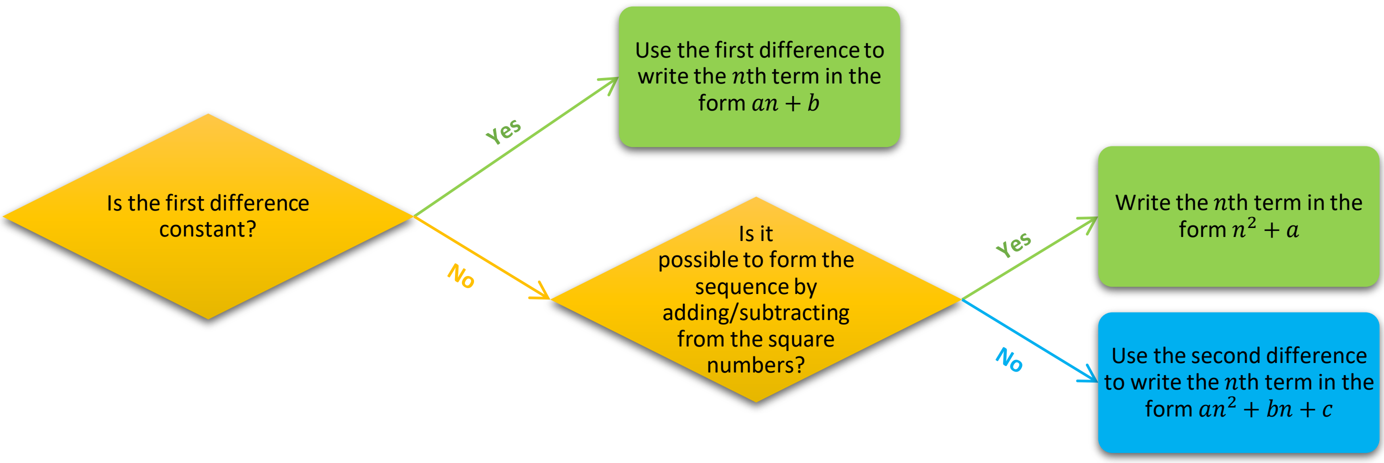
The nth term of the linear sequence is $-5n + 4$, so the nth term of the quadratic sequence is $3n^2 - 5n + 4$.
(We can verify this by substituting into the formula, or by using *Table Mode* on a calcuator.)

Exercise 20

Find the nth term of the following quadratic sequences.

(a) 6, 11, 18, 27, 38, ...
(b) 0, 5, 12, 21, 32, ...
(c) 2, 3, 6, 11, 18, ...
(d) −4, −3, 0, 5, 12, ...
(e) 11, 22, 37, 56, 79, ...
(f) 1, 2, 7, 16, 29
(g) 9, 18, 31, 48, 69, ...
(h) 6, 11, 20, 33, 50, ...
(i) 4, 18, 38, 64, 96, ...
(j) 6, 15, 32, 57, 90, ...
(k) −3, 8, 29, 60, 101, ...
(l) 10, 31, 64, 109, 166, ...
(m) 10, 40, 90, 160, 250, ...
(n) 8, 14, 24, 38, 56, ...
(o) 8, 22, 42, 68, 100, ...
(p) 9, 12, 13, 12, 9, ...
(q) 10, 9, 4, −5, −18, ...
(r) 3, −10, −29, −54, −85, ...
(s) 4, −8, −30, −62, −104, ...
(t) −15, −28, −49, −78, −115, ...
(u) 10.5, 17, 27.5, 42, 60.5, ...

Flow Chart: Finding the nth term of a linear or quadratic sequence



Exercise 21

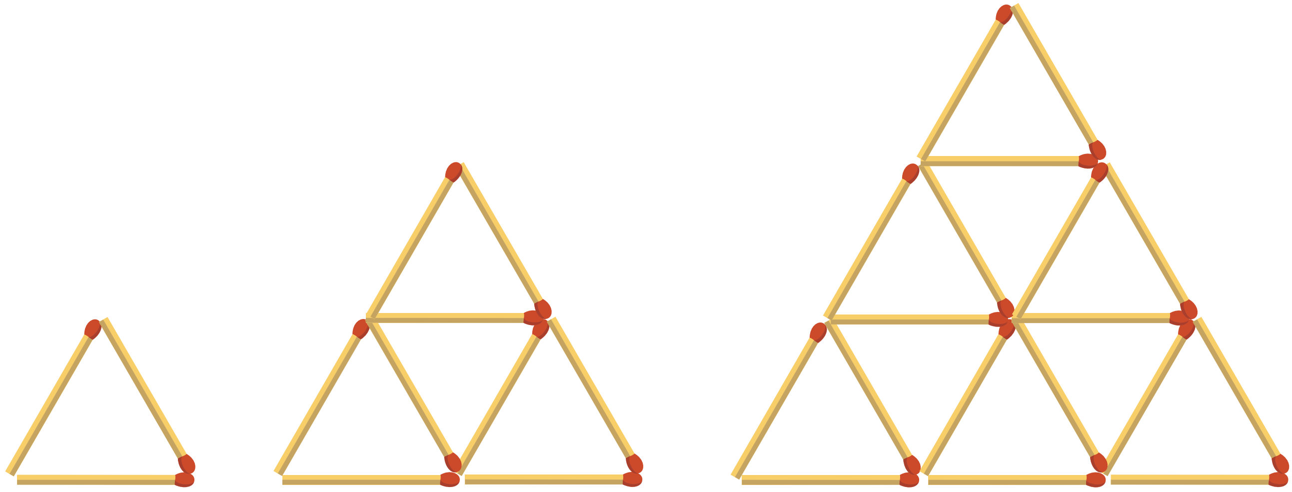
Write the first 5 terms of the sequences with the following nth terms.

(a) $4n + 3$
(b) $n^2 + 9$
(c) $4n^2$
(d) $2n^2 + 6n + 5$
(e) $5n^2 - 3n + 7$
(f) $-3n^2 + 10n - 4$
(g) $n^3 + 2$
(h) n^4
(i) $\frac{1}{n}$

Exercise 22

Applying I

Here is a matchstick pattern.



Pattern 1 Pattern 2 Pattern 3

(a) Draw Pattern 4 in your book.

(b) Copy and complete the following table.

Pattern number	1	2	3	4	5	6
Number of triangles	1	4				
Number of matches	3	9				

(c) Consider the sequence for the number of triangles. What is the nth term of this sequence?

(d) Consider the sequence for the number of matches. What is the nth term of this sequence?

(e) How many matches are required in order to make Pattern 20?

(f) What is the number of the pattern that contains 100 triangles?

(g) Steffan has 1,000 matches. What is the number of the biggest pattern that Steffan can create?

(h) Lisa creates a pattern that contains 225 triangles. How many matches are in Lisa’s pattern?

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ❑ Target ❑

Inequalities

If you want to buy a bag of sweets that costs 79 pence, you require **at least** 79 pence.

Perhaps you have more than this amount in your pocket. The sum in your pocket must be greater than or equal to 79 pence.

If x represents the sum of money in your pocket, then we can write the **inequality** $x \geq 79$ to show when we would be able to buy the bag of sweets.

Inequality Symbols

- The meaning of the symbol $\geq$ is 'greater than or equal to'.
- The meaning of the symbol $>$ is 'greater than'.
- The meaning of the symbol $\leq$ is 'less than or equal to'.
- The meaning of the symbol $<$ is 'less than'.

In handwriting, we can write the symbols $\geq$ and $\leq$ as $\geqslant$ and $\leqslant$.

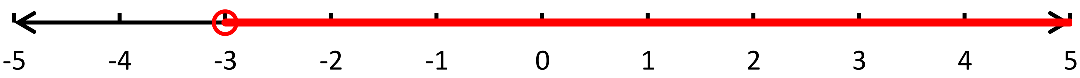
Inequalities on a Number Line

Example

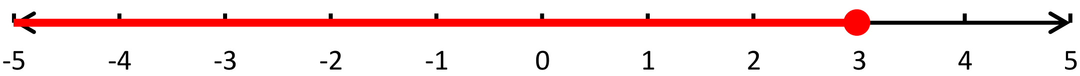
Display the following inequalities on a number line.

(a) $x > -3$ (b) $y \leq 3$ (c) $2 \leq x < 4$

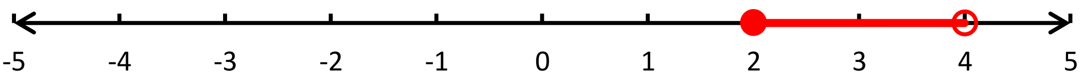
Answer: (a)



(b)



(c)

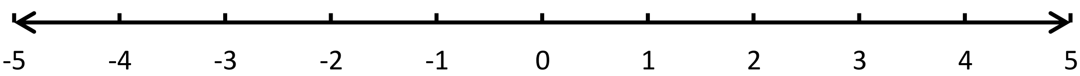


Exercise 23

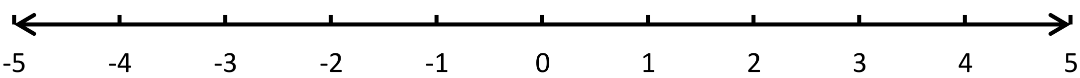
Skill I

Use the number lines below to display the following inequalities.

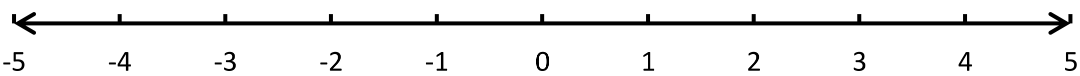
(a) $x < 4$



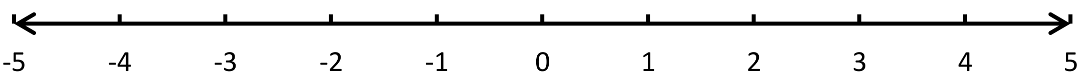
(b) $x \geq -2$



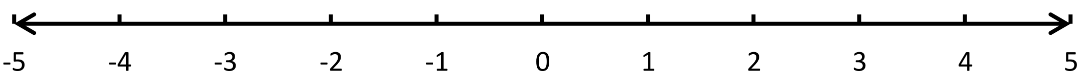
(c) $-4 < x \leq 1$



(d) $-2.5 \leq x < 3$



(e) $x > 0$

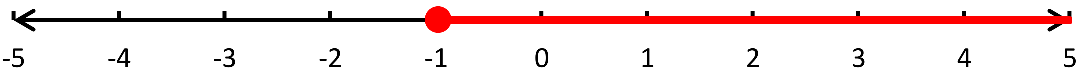


Exercise 24

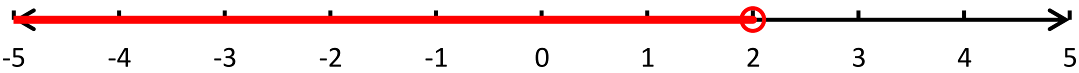
I

Write the inequalities that are shown on the following number lines. (Use x as the variable.)

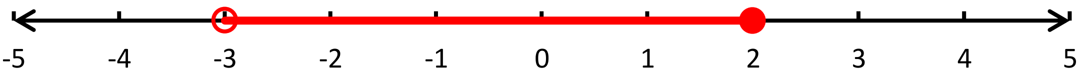
(a)



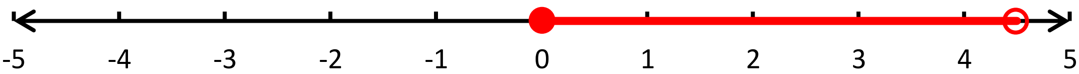
(b)



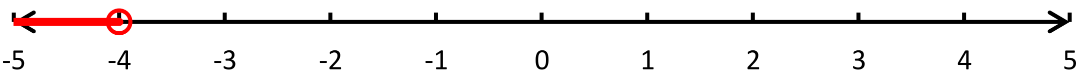
(c)



(d)



(e)



Solving Equations

Because solving inequalities is very similar to solving equations, it is appropriate now to revise some of the equation solving work from previous years.

Exercise 25

Revision

I

Solve the following equations.

One-step equations

(a) $x + 7 = 9$ (b) $3x = 15$ (c) $x - 4 = 14$

(d) $\frac{x}{2} = 10$ (e) $7y = 42$ (f) $\frac{12}{w} = -4$

Two-step equations

(g) $2x + 3 = 19$ (h) $3x - 1 = 17$ (i) $5y + 9 = 64$

Three-step equations

(j) $5x + 2 = 3x + 32$ (k) $4x - 5 = x + 16$ (l) $4x + 4 = 7x - 11$

Equations which require expansion first

(m) $2(x + 7) = 22$ (n) $3(y - 4) = 24$ (o) $20 = 4(x - 2)$

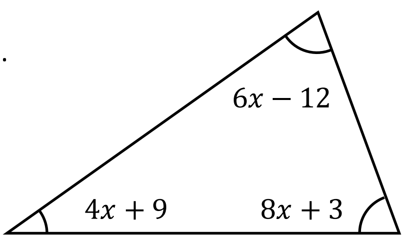
(p) $4(x + 2) = 2(x + 7)$ (q) $4(x - 12) + 2x = 0$ (r) $3(x - 4) = 2(x + 4) + 8$

Equations involving fractions

(s) $\frac{x}{2} + 5 = 9$ (t) $\frac{x+5}{2} = 4$ (u) $\frac{18}{x-2} = 3$

Equations in context

(v) Each angle in the triangle on the right is measured in degrees.
Calculate the size of the smallest angle.



Solving Inequalities

Solving inequalities is exactly the same as solving equations, but there is one important additional rule:

We must **change the symbol** in the middle of an inequality if we
(a) **swap sides**; (b) **multiply or divide by a negative number**.

If we need to change the symbol in the middle of an inequality, then the symbol $\geq$ changes to be $\leq$; the symbol $>$ changes to be $<$; the symbol $\leq$ changes to be $\geq$; and the symbol $<$ changes to be $>$.

Why do we need to change the symbol?

(a) Consider the inequality $5 > 2$, which shows something that is true.
If we swap the inequality's sides **without** changing the symbol in the middle, then we will finish with something that is false: $2 > 5$. We must therefore change the symbol in the middle of an inequality if we swap an inequality's sides. (In the example, the correct inequality after swapping sides would be $2 < 5$.)

(b) Consider again the inequality $5 > 2$. If we multiply both sides of the inequality by -2, we will finish with something that is false: $-10 > -4$. We must therefore change the symbol in the middle of an inequality if we multiply an inequality by a negative number. (In the example, the correct inequality after multiplying by -2 is $-10 < -4$.) The same is true if we divide an inequality by a negative number.

Example

Solve the following inequalities.

(a) $4x + 1 \geq 13$ (b) $7 - 3x < 1$ (c) $2(x + 4) \leq 5(x + 1)$ (d) $\frac{x}{2} > 6 + 2x$

Answer:

(a) $4x + 1 \geq 13$
$4x \geq 12$ [Subtract 1]
$x \geq 3$ [Divide by 4]

(b) $7 - 3x < 1$
$-3x < -6$ [Subtract 7]
$x > 2$ [Divide by –3]

(c) $2(x + 4) \leq 5(x + 1)$
$2x + 8 \leq 5x + 5$ [Expand brackets]
$2x \leq 5x - 3$ [Subtract 8]
$-3x \leq -3$ [Subtract $5x$]
$x \geq 1$ [Divide by –3]

(d) $\frac{x}{2} > 6 + 2x$
$x > 2(6 + 2x)$ [Multiply by 2]
$x > 12 + 4x$ [Expand brackets]
$-3x > 12$ [Subtract $4x$]
$x < -4$ [Divide by –3]

Exercise 26

Skill I

Solve the following inequalities.

(a) $x + 2 > 5$ (b) $5x \geq 20$ (c) $\frac{x}{3} < 6$

(d) $y - 4 \leq 10$ (e) $-2x < 8$ (f) $\frac{x}{-2} \leq 4$

(g) $2x + 5 > 37$ (h) $3y - 2 < 7$ (i) $4x - 4 \geq 4$

(j) $6 - 2x \geq 10$ (k) $10 - 3x < 22$ (l) $1 - x \leq 7$

(m) $4x + 6 > 2x + 18$ (n) $5x - 1 \geq 2x + 32$ (o) $3y + 4 < 2y - 10$

(p) $2x + 3 > 4x + 23$ (q) $3x - 8 \leq 5x + 20$ (r) $5y + 7 \geq y - 29$

Exercise 27

Solve the following inequalities.

(a) $3(x-1) < 9$ (b) $2(x+3) \leq 22$ (c) $5(3-y) > 10$

(d) $4(x+2) \geq 2(x+6)$ (e) $5(x-1) < 3(x+5)$ (f) $5(1-2x) > 4(2-3x)$

(g) $2x + 3(x-2) \geq 3x - 4$ (h) $z + 3(z-4) \leq 4$ (i) $7(3+x) < 7(3-x)$

(j) $3x - 2(x-1) > 4(x+2)$ (k) $2 - 2(3-y) \geq 6(2-y)$ (l) $5t - 3(2-t) < 2(3t+10)$

Exercise 28

Solve the following inequalities.

(a) $\frac{x}{2} + 3 > 8$ (b) $\frac{x}{3} - 2 \leq 4$ (c) $\frac{x}{-4} + 1 > 10$

(d) $\frac{x-12}{3} > 5$ (e) $\frac{x+4}{2} \leq -4$ (f) $\frac{y-4}{3} < 2$

(g) $\frac{x}{2} < 10 - 2x$ (h) $\frac{x}{3} \geq 4 + x$ (i) $\frac{2x}{5} < x - 9$

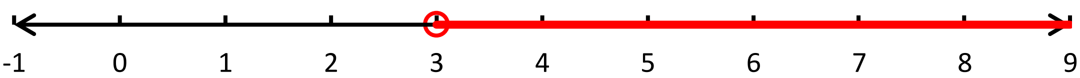
Example

Find the **least** whole number that satisfies the inequality $3x + 9 > x + 15$.

Answer: To begin, let us solve the inequality:

$3x + 9 > x + 15$
$3x > x + 6$ [Subtract 9]
$2x > 6$ [Subtract x]
$x > 3$ [Divide by 2]

We can show this solution on a number line:



The **least** whole number that is greater than 3 is 4, so 4 is the answer to the question.

Exercise 29

Find the **least** whole number that satisfies the following inequalities.

(a) $x > 8$ (b) $x \geq 4$ (c) $x - 4 > 9$

(d) $2x + 6 \geq 24$ (e) $6x + 5 > 4x + 13$ (f) $6x + 4 > 4x + 13$

(g) $3x + 9 \geq x + 3$ (h) $4x - 8 \leq 5x + 5$ (i) $\frac{x+1}{2} > 5$

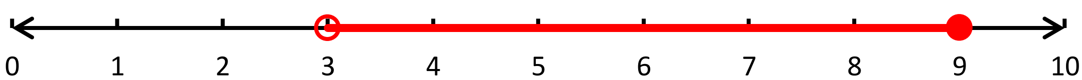
Example

List the whole numbers that satisfy the inequality $5 < 2x - 1 \leq 17$.

Answer: To begin, let us solve the inequality:

$5 < 2x - 1 \leq 17$
$6 < 2x \leq 18$ [Add 1]
$3 < x \leq 9$ [Divide by 2]

We can show this solution on a number line:



The whole numbers that satisfy the inequality are 4, 5, 6, 7, 8 and 9.

Exercise 30

I

List the whole numbers that satisfy the following inequalities.

(a) $5 \leq x \leq 8$ (b) $5 < x < 8$ (c) $5 < x \leq 8$

(d) $-4 \leq x \leq 2$ (e) $-4 < x < 2$ (f) $-4 \leq x < 2$

(g) $6 < 2x < 10$ (h) $6 \leq 3x < 18$ (i) $4 < 4y \leq 20$

(j) $3 \leq 2x + 1 \leq 13$ (k) $3 < 2x - 1 < 17$ (l) $5 \leq 3x - 1 < 11$

(m) $3 < 2x \leq 9$ (n) $5 \leq 2x + 4 < 15$ (o) $7 < 5x + 1 \leq 21$

Exercise 31

Applying

I

(a) Four times a number n take away 3 is less than twice the number n add 5.
Write an inequality satisfied by n and solve it to find the possible values for n.

(b) Vincent and Rowena start to rent television sets at the same time. Vincent pays £14 per month for his rental television. Rowena uses a different method; she pays an upfront payment of £50 then pays rent at £8 a month. Let x represent the number of months both Vincent and Rowena have been renting their televisions.

(i) Write an inequality that is satisfied by x for the number of months the total amount payed by Vincent is **less** than the total amount payed by Rowena.
(ii) Solve the inequality. Explain what your solution tells you about Vincent and Rowena.

(c) Sali has mathematics and science homework. Let m and s represent the time that Sali intends to spend completing each of the homework tasks.
(i) Sali intends to spend less than 3 hours on her mathematics homework. Write this as an inequality.
(ii) What is the meaning of $1 < s < 2$?
(iii) What is the meaning of $m > s$?

(d) A bus can hold up to 46 people. A school intends to transport 5 adults and as many groups of 4 children as is possible to fit on the bus.

(i) Which of the following inequalities is true about the bus?
$4n + 5 > 46$ $4n + 5 \leq 46$ $4n - 5 < 46$ $4n - 5 \geq 46$
(ii) Solve the correct inequality from part (i) to find the maximum number of groups of four children that can be transported on the bus.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Regions of Graphs

Higher Tier

In this chapter, we will discuss how to shade a region of graph paper defined by a set of inequalities. In order to do this, we must revise how to plot graphs of the form $x = a$; $y = b$ and $y = mx + c$, and learn a new technique for plotting graphs of the form $ax + by + c = 0$.

Revising plotting graphs of the form $x = a$ and $y = b$

- The graph of $x = a$ is a vertical line passing through the point $(a, 0)$.
- The graph of $y = b$ is a horizontal line passing through the point $(0, b)$.

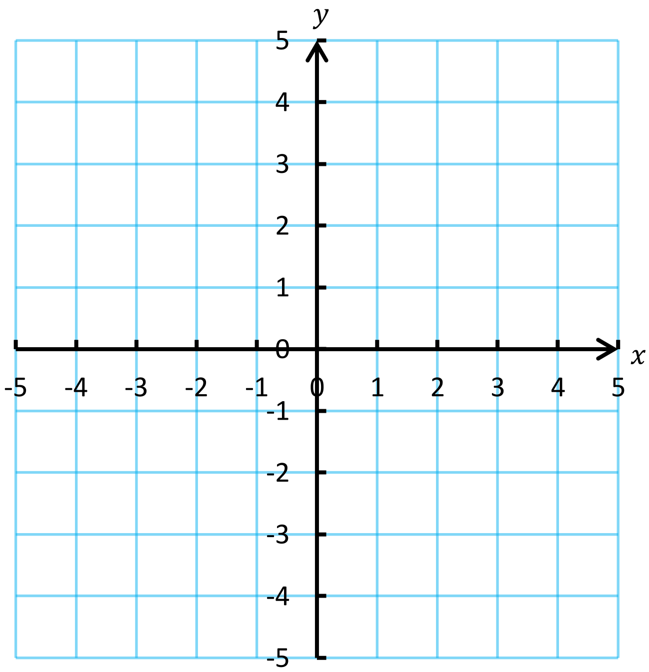
Exercise 32

Revision

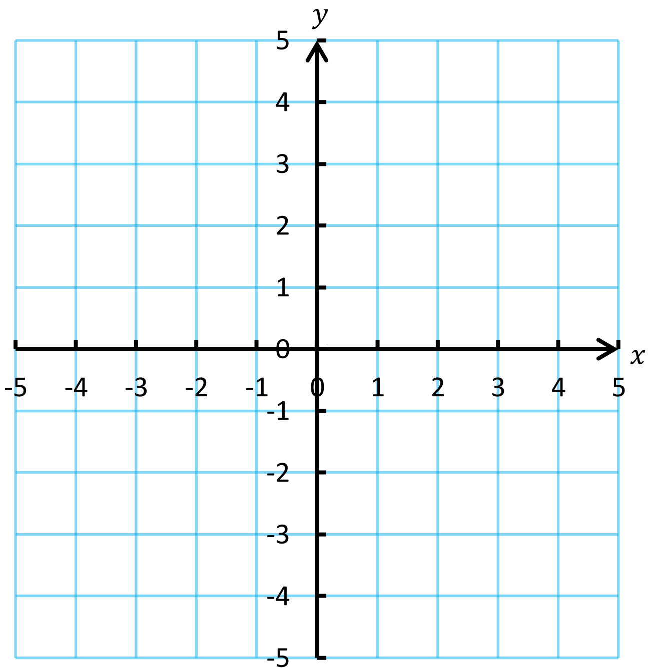
F

Use the graph paper below to plot the following lines.

(a) $x = 3, y = 4, x = -2, y = -3$



(b) $y = 2, x = -3, y = 1.5, x = -\frac{5}{2}$



Revising plotting graphs of the form $y = mx + c$

Given a straight line of the form $y = mx + c$, for example $y = 3x - 2$, here are two ways of plotting the line on graph paper.

Method 1: Using a table

(a) Substitute different values of x into the equation in order to create a table of values.

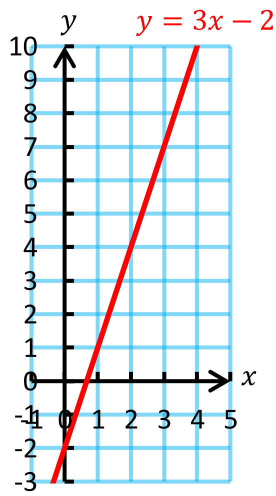
x	0	1	2	3
y	−2	1	4	7
	↑	↑	↑	↑
	3 × 0 − 2 = 0 − 2 = −2	3 × 1 − 2 = 3 − 2 = 1	3 × 2 − 2 = 6 − 2 = 4	3 × 3 − 2 = 9 − 2 = 7

(b) Plot the values from the table on graph paper before connecting the points with a straight line.

Method 2: Using the gradient and y–intercept

(a) For the line $y = 3x - 2$, the y–intercept is –2, so the line passes through the point (0, –2). Plot this point on the graph paper.

(b) The gradient is 3, so for each **one** unit we move to the right (starting from the point (0, –2)), we must move **three** units up. Plot some of these points before connecting the points with a straight line.

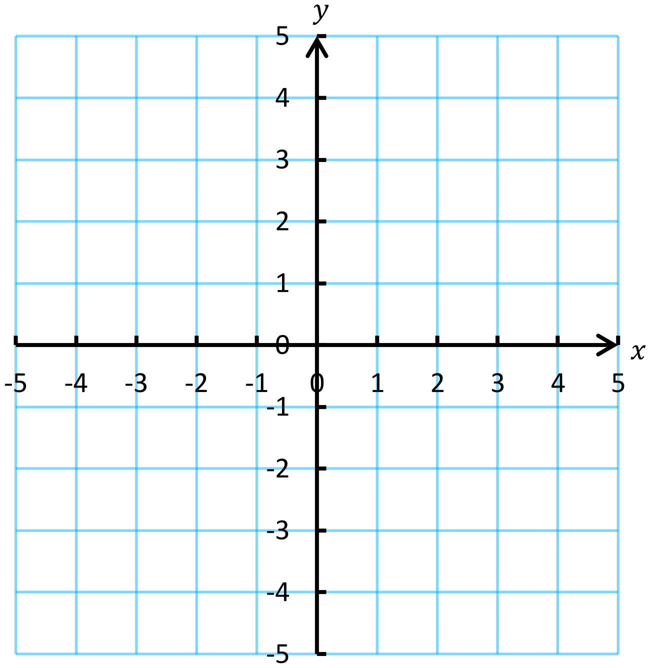


Exercise 33

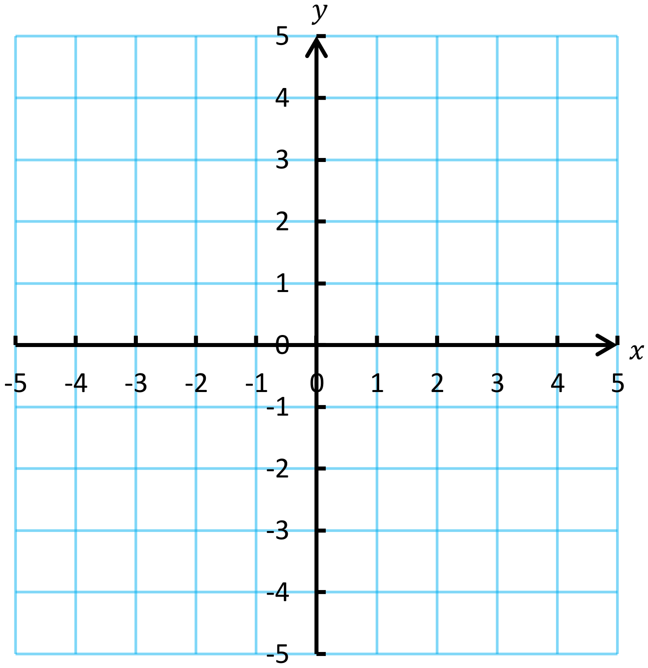
F

Use the graph paper below to plot the following lines.

(a) $y = 2x - 3$, $y = -\frac{1}{2}x + 1$



(b) $y = x$, $y = -x$



Plotting graphs of the form $ax + by + c = 0$

See the Developing Algebra 2 workbook to revise this topic.

Example

Plot a straight line for the equation $2x + 3y - 12 = 0$.

Method 1: The Hiding Method

To find the value of y when $x = 0$, **hide** the term in x using your finger, and solve the equation that remains.

$2x + 3y - 12 = 0 \longrightarrow 3y - 12 = 0$

$3y = 12$ [Add 12]

$y = 4$ [Divide by 3]

So the line goes through the point $(0, 4)$.

To find the value of x when $y = 0$, **hide** the term in y using your finger, and solve the equation that remains.

$2x + 3y - 12 = 0 \longrightarrow 2x - 12 = 0$

$2x = 12$ [Add 12]

$x = 6$ [Divide by 2]

So the line goes through the point $(6, 0)$.

To plot the line for the equation $2x + 3y - 12 = 0$, plot the two points $(0, 4)$ and $(6, 0)$ on graph paper before connecting them with a straight line.

We can "hide" the terms because they disappear when substituting in 0.

Method 2: The Rearranging Method

Re-arrange the equation in order to make y the subject of the equation.

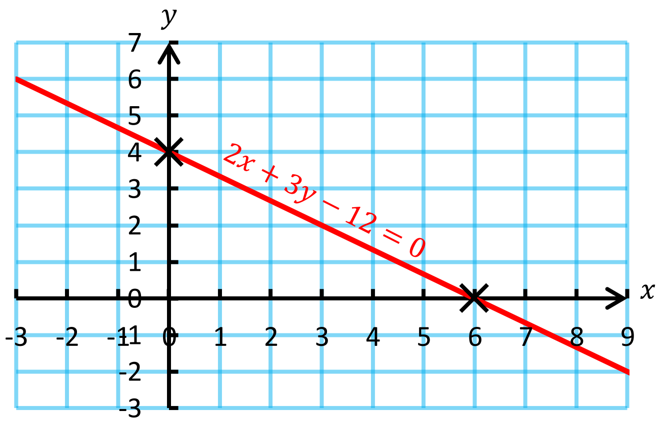
$2x + 3y - 12 = 0$

$3y - 12 = -2x$ [Subtract $2x$]

$3y = -2x + 12$ [Add 12]

$y = -\frac{2}{3}x + 4$ [Divide by 3]

We can now plot the equation, using the techniques for plotting an equation of the form $y = mx + c$.

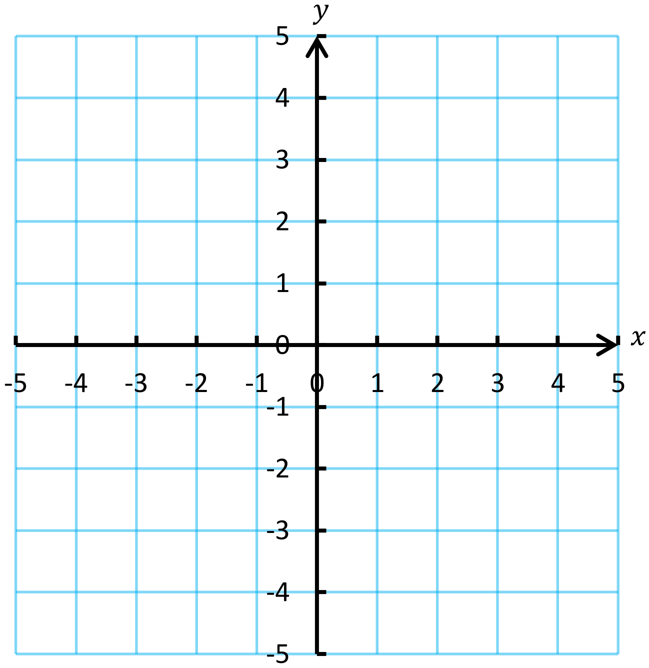


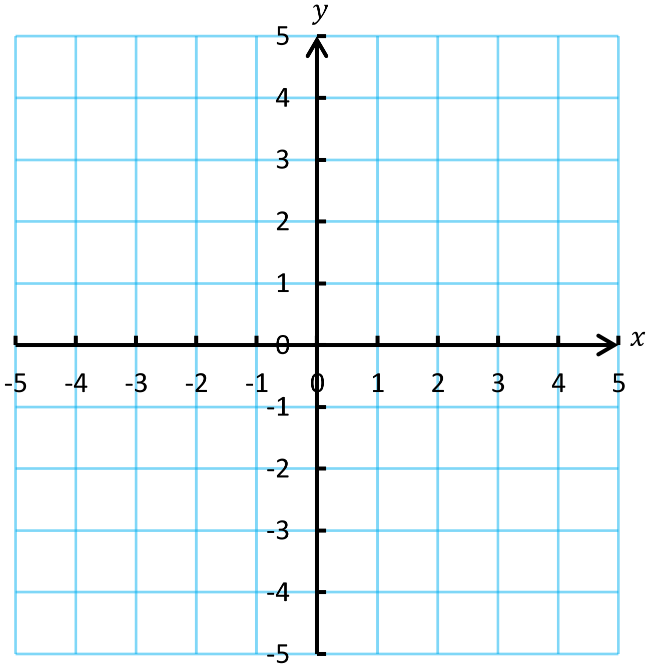
Exercise 34

Skill I

Use the graph paper below to plot the following lines.

(a) $2x + 3y - 6 = 0$ (b) $4x - 2y - 8 = 0$





Shading Regions

We can now consider how to use a set of inequalities to shade a region on graph paper.

Example

Shade the region defined by the following inequalities.

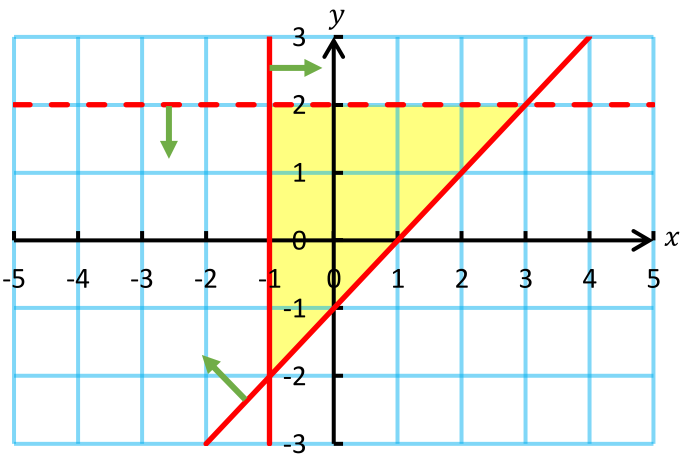
$$y < 2,\ x \geq -1,\ y \geq x - 1$$

Answer: **Step 1:** Plot the graphs of $y = 2,\ x = -1, y = x - 1$.

Rule: Use a solid line (——) for inequalities containing ≥ or ≤; and a dotted line (- - - -) for inequalities containing > or <.

Step 2: Show, using an arrow, a region corresponding to each of the three lines.

Step 3: Shade the region that is satisfied by all of the arrows / inequalities.



For lines that aren't vertical or horizontal, **substitute a point that doesn't lie on the line** to decide **which way the arrow should point**. For example, considering $y \geq x - 1$, substitute the point $(0, 0)$:

$$0 \geq 0 - 1$$
$$0 \geq -1$$

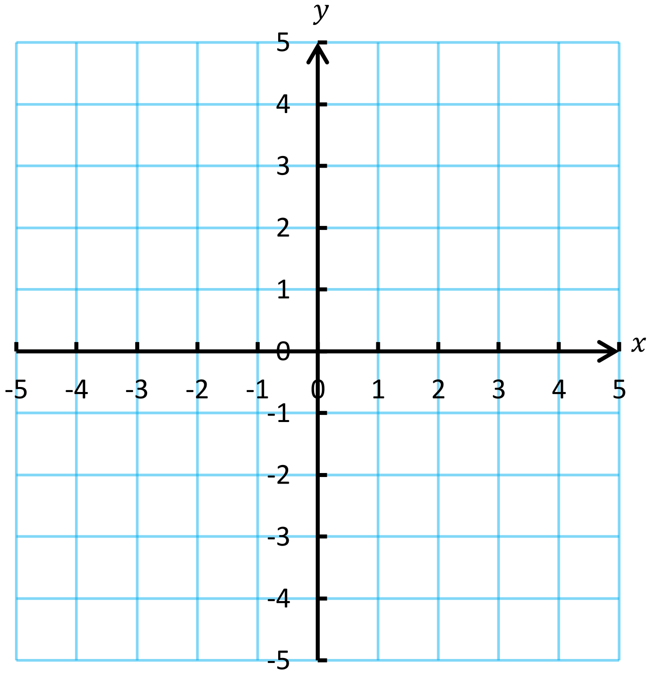
This inequality is **true**, so the arrow should point **towards** the point (0, 0).

Exercise 35

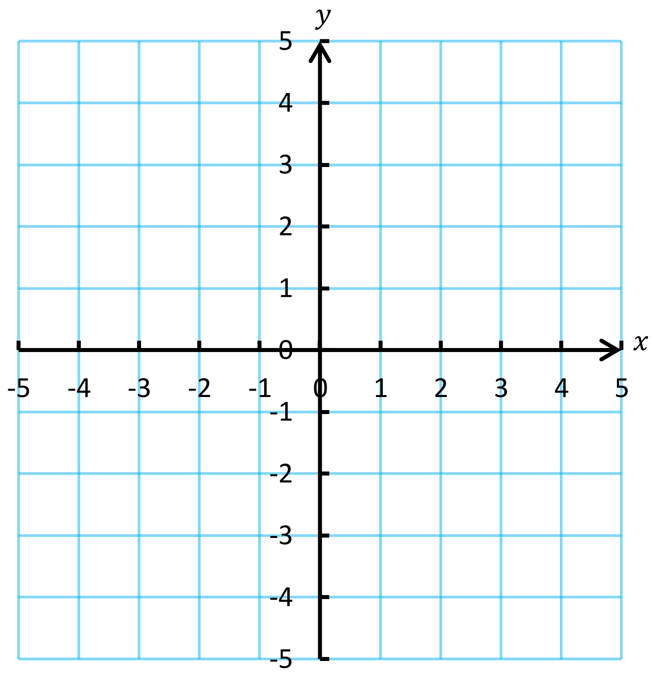
H

Shade the region that is defined by the following sets of inequalities.

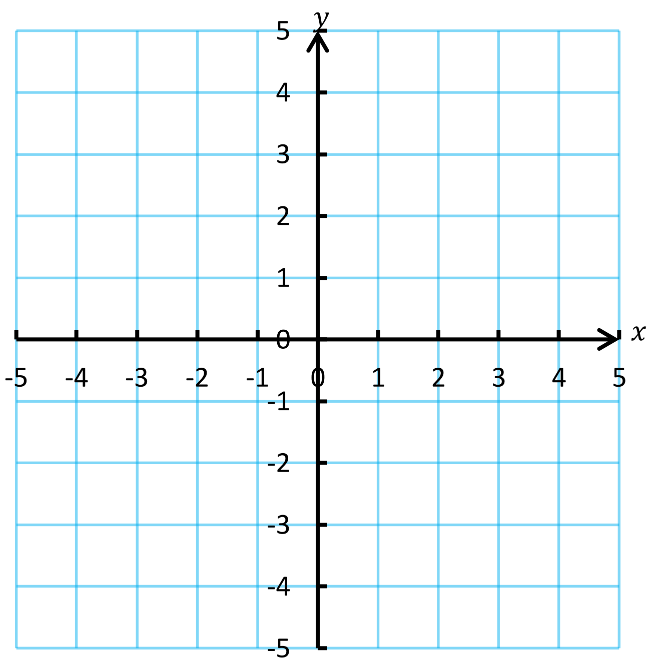
(a) $y < 4, x < 3, y \geq -2, x \geq -1$



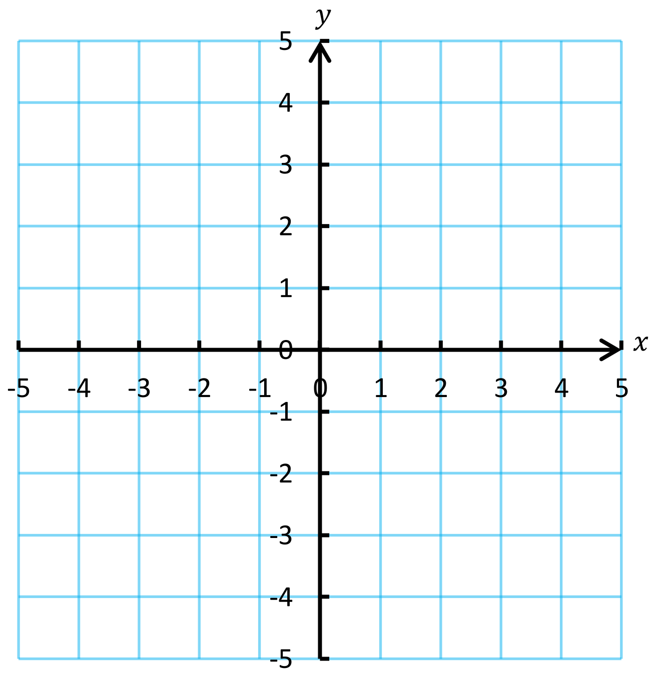
(b) $y \leq 3, x \geq 0, y \geq 2x - 3$



(c) $y > 1, x \geq 1, x + 2y - 4 \leq 0$



(d) $y > -2, y < x + 1, y < -2x + 3$



Exercise 36

H

Draw suitable axes in order to shade the regions that are defined by the following sets of inequalities.

(a) $x + y \leq 4, y \leq 2x + 4, y \geq 1$

(b) $y \geq 0, x < -1, y \leq x + 3$

(c) $x \geq -1, y < 4, y \geq 3x - 1$

(d) $y > -4, x < -1, y \leq 2x + 1$

(e) $y < 2, x \leq 1, y > -x + 2$

(f) $y > -3, x \geq -2, x \leq 1.5, y \leq -\frac{1}{2}x + 1$

Exercise 37

H

Draw suitable axes in order to shade the regions that are defined by the following sets of inequalities.

(a) $x \leq 2,\ y > -4,\ y \geq 2x - 2.5$

(b) $y < 2,\ x \geq -3,\ y \geq x - 1,\ y \geq -x - 4$

(c) $x + y < 1,\ x \geq -3$

(d) $y \geq x - 2,\ y < x + 4$

(e) $y < 2x,\ y \geq x + 1$

(f) $x + y > 2,\ y > x - 3$

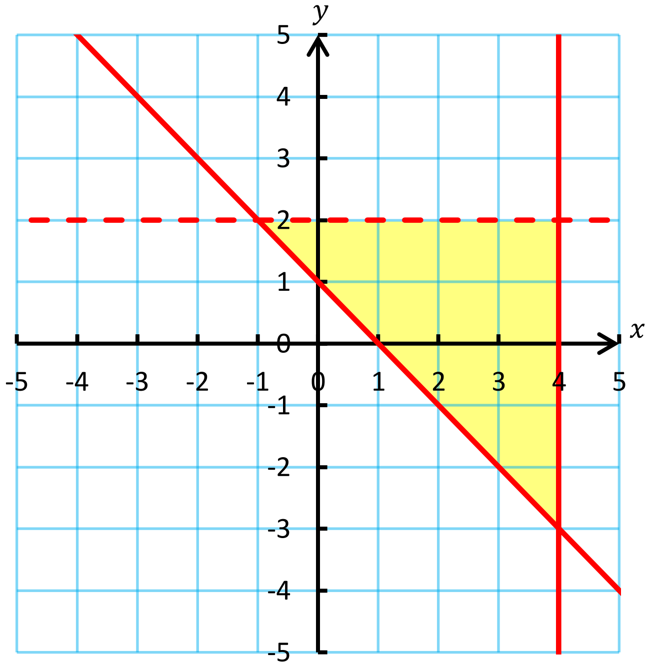
(g) $x - 2y < 4,\ y \leq x$

(h) $2x - 3y \leq 6,\ 2x + 2y < 0$

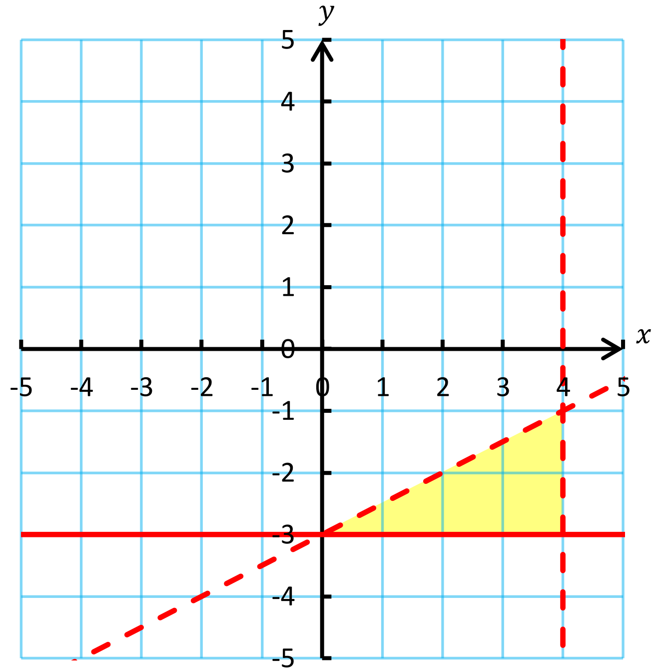
Exercise 38

What inequalities define the following shaded regions?

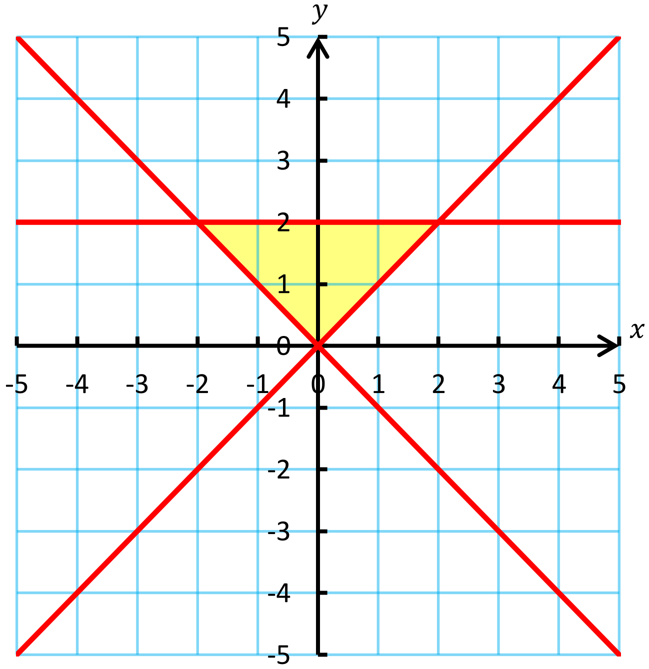
(a)



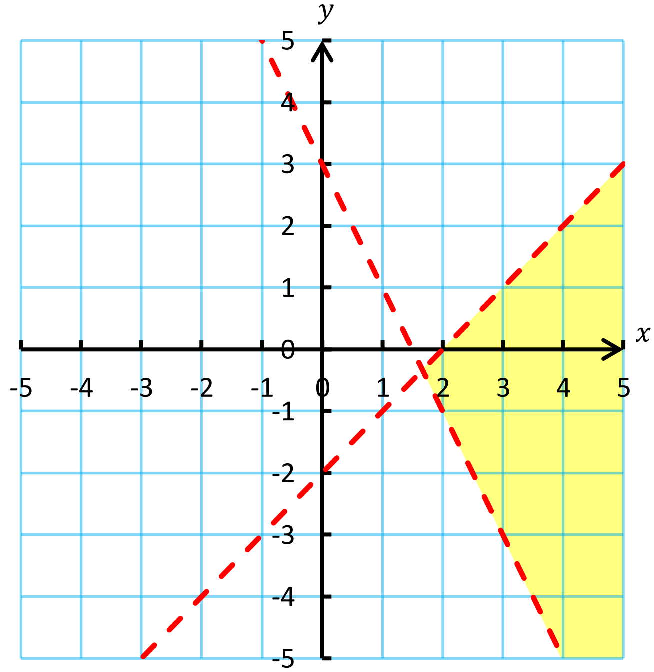
(b)



(c)



(d)



Exercise 39

Applying

H

A shop has asked a manufacturer to produce skirts and jackets.

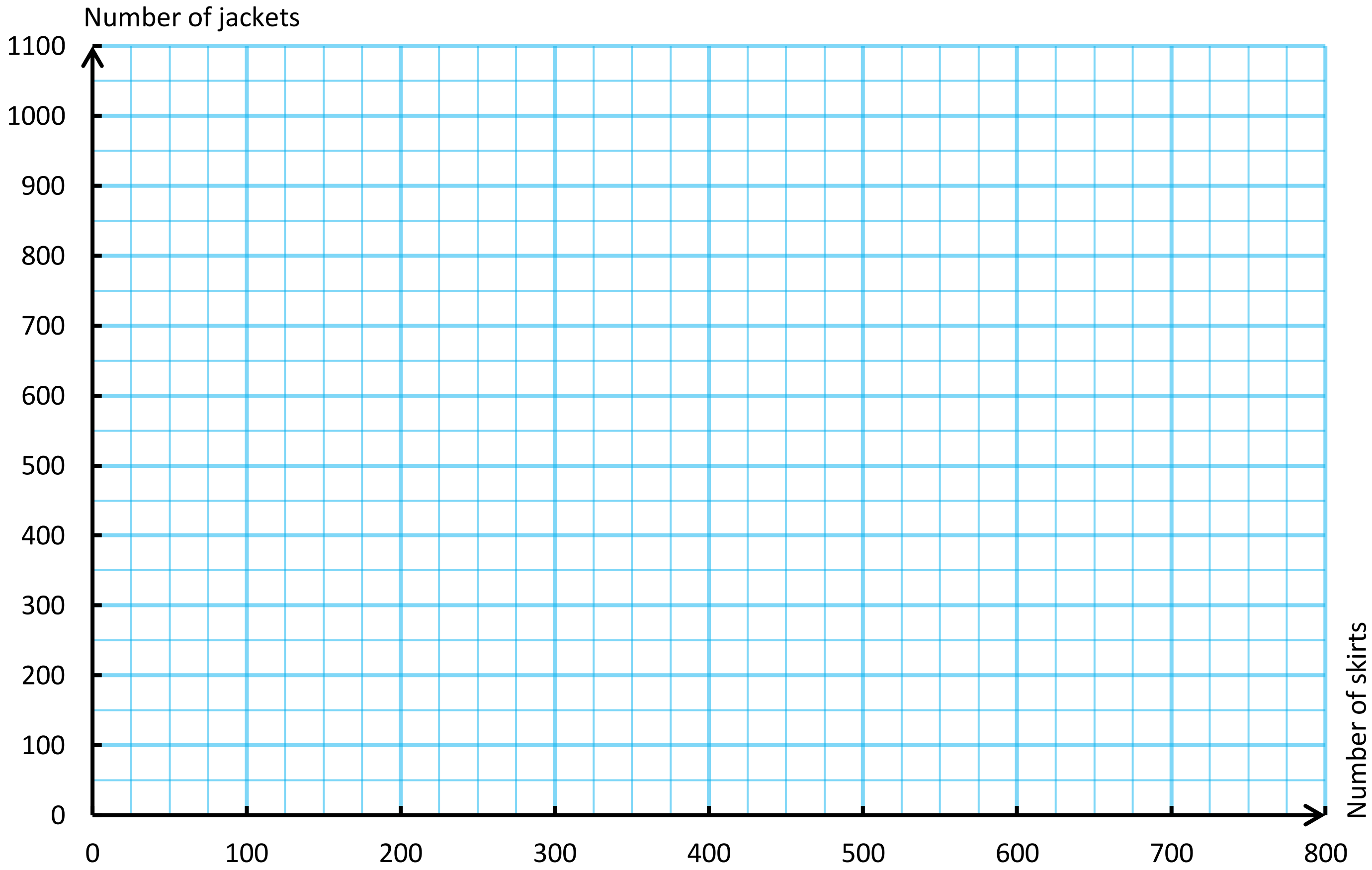
As raw materials, the manufacturer has 750 m² of cotton and 1,000 m² of polyester.

Each skirt requires 1 m² of cotton and 2 m² of polyester.

Each jacket requires 1.5 m² of cotton and 1 m² of polyester.

The price of a skirt is £50 and the price of a jacket is £40.

Assuming that everything is sold, how many skirts and jackets should the shop buy in order to maximise their profit?



Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Developing

Algebra 3

Reflection

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I know how to recognise whether the connection between two measures is a **direct proportion** or an **inverse proportion**.			2, 5, 7, 8, 9	
If two measures are in direct proportion, I can **calculate one of the missing measures**.			5, 8	
If two measures are in inverse proportion, I **can calculate one of the missing measures**.			7, 9	
I can work with **more than one proportion**.			10	
I can **recognise and use the graphs** of direct proportion and inverse proportion.			1	
I can **write the nth term** for **simple quadratic sequences**, e.g. $n^2 + 9$.			11	
I can **write the nth term** for **more complex quadratic sequences**, e.g. $4n^2 + 2n - 1$.			11	
I can **illustrate an inequality on a number line**.			12	
I can **solve inequalities**.			3, 13	
I can **find the least whole number** (or the **greatest**) that satisfies an inequality.			4	
I can **list all the whole numbers** that satisfy an inequality.			14	

Am I ready for the test?
Tick the boxes below...

- [ ] I have revised the work in my mathematics book.
- [ ] I have revised the workbook.
- [ ] I have watched the relevant videos on YouTube.
- [ ] I have completed the Diagnostic Questions quiz.
- [ ] I have completed at least 4 pages in my revision book.

Developing

Algebra 3

Reflection

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I know how to recognise whether the connection between two measures is a **direct proportion** or an **inverse proportion**.			1, 2, 3	
If two measures are in direct proportion, I can **calculate one of the missing measures**.			2	
If two measures are in inverse proportion, I **can calculate one of the missing measures**.			1, 3	
I can work with **more than one proportion**.			4	
I can **recognise and use the graphs** of direct proportion and inverse proportion.				
I can write and use **proportion equations**.			5	
Given a set of data for two measurements, I can **find the equation** that describes the connection between the measurements.				
I can **write the nth term** for **simple quadratic sequences**, e.g. $n^2 + 9$.			6	
I can **write the nth term** for **more complex quadratic sequences**, e.g. $4n^2 + 2n - 1$.			6	
I can **illustrate an inequality on a number line**.			7	
I can **solve inequalities**.			8	
I can **find the least whole number** (or the **greatest**) that satisfies an inequality.				
I can **list all the whole numbers** that satisfy an inequality.			9	
I can **plot lines of the form $ax + by + c = 0$**.				
Given a set of inequalities, I can **shade the region** that is defined by those inequalities.			10	
Given a region on graph paper, I can **find the set of inequalities** that define the region.				

Am I ready for the test?
Tick the boxes below...

I have revised the work in my mathematics book.	☐
I have revised the workbook.	☐
I have watched the relevant videos on YouTube.	☐
I have completed the Diagnostic Questions quiz.	☐
I have completed at least 4 pages in my revision book.	☐

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The Mathematics Department

11

Measuring Shapes 4

Name:

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Chapter	Mathematics	Page Number
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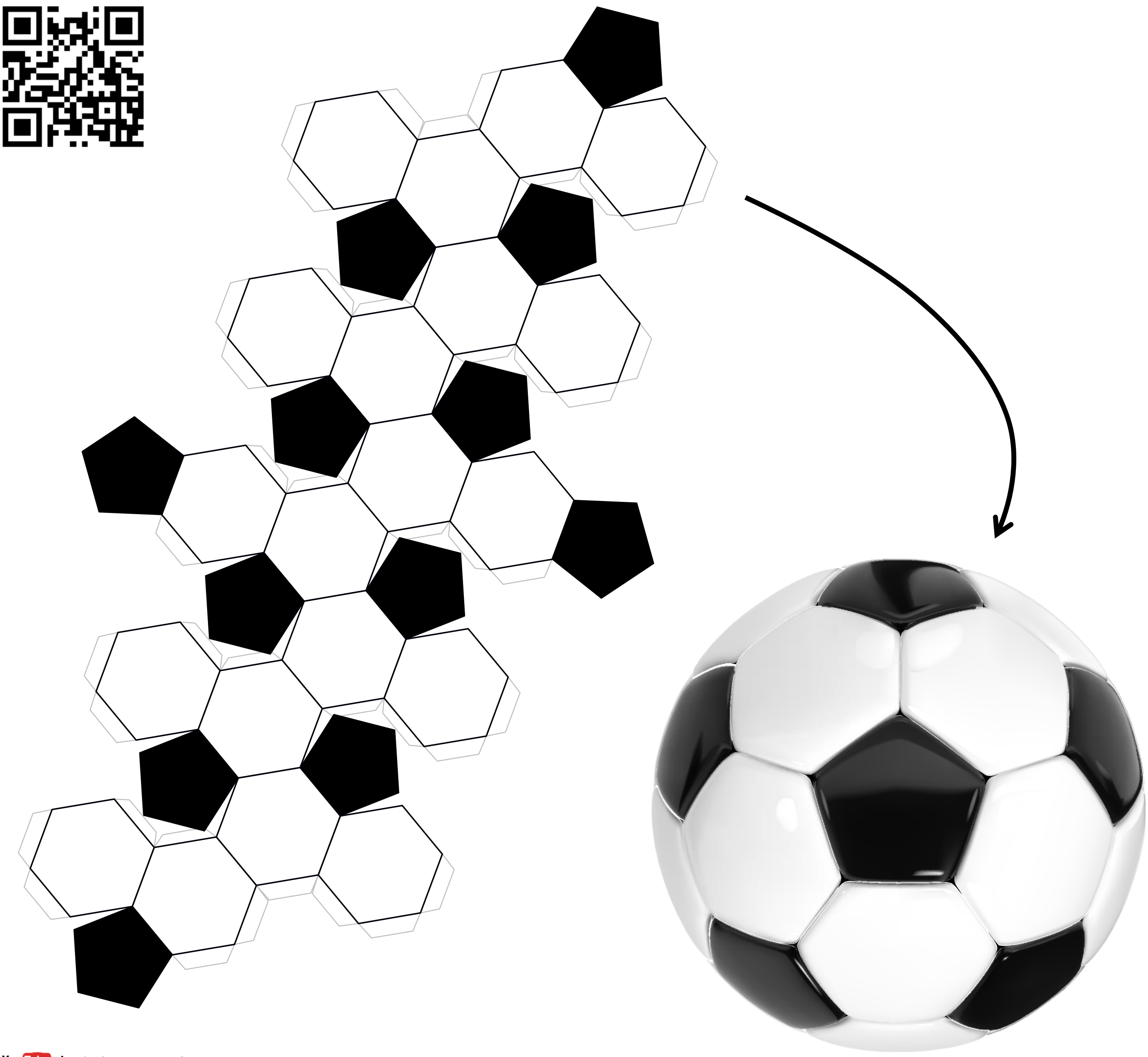
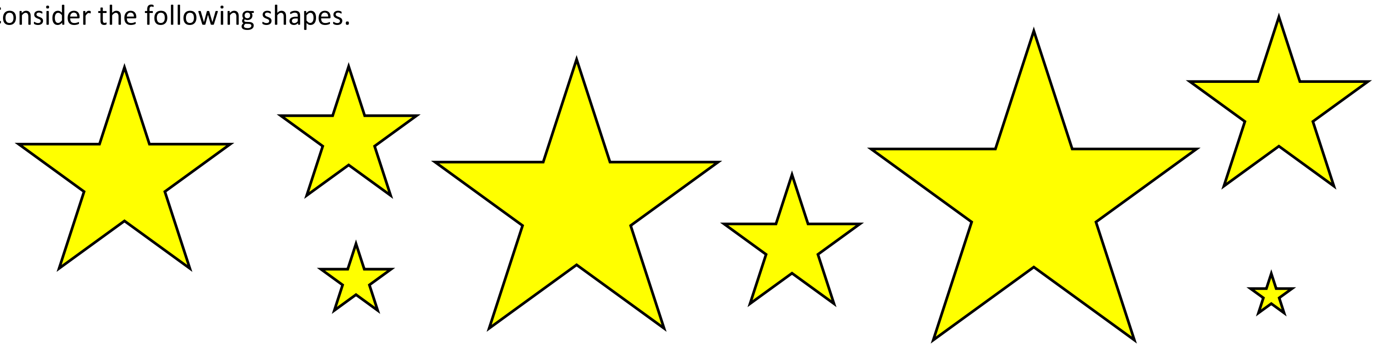
Angles in Polygons	The interior angles of a polygon. The exterior angles of a polygon. Tessellations.	8
Circle Theorems	Intermediate tier circle theorems. **Higher tier circle theorems.**	12
Transformations	Translation. Rotation. Reflection. Enlargement. Combinations.	18



Congruent Shapes

Consider the following shapes.



