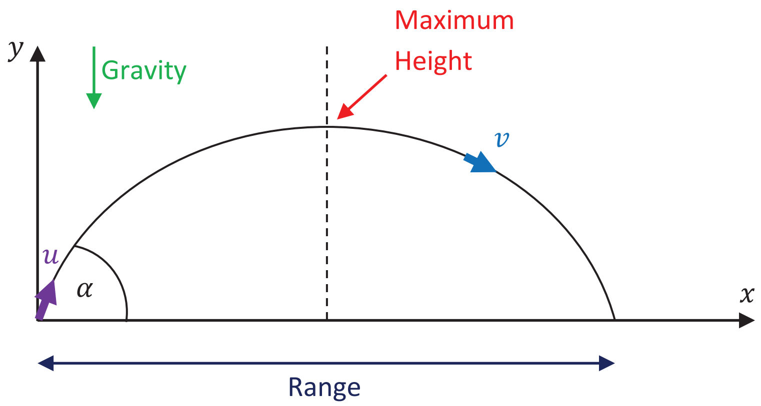
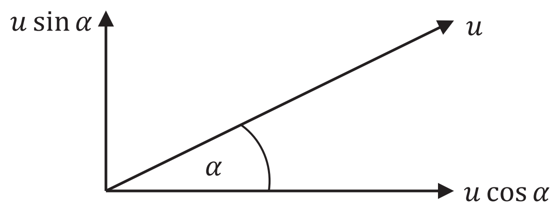
5. Motion under Gravity in Two Dimensions.

A projectile is a particle travelling through space that has been launched in some way. We will consider its motion in two dimensions – *horizontally* and *vertically*. Air resistance will be **ignored**. Once launched, there is only one force acting on the particle, namely gravity.



If the initial velocity is u, projected at an angle α to the horizontal, then the initial horizontal velocity is $u_x = u\cos\alpha$ and the initial vertical velocity is $u_y = u\sin\alpha$.



At time t, the horizontal velocity will still be $u\cos\alpha$ because we assume there is no air resistance. Vertically, we must consider the effect of gravity, so that the vertical velocity at time t is given by the equation of motion

$$v = u + at$$
$$v_y = u\sin\alpha - gt$$

As there is no acceleration horizontally, the horizontal distance from the origin at time t is given by

Distance = Velocity × Time

$$S_x = (u\cos\alpha)t$$

The vertical distance from the origin at time t is given by the equation of motion

$$S = ut + \frac{1}{2}at^2$$
$$S_y = (u\sin\alpha)t - \frac{1}{2}gt^2.$$

We can summarise the above results in the following table.

Symbol	*Horizontal (x)*	*Vertical (y)*
u	$u\cos\alpha$	$u\sin\alpha$
a	0	−g
v	$u\cos\alpha$	$u\sin\alpha - gt$
S	$(u\cos\alpha)t$	$(u\sin\alpha)t - \frac{1}{2}gt^2$

Enghraifft (Haf 2006)

A stone is projected in a direction which makes an angle of 45° above the horizontal. It strikes a small target whose horizontal and vertical distances from the point of projection are 120m and 41.6m respectively. The target is above the level of the point of projection.
(a) Find the speed of projection and show that the time taken for the stone to reach the target is 4s.
(b) Determine, correct to two decimal places, the speed and direction of motion of the stone as it hits the target.

Ateb

(a) We are given $S_x = 120$m, $S_y = 41.6$m.

So $(u\cos\alpha)t = 120$ and $(u\sin\alpha)t - \frac{1}{2}gt^2 = 41.6$

$(u\cos 45°)t = 120$ $\quad$ $(u\sin 45°)t - \frac{1}{2}gt^2 = 41.6$

$\frac{ut}{\sqrt{2}} = 120$ $\quad$ $\frac{ut}{\sqrt{2}} - \frac{1}{2}gt^2 = 41.6$

Substituting from the first equation into the second equation, we obtain

$120 - \frac{1}{2}gt^2 = 41.6$

$120 - 41.6 = \frac{1}{2}gt^2$

$16 = t^2$

$t = 4$s.