The shapes are all **similar**, so that the same shape is seen each time, but only two of the shapes are **congruent**, that is to say they have the same *size*.

Congruent shapes are the same **shape**, and the same **size**.

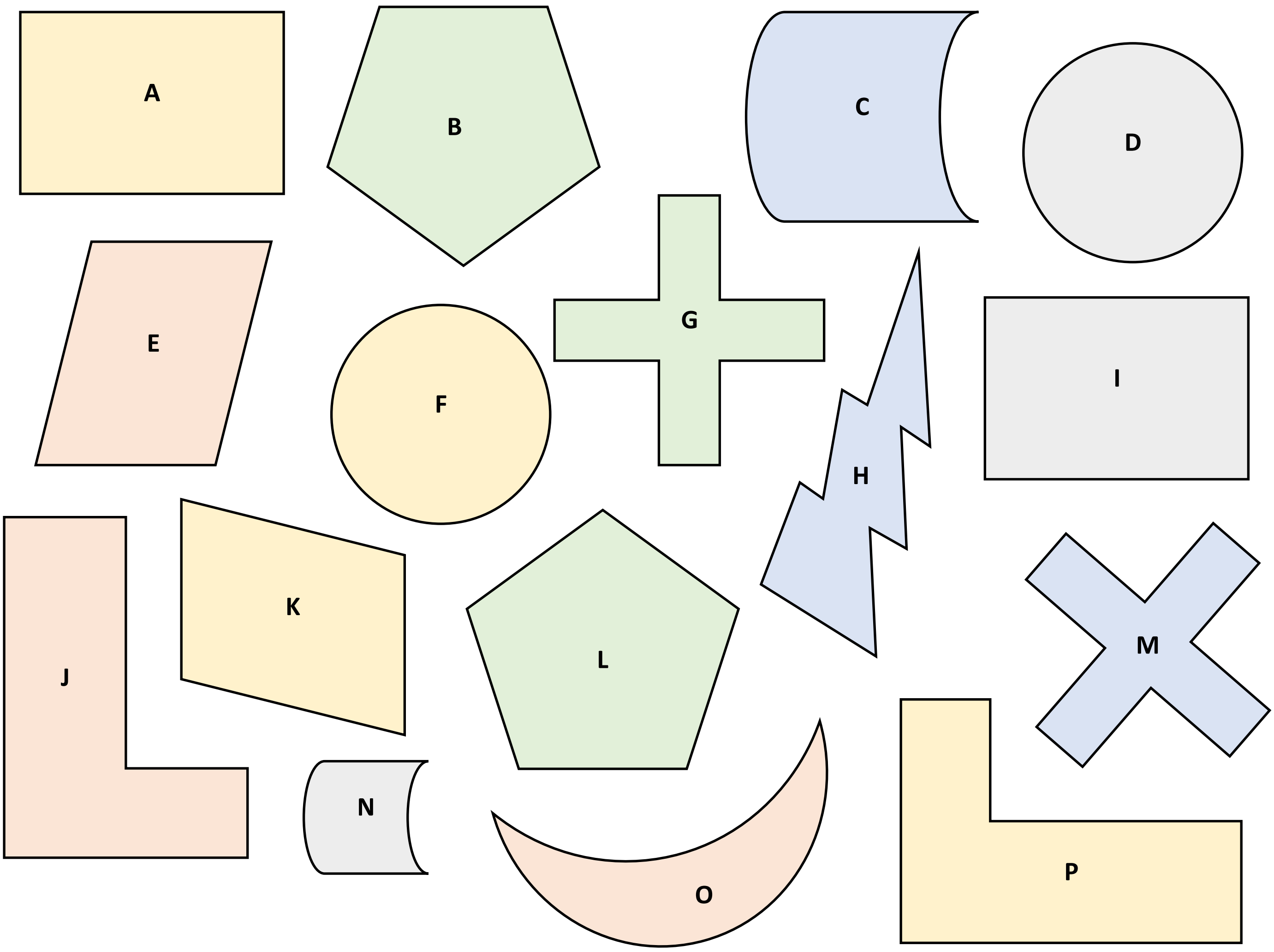
Skill I

Exercise 1

Tick the two shapes that are congruent above.

Exercise 2

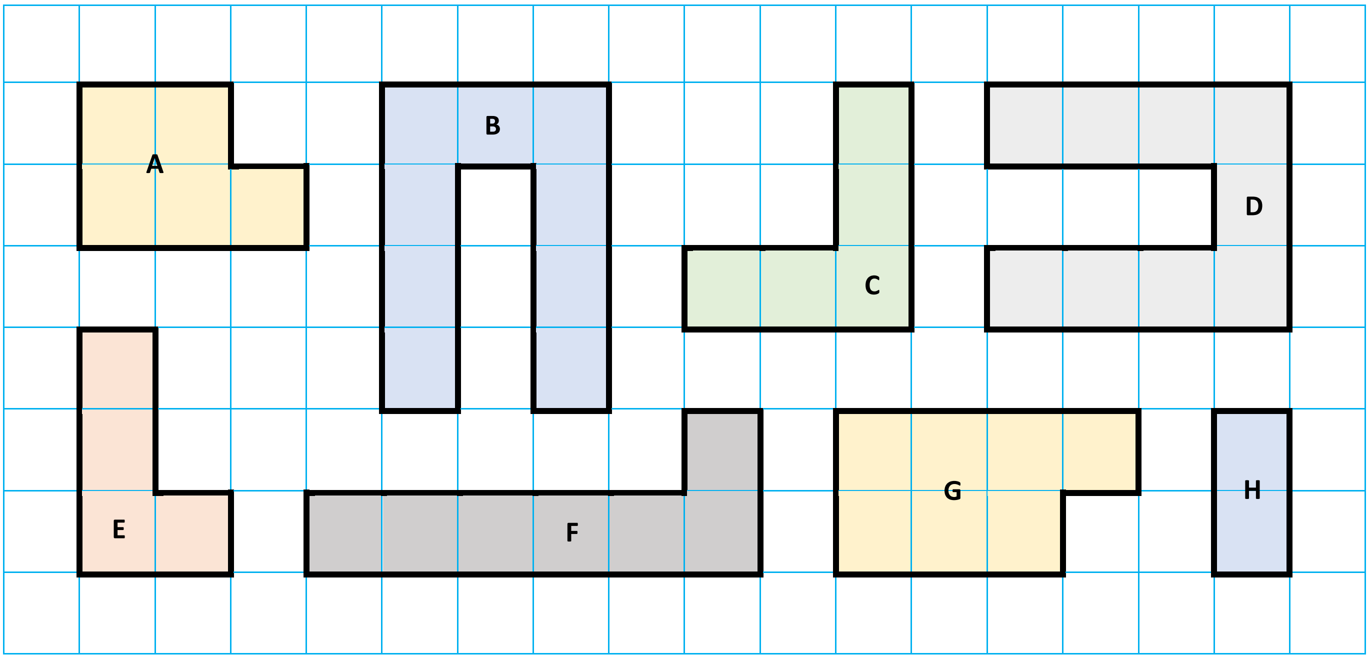
Look at the following shapes. Which pairs of shapes are congruent?



Exercise 3

I

Below there is a collection of shapes drawn on a squared centimetre grid.

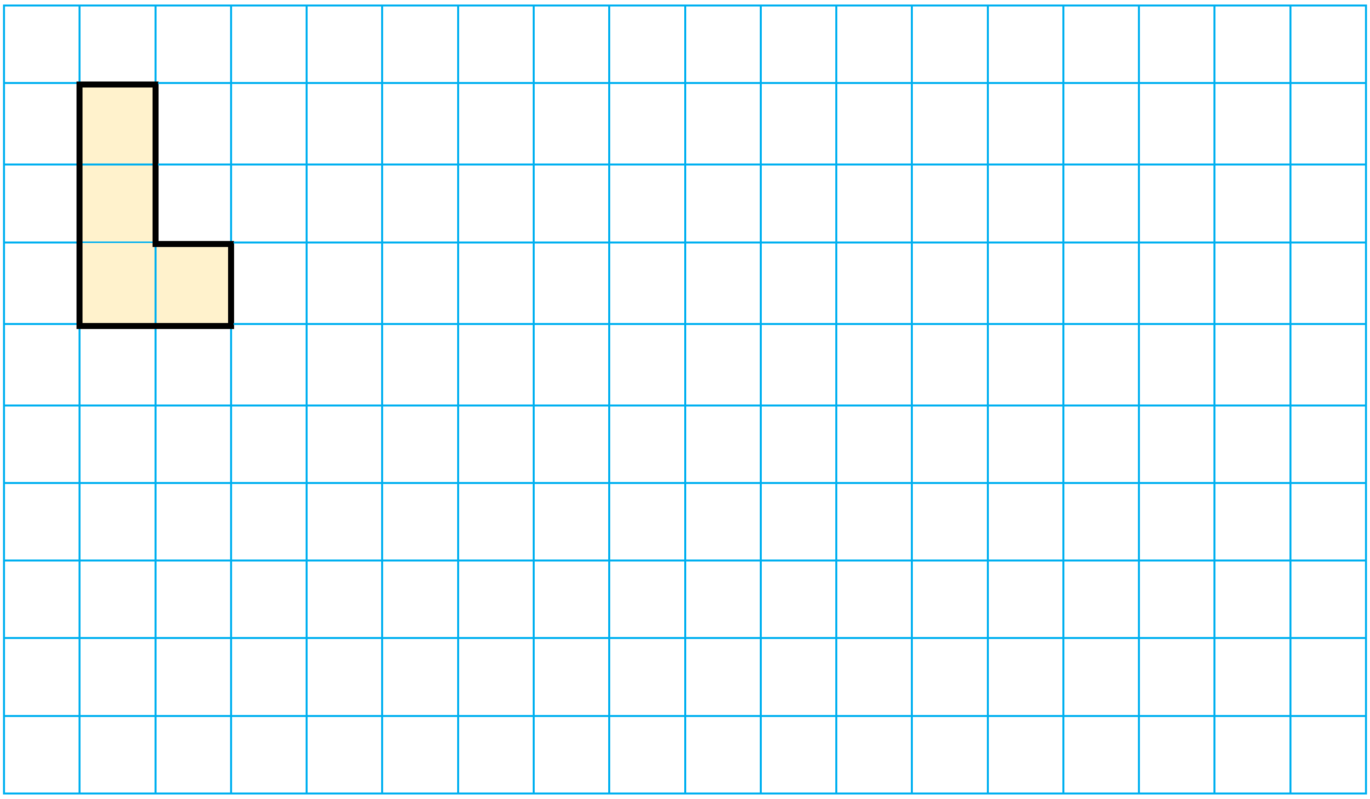


(a) Which two shapes are congruent?
(b) Which two shapes have an area of 5 cm^2?
(c) Which two shapes have a perimeter of 12 cm?
(d) Which two shapes have an area of 7 cm^2?
(e) Which two shapes have a perimeter of 10 cm?

Exercise 4

I

On the following grid, draw shapes that are congruent to the shown shape, but have different orientations. How many different orientations can be drawn?

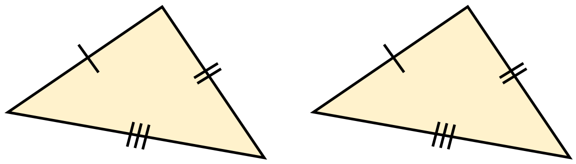


Congruent Triangles Proofs

Higher Tier

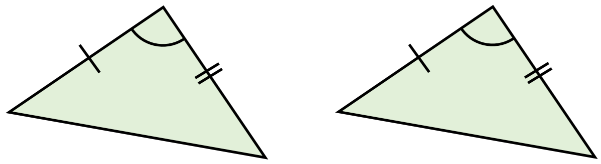
There are four ways of proving that two triangles are congruent.

(1) Side, Side, Side (SSS)



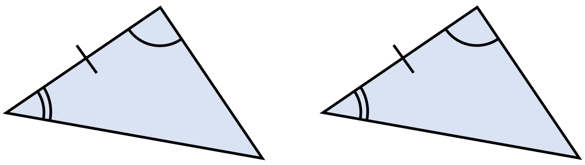
The lengths of the sides of the first triangle correspond to the lengths of the sides in the second triangle.

(2) Side, Angle, Side (SAS)



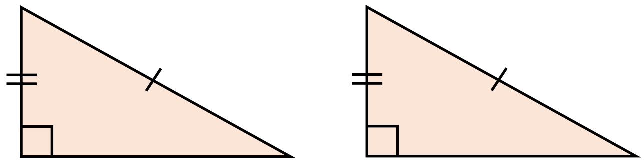
The lengths of two of the sides in the first triangle correspond to lengths of two of the sides in the second triangle, and the angles **between** the sides are equal.

(3) Angle, Side, Angle (ASA)



The size of two of the angles in the first triangle correspond to the size of two of the angles in the second triangle, and the length of the sides **between** the angles are equal.

(4) Right Angle, Hypotenuse, Side (RHS)

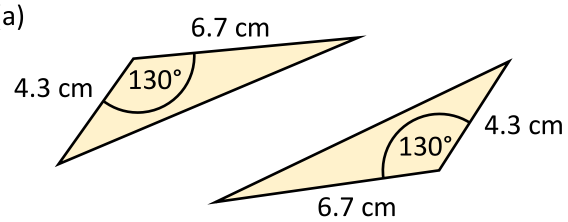


The two triangles are right-angled triangles; the lengths of the hypotenuse are equal; and the lengths of another side are equal.

Example

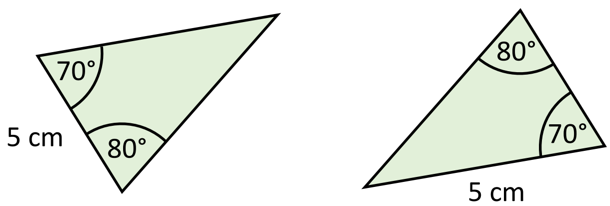
Explain, **noting your reasons**, if the following pairs of triangles are congruent or not.

(a)



Answer: The lengths of two of the sides in the first triangle are equal to the lengths of two of the sides in the second triangle (4.3 cm, 6.7 cm). The angle between the sides (130°) is also equal, so the triangles are congruent due to the SAS rule.

(b)



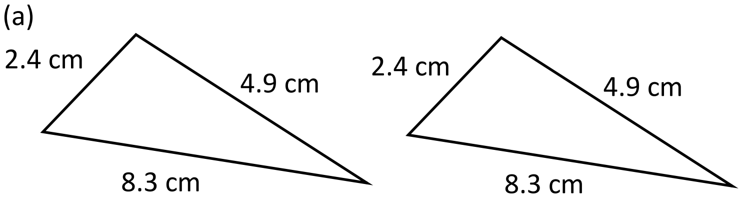
Answer: These triangles are not congruent. There is an angle of 70°, an angle of 80° and a side length of 5 cm in each triangle, but the 5 cm length is not **between** the angles in the triangle on the right. Because the longest length in a triangle is always opposite the greatest angle (80° in this case), the side between the angles must have a length less than 5 cm. So, we cannot use the ASA rule to prove that these triangles are congruent.

Exercise 5

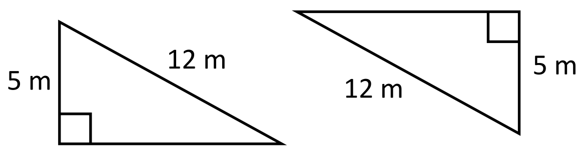
H

Explain, **noting your reasons**, if the following pairs of triangles are congruent or not.
(The diagrams are not drawn to scale.)

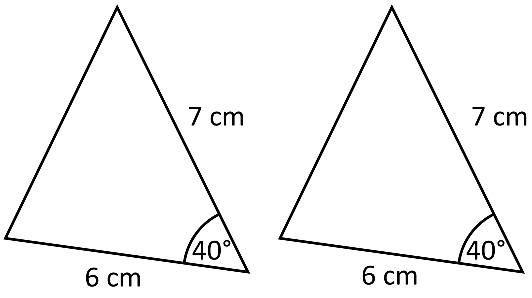
(a)



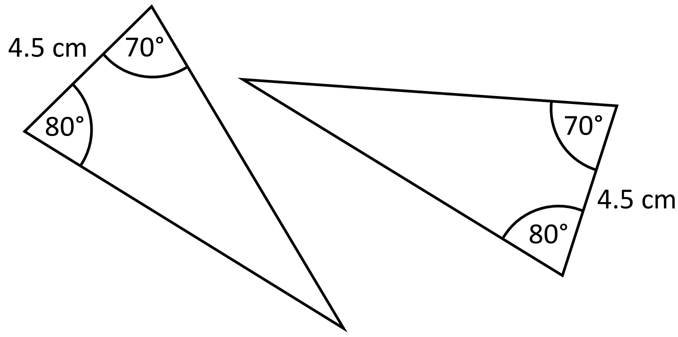
(b)



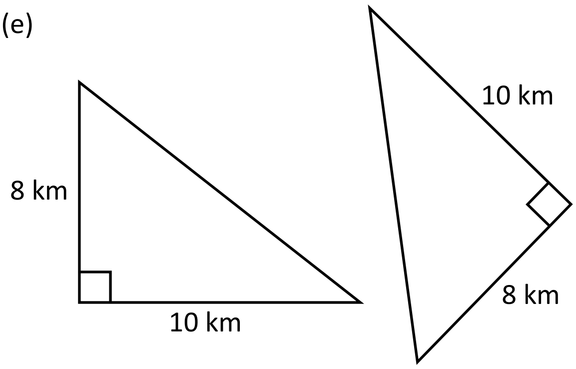
(c)


7 cm
40°
6 cm
7 cm
40°
6 cm

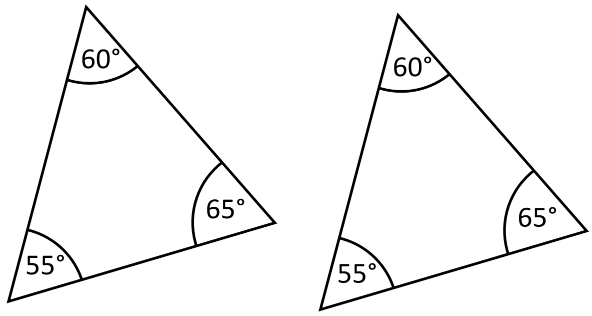
(d)


4.5 cm
70°
80°
70°
4.5 cm
80°

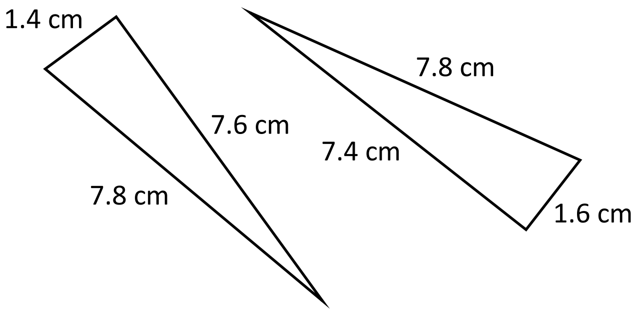
(e)


8 km
10 km
10 km
8 km

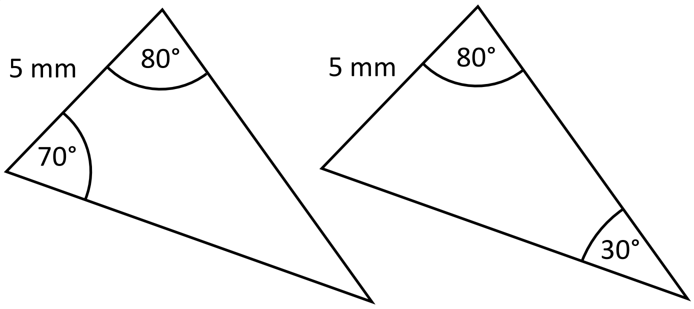
(f)


60°
65°
55°
60°
65°
55°

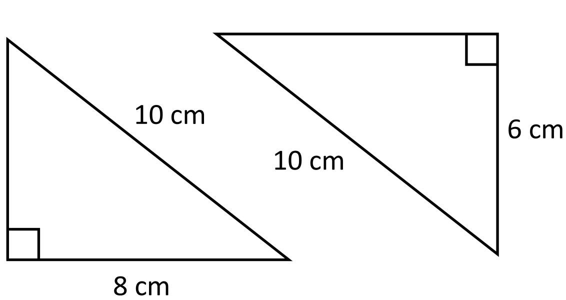
(g)


1.4 cm
7.6 cm
7.8 cm
7.8 cm
7.4 cm
1.6 cm

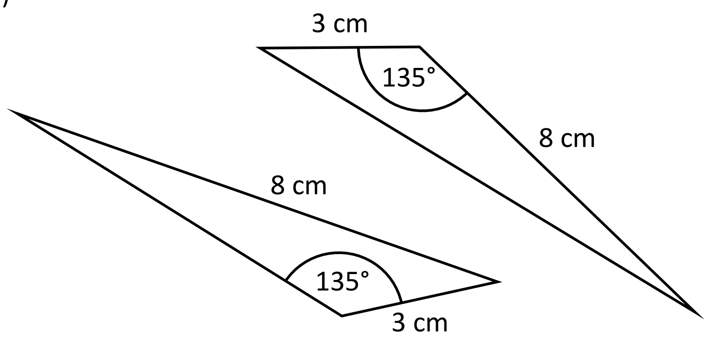
(h)


5 mm
80°
70°
5 mm
80°
30°

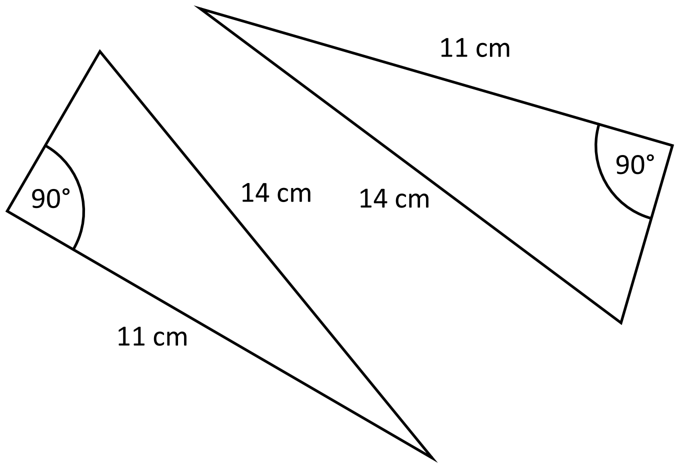
(i)


10 cm
8 cm
10 cm
6 cm

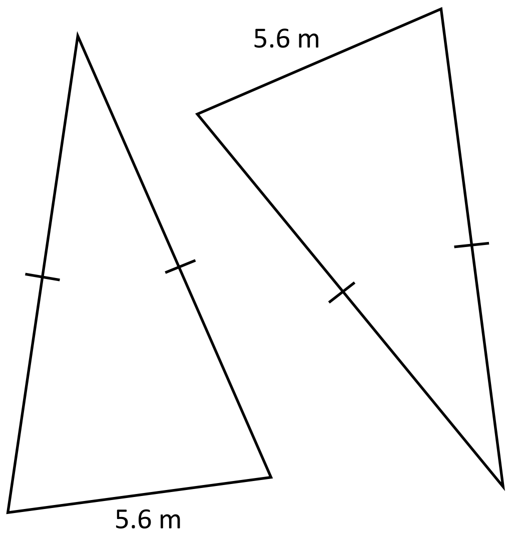
(j)


3 cm
135°
8 cm
8 cm
135°
3 cm

(k)


90°
14 cm
11 cm
11 cm
14 cm
90°

(l)


5.6 m
5.6 m

Exercise 6

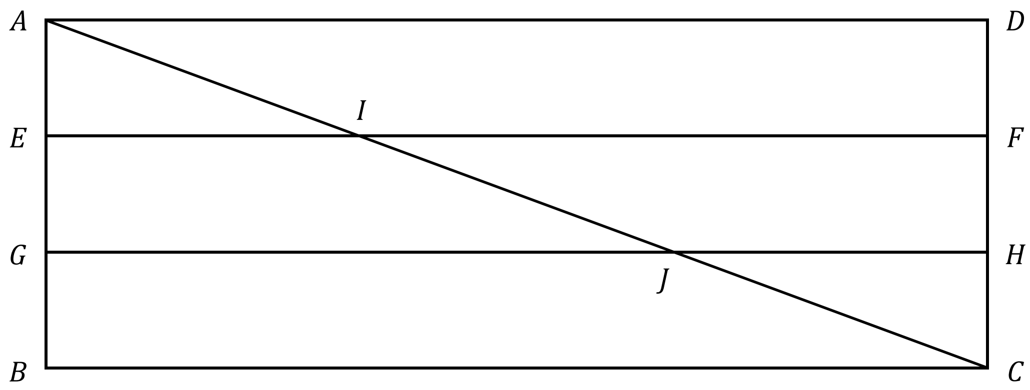
Applying

H

The following diagram shows a wooden fence, $ABCD$. The framework is strengthened by the addition of three wooden bars, EF, GH and AC.

The beams AD, EF, GH and BC are parallel to each other, with equal spaces between them.

The bar AC meets EF and GH at I and J, respectively.



(a) Name a triangle that is congruent to the triangle AGJ.
(b) Explain clearly why these triangles are congruent.

Exercise 7

I

Circle either TRUE or FALSE for the following statements.

STATEMENT		
Every rectangle is congruent.	TRUE	FALSE
Circles with equal area are congruent.	TRUE	FALSE
Every regular pentagon is congruent.	TRUE	FALSE
Using a 100% setting, a photocopier produces congruent shapes.	TRUE	FALSE
Each triangle with a base of 5 cm and a height of 4 cm is congruent.	TRUE	FALSE
Every semicircle with a diameter of 6 cm is congruent.	TRUE	FALSE

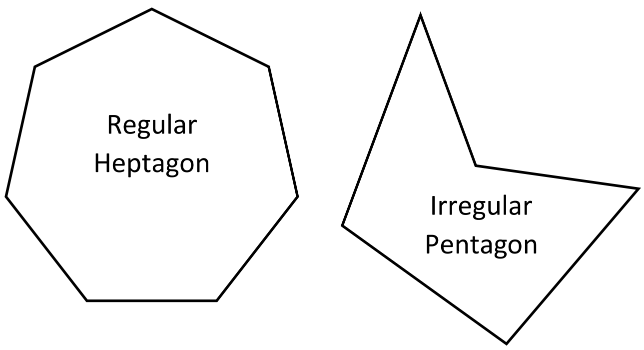
Evaluation

Key Words	**Further Questions**	**What went well?**	**To reach my target grade I will…**
			Grade ❏ **Target** ❏

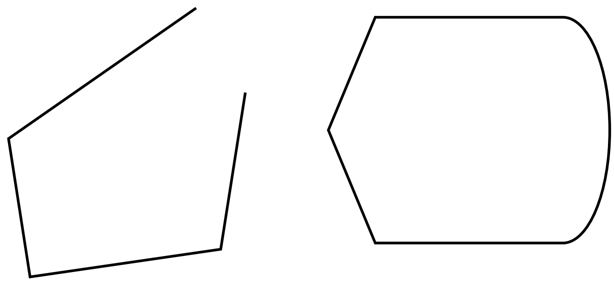
Angles in Polygons

A shape which uses straight lines only is called a **polygon**. The polygon is **regular** if its sides all have the same length and its angles are also all equal. If the polygon is not regular, then it is an **irregular** polygon.

Examples of polygons



Non-examples of polygons



Exercise 8

Skill

F

Complete the following table.

Number of edges	Polygon name	Total interior angles	Interior angle for a regular polygon
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
n	Polygon with n edges		

Exercise 9

F

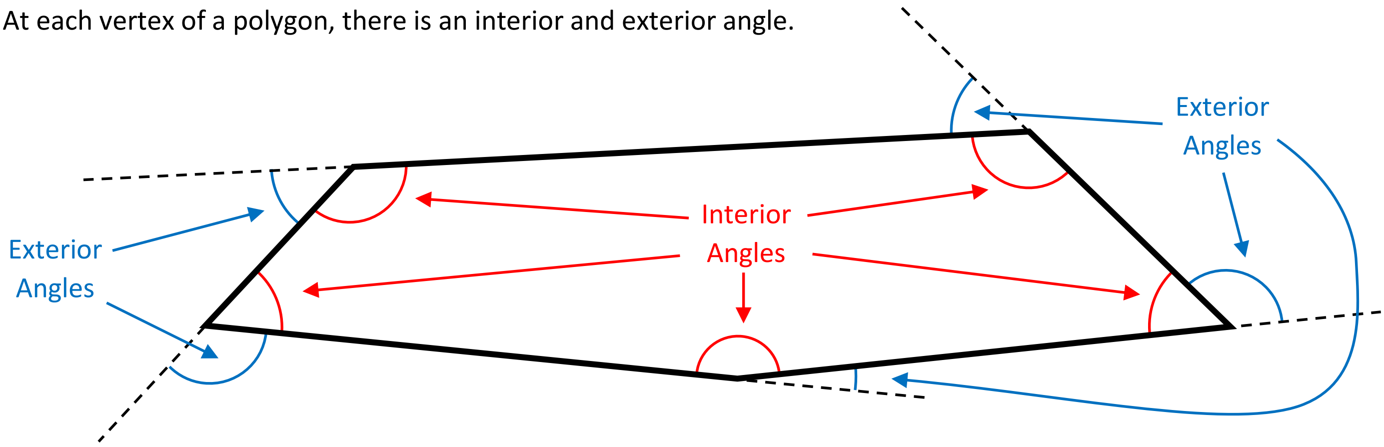
Use a ruler and a protractor to draw (a) a regular pentagon; (b) a regular hexagon; (c) a regular decagon.

Challenge!

Use an Excel spreadsheet to investigate the size of the interior angles of different regular polygons. As the number of edges increases, what happens to the interior angle? Does this pattern continue forever? What type of shape is a polygon with an ∞ number of edges?

The Exterior Angle of a Polygon

At each vertex of a polygon, there is an interior and exterior angle.



When walking along the exterior perimeter of a polygon, the exterior angle is the angle that you must turn through to continue travelling along the perimeter. For example, imagine walking around the exterior perimeter of the *Pentagon*, the headquarters of the United States Department of Defense.

The total of the interior angles is dependent upon the type of polygon. Above, the polygon is a pentagon, so the total interior angles is 540°. What is the total of the exterior angles? Again, imagine walking along the exterior perimeter. On returning to your original position, you will have rotated around a full turn, or 360°. The type of polygon is not important here, so the total exterior angles of any polygon is 360°.

Summary

For a polygon with n edges,

$$\text{Total of the exterior angles} = 360°$$
$$\text{Total of the interior angles} = 180°(n-2)$$

For any vertex in a polygon,

$$\text{Interior angle + exterior angle} = 180°$$

If the polygon is a regular polygon,

$$\text{One exterior angle} = \frac{360°}{n}$$
$$\text{One interior angle} = \frac{180°(n-2)}{n} \text{ or } 180° - \frac{360°}{n}$$

Challenge!

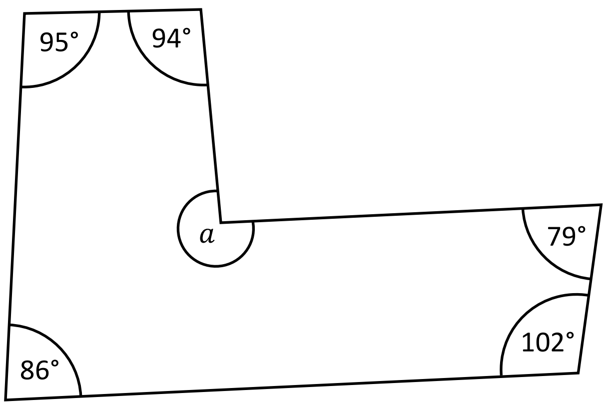
Prove that $\frac{180°(n-2)}{n} \equiv 180° - \frac{360°}{n}$.

Exercise 10

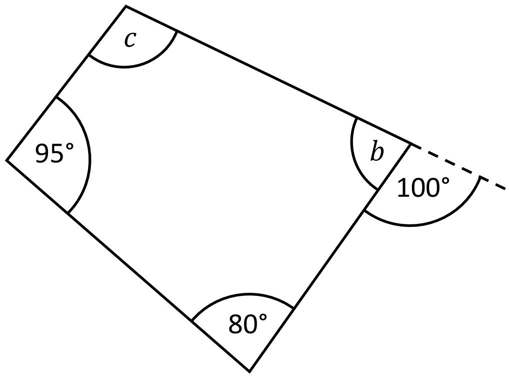
F

Calculate the size of the missing angles. (The diagrams are not drawn to scale.)

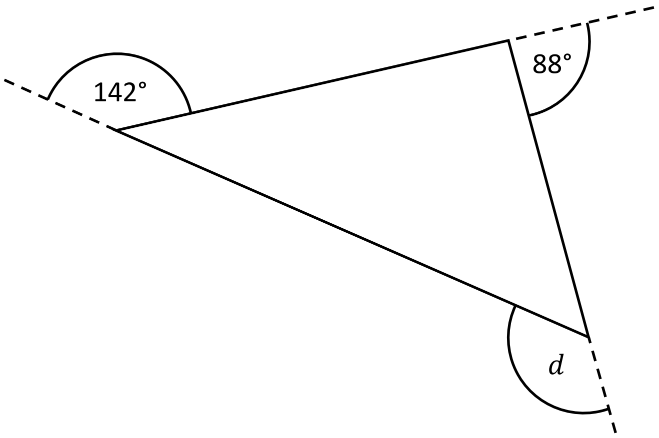
(a)



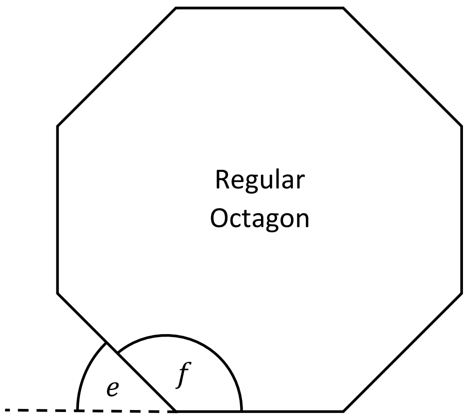
(b)



(c)



(d)



Exercise 11

F

(a) What is the total interior angles of any heptagon?

(b) What is the exterior angle of any equilateral triangle?

(c) What is the interior angle of any regular nonagon?

(d) Four of the exterior angles of a pentagon are 110°, 90°, 70°, 50°. What is the size of the fifth exterior angle?

(e) Four of the interior angles of a pentagon are 150°, 130°, 110°, 90°. What is the size of the fifth interior angle?

Exercise 12

I

(a) The size of the exterior angles of a regular polygon are 18°? How many edges does the regular polygon have?

(b) The size of the interior angles of a regular polygon are 156°. How many edges does the regular polygon have?

(c) Three of the exterior angles of a hexagon are 100°. The other three exterior angles are equal. Calculate the size of each of these exterior angles.

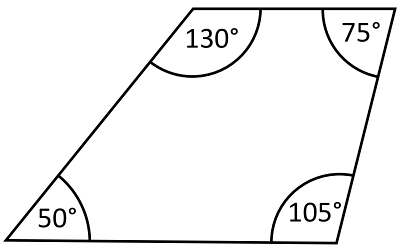
(d) Why is it not possible to draw a triangle with exterior angles 170°, 160°, 150°?

(e) Four of the six interior angles of a hexagon are 130°, 140°, 150°, 160°. The other two interior angles are equal. Calculate the size of the largest **exterior** angle of the hexagon.

Exercise 13

F

Draw polygons in the spaces below, showing clearly the size of each angle.

		Size of the least interior angle		
		Less	Equal	More
Total of the interior angles	More			
	Equal			
	Less			

Exercise 14

F

Squares (or regular quadrilaterals) tessellate, as can be seen in the picture on the right of tiles placed on a floor.

Use the ATM mats to find which two other regular polygons tessellate.

Prove that only these three regular polygons tessellate. (Hint: use the factors of 360 and the list of interior angles of regular polygons.)

Exercise 15

F

A **semi-regular tessellation** uses two or more regular polygons to fill the plane.

For example, the tessellation shown on the right uses squares (black and white) and octagons (red and white).

Use the ATM mats to find the **eight** semi-regular tessellations. Record the tessellations in the following table. (The first row has been completed for you.)

Tessellation	Polygons	Angles around any point
1	Octagon, Octagon, Square	135° + 135° + 90° = 360°
2		
3		
4		
5		
6		
7		
8		

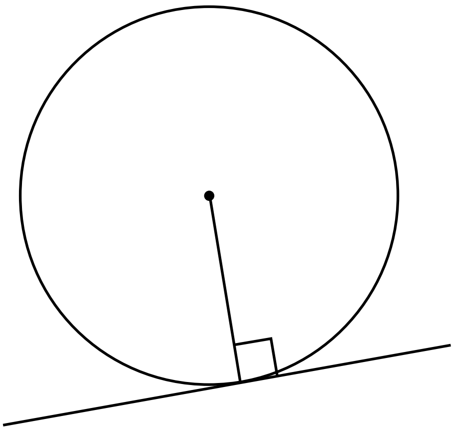
Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Circle Theorems

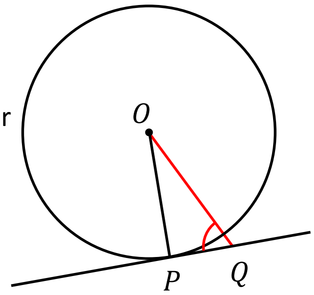
A number of facts related to angles in a circle must be learnt. (You do not have to learn the proofs.)

(1) Tangent and radius meet at a right angle.



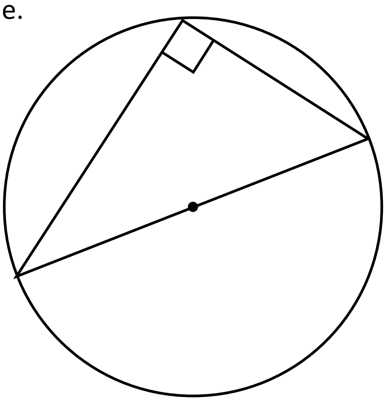
Proof

Assume that the tangent and radius do not meet at a right angle. Then we can draw the perpendicular line from the centre O to the point Q on the tangent (a point that is outside the circle), so that the angle $O\hat{Q}P = 90°$.



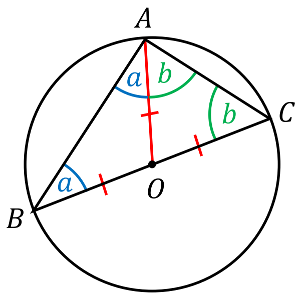
It follows that the triangle OQP is a right-angled triangle where the radius OP is the hypotenuse of the triangle. But we see that the line OQ must be longer than the line OP (as Q lies outside the circle). This goes against the mathematical fact that the hypotenuse of a right-angled triangle is the longest side, so tangent and radius must meet at a right angle.

(2) The angle in a semicircle is a right angle.



Proof

Split the triangle into two isosceles triangles, by adding a radius from the centre O to the vertex A.



The total of the angles of triangle ABC is 180°, so

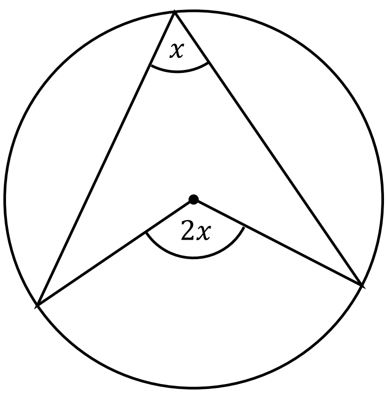
$a + a + b + b = 180°$

$2a + 2b = 180°$ [Collect like terms]

$a + b = 90°$ [Divide by 2]

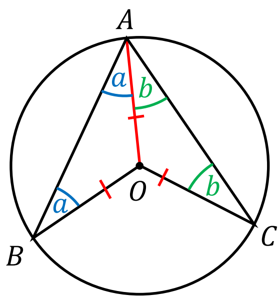
So, the angle $B\hat{A}C$ is a right angle.

(3) The angle in the centre of a circle is twice the angle on the circumference.



Proof

Split the quadrilateral into two isosceles triangles, by adding a radius from the centre O to the vertex A.



In triangle ABO, $A\hat{O}B = 180° - 2a$.

In triangle ACO, $A\hat{O}C = 180° - 2b$.

Using the angles around the centre O,

$B\hat{O}C = 360° - A\hat{O}B - A\hat{O}C$

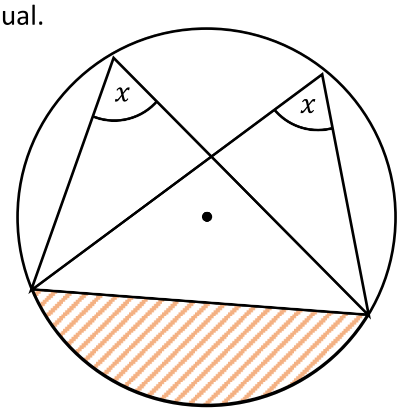
$B\hat{O}C = 360° - (180° - 2a) - (180° - 2b)$

$B\hat{O}C = 2a + 2b$

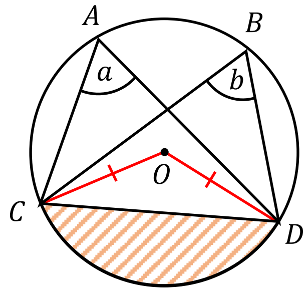
$B\hat{O}C = 2(a + b)$

So, the angle in the centre of the circle ($B\hat{O}C$) is twice the angle on the circumference ($B\hat{A}C$ or $a + b$).

(4) Angles in the same segment are equal.



Proof

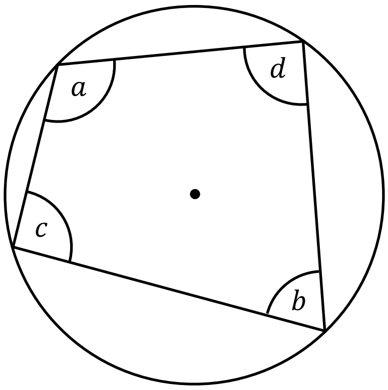


Add two radii from the centre O to the vertices C and D.

From the previous circle theorem, the size of angle $C\hat{O}D$ is twice the angle $C\hat{A}D$. But we can also state that the angle $C\hat{O}D$ is twice the angle $C\hat{B}D$.

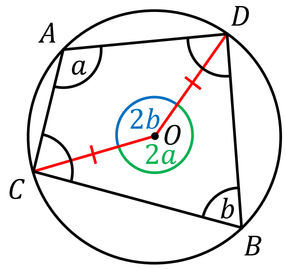
It follows that the angles $C\hat{A}D$ and $C\hat{B}D$ are equal. So, angles in the same segment are equal.

(5) Opposite angles in a cyclic quadrilateral sum to 180°.



$$a + b = 180°$$
$$c + d = 180°$$

Proof



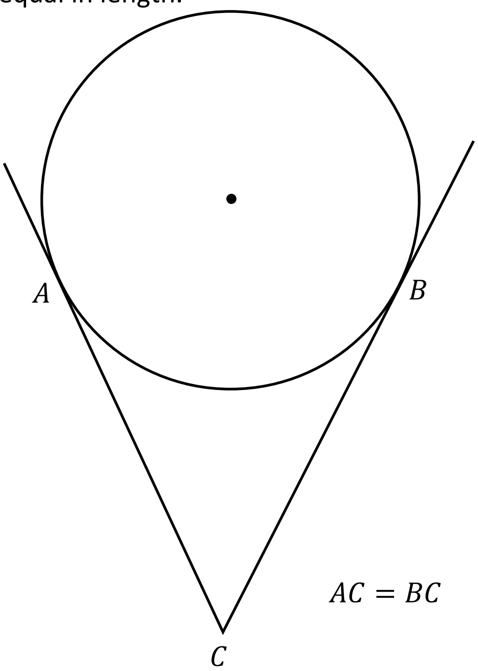
Add two radii from the centre O to the vertices C and D.

Because the angle in the centre is twice the angle on the circumference, we can state that $C\hat{O}D = 2b$, and $C\hat{O}D$ reflex $= 2a$.

Angles around any point sum to 360°, so
$2a + 2b = 360°$
$a + b = 180°$ [Divide by 2]

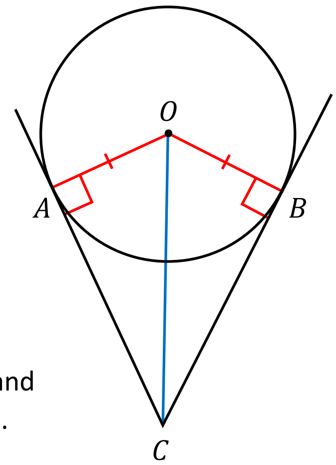
So, opposite angles in a cyclic quadrilateral sum to 180°.

(6) Tangents from an external point are equal in length.



$AC = BC$

Proof (Higher Tier)



Add two radii from the centre O to the vertices A and B.
Then, add a line from the centre O to the vertex C.

Because tangent and radius meet at a right angle, we have $O\hat{A}C = O\hat{B}C = 90°$.

The two right-angled triangles OAC and OBC share the same hypotenuse OC.

We have $OA = OB$, because they are both radii.

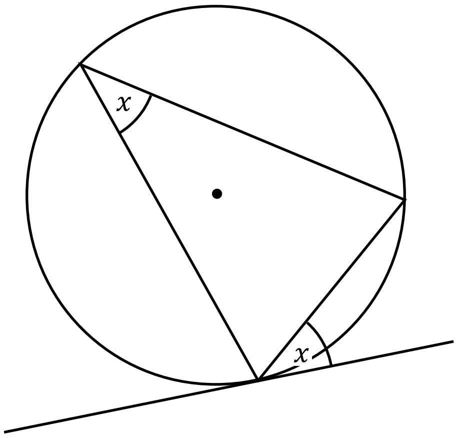
Using the RHS rule, we can state that the triangles OAC and OBC are congruent.

It follows that $AC = BC$, and so tangents from an external point are equal in length.

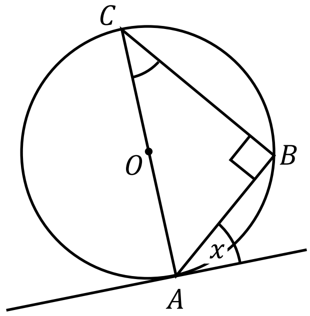
The final two theorems appear in the higher tier only.

Higher Tier

(7) The angle between chord and tangent is equal to the angle in the alternate segment.



Proof



Because angles in the same segment are equal, we can choose to prove the case where the line AC is a diameter to the circle.

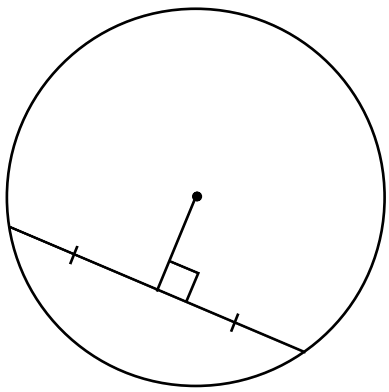
Tangent and radius meet at a right angle, so $C\hat{A}B = 90° - x$.

The angle in a semicircle is a right angle, so $A\hat{B}C = 90°$.

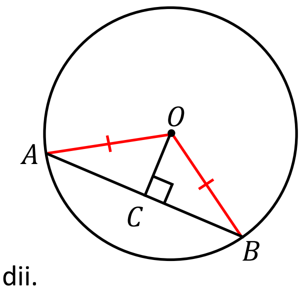
Using the triangle ABC,
$A\hat{C}B = 180° - A\hat{B}C - C\hat{A}B$
$A\hat{C}B = 180° - 90° - (90° - x)$
$A\hat{C}B = x$

So, the angle between chord and tangent is equal to the angle in the alternate segment.

(8) The perpendicular from the centre to a chord bisects the chord.



Proof



Add two radii from the centre O to the vertices A and B.

The triangles OAC and OBC are right-angled triangles.

The hypotenuse of both triangles are equal in length, since they are both radii.

The two triangles also share a side OC.

Using the RHS rule, we can state that the triangles OAC and OBC are congruent.

It follows that $AC = BC$, and so the perpendicular from the centre to a chord bisects the chord.

Exercise 16

I

Take some time to become familiar with the circle theorems. Here are some ideas:

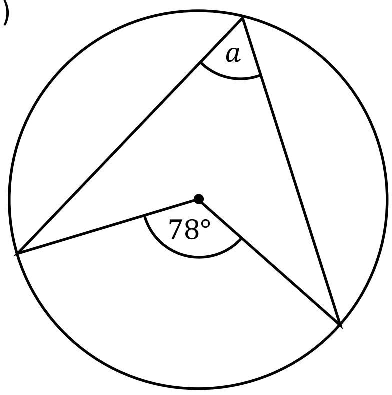
- Try to re-create the circle theorems using the GeoGebra software.
- Draw examples of the circle theorems in your revision book.
- Try to re-create the circle theorems using paper plates, string and colouring materials.
- Verify that the theorems are true by drawing examples using a compass, ruler and protractor.

Exercise 17

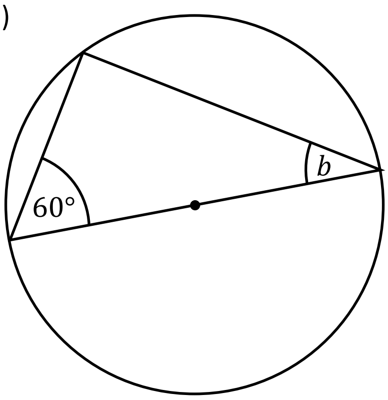
Skill I

Use the circle theorems to find the size of the marked angles in the following diagrams. (The diagrams are not drawn to scale.)

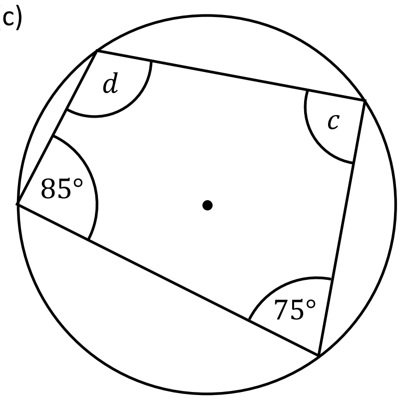
(a)



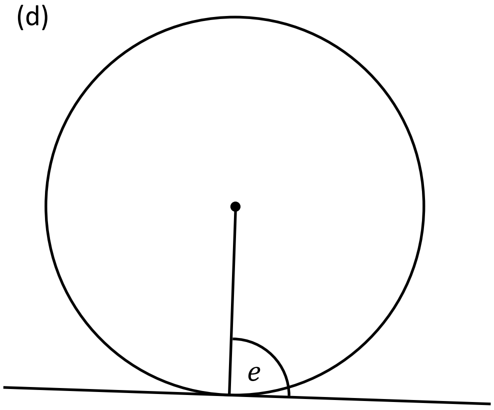
(b)



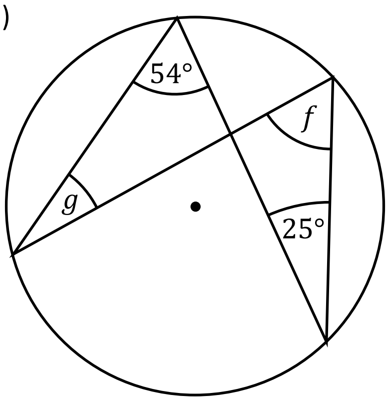
(c)



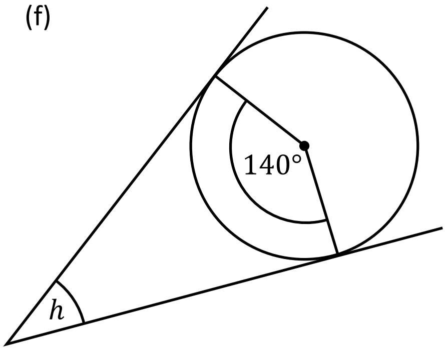
(d)



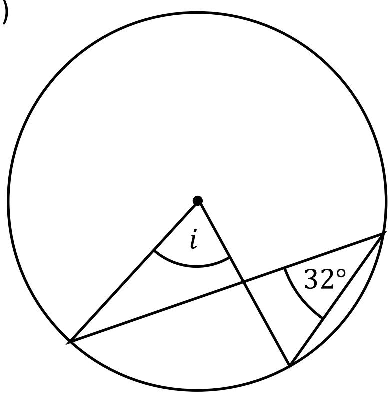
(e)



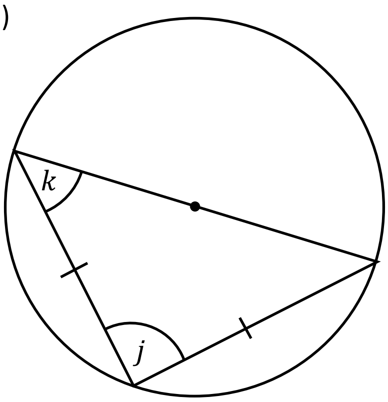
(f)



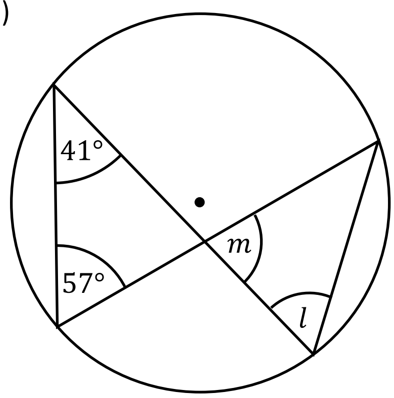
(g)



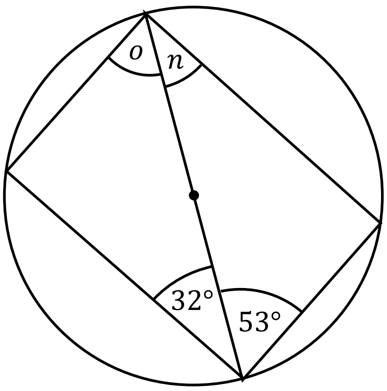
(h)



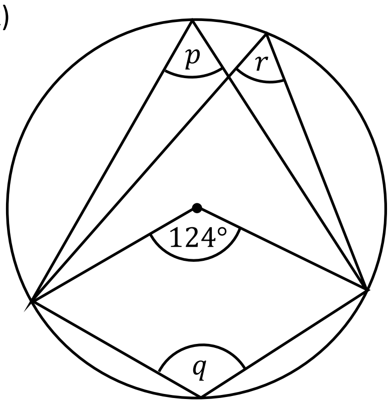
(i)



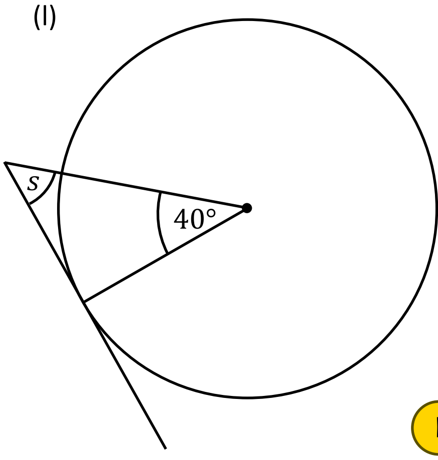
(j)



(k)



(l)



Exercise 18

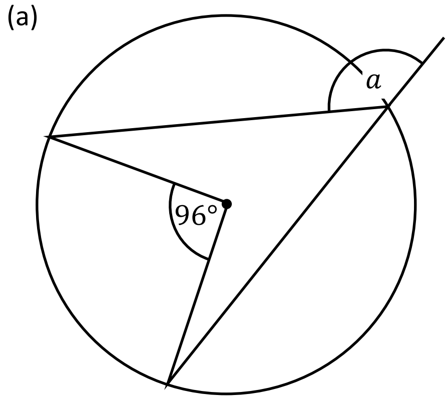
I

For each question in Exercise 17 above, note which circle theorem(s) you used in finding the missing angle(s).

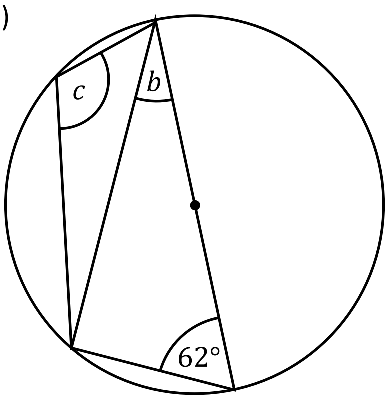
Exercise 19 I

Use the circle theorems to find the size of the marked sides or angles in the following diagrams.
(The diagrams are not drawn to scale.)

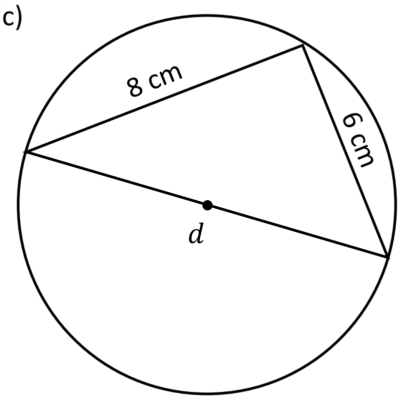
(a)



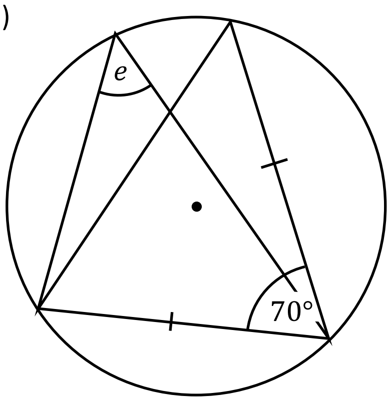
(b)



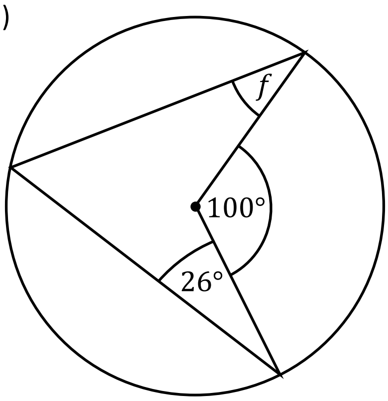
(c)



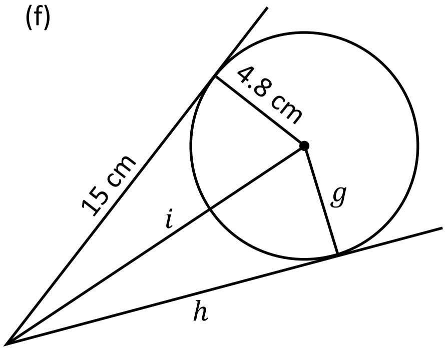
(d)



(e)



(f)



Exercise 20 I

For each question in Exercise 19 above, note which circle theorem(s) you used in finding the missing value(s).

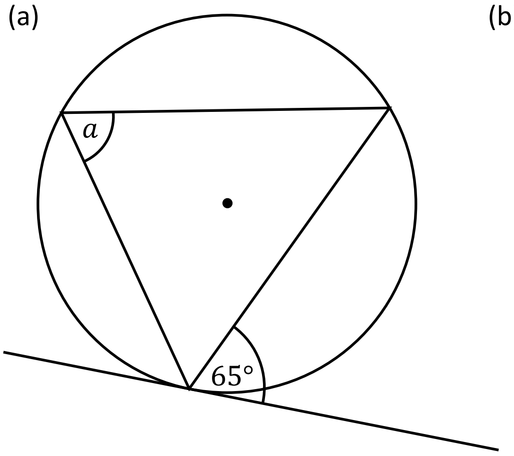
Challenge!

Imagine a circle with radius 1 m. 6 equilateral triangles are placed in the circle, with one vertex of each triangle in the centre of the circle, and the two other vertices on the circumference. The triangles do not overlap. What is the difference between the area of the circle and the area of the six equilateral triangles?

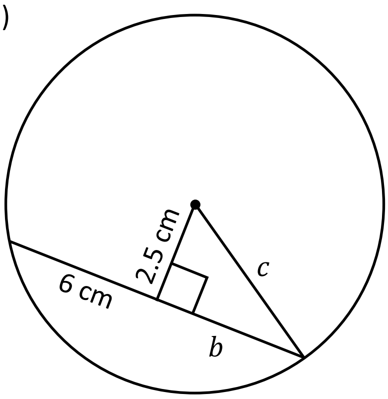
Exercise 21 H

Use the circle theorems to find the size of the marked sides or angles in the following diagrams. (The diagrams are not drawn to scale.)

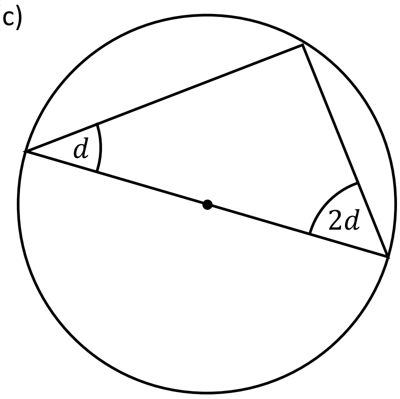
(a)



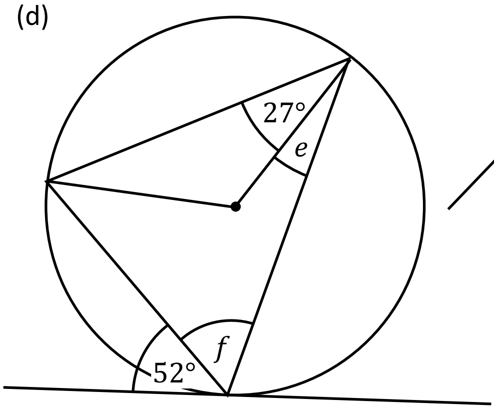
(b)



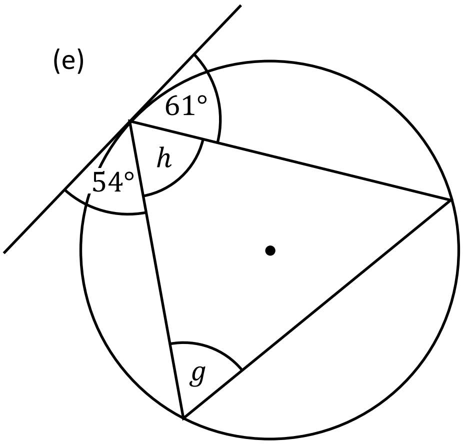
(c)



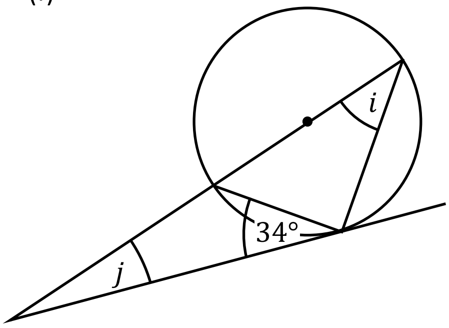
(d)



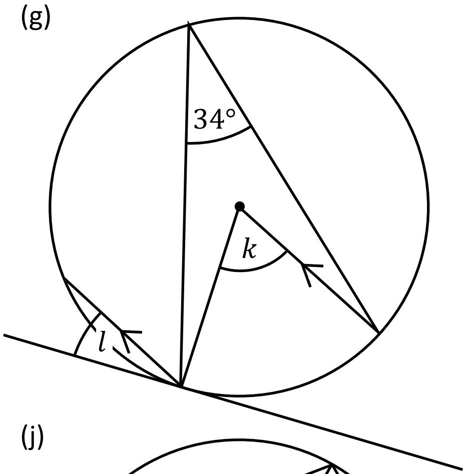
(e)



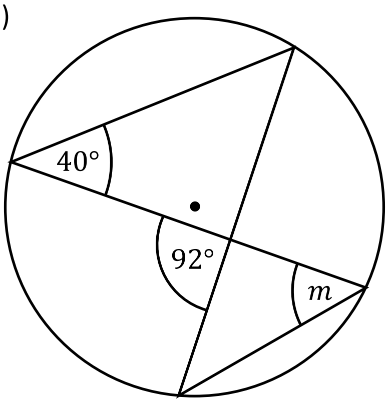
(f)



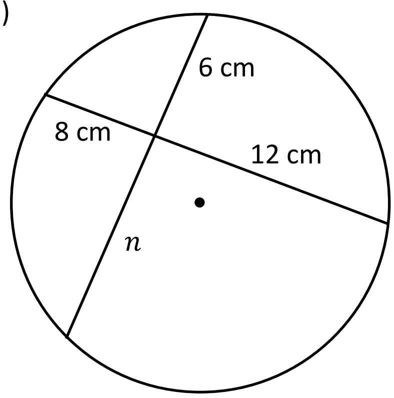
(g)



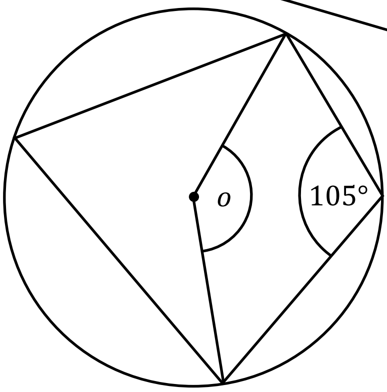
(h)



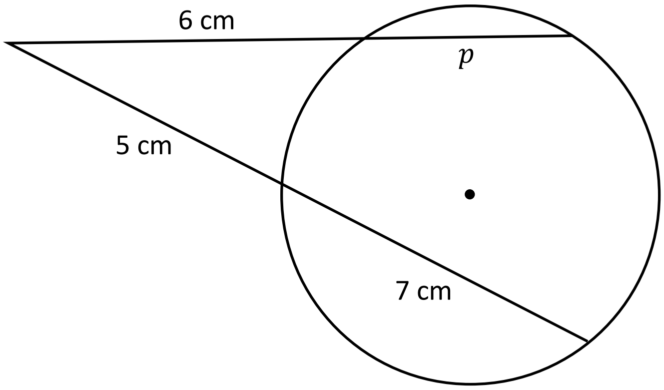
(i)



(j)



(k)



Exercise 22

H

For each question in Exercise 21 above, note which circle theorem(s) you used in finding the missing value(s).

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ❑ Target ❑

Transformations

Over the years, we have seen four types of transformation.

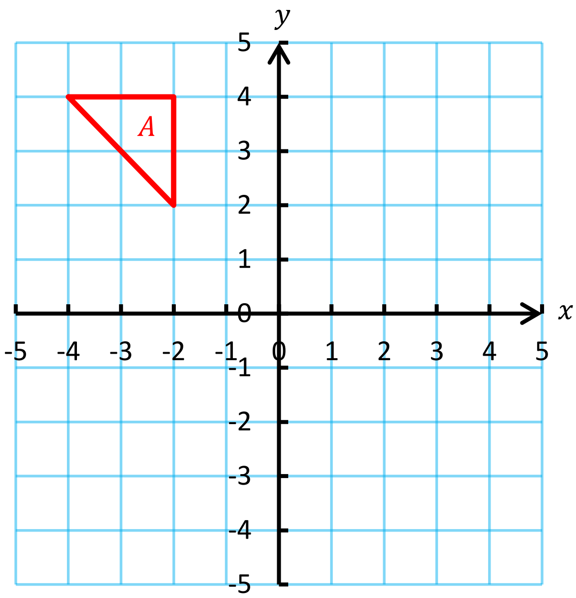
Year 7	Year 8	Year 9	Year 10
Translation (moving a shape)	Rotation (turning a shape)	Reflection (reflecting a shape)	Enlargement (changing the size of a shape)

In this chapter, we will revise these transformations, before combining them in different ways.

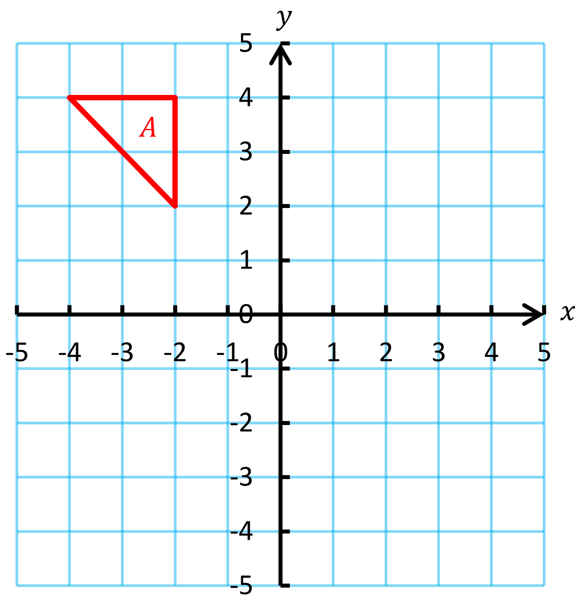
Exercise 23

Revision I

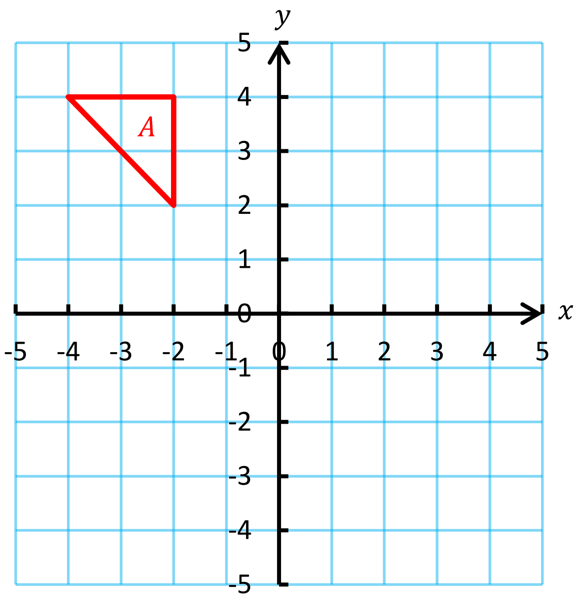
(a) Translate the triangle A using the column vector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$.



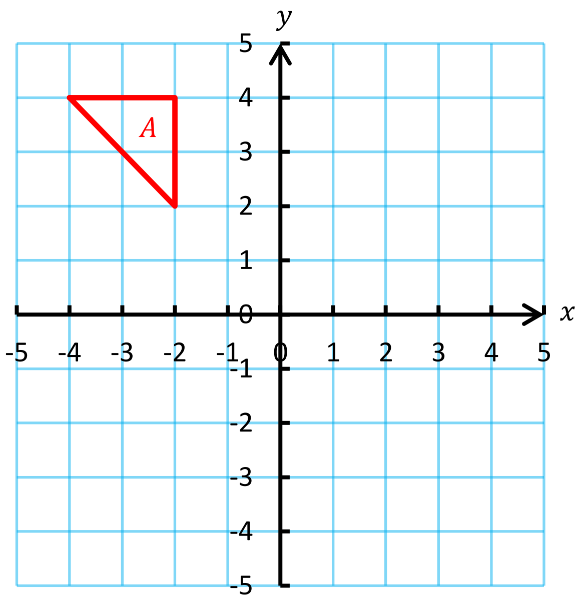
(b) Rotate the triangle A 90° clockwise around the point (–1, 1).



(c) Reflect the triangle A in the line $y = 1$.



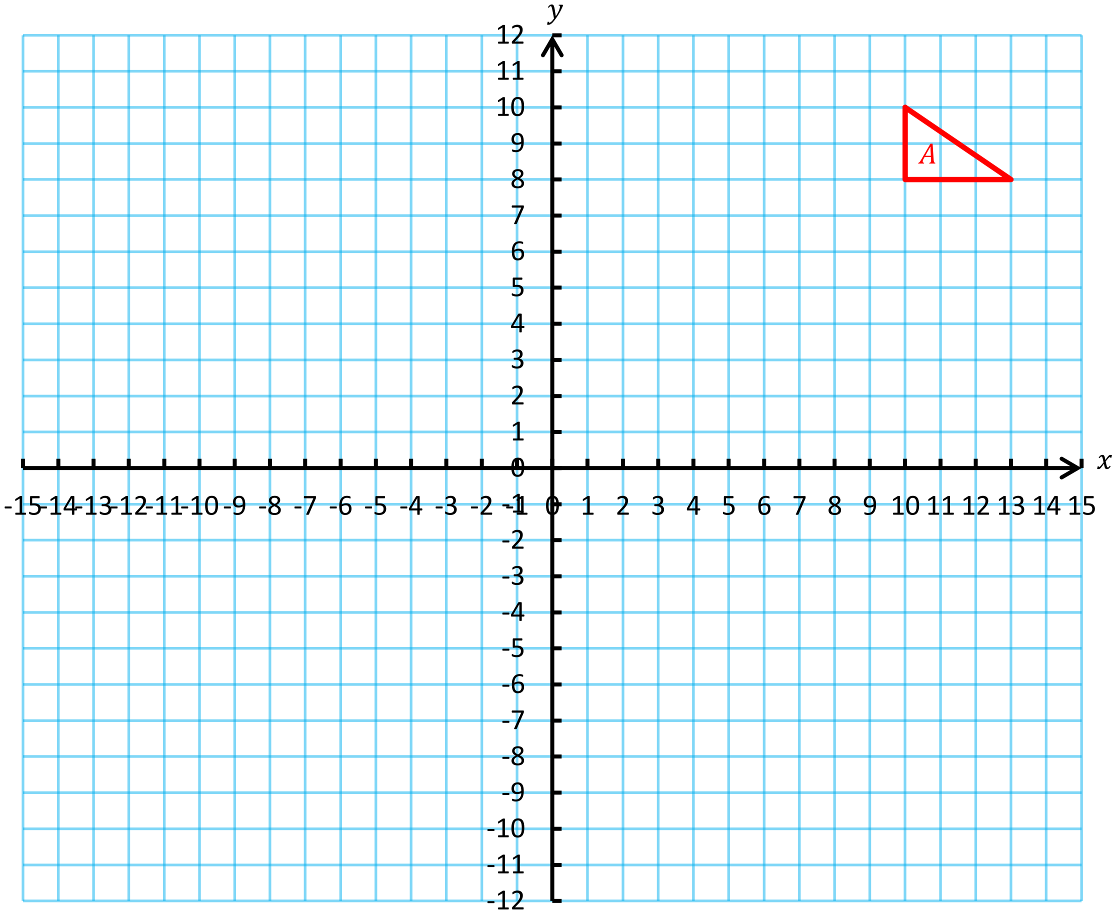
(d) Enlarge the triangle A using scale factor 2 and centre of enlargement (–5, 3).



Exercise 24

Skill I

(a) Translate the triangle A using the column vector $\begin{pmatrix}-2\\-3\end{pmatrix}$. Label the new triangle B.
(b) Reflect the triangle B in the line $x = 4$. Label the new triangle C.
(c) Rotate the triangle C 90° anticlockwise around the point (–4, 3). Label the new triangle D.
(d) Reflect the triangle D in the line $y = 1$. Label the new triangle E.
(e) Enlarge the triangle E using scale factor 3 and centre of enlargement (–10, –3). Label the new triangle F.
(f) Reflect the triangle F in the line $x = 4$. Label the new triangle G.
(g) Translate the triangle G using the column vector $\begin{pmatrix}-20\\11\end{pmatrix}$. Label the new triangle H.



Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Mesur

Siapiau 4

Reflection

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I can **recognise congruent shapes**.			1, 2	
I can calculate the **total interior angles** of any polygon.			3, 7	
I can calculate the **interior angle of any regular polygon**.			6, 7	
I know the **total exterior angles of any polygon**.			3, 5	
I can calculate the **exterior angle of any regular polygon**.			4	
I know the **connection between the interior and exterior angles** for any vertex in a polygon.			3, 5	
I know when regular polygons **tessellate**, and when they do not tessellate.			7	
I can recite the names of the **circle theorems**.			8, 9, 10	
I can **use the circle theorems** to find missing angles and sides.			8, 9, 10	
I can **combine the four transformations** to transform different shapes.			11	

Am I ready for the test?
Tick the boxes below...

I have revised the work in my mathematics book.	☐
I have revised the workbook.	☐
I have watched the relevant videos on YouTube.	☐
I have completed the Diagnostic Questions quiz.	☐
I have completed at least 4 pages in my revision book.	☐

Mesur

Siapiau 4

Reflection

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I can **recognise congruent shapes**.			1	
I can use the **SSS, SAS, ASA and RHS rules to prove when two triangles are congruent.**			2, 3	
I can calculate the **total interior angles** of any polygon.			4, 6, 8	
I can calculate the **interior angle of any regular polygon**.			7, 8	
I know the **total exterior angles of any polygon**.			4	
I can calculate the **exterior angle of any regular polygon**.			5	
I know the **connection between the interior and exterior angles** for any vertex in a polygon.			4, 6	
I know when regular polygons **tessellate**, and when they do not tessellate.			8	
I can recite the names of the **circle theorems**.			9, 10, 11	
I can **use the circle theorems** to find missing angles and sides.			9, 10, 11	
I can **combine the four transformations** to transform different shapes.			12	

Am I ready for the test?
Tick the boxes below...

I have revised the work in my mathematics book.	☐
I have revised the workbook.	☐
I have watched the relevant videos on YouTube.	☐
I have completed the Diagnostic Questions quiz.	☐
I have completed at least 4 pages in my revision book.	☐

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The Mathematics Department

11

Developing Probability

Name:

Contents

Chapter	Mathematics	Page Number
Relative Frequency	Revision. Relative Frequency.	3
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Tree Diagrams	Displaying combinations of two or more events. Independent events. **Dependent events.**	15

Relative Frequency

Exercise 1 (Revision of previous work on probability)

Revision

F

(a) Use “no chance”; “low chance”; “even chance”; “good chance” or “certain” to describe the probability of the following events.

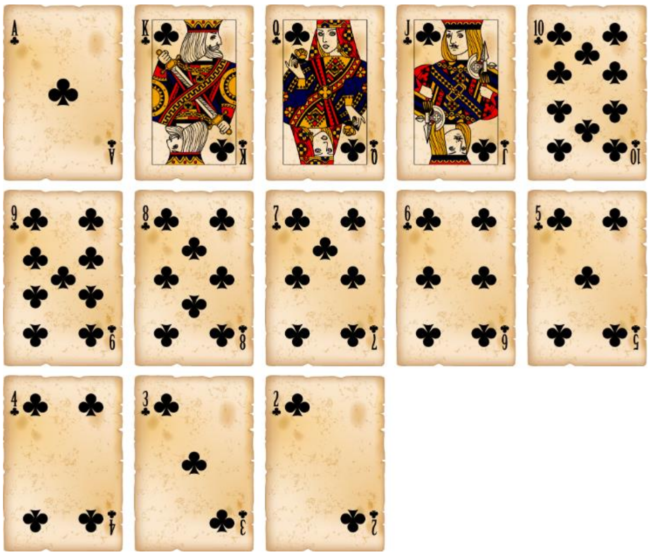
(i) You land on “tails” when throwing a fair coin.
(ii) The next person you meet writes with their left hand.
(iii) You obtain a number less than 5 when rolling a normal fair die.
(iv) St. David’s Day will be on March 1st next year.
(v) You will obtain 101% in your end of year mathematics examination.

(b) Draw a probability scale. Mark the points i, ii, iii, iv in order to show how probable, in your opinion, the following events are.

(i) A man will be driving the next car you see.
(ii) It will snow during the day tomorrow.
(iii) A story about politics will feature on the news tonight.
(iv) Germany will win the next football world cup.

(c) Answer with a number between 0 and 1: what is the probability that someone will walk to the top of Snowdon tomorrow?

(d) Answer the following questions using fractions.
What is the probability of obtaining…



(i) The number 4 when rolling a normal fair die?
(ii) “Heads” when throwing a fair coin?
(iii) A square number when spinning a spinner showing the numbers 1 to 8?

(e) Rheinallt shuffles the 52 cards in a standard deck of playing cards and chooses one card randomly from the deck. What is the probability that the chosen card is:

(i) a diamond?
(ii) 6?
(iii) a face card?
(iv) a spade showing an even number?
(v) a red card less than 5?

(f) The probability that Meira goes to the shop to buy a loaf of bread tomorrow is 0.4. What is the probability that Meira does not go to the shop to buy a loaf of bread tomorrow?

(g) A red die and a blue die are labelled from 1 to 6.
Gethin rolls both dice and adds the scores obtained.

(i) Use a sample space diagram to list all the possible outcomes.
(ii) What is the probability that the sum of both numbers is 12?
(iii) What is the probability that the sum of both numbers is a one-digit number?

(h) The probability that Ellie goes on a training run on any day is 0.7.
There are 30 days in April. On how many days can you expect Ellie to go running?

Relative Frequency

The **frequency** of an event refers to how many times the event has occurred during a number of trials.

The **relative frequency** of an event compares the frequency to the number of trials.

$$\text{Relative frequency of an event} = \frac{\text{How many times the event has occurred}}{\text{Number of trials}}$$

It is possible to use relative frequency to **estimate the probability of an event**.

Exercise 2

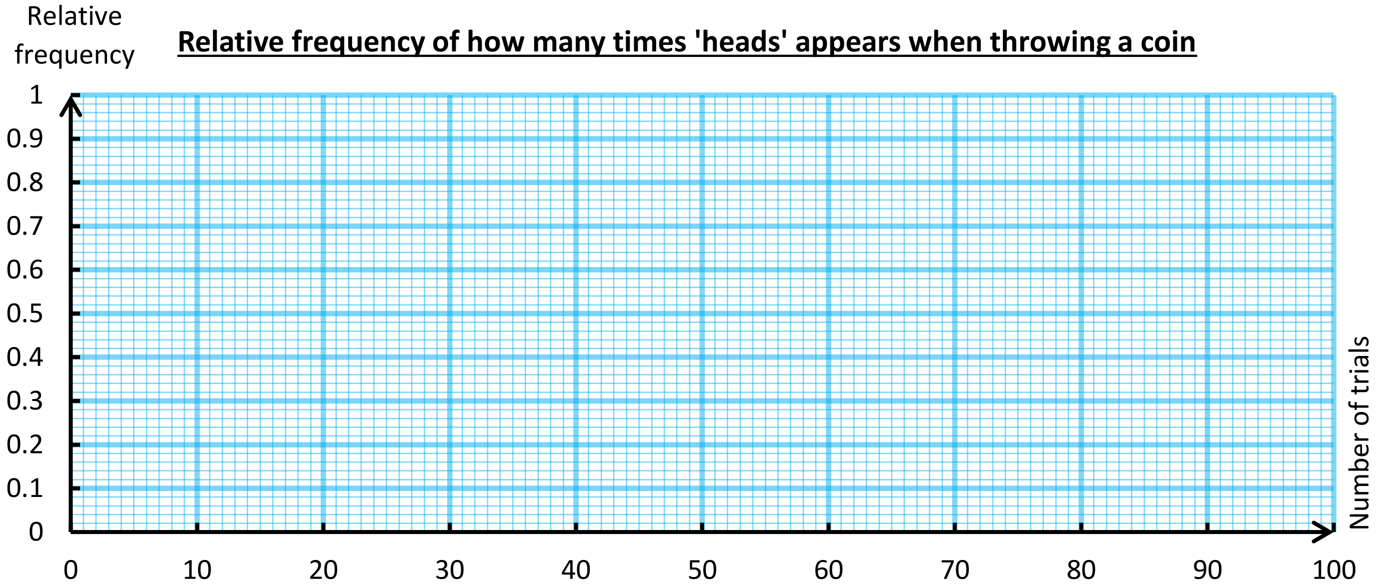
Applying I

You will need a coin for this exercise.
(a) Throw the coin 100 times, recording in the following table, after each set of 10 throws, how many times the coin has landed showing 'heads'.

Total number of trials	Number of heads in these 10 throws	Total number of heads so far	Relative frequency of the number of heads, as a fraction	Relative frequency of the number of heads, as a decimal
10			$\frac{\ }{10}$	
20			$\frac{\ }{20}$	
30			$\frac{\ }{30}$	
40				
50				
60				
70				
80				
90				
100				

(b) Plot, on the following graph paper, a line graph showing what happens to the relative frequency as the number of trials increases.

Relative frequency of how many times 'heads' appears when throwing a coin



(c) If a great many more trials were held, how would you expect the graph to change?

Exercise 3 (Buffon's Needle Experiment)

I

For this exercise, you will need a plastic straw of length 3 cm, and lined paper where the lines are separated by 6 cm.

In 1777, a Frenchman called Georges-Louis Leclerc (Comte de Buffon) devised an experiment that can be used to estimate the value of π.

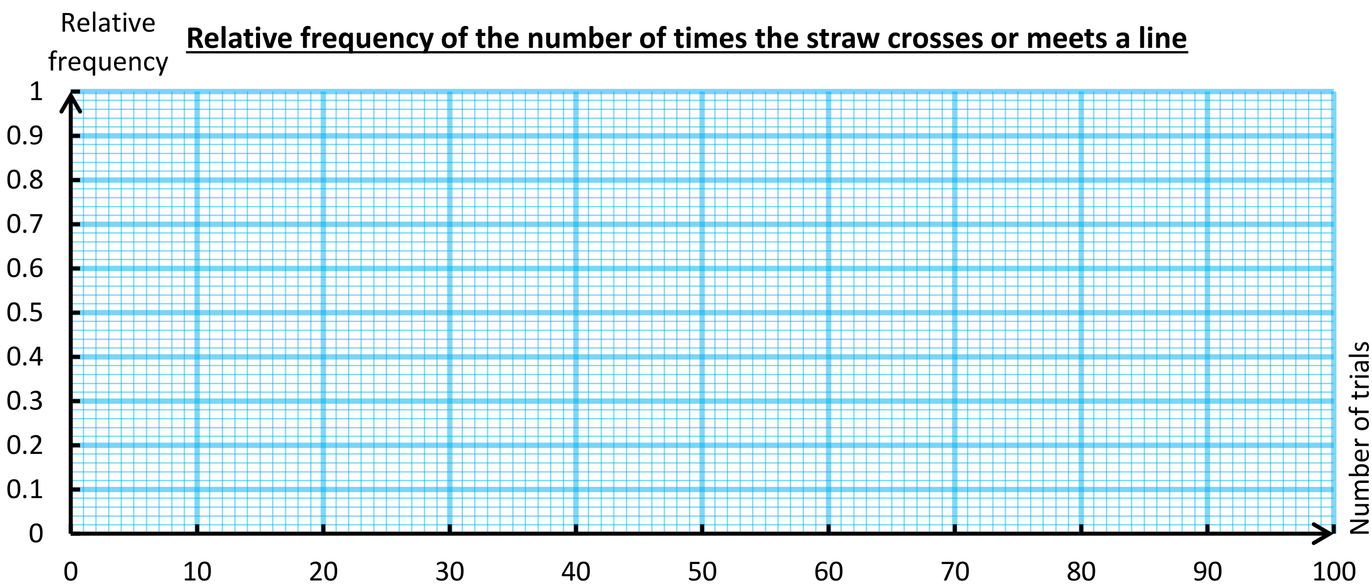
In each trial, a straw is dropped on a piece of paper. The straw should be dropped above the centre of the paper, from arm height. You should record whether or not the straw crosses (or meets) one of the lines.

(a) Perform the experiment 100 times. Record, after each set of 10 trials, how many times the straw crosses (or meets) one of the lines.

Total number of trials	Number of times the straw has crossed a line in these 10 trials	Total number of crossings so far	Relative frequency of the number of crossings, as a fraction	Relative frequency of the number of crossings, as a decimal
10			$\frac{}{10}$	
20			$\frac{}{20}$	
30			$\frac{}{30}$	
40				
50				
60				
70				
80				
90				
100				

(b) Plot, on the graph paper below, a line graph showing what happens to the relative frequency as the number of trials increases.

Relative frequency of the number of times the straw crosses or meets a line



(c) Calculate the **reciprocal** of your final relative frequency. How close is this value to π?

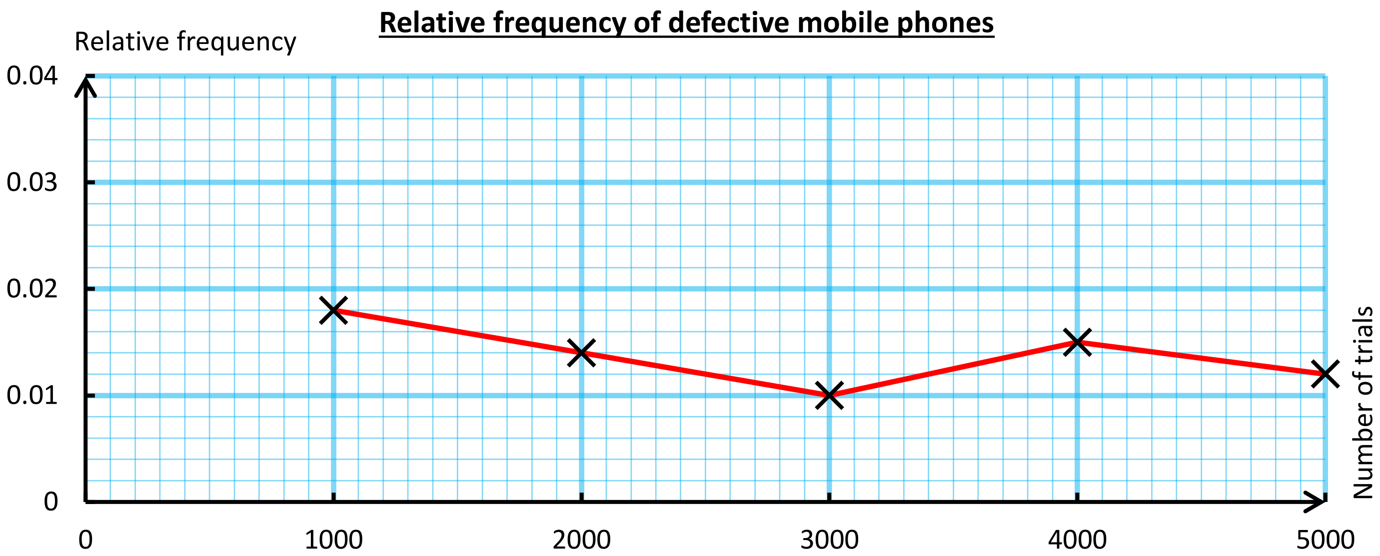
In any experiment,

The more trials are held, the better the relative frequency is as an estimate of the probability.

Exercise 4 I

A factory produces mobile phones. The manager holds a survey to investigate the probability that the factory produces a defective mobile phone.

The relative frequency of defective mobile phones is recorded after testing a total of 1,000, 2,000, 3,000, 4,000 and 5,000 mobile phones. The results are shown in the following graph.



(a) How many of the first 2,000 mobile phones tested were defective?

(b) Write the best estimate for the probability that a randomly selected mobile phone is defective.
You must give a reason for your answer.

Exercise 5 F

(a) Fred throws a die 200 times and records how many times each score occurs.

Score	1	2	3	4	5	6
Frequency	29	34	35	32	34	36

(i) Calculate the relative frequency of each score.
(ii) In your opinion, is Fred's die fair? Give a reason for your answer.

(b) Rhys recorded the results of his favourite football team.

Win	Draw	Lose
32	11	7

(i) Calculate the relative frequency of each of the three possible results.
(ii) Are your answers to part (i) a good estimate for the probability of the result of the next game?
Explain your answer.

(c) A petrol station owner noticed that 287 customers out of a total of 340 customers spent more than £30 filling up their cars. Use these figures to estimate the probability that the next customer spends

(i) more than £30, (ii) £30 or less.

(d) In a survey, 600 people were asked about their favourite crisp flavour.
The following table shows the results.

Flavour	Frequency
Ready Salted	166
Salt & Vinegar	130
Cheese & Onion	228
Other	76

(i) Calculate the relative frequency of each flavour. Give your answers correct to 2 decimal places.
(ii) Explain why it is reasonable to use these figures to estimate the probability of the favourite crisp flavour of the next person to be asked.

(e) A card was picked from a standard pack of cards, and its suit was noted. The card was put back and the pack was shuffled. This was repeated 250 times. The results are shown in the following table.

Suit	Frequency
Spades	52
Hearts	67
Diamonds	61
Clubs	70

Find the relative frequency of
(i) Spades (ii) Hearts (iii) Diamonds (iv) Clubs.
(v) What is the sum of all the relative frequencies?

(f) The following table shows the number of pictures on each page of a newspaper.

Number of pictures	Tally Marks	Frequency
0	卌 卌	10
1	卌 ||	7
2	卌 ||||	9
3	卌 |	6
4	|||| |	6
5	卌	4
6	卌 |	6

(i) How many pages does the newspaper have?
(ii) Find the relative frequency of one picture appearing on a page of the newspaper.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

Combined Events

Combined events involve two or more events that occur together.
For example,

- Throwing a penny and rolling a die at the same time;
- Rolling a die and then rolling it again;
- Spinning a spinner that shows the numbers 1-8, and then randomly choosing a card from a standard deck of playing cards.

Independent Events (Intermediate Tier)

Two events are **independent** if the result of the first event does *not* affect the probability of the second event.
For example, when throwing the same die twice, the fact that the die has landed on 6 the first time does not affect the probability of obtaining a 6 the second time.

For independent events A and B,

$$P(A \cap B) = P(A) \times P(B)$$

The **multiplication** rule for independent events

This means the probability of A **and** B occurring is the product of A's probability and B's probability.

For example, the probability of obtaining heads when throwing a coin and 4 when rolling a die is $\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$.

Dependent Events (Higher Tier)

Two events are **dependent** if the result of the first event *does* affect the probability of the second event.
For example, when choosing two cards from a standard deck of playing cards, **without replacement**, the first card chosen affects the probability of the second card chosen. If the first card is a king, then the probability of the second card being a king is $\frac{3}{51}$, not $\frac{4}{52}$ as for the first card.

For dependent events A and B,

$$P(A \cap B) = P(A) \times P(B|A)$$

The **multiplication** rule for dependent events

This means the probability of A **and** B occurring is the product of A's probability with the probability of B occurring, given that A has already occurred. (The probability $P(B|A)$ is a **conditional** probability.)

For example, the probability of choosing two kings from a standard deck of playing cards is $\frac{4}{52} \times \frac{3}{51} = \frac{12}{2652}$, or $\frac{1}{221}$.

Mutually Exclusive Events (Intermediate Tier)

Two events are **mutually exclusive** if they cannot happen at the same time. For example, when rolling a normal fair die, the events "landing on an odd number" and "landing on 6" are mutually exclusive, because 6 is not an odd number. On the other hand, the events "landing on an odd number" and "landing on a prime number" are not mutually exclusive, because the numbers 3 and 5 are both odd and prime.

For mutually exclusive events A and B,

$$P(A \cup B) = P(A) + P(B)$$

The **addition** rule for mutually exclusive events

This means the probability of A **or** B occurring is the sum of A's probability and B's probability.

For example, the probability of obtaining an odd number or a 6 when rolling a normal fair die is $\frac{3}{6} + \frac{1}{6} = \frac{4}{6}$, or $\frac{2}{3}$.

Exercise 6

For the following pairs of events, decide whether the events are independent or dependent.

(a) Obtaining 'heads' when throwing a coin and obtaining '6' when rolling a normal fair die.

(b) Choosing two queens when choosing two cards from a standard deck of playing cards, without replacement.

(c) Choosing two queens when choosing a card from a standard deck of playing cards, returning the card to the deck, and then choosing another card.

(d) Choosing two red balls when one ball is chosen from a bag containing 4 red balls and 5 blue balls, and the other is chosen from a bag containing 5 red balls and 4 blue balls.

(e) Choosing two red balls from a bag containing 4 red balls and 5 blue balls, without replacement.

Exercise 7

For the following pairs of events, decide whether the events are mutually exclusive or not.

(a) Obtaining 'heads' when throwing a coin and obtaining '6' when rolling a normal fair die.

(b) When rolling a normal fair die, obtaining

(i) a number less than 3 and a number greater than 3.
(ii) an even number and a number greater than 4.
(iii) an odd number and a square number.
(iv) an even number and a cube number.
(v) an even number and a prime number.

(c) Obtaining 'heads' when throwing a coin and 'heads' when throwing another coin.

(d) On a spinner showing the numbers 1-10, landing on a multiple of 3 and landing on a multiple of 4.

(e) On a spinner showing the numbers 1-12, landing on a multiple of 3 and landing on a multiple of 4.

Example

Christine has a normal fair die. She rolls the die twice.
Calculate the probability that the die lands on 3 each time.

Answer: The result of the first roll does not affect the second roll, so the events are independent.
We can therefore use the multiplication rule for independent events.

$$\begin{aligned} P(\text{3 the first time, 3 the second time}) &= P(\text{3 the first time}) \times P(\text{3 the second time}) \\ &= \frac{1}{6} \times \frac{1}{6} \\ &= \frac{1}{36} \end{aligned}$$

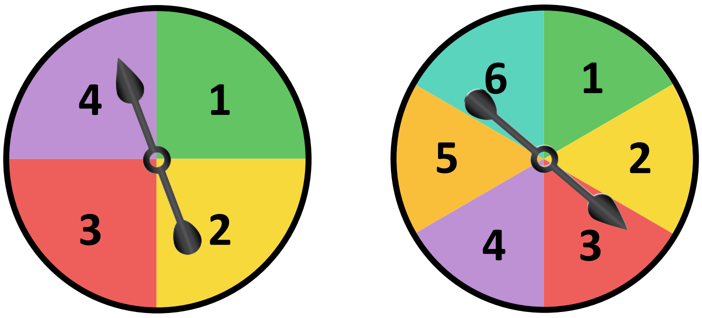
Exercise 8

(a) David has a normal fair die. He rolls it twice.
Calculate the probability that he rolls 5 the first time and 1 the second time.

(b) Fiona has a fair coin. She throws it 3 times.
Calculate the probability that she obtains 3 'tails'.

(c) Rachel has a normal fair die and a fair coin. She rolls the die and throws the coin.
Calculate the probability that the die lands on 4 and the coin lands on 'tails'.

(d) A game is played where the two spinners below are spun at the same time.



(i) What is the probability that the spinner on the left stops at 4 and the spinner on the right stops at 3?
(ii) What is the probability that the spinner on the left stops at an even number and the spinner on the right stops at an odd number?
(iii) What is the probability that both spinners stop at the same number?

(e) A bag contains 12 counters. 3 are red, 4 are blue and the rest are green. Another bag contains 15 counters. 7 are red, 2 are blue and rest are green. What is the probability of choosing

(i) A red counter from the first bag and a blue counter from the second bag?
(ii) A blue counter from the first bag and a red counter from the second bag?
(iii) Two green counters?

H

Exercise 9

(a) A bag contains 8 counters. 3 are red and 5 are blue. 2 counters are chosen randonly from the bag, without replacement. What is the probability of choosing

(i) Two red counters?
(ii) A red counter first then a blue counter?
(iii) A blue counter first then a red counter?
(iv) Two blue counters?

(b) Tom shuffles a standard pack of playing cards before choosing, at random, two cards from the pack without replacement. What is the probability that Tom chooses

(i) The king of hearts first and then the queen of diamonds?
(ii) Two hearts?
(iii) Two cards showing 7?
(iv) A red card first and then a black card?

(c) A class in a school has 15 girls and 12 boys. Two names are chosen at random from the register in order to represent the class in a survey. What is the probability of choosing two girls?

(d) The probability that James watches television tonight is 0.6. If James watches television tonight, the probability that he reads a book tonight is 0.2. If James does not watch television tonight, the probability that he reads a book tonight is 0.7. What is the probability that James, tonight,

(i) Watches television and reads a book?
(ii) Does not watch television and reads a book?
(iii) Watches television and does not read a book?
(iv) Does not watch television and does not read a book?

(e) An office has 20 workers. 7 of the workers wear spectacles. Two workers are chosen at random. What is the probability that the workers chosen do **not** wear spectacles?

Exercise 10

(a) Mariel rolls a normal fair die. What is the probability that Mariel's die lands on

(i) 2 or 3?
(ii) An even number or 5?
(iii) A number less than 3 or a number greater than 4?
(iv) A prime number or a square number?

(b) Heulwen shuffles a standard deck of playing cards before randoomly choosing one card from the pack. What is the probability that Heulwen chooses

(i) Hearts or spades?
(ii) 3 or 5?
(iii) A face card or a card less than 6?
(iv) A black card or a red card?

(c) One number is chosen at random from the grid on the right. What is the probability that the number is:

(i) 4 or 5?
(ii) A multiple of 5 or a multiple of 7?
(iii) A factor of 8 or a two-digit number?
(iv) A cube number or a prime number?

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20

(d) Gareth rolls a fair die with 12 faces.
What is the probability that the die lands on

(i) 1 or 12?
(ii) An odd number or a multiple of 4?
(iii) A square number or 7?
(iv) A multiple of 3 or a factor of 11?

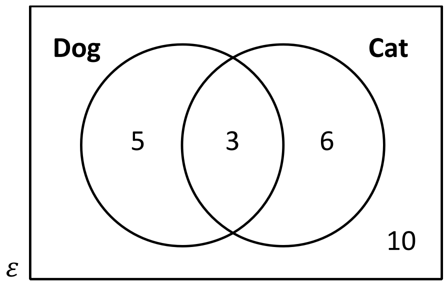
(e) A bag contains 10 counters. 3 are red, 2 are blue and the rest are purple. One counter is chosen at random from the bag. What is the probability that the chosen counter is red or purple?

Venn Diagrams

Example

In a class of 24 learners, 8 own a dog, 9 own a cat and 3 own both a dog and a cat. One learner is chosen at random from the class. What is the probability that the chosen learner does not own either a dog or a cat?

Answer: To begin, let us draw a Venn diagram to illustrate the situation.



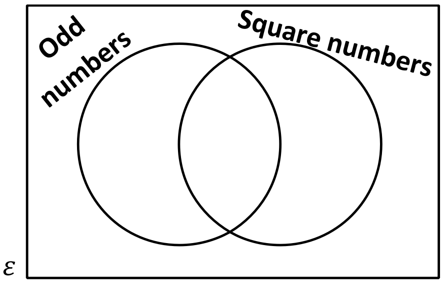
We see from the Venn diagram that 10 learners in the class do not own either a dog or a cat. Therefore, the probability of choosing a learner that does not own either a dog or a cat is $\frac{10}{24}$, or $\frac{5}{12}$.

Exercise 11

F

(a) In a survey someone asked 40 pupis whether they liked football or rugby.
32 pupils liked football.
25 pupils liked rugby.
22 pupils liked both sports.
(i) Draw a Venn diagram to illustrate this information.
(ii) What is the probability that a randomly chosen pupil from this group likes rugby only?

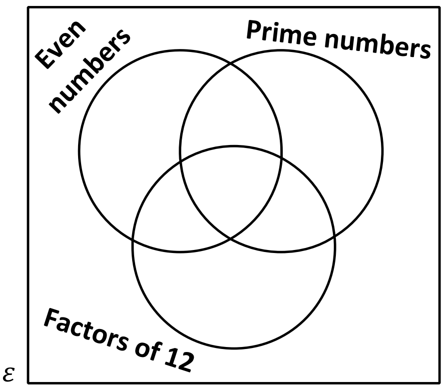
(b) (i) Place the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 into the correct positions in the following Venn diagram.



(ii) A number is chosen at random from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
Find the probability that the chosen number
(I) is an odd number; (II) is an odd number and a square number; (III) is neither odd nor a square number.

(c) An ice cream company conducted a taste test in a supermarket. 110 people took part in the survey, where they were asked to taste strawberry, vanilla and chocolate ice creams.
65 people stated that they liked the strawberry ice cream.
80 people stated that they liked the vanilla ice cream.
60 people stated that they liked the chocolate ice cream.
55 people stated that they liked both the strawberry and vanilla flavours.
50 people stated that they liked both the vanilla and chocolate flavours.
45 people stated that they liked both the strawberry and chocolate flavours.
40 people stated that they liked all 3 flavours.
(i) Draw a Venn diagram to illustrate this information.
(ii) What is the probability that a randomly selected person from this group liked
(I) Vanilla only; (II) None of the three flavours; (III) Vanilla or Strawberry?

(d) (i) Place the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 into the correct positions in the following Venn diagram.



(ii) A number is chosen at random from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
Find the probability that the chosen number is
(I) an even number; (II) a prime number and a factor of 12; (III) a prime number but not an even number.

Sample Space Diagrams and Dependent Events

Higher Tier

Example

Each one of the numbers 1, 2, 3, 4, 5, 6 are written on cards.

1 2 3 4 5 6

Two out of the six cards are chosen at random, **without** replacement.
Find the probability that the **sum** of the numbers shown on the chosen cards is less than 10.

Answer: We can show all the possible combinations in a sample space diagram.

		Number on the second card					
	+	**1**	**2**	**3**	**4**	**5**	**6**
Number on the first card	**1**	Impossible	3	4	5	6	7
	2	3	Impossible	5	6	7	8
	3	4	5	Impossible	7	8	9
	4	5	6	7	Impossible	9	10
	5	6	7	8	9	Impossible	11
	6	7	8	9	10	11	Impossible

There are 6 × 5 = 30 different combinations, and of these, 26 give a sum that is less than 10. Therefore, the answer to the question is $\frac{26}{30}$, or $\frac{13}{15}$.

Exercise 12

H

(a) Each one of the numbers 1, 2, 3, 4, 5, 6 are written on cards.

1 2 3 4 5 6

Two out of the six cards are chosen at random, **without** replacement.
Find the probability that the **product** of the numbers shown on the chosen cards is less than 10.

(b) Each one of the numbers 1, 2, 2, 3, 3, 3 are written on cards.

1 2 2 3 3 3

Two out of the six cards are chosen at random, **without** replacement.

(i) Find the probability that the sum of the numbers shown on the chosen cards is less than 6.
(ii) Find the probability that the sum of the numbers shown on the chosen cards is exactly 3.
(iii) Find the probability that the two chosen numbers are the same.
(iv) Find the probability that the two chosen numbers are different.
(v) Find the probability that the card showing 1 is chosen.

(c) A different factor of 24 is shown on each of 8 cards.

2 cards are chosen at random **without** replacement.

Find the probability that the **positive difference** between the two numbers on the chosen cards is
(i) 4;
(ii) an odd number;
(iii) a one-digit number.

(d) A different factor of 18 is shown on each of 6 cards.

In a game, a player chooses two of the above cards at random, **without** replacement.
The score for the game is the **largest** of the two chosen numbers.

(i) Draw a sample space diagram to show all the possible outcomes.
(ii) Find the probability that the score is 18.
(iii) A player wins the game if the score is 18.
If 120 people play the game once each, how many of them would you expect to win?
(iv) It costs 20p to play the game once.
The prize for winning the game is 50p.
If 120 people play the game once each, how much profit would you expect the game to make?

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ Target ☐

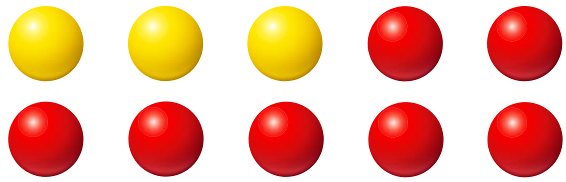
Tree Diagrams

Tree diagrams are used to show **combinations of two or more events**.
Each branch is labelled on the right with the **result** and in the middle with the **probability**.

It is possible to use tree diagrams to show independent events (intermediate tier) or dependent events (higher tier).

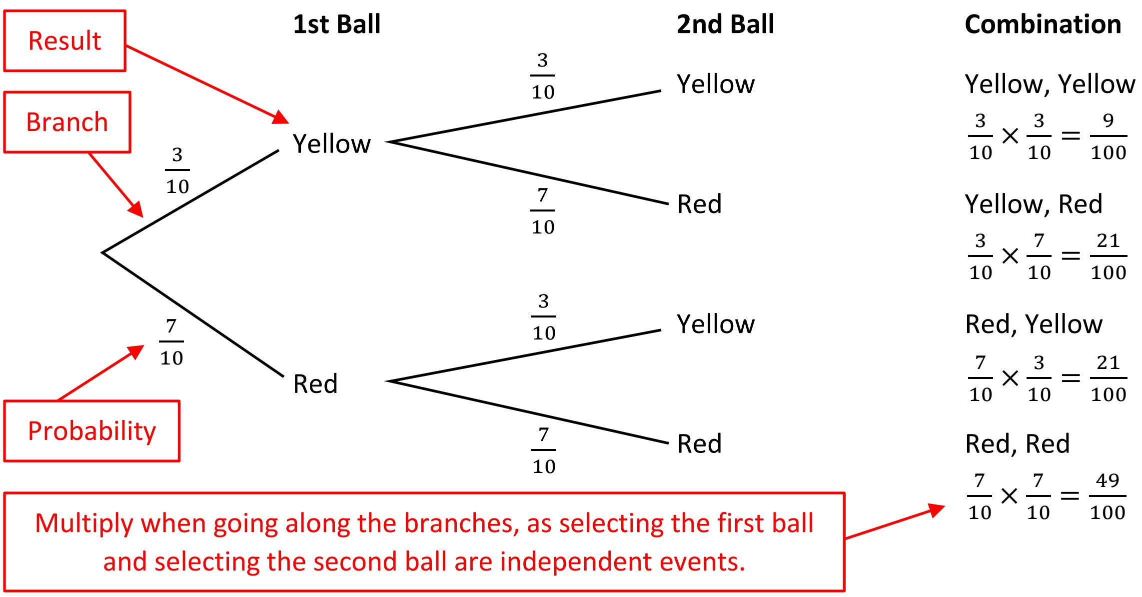
Example

A bag contains ten balls that are indentical apart from their colour. Three of the balls are yellow and the rest red. A ball is selected at random from the bag, its colour noted and returned to the bag. A second ball is chosen and its colour is also noted.



Use a tree diagram to calculate the probability that
(a) the two selected balls are yellow;
(b) the two selected balls have different colours;
(c) the two selected balls are not both yellow.

Answer: Step 1: Draw a tree diagram to illustrate the situation.



Step 2: Consider which paths in the tree diagram need to be considered in order to answer the questions.

(a) The first path in the tree diagram (yellow, yellow) shows the situation in which both selected balls are yellow. The probability for this is $\frac{9}{100}$.

(b) The paths that show balls of different colours are the second path (yellow, red) or the third path (red, yellow). Because these paths are mutually exclusive, it is possible to add the probabilities to obtain the final answer: $\frac{21}{100} + \frac{21}{100} = \frac{42}{100}$. Note that it would be possible to simplify this fraction to obtain $\frac{21}{50}$ but, unless the question notes otherwise, fractions do not have to be simplified in questions on probability.

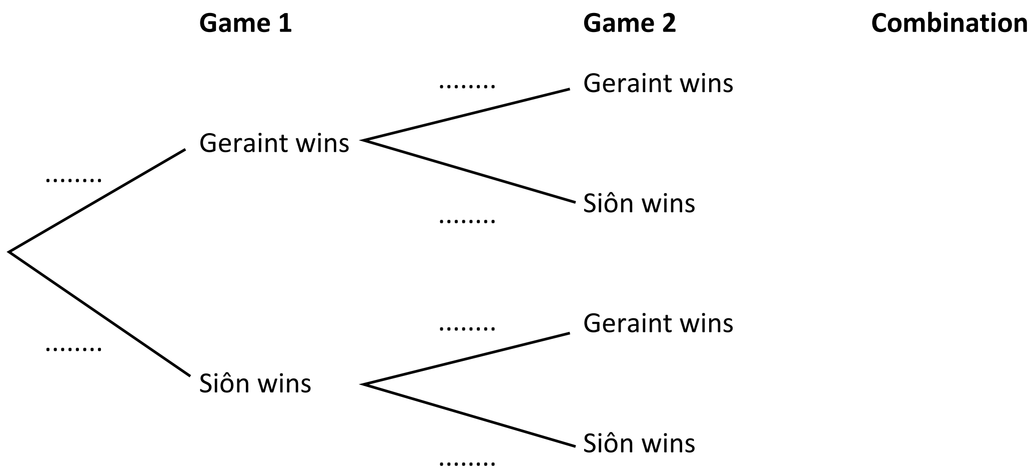
(c) There are two ways of answering this question:
(i) Consider all paths in the tree diagram that do **not** give two yellow balls: $\frac{21}{100} + \frac{21}{100} + \frac{49}{100} = \frac{91}{100}$;
(ii) Consider the probability of selecting two yellow balls ($\frac{9}{100}$, the answer to part (a)) and subtract from 1: $1 - \frac{9}{100} = \frac{91}{100}$.

Exercise 13

Applying

Whenever Geraint and Siôn play a game of 'FIFA' on a games console, the probability that Gerains wins is 0.4.

(a) Complete the following tree diagram to show the probabilities of what can happen when Geraint and Siôn play two games of 'FIFA'.



(b) Calculate the probability that Siôn wins both games.

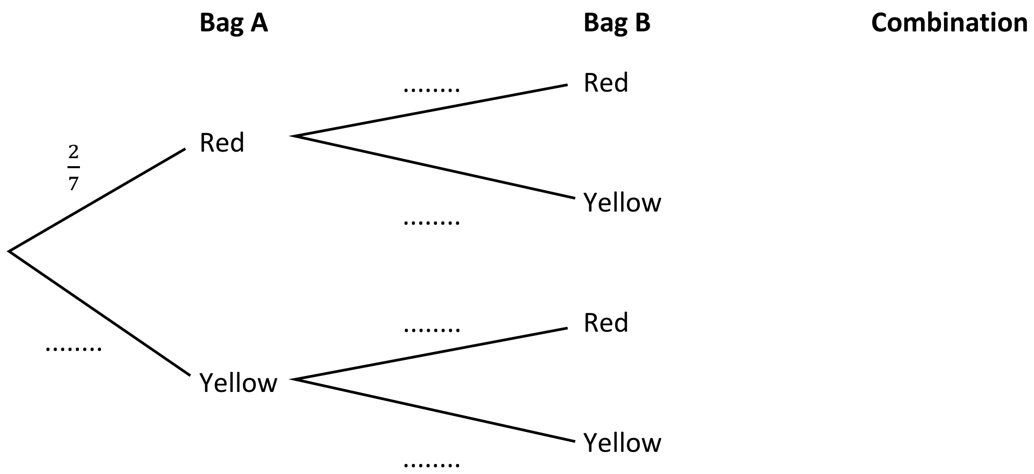
(c) Calculate the probability that Siôn wins exactly one of the games.

(d) Calculate the probability that Siôn wins neither of the games.

Exercise 14

There are two bags in a game, and both bags contain coloured balls.
Bag A contains 2 red balls and 5 yellow balls.
Bag B contains 3 red balls and 2 yellow balls.
A player randomly chooses one ball from each bag.

(a) Complete the following tree diagram.



(b) Find the probability of choosing two red balls.

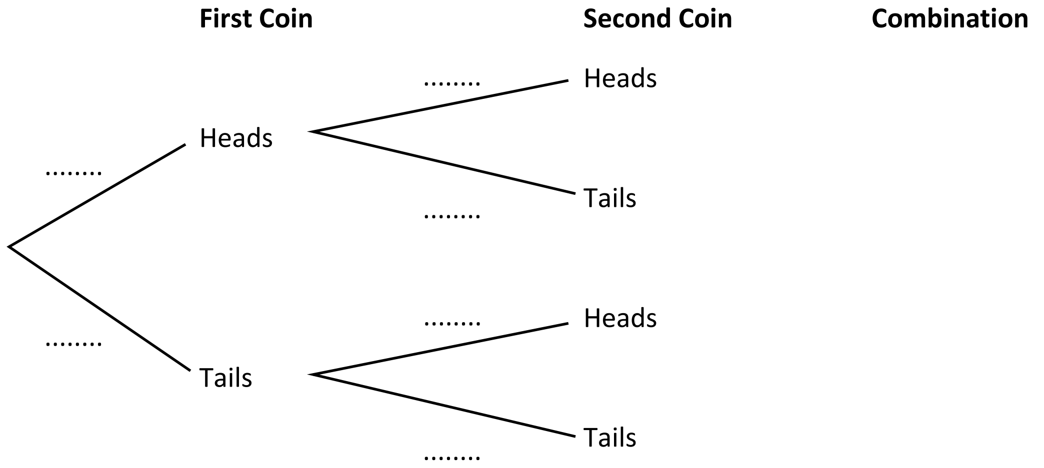
(c) Find the probability of choosing a ball of each colour.

(d) Find the probability of not choosing two red balls.

Exercise 15 I

Two biased coins are thrown. The probability of obtaining heads with the first coin is 70%.
The probability of obtaining heads with the second coin is 60%.

(a) Complete the following tree diagram.



(b) Calculate the probability that one coin shows ‘Heads’ and the other coin shows ‘Tails’.

Example

The probability that Elin posts a picture on *Instagram* over the weekend is 0.4
The probability that Elin goes shopping over the weekend is independent of her posting a picture on Instagram over the weekend. The probability of Elin posting a picture on Instagram over the weekend, and going shopping over the weekend, is 0.12.

(a) Complete the following tree diagram.
(b) Find the probability that Elin posts a picture on Instagram over the weekend but does not go shopping over the weekend.

Answer: (a) The red text is the text that has been added.
Note in this example that the probability of one of the **combinations** has been given, so we must perform a **division** sum in order to calculate the probability of shopping over the weekend.

(b) 0.4 × 0.7 = 0.28 (the second path).

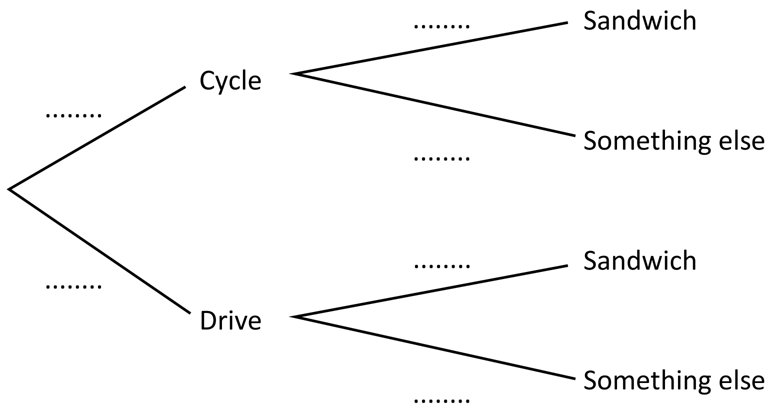
Exercise 16

Megan lives 5 miles from her work, and either cycles or drives to work.
The probability that she cycles to work is 0.3.
The probability that she cycles to work and has a sandwich for lunch is 0.24.
In Megan's case, her method of transport to work is independent of what she has for lunch.

(a) Find the probability that Megan has a sandwich for lunch.

(b) Complete the following tree diagram.

Method of Transport to Work **Lunch** **Combination**



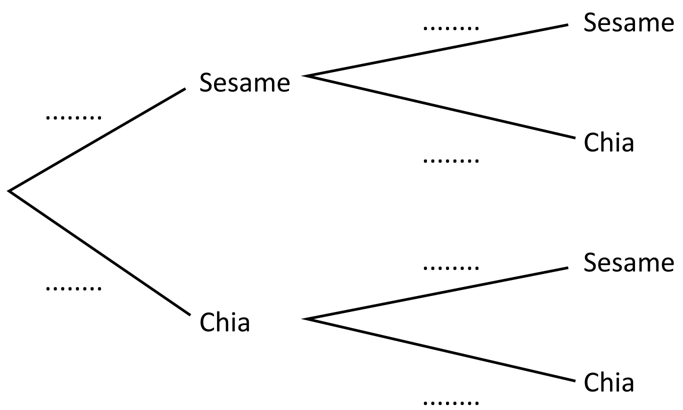
(c) Find the probability that Megan drives to work and eats something apart from a sandwich for lunch.

Exercise 17

A bag contains a large number of two types of seed, namely sesame and chia.
Two seeds are randomly chosen from the bag.
The probability that two sesame seeds are chosen is 0.49.

(a) Complete the following tree diagram.

1st Seed **2nd Seed** **Combination**



(b) Calculate the probability that one seed of each type is selected.

(c) Calculate the probability that two chia seeds are chosen.

(d) Why are the words "large number" needed at the start of the question?

(e) If three seeds were chosen from the bag, what would be the probability of choosing 3 sesame seeds?

Higher Tier

Dependent Events

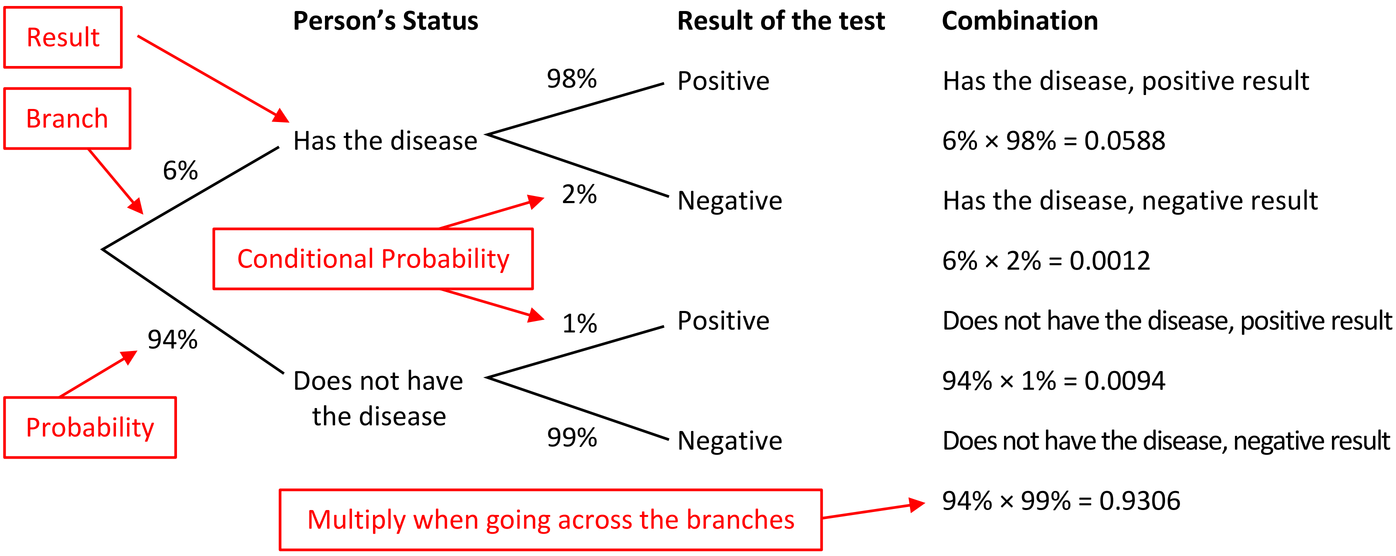
Example

A hospital tests patients for a specific disease. If the person has the disease, the test returns a “positive” result. If the person does not have the disease, the test returns a “negative” result. The test isn’t perfect however:

- 98% of patients with the disease receive a “positive” result;
- 1% of patients without the disease receive a “positive” result;
- 6% of the population have the disease under consideration.

Use a tree diagram to calculate the probabilty
(a) that a randomly chosen person receives a “positive” result in the test;
(b) that the incorrect result is given to a person taking the test;
(c) that the correct result is given to a person taking the test.

Answer: Step 1: Draw a tree diagram to illustrate the situation.



Step 2: Consider which paths in the tree diagram need to be considered in order to answer the questions.

(a) Two paths give a positive result: either “has the disease, positive” (probability 0.0588) or “does not have the disease, positive” (probability 0.0094). We add these two probabilities to obtain the answer (they are mutually exclusive events): 0.0588 + 0.0094 = 0.0682.