Substituting back into the first equation,

$\frac{4u}{\sqrt{2}} = 120$

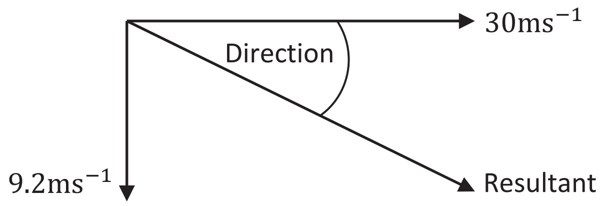
$u = 30\sqrt{2}\text{ms}^{-1}$.

$u = 42.43\text{ms}^{-1}$, correct to 2 d.p.

(b) When $t = 4s$, we have $v_x = u\cos\alpha$ and $v_y = u\sin\alpha - gt$

$v_x = 30\sqrt{2}\cos 45°$ $\quad$ $v_y = 30\sqrt{2}\sin 45° - 9.8 \times 4$

$v_x = 30\text{ms}^{-1}$ $\quad$ $v_y = -9.2\text{ms}^{-1}$



The resultant of these horizontal and vertical velocities is given by $\sqrt{30^2 + 9.2^2} = 31.38\text{ms}^{-1}$, to 2 d.p.

The direction of the resultant velocity is given by $\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{9.2}{30}\right) = 17.05°$, to 2 d.p.

It follows that, as the stone hits its target, it is moving with a velocity of 31.38ms^{-1} at an angle of $17.05°$ below the horizontal.

Atebion yr Hen Gwestiynau Arholiad

Haf 2007 (a) 5.3ms^{-1} (b) t = 6s.
Haf 2008 (a) S_x = 24m. (b) Speed = 13.24ms^{-1}, to 2 d.p.; Direction = 25.02° below the horizontal, to 2 d.p.
Haf 2009 (a) t = 2.4s. (b) S_y = 5.376m. (c) t = $\frac{10}{7}$s.
Haf 2010 (b) t = 4s (c) 25.9ms^{-1}.
Haf 2011 (a) 4.78s, to 2 d.p. (b) 28.68m, to 2 d.p. (c) Magnitude = 44.75ms^{-1}, to 2 d.p.; Direction = 82.29° below the horizontal, to 2 d.p.
Haf 2012 (a) Horizontal $\frac{4}{5}V$; Vertical $\frac{3}{5}V$ (c) $T = \frac{12}{7}$s; $V = 8.75\text{ms}^{-1}$ (d) 13.5ms^{-1} to 1 d.p.
Haf 2013 (a) (i) t = 0.75s (ii) 4.99375m (b) 15.64ms^{-1}, to 2 d.p.
Haf 2014 (a) Horizontal 16.7ms^{-1}; Vertical 13.45ms^{-1} (b) 20.11ms^{-1}, to 2 d.p.; 33.33° to 2 d.p. (c) 4.11m to 2 d.p.
Haf 2015 (a) The ball does **not** fall into the lake as the range (120m) is greater than 117.5m. (b) Magnitude = 28.89ms^{-1}, to 2 d.p.; Direction = 43.36° above the horizontal, to 2 d.p.
Haf 2016 (a) 53.04m to 2 d.p. (b) 7.66m to 2 d.p. (c) Magnitude = 24.5ms^{-1}; Direction = 30° below the horizontal.
Haf 2017 (a) v = $15\sqrt{3}\text{ms}^{-1}$ (b) t = 0.6s ✓ (c) 10.33ms^{-1} to 2 d.p.
Haf 2018 (b) θ = 27.82° to 2 d.p.; v = 36.79ms^{-1} to 2 d.p.

Derivations of Formulae

You may be expected to derive the following equations in an examination. In questions where derivation of formulae has not been requested, the quoting of these formulae will **not** gain full credit.

(a) *Greatest Height.* (This is the maximum height of the particle during its flight.)

At the greatest height of the particle, the vertical velocity, v_y, is zero.

(i) Greatest Height.

Using $v^2 = u^2 + 2aS$

$$v_y^2 = u_y^2 - 2gS_y$$
$$0 = (u\sin\alpha)^2 - 2gS_y$$
$$S_y = \frac{(u\sin\alpha)^2}{2g}.$$

(ii) Time at Greatest Height.

Using $v = u + at$

$$v_y = u_y - gt$$
$$0 = u\sin\alpha - gt$$
$$t = \frac{u\sin\alpha}{g}.$$

(b) *Time of Flight.* (This is the time taken to return to the ground.)

Method (i): Double the time at the greatest height (above) to get $\frac{2u\sin\alpha}{g}$.