(b) Two paths give an incorrect result: either “has the disease, negative” (probability 0.0012) or “does not have the disease, positive” (probability 0.0094). We add these two probabilities to obtain the answer (they are mutually exclusive events): 0.0012 + 0.0094 = 0.0106.

(c) There are two ways of answering this question:
(i) Consider all the paths in the tree diagram that give a correct result (“has the disease, positive” or “does not have the disease, negative”): 0.0588 + 0.9306 = 0.9894;
(ii) Consider the probability of obtaining an incorrect result (0.0106, the answer to part (b)) and subtract from 1:
1 – 0.0106 = 0.9894.

Challenge!

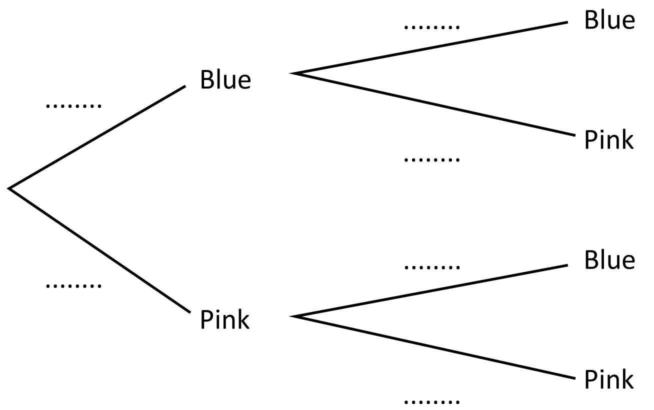
Use the internet to investigate the meaning of the terms *Type I Error* and *Type II Error*.
What is the connection between these statistical terms and the above example?

Exercise 18

A bag contains 4 blue balls and 6 pink balls.
Billy randomly chooses two balls from the bag **without replacement**.

(a) Complete the following tree diagram.

1st Ball **2nd Ball** **Combination**



(b) Find the probability of choosing two pink balls.

(c) Find the probability of choosing one ball of each colour.

Exercise 19

A bag contains 7 yellow beads, 3 white beads and one black bead.
Two beads are chosen randomly from the bag **without replacement**.

(a) Calculate the probability that both beads are yellow.

(b) Calculate the probability that at least one white bead is chosen.

Exercise 20

A box contains 3 banana yogurts, 4 blueberry yogurts and 5 cherry yogurts.
Three yogurts are randomly chosen from the box **without replacement**.
Calculate the probability that at least one of the chosen yogurts is a cherry yogurt.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ❑ Target ❑

Developing Probability

Reflection

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I have revised the **previous work on probability**, e.g. sample space diagrams, expected frequency.			1, 3, 7	
I know how to **calculate relative frequency from experimental data**.			2	
I know how to **draw a graph** showing relative frequency against the number of trials.			2	
I understand the **more** trials are held, the **better** the relative frequency is as an estimate of the probability.			2	
I can tell the difference between **independent events** and **dependent events**.			4	
I can tell the difference between **mutually exclusive events** and events that are **not mutually exclusive**.			4	
I can use the **multiplication rule** for **independent events**.			4	
I can use the **addition rule** for **mutually exclusive events**.			5	
I can answer questions on probability that involve **Venn diagrams**.			6	
I can draw and use **tree diagrams**.			8	
Given the probability of one of the **combinations** in a tree diagram, I can work backwards to fill in all the probabilities of the tree diagram.			9	

Am I ready for the test?
Tick the boxes below...

- I have revised the work in my mathematics book. ☐
- I have revised the workbook. ☐
- I have watched the relevant videos on YouTube. ☐
- I have completed the Diagnostic Questions quiz. ☐
- I have completed at least 4 pages in my revision book. ☐

Developing Probability

Reflection

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I have revised the **previous work on probability**, e.g. sample space diagrams, expected frequency.			2, 6	
I know how to **calculate relative frequency from experimental data**.			1	
I know how to **draw a graph** showing relative frequency against the number of trials.			1	
I understand the **more** trials are held, the **better** the relative frequency is as an estimate of the probability.			1	
I can tell the difference between **independent events** and **dependent events**.			3	
I can tell the difference between **mutually exclusive events** and events that are **not mutually exclusive**.			3	
I can use the **multiplication rule** for **independent events**.			3	
I can use the **multiplication rule** for **dependent events**.			9	
I can use the **addition rule** for **mutually exclusive events**.			4	
I can answer questions on probability that involve **Venn diagrams**.			5	
I can answer questions on probability that involve **sample space diagrams** showing **dependent events**.			6	
I can draw and use **tree diagrams** for **independent** events and **dependent** events.			7, 9	
Given the probability of one of the **combinations** in a tree diagram, I can work backwards to fill in all the probabilities of the tree diagram.			8	

Am I ready for the test?
Tick the boxes below...

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Tracking Sheet Year 11

Sheet 2

Mock Examination 1 Grade: _____

Target Grade: _____

Attainment for Term 2: _____

Developing Algebra 4

%

Measuring Shapes 5

%

End of Year 11

%

Year 11 Mock Examination 2

%

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The Mathematics Department

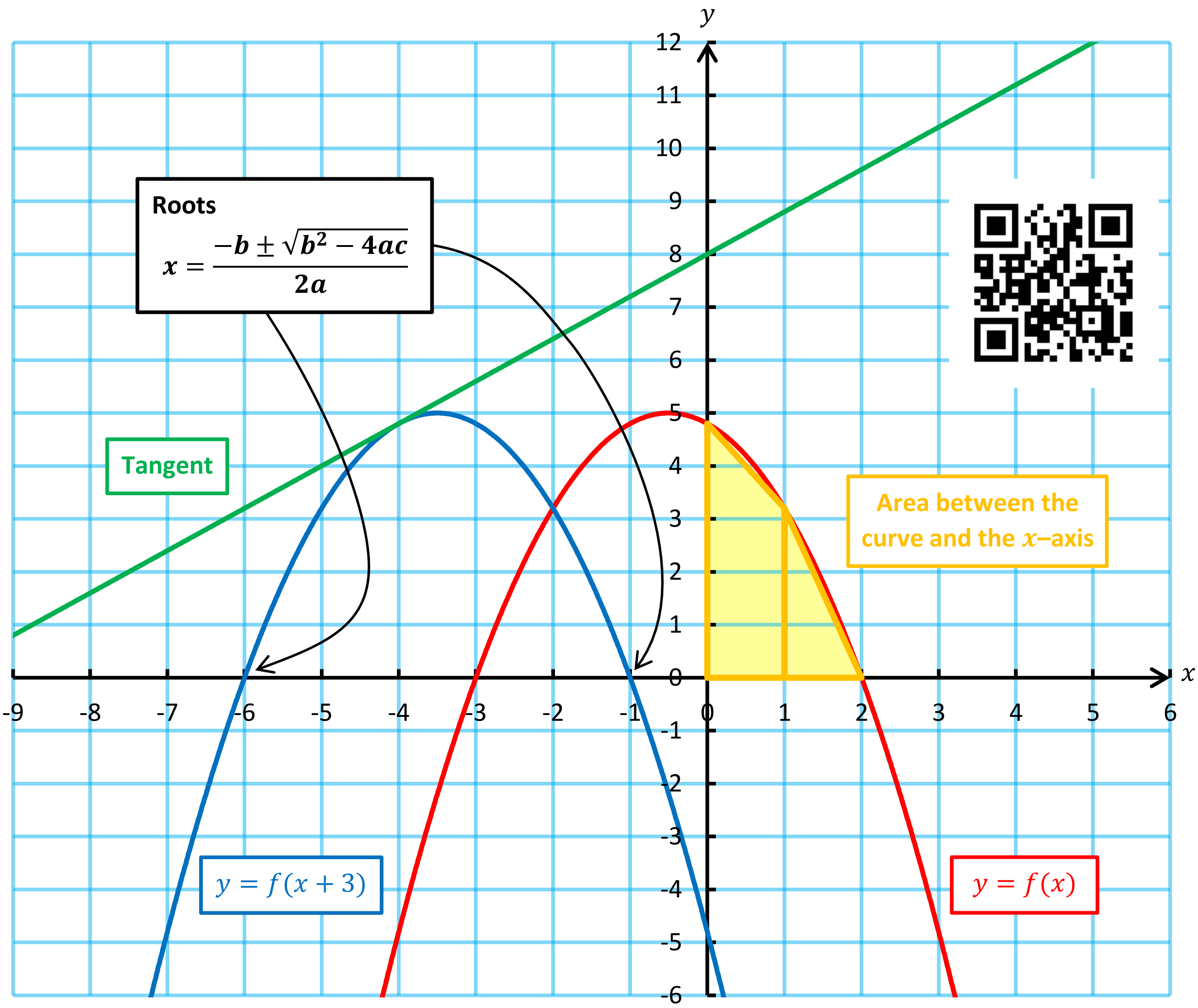
11

Developing Algebra 4

Name:

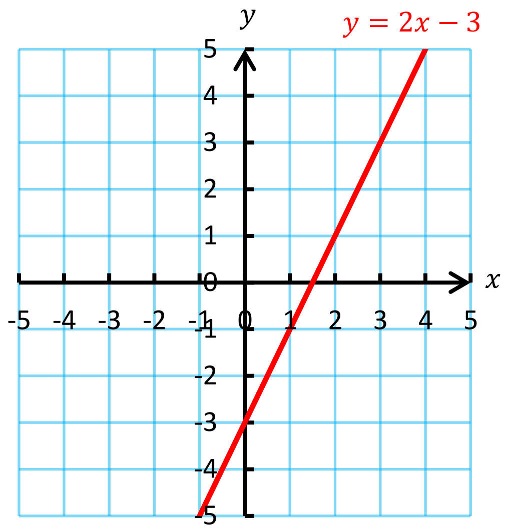
Contents

Chapter	Mathematics	Page Number
Transformations of Functions	**Function notation. Translation. Stretching. Reflection.**	3
Pre-Calculus	**Tangent to a function. Travel graphs. Displacement, Velocity and Acceleration. The area between a graph and the x–axis. The Trapezium Rule.**	9
Further Changing the Subject	**Revision of previous work. Changing the subject where the subject appears more than once.**	18
The Quadratic Formula	**Solving quadratic equations. Solving problems.**	20
Algebraic Fractions	**Numerical fractions. Algebraic fractions. Solving equations involving algebraic fractions.**	23



Transformations of Functions

Function Notation



A graph for $y = 2x - 3$ is shown on the right.

We can see from the graph that the value of y for $x = 2$ is 1.
In mathematics, a special notation is used to refer to values such as these. We write

$$f(2) = 1,$$

where f represents the **function** $f(x) = 2x - 3$,
2 is the **input** of the function,
and 1 is the **output** of the function.
We say that 'f of two is one'.

Skill

H

Exercise 1

Use the graph for $f(x) = 2x - 3$ to write
(a) $f(3)$ (b) $f(1)$ (c) $f(0)$

Example

If $f(x) = 5x - 2$, then it is possible to calculate $f(3)$ by substituting $x = 3$ into $5x - 2$:

$$f(3) = 5 \times 3 - 2$$
$$f(3) = 13$$

Exercise 2

H

(a) If $f(x) = 2x - 3$, calculate
(i) $f(5)$ (ii) $f(-2)$ (iii) $f(20)$

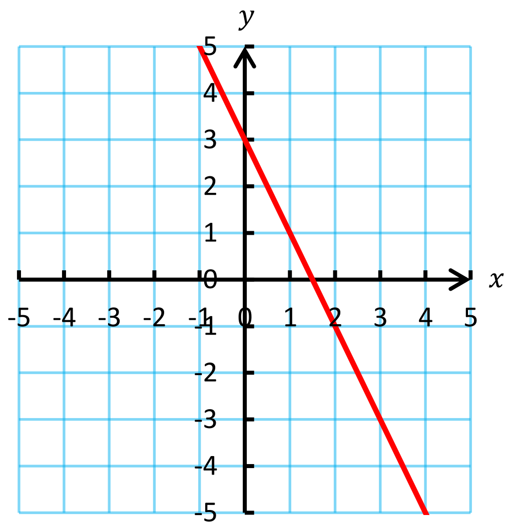
(b) If $f(x) = x^2 + 4x + 2$, calculate
(i) $f(2)$ (ii) $f(5)$ (iii) $f(0)$ (iv) $f(-2)$

(c) If $f(x) = -4x + 15$, complete the following table.

x	-3	-2	-1	0	1	2	3
$f(x)$							

(d) If $f(x) = 3x - 2$, find the value of x so that $f(x) = 19$.

(e) If $f(x) = x^2$, find the values of x so that $f(x) = 25$.



(f) The graph on the right shows a function $f(x)$ plotted on graph paper. What is the function $f(x)$?

(g) For the function from part (f), what is the value of $f(5)$?

(h) If $g(x) = 2x^2 + 5$, calculate
(i) $g(4)$ (ii) $g(0)$ (iii) $g(-2)$

(i) If $h(x) = \frac{2}{x-3}$, calculate
(i) $h(11)$ (ii) $h(-7)$ (iii) $h\left(\frac{1}{2}\right)$

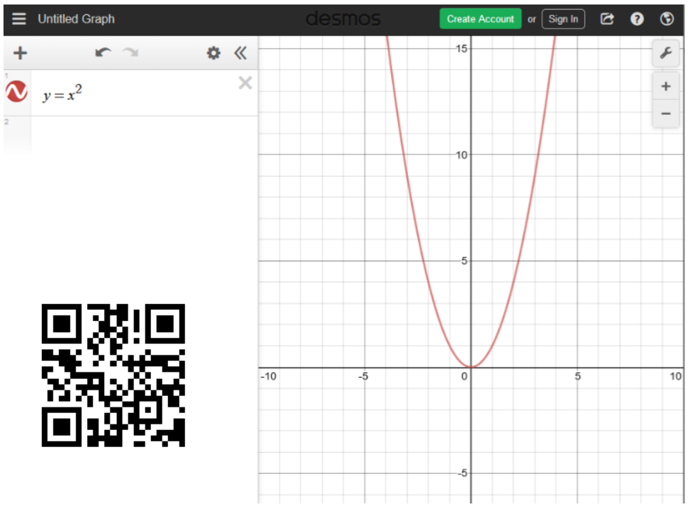
Transformations of Functions

Function notation is useful to describe the effect a transformation has on the graph of a function.

Exercise 3

H

Go to the website www.desmos.com/calculator.
Type $y = x^2$ into the first box.



(a) Type $y = x^2 + a$ into the second box. When the "add slider" option appears click on a. What effect does changing a have on the graph?

(b) Change box 2 to show $y = (x - a)^2$ instead of $y = x^2 + a$. What effect does changing a now have on the graph?

(c) What type of transformations are the transformations from parts (a) and (b) of this question?

(d) Change box 2 to show $y = ax^2$. What effect does changing a now have on the graph?

(e) Change box 2 to show $y = (ax)^2$. What effect does changing a now have on the graph?

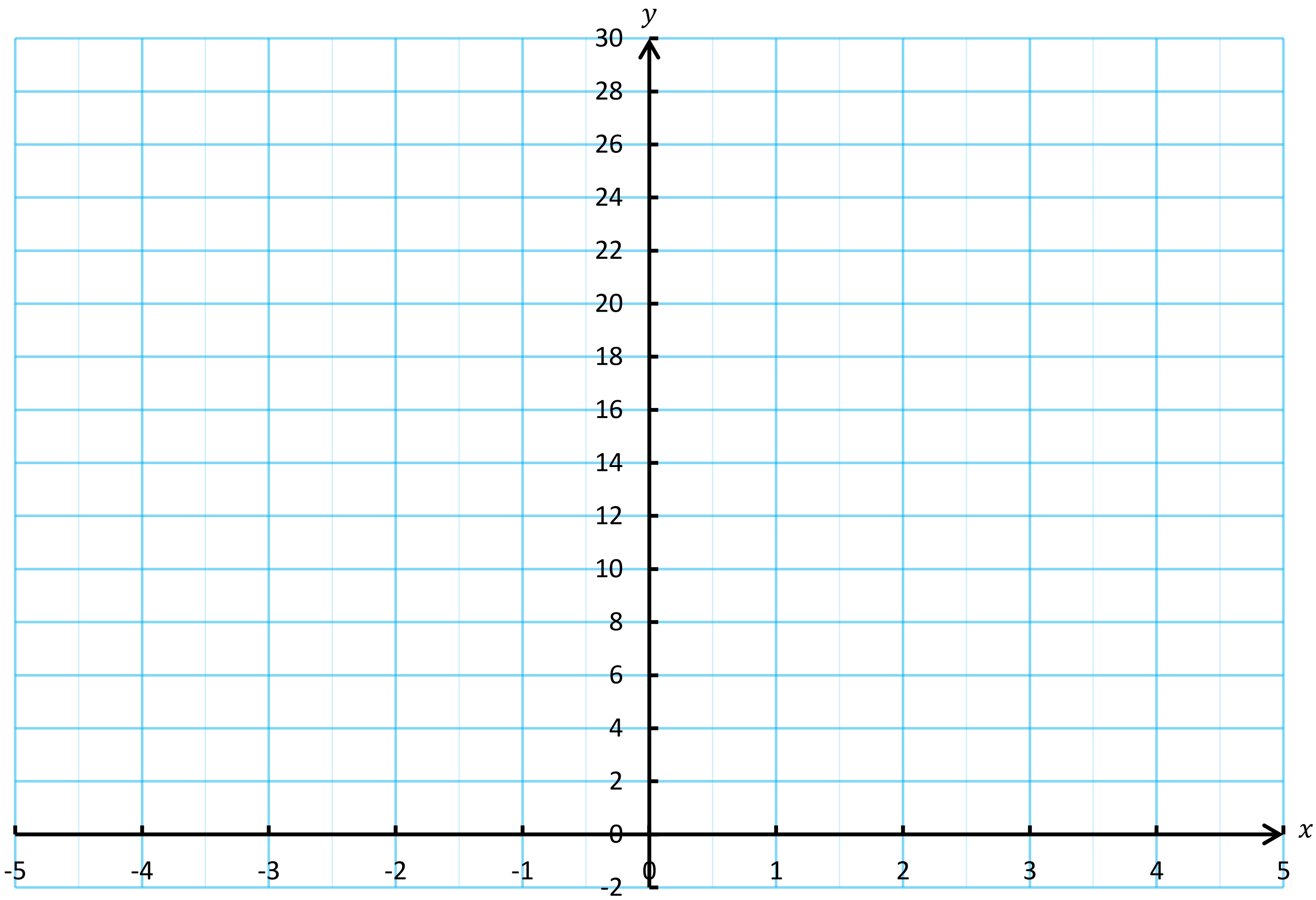
(f) What is the difference between the way changing a changes the graph in parts (d) and (e) of this question?

Exercise 4

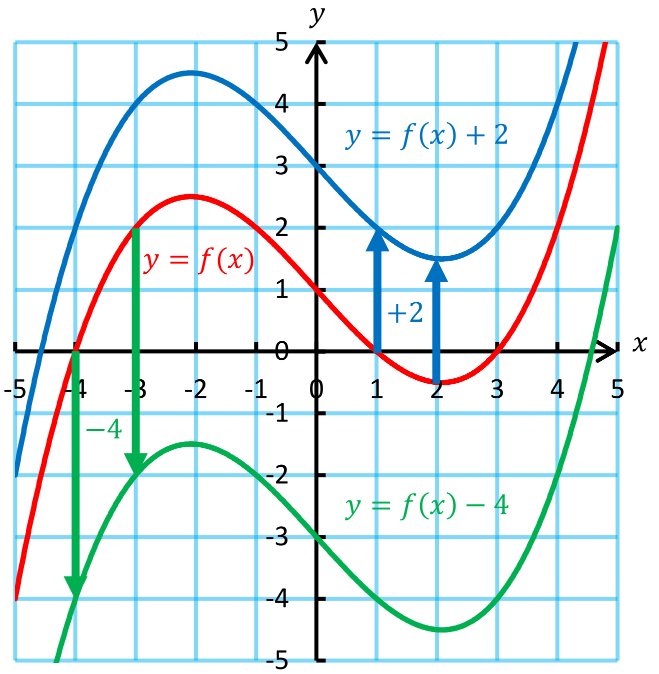
H

Draw, on the axes shown below, graphs for the following.

(a) $y = x^2$ (b) $y = x^2 + 3$ (c) $y = (x - 2)^2$ (d) $y = 2x^2$

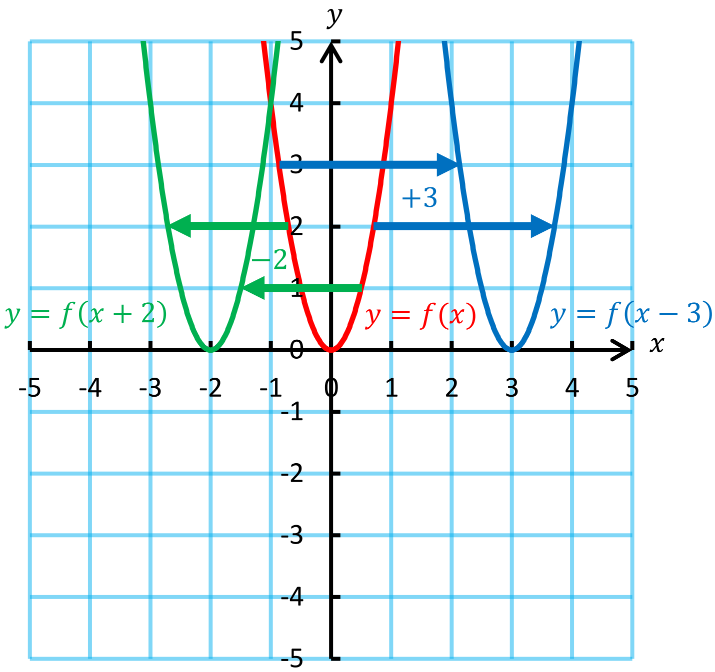


$y = f(x) + a$



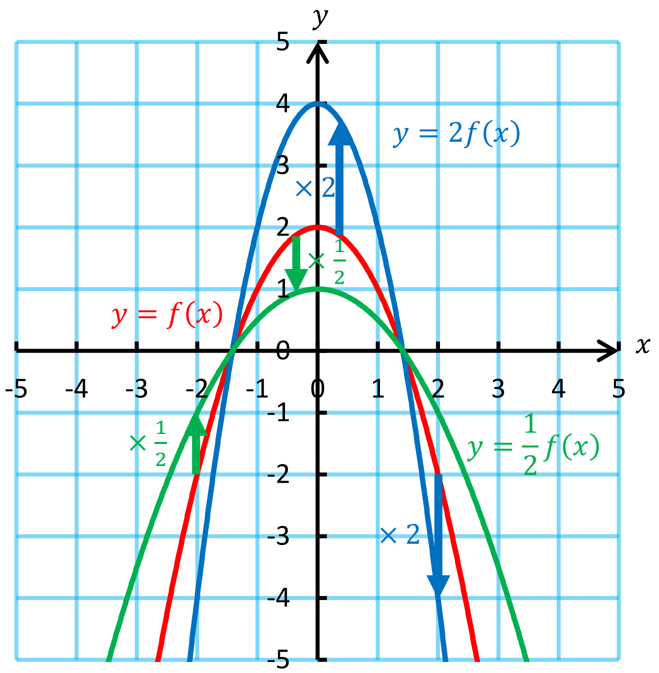
The transformation $y = f(x) + a$ translates the graph a units up (if a is positive) or a units down (if a is negative).

$y = f(x + a)$



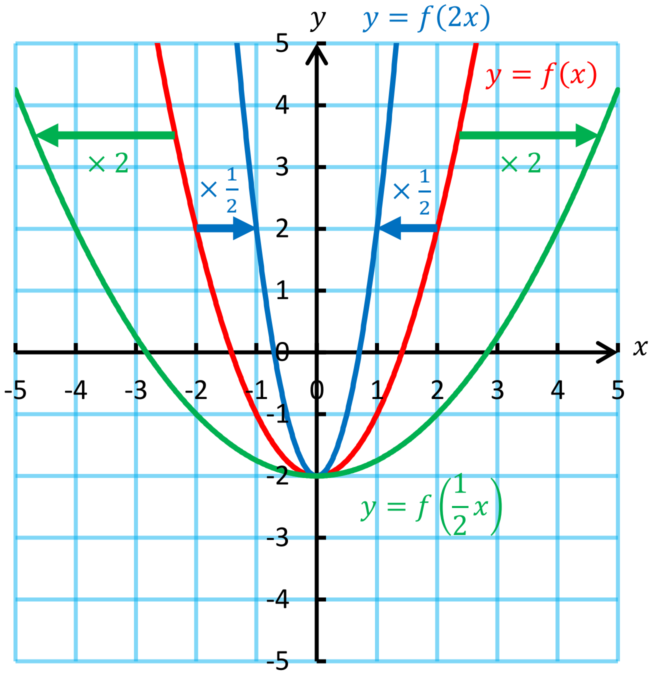
The transformation $y = f(x + a)$ translates the graph a units to the left (if a is positive) or a units to the right (if a is negative).

$y = af(x)$



The transformation $y = af(x)$ stretches the graph in the direction of the y–axis (if $a > 1$) or compresses the graph in the direction of the y–axis (if $0 < a < 1$).

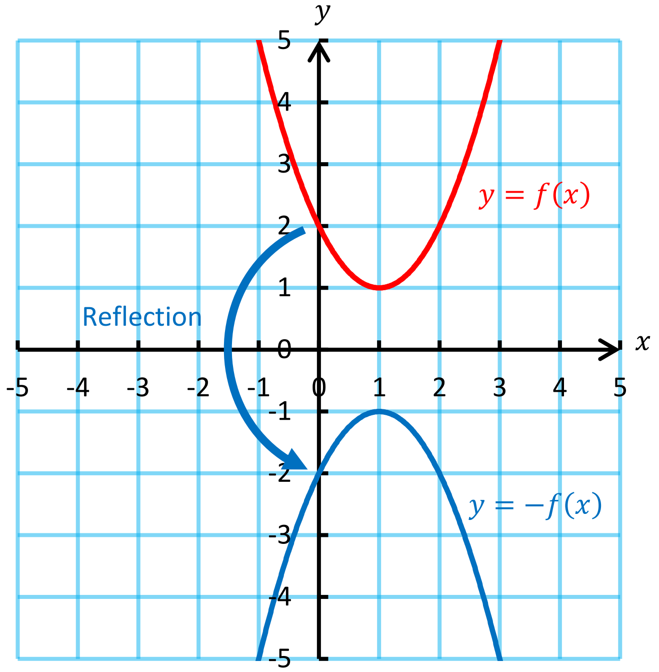
$y = f(ax)$



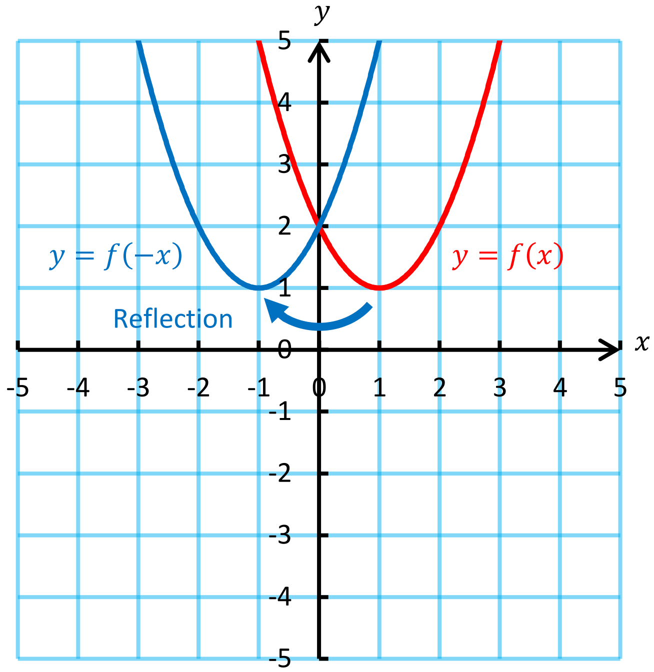
The transformation $y = f(ax)$ compresses the graph in the direction of the x–axis (if $a > 1$) or stretches the graph in the direction of the x–axis (if $0 < a < 1$).

As general advice, we need to **follow** anything that takes place **outside** parentheses, and **undo** anything that takes place **inside** parentheses. So, for example, we follow $y = f(x) + 3$ and translate the graph 3 units up, but undo $y = f(x + 3)$ by translating the graph 3 units to the left.

$\boldsymbol{y = -f(x)}$



$\boldsymbol{y = f(-x)}$



The transformation $y = -f(x)$ reflects the graph $y = f(x)$ in the x–axis.

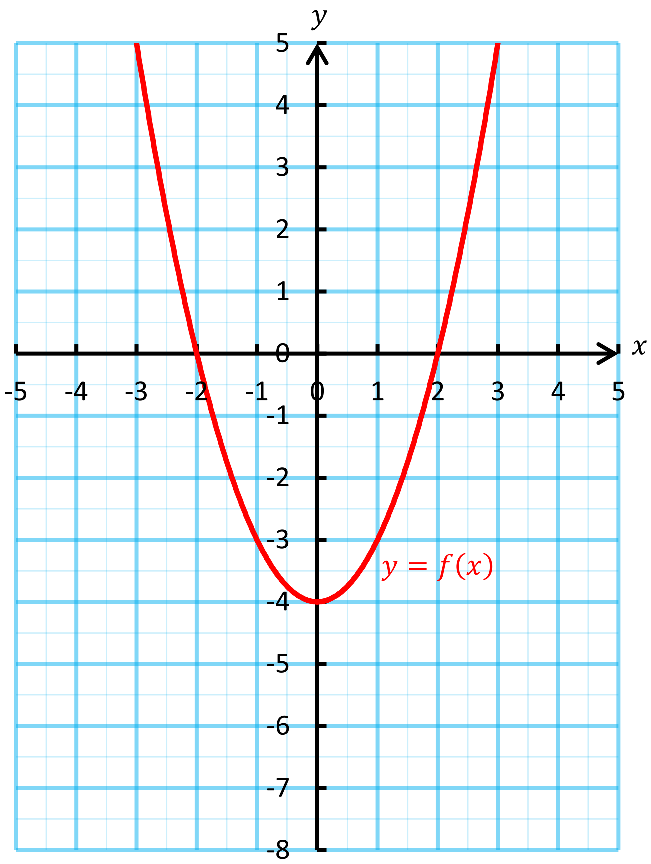
The transformation $y = f(-x)$ reflects the graph $y = f(x)$ in the y–axis.

Exercise 5

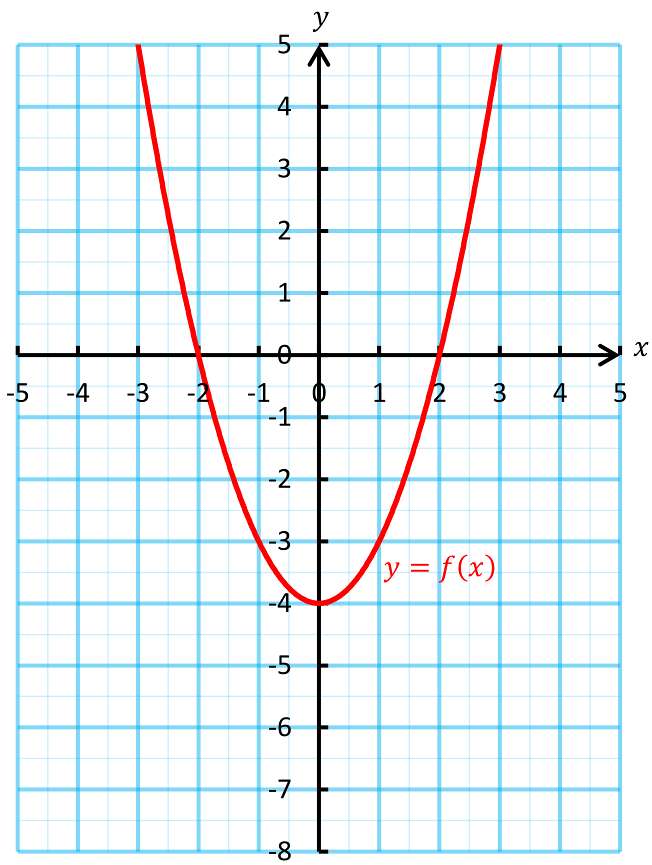
H

Draw, on the following graphs, the following transformations.

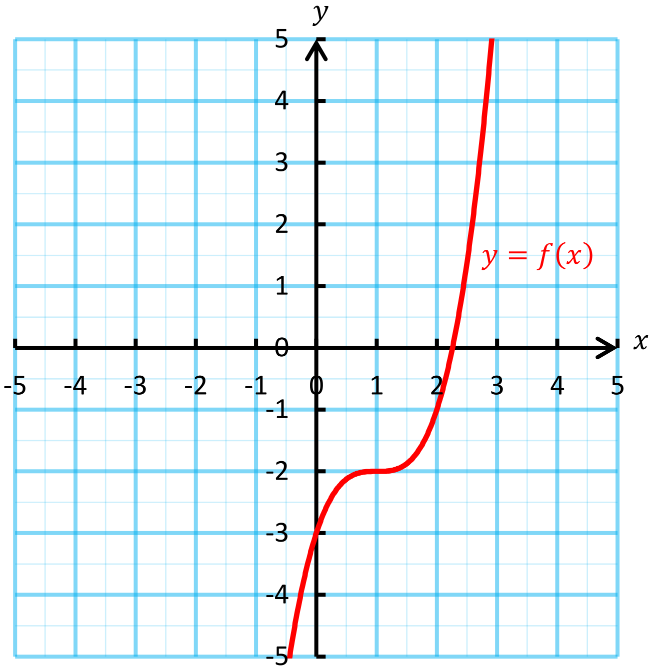
(a) $y = f(x) + 2$



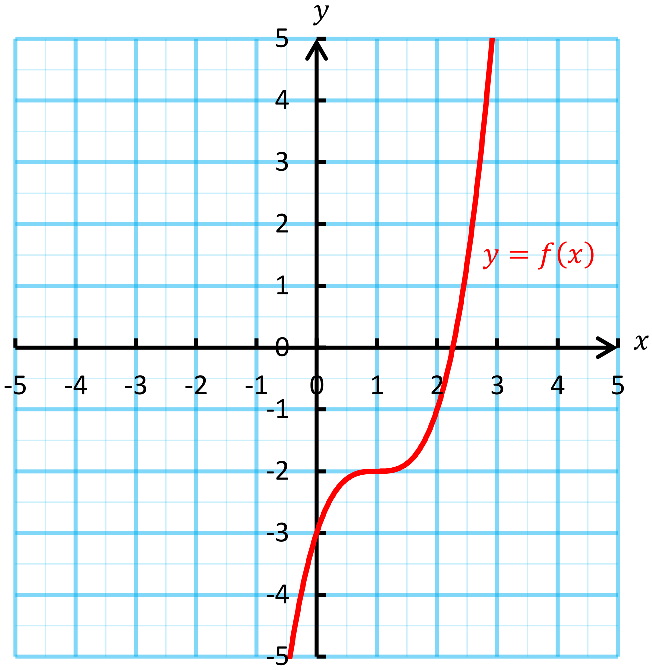
(b) $y = f(x + 2)$



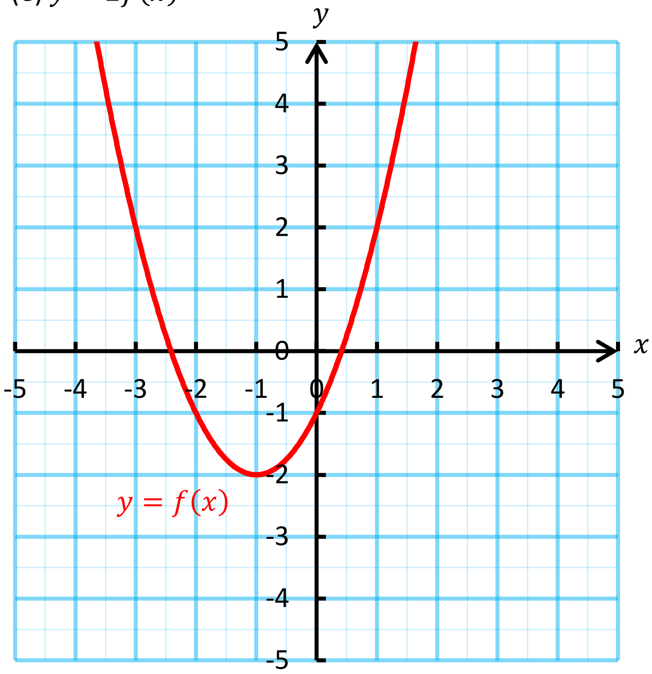
(c) $y = f(-x)$



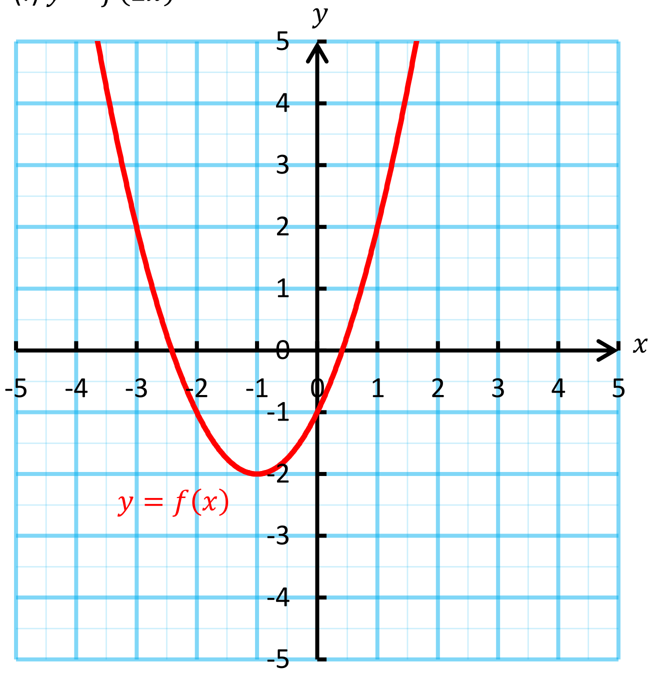
(d) $y = -f(x)$



(e) $y = 2f(x)$



(f) $y = f(2x)$



Exercise 6

H

(a) Draw a set of axes with x and y values going from –10 to 10.
Plot the graph of $y = 2^x$ on the axes, before using transformations of functions to plot the following graphs.
(i) $y = 2^x - 4$ (ii) $y = 2^{x-3}$ (iii) $y = 2^{-x}$

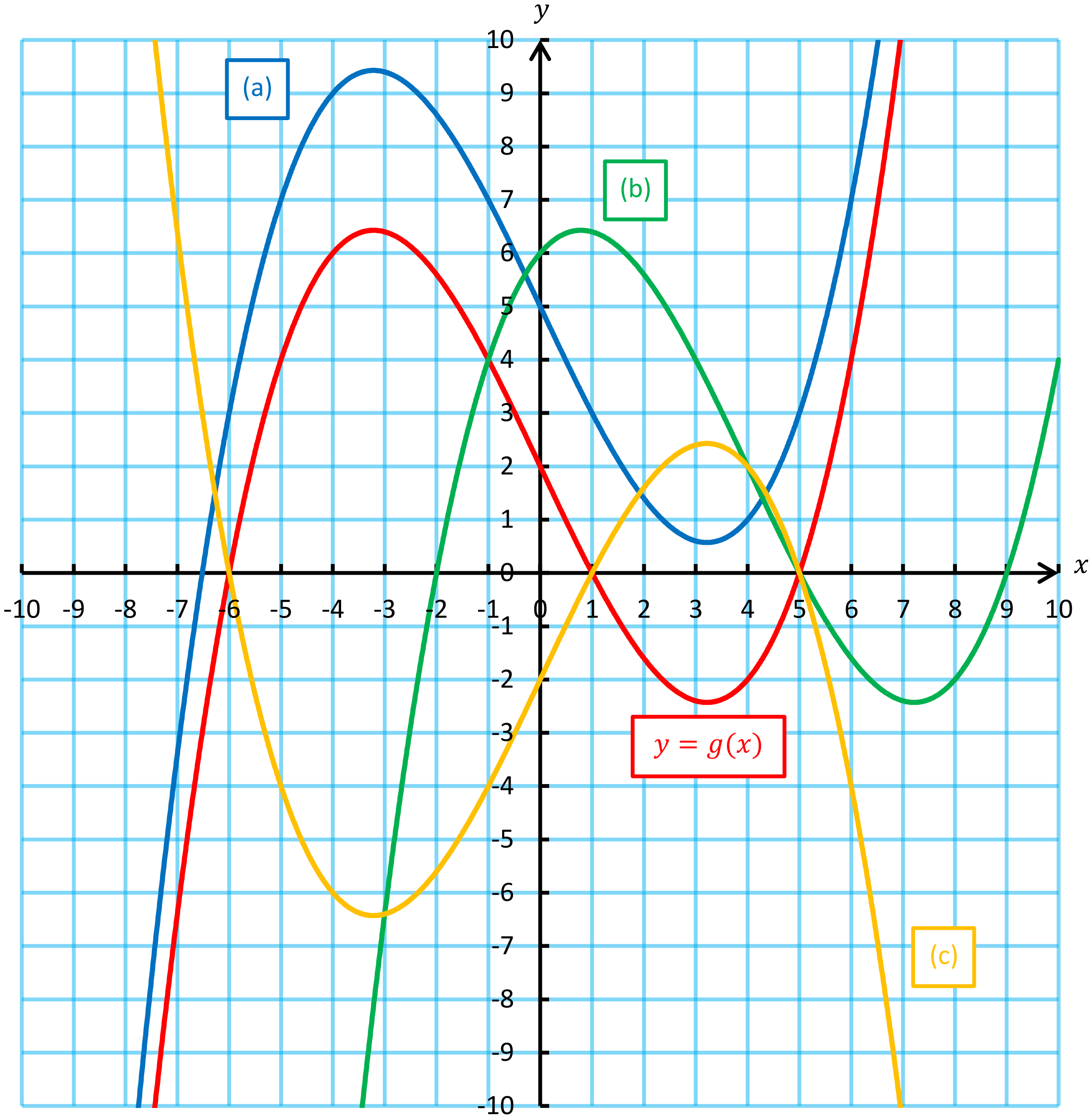
(b) Draw a set of axes with x and y values going from –10 to 10.
Plot the graph of $y = \frac{1}{x}$ on the axes, before using transformations of functions to plot the following graphs.
(i) $y = -\frac{1}{x}$ (ii) $y = \frac{1}{x} - 5$ (iii) $y = \frac{1}{x+4} - 5$

(c) Draw a set of axes with x and y values going from –10 to 10.
Plot the graph of $y = x^3$ on the axes, before using transformations of functions to plot the following graphs.
(i) $y = (x+4)^3$ (ii) $y = \frac{1}{4}x^3$ (iii) $y = \left(\frac{1}{4}x\right)^3$

Exercise 7

H

The following graph shows the function $y = g(x)$.
Write, in terms of $g(x)$, a function for each of the other graphs that are shown.



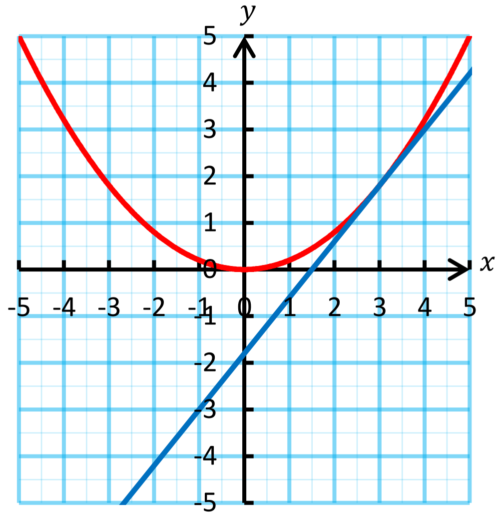
Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ❑ Target ❑

Pre–Calculus

Tangent to a Function

Given a **non-linear function** (a function that is not a straight line), a **tangent** to a specific point on the function is a straight line that meets the point so that the gradient of the tangent is equal to the gradient of the function at that point.

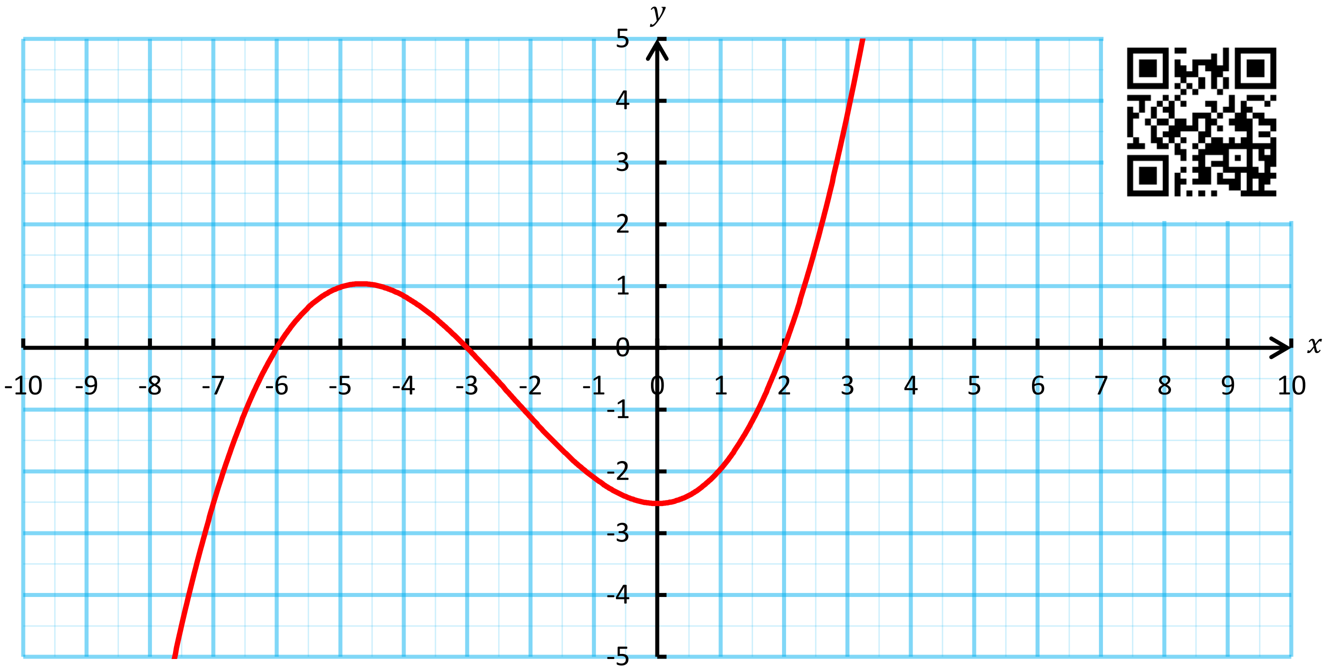


Example

The graph on the right shows a tangent to the red curve at the point where $x = 3$. This tangent has a positive gradient.

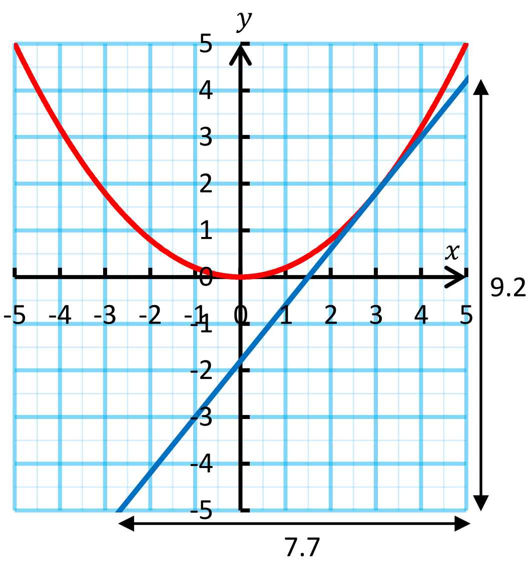
Exercise 8 H

For the curve shown below, draw (by eye) a tangent for the points where
(a) $x = -5$ (b) $x = 2.5$ (c) $x = 0$ (d) $x = -2$



The gradient of a tangent gives the **rate of change** for the point under consideration. That is, it represents how much the variable on the vertical axis (y) changes with respect to one unit of the unit on the horizontal axis (x). The steeper the tangent, the greater the rate of change.

For the example on the right, the gradient of the tangent is the change in the vertical distance divided by the change in the horizontal distance.



Gradient of the tangent $= \frac{9.2}{7.7}$
Gradient of the tangent $= 1.19$ to 2 decimal places.

Exercise 9 H

Calculate the gradient of your four tangents from Exercise 8.

Travel Graphs

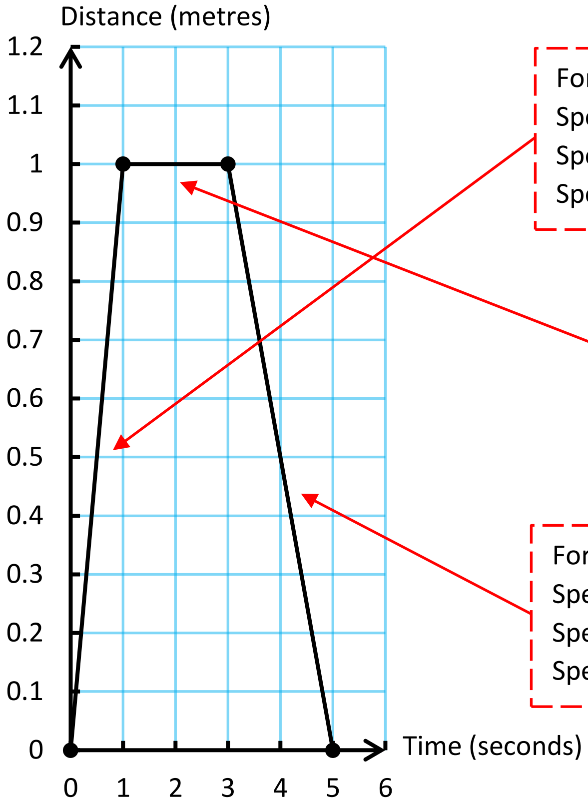
We saw travel graphs for the first time in the *Movement with Sphero* workbook in year 9.

Example

The following graph shows the journey of a sphero over 5 seconds.

Watch the journey of the sphero here:

Distance-time graph for the motion of the sphero



For the **first** part of the journey,
Speed = Change in distance ÷ Change in time
Speed = 1 ÷ 1
Speed = 1 metre per second.

For the **second** part of the journey,
Speed = Change in distance ÷ Change in time
Speed = 0 ÷ 2
Speed = 0 metres per second.

For the **third** part of the journey,
Speed = Change in distance ÷ Change in time
Speed = 1 ÷ 2
Speed = 0.5 metres per second.

Speed is an example of a **scalar** measure, where the **direction of travel** is ignored. So, during the third part of the journey, where the sphero returns to its original position, and the gradient of the distance-time graph is negative, the speed remains positive. In order to consider the direction of travel, and therefore differentiate between positive and negative gradients in a distance-time graph, we need to consider new measures that are **vector** measures.

Displacement and Velocity

Distance and **displacement** are measures that measure how far an object is from a specific origin, but displacement takes into account the **direction of travel** as well as the distance from the origin. For example, at the end of the first part of the above journey, during which time the sphero moves away from the origin (the starting point), the distance and displacement are both 1 metre. For the final part of the journey however, where the sphero travels back to the origin, the distance travelled is 1 metre, but the displacement is –1 metre. The negative sign represents the direction of travel of the sphero, and also reflects the negative gradient of the distance-time graph for this part of the journey.

Velocity is the vector that corresponds to the scalar measure speed, and is calculated using the formula

$$\text{Velocity} = \frac{\text{Change in displacement}}{\text{Change in time}}$$

For the final part of the above journey, where the speed of the sphero is 1 ÷ 2 = 0.5 metres per second, the velocity is –1 ÷ 2 = –0.5 metres per second. Again, the velocity takes into account that the distance-time graph has a negative gradient during the third part of the journey.

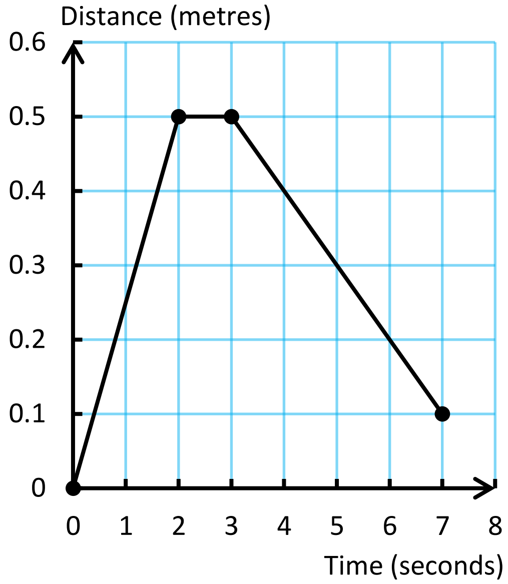
Exercise 10

Applying H

The graph on the right shows the journey of a sphero over 7 seconds. Complete the following table.

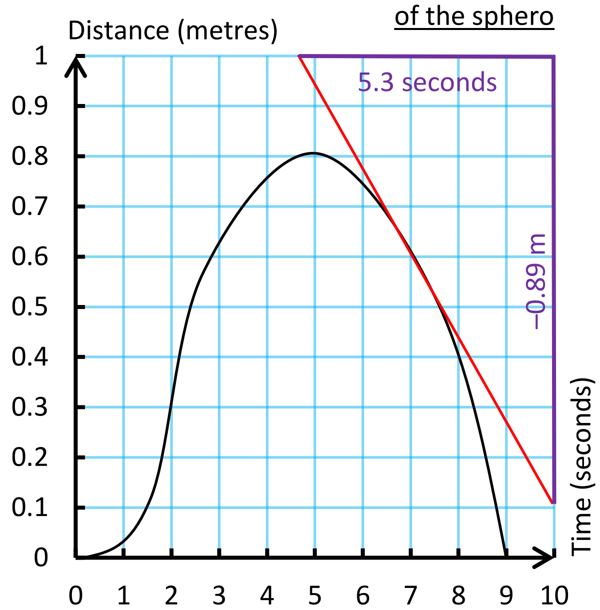
Part of the journey	Change in distance	Change in displacement	Change in time	Speed (scalar)	Velocity (vector)
1st					
2nd					
3rd					

Distance-time graph for Exercise 10



In any **non-linear** distance-time graph, it is possible to estimate the velocity at any point by calculating the gradient of the tangent at that point.

Distance-time graph to show the journey of the sphero



Example

The graph on the left shows the journey of a sphero over 9 seconds. Estimate the velocity of the sphero at time 7 seconds.

Answer: To begin, we draw (by eye) a tangent to the curve at 7 seconds. This tangent is shown in red on the graph.

Next, we complete a right-angled triangle around the tangent, in order to measure the change in displacement and change in time. From these we can calculate an estimate for the velocity:

$$\text{Velocity} = \frac{\text{Change in displacement}}{\text{Change in time}}$$

$$\text{Velocity} = \frac{-0.89}{5.3}$$

Velocity = −0.17 metres per second, to 2 decimal places.

Exercise 11

H

The graph on the right shows the journey of a sphero over 10 seconds.

(a) Estimate the velocity of the sphero at time 2 seconds.

(b) Estimate the velocity of the sphero at time 8 seconds.

(c) Estimate the speed of the sphero at time 2 seconds.

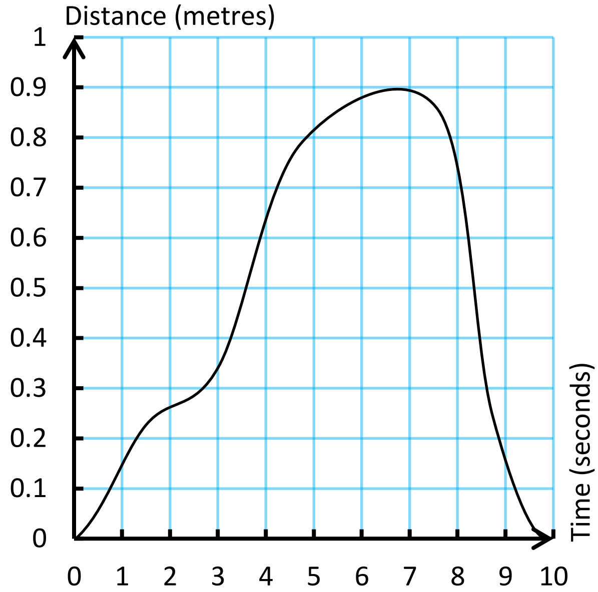
(d) Estimate the speed of the sphero at time 8 seconds.

(e) What is the total distance travelled by the sphero during its journey?

(f) What is the average speed for the entire journey?

(g) What is the average velocity for the entire journey?

Distance-time graph for Exercise 11



Acceleration

Whilst velocity is a vector that measures the rate of change of displacement with respect to time, **acceleration** is a vector that measures the rate of change of velocity with respect to time.
We can use the formula

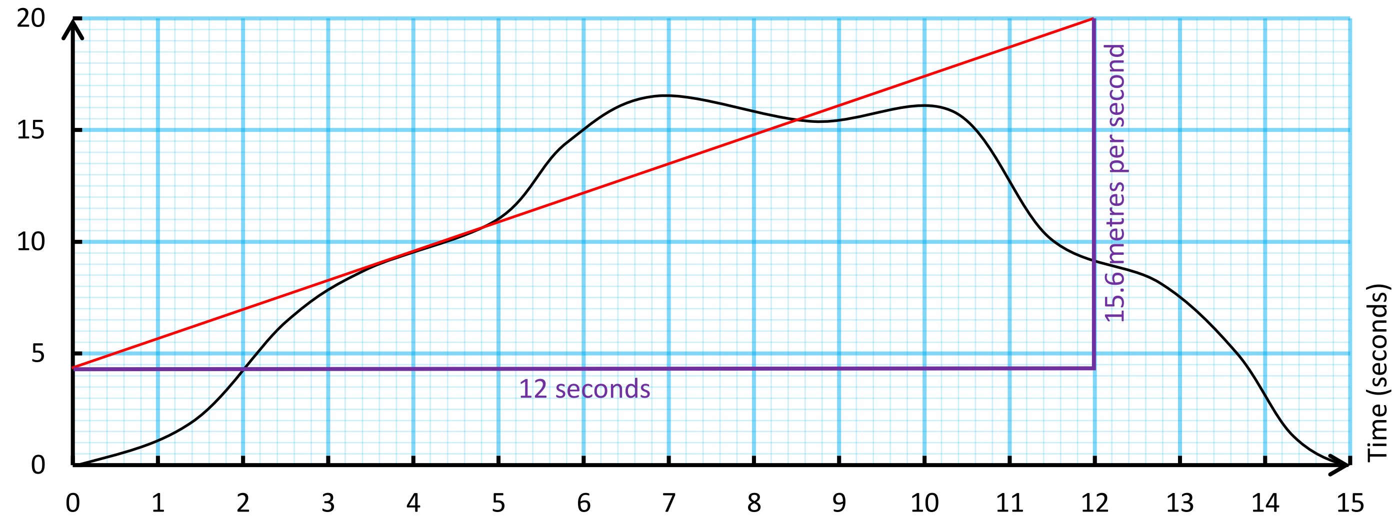
$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Change in time}}$$

to calculate acceleration, or we can estimate the gradient of a tangent to a point on a velocity-time graph.

Example

The following graph shows the velocity of a cyclist on a straight road between two sets of traffic lights. Calculate the acceleration of the cyclist at time 4 seconds. Give the units of your answer.

<u>Velocity-time graph to show the journey of a cyclist between two sets of traffic lights</u>



Answer: To begin, we draw (by eye) a tangent to the curve at time 4 seconds. This tangent is shown in red on the graph.

Next, we complete a right-angled triangle around the tangent, in order to measure the change in velocity and change in time. From these we can calculate an estimate for the acceleration:

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Change in time}}$$

$$\text{Acceleration} = \frac{15.6}{12}$$

Acceleration = 1.3 metres per second squared, or m/s².

Units of Acceleration

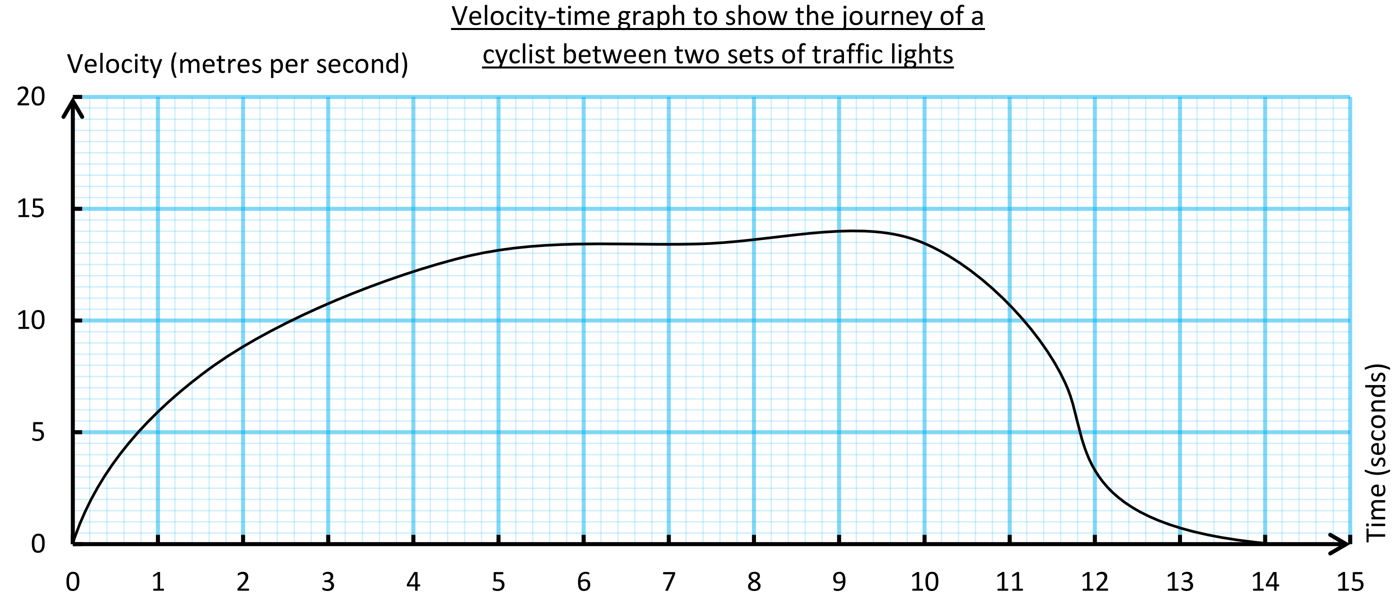
The following table shows some common units for measuring acceleration.

Displacement Unit	Time Unit	Velocity Unit	Acceleration Unit
Metres (m)	Seconds (s)	Metres per second (m/s)	Metres per second per second, or metres per second squared (m/s^2)
Kilometres (km)	Hours (h)	Kilometres per hour (km/h)	Kilometres per hour squared (km/h^2) or kilometres per hour per second
Miles (mi)	Hours (h)	Miles per hour (mph)	Miles per hour squared (mph^2) or miles per hour per second

The acceleration unit can depend on the horizontal axis unit of a velocity-time graph

Exercise 12 H

The following graph shows the velocity of a cyclist on a straight road between two sets of traffic lights.



(a) Calculate the acceleration of the cyclist at time 2 seconds. Give the unit of your answer.

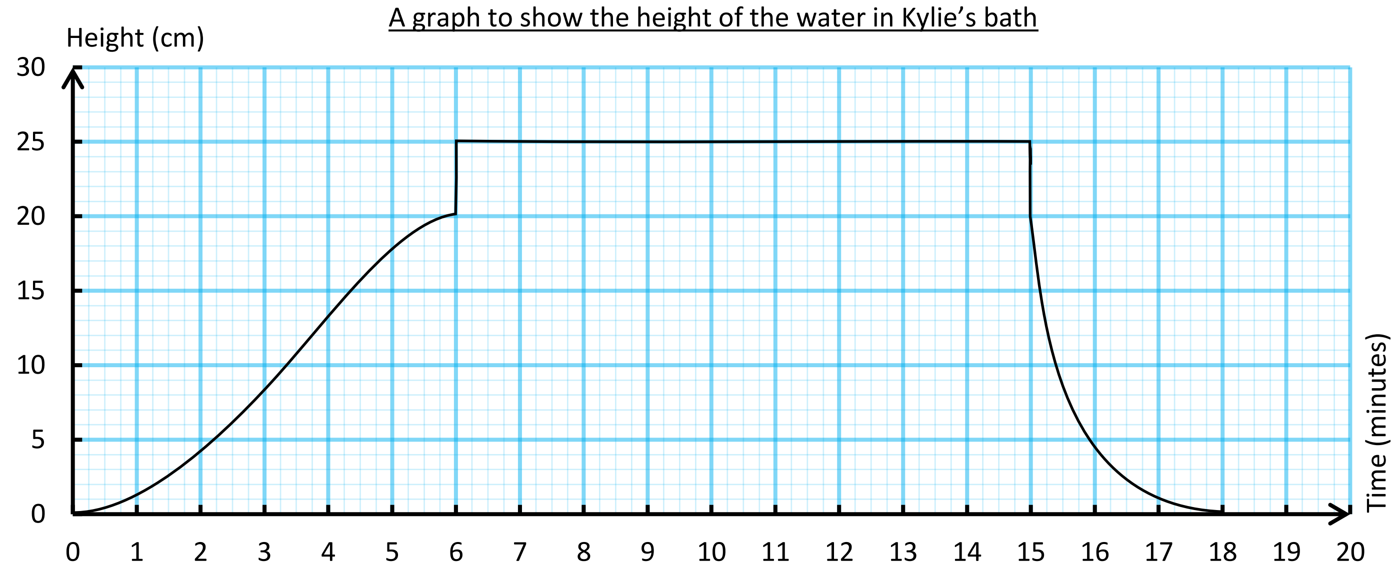
(b) Calculate the acceleration of the cyclist at time 11 seconds. Give the unit of your answer.

(c) What is the maximum velocity of the cyclist during the journey? At which time does this happen?

(d) Give the times at which the cyclist is travelling at a velocity of 5 metres per second.

Exercise 13 H

The following graph shows the height of the water in a bath as Kylie takes a bath one Sunday night.



(a) What is the height of the water in the bath at time 2 minutes?

(b) What happens at time 6 minutes?

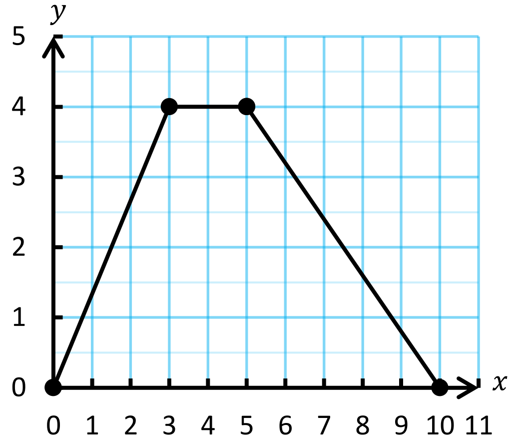
(c) How many minutes did Kylie spend in the bath?

(d) What is the rate of change of the height of the water in the bath at time 4 minutes?

(e) What is the rate of change of the height of the water in the bath at time 15 minutes 30 seconds?

The Area Between a Graph and the x–axis

If a graph uses straight lines only, then it is possible to find the area between the lines and the x–axis using formulae for the area of common 2-D shapes.



For example, considering the graph on the left, we can calculate the area between the lines of the graph and the x–axis by calculating the area of the trapezium:

Formula for the area of a trapezium = $\frac{1}{2}(a+b)h$

2 + 10 = 12	Calculating $a + b$
12 ÷ 2 = 6	Calculating $\frac{1}{2}(a+b)$
6 × 4 = 24	Calculating $\frac{1}{2}(a+b)h$

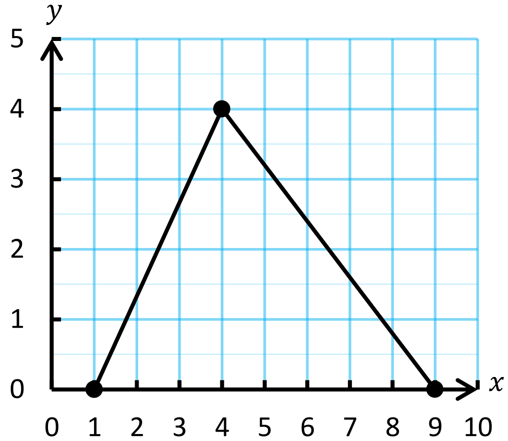
So, the required area is 24 square units.

Exercise 14

Skill

H

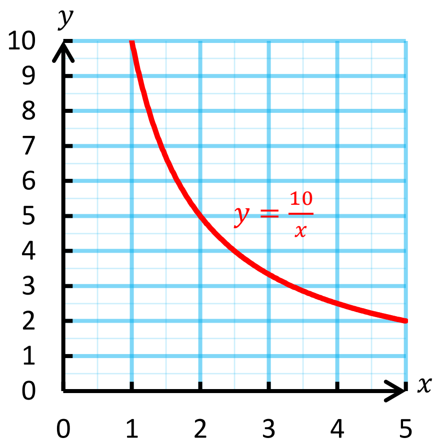
For the graph shown on the right, find the area between the lines of the graph and the x-axis.



The Trapezium Rule

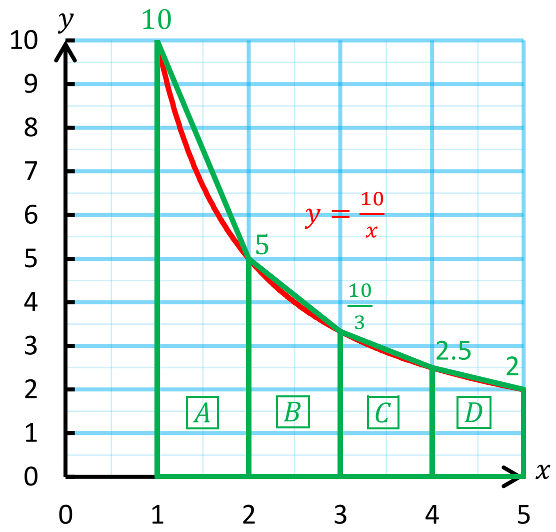
If we have a non-linear graph, then we cannot use a method similar to the above method to find the area between the graph and the x–axis – we must estimate the area using the **trapezium rule**.

Example



The graph on the left shows the function $y = \frac{10}{x}$ between $x = 1$ and $x = 5$. Use the ordinates $x = 1$, $x = 2$, $x = 3$, $x = 4$ and $x = 5$ to estimate the area enclosed by the curve $y = \frac{10}{x}$ and the x–axis, between $x = 1$ and $x = 5$.

Answer: The first step is to add verical lines to the graph corresponding to the 5 ordinates, and then to complete the 4 trapezia formed by the ordinates. We can then read from the graph (or calculate) the values of the function for each of the five ordinates.



The estimate of the area between the curve and the x–axis is the total area of the four trapezia.

$\boxed{A}$

$5 + 10 = 15$

$15 \div 2 = 7.5$

$7.5 \times 1 = 7.5$

$\boxed{B}$

$5 + \frac{10}{3} = \frac{25}{3}$

$\frac{25}{3} \div 2 = \frac{25}{6}$

$\frac{25}{6} \times 1 = \frac{25}{6}$

$\boxed{C}$

$\frac{10}{3} + 2.5 = \frac{35}{6}$

$\frac{35}{6} \div 2 = \frac{35}{12}$

$\frac{35}{12} \times 1 = \frac{35}{12}$

$\boxed{D}$

$2.5 + 2 = 4.5$

$4.5 \div 2 = 2.25$

$2.25 \times 1 = 2.25$

Final answer: $7.5 + \frac{25}{6} + \frac{35}{12} + 2.25 = \frac{101}{6}$ square units (or $16.8\dot{3}$)

The Trapezium Rule Formula

Instead of calculating the area of all the individual trapezia, we can use the following formula to estimate the area between the curve and the x–axis:

$$\frac{1}{2}h\{y_0 + y_n + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\}$$

Here, h is the height of each individual trapezium, and y_0, y_1, y_2, …, y_n are the values of the function for each of the ordinates x_0, x_1, x_2, …, x_n.

For the example on the previous page, we have $h = 1$, and the following table summarises the values of x_n and y_n.

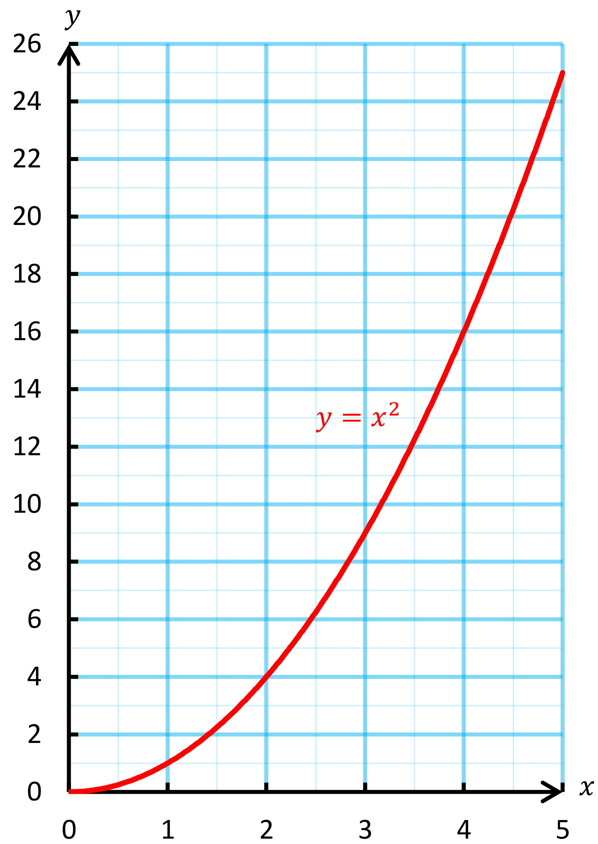
n	0	1	2	3	4
x_n	1	2	3	4	5
y_n	10	5	$\frac{10}{3}$	2.5	2

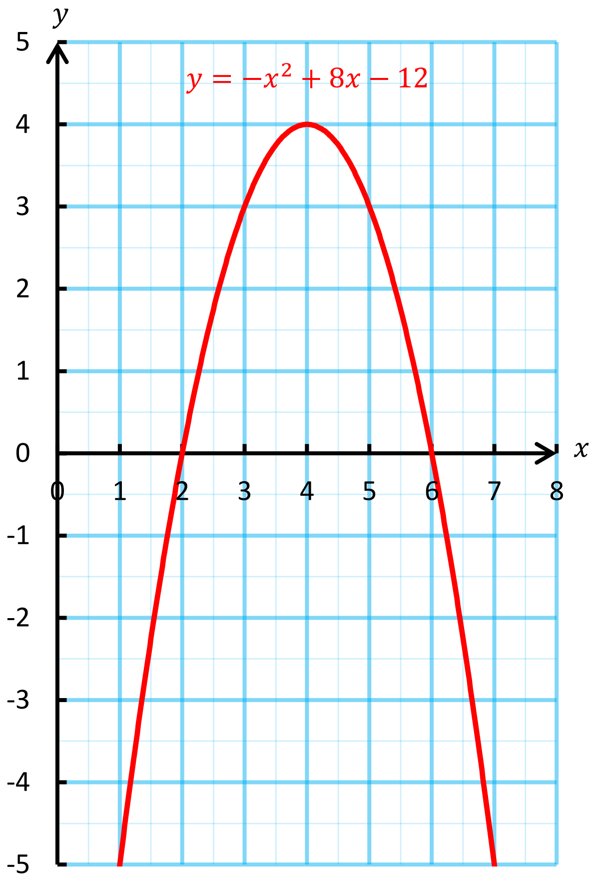
Using the formula for the trapezium rule, the estimate of the area between the curve and the x–axis is given by

$$\frac{1}{2} \times 1 \times \left\{10 + 2 + 2\left(5 + \frac{10}{3} + 2.5\right)\right\} = \frac{101}{6} \text{ square units (or } 16.8\dot{3}\text{)}$$

Exercise 15 H

The graph on the right shows the function $y = x^2$ between $x = 0$ and $x = 5$. Use the trapezium rule and the ordinates $x = 1$, $x = 2$, $x = 3$, $x = 4$ and $x = 5$ to estimate the area enclosed by the curve $y = x^2$ and the x–axis, between $x = 1$ and $x = 5$.





Exercise 16 H

The graph on the left shows the function $y = -x^2 + 8x - 12$ between $x = 1$ and $x = 7$. Use the trapezium rule and the ordinates $x = 2$, $x = 3$, $x = 4$, $x = 5$ and $x = 6$ to estimate the area enclosed by the curve $y = -x^2 + 8x - 12$ and the x–axis, between $x = 2$ and $x = 6$.

Applications

Sometimes, the area between a graph and the x–axis has a special meaning.

- In a **velocity-time graph**, the area between the graph and the x–axis gives the **distance travelled**.
- In a graph that plots **time** (in years) against **yearly salary**, the area between the graph and the x–axis gives the **total earnings**.
- In a graph that plots **time** (in minutes) against the **rate of filling a petrol tank** (in litres per minute), the area between the graph and the x–axis gives the **volume of petrol** that has been added to the petrol tank.

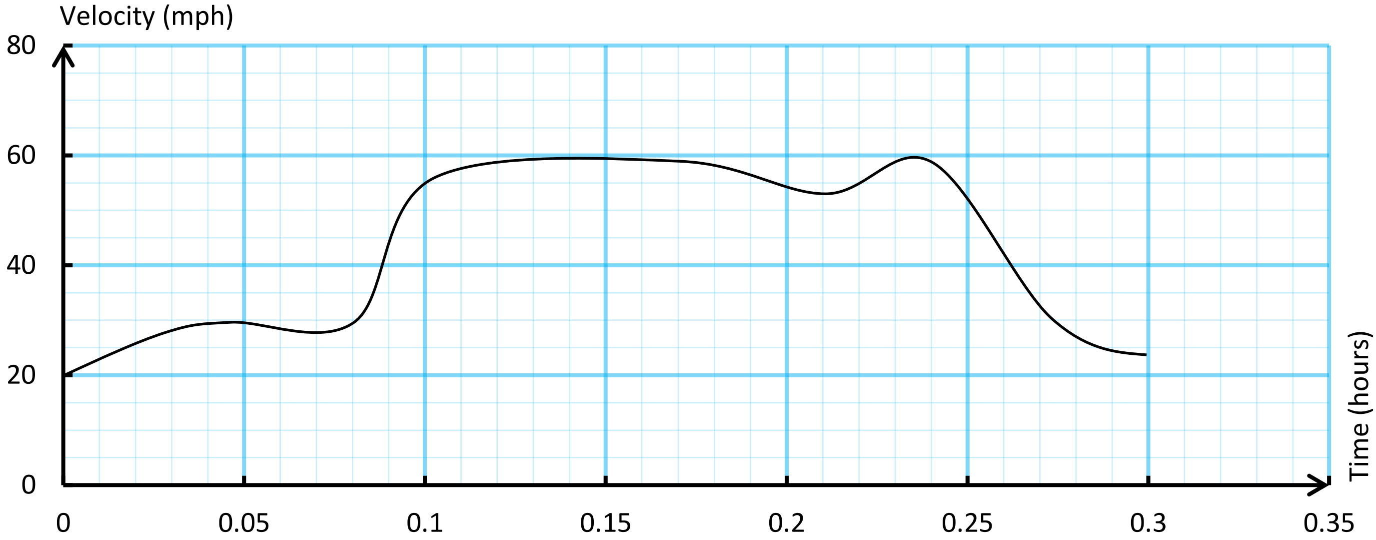
In general, the area between the graph and the x–axis represents the measure that has unit given by multiplying the unit on the vertical axis by the unit on the horizontal axis.

Exercise 17

Applying H

The following graph shows the velocity of a car as it travels between Bangor and Caernarfon.

Velocity-time graph to show the journey of a car travelling between Bangor and Caernarfon



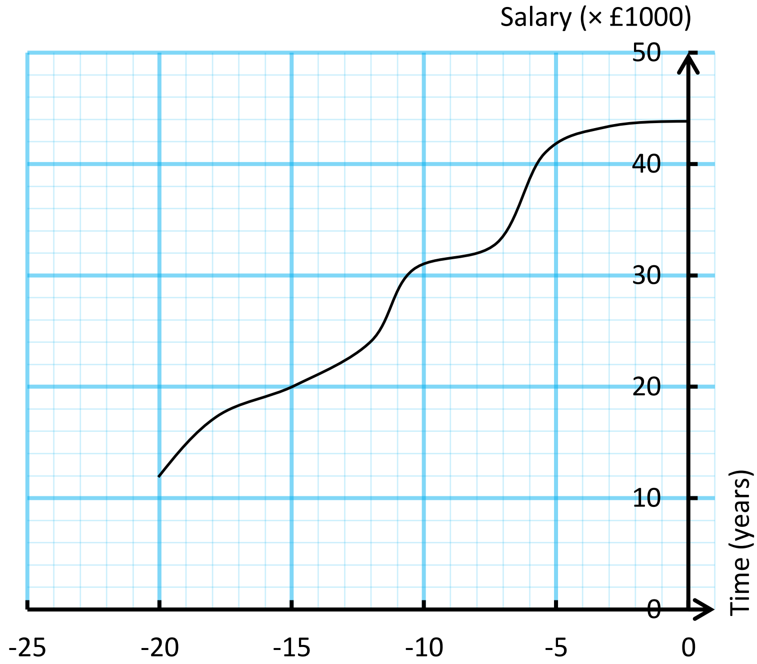
Use the trapezium rule and the ordinates $x = 0$, $x = 0.1$, $x = 0.2$ and $x = 0.3$ to estimate the distance that the car has travelled during this journey.

Exercise 18 H

The graph on the right shows how Trefor's yearly salary has increased during the past 20 years.

Use the trapezium rule and the ordinates $x = -20$, $x = -15$, $x = -10$, $x = -5$ and $x = 0$ to estimate Trefor's total earnings over the past 20 years.

A graph to show Trefor's salary during the past 20 years



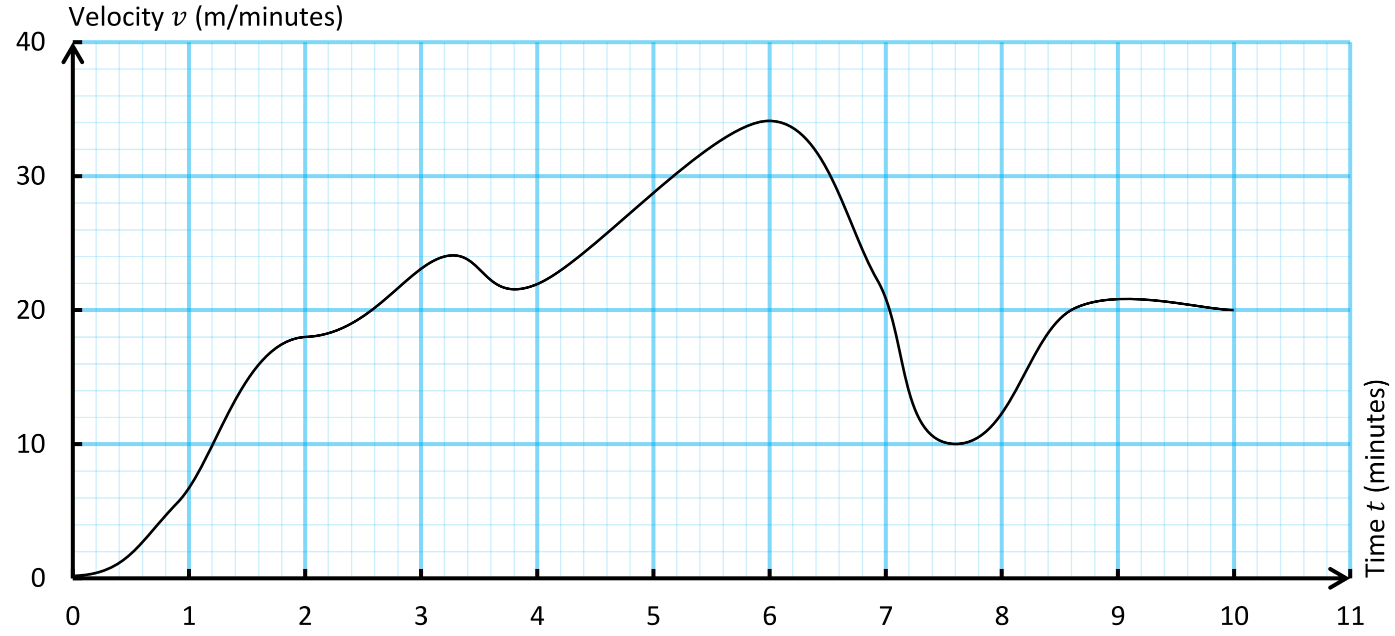
Exercise 19 (Revision)

H

Non conducted an experiment. She used equipment to measure the velocity, v, of an object during the first 10 minutes of the experiment.

The velocity-time graph for the experiment is shown below.

A graph to show the velocity of an object during an experiment



(a) Write down the gradient of the curve when the time is 7.6 minutes.

(b) Find an estimate for the acceleration of the object when the time is 4.5 minutes.

(c) Use the trapezium rule and the ordinates $t = 0$, $t = 2$, $t = 4$, $t = 6$, $t = 8$ and $t = 10$ to estimate the area enclosed by the curve, the positive time axis and the line $t = 10$.

(d) Calculate an estimate of the distance travelled by the object during the first 10 minutes of Non's experiment, giving your answer in kilometres.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ❑ Target ❑

Further Changing the Subject

The purpose of **changing the subject** is to re-arrange a formula so that a particular **variable** appears on its own on the left-hand side of the formula.

Revision I

Exercise 20

(a) Make x the subject of the formula $2y = 3z + 5x$

(b) Make s the subject of the formula $t = \frac{s}{3}$

(c) Make a the subject of the formula $F = c(a - b)$

(d) Make z the subject of the formula $\frac{x}{z} = y$

(e) Make r the subject of the formula $C = \frac{4}{3}\pi r^3$

(f) Make e the subject of the formula $g = \frac{f}{e} + c$

Changing the subject where the subject appears more than once

Example

(a) Make x the subject of the formula $3z + 4x = yx + 6y$.

Answer: $3z + 4x = yx + 6y$

$4x - yx = 6y - 3z$ [Subtract $3z$ from both sides; subtract yx from both sides]

$x(4 - y) = 6y - 3z$ [Factorise x on the left-hand side]

$x = \frac{6y-3z}{4-y}$ [Divide both sides by $4 - y$]

(b) Make x the subject of the formula $\frac{2x+y}{3-5x} = 2$.

Answer: $\frac{2x+y}{3-5x} = 2$

$2x + y = 2(3 - 5x)$ [Multiply both sides by $3 - 5x$]

$2x + y = 6 - 10x$ [Expand the bracket]

$2x + 10x = 6 - y$ [Subtract y from both sides; add $10x$ to both sides]

$12x = 6 - y$ [Collect like terms]

$x = \frac{6-y}{12}$ [Divide both sides by 12]

Exercise 21

Skill H

(a) Make x the subject of the formula $5z + 3x = xz + 3y$

(b) Make y the subject of the formula $5y - 3x = 2y + 3z$

(c) Make z the subject of the formula $\frac{4z-5y}{6-3z} = 6$

(d) Make f the subject of the formula $11f - 1 = 4g(3f + e)$

(e) Make f the subject of the formula $7f - 5 = 3g(2f + h)$

(f) Make k the subject of the formula $5(2k - m) = ck + 5$

(g) Make w the subject of the formula $8(w - 3y) = 3(w + 2y)$

(h) Make u the subject of the formula $\frac{8u+3y}{3-5u} = 3z$

(i) Make r the subject of the formula $5(r + 3t) = 7(2 - 6r)$

(j) Make v the subject of the formula $\frac{2}{3}v + 5w = \frac{1}{3}(3 - 2v)$

Example

(a) Make x the subject of the formula $z = \frac{1}{x} + \frac{1}{y}$.

Answer: $z = \frac{1}{x} + \frac{1}{y}$

$\frac{1}{x} + \frac{1}{y} = z$	[Swap sides]
$\frac{1}{x} = z - \frac{1}{y}$	[Subtract $\frac{1}{y}$ from both sides]
$x = \frac{1}{z - \frac{1}{y}}$	[Take reciprocal of both sides]

(b) Make x the subject of the formula $z = y + \sqrt{\frac{w}{x}}$.

Answer: $z = y + \sqrt{\frac{w}{x}}$

$y + \sqrt{\frac{w}{x}} = z$	[Swap sides]
$\sqrt{\frac{w}{x}} = z - y$	[Subtract y from both sides]
$\frac{w}{x} = (z-y)^2$	[Square both sides]
$w = x(z-y)^2$	[Multiply both sides by x]
$x(z-y)^2 = w$	[Swap sides]
$x = \frac{w}{(z-y)^2}$	[Divide by $(z-y)^2$]

Exercise 22 H

(a) Make x the subject of the formula $z = \frac{1}{x} - \frac{1}{y}$

(b) Make x the subject of the formula $z = \sqrt{\frac{w}{x}} - y$

(c) Make x the subject of the formula $z = \frac{y-w}{x}$

(d) Make x the subject of the formula $z = y + \sqrt[3]{\frac{x}{w}}$

(e) Make f the subject of the formula $\frac{a}{2f+1} = \frac{b}{3f-1}$

(f) Make p the subject of the formula $y = \frac{q}{p} - x$

(g) Make p the subject of the formula $q = \frac{y}{x^2-p}$

(h) Make x the subject of the formula $w = \sqrt{\frac{x-y}{x+y}}$

(i) Make h the subject of the formula $A = \pi r\sqrt{h^2 + r^2}$

(j) Make l the subject of the formula $t = 2\pi\sqrt{\frac{l}{g}}$

(k) Make b the subject of the formula $m = \frac{ax+by}{a+b}$

(l) Make y the subject of the formula $s = \sqrt{\frac{x^2+y^2}{n}}$

(m) Make q the subject of the formula $y = \frac{x-np}{\sqrt{npq}}$

(n) Make x the subject of the formula $F = \frac{x^2}{1-x^2}$

(o) Make x the subject of the formula $y = \frac{2}{x+3} - 5$

(p) Make a the subject of the formula $a - b = \frac{a+2}{b}$

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

The Quadratic Formula

Factorise using the splitting method or the detective method

Difference between two squares

It is possible to solve some quadratic equations by factorisation.

Example

Solve the following equations.

(a) $x^2+9x+14=0$

$x^2+9x+14=0$
$(x+2)(x+7)=0$
Either $x+2=0$ or $x+7=0$
$x=-2$ $x=-7$

(b) $3x^2+13x+4=0$

$3x^2+13x+4=0$
$(3x+1)(x+4)=0$
Either $3x+1=0$ or $x+4=0$
$3x=-1$ $x=-4$
$x=-\frac{1}{3}$

(c) $4x^2-9=0$

$4x^2-9=0$
$(2x+3)(2x-3)=0$
Either $2x+3=0$ or $2x-3=0$
$2x=-3$ $2x=3$
$x=-\frac{3}{2}$ $x=\frac{3}{2}$

Exercise 23

Revision

H

Solve the following equations.

(a) $x^2+10x+24=0$ (b) $2x^2+21x+40=0$ (c) $x^2-16=0$

(d) $x^2+3x-54=0$ (e) $2x^2+7x-15=0$ (f) $25x^2-36=0$

(g) $x^2-8x+16=0$ (h) $4x^2-14x+6=0$ (i) $8y^2-98=0$

If we cannot solve a quadratic equation through factorisation, then we can attempt to solve the equation using the **quadratic formula**.

THE QUADRATIC FORMULA

The solutions of $ax^2+bx+c=0$ where $a\neq 0$ are given by the quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The symbol $\pm$ means 'plus or minus'.

If $b^2-4ac<0$, there are no real solutions.

Round off your answers to 2 decimal places at the end.

@MATHEMATEG

Exercise 24

Skill

H

The general form of a quadratic equation is $ax^2 + bx + c = 0$.
Write a, b and c for each of the following quadratic equations.

(a) $x^2 + 3x + 7 = 0$

(b) $2x^2 - 8x + 11 = 0$

(c) $9x^2 - 4 = 0$

(d) $5x - 3x^2 + 17 = 0$

(e) $23 + 5x + 3x^2 = 0$

(f) $4x^2 - 3x = 0$

Example

Using the quadratic formula, solve the equation $5x^2 + 10x - 3 = 0$.

Answer: In this case, we have $a = 5$, $b = 10$ and $c = -3$.
Substituting these values into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

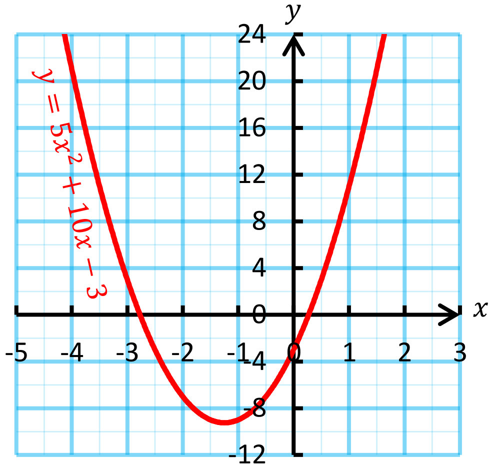
$$x = \frac{-10 \pm \sqrt{10^2 - 4 \times 5 \times -3}}{2 \times 5}$$

$$x = \frac{-10 \pm \sqrt{100 + 60}}{10}$$

$$x = \frac{-10 \pm \sqrt{160}}{10}$$

Either $x = \frac{-10 + \sqrt{160}}{10}$ or $x = \frac{-10 - \sqrt{160}}{10}$

Either $x = 0.26$ to 2 d.p., or $x = -2.26$ to 2 d.p.



Exercise 25

H

Using the quadratic formula, solve the following equations.
Round off your answers correct to two decimal places.
If there are no real solutions, state this.

(a) $x^2 + 8x + 5 = 0$

(b) $x^2 - 8x + 5 = 0$

(c) $x^2 + 8x - 5 = 0$

(d) $x^2 - 8x - 5 = 0$

(e) $x^2 + 8x + 25 = 0$

(f) $2x^2 + 10x + 5 = 0$

(g) $3x^2 + 8x + 1 = 0$

(h) $3x^2 - 6x - 4 = 0$

(i) $2x^2 - 9x + 8 = 0$

(j) $2x^2 + x - 20 = 0$

(k) $7x^2 + 3x - 1 = 0$

(l) $x^2 - 2x - 100 = 0$

(m) $5x^2 + 11x + 3 = 0$

(n) $4x^2 + 7x + 6 = 0$

(o) $4x^2 - 12 + 3x = 0$

Exercise 26

Applying

H

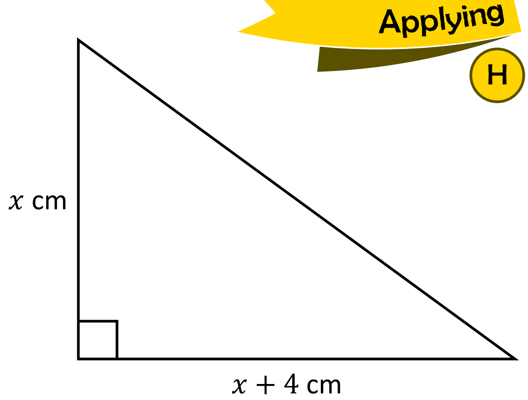
The area of the right-angled triangle shown on the right is 45 cm².

(a) Show that x satisfies the equation $x^2 + 4x - 90 = 0$.

(b) Use the quadratic formula to solve the equation $x^2 + 4x - 90 = 0$, giving your answers correct to two decimal places.

(c) Using your answer to (b), find the base and height of the triangle.

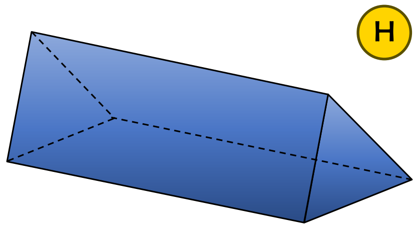
(d) Calculate the length of the hypoteuse of the triangle.
Give your answer correct to one decimal place.



Challenge!

Solve the equation $5x^3 + 2x^2 - 11x = 0$

Exercise 27



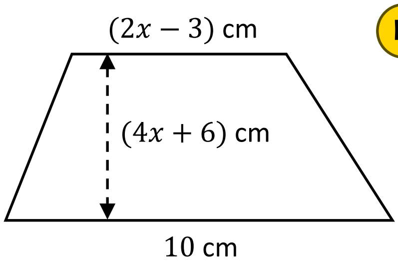
The diagram shows a triangular prism.

The cross-sectional area of the triangular prism is $3x$ cm² and its length is $(x+4)$ cm. The volume of the prism is 89 cm³.

(a) Show that x satisfies the equation $3x^2 + 12x - 89 = 0$.

(b) Use the formula method to solve the equation $3x^2 + 12x - 89 = 0$, giving your answer correct to one decimal place. Hence, write the length of the prism correct to one decimal place.

Exercise 28



The diagram shows a trapezium.

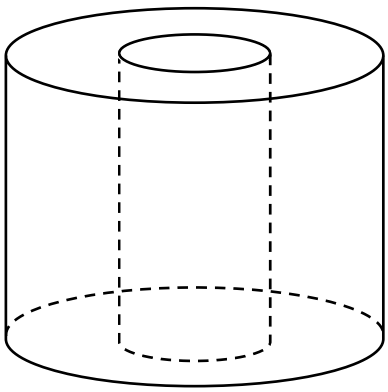
The lengths of the parallel sides of the trapezium are 10 cm and $(2x-3)$ cm. The height of the trapezium is $(4x+6)$ cm and its area is 70 cm².

(a) Show that $4x^2 + 20x - 49 = 0$.

(b) Use the quadratic formula to solve the equation $4x^2 + 20x - 49 = 0$.
Give your answers correct to two decimal places.
Hence, write the height of the trapezium correct to two decimal places.

Exercise 29

A length of plastic tube has a uniform circular cross-section.
The radius of the circular hole in the middle is x cm.
The thickness of the plastic is 3 cm and the length of the plastic tube is $5x$ cm.



(a) Show that the volume of the plastic used to make the tube is $(30\pi x^2 + 45\pi x)$ cm³.

(b) Given that the volume of plastic used to make the tube is 88π cm³, find the length of the tube correct to one decimal place.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ❑ Target ❑

Algebraic Fractions

Numerical Fractions

In order to be able to work with fractions that include algbebraic expressions, it is a good idea to first revise how to work with numerical fractions.

Example

Here are two methods of calculating $\frac{3}{4}+\frac{1}{6}$.

The Traditional Method

Step 1: Find the **lowest common denominator** of the two fractions. Here, the lowest common multiple of 4 and 6 is 12.

Step 2: Write **equivalent fractions** for $\frac{3}{4}$ and $\frac{1}{6}$, using 12 as the common denominator.

$$\frac{3}{4} \xrightarrow{\times 3} \frac{9}{12} \qquad \frac{1}{6} \xrightarrow{\times 2} \frac{2}{12}$$

Step 3: **Add** the two new fractions.

$$\frac{9}{12}+\frac{2}{12}=\frac{11}{12}$$

Step 4: Check to see if it is possible to simplify the answer. (It is not possible in this case.)

Step 1: You can find a common denominator by multiplying the two original denominators, however this might not be the ***lowest*** *common denominator.*

The Peanut Method

Step 1: Draw the following template. (Note that the first fraction goes on the top and the second fraction goes on the left.)

	3	4
1	×	
6		

Step 2: Fill in the gaps in the table by **multiplying** the numbers.

	3	4
1	×	4
6	18	24

Step 3: **Add** the two numbers that form the peanut shape.

	3	4
1	×	4
6	18	24

18 + 4 = 22

The answer is $\frac{22}{24}$.

Step 4: Check to see if it is possible to simplify the answer. Here, $\frac{22}{24}$ simplifies to give the final answer $\frac{11}{12}$.

Revision

F

Exercise 30

Calculate the following. Give your answers in their simplest form.

(a) $\frac{2}{5}+\frac{1}{3}$ (b) $\frac{1}{4}+\frac{3}{8}$ (c) $\frac{9}{10}-\frac{3}{5}$ (d) $\frac{3}{7}+\frac{1}{6}$

(e) $\frac{7}{8}-\frac{2}{5}$ (f) $\frac{2}{9}+\frac{2}{3}$ (g) $\frac{11}{12}-\frac{1}{2}$ (h) $2\frac{1}{3}+4\frac{3}{4}$

Algebraic Fractions

Algebraic fractions contain at least one numerator or denominator that is an algebraic expression.

Examples of numerical fractions	$\frac{3}{4}$	$\frac{4}{9}$	$\frac{12}{5}$	$-\frac{4}{5}$
Examples of algebraic fractions	$\frac{x+2}{3}$	$\frac{5}{x-2}$	$\frac{x+7}{x^2+x-1}$	$-\frac{1}{y}$

Example

Here are two methods of writing $\frac{2}{3x+4}+\frac{3}{x+2}$ as a single fraction in its simplest form.

The traditional method

Step 1: Find the **lowest common denominator** of $3x+4$ and $x+2$. Here, the smallest expression that is a multiple of both $3x+4$ and $x+2$ is the product $(3x+4)(x+2)$.

Step 2: Write **equivalent fractions** for $\frac{2}{3x+4}$ and $\frac{3}{x+2}$, with $(3x+4)(x+2)$ appearing in the denominator.

$$\frac{2}{3x+4} \xrightarrow{\times (x+2)} \frac{2(x+2)}{(3x+4)(x+2)}$$

$$\frac{3}{x+2} \xrightarrow{\times (3x+4)} \frac{3(3x+4)}{(3x+4)(x+2)}$$

Step 3: **Add** the two new fractions.

$$\begin{aligned} &\frac{2(x+2)}{(3x+4)(x+2)}+\frac{3(3x+4)}{(3x+4)(x+2)} \\ =&\frac{2(x+2)+3(3x+4)}{(3x+4)(x+2)} \\ =&\frac{2x+4+9x+12}{(3x+4)(x+2)} \\ =&\frac{11x+16}{(3x+4)(x+2)} \end{aligned}$$

Step 4: Check to see if it is possible to **simplify** the answer, e.g. through factorising. (This is not possible in this case.)

The peanut method

Step 1: Draw the following template. (Note that the first fraction goes on the top and the second fraction goes on the left.)

	2	$3x+4$
3	$\times$	
$x+2$		

Step 2: Fill in the gaps in the table by **multiplying** the expressions.

	2	$3x+4$
3	$\times$	$3(3x+4)$
$x+2$	$2(x+2)$	$(x+2)(3x+4)$

Cam 3: **Add** the two expressions that form the peanut shape.

$$\begin{aligned} &2(x+2)+3(3x+4) \\ =&2x+4+9x+12 \\ =&11x+16 \end{aligned}$$

The answer is $\frac{11x+16}{(3x+4)(x+2)}$

Step 4: Check to see if it is possible to **simplify** the answer, e.g. through factorising. (This is not possible in this case.)

Exercise 31

Write each of the following as a single fraction. Give your answers in their simplest form.

(a) $\frac{5}{3x+2}+\frac{4}{x+1}$ (b) $\frac{7}{2x+3}+\frac{6}{x+4}$ (c) $\frac{1}{x+5}+\frac{2}{2x+7}$

(d) $\frac{5}{3x+2}+\frac{4}{x-1}$ (e) $\frac{7}{2x-3}+\frac{6}{x+4}$ (f) $\frac{1}{x+5}-\frac{2}{2x+7}$

(g) $\frac{5}{4x+2}-\frac{3}{x-8}$ (h) $\frac{3}{x-5}-\frac{4}{x+2}$ (i) $\frac{9}{x-3}+\frac{6}{2x-5}$

(j) $\frac{x+1}{2}+\frac{x+4}{3}$ (k) $\frac{x+3}{5}-\frac{x+1}{2}$ (l) $\frac{x-2}{4}+\frac{x-5}{3}$

(m) $\frac{3x+1}{4}+\frac{2x-5}{2}$ (n) $\frac{x+2}{3}+\frac{2}{x-1}$ (o) $\frac{1}{x-2}+\frac{x+3}{4}$

(p) $\frac{2x+5}{3}-\frac{2}{x-5}$ (q) $\frac{3x+2}{4x-1}+\frac{2}{7}$ (r) $\frac{5+3x}{2x-3}+\frac{1}{4}$

(s) $\frac{2}{x+3}+\frac{x+1}{x}$ (t) $\frac{x+2}{3x}+\frac{x}{x+3}$ (u) $\frac{x+3}{x-4}+\frac{x-3}{x+4}$

(v) $\frac{2x+3}{x-3}-\frac{x-2}{x-5}$ (w) $\frac{x+2}{3x-4}+\frac{x-3}{x+2}$ (x) $\frac{2}{2x+1}+\frac{3x+5}{x+2}$

(y) $\frac{4x+17}{x+3}-\frac{2x-15}{x-3}$ (z) $\frac{2x+3}{x-1}+\frac{4-x}{3x-5}$ (α) $\frac{3x-4}{x+1}-\frac{x+2}{5x+3}$

Solving Equations involving Algebraic Fractions

One fraction equal to another fraction

Technique: Multiply by each denominator in turn.

Example

Solve $\frac{4x}{3x+1} = \frac{3}{x+5}$.

Answer: $\frac{4x}{3x+1} = \frac{3}{x+5}$

$\frac{4x(x+5)}{3x+1} = 3$ [Multiply both sides by $x+5$]

$4x(x+5) = 3(3x+1)$ [Multiply both sides by $3x+1$]

$4x^2 + 20x = 9x + 3$ [Expand the brackets]

$4x^2 + 11x - 3 = 0$ [Subtract $9x$ from both sides; subtract 3 from both sides]

$(4x-1)(x+3) = 0$ [Factorise]

Either $4x - 1 = 0$ or $x + 3 = 0$ [Solve]

$4x = 1$ $\quad x = -3$

$x = \frac{1}{4}$

Exercise 32 H

Solve the following equations.

(a) $\frac{x+4}{x-1} = \frac{x}{x-3}$ (b) $\frac{6}{x-4} = \frac{5}{x-3}$ (c) $\frac{1}{2x+3} = \frac{1}{3x-2}$

(d) $\frac{3}{2(2x-1)} = \frac{4}{3x+2}$ (e) $\frac{x}{4x+3} = \frac{1}{2x+9}$ (f) $\frac{3x}{4-x} = \frac{2}{x-4}$

(g) $\frac{3x}{x+1} - \frac{x+4}{3} = 0$ (h) $\frac{1}{3x-2} - \frac{2x+3}{x-4} = 0$ (i) $\frac{2}{3x+1} - \frac{5}{x+3} = 0$

Fractions where all the denominators are numbers

Technique: Multiply by the lowest common denominator of the fractions.

Example

Solve $\frac{2x+3}{6} + \frac{x-1}{3} = \frac{7}{12}$.

Answer: $\frac{2x+3}{6} + \frac{x-1}{3} = \frac{7}{12}$

$12\left(\frac{2x+3}{6}\right) + 12\left(\frac{x-1}{3}\right) = 12\left(\frac{7}{12}\right)$ [Multiply each term by 12]

$2(2x+3) + 4(x-1) = 7$ [Simplify]

$4x + 6 + 4x - 4 = 7$ [Expand the brackets]

$8x + 2 = 7$ [Collect like terms]

$8x = 5$ [Subtract 2 from both sides]

$x = \frac{5}{8}$ [Divide both sides by 8]

Exercise 33 H

Solve the following equations.

(a) $\frac{x+3}{6} + \frac{2x-5}{3} = \frac{2}{9}$ (b) $\frac{x-2}{5} - \frac{2x+5}{4} = \frac{1}{4}$ (c) $\frac{3-x}{4} + \frac{2x+5}{3} = 1$

(d) $\frac{2x-1}{3} + \frac{x-2}{6} = \frac{3x}{4}$ (e) $\frac{x+2}{3} - \frac{x-3}{2} = 2$ (f) $\frac{2x-3}{4} - \frac{x+1}{3} = \frac{3}{4}$

(g) $\frac{2x+5}{4} + \frac{x-4}{6} = \frac{x}{3}$ (h) $\frac{3-x}{8} + \frac{3}{16} = \frac{3x+2}{4}$ (i) $\frac{3x+5}{10} - \frac{2-x}{25} = \frac{3}{5}$

Example

Technique: Multiply by the lowest common denominator of the fractions.

Solve the equation $\frac{2x}{x+3}+\frac{3x+1}{2x-1}=3$.

Answer: $\frac{2x}{x+3}+\frac{3x+1}{2x-1}=3$

$(x+3)(2x-1)\times\frac{2x}{x+3}+(x+3)(2x-1)\times\frac{3x+1}{2x-1}=3(x+3)(2x-1)$ [Multiply by $(x+3)(2x-1)$]

$(2x-1)2x+(x+3)(3x+1)=3(x+3)(2x-1)$ [Simplify]

$4x^2-2x+(3x^2+x+9x+3)=3(2x^2-x+6x-3)$ [Expand the brackets]

$4x^2-2x+(3x^2+10x+3)=3(2x^2+5x-3)$ [Collect like terms]

$7x^2+8x+3=6x^2+15x-9$ [Collect like terms; expand]

$x^2-7x+12=0$ [Subtract $6x^2+15x-9$]

$(x-3)(x-4)=0$ [Factorise]

Either $x-3=0$ or $x-4=0$ [Solve]

$x=3$ $\quad$ $x=4$

Exercise 34

H

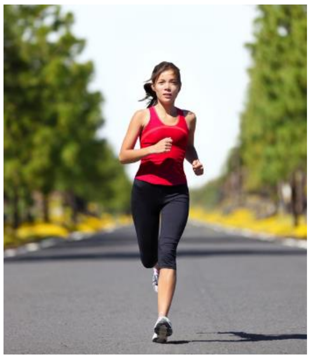
Solve the following equations.

(a) $\frac{2}{2x+3}+\frac{1}{x+2}=3$ (b) $\frac{3x}{x+4}+\frac{2x}{5x-2}=\frac{3}{2}$ (c) $\frac{2x}{x-3}-\frac{x}{x-2}=6$

(d) $\frac{6x}{3x-1}+\frac{15}{2x+3}=5$ (e) $\frac{4x}{5x-2}+\frac{3}{3x+1}=3$ (f) $\frac{x+3}{x+1}+\frac{3}{x-3}=2$

(g) $\frac{8x}{4x-3}+\frac{20}{3x+2}=10$ (h) $\frac{2x}{x-5}+\frac{x-1}{3x}=2$ (i) $\frac{2}{x}+\frac{1}{x+1}=5$

Challenge!



Ava runs a distance of 26 miles at an average speed of x mph.
Delyth runs the same distance at an average speed that is 2 mph slower than Ava.
The difference between their times is exactly 1 hour.

(a) Show that x satisfies the equation $x^2-2x-52=0$.

(b) Use the quadratic formula to find Ava's speed.
Give your answer correct to 2 decimal places.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Developing Algebra 4

Reflection

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I am familiar with using **function notation**, e.g. $f(x) = 3x^2 - 4x + 5$.			1	
I can **transform the graphs of functions** using the transformations $y = f(x) + a$, $y = f(x + a)$, $y = af(x)$, $y = f(ax)$, $y = -f(x)$ and $y = f(-x)$.			1	
I know how to draw a **tangent to a function** and **measure its gradient**.			2	
I know and can use the definitions for **velocity** and **acceleration**.			2	
I know how to find the **area between a graph and the x–axis** using the **trapezium rule**.			2	
I know when the area between a graph and the x–axis has a **special meaning**.			2	
I can change the subject of a formula when the **subject appears more than once** in the formula.			3	
I can change the subject of a formula in questions involving **fractions** and **roots**.			4, 5	
I can use the **quadratic formula** to solve quadratic equations.			6, 8	
I can **combine two algebraic fractions** to give a single fraction in its simplest form.			7	
I can **solve equations that include algebraic fractions**.			8, 9, 10	

Am I ready for the test?
Tick the boxes below...

- [ ] I have revised the work in my mathematics book.
- [ ] I have revised the workbook.
- [ ] I have watched the relevant videos on YouTube.
- [ ] I have completed the Diagnostic Questions quiz.
- [ ] I have completed at least 4 pages in my revision book.

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The Mathematics Department

11

Measuring Shapes 5

Name:

Contents

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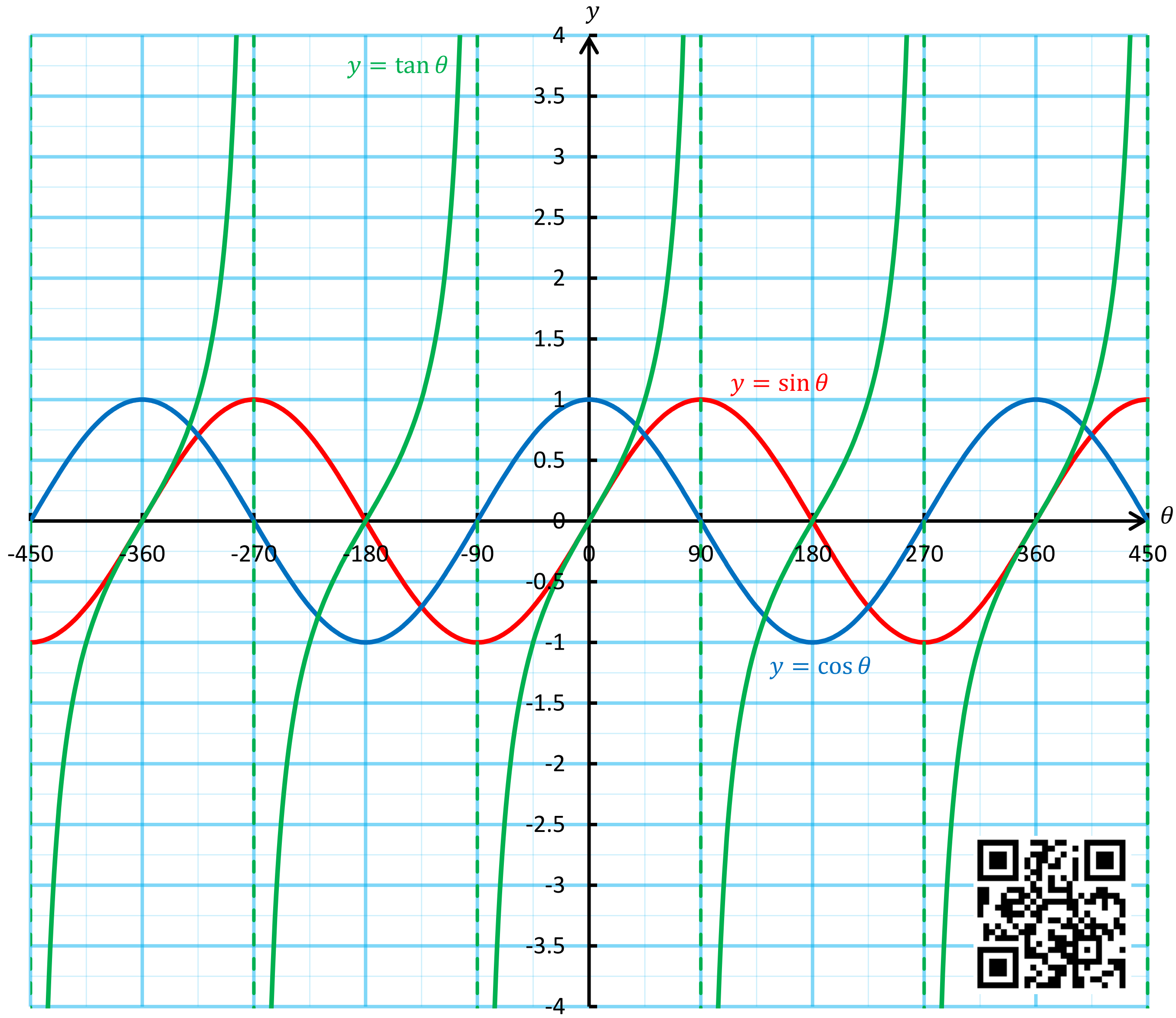
The Sine Rule, The Cosine Rule	**The Sine Rule. The Cosine Rule. Sine Rule or Cosine Rule?**	6
Area of a Triangle	**Calculating the area of a triangle.**	14
Trigonometric Graphs	**The unit circle. Trigonometric graphs. Solving trigonometric equations. Function transformations.**	17



3–D Trigonometry

2–D Trigonometry (Revision)

In mathematics, we worship the King SOHCAHTOA.

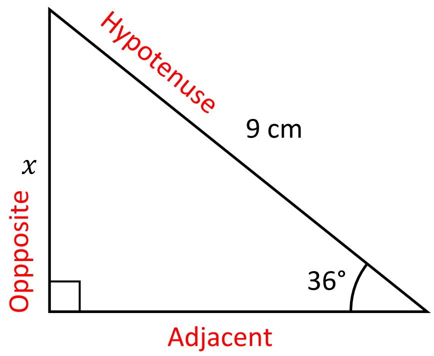
$$S\frac{O}{H}\quad C\frac{A}{H}\quad T\frac{O}{A}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

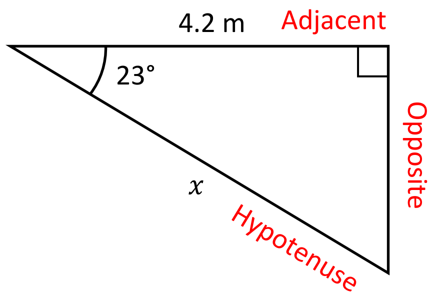
Example

Calculate the missing side x or the missing angle θ. (The diagrams are not drawn to scale.)

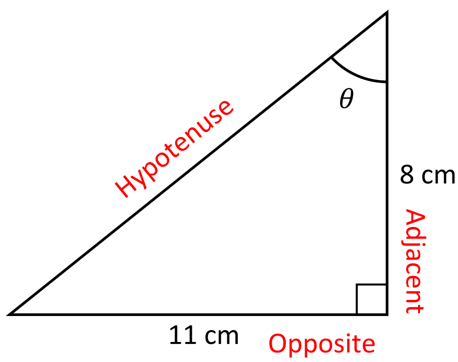
(a)



(b)



(c)



(a) *Answer:* Label the sides in red.

We know the hypotenuse, and need to find the opposite: choose

$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$

$\sin 36° = \frac{x}{9}$

$x = 9 \times \sin 36°$

$x = 5.29$ cm to 2 decimal places.

(b) *Answer:* Label the sides in red.

We know the adjacent, and need to find the hypotenuse: choose

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\cos 23° = \frac{4.2}{x}$

$x = 4.2 \div \cos 23°$

$x = 4.56$ m to 2 decimal places.

(c) *Answer:* Label the sides in red.

We know the opposite and adjacent: choose

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

$\tan\theta = \frac{11}{8}$

$\theta = \tan^{-1}\left(\frac{11}{8}\right)$

$\theta = 53.97°$ to 2 decimal places.

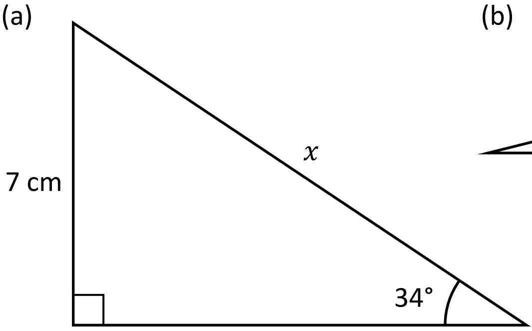
x on **top** of the fraction leads to a **multiplication** sum.

x at the **botton** of the fraction leads to a **division** sum.

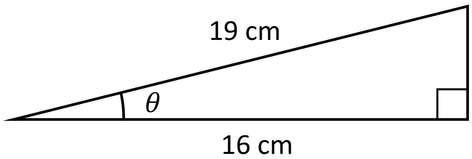
Exercise 1 (Revision)

Calculate the missing side x or the missing angle θ. (The diagrams are not drawn to scale.)

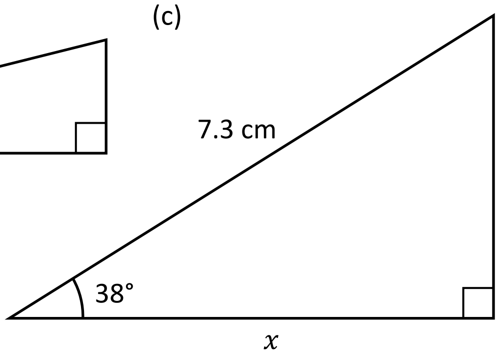
(a)



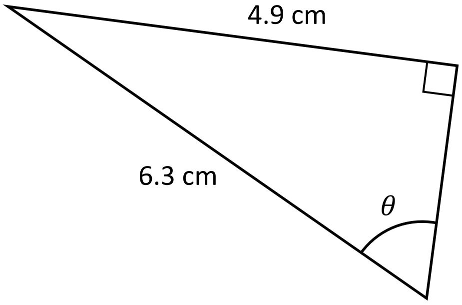
(b)



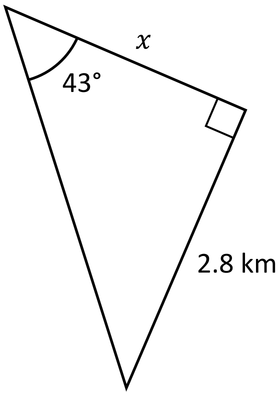
(c)



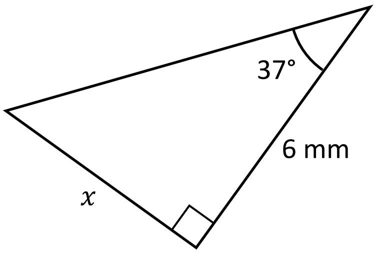
(d)



(e)



(f)

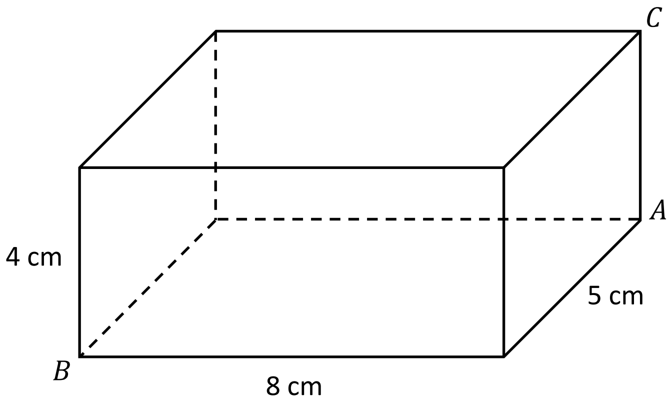


3–D Trigonometry

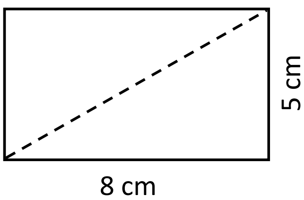
It is possible to use trigonometry to find missing sides and angles in three dimensional shapes.

Example

For the cuboid shown on the right, calculate the size of the angle $A\hat{B}C$.