Method (ii): Use the equation of motion $S = ut + \frac{1}{2}at^2$

$$S_y = u_y t - \frac{1}{2}gt^2$$
$$0 = (u\sin\alpha)t - \frac{1}{2}gt^2$$
$$0 = t(u\sin\alpha - \frac{1}{2}gt)$$

Either $t = 0$ or $t = \frac{2u\sin\alpha}{g}$.

(c) *Range*. (This is the horizontal distance travelled.)

Horizontally, we have constant velocity, so

Distance = Velocity × Time

Range = Initial Horizontal Velocity × Time of Flight

Range = $u\cos\alpha \times \left(\frac{2u\sin\alpha}{g}\right)$

Range = $\frac{u^2 \times 2\cos\alpha\sin\alpha}{g}$

Range = $\frac{u^2\sin 2\alpha}{g}$.

Double angle formulae

(d) *Equation of path.* (This is the equation of the path the particle takes during its flight.)

Consider the motion of the particle from the origin O to a general point (x, y). Vertically, using the equation of motion $S = ut + \frac{1}{2}at^2$, we find that $S_y = u_y t - \frac{1}{2}gt^2$

$$S_y = (u\sin\alpha)t - \frac{1}{2}gt^2.$$

Therefore, for the general point (x, y), we have $y = (u\sin\alpha)t - \frac{1}{2}gt^2$.

Horizontally, we have $S_x = u_x t$

$$S_x = (u\cos\alpha)t.$$

Therefore, for the general point (x, y), we have $x = (u\cos\alpha)t$, so it follows that $t = \frac{x}{u\cos\alpha}$.

Substituting for t into our equation for y, we find that

$$y = (u\sin\alpha)\left(\frac{x}{u\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{1}{2}g\left(\frac{x^2}{u^2\cos^2\alpha}\right)$$

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}.$$

Recall that $\sec\alpha = \frac{1}{\cos\alpha}$

Using $\sec^2\alpha = 1 + \tan^2\alpha$,

$$y = x\tan\alpha - \frac{gx^2(1+\tan^2\alpha)}{2u^2}.$$

This is the **equation of path**. Given a general point (x, y) on the path and the initial velocity u, this can be used to find the angle of projection α.

Enghraifft

A particle is projected at 25ms^{-1} at an angle α to the horizontal, where $\tan\alpha = \frac{4}{3}$. Find the equation of path and the direction of motion at $t = 3$s.

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Vertically, we have $S_y = u_y y - \frac{1}{2}gt^2$

$$S_y = (25\sin\alpha)t - \frac{1}{2}gt^2.$$

Horizontally, we have $S_x = u_x t$

$$S_x = (25\cos\alpha)\,t$$

$$\frac{S_x}{25\cos\alpha} = t.$$

Substituting for t from the second equation into the first equation,

$$S_y = (25\sin\alpha)\left(\frac{S_x}{25\cos\alpha}\right) - \frac{1}{2}g\left(\frac{S_x}{25\cos\alpha}\right)^2$$

$$S_y = S_x\tan\alpha - \frac{gS_x^2}{2(625)\cos^2\alpha}$$

$$S_y = S_x\left(\frac{4}{3}\right) - \frac{gS_x^2\sec^2\alpha}{1250}$$

$$S_y = \frac{4}{3}S_x - \frac{gS_x^2(1+\tan^2\alpha)}{1250}$$

$$S_y = \frac{4}{3}S_x - \frac{gS_x^2(1+\frac{16}{9})}{1250}$$

$$S_y = \frac{4}{3}S_x - \frac{49}{2250}S_x^2.$$

It follows that $y = \frac{4}{3}x - \frac{49}{2250}x^2$ is the equation of path.

When $t = 3$s, we have $v_x = 25\cos\alpha$ and $v_y = u_y - gt$

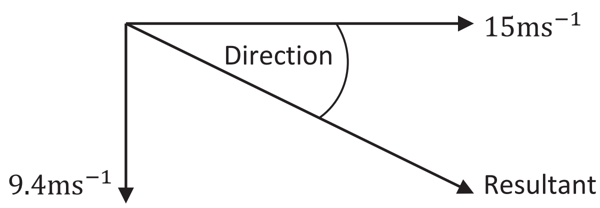
$$v_x = 24(\tfrac{3}{5})$$

$$v_x = 15\text{ms}^{-1}$$

$$v_y = 25\sin\alpha - g(3)$$

$$v_y = 25\left(\frac{4}{5}\right) - 9.8\times 3$$

$$v_y = -9.4\text{ms}^{-1}$$



The direction of the resultant velocity is given by $\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{9.4}{15}\right) = 32.07°$, to 2 d.p.