Answer: To start, let us use Pythagoras' Theorem to calculate the length of the diagonal on the base of the cuboid, which is the diagonal of this rectangle:

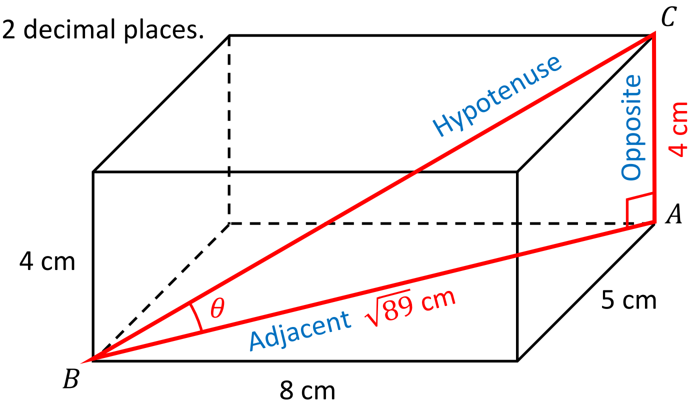


a^2
b^2
c^2

$$\begin{aligned} 5^2 &= 25 \\ 8^2 &= +\,64 \\ & \,\,\overline{89} \end{aligned}$$

$\sqrt{89} = 9.43$ cm to 2 decimal places.

Next, we need to consider the red right-angled triangle shown on the right.



$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

$\tan\theta = \frac{4}{\sqrt{89}}$

$\theta = \tan^{-1}\left(\frac{4}{\sqrt{89}}\right)$

$\theta = 22.98°$ to 2 decimal places.

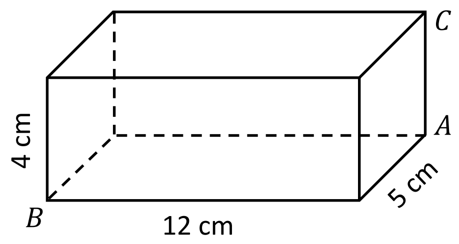
Exercise 2

Skill

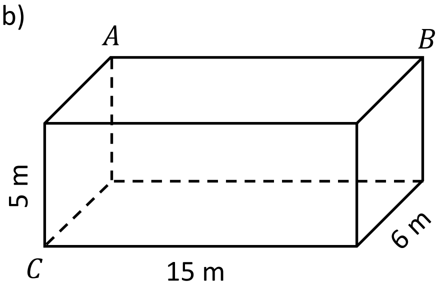
H

For the following cuboids, calculate the size of the angle $A\hat{B}C$.

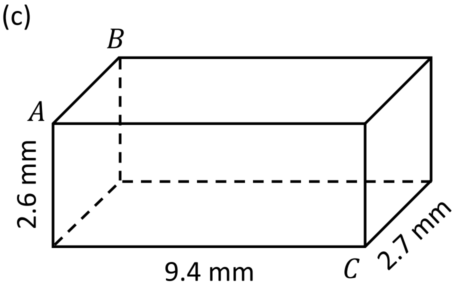
(a)



(b)



(c)

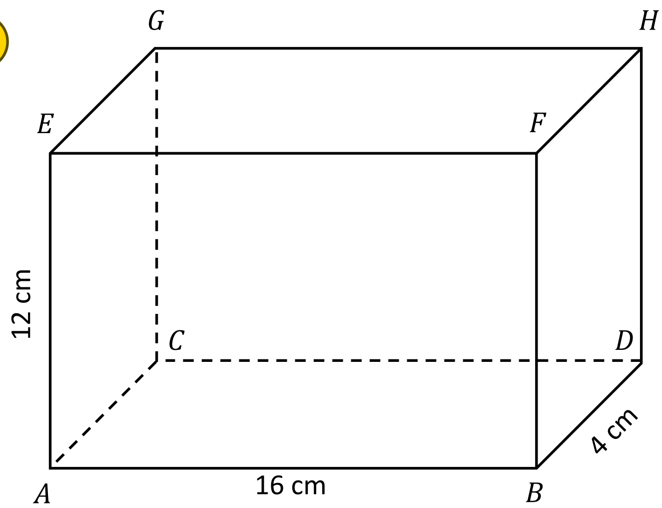


Exercise 3

H

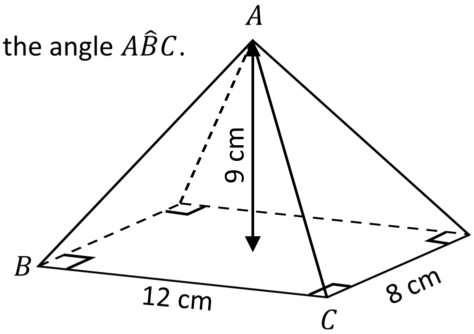
The diagram on the right shows a cuboid.
Calculate the size of the following angles.

(a) $A\hat{B}C$
(b) $A\hat{E}C$
(c) $A\hat{F}E$
(d) $G\hat{A}B$
(e) $D\hat{A}H$
(f) $G\hat{D}E$

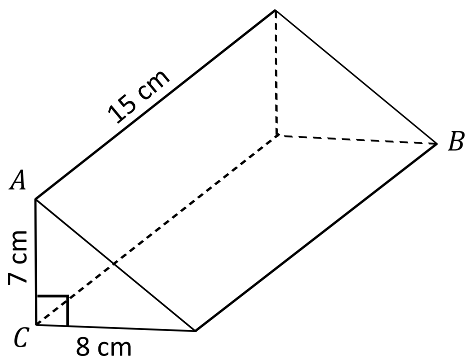


Exercise 4

(a) Calculate the size of the angle $A\hat{B}C$.



(b) Calculate the size of the angle $A\hat{B}C$.



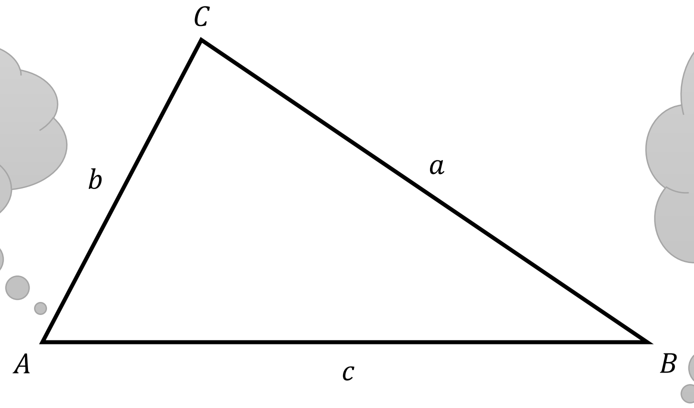
Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

The Sine Rule, The Cosine Rule

The **Sine Rule** and the **Cosine Rule** are used to calculate the size of angles and sides in triangles that are not right-angled triangles.

A general triangle with sides a, b, c and angles A, B, C.



The formulae for *sides* are given on page 2 of a GCSE examination paper.

Sine Rule for finding sides:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Sine Rule for finding angles:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ [Take the reciprocal]

Cosine Rule for finding sides:

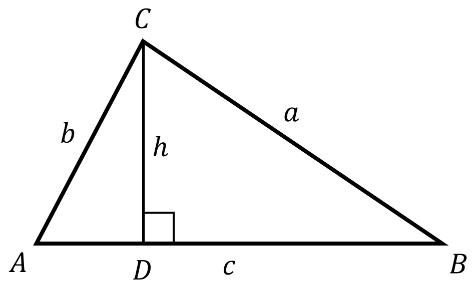
$$a^2 = b^2 + c^2 - 2bc\cos A$$

Cosine Rule for finding angles:

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$ [Re-arrange the formula]

Proof (for triangles without an obtuse angle[1])

Draw the perpendicular from C to the base AB.



Using the triangle CDB, $\sin B = \frac{h}{a}$ so that $h = a\sin B$.

Using the triangle CAD, $\sin A = \frac{h}{b}$ so that $h = b\sin A$.

Using the two expressions for the height of the triangle h, we have $a\sin B = b\sin A$.

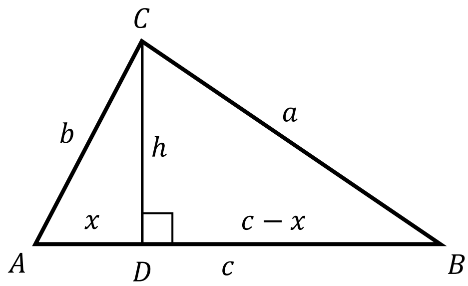
Therefore, $\frac{a}{\sin A} = \frac{b}{\sin B}$ [Divide by $\sin A$ and $\sin B$]

It would be possible to repeat the above ("Draw the perpendicular from A to the base BC...") to obtain $\frac{b}{\sin B} = \frac{c}{\sin C}$. We can combine the two formulae that use fractions to obtain the Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof (for triangles without an obtuse angle[1])

Draw the perpendicular from C to the base AB.



Pythagoras' Theorem for triangle BCD:
$a^2 = (c-x)^2 + h^2$ so that $h^2 = a^2 - (c-x)^2$.

Pythagoras' Theorem for triangle ACD:
$b^2 = x^2 + h^2$ so that $h^2 = b^2 - x^2$.

Using the two expressions for h^2:

$$a^2 - (c-x)^2 = b^2 - x^2$$
$$a^2 - (c-x)(c-x) = b^2 - x^2$$
$$a^2 - (c^2 - cx - cx + x^2) = b^2 - x^2$$
$$a^2 - c^2 + 2cx \cancel{-x^2} = b^2 \cancel{-x^2}$$
$$a^2 = b^2 + c^2 - 2cx$$

Using the triangle ACD, $\cos A = \frac{x}{b}$ so that $x = b\cos A$.

So, $a^2 = b^2 + c^2 - 2c(b\cos A)$ which gives the Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

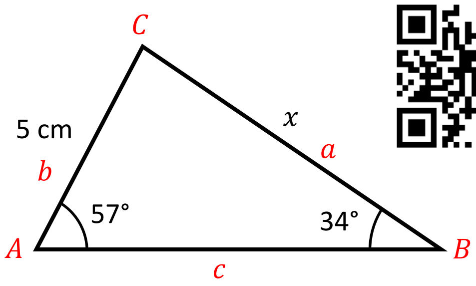
[1] Search on the internet for a proof in the case of a triangle that includes an obtuse angle.

The Sine Rule

Example

Calculate the missing side x or the missing angle θ. (The diagrams are not drawn to scale.)

(a)



Answer: To start, label the angles and then the corresponding sides. Becasuse we want to find the length of the side x, we write the Sine Rule for finding sides:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

We do not know the length of the side c nor the size of the angle C, so we cross out this fraction from the formula:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \cancel{\frac{c}{\sin C}}$$

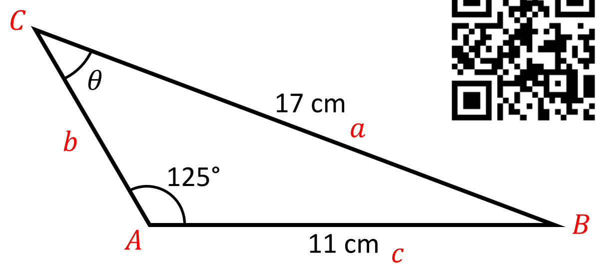
By substituting into the two fractions that ae left, we obtain

$$\frac{x}{\sin 57°} = \frac{5}{\sin 34°}$$

We can solve this equation by multiplying each side by $\sin 57°$:

$$x = \left(\frac{5}{\sin 34°}\right) \times \sin 57°$$
$x =$ 7.50 cm to 2 decimal places.

(b)



Answer: To start, label the angles and then the corresponding sides. Because we want to find the size of the angle θ, we write the Sine Rule for finding angles:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

We do not know the length of the side b nor the size of the angle B, so we cross out this fraction from the formula:

$$\frac{\sin A}{a} = \cancel{\frac{\sin B}{b}} = \frac{\sin C}{c}$$

By substituting into the two fractions that ae left, we obtain

$$\frac{\sin 125°}{17} = \frac{\sin \theta}{11}$$

We can solve this equation by multiplying each side by 11:

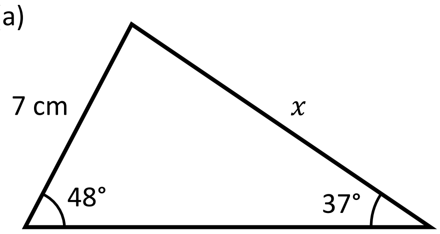
$$\sin\theta = \left(\frac{\sin 125°}{17}\right) \times 11$$
$$\theta = \sin^{-1}\left(\left(\frac{\sin 125°}{17}\right) \times 11\right)$$
$\theta =$ 32.01° to 2 decimal places.

Skill H

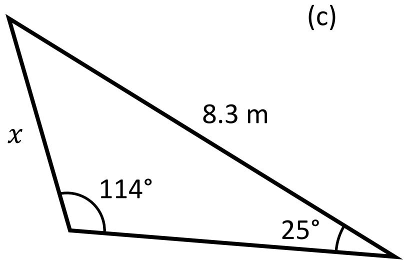
Exercise 5

Calculate the length of the missing side x. (The diagrams are not drawn to scale.)

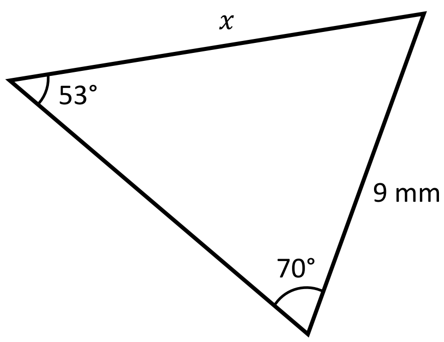
(a)



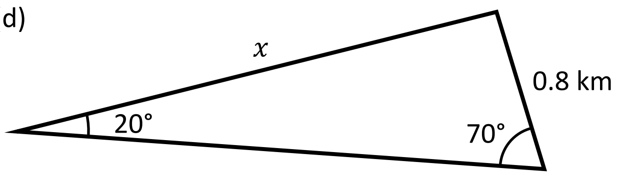
(b)



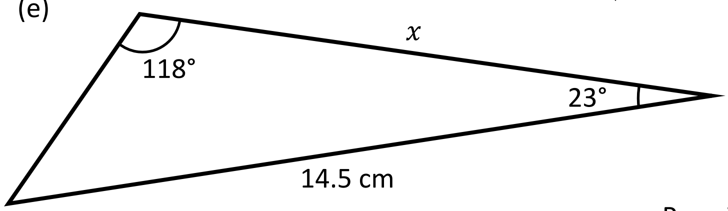
(c)



(d)



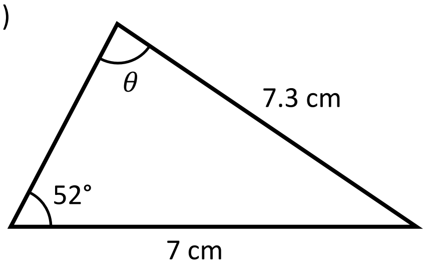
(e)



Exercise 6 H

Calculate the size of the missing angle θ. (The diagrams are not drawn to scale.)

(a)



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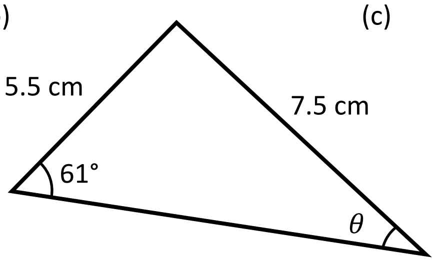
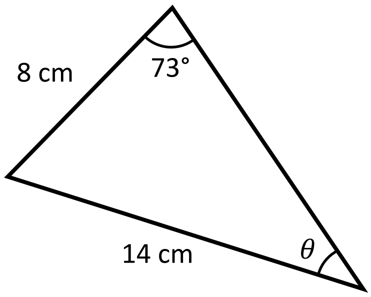


(c)



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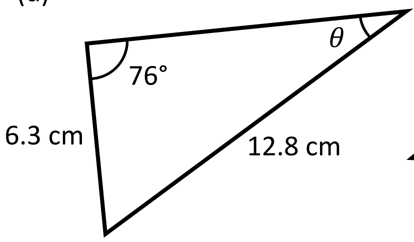
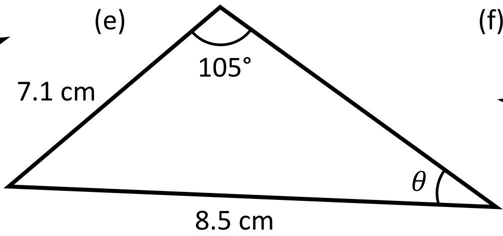
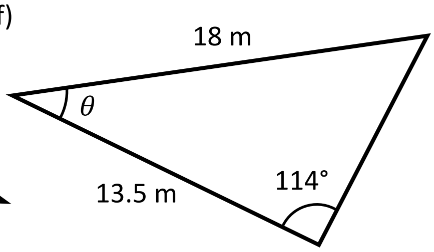


(e)



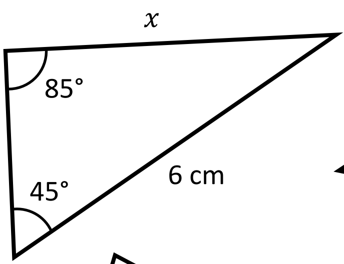
(f)



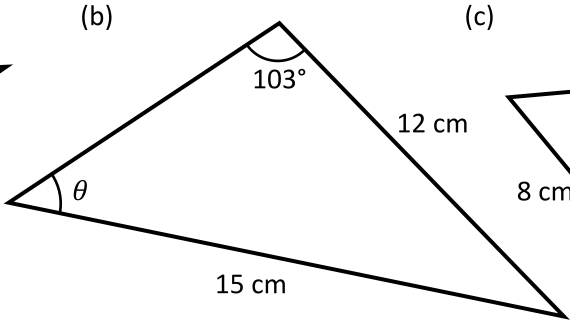
Exercise 7 H

Calculate the missing side x or the missing angle θ. (The diagrams are not drawn to scale.)

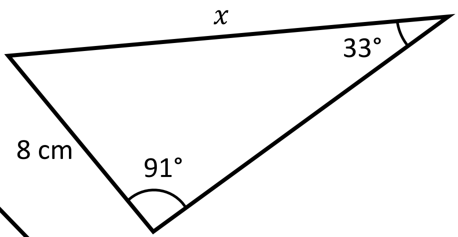
(a)



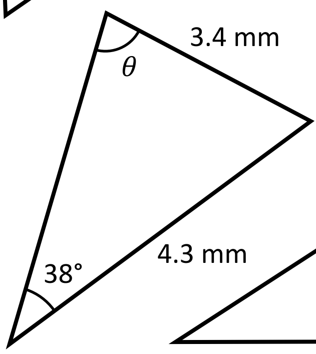
(b)



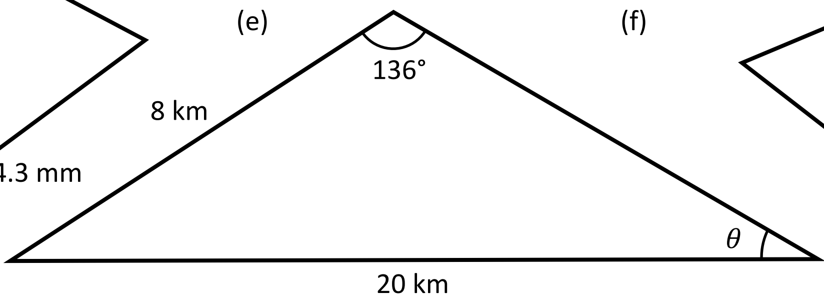
(c)



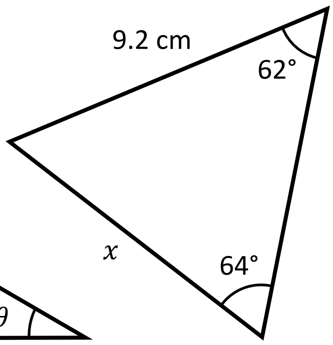
(d)



(e)



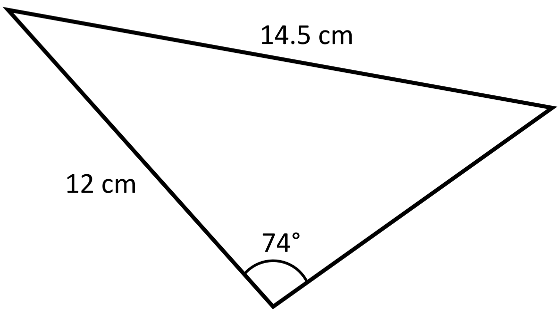
(f)



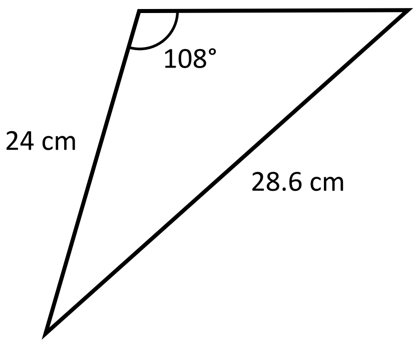
Exercise 8 H

Calculate the size of each of the missing sides and angles in the following diagrams.
(The diagrams are not drawn to scale.)

(a)



(b)

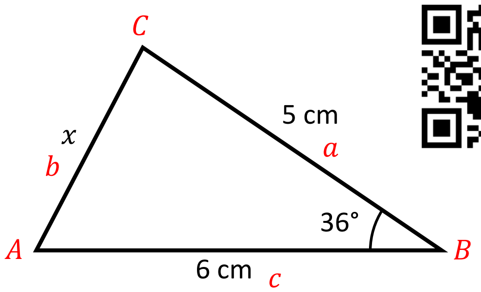


The Cosine Rule

Example

Calculate the missing side x or the missing angle θ. (The diagrams are not drawn to scale.)

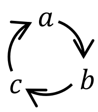
(a)



Answer: To start, label the angles and then the corresponding sides. Because we want to find the length of the side x, we write the Cosine Rule for finding sides:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

The formula does not fit the labels we have chosen, so we change the variables in the Cosine Rule by cycling around the following circle once.

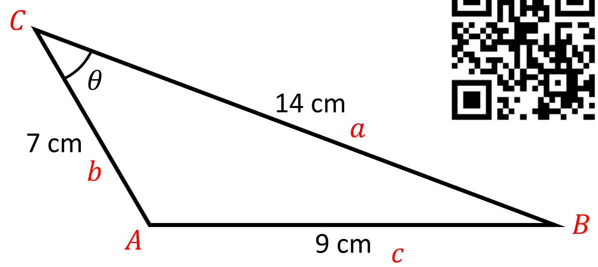


$$b^2 = c^2 + a^2 - 2ca\cos B$$

We can now substitute in the values from the triangle:

$$x^2 = 6^2 + 5^2 - 2 \times 6 \times 5 \times \cos 36°$$
$$x = \sqrt{12.45898034}$$
$$x = 3.53 \text{ cm to 2 decimal places.}$$

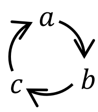
(b)



Answer: To start, label the angles and then the corresponding sides. Because we want to find the size of the angle θ, we write the Cosine Rule for finding angles:

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

The formula does not fit the labels we have chosen, so we change the variables in the Cosine Rule by cycling around the following circle twice.



$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

We can now substitute in the values from the triangle:

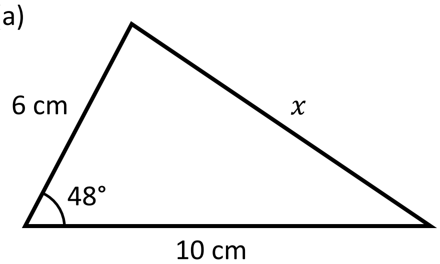
$$\cos\theta = \frac{14^2+7^2-9^2}{2\times14\times7}$$
$$\theta = \cos^{-1}\left(\frac{41}{49}\right)$$
$$\theta = 33.20° \text{ to 2 decimal places.}$$

Exercise 9

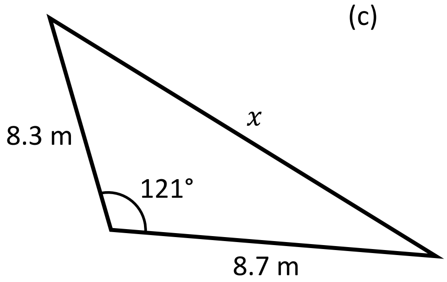
H

Calculate the length of the missing side x. (The diagrams are not drawn to scale.)

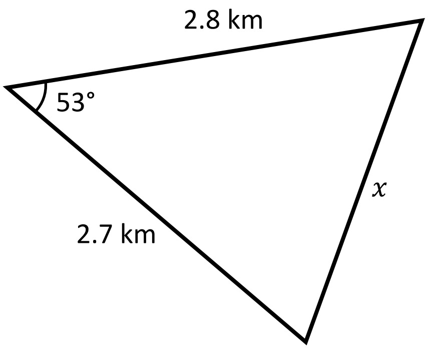
(a)



(b)



(c)



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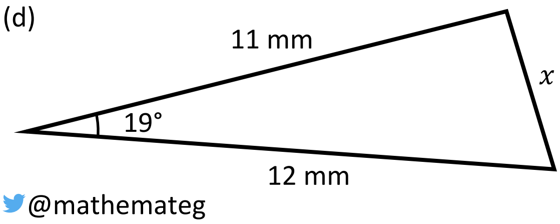
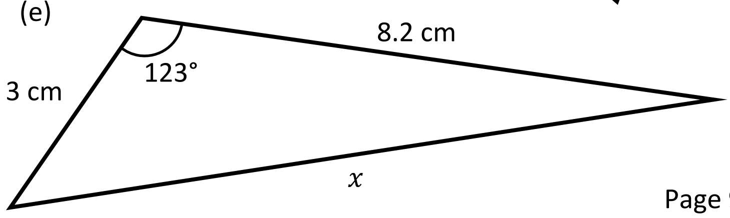


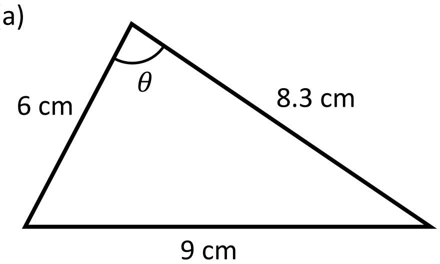
(e)



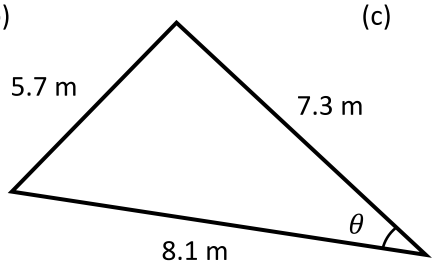
Exercise 10 H

Calculate the size of the missing angle θ. (The diagrams are not drawn to scale.)

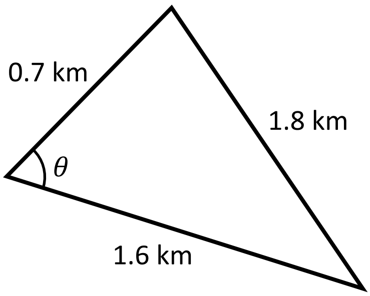
(a)



(b)



(c)



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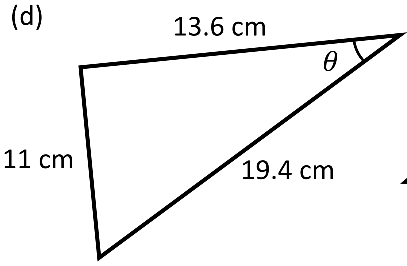
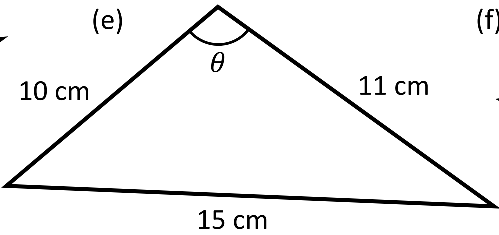


(e)



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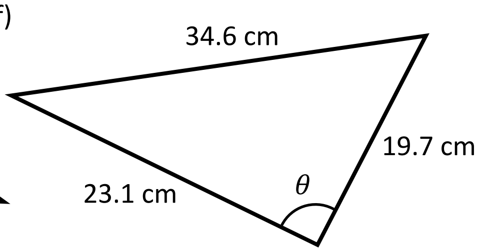
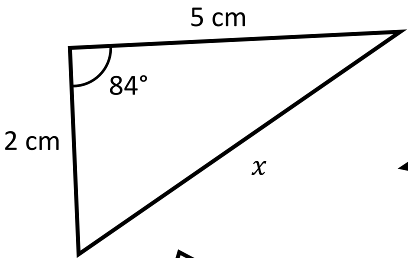


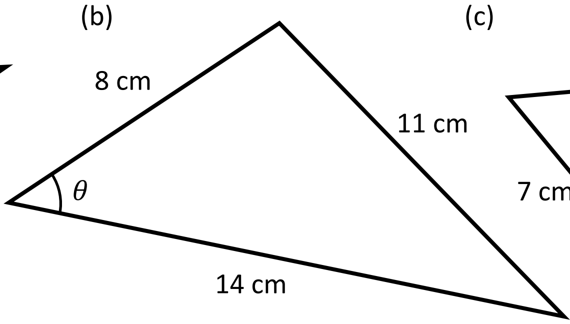
Exercise 11 H

Calculate the missing side x or the missing angle θ. (The diagrams are not drawn to scale.)

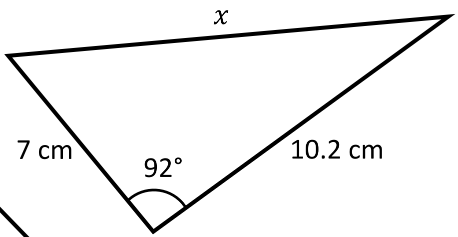
(a)



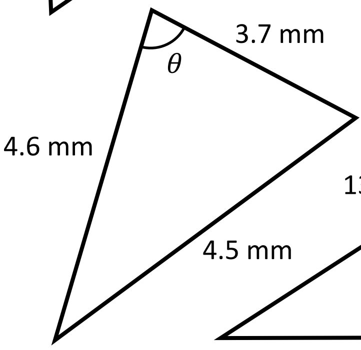
(b)



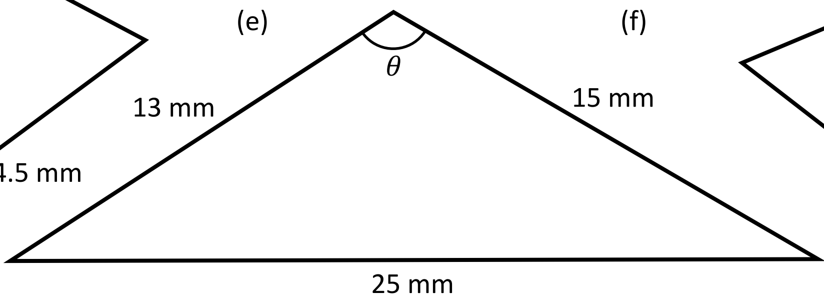
(c)



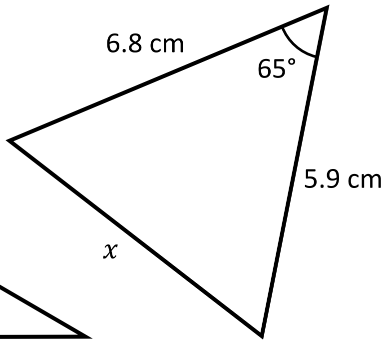
(d)



(e)



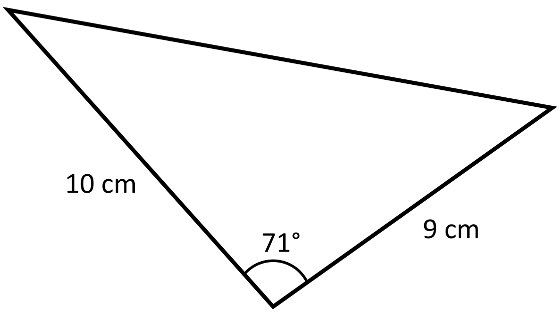
(f)



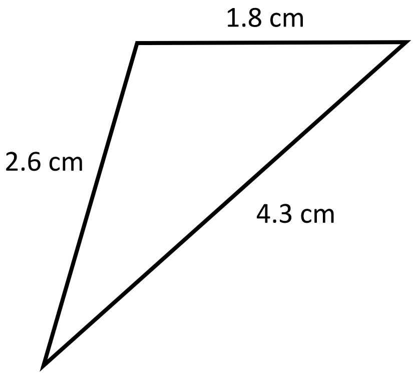
Exercise 12 H

Calculate the size of each of the missing sides and angles in the following diagrams.
(The diagrams are not drawn to scale.)

(a)



(b)



Sine Rule or Cosine Rule?

The Sine Rule works for any triangle where we know

- The lengths of two sides and the size of an angle that is not between the two sides
- The length of one side and the size of any two angles (because that means in reality we know all three angles).

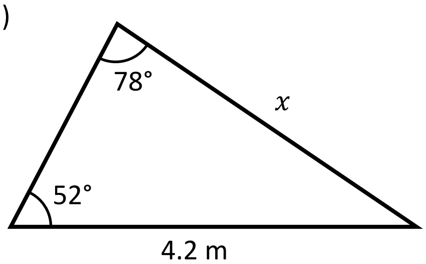
The Cosine Rule works for any triangle where we know

- The lengths of the three sides
- The lengths of two of the sides and the size of the angle between the sides.

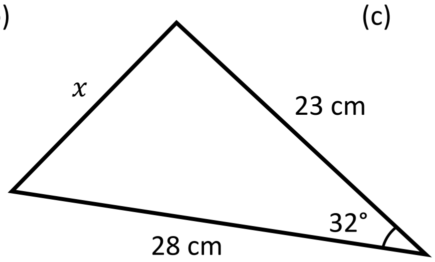
Exercise 13

For the following triangles, decide whether we need to use the Sine Rule or the Cosine Rule to calculate the length of the missing side x or the size of the missing angle θ.

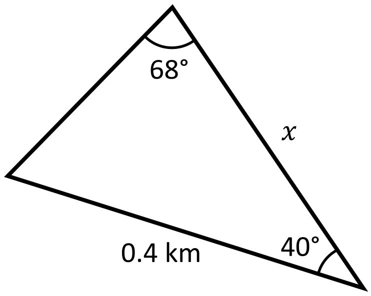
(a)



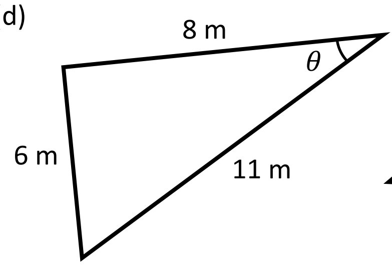
(b)



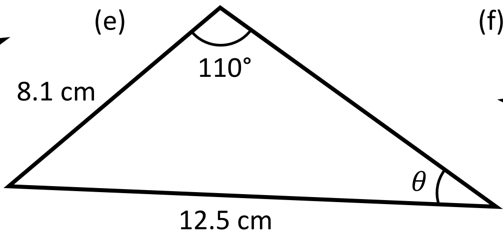
(c)



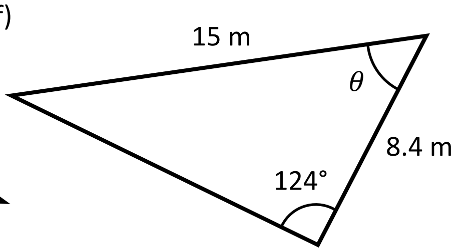
(d)



(e)



(f)

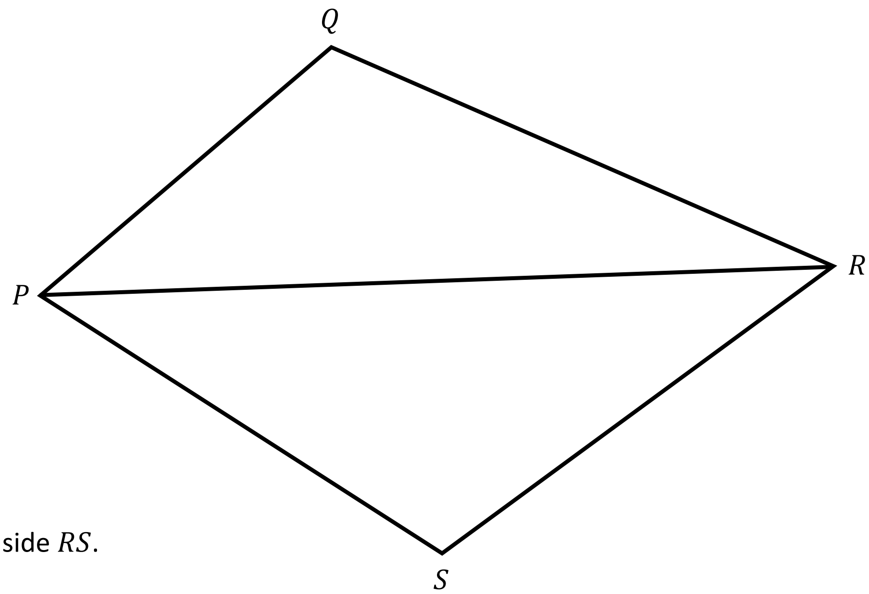


Exercise 14

For the triangles in Exercise 13, calculate the missing side x or the missing angle θ.
(The diagrams are not drawn to scale.)

Exercise 15

The following diagram shows two triangles PQR and PRS with $PQ = 24$ cm, $QR = 18$ cm, $P\hat{Q}R = 124°$, $S\hat{P}R = 36°$ and $P\hat{S}R = 112°$.

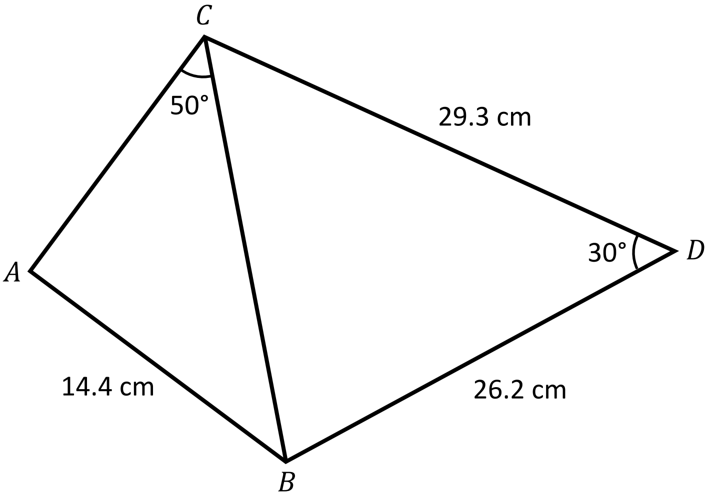


Find the length of the side RS.

Exercise 16 (The diagrams are not drawn to scale)

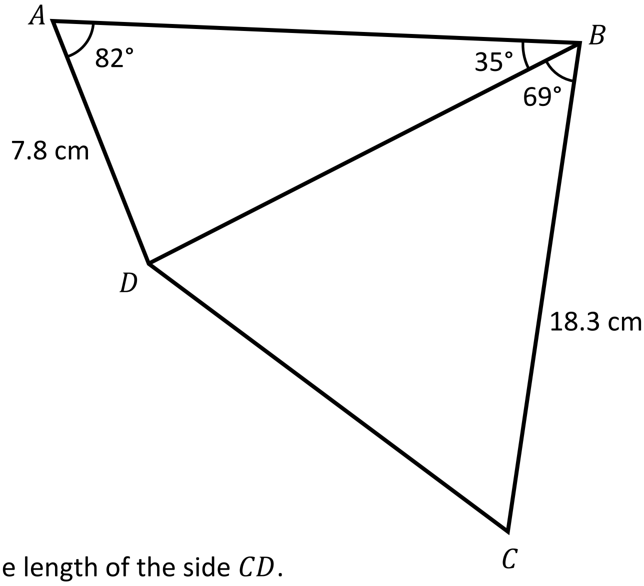
H

(a)



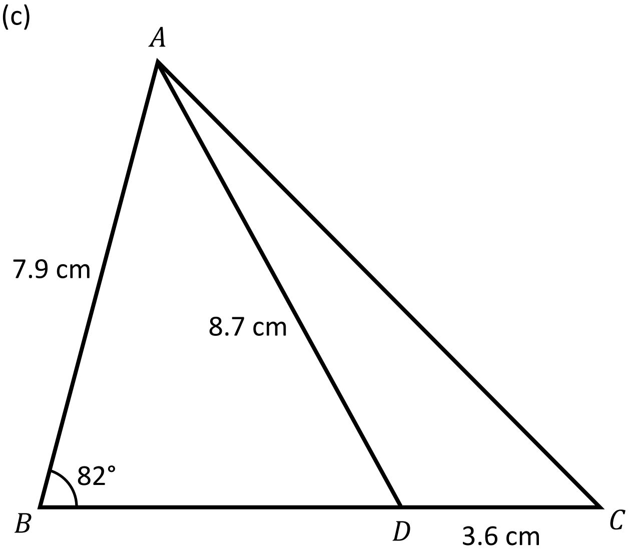
Find the size of the angle $C\hat{A}B$.

(b)



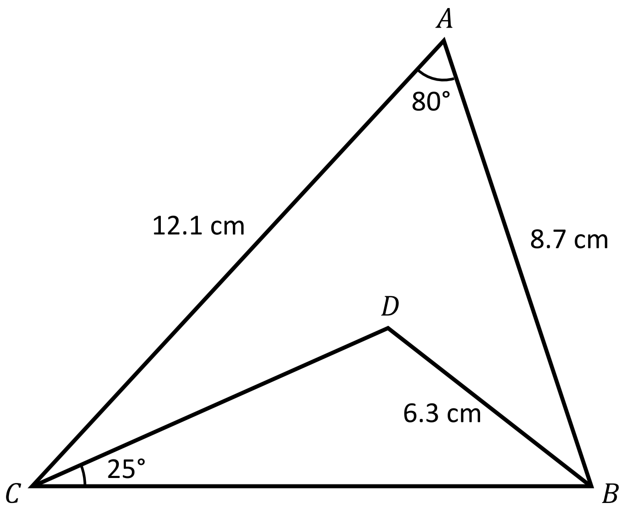
Find the length of the side CD.

(c)



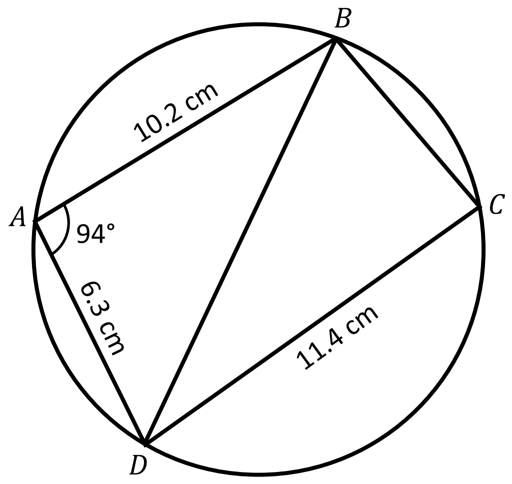
Given that BC is a straight edge, find the length AC.

(d)



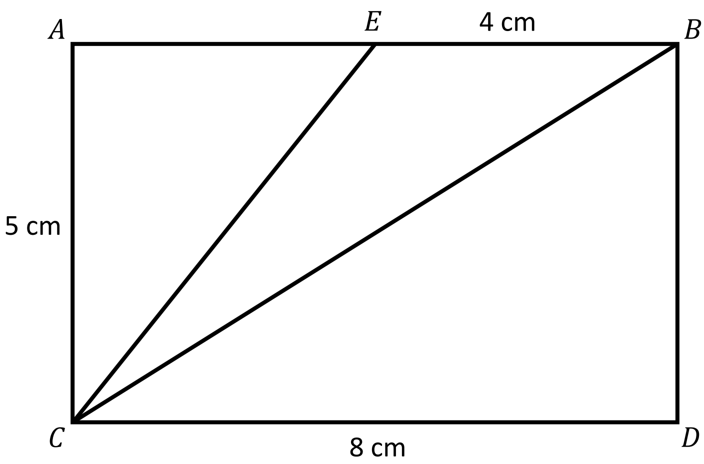
Find the size of the angle $C\widehat{D}B$.

(e)



Find the size of the angle $D\hat{B}C$.

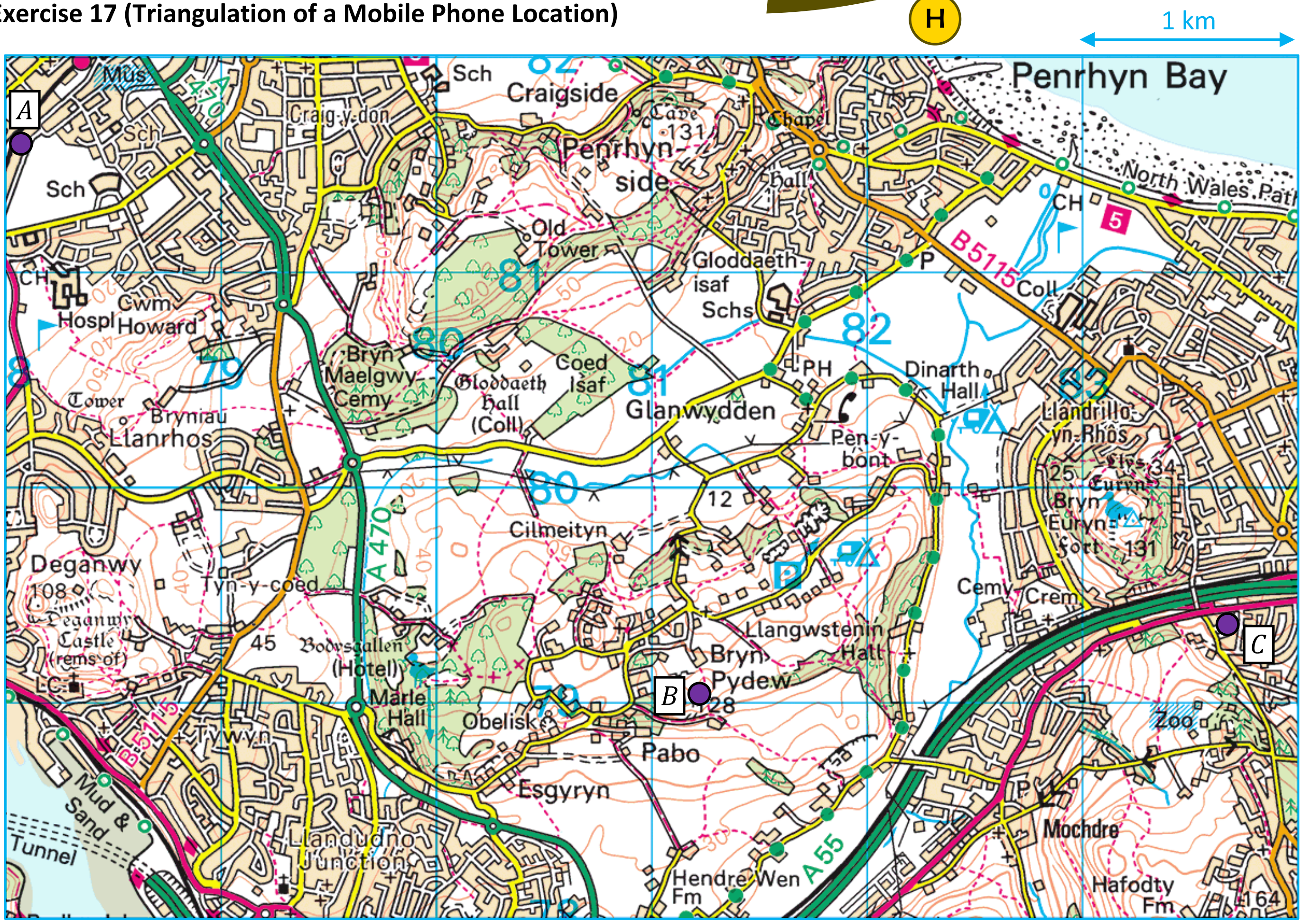
(f)



Given that $ABCD$ is a rectangle, calculate the size of $C\hat{E}B$.

Exercise 17 (Triangulation of a Mobile Phone Location)

H



The above map shows the location of three mobile phone masts A, B, C belonging to Vodafone.

(a) Steve's mobile phone reports that it is exactly 3 km from mast A. Use a compass to plot Steve's possible locations on the map.

(b) Steve's mobile phone also reports that it is exactly 2 km from mast B. Use a compass to find Steve's two possible locations.

(c) Given that Steve is around 3 km from mast C, mark Steve's location on the map with the letter S.

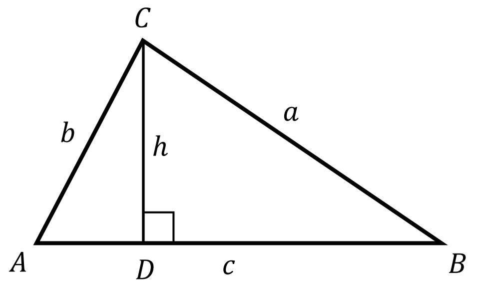
(d) Given that the distance between masts A and B is 4,064 m, find the size of the angle $A\hat{S}B$.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ❑ Target ❑

Area of a Triangle

The following diagram shows a general triangle with sides a, b, c and angles A, B, C.

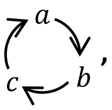


Let us draw the perpendicular from the vertex C to the base AB. Using the triangle CAD that is formed, we see that $\sin A = \frac{h}{b}$, and so $h = b \sin A$. Using the formula Area of a Triangle $= \frac{\text{base} \times \text{height}}{2}$,

$$\text{Area of a Triangle} = \frac{c \times b \sin A}{2}$$

$$\text{Area of a Triangle} = \frac{1}{2} bc \sin A$$

This is the version given on page 2 of a GCSE examination paper.

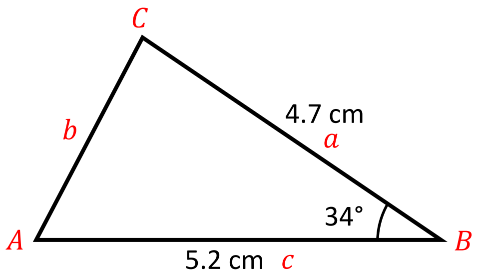
Or, changing the variables using ,

$$\text{Area of a Triangle} = \frac{1}{2} ac \sin B$$

$$\text{Area of a Triangle} = \frac{1}{2} ab \sin C$$

Example

(a) Calculate the area of the triangle below.



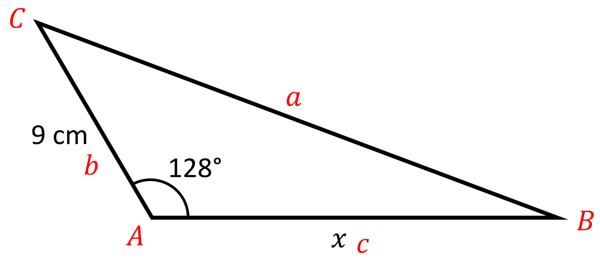
Answer: To start, we label the angles and then the corresponding sides. Using the formula

$$\text{Area of a Triangle} = \frac{1}{2} ac \sin B$$

$$\text{Area of a Triangle} = \frac{1}{2} \times 4.7 \times 5.2 \times \sin 34°$$

Area of a Triangle $= 6.83$ cm² to 2 decimal places.

(b) Given that the area of the triangle below is 27 cm², calculate the length x of the base of the triangle.



Answer: To start, we label the angles and then the corresponding sides. Using the formula

$$\text{Area of a Triangle} = \frac{1}{2} bc \sin A$$

$$27 = \frac{1}{2} \times 9 \times x \times \sin 128°$$

$$27 \times 2 = 9 \times x \times \sin 128°$$

$$\frac{54}{9 \times \sin 128°} = x$$

$x = 7.61$ cm to 2 decimal places.

We can use the formula

$$\text{Area of a Triangle} = \frac{1}{2} ab \sin C$$

if we know the length of two of the sides of a triangle, and the size of the angle between the sides.

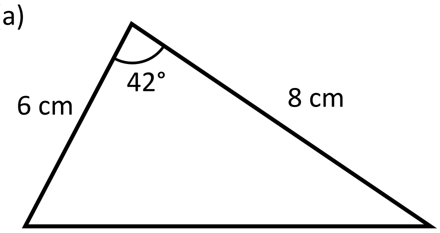
Exercise 18

Skill

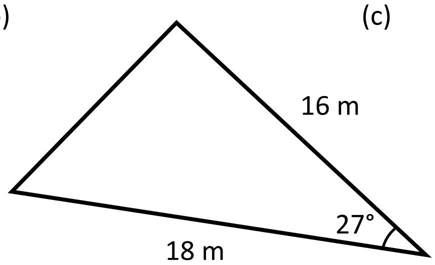
H

Calculate the area of the following triangles. (The diagrams are not drawn to scale.)

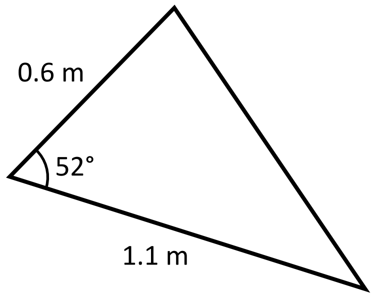
(a)



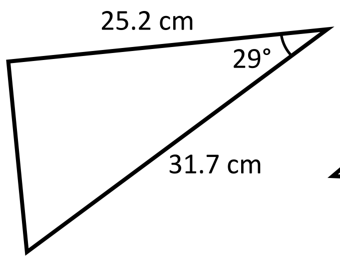
(b)



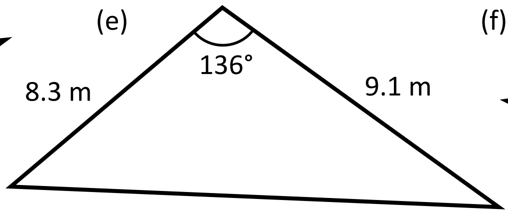
(c)



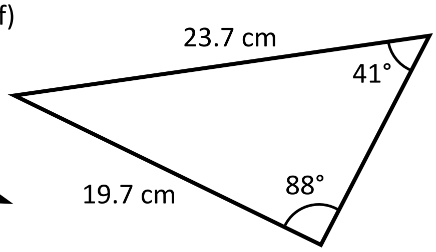
(d)



(e)



(f)

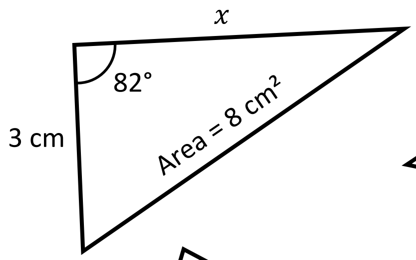


Exercise 19

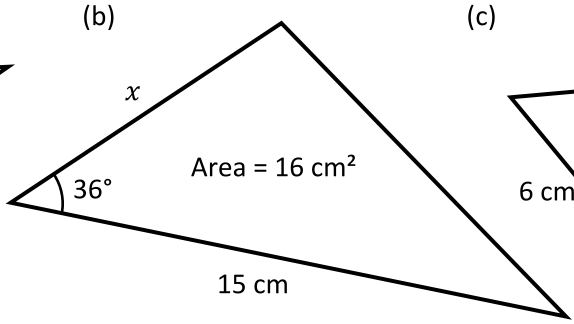
H

Calculate the length of the missing side x. (The diagrams are not drawn to scale.)

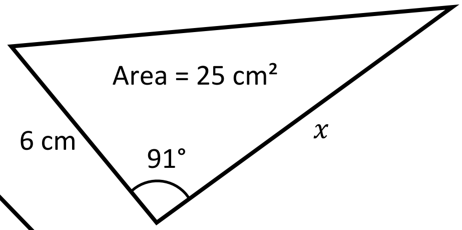
(a)



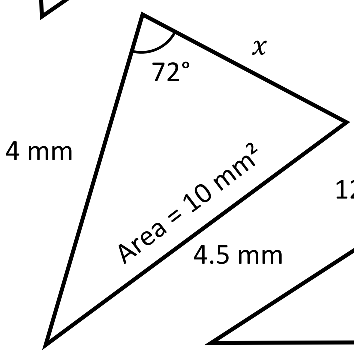
(b)



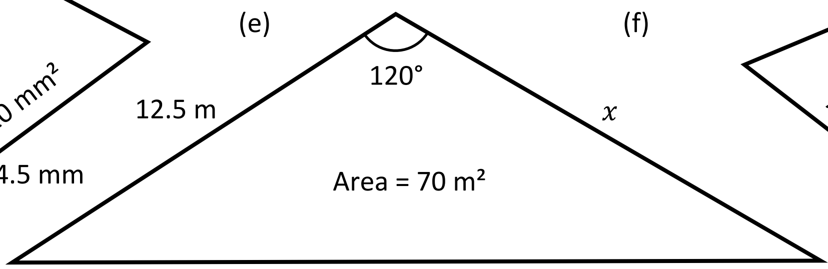
(c)



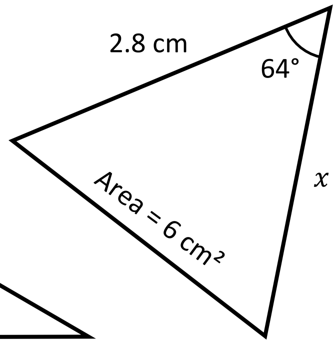
(d)



(e)

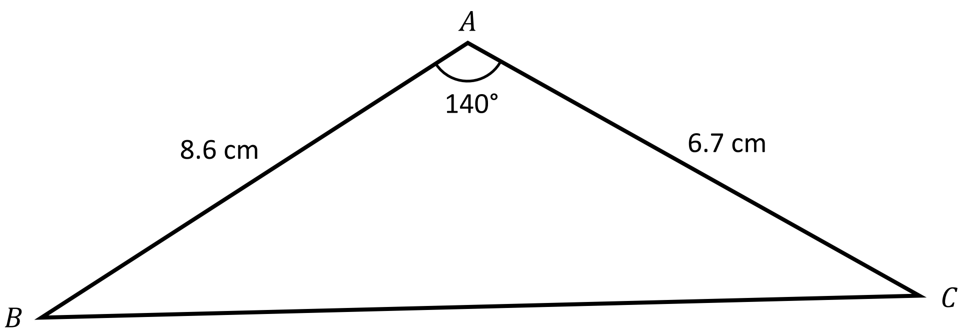


(f)



Exercise 20

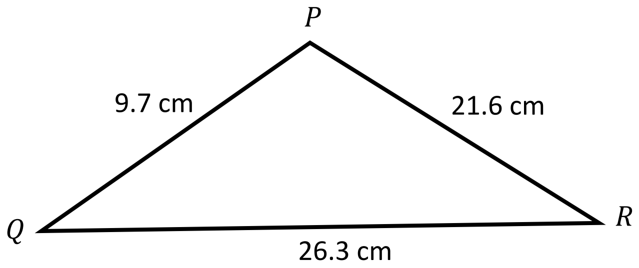
H



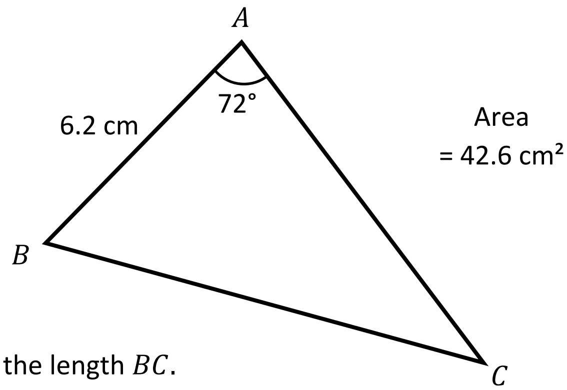
(a) Find the length of the side BC.
(b) Calculate the area of the triangle ABC.
(c) Hence, find the perpendicular distance between A and BC.

Exercise 21 (The diagrams are not drawn to scale.) H

(a) Calculate the area of the triangle PQR.

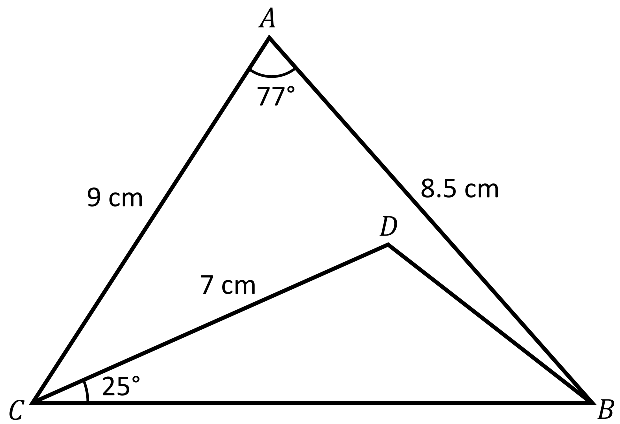


(b) The following diagram shows the triangle ABC.

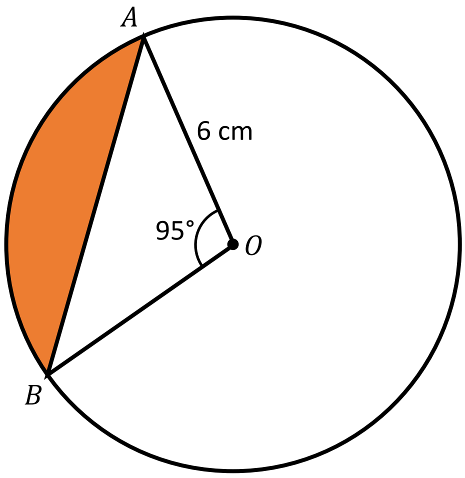


Calculate the length BC.

(c) Calculate the area of the quadrilateral $ABCD$.

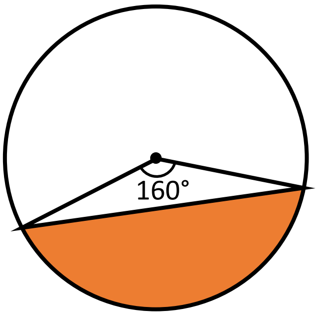


(d) Calculate the area of the minor segment AB.



Challenge! ⚠

The area of the segment on the right is 50 cm².
Calculate the length of the radius of the circle.



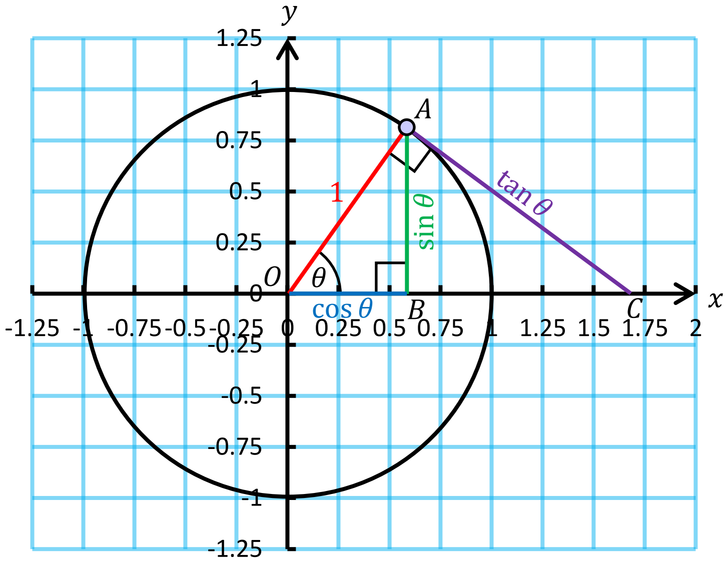
Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will…
			Grade ☐ **Target** ☐

Trigonometric Graphs

The Unit Circle

Let us consider a unit circle (a circle where the radius is 1 unit) where the centre of the circle O is located at the origin of a set of x and y axes.



If A represents a general point on the circle's circumference, let θ represent the angle between the radius OA and the positive x–axis. Then

- $\sin\theta$ is the vertical displacement from the x–axis to the point A;
- $\cos\theta$ is the horizontal displacement from the origin to the x–coordinate of A;
- $\tan\theta$ is the length of the tangent to the point A, measuring from A to the x–axis.

Why is this true?

Let us consider the right-angled triangle OAB to begin with.

$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$

$\sin\theta = \frac{AB}{1}$

$1 \times \sin\theta = AB$

$AB = \sin\theta$

So, the height of the triangle is $\sin\theta$.

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\cos\theta = \frac{OB}{1}$

$1 \times \cos\theta = OB$

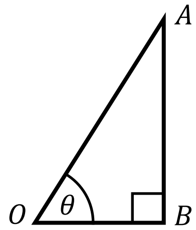
$OB = \cos\theta$

So, the base of the triangle is $\cos\theta$.

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

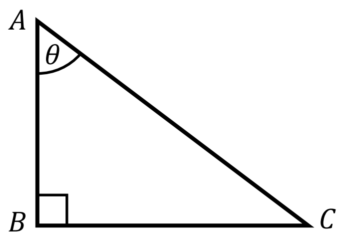
$\tan\theta = \frac{AB}{OB}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$



Next, let us consider the right-angled triangle ABC.

We have $B\hat{A}C = \theta$ because the triangles OAB, OAC and ABC are all similar triangles (they share the same angles).



$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\cos\theta = \frac{AB}{AC}$

$AC \times \cos\theta = AB$

$AC = \frac{AB}{\cos\theta}$

$AC = \frac{\sin\theta}{\cos\theta}$ [from the triangle OAB]

But the triangle OAB also tells us that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, so we must have $AC = \tan\theta$.

Exercise 22

Skill H

Experiment with the unit circle using GeoGebra: https://www.geogebra.org/m/fGsz9sfN

Move the point A around the circle.

What happens to the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ as you move A?

When are $\sin\theta$, $\cos\theta$ and $\tan\theta$ positive, and when are they negative? What are their lowest and highest values?

Write a paragraph summarising your findings.

Exercise 23

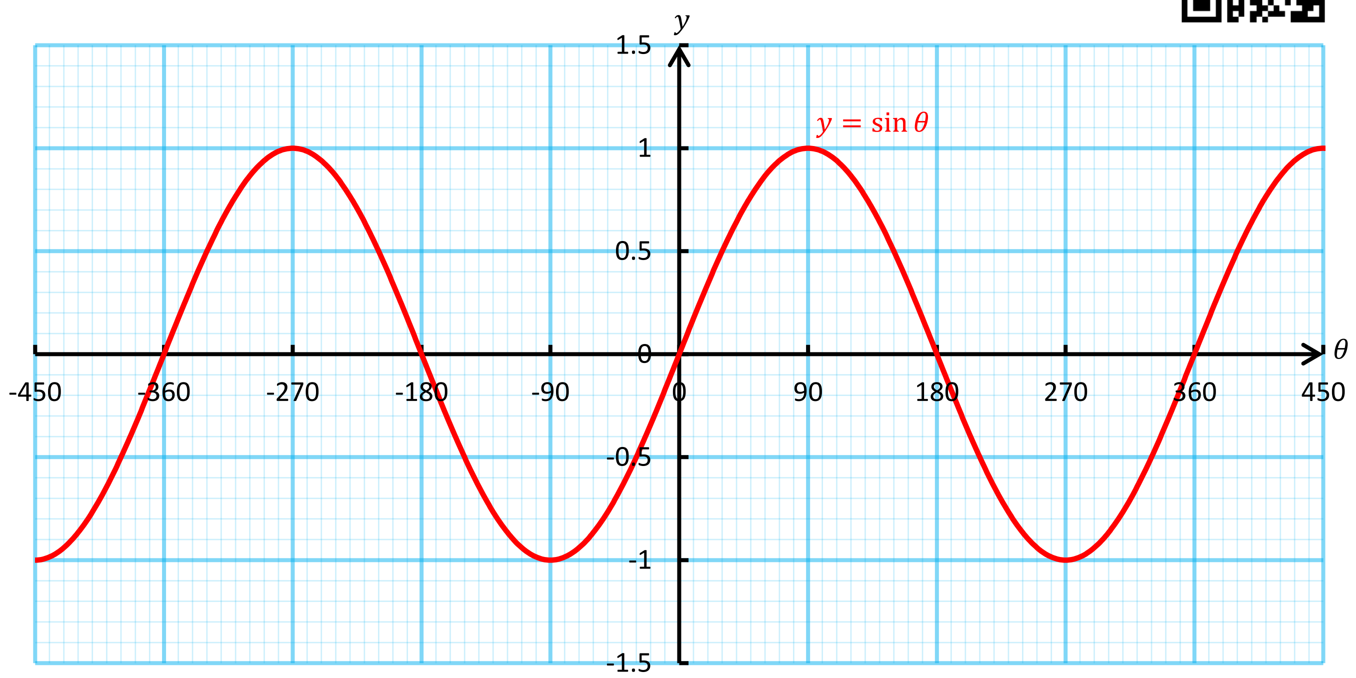
H

Use your calculator to complete the following table. Give your answers correct to 4 decimal places.

Angle θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0		
30°		0.8660	
60°			1.7321
90°			
120°	0.8660		
150°		–0.8660	
180°			0
210°			
240°	–0.8660		
270°		0	
300°			–1.7321
330°			
360°	0		

Trigonometric Graphs

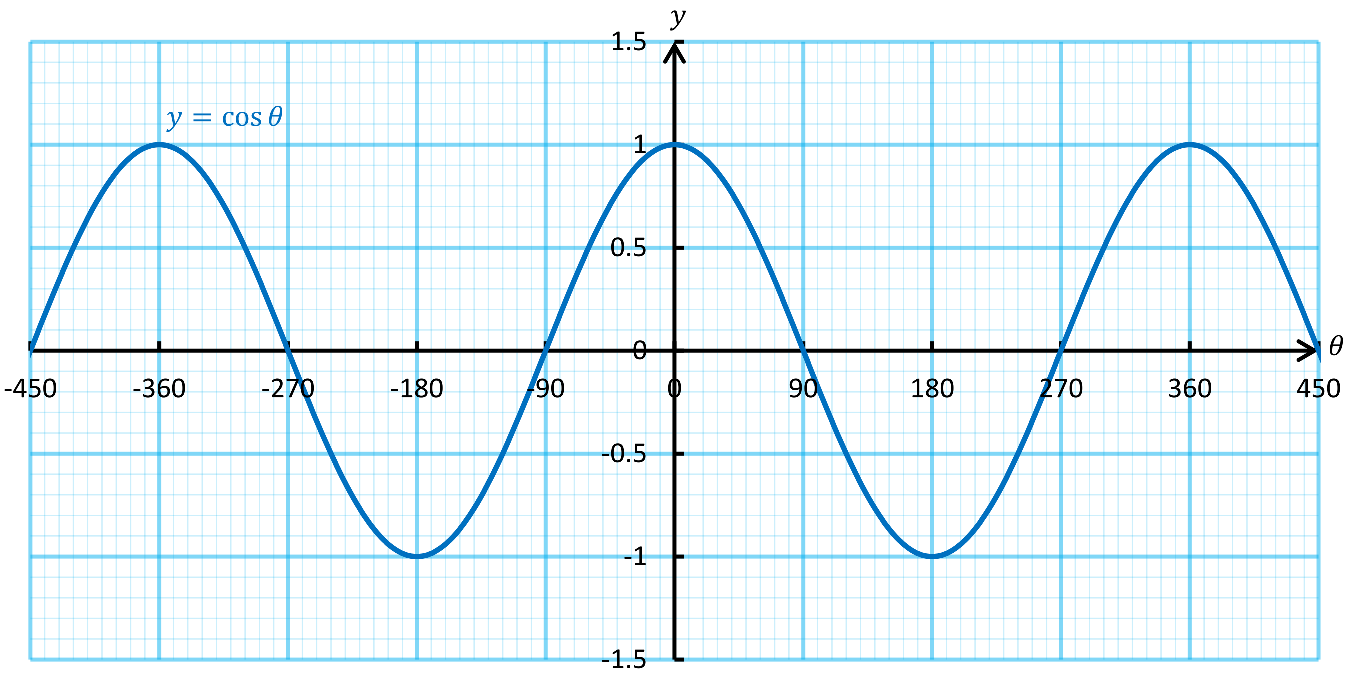
$y = \sin\theta$



Properties

- The greatest **amplitude** (height) of the graph is 1 unit. The graph varies between –1 and 1.
- The **period** of the graph is 360°. (The graph repeats every 360°.)

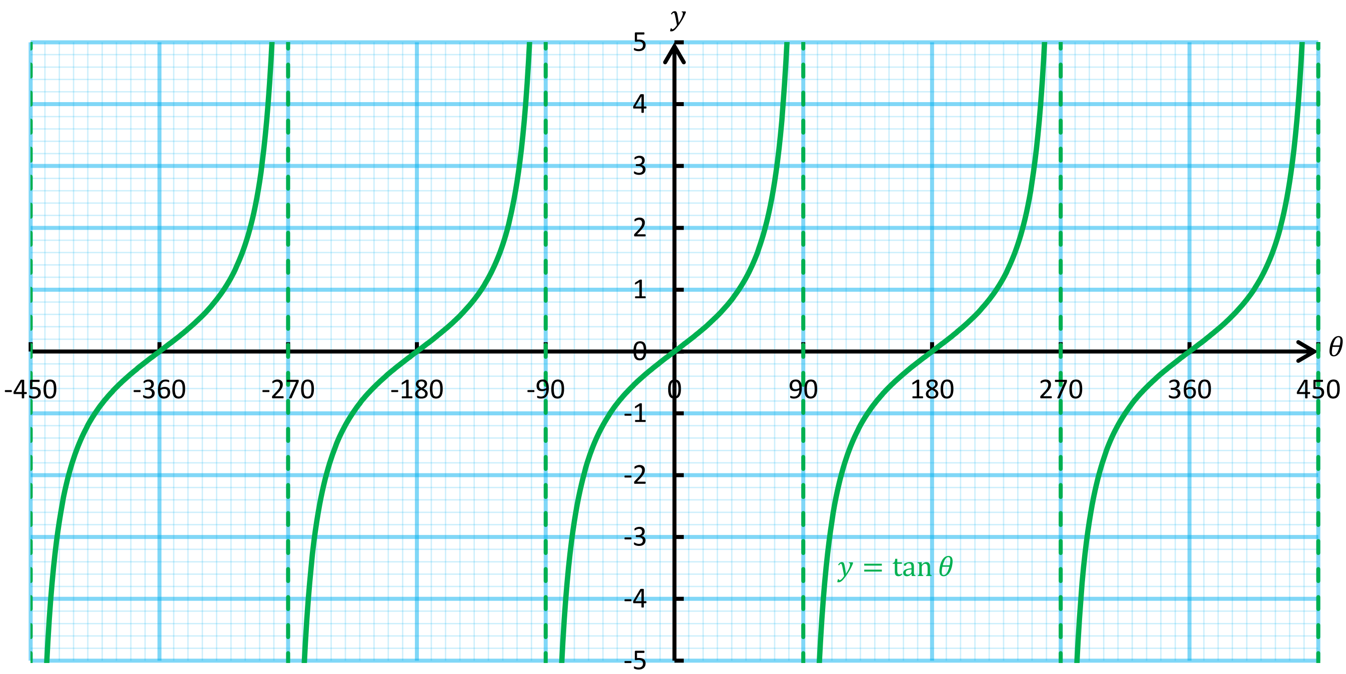
$y = \cos\theta$



Properties

- The greatest **amplitude** (height) of the graph is 1 unit. The graph varies between –1 and 1.
- The **period** of the graph is 360°. (The graph repeats every 360°.)

$y = \tan\theta$



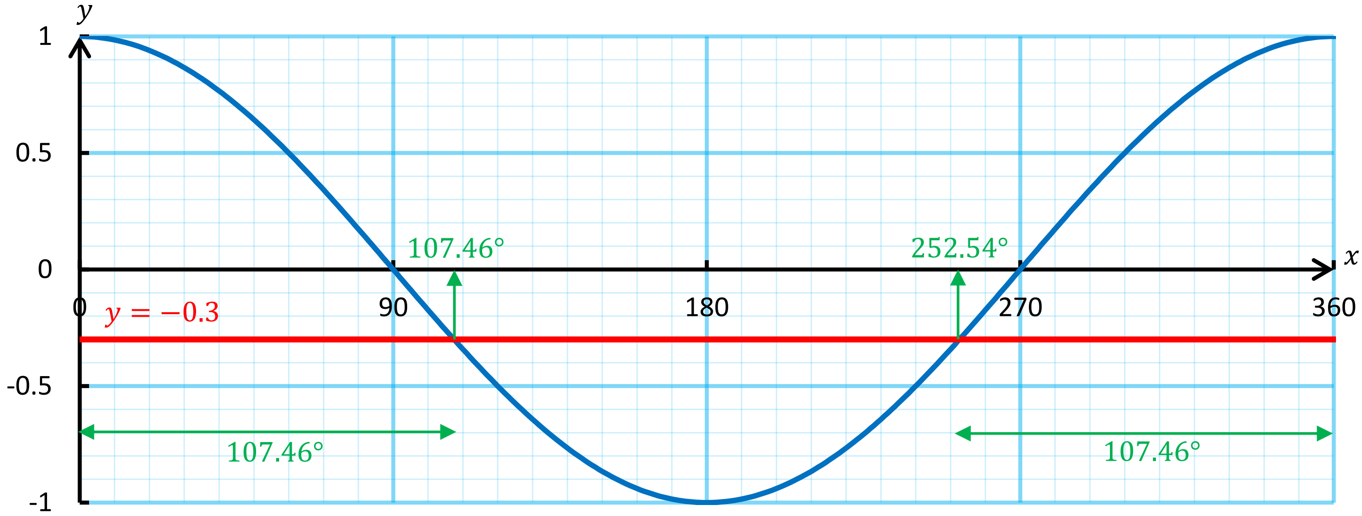
Properties

- The **amplitude** (height) of the graph is not defined. The graphs varies between $-\infty$ and ∞.
- The **period** of the graph is 180°. (The graph repeats every 180°.)
- The graph has **asymptotes** every 180°, e.g. at –90° or at 90°. $y = \tan\theta$ is not defined at these angles.

Solving Trigonometric Equations

Example

The following diagram shows a sketch of $y = \cos x$ for values of x between 0° and 360°.



Find all the solutions of the following equation between 0° and 360°.

$$\cos x = -0.3$$

Answer: To begin, draw the horizontal line $y = -0.3$ on the graph. This line intersects the blue curve at two different points, so there are two solutions to the equation between 0° and 360°. We can find one of the solutions by using a calculator:

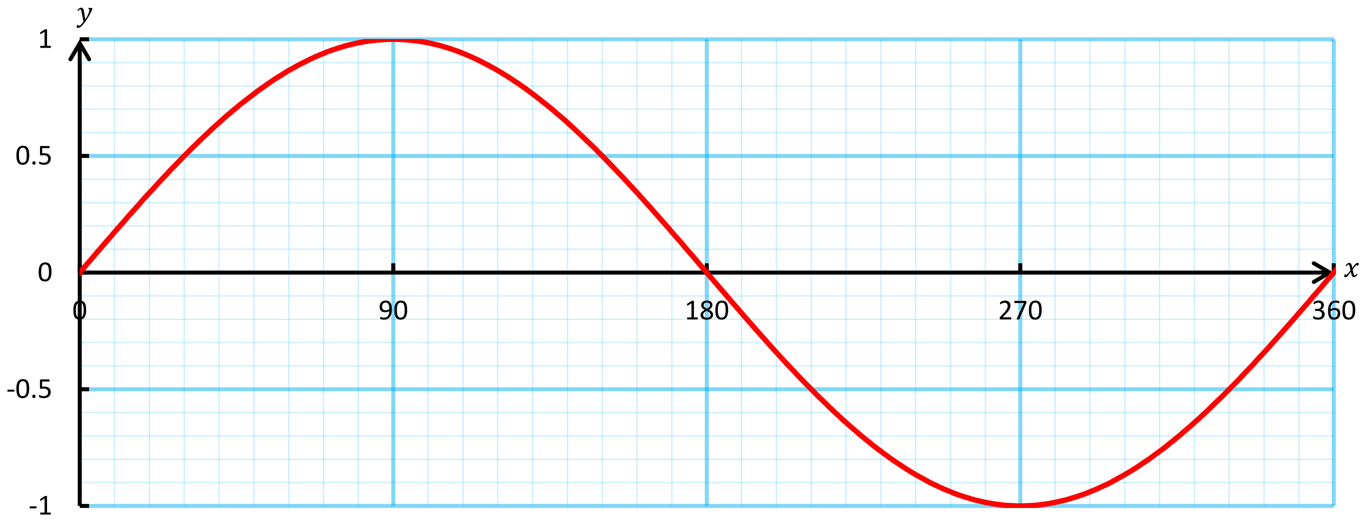
$$x = \cos^{-1}(-0.3)$$
$$x = 107.46° \text{ to 2 decimal places}$$

We can find the second solution by using the symmetry of the graph of $y = \cos x$. If 107.46° is a solution, then 360° – 107.46° = 252.54° must also be a solution. So, the answers (to 2 decimal places) are 107.46° and 252.54°.

Exercise 24

H

The following diagram shows a sketch of $y = \sin x$ for values of x between 0° and 360°.

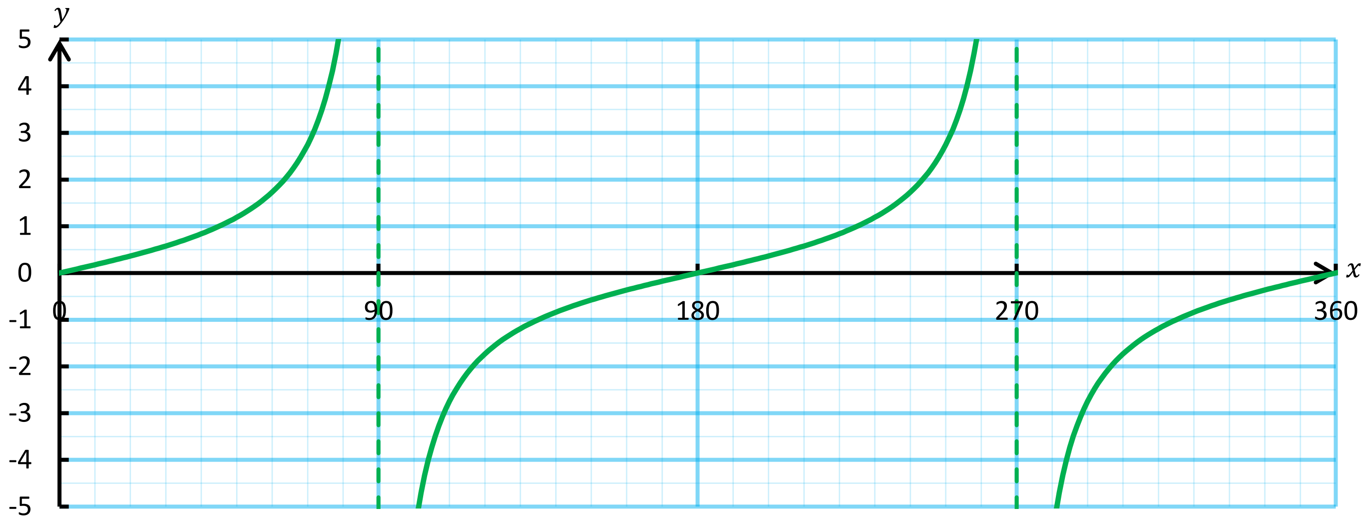


Find all the solutions of the following equation between 0° and 360°.

$$\sin x = 0.75$$

Exercise 25 H

The following diagram shows a sketch of $y = \tan x$ for values of x between 0° and 360°.

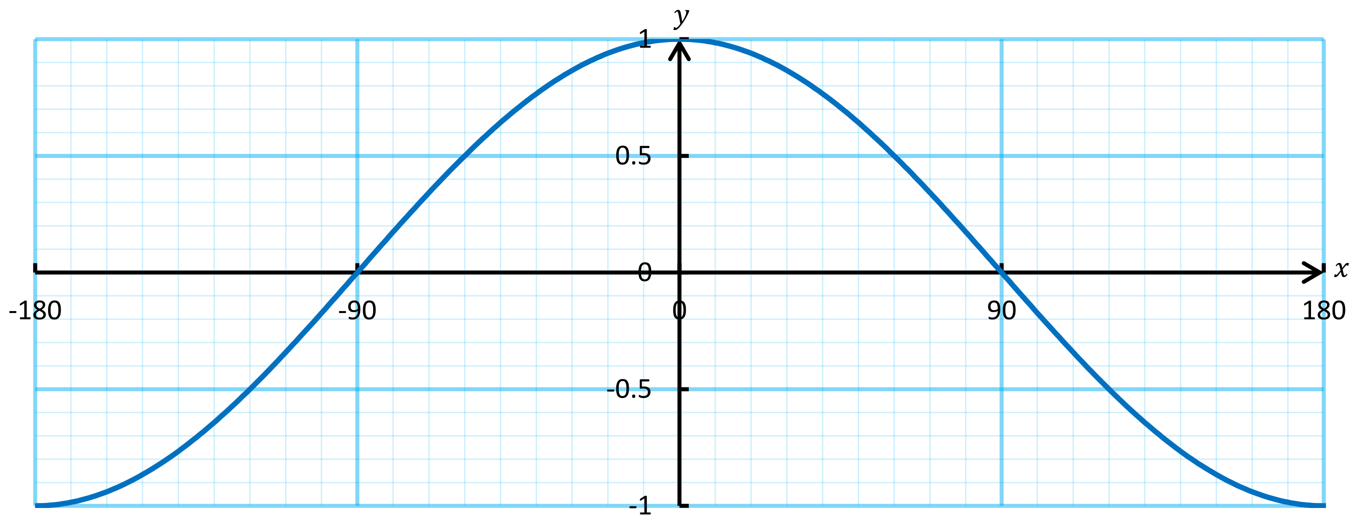


Find all the solutions of the following equation between 0° and 360°.

$$\tan x = 3$$

Exercise 26 H

The following diagram shows a sketch of $y = \cos x$ for values of x between –180° and 180°.



Find all the solutions of the following equation between –180° and 180°.

$$\cos x = -0.8$$

Exercise 27 H

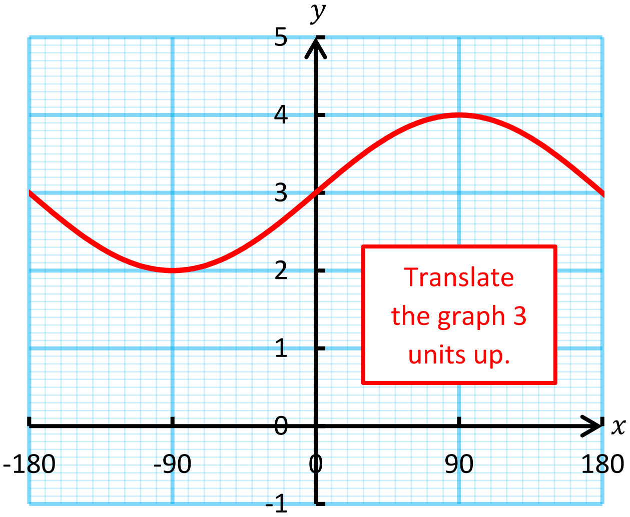
Use suitable graphs to solve the following trigonometric equations.

(a) $\sin x = 0.1$ between 0° and 360°
(b) $\cos x = 0.83$ between 0° and 360°
(c) $\tan x = 3.14$ between 0° and 360°
(d) $\sin x = 0.36$ between –180° and 180°
(e) $\cos x = -0.4$ between –180° and 180°
(f) $\tan x = -1.3$ between –180° and 180°

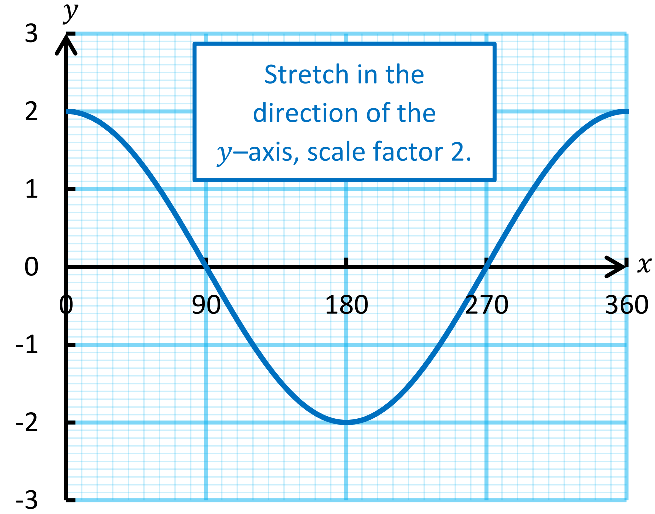
Function Transformations

Here are some examples of transformations that use trigonometric functions.

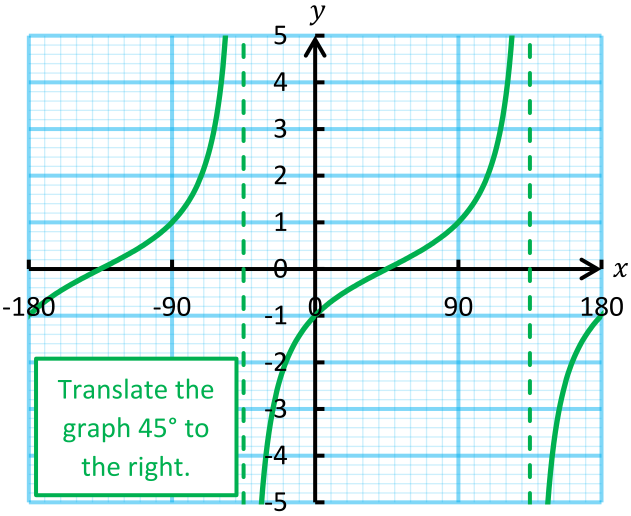
(a) $y = \sin(x) + 3$ between –180° and 180°.



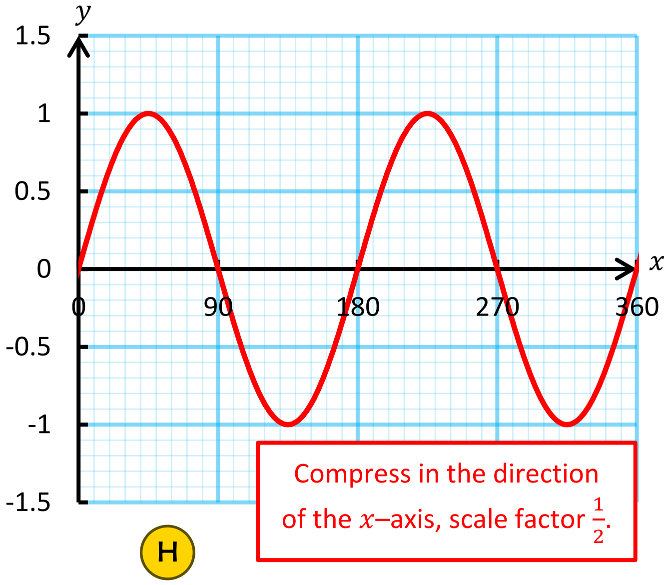
(b) $y = 2\cos(x)$ between 0° and 360°.



(c) $y = \tan(x - 45°)$ between –180° and 180°.



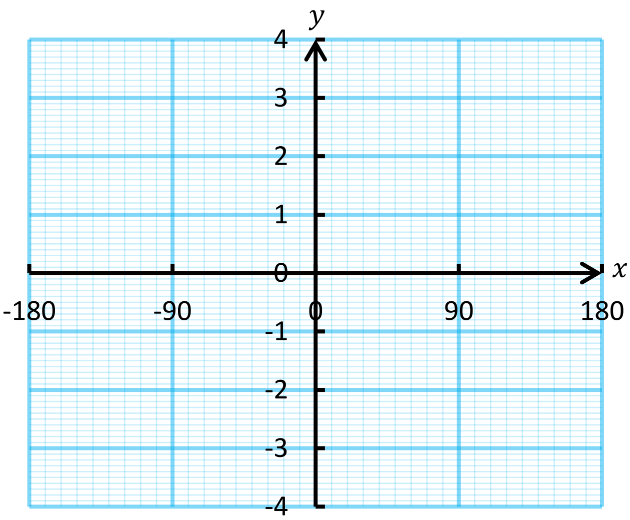
(d) $y = \sin(2x)$ between 0° and 360°.



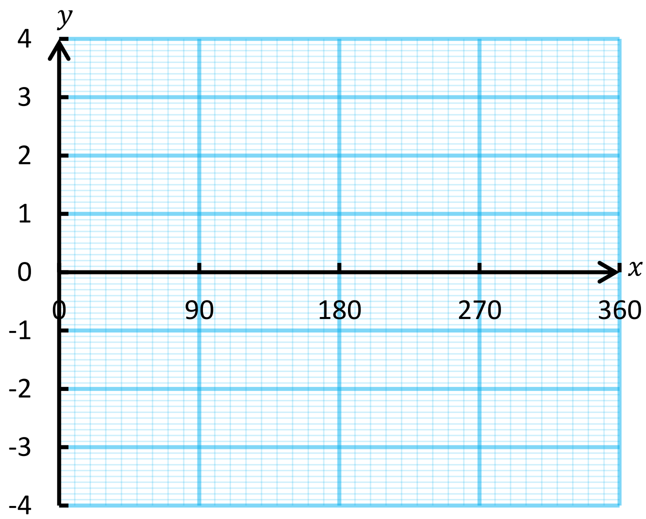
Exercise 28

Draw, on the graph paper provided, graphs for the following functions.

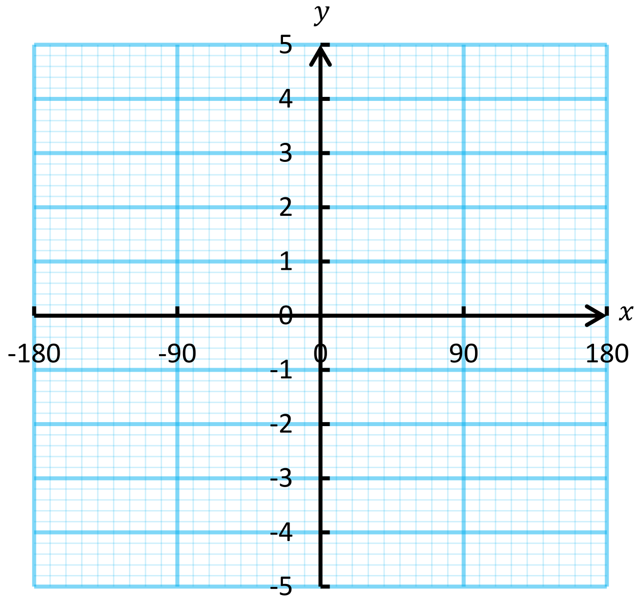
(a) $y = \cos(x) - 2$ between –180° and 180°.



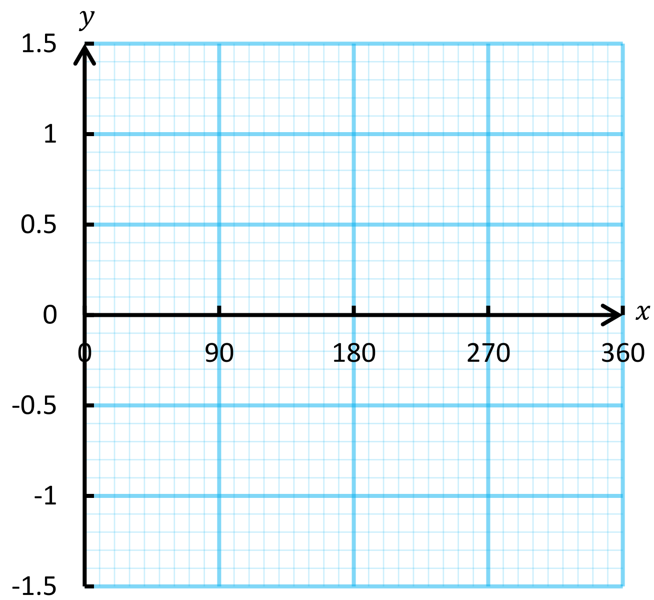
(b) $y = -3\sin(x)$ between 0° and 360°.



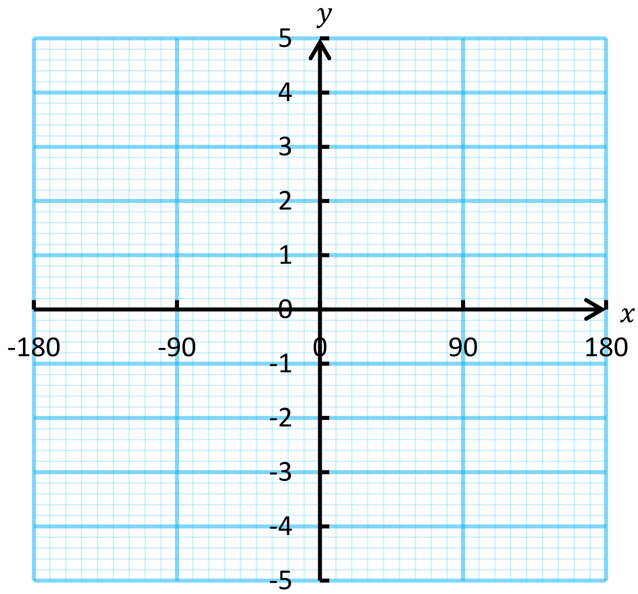
(c) $y = \tan(2x)$ between $-180°$ and $180°$.



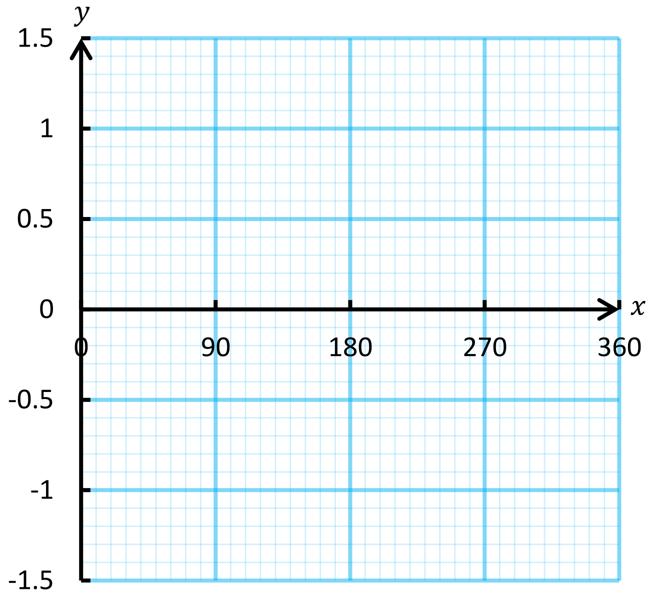
(d) $y = \cos(x + 45°)$ between $0°$ and $360°$.



(e) $y = -\tan(x)$ between $-180°$ and $180°$.



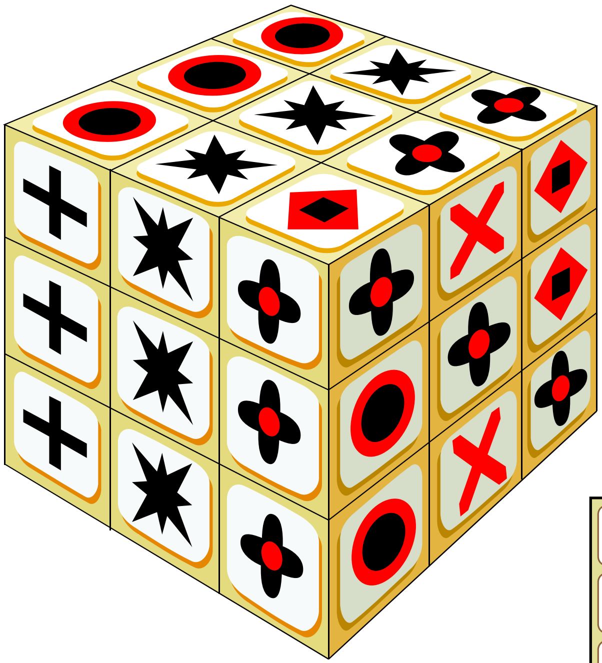
(f) $y = \sin(-x)$ between $0°$ and $360°$.

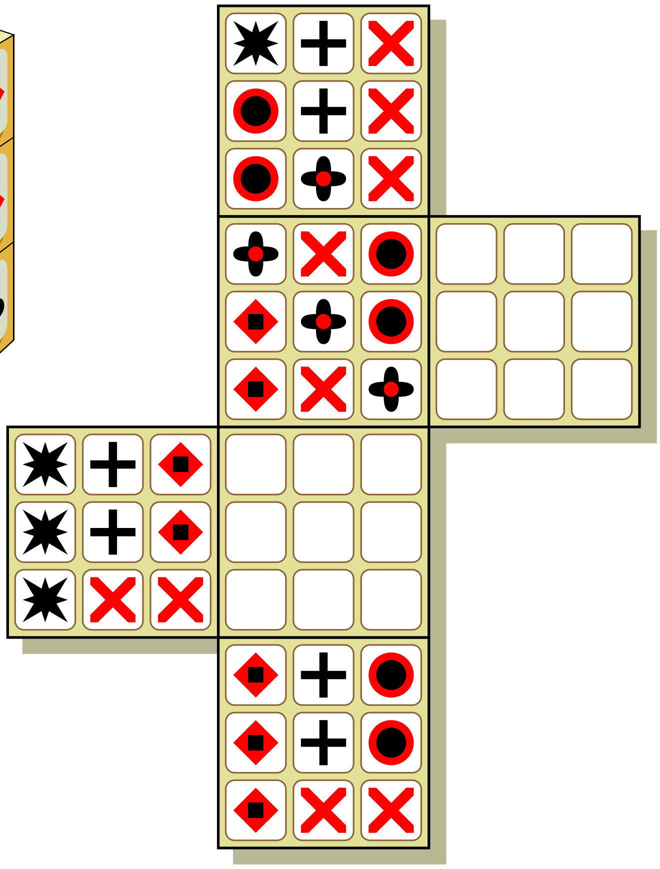


Evaluation

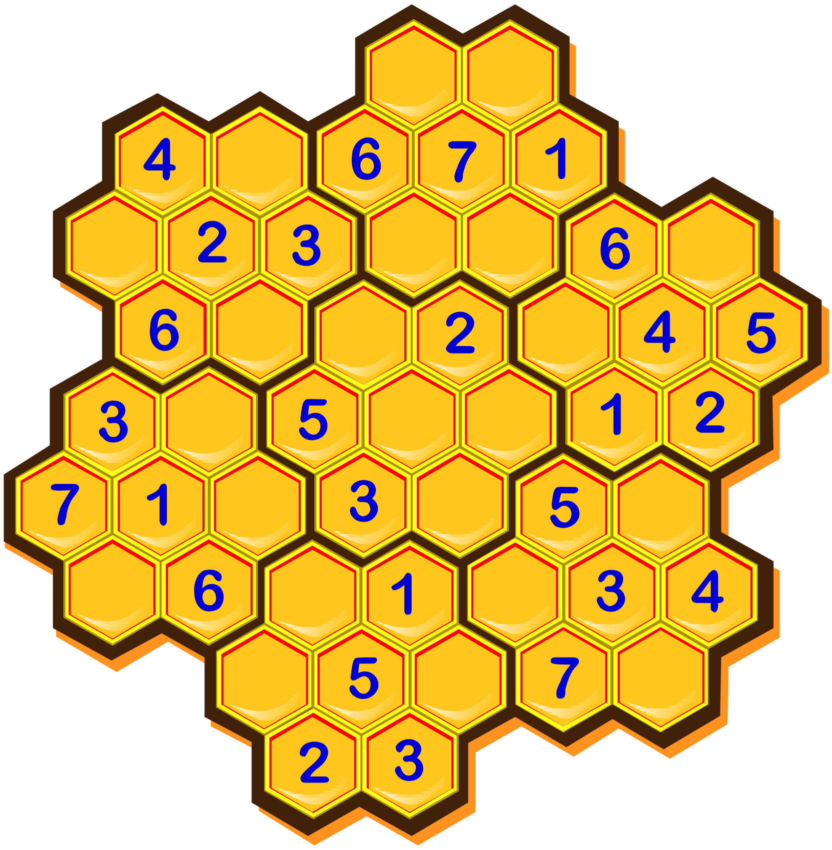
Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Puzzle 1: Complete the net so that it corresponds to the cube.





Puzzle 2: Write the numbers between 1 and 9 in each row, diagonal and honeycomb cell.



Measuring Shapes 5

Reflection

Name: ....

Percentage in the test: ....

	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I know how to use trigonometry to find angles and lengths in **three dimensional shapes**.			1	
I know how to use the **Sine Rule** to find **lengths** in triangles.			3	
I know how to use the **Sine Rule** to find **angles** in triangles.			4, 7	
I know how to use the **Cosine Rule** to find **lengths** in triangles.			4	
I know how to use the **Cosine Rule** to find **angles** in triangles.			2, 7	
I know **when** to use the Sine Rule or Cosine Rule to solve problems.			2, 3, 4, 7	
I know how to use the formula $\frac{1}{2}ab\sin C$ to find the **area of a triangle**.			2, 3, 7	
I know how to find the **area of a segment of a circle**.			3	
I can recognise and sketch the **trigonometric graphs** $y=\sin\theta$, $y=\cos\theta$ and $y=\tan\theta$.			5	
I can **solve trigonometric equations** with the help of a trigonometric graph.			5	
I can draw **transformations** of trigonometric functions.			6	

Am I ready for the test?
Tick the boxes below...

- I have revised the work in my mathematics book. ☐
- I have revised the workbook. ☐
- I have watched the relevant videos on YouTube. ☐
- I have completed the Diagnostic Questions quiz. ☐
- I have completed at least 4 pages in my revision book. ☐

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DAWN DYSG DAIONI

The Mathematics Department

11

The End of Year 11

Name:

Contents

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Histograms	Frequency density. Drawing a histogram. Interpreting a histogram. Estimating the median from a histogram. Estimating the quartiles from a histogram. Comparing histograms.	13

Surds

Rational and Irrational Numbers

A number is a **rational number** if it can be written in the form of a fraction $\frac{a}{b}$, where a and b are integers, and $b \neq 0$. For example, $\frac{4}{5}$, $6 = \frac{6}{1}$, $3\frac{1}{2} = \frac{7}{2}$ and $0.\dot{3} = \frac{1}{3}$ are rational numbers.

A number is an **irrational number** if it **cannot** be written in the form of a fraction $\frac{a}{b}$, where a and b are integers, and $b \neq 0$. For example, π, $\sqrt{2}$ and 0.202002000200002.... are irrational numbers.

Exercise 1

Skill

H

Circle the rational numbers below.

8	$\sqrt{3}$	$\frac{5}{6}$	$\sqrt{4}$	π	$0.\dot{4}5\dot{2}$	0.45445444544445...
$\sqrt[3]{2}$	$4\frac{2}{3}$	$\sqrt[3]{27}$	π^2	$(\sqrt{2})^2$	$\frac{2}{0}$	0.27277277727777

Surds

A **surd** is a number that contains a root that does not correspond to a rational number. Here are some examples.

$\sqrt{2}$	Surd (it is not possible to write $\sqrt{2}$ as a fraction $\frac{a}{b}$).
$3\sqrt{2}$	Surd (a multiple of the surd $\sqrt{2}$).
$\sqrt{9}$	Not a surd (corresponds to 3).
$\sqrt[3]{4}$	Surd (it is not possible to write $\sqrt[3]{4}$ as a fraction $\frac{a}{b}$).
$\sqrt[3]{64}$	Not a surd (corresponds to 4).

Exercise 2

H

Circle the surds below.

$\sqrt{5}$	$\sqrt{16}$	$\sqrt{49}$	$\sqrt{32}$	$\sqrt[3]{4}$	$\sqrt[3]{1}$	$\sqrt[3]{25}$
$\sqrt[4]{2}$	$\sqrt[4]{16}$	$9\sqrt{2}$	$2\sqrt{4}$	$3\sqrt{10}$	$6\sqrt[3]{8}$	$4\sqrt[3]{36}$

Working with surds

It is possible to collect like surds together, exactly as we can collect like terms in algebra.

Example

$2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$

$7\sqrt{2} - \sqrt{2} = 6\sqrt{2}$

$4\sqrt[3]{5} + 2\sqrt{5} + 6\sqrt[3]{5} + 7\sqrt{5} = 10\sqrt[3]{5} + 9\sqrt{5}$

$5\sqrt{6} + \sqrt{16} - 2\sqrt{6} - 1 = 3\sqrt{6} + 3$

Exercise 3

H

Simplify the following.

(a) $5\sqrt{2} + 3\sqrt{2}$

(b) $7\sqrt{2} + \sqrt{2}$

(c) $6\sqrt{2} - 2\sqrt{2}$

(d) $8\sqrt[3]{7} + 2\sqrt[3]{7}$

(e) $10\sqrt[4]{2} - 3\sqrt[4]{2}$

(f) $\sqrt{3} + 2 + \sqrt{3} + 9$

(g) $5\sqrt{7} + 2\sqrt[3]{4} + 2\sqrt{7} + 6\sqrt[3]{4}$

(h) $7\sqrt[3]{5} + \sqrt{11} - 4\sqrt[3]{5} + 2\sqrt{11}$

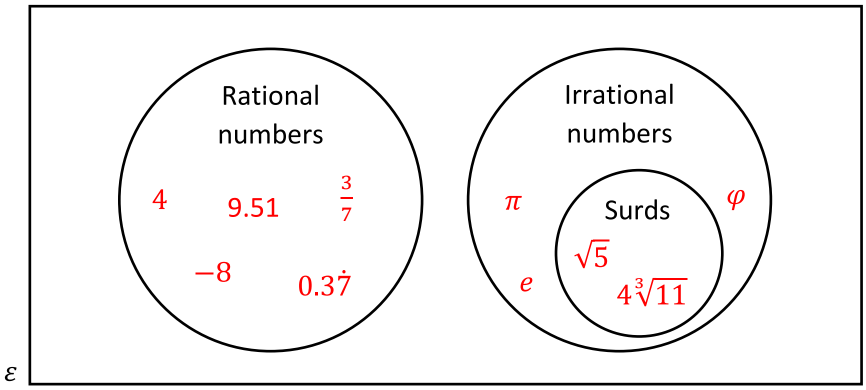
(i) $3\sqrt{4} + 2\sqrt{6} + 8$

(j) $\sqrt{10} + 3\sqrt[3]{3} - 3\sqrt{10} + \sqrt[4]{3}$

(k) $\sqrt{7} + \sqrt[3]{7} - \sqrt{7} + \sqrt[3]{7}$

(l) $\sqrt[100]{10} + \sqrt[100]{1}$

A Venn diagram showing different types of numbers



Simplifying Surds

It is possible to use the following rules to simplify surds.

$$\sqrt{a} \times \sqrt{a} = a, \qquad \sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

Example

$\sqrt{3} \times \sqrt{3} = 3$

$$\begin{aligned} \sqrt{8} &= \sqrt{4 \times 2} \\ &= \sqrt{4} \times \sqrt{2} \\ &= 2 \times \sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \sqrt{15} \times \sqrt{12} &= \sqrt{5 \times 3} \times \sqrt{3 \times 4} \\ &= \sqrt{5} \times \sqrt{3} \times \sqrt{3} \times \sqrt{4} \\ &= \sqrt{5} \times 3 \times 2 \\ &= 6\sqrt{5} \end{aligned}$$

Exercise 4 H

Complete the following table to show the surds in their simplest form. (Hint: look for a factor that is a square number.)

$\sqrt{1}$	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{4}$	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	$\sqrt{8}$ $= 2\sqrt{2}$	$\sqrt{9}$ $= 3$	$\sqrt{10}$
$\sqrt{11}$	$\sqrt{12}$	$\sqrt{13}$	$\sqrt{14}$	$\sqrt{15}$	$\sqrt{16}$	$\sqrt{17}$	$\sqrt{18}$	$\sqrt{19}$	$\sqrt{20}$
$\sqrt{21}$	$\sqrt{22}$	$\sqrt{23}$	$\sqrt{24}$	$\sqrt{25}$	$\sqrt{26}$	$\sqrt{27}$	$\sqrt{28}$	$\sqrt{29}$	$\sqrt{30}$
$\sqrt{31}$	$\sqrt{32}$	$\sqrt{33}$	$\sqrt{34}$	$\sqrt{35}$	$\sqrt{36}$	$\sqrt{37}$	$\sqrt{38}$	$\sqrt{39}$	$\sqrt{40}$
$\sqrt{41}$	$\sqrt{42}$	$\sqrt{43}$	$\sqrt{44}$	$\sqrt{45}$	$\sqrt{46}$	$\sqrt{47}$	$\sqrt{48}$	$\sqrt{49}$	$\sqrt{50}$
$\sqrt{51}$	$\sqrt{52}$	$\sqrt{53}$	$\sqrt{54}$	$\sqrt{55}$	$\sqrt{56}$	$\sqrt{57}$	$\sqrt{58}$	$\sqrt{59}$	$\sqrt{60}$
$\sqrt{61}$	$\sqrt{62}$	$\sqrt{63}$	$\sqrt{64}$	$\sqrt{65}$	$\sqrt{66}$	$\sqrt{67}$	$\sqrt{68}$	$\sqrt{69}$	$\sqrt{70}$
$\sqrt{71}$	$\sqrt{72}$	$\sqrt{73}$	$\sqrt{74}$	$\sqrt{75}$	$\sqrt{76}$	$\sqrt{77}$	$\sqrt{78}$	$\sqrt{79}$	$\sqrt{80}$
$\sqrt{81}$	$\sqrt{82}$	$\sqrt{83}$	$\sqrt{84}$	$\sqrt{85}$	$\sqrt{86}$	$\sqrt{87}$	$\sqrt{88}$	$\sqrt{89}$	$\sqrt{90}$
$\sqrt{91}$	$\sqrt{92}$	$\sqrt{93}$	$\sqrt{94}$	$\sqrt{95}$	$\sqrt{96}$	$\sqrt{97}$	$\sqrt{98}$	$\sqrt{99}$	$\sqrt{100}$

Exercise 5 H

Simplify the following.

(a) $\sqrt{2}\times 5\sqrt{2}$ (b) $4\sqrt{2}\times\sqrt{8}$ (c) $\sqrt{125}$

(d) $5\sqrt{32}$ (e) $\sqrt{12}+4\sqrt{3}$ (f) $7\sqrt{5}-\sqrt{45}$

(g) $\sqrt{60}\times 2\sqrt{3}$ (h) $\sqrt{300}$ (i) $\sqrt{32}\times\sqrt{18}$

(j) $5\sqrt{30}\times\sqrt{60}$ (k) $6\sqrt{5}-\sqrt{20}$ (l) $\sqrt{48}+4\sqrt{3}$

(m) $\sqrt{18}+8\sqrt{2}$ (n) $\sqrt{8}\times\sqrt{24}$ (o) $5\sqrt{15}\times\sqrt{3}$

Exercise 6 H

Simplify the following.

(a) $\frac{\sqrt{2}\times\sqrt{2}\times\sqrt{5}}{\sqrt{5}}$ (b) $\frac{\sqrt{12}\times\sqrt{3}}{2}$ (c) $\frac{3\times\sqrt{7}\times\sqrt{2}\times\sqrt{7}}{7\sqrt{2}}$

(d) $\sqrt{7}\times 7^{\frac{1}{2}}$ (e) $(4\sqrt{2})^2$ (f) $(\sqrt{3})^5$

Expanding with surds

Example

$$\begin{aligned}\sqrt{5}(4+\sqrt{5}) &= \sqrt{5}\times 4+\sqrt{5}\times\sqrt{5}\\ &= 4\sqrt{5}+5\end{aligned}$$

$$\begin{aligned}2\sqrt{3}(5\sqrt{3}-4) &= 2\times\sqrt{3}\times 5\times\sqrt{3}-2\times\sqrt{3}\times 4\\ &= 10\times 3-8\times\sqrt{3}\\ &= 30-8\sqrt{3}\end{aligned}$$

$$\begin{aligned}(\sqrt{3}+2)(\sqrt{7}-5) &= \sqrt{3}\times\sqrt{7}-\sqrt{3}\times 5+2\times\sqrt{7}-2\times 5\\ &= \sqrt{21}-5\sqrt{3}+2\sqrt{7}-10\end{aligned}$$

$$\begin{aligned}(4-2\sqrt{3})^2 &= (4-2\sqrt{3})(4-2\sqrt{3})\\ &= 4\times 4-4\times 2\times\sqrt{3}-2\times\sqrt{3}\times 4+2\times\sqrt{3}\times 2\times\sqrt{3}\\ &= 16-8\sqrt{3}-8\sqrt{3}+4\times 3\\ &= 28-16\sqrt{3}\end{aligned}$$

Use the acronym FOIL

Exercise 7 H

Expand and simplify the following.

(a) $\sqrt{2}(3+\sqrt{2})$ (b) $4\sqrt{2}(3+\sqrt{2})$ (c) $4\sqrt{2}(3+2\sqrt{2})$

(d) $\sqrt{3}(\sqrt{5}+\sqrt{3})$ (e) $2\sqrt{5}(\sqrt{2}+\sqrt{5})$ (f) $4\sqrt{7}(3+\sqrt{14})$

(g) $5\sqrt{2}(\sqrt{8}+\sqrt{6})$ (h) $2\sqrt{11}(1+\sqrt{22})$ (i) $3\sqrt{3}(\sqrt{6}+\sqrt{12})$

Exercise 8 H

Expand and simplify the following.

(a) $(2+\sqrt{3})(4+\sqrt{2})$ (b) $(2+\sqrt{3})(4+\sqrt{3})$ (c) $(2+3\sqrt{3})(4+\sqrt{3})$

(d) $(2+\sqrt{3})(4-\sqrt{2})$ (e) $(4-\sqrt{2})(1+\sqrt{2})$ (f) $(4-3\sqrt{5})(1+\sqrt{5})$

(g) $(\sqrt{7}+5)(\sqrt{7}-5)$ (h) $(2+3\sqrt{7})(3\sqrt{2}-5)$ (i) $(10-5\sqrt{7})(\sqrt{7}+4)$

(j) $(3+\sqrt{5})^2$ (k) $(5-\sqrt{2})^2$ (l) $(4+9\sqrt{5})^2$

Exercise 9 H

(a) Given that $a = \sqrt{2}$, $b = \sqrt{3}$ and $c = \sqrt{12}$, find the value of abc. Write your answer in the form $n\sqrt{2}$ where n is a whole number.

(b) Given that $p = \sqrt{5}$, $q = \sqrt{10}$, $r = \sqrt{50}$, find the value of the following. Note clearly whether your answers are rational or irrational.

(i) pr (ii) $\frac{pq}{r}$ (iii) $pq + r$

(c) Evaluate $\frac{(3+\sqrt{5})(3-\sqrt{5})}{2}$. Note clearly whether your answer is rational or irrational.

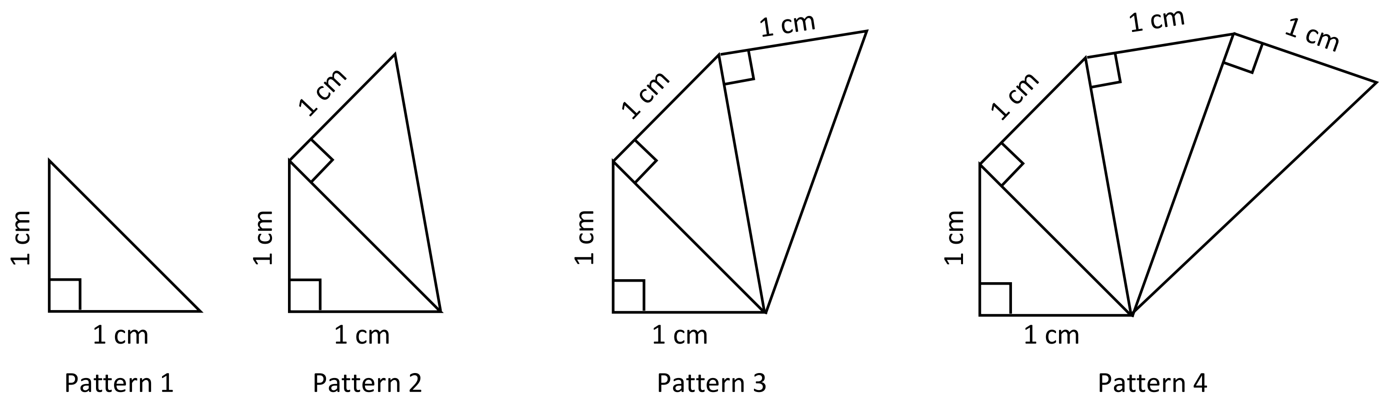
(d) Simplify $\frac{(2\sqrt{5})^2 - \frac{3\sqrt{12}}{\sqrt{3}}}{2}$ and note whether your answer is rational or irrational.

(e) Simplify $(\pi\sqrt{24} - \pi\sqrt{6})^2$, leaving your answer in terms of π.

(f) Write a value for x (with $x > 1$) so that $x^{\frac{3}{2}}$ is rational.

Challenge!

Patterns are produced as shown in the following diagrams.



The diagrams are not drawn to scale.

Find the perimeter of Pattern 6 in the form $a + \sqrt{b}$, where a and b are whole numbers. Show all your working.

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ **Target** ☐

AER, APR

When **borrowing** or **investing** money, it is important to consider the **interest rate** that is used to calculate the **interest** that is added to the loan or investment. Another factor is important however – **how frequently** interest is added. This makes it difficult to directly compare interest rates if the periods for adding interest are different. For example, consider that you want to invest a sum of money. Which option is best for you: an interest rate of 4% paid every six months, or an interest rate of 2% paid every quarter? In order to compare interest rates of this type fairly, we use the special percentages **AER** and **APR**.

AER = Annual Equivalent Rate

AER is used to note the percentage of interest earned in a period of **one year**. It allows you to compare different accounts that pay interest at different times, e.g. every month, every quarter, every six months. The following method for calculating AER is given on page 2 of the examination paper.

AER, as a decimal, is calculated using the formula $\left(1+\frac{i}{n}\right)^n-1$, where i is the nominal interest rate per annum as a decimal and n is the number of compounding periods per annum.

Note that sometimes the 'nominal interest rate' is referred to as the 'gross interest rate'.

Exercise 10

Skill H

Complete the following table. (The first row has been completed for you.)

Interest rate	Number of compounding periods per annum	Nominal interest rate per annum
3%	4	12%
4%	2	
2.5%		7.5%
	12	6%

Example

Calculate the AER for the following two savings accounts: one with an interest rate of 4% paid every six months, and another with an interest rate of 2% paid every quarter.

Interest rate 4% paid every six months

There are 2 compounding periods during the year.
The nominal interest rate per annum is 4% × 2 = 8%.
As a decimal, this is 0.08.

$\text{AER} = \left(1+\frac{0.08}{2}\right)^2-1$
$\text{AER} = 0.0816$

As a percentage, the AER is 8.16%.

Interest rate 2% paid every quarter

There are 4 compounding periods during the year.
The nominal interest rate per annum is 2% × 4 = 8%.
As a decimal, this is 0.08.

$\text{AER} = \left(1+\frac{0.08}{4}\right)^4-1$
$\text{AER} = 0.08243216$

As a percentage, the AER is 8.24%, correct to 2 decimal places.

By comparing the two AER values, it is possible to see that the second account (2% interest paid every quarter) is the better option, as the AER is higher.

Exercise 11

H

Calculate the AER for the following savings accounts.

(a) An interest rate of 5% paid every six months.

(b) An interest rate of 3% paid every quarter.

(c) An interest rate of 7% paid every 4 months.

(d) An interest rate of 2% paid every month.

(e) An interest rate of 8.4% paid every quarter.

(f) An interest rate of 0.25% paid every quarter.

Example

Susan intends to invest £2,500 into a savings account for one year.
HSBC bank offer a nominal interest rate of 3% a year, with interest paid every quarter.

(a) Calculate the AER for HSBC's account.
(b) If Susan decides to invest her money with HSBC for one year, how much money will be in her account at the end of the year?

Answer: (a) With interest paid every quarter, there are 4 compounding periods during the year.
AER = $\left(1+\frac{0.03}{4}\right)^4 - 1$
AER = $0.03033919066 \ldots$
AER = 3.03%, to 2 decimal places.

(b) Method 1: Use the AER.
£2,500 × 103.033919066% = £2,575.85, to the nearest penny.

Use the percentage before rounding off for an accurate answer.

Method 2: Use the nominal interest rate.
3% a year so 3 ÷ 4 = 0.75% a quarter.
£2,500 × 100.75%^{4} = £2,575.85, to the nearest penny.

The £2,500 increases by 0.75% four times.

Exercise 12

Applying

H

(a) Dave intends to invest £4,000 into a savings account for one year. Barclays bank offer a nominal interest rate of 2% a year, with interest to be paid every quarter.

(i) Calculate the AER for Barclays' account.
(ii) If Dave decides to invest the money with Barclays for one year, how much money will be in his account at the end of the year?

(b) Victoria intends to invest £2,500 into a savings account for one year. HSBC bank offer a nominal interest rate of 5% a year, with interest to be paid every month.

(i) Calculate the AER for HSBC's account.
(ii) If Victoria decides to invest the money with HSBC for one year, how much money will be in her account at the end of the year?

(c) Which is better: investing money into an account that offers AER at a rate of 4%, or investing money into an account that offers an interest rate of 1% paid every three months?

(d) Always / sometimes / never: AER is always greater than the nominal interest rate.

Example

(a) Morgan invests £400 with Barclays bank at an AER of 2.4%.
How much money will Morgan have in the bank after 3 years?

(b) Four years ago, Mari invested a sum of money into HSBC bank at an AER of 4.5%. The money is now worth £4,000. What is the minimum amount of money that Mari had to invest in order to accomplish this?

Answer: (a) £400 × 102.4%³ = £429.50, to the nearest penny.

(b) ? × 104.5%⁴ = £4,000
? = £4,000 ÷ 104.5%⁴
? = £3,354.25, to the nearest penny.

Exercise 13

(a) Ffion invests £800 into Lloyds bank at an AER of 3.1%.
How much money will Ffion have in the bank after 5 years?

(b) Three years ago, Jac invested a sum of money into Santander bank at an AER of 2.3%. The money is now worth £1,400. What is the minimum amount of money that Jac had to invest in order to accomplish this?

(c) Meical invests £6,500 into Halifax bank at an AER of 1.7%.
How much money will Meical have in the bank after 2 years?

(d) Nine years ago, Catrin invested a sum of money into Barclays bank at an AER of 6.25%. The money is now worth £20,000. What is the minimum amount of money that Catrin had to invest in order to accomplish this?

(e) Megan has £5,000 to invest in HSBC bank at an AER of 6.4%. In how many years will Megan's money be worth more than £7,000?

Alternative method of calculating AER

As well as the method shown on page 2 of a GCSE examination paper, it is possible to use the following method for calculating AER.

$$\text{AER} = \frac{\text{Interest accrued over one year}}{\text{Initial value}} \times 100\%$$

Example

Calculate the AER for a savings account that offers an interest rate of 4% paid every quarter.

Answer: Imagine that we decide to invest £1,000 into this savings account. After one year, the money will be worth £1,000 × 104%⁴ = £1,169.86 (to the nearest penny), so the interest accrued over one year is £169.86 (to the nearest penny). So, the AER is $\frac{169.86}{1000} \times 100\% = 16.99\%$, to 2 decimal places.

(The previous method gives the same answer, as $\left(1+\frac{0.16}{4}\right)^4 - 1 = 0.16985856 = 16.99\%$, to 2 decimal places.)

Exercise 14

Use the alternative method of calculating AER to calculate the AER for the following savings accounts.

(a) An interest rate of 5% paid every six months.

(b) An interest rate of 3% paid every quarter.

(c) An interest rate of 7% paid every 4 months.

(d) An interest rate of 2% paid every month.

(e) An interest rate of 8.4% paid every quarter.

(f) An interest rate of 0.25% paid every quarter.

APR = Annual Percentage Rate

APR is used to compare accounts where there is a charge for the account, or these are additional costs associated with the account.

For a savings account,

$$\text{APR} = \frac{\text{Interest accrued over one year} - \text{costs}}{\text{Initial value}} \times 100\%$$

For a borrowing account,

$$\text{APR} = \frac{\text{Interest accrued over one year} + \text{costs}}{\text{Initial value}} \times 100\%$$

In most cases, there are **no costs** associated with a **savings** account, so the AER and APR rates are equal to each other. This explains why we see AER rates advertised alongside savings accounts.

In most cases, there **are** costs associated with a **borrowing** account, so we must use the APR rate. This explains why we see APR rates advertised alongside borrowing accounts such as mortgages, credit cards and loans from the bank.

Example

Huw intends to borrow £4,800 from the company *Loans4U*. The company offers an interest rate of 4% a month, and charges an annual fee of £150 to use the account.

(a) How much interest will this loan accrue over a period of one year?
(b) Calculate the APR for this loan.

Answer: (a) There are 12 compounding periods during the year. £4,800 × $104\%^{12}$ = £7,684.95, to the nearest penny. So, £7,684.95 – £4,800 = £2,884.95 of interest is accrued during the year.

(b) $\text{APR} = \frac{\text{Interest accrued over one year} + \text{costs}}{\text{Initial value}} \times 100\%$

$\text{APR} = \frac{2884.95+150}{4800} \times 100\%$

APR = 63.2%, to one decimal place.

Applying H

Exercise 15

(a) Lisa intends to borrow £7,000 from the company *BestLoans*. The company offers an interest rate of 2% a month, and charges an annual fee of £200 to use the account.

(i) How much interest will this loan accrue over a period of one year?
(ii) Calculate the APR for this loan.

(b) Deiniol intends to borrow £24,000 from the company *LoanKing*. The company offers an interest rate of 3% every six months, and charges a monthly fee of £15 for using the account.

(i) How much interest will this loan accrue over a period of one year?
(ii) Calculate the APR for this loan.

(c) Sophie intends to borrow £154,000 from the company *MorgaisGorau*. The company offers an interest rate of 0.4% a month, and charges an annual fee of £300 for using the account.

(i) How much interest will this loan accrue over a period of one year?
(ii) Calculate the APR for this loan.

Example

Calculate the AER or APR for each of the following situations.

Situation 1: Savings account with no costs.

Method 1: Use the formula $\text{AER} = \left(1 + \frac{i}{n}\right)^n - 1$

Nominal interest rate per annum 3% × 4 = 12%.

$\text{AER} = \left(1 + \frac{0.12}{4}\right)^4 - 1$

$\text{AER} = 0.12550881 \ldots$

$\text{AER} = 12.55\%$ to 2 decimal places.

Method 2: Use the formula

$$\text{AER} = \frac{\text{Interest accrued over one year}}{\text{Initial value}} \times 100\%$$

Value at the end of the year

$= £2{,}500 \times 103\%^4$

$= £2{,}813.77$ to the nearest penny.

Interest accrued over one year

$= £2{,}813.77 - £2{,}500$

$= £313.77$

$\text{AER} = \frac{£313.77}{£2{,}500} \times 100\%$

$\text{AER} = 12.55\%$ to 2 decimal places.

Situation 2: Savings account with costs of £40 a year.

We must use the formula

$$\text{APR} = \frac{\text{Interest accrued over one year} - \text{costs}}{\text{Initial value}} \times 100\%$$

Value at the end of the year

$= £2{,}500 \times 103\%^4$

$= £2{,}813.77$ to the nearest penny.

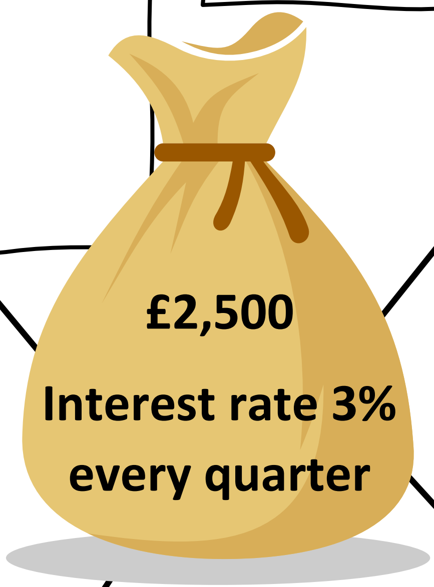
Interest accrued over one year

$= £2{,}813.77 - £2{,}500$

$= £313.77$

$\text{APR} = \frac{£313.77 - £40}{£2{,}500} \times 100\%$

$\text{APR} = 10.95\%$ to 2 decimal places.



Situation 3: Borrowing account with no charges.

The calculations are exactly the same as for situation 1. So, the AER is 12.55% to 2 decimal places.

Because there are no costs, the APR is also 12.55% to 2 decimal places.

Situation 4: Borrowing account with costs of £40 a year.

We must use the formula

$$\text{APR} = \frac{\text{Interest accrued over one year} + \text{costs}}{\text{Initial value}} \times 100\%$$

Loan at the end of the year

$= £2{,}500 \times 103\%^4$

$= £2813.77$ to the nearest penny.

Interest accrued over one year

$= £2{,}813.77 - £2{,}500$

$= £313.77$

$\text{APR} = \frac{£313.77 + £40}{£2{,}500} \times 100\%$

$\text{APR} = 14.15\%$ to 2 decimal places.

Exercise 16 H

Calculate the AER or APR for each of the following situations.

(a) An investment of £1,500 into a savings account that offers an interest rate of 2% per quarter.

(b) An investment of £2,400 into a savings account that offers an interest rate of 5% per year and annual costs of £50.

(c) A loan of £3,500 from an account that offers an interest rate of 3.2% per quarter.

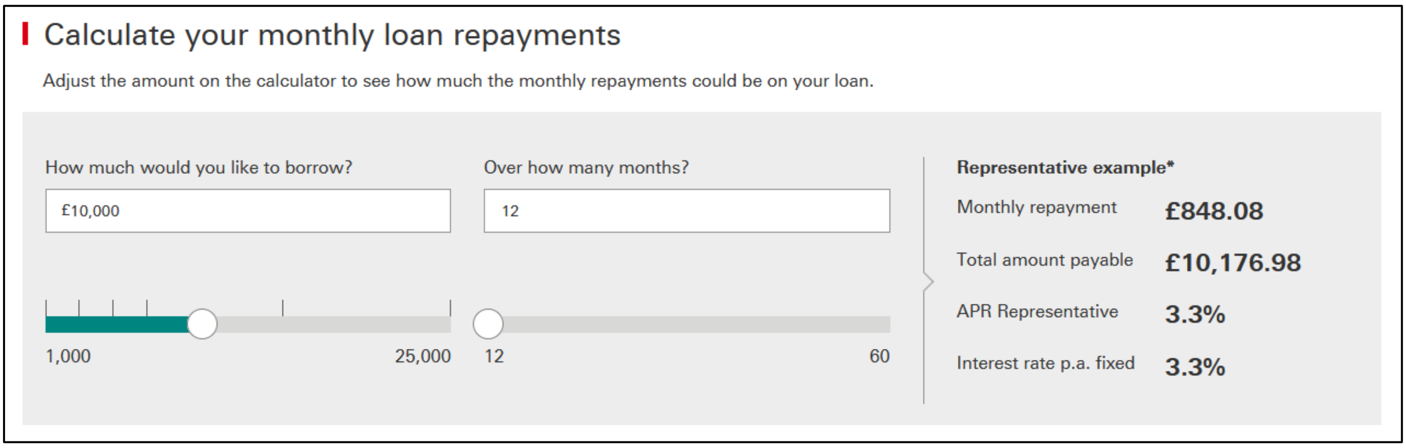
(d) A loan of £15,000 from an account that offers an interest rate of 1.2% per month and annual costs of £150.

(e) A loan of £140,000 from an account that offers an interest rate of 1.8% per quarter and quarterly costs of £50.

(f) An investment of £250,000 into a savings account that offers an interest rate of 0.4% per month and monthly costs of £5.

Challenge!

HSBC's website shows the following information for a personal loan of £10,000 taken over 12 months.



https://www.hsbc.co.uk/loans/products/personal/ , 30/12/2019

3.3% of £10,000 is £330. Why is the total amount payable not £10,330? Investigate...

Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

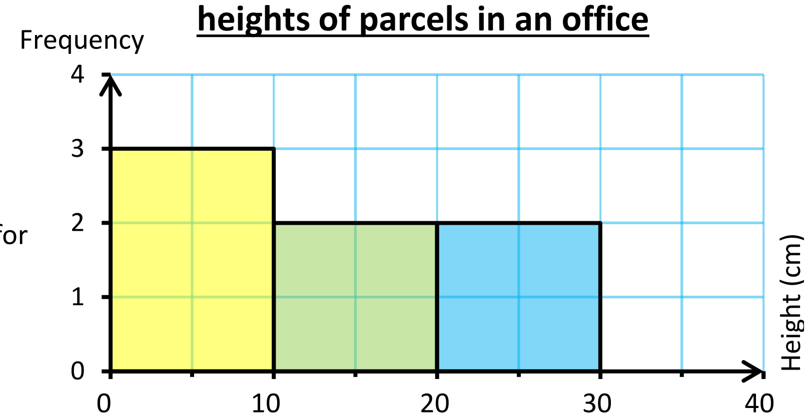
Histograms

Consider the following data that shows the heights of parcels in an office one morning.

Height (h cm)	Frequency
$0 < h \leqslant 10$	3
$10 < h \leqslant 20$	2
$20 < h \leqslant 30$	2

It would be possible to draw a frequency diagram for this data; this is shown on the right.

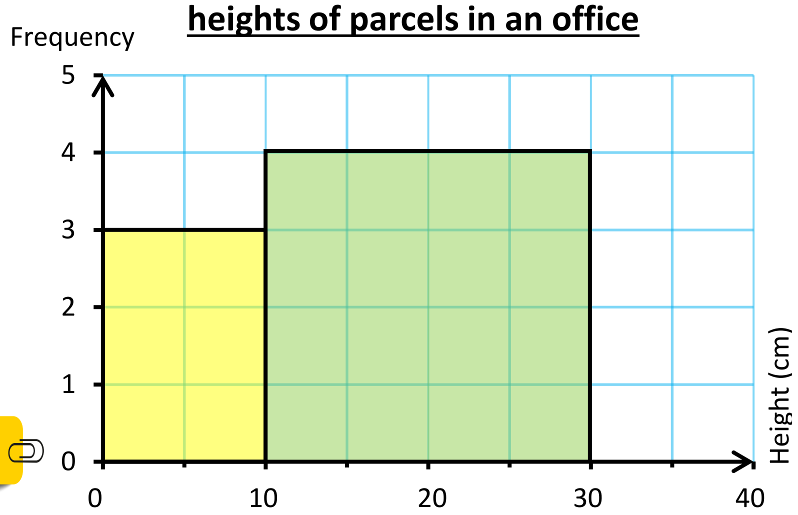
Frequency diagram to show the heights of parcels in an office



Consider now that the same data is grouped as shown below.

Height (h cm)	Frequency
$0 < h \leqslant 10$	3
$10 < h \leqslant 30$	4

Frequency diagram to show the heights of parcels in an office



On drawing a frequency diagram for this data (shown on the right), the diagram does not give a fair reflection of the data – it is unfair to compare classes of different widths. To deal with this, we introduce the idea of drawing a **histogram** to show the data, where we plot not the height against the frequency, but the height against the **frequency density**.

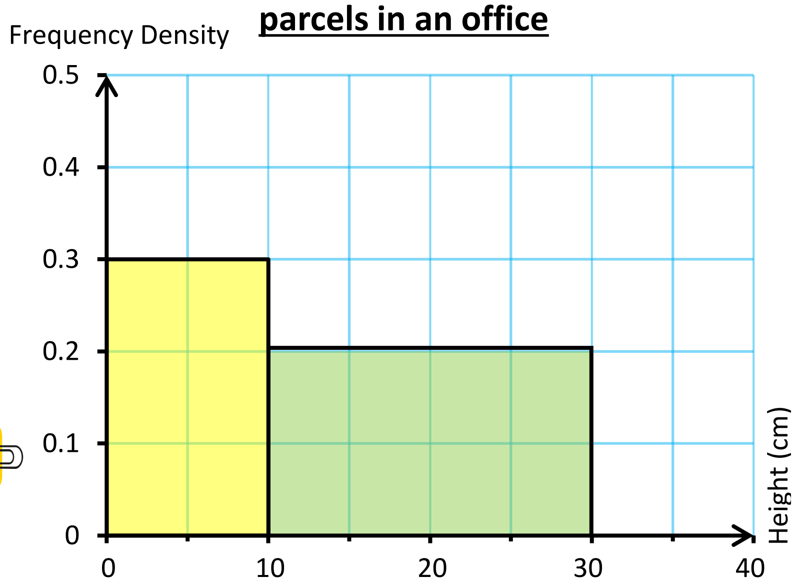
$$\textbf{Frequency Density} = \frac{\text{Frequency}}{\text{Class width}}$$

For the second set of data above, we can calculate the frequency density using the table below.

Height (h cm)	Frequency	Frequency Density
$0 < h \leqslant 10$	3	$3 \div 10 = 0.3$
$10 < h \leqslant 30$	4	$4 \div 20 = 0.2$

We can then draw the histogram on the right to illustrate the data. The histogram gives a fairer reflection of the data, as it takes into account that the second class is wider than the first class.

Histogram to show the heights of parcels in an office



The **area** of a bar in a histogram gives the frequency of the class under consideration.

Frequency = Class width × Frequency Density

For the histogram shown on the right, the frequency corresponding to the first class is $10 \times 0.3 = 3$, and the frequency corresponding to the second class is $20 \times 0.2 = 4$.

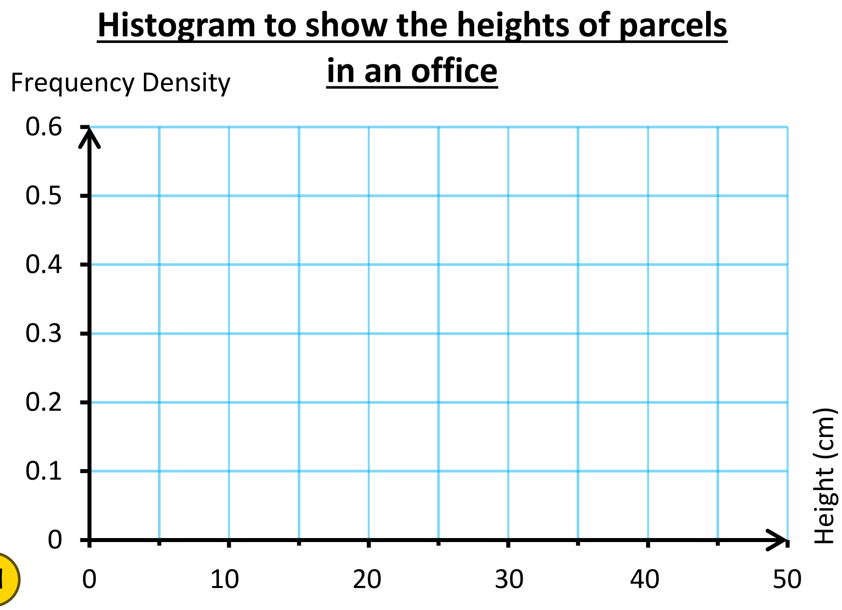
Exercise 17 Skill H

The following data shows the heights of parcels in an office one morning.

Height (h cm)	Frequency	Frequency Density
$0 < h \leqslant 10$	5	
$10 < h \leqslant 30$	6	
$30 < h \leqslant 40$	2	

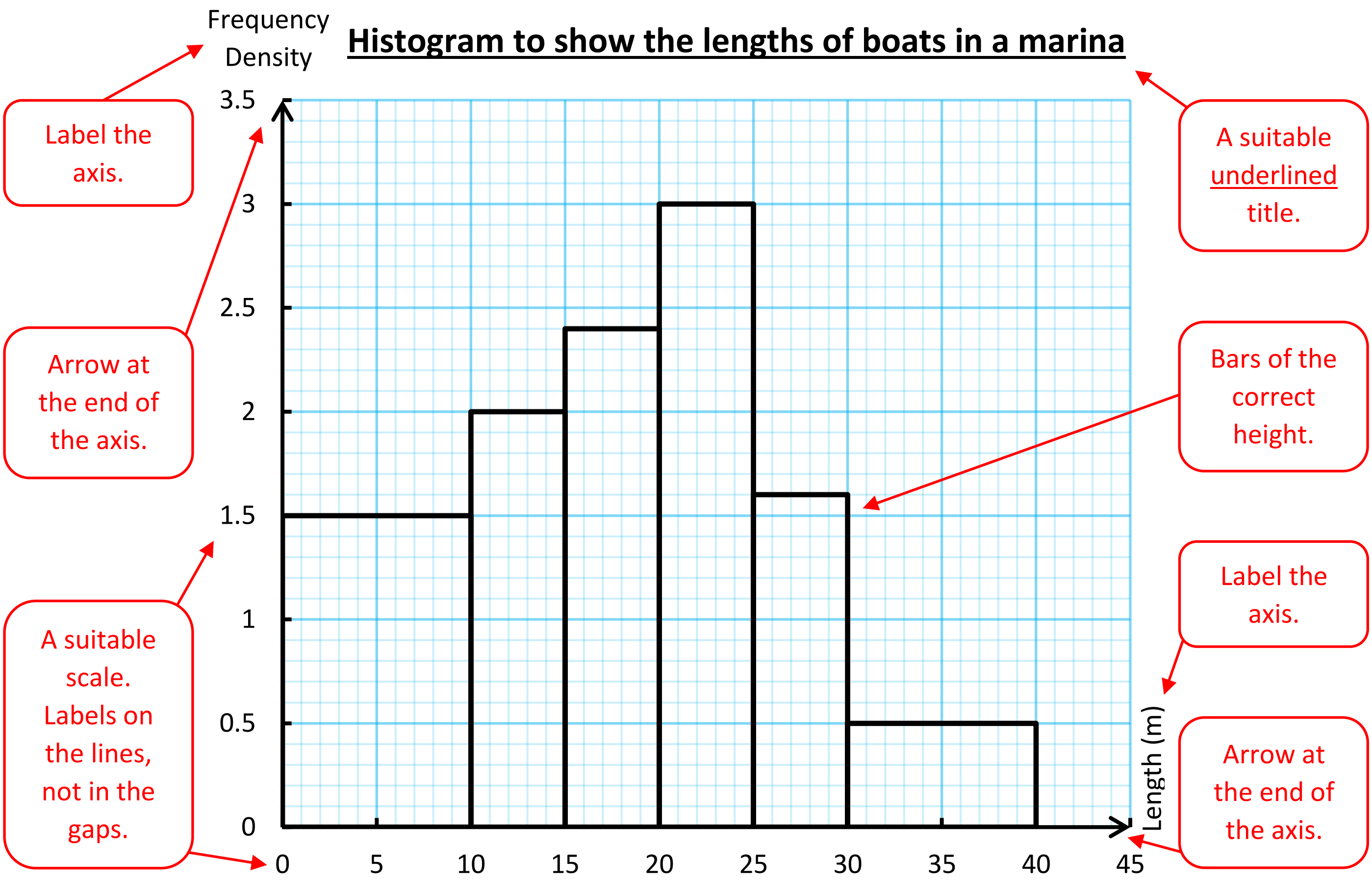
(a) Complete the 'Frequency Density' column in the table.

(b) Use the squared paper to draw a histogram for the data.



Exercise 18 H

The histogram below shows the length of boats in a marina.



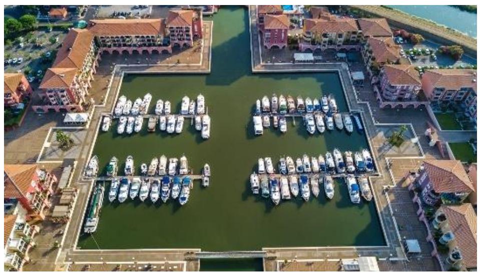
(a) How many boats had a length between 0 m and 10 m?

(b) Complete the frequency table:

Length (l m)	Frequency
$0 < l \leqslant 10$	
$10 < l \leqslant 15$	
$15 < l \leqslant 20$	
$20 < l \leqslant 25$	
$25 < l \leqslant 30$	
$30 < l \leqslant 40$	

(c) How many boats had a length of less than or equal to 20 m?

(d) How many boats were measured in total?



Care with the class width.

H

Exercise 19

Draw histograms for the following sets of data.

(a) The sum raised by a group of people for a charity.

Sum raised (£*s*)	Frequency
$0 < s \leqslant 50$	6
$50 < s \leqslant 100$	22
$100 < s \leqslant 200$	31
$200 < s \leqslant 500$	42
$500 < s \leqslant 1{,}000$	15

(b) The age of people in a hockey club.

Age in years	Frequency
11–15	7
16–18	10
19–24	15
25–34	20
35–49	12
50–64	7

(c) The earnings of a group of students during one week.

Earnings (£*e*)	Frequency
$0 < e \leqslant 20$	5
$20 < e \leqslant 40$	15
$40 < e \leqslant 70$	27
$70 < e \leqslant 100$	30
$100 < e \leqslant 150$	6

(d) The weights of passengers' bags on an aircraft.

Weight (*w* kg)	Frequency
$0 < w \leqslant 5$	7
$5 < w \leqslant 10$	12
$10 < w \leqslant 20$	24
$20 < w \leqslant 40$	15
$40 < w \leqslant 50$	3

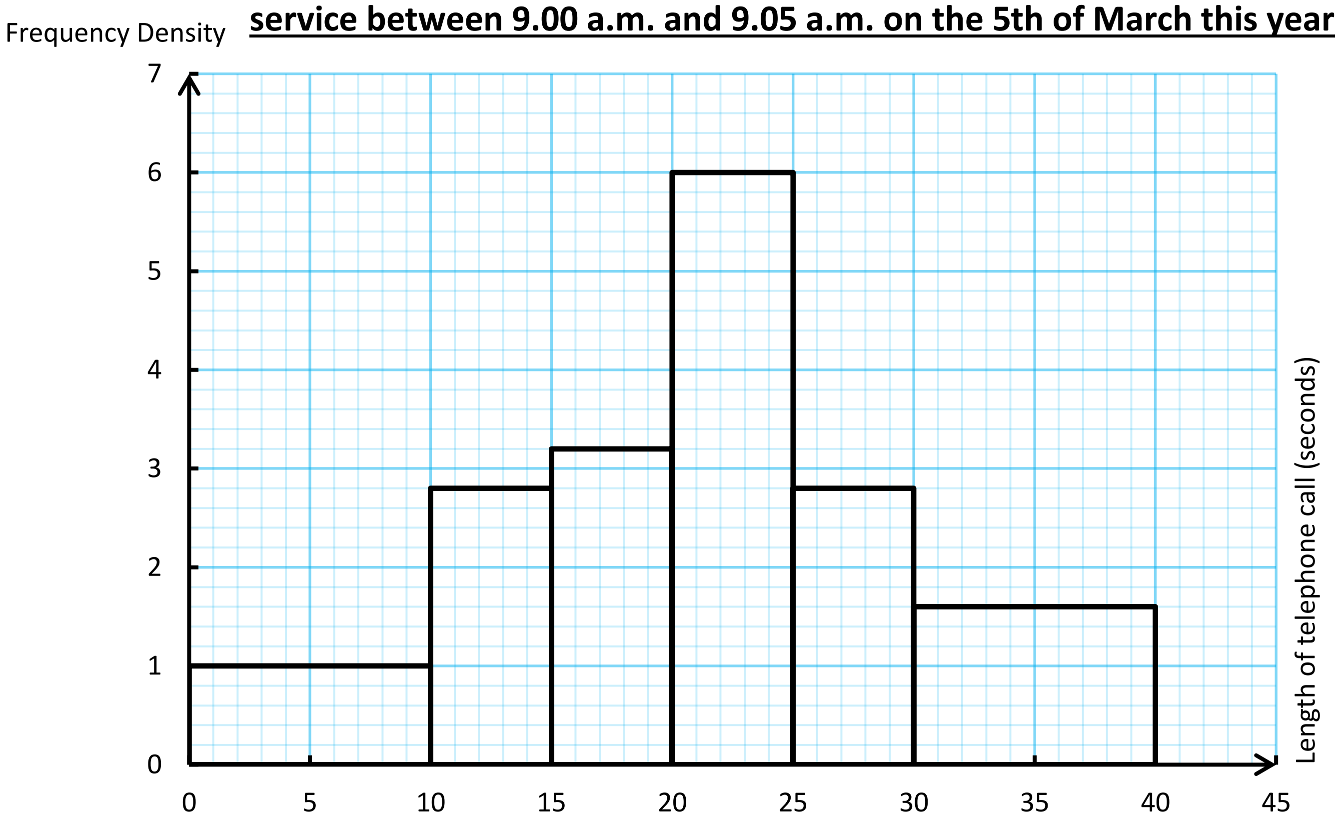
Exercise 20

Applying

H

The histogram below shows the lengths of telephone calls made to a directory enquiries service between 9.00 a.m. and 9.05 a.m. on the 5th of March this year.

Histogram to show the lengths of telephone calls made to a directory enquiries service between 9.00 a.m. and 9.05 a.m. on the 5th of March this year

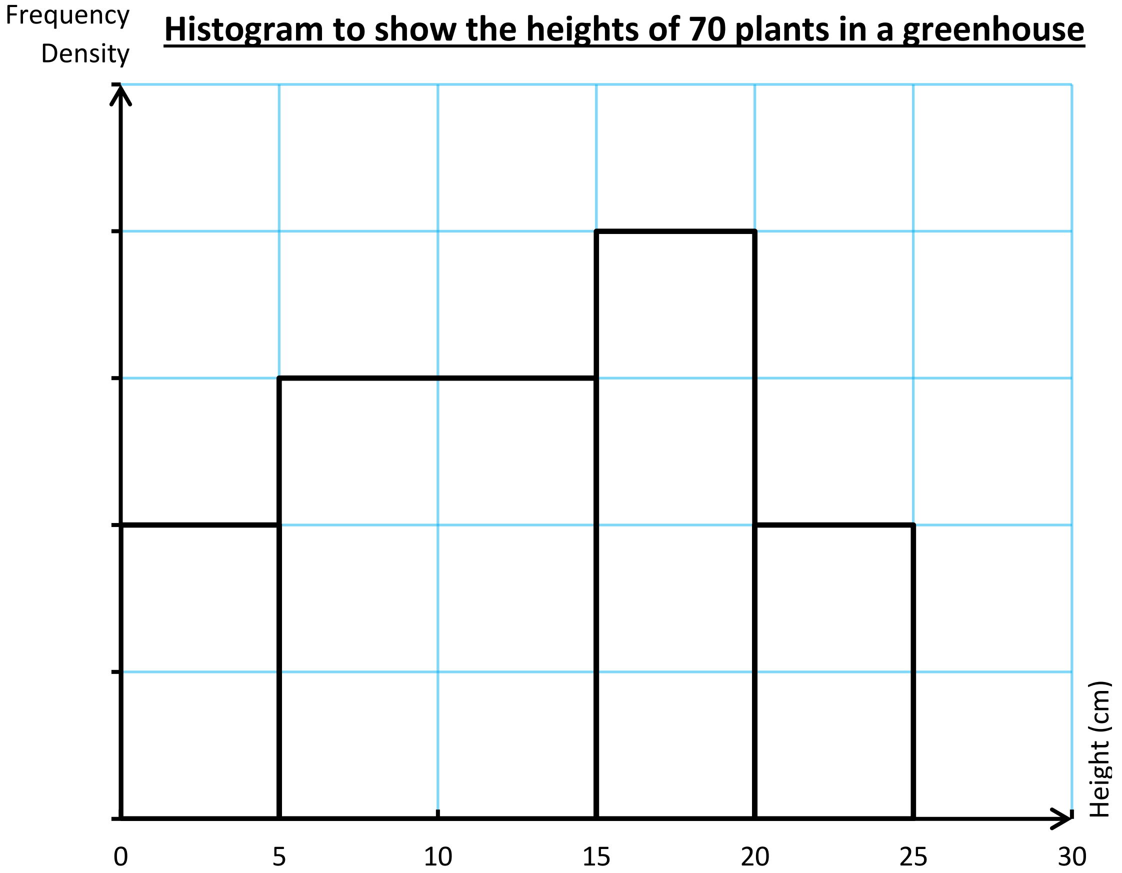


Use the histogram to calculate how many telephone calls were made to the directory enquiries service between 9.00 a.m. and 9.05 a.m. on the 5th of March this year.

Exercise 21

H

The following histogram shows the height distribution of 70 plants in a greenhouse.



(a) Complete the missing scale on the vertical axis.

(b) How many plants had a height of between 15 cm and 20 cm?

(c) Complete the following frequency table.

Height (h cm)	Frequency
$0 < h \leqslant 5$	
$5 < h \leqslant 15$	
$15 < h \leqslant 20$	
$20 < h \leqslant 25$	

(d) Calculate an estimate of the amount of plants of height less than 10 cm.

(e) What is the modal class of the data?

(f) Calculate an estimate of the mean height of the plants.

(g) Calculate an estimate of the range of the heights of the plants.

(h) What percentage of all the plants have a height of more than 20 cm?

(i) What fraction of all the plants have a height of more than 5 cm? Give your answer in its simplest form.

(j) Which class is the median class of the data?

Estimating the median from a histogram

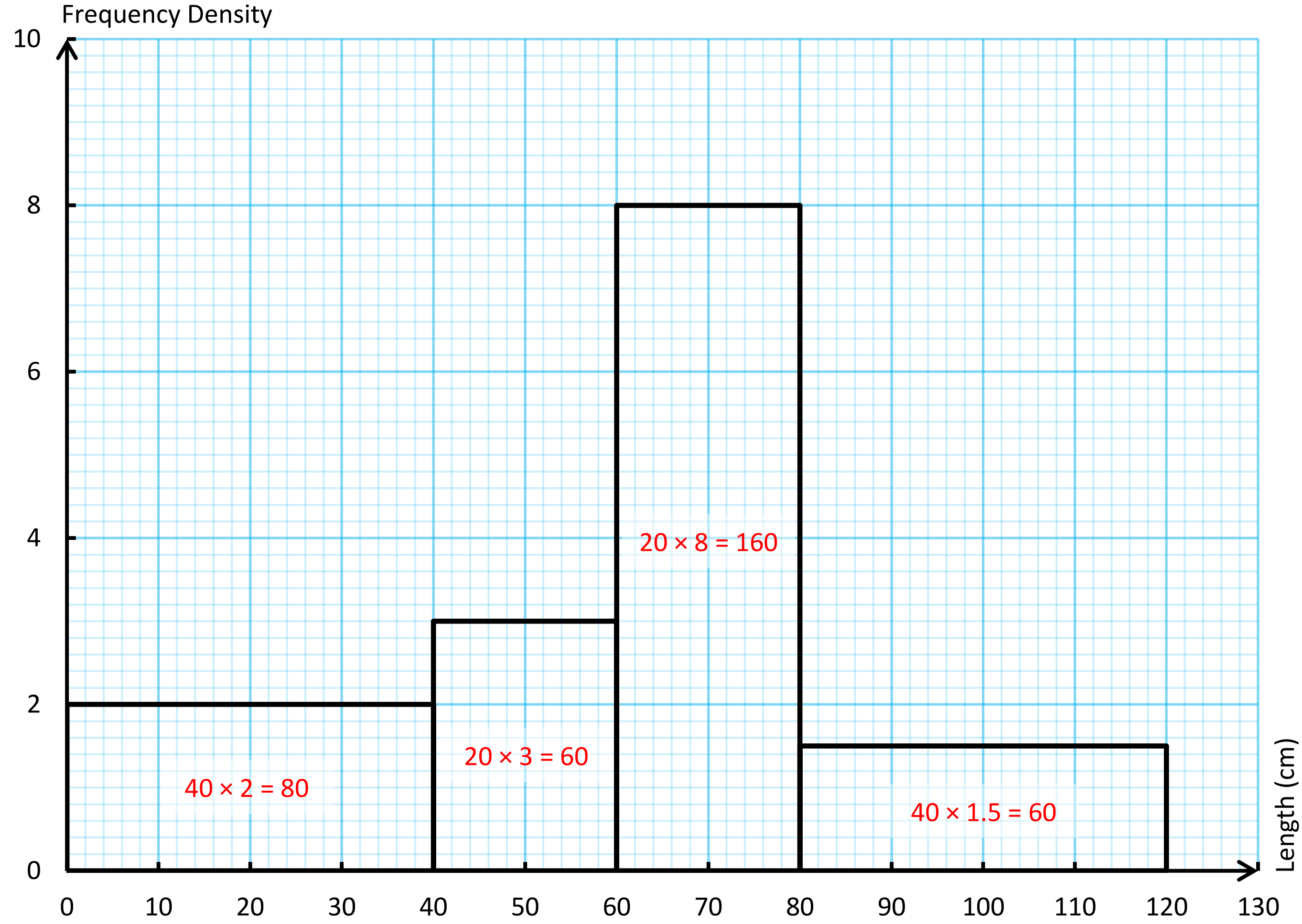
For any histogram,

The estimate of the median is the vertical line in the histogram that halves the total area of the histogram.

Example

Let us consider the following histogram that represents the results of collecting and measuring the lengths of driftwood on a beach.

Histogram to show the lengths of driftwood on a beach



By calculating the area of each bar in the histogram (shown above in red), and adding the results, we see that a total of 80 + 60 + 160 + 60 = 360 pieces of driftwood were collected and measured.

To estimate the median length of a piece of driftwood, we need to draw a vertical line in the histogram that halves the total area of the histogram. Because 360 ÷ 2 = 180, we need to draw a vertical line in the histogram so that an area of 180 squared units is found on either side of the vertical line. This line must be in the third bar, as 80 + 60 = 140 is less than 180, and 80 + 60 + 160 = 300 is greater than 180.

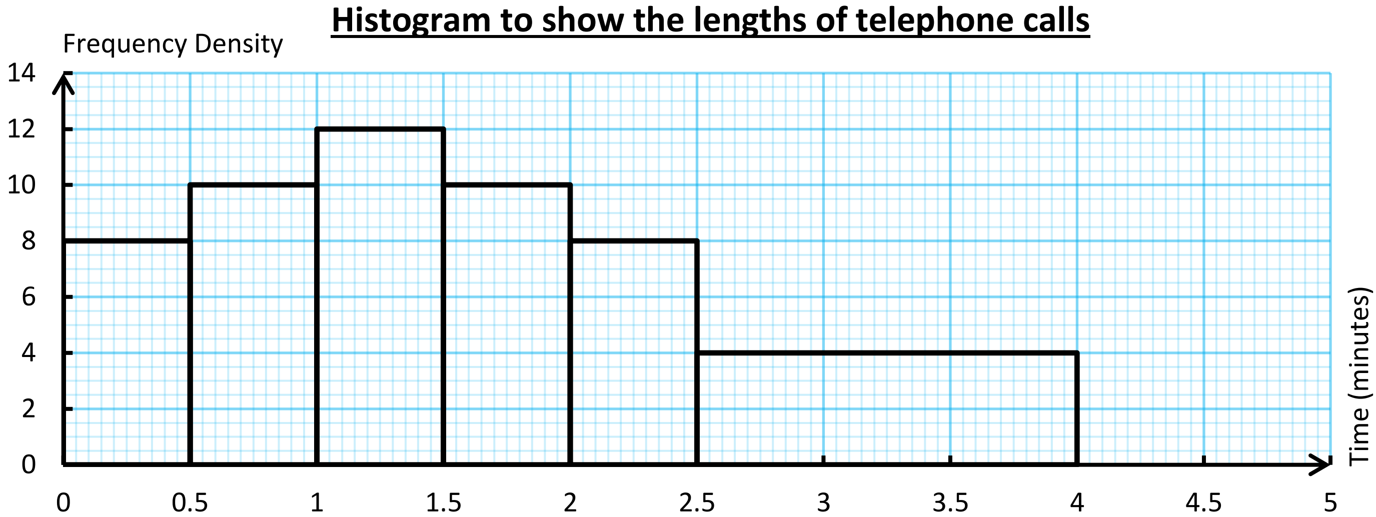
We need to travel across the third bar by the fraction $\frac{180-140}{160} = \frac{40}{160} = \frac{1}{4}$.

The width of the third bar is 20 cm, so we need to travel across the third bar by a distance of $20 \times \frac{1}{4} = 5$ cm.

So, the estimate of the median length of a piece of driftwood is 60 + 5 = **65 cm**.

Exercise 22 H

The histogram below represents the results of recording the length of a number of telephone calls.

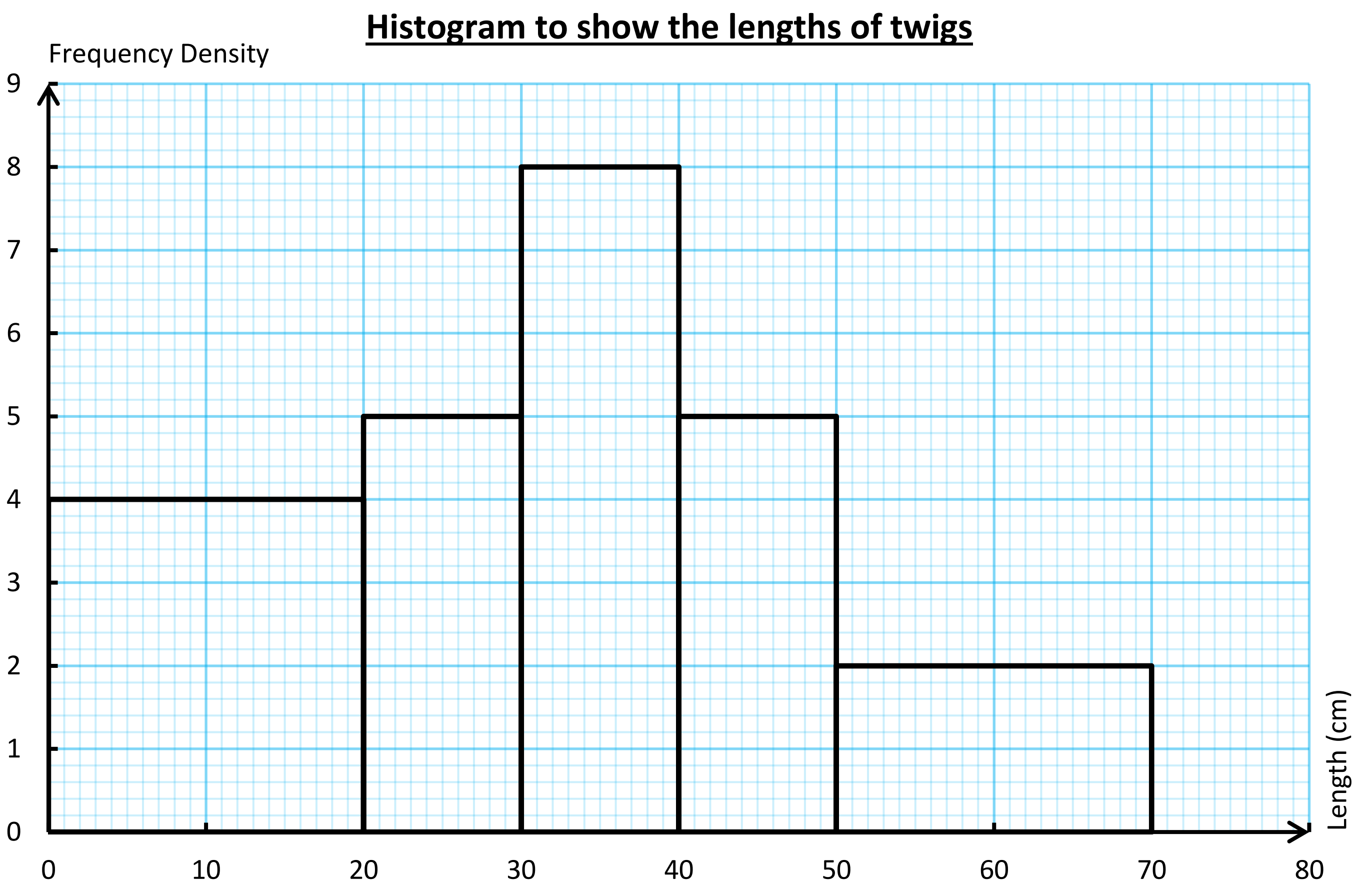


(a) Use the histogram to calculate the total number of telephone calls.

(b) Find an estimate for the median length of a telephone call, in minutes.

Exercise 23 H

The histogram below represents the results of recording the length of a number of twigs.



(a) Use the histogram to calculate the total number of twigs.

(b) Find an estimate for the median length of a twig, in cm.

Estimating the quartiles from a histogram

For any histogram,

The estimate of the lower quartile is the vertical line in the histogram that splits the histogram's area into the ratio 1 : 3.

The estimate of the upper quartile is the vertical line in the histogram that splits the histogram's area into the ratio 3 : 1.

Example

Let us again consider the histogram from page 17 that represents the results of collecting and measuring the lengths of driftwood on a beach.

By calculating the area of each bar in the histogram, and adding the results, we see that a total of 80 + 60 + 160 + 60 = 360 pieces of driftwood were collected and measured.

To estimate the lower quartile, we need to draw a vertical line in the histogram that splits the histogram's area into the ratio 1 : 3. Because 360 ÷ 4 = 90, we need to draw a vertical line in the histogram so that an area of 90 squared units lies to the left of the vertical line, and an area of 90 × 3 = 270 squared units lies to the right of the vertical line. This line must be in the second bar, as 80 is less than 90, and 80 + 60 = 140 is greater than 90.

We need to travel across the second bar by the fraction $\frac{90-80}{60} = \frac{10}{60} = \frac{1}{6}$.

The width of the second bar is 20 cm, so we need to travel across the second bar by a distance of $20 \times \frac{1}{6} = \frac{10}{3} = 3\frac{1}{3}$ cm.

So, the estimate of the lower quartile is $40 + 3\frac{1}{3} = \mathbf{43\frac{1}{3}}$ **cm**.

To estimate the upper quartile, we need to draw a vertical line in the histogram that splits the histogram's area into the ratio 3 : 1. Because 360 ÷ 4 = 90, and 90 × 3 = 270, we need to draw a vertical line in the histogram so that an area of 270 squared units lies to the left of the vertical line, and an area of 90 squared units lies to the right of the vertical line. This line must be in the third bar, as 80 + 60 = 140 is less than 270, and 80 + 60 + 160 = 300 is greater than 270.

We need to travel across the third bar by the fraction $\frac{270-140}{160} = \frac{130}{160} = \frac{13}{16}$.

The width of the third bar is 20 cm, so we need to travel across the third bar by a distance of $20 \times \frac{13}{16} = 16.25$ cm.

So, the estimate of the upper quartile is 60 + 16.25 = **76.25 cm**.

Exercise 24 H

For the histogram in Exercise 22,

(a) Find an estimate for the lower quartile;

(b) Find an estimate for the upper quartile.

Exericse 25 H

For the histogram in Exercise 23,

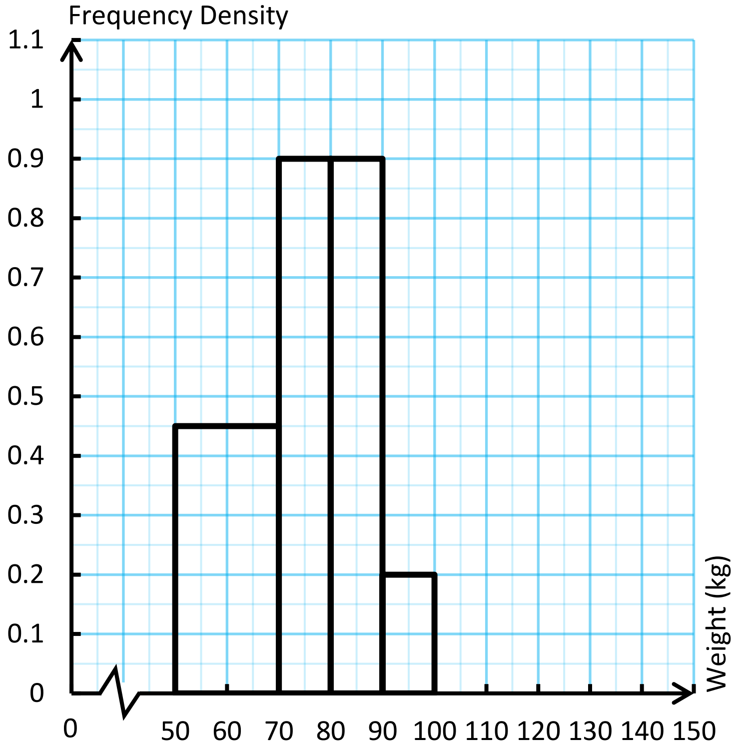
(a) Find an estimate for the lower quartile;

(b) Find an estimate for the upper quartile.

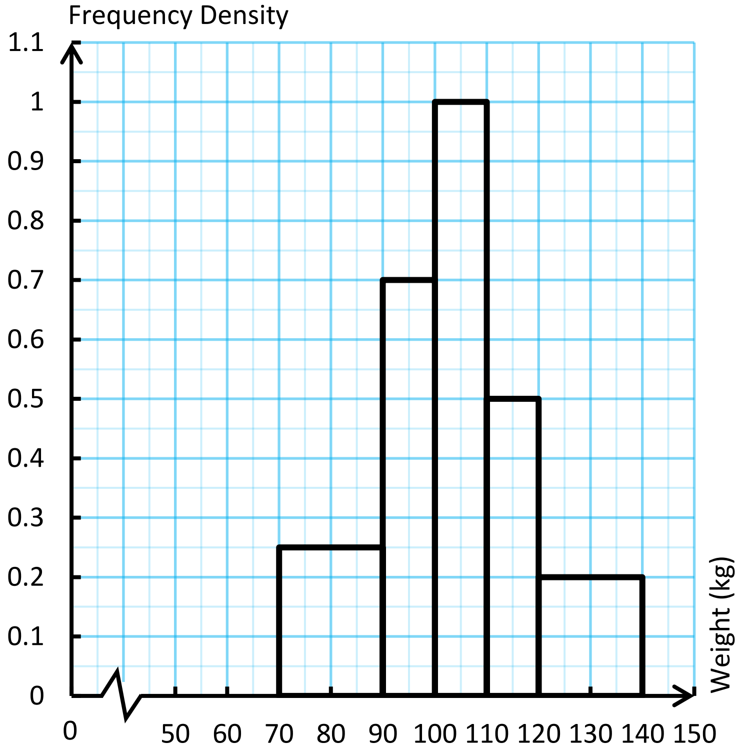
Comparing histograms

Exercise 26

<u>A histogram to show the weight of Wales' rugby squad at the 2017 World Cup (women)</u>



<u>A histogram to show the weight of Wales' rugby squad at the 2019 World Cup (men)</u>



The two histograms on the left show information about the weights of Wales' rugby squads at the 2017 World Cup (women) and the 2019 World Cup (men).

(a) How many women weighed between 70 kg and 80 kg?

(b) How many men weighed between 110 kg and 120 kg?

(c) Complete the following frequency table for the women.

Weight (w kg)	Frequency
$50 < w \leqslant 70$	
$70 < w \leqslant 80$	
$80 < w \leqslant 90$	
$90 < w \leqslant 100$	

(d) Complete the following frequency table for the men.

Weight (w kg)	Frequency
$70 < w \leqslant 90$	
$90 < w \leqslant 100$	
$100 < w \leqslant 110$	
$110 < w \leqslant 120$	
$120 < w \leqslant 140$	

(e) How many women were in the squad in total?

(f) How many men were in the squad in total?

(g) Find an estimate for the median weight of a woman in the 2017 rugby squad.

(h) Find an estimate for the median weight of a man in the 2019 rugby squad.

(i) On average, which squad was heaviest?

(j) What is the greatest possible range of the women's rugby squad?

(k) What is the greatest possible range of the men's rugby squad?

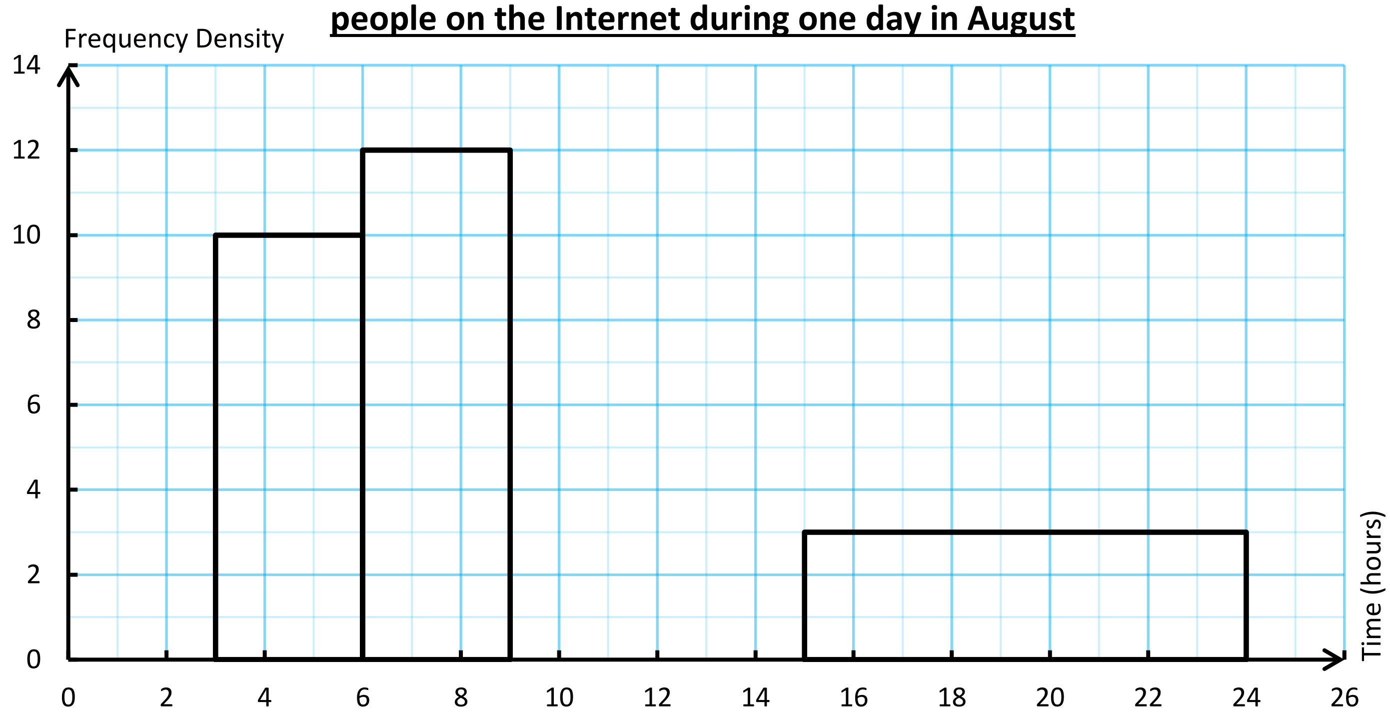
(l) Use your answers to (j) and (k) above to comment on which squad had the most consistent weight.

Exercise 27 (Revision) H

The following histogram and frequency table shows some information about the time each person, in a group of people, spent on the Internet during one day in August.

Time (*t* hours)	Frequency
$0 < t \leqslant 3$	24
$3 < t \leqslant 6$	
$6 < t \leqslant 9$	36
$9 < t \leqslant 15$	30
$15 < t \leqslant 24$	

<u>Histogram to show the time spent by a group of people on the Internet during one day in August</u>



(a) Complete the frequency table and histogram shown above.

(b) Calculate an estimate for the median time spent on the Internet by the group of people during the day in August.

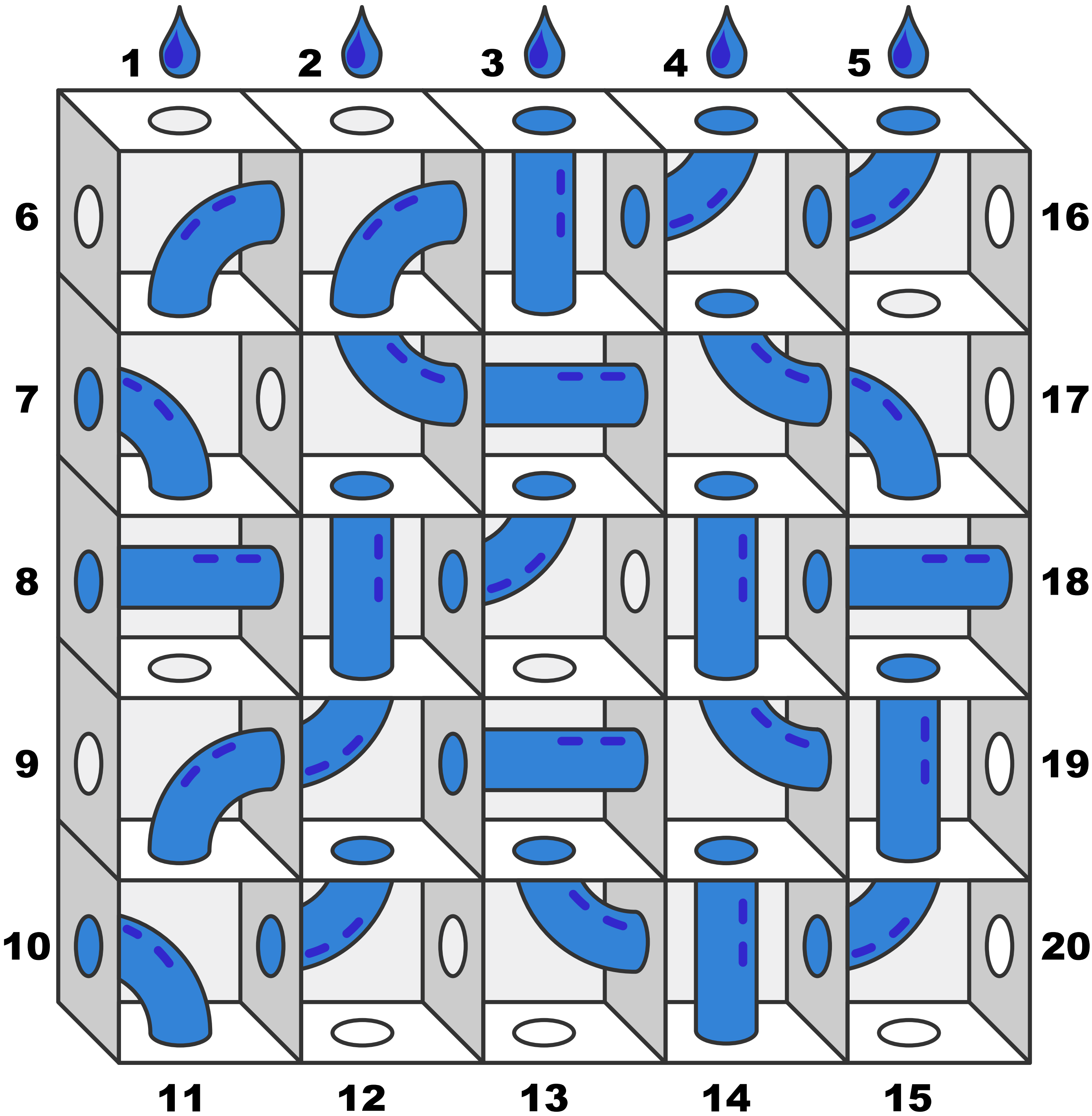
Evaluation

Key Words	Further Questions	What went well?	To reach my target grade I will...
			Grade ☐ Target ☐

Puzzle

The front of the tank below is solid and transparent.

Where will the liquid pour out if it is poured into hole 1?
What about hole 2? Hole 3? Hole 4? Hole 5?



The End of Year 11

Reflection

Name: .... Percentage in the test:	I know this.	I need to revise this.	Question in the test:	Correct in the test?
I can differentiate between **rational numbers** and **irrational numbers**.			1	
I can **collect like surd terms**, e.g. $3\sqrt{2}+7\sqrt{2}$.			2	
I can **simplify surds**, e.g. $\sqrt{32}$, $\frac{\sqrt{8}\times\sqrt{2}}{2}$.			2, 4	
I can **expand using surds**, e.g. $4\sqrt{7}(\sqrt{7}+3)$, $(4+\sqrt{5})(\sqrt{5}-2)$.			3	
I can **calculate the AER** for a savings account using the formula $\left(1+\frac{i}{n}\right)^n-1$.			5	
I can use the AER to calculate **how much money** is in a savings account, at the end or start of an investment.			5, 6, 7	
I can **calculate the APR** for savings or borrowing accounts where there is an additional charge associated with the account.			8	
I can **draw a histogram** for a specific data set.			9	
I can **re-create the frequency table** for a specific histogram.			9	
I can **estimate the median** for a specific histogram.			9	
I can estimate the **lower quartile and upper quartile** for a specific histogram.				
I can **compare two histograms** using averages and measures of spread.				

Am I ready for the test?

Tick the boxes below...

I have revised the work in my mathematics book.	☐
I have revised the workbook.	☐
I have watched the relevant videos on YouTube.	☐
I have completed the Diagnostic Questions quiz.	☐
I have completed at least 4 pages in my revision book.	☐