It follows that, when $t = 3$s, the direction of motion is 32.07° below the horizontal.

Enghraifft

A tennis ball is projected at 40ms^{-1} at an angle 5° below the horizontal from a height of 2.4m. The ball should pass a net 12m away (the net is 0.9m high). The ball is modelled as a particle and there is negligible air resistance.

(a) Show that at $t = 0.3s$ the ball is directly above the net.

(b) Find the clearance above the net.

(c) Find the magnitude and direction of the velocity as the ball reaches the ground.

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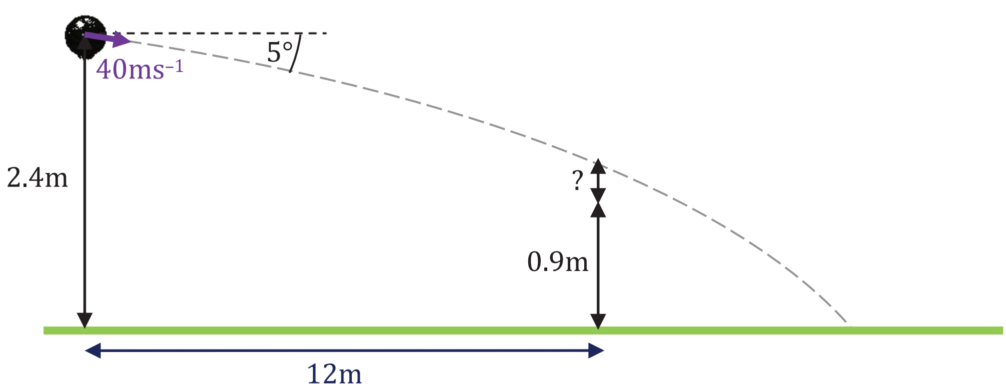
We start by sketching a diagram of the situation as described in the question.



(a) Using $S_x = (u\cos\alpha)t$ at the net, we find that $12 = 40\cos 5° \times t$. It follows that $t = \frac{12}{40\cos 5°} = 0.3011$s, to 4 d.p. Therefore at $t = 0.3s$ the ball is more or less directly above the net.

(b) *Vertically*, we have $S_y = u_y t + \frac{1}{2}at^2$.

In projectile questions, the usual convention is to take the positive direction as upwards, so we must remember here that the vertical component of the initial velocity (u_y) will be negative.

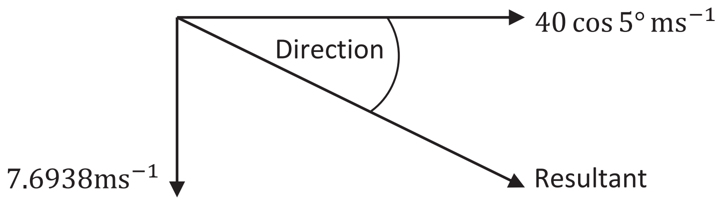
When $t = 0.3$s, we find that $S_y = (-40\sin 5°)(0.3) - \frac{1}{2}g(0.3)^2$

$S_y = -1.4869$m, to 4 d.p.

So the clearance above the net is 2.4 – 0.9 – 1.4869 = 0.0131m, to 4 d.p.

(c) We require $S_y = -2.4$m. Using $v_y^2 = u_y^2 + 2aS_y$

$$v_y^2 = (-40\sin 5°)^2 - 2 \times 9.8 \times -2.4$$
$$v_y^2 = 59.19$$
$$v_y = \pm 7.6938\text{ms}^{-1}, \text{ to 4 d.p.}$$



The resultant of these horizontal and vertical velocities is given by $\sqrt{(40\cos 5°)^2 + 7.6938^2}$ = 40.58ms^{-1}, to 2 d.p. The direction of the resultant velocity is given by $\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{7.6938}{40\cos 5°}\right) = 10.93°$, to 2 d.p. It follows that the ball travels at a velocity of 40.58ms^{-1} at an angle of 10.93° below the horizontal as it reaches the ground